Problem Set #2

Due Date: 23:59 February 14$^{\text{th}}$, 2025

Answer each of the problems below to the best of your ability. Problem sets are to be delivered electronically via Blackboard by the posted deadline. Late submissions will not be accepted. Show all your work and state all your assumptions. If a problem requires coding or plotting, include all source code with your submission.

Problem 1 - Transport equation for the mean flow kinetic energy

a) The *mean-flow kinetic energy* per unit mass is

$$\overline{K} = \frac{1}{2}U_i U_i.$$

Starting from the Reynolds averaged momentum equation for an incompressible fluid, derive the transport equation for the mean flow kinetic energy as shown below:

$$\frac{\partial \overline{K}}{\partial t} + U_j \frac{\partial \overline{K}}{\partial x_j} = \overline{u_i' u_j'} \frac{\partial U_i}{\partial x_j} - \frac{1}{\rho}\frac{\partial U_j P}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'} U_i\right) + \nu \frac{\partial^2 \overline{K}}{\partial x_j \partial x_j} - \nu \frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j},$$

where the Reynolds decomposition of the pressure and velocity are: $p = P + p'$, $u_i = U_i + u_i'$. Note that $\overline{K}$ is the kinetic energy of the mean flow per unit mass and should not be confused with turbulent kinetic energy, i.e., $k = \frac{1}{2}\overline{u_i' u_i'}$.

Hint: you may want to begin with multiplying the Reynolds averaged momentum equation by U_i and then rearrange the terms.

b) There are seven different terms in the above equation. Name each of them.

c) Which term directly relates this equation to the turbulent kinetic energy equation? Explain the relation.

Problem 2 - Turbulent kinetic energy equation

a) Show that the turbulent kinetic energy production can be rewritten in terms of the mean strain rate tensor as:

$$\mathcal{P} = -\overline{u_i' u_j'}\,\overline{S_{ij}},$$

where the Reynolds decomposition of the strain rate tensor is: $S_{ij} = \overline{S_{ij}} + s_{ij}'$.

Hint: you might need to use the algebraic analogy that the double-dot product of a symmetric and an antisymmetric tensor is zero, e.g., $A_{ij}B_{ij} = 0$, where $A_{ij} = A_{ji}$ and $B_{ij} = -B_{ji}$.

b) Starting from the momentum equation for an incompressible fluid, derive a new form of the turbulent kinetic energy equation in terms of the mean and fluctuating components of the strain rate tensor as shown below:

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = -\overline{u_i' u_j'}\,\overline{S_{ij}} - 2\nu \overline{s_{ij}' s_{ij}'} - \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{u_j' p'} + \frac{1}{2}\overline{u_i' u_i' u_j'} - 2\nu \overline{u_i' s_{ij}'}\right] = \mathcal{P} - \epsilon - \frac{\partial J_j}{\partial x_j},$$

where $\mathcal{P}$, ϵ, and $\frac{\partial J_j}{\partial x_j}$ are the turbulent kinetic energy production, dissipation and transport terms.

Hint: you may want to begin with replacing the viscous term in the Navier-Stokes momentum equation by $2\nu \frac{\partial S_{ij}}{\partial x_j}$, and then multiply it by u_i' and proceed.

Problem 3 - Reynolds stresses and vorticity

You have been given some processed Particle Image Velocimetry (PIV) data. PIV is an optical technique that measures the velocity inside a plane or volume by correlating particle images taken in rapid succession. This particular dataset shows the flow field around a model wind turbine inside a wind tunnel subjected to a turbulent shear flow. You are given 2-D planar data for U_1, U_3, $\overline{u_1'^2}$, and $\overline{u_3'^2}$. The x_1 and x_3 coordinates are also provided. Note that the dataset is dimensional and in SI units. The diameter D of the wind turbine rotor is 0.21 m, and the velocity along the rotor centreline upstream of the turbine is 9.91 m/s, which is denoted U_0. The origin is taken as the tip of the rotor hub. Figure 1 shows the U_1 velocity field, normalized by U_0. The white rectangle masks the position of the model turbine, where no useful data was collected. The flow is travelling from left to right.

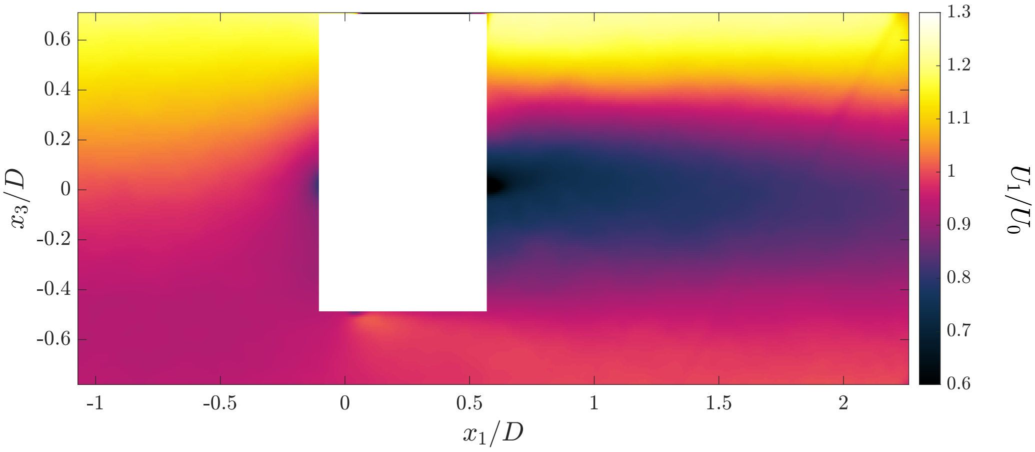

Figure 1: U_1/U_0 field

Perform the following tasks with the provided data. When plotting, keep the spatial orientation the same as Figure 1 and normalize x_1 and x_3 by D.

a) Plot the U_3/U_0 field similar to the U_1/U_0 field shown in the figure above. Note that you may need to change the colorbar limits manually to show the contrast between positive and negative U_3/U_0 values.

b) Calculate and plot the local turbulence intensity fields, i.e., u_1'/U_1 and u_3'/U_1. Note that you may need to change the colorbar limits manually.

c) You are also provided with two time-series of u_1 and u_3 for the point $[x_1/D, x_3/D] = [1, 0.3]$. Calculate the Reynolds stresses $\overline{u_1'u_1'}$, $\overline{u_3'u_3'}$, and $\overline{u_1'u_3'}$ for this point. Normalize your answers by U_0^2.

d) Extract U_1 between $-0.75 \leq x_1/D \leq -0.7$, then compute the averaged velocity profile $U_1(x_3)$ for this strip upstream of the turbine by averaging the extracted field in x_1-direction. Plot the resulting velocity profile versus x_3. Calculate the averaged shear rate $\dfrac{\partial U_1}{\partial x_3}$ of this velocity profile for $0 \leq x_3/D \leq 0.6$.

e) Calculate and plot the 2-D vorticity field $\dfrac{\partial U_1}{\partial x_3} - \dfrac{\partial U_3}{\partial x_1}$. You will need to use a numerical differentiation scheme. Is positive vorticity clockwise or counter-clockwise in this spatial orientation? Relating it back to Figure 1, is the sign of the vorticity upstream of the rotor consistent with the shear flow profile? Comment on the distribution of vorticity in the wake of the rotor.