Traffic flow: Nonlinear Scalar Conservation Laws

After Randall LeVeque: "*Finite-Volume Methods for Hyperbolic Problems*", 2002.

Consider the flow of cars on a one-lane highway. Let ρ be the density of cars; $\rho = 0$ is no cars and $\rho = 1$ is bumper to bumper.

Let U be the velocity of the cars. As a simple model we use $U = U(\rho) = u_{max}(1- \rho)$ where u_{max} is the speed limit. For simplicity, we choose u_{max} = 1m/s.

Individual car simulation "*traffic.m*"
We want to track the motion of individual cars $X_k(t)$. For driver of car k the density ρ_k is given by his own car and the one in front of him:

$$\rho_k(t) = \frac{\text{car length}}{X_{k+1} - X_k}$$

For simplicity, assume car length = 1m.

The car distribution for a given density ρ_k can then be calculated recursively as $X_{k+1} = X_k + 1/\rho_k$.

The motion of the cars now becomes a coupled set of nonlinear ordinary differential equations:

$$\frac{dX_k(t)}{dt} = U(\rho_k(t)) = U\left(\frac{1}{X_{k+1} - X_k}\right) = u_{\max}\left(1 - \frac{1}{X_{k+1} - X_k}\right)$$

Conservation law "*burger_FTCS.m*" and "*burger_Upwind.m*"
The nonlinear conservation law for car traffic can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial\left(\rho \cdot U(\rho)\right)}{\partial x} = 0$$

where $\rho \cdot U(\rho)$ is the nonlinear flux funcion f. This is often called the LWR model for traffic flow, after Lighthill & Whitham and Richards.

The FTCS-scheme becomes:

$$\frac{\rho_k^{n+1} - \rho_k^n}{\Delta t} + \frac{\rho_{k+1}^n u_{\max}\left(1 - \rho_{k+1}^n\right) - \rho_{k-1}^n u_{\max}\left(1 - \rho_{k-1}^n\right)}{2\Delta x} = 0$$

which is of course unconditionally unstable.

For the upwind scheme we need something (a transport velocity) in front of the d/dx term, so we use the chain rule:

$$\frac{\partial}{\partial x}\left(f\left(\rho\right)\right)=\frac{\partial f}{\partial \rho}\cdot\frac{\partial \rho}{\partial x}=u_{\max}\left(1-2\rho\right)\frac{\partial \rho}{\partial x}$$

The first order upwind scheme then becomes:

`if` $u_{\max}(1-2\rho_k)>0$

$$\frac{\rho_k^{n+1}-\rho_k^n}{\Delta t}+u_{\max}(1-2\rho_k^n)\frac{\rho_k^n-\rho_{k-1}^n}{\Delta x}=0$$

`else`

$$\frac{\rho_k^{n+1}-\rho_k^n}{\Delta t}+u_{\max}(1-2\rho_k^n)\frac{\rho_{k+1}^n-\rho_k^n}{\Delta x}=0$$

`end`

The solutions given by LeVeque:

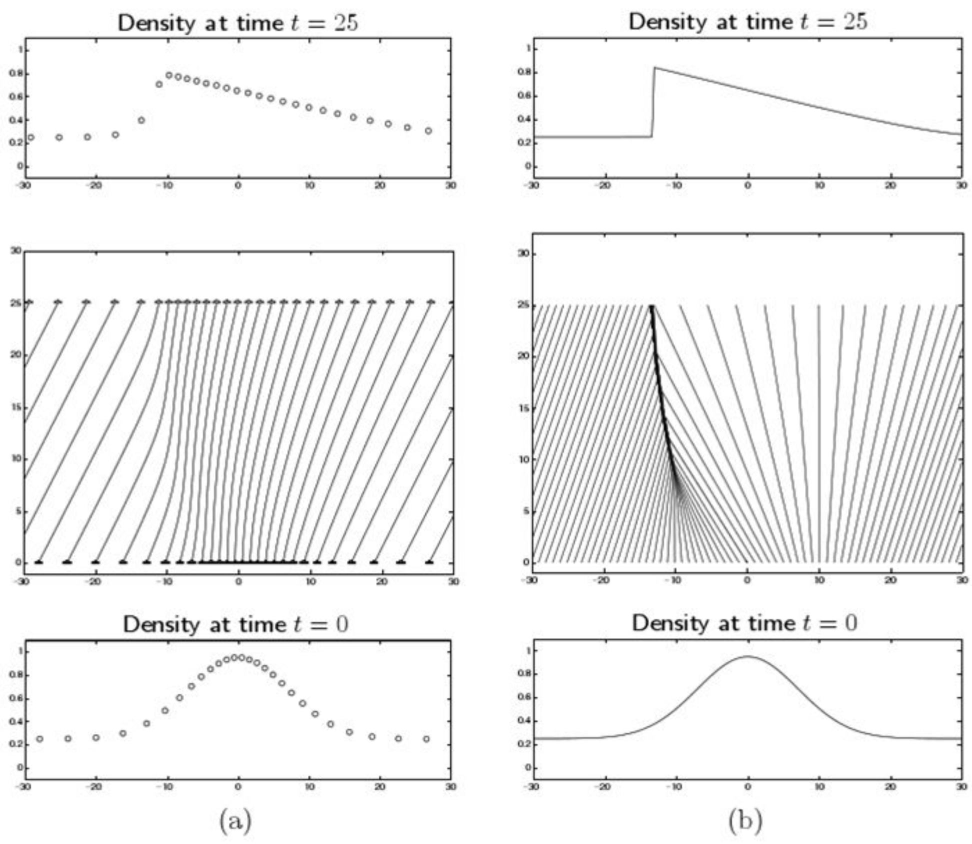

Fig. 11.1. Solution to the traffic flow model starting with a bulge in the density. (a) Trajectories of individual vehicles and the density observed by each driver. (b) The characteristic structure is shown along with the density as computed by CLAWPACK. [claw/book/chap11/congestion]

The solutions as given by ”*traffic.m*” and ”*burger_Upwind.m*”: