

The Three Utilities Problem

Markus Hoehn



Connect each house to each utility without crossing lines.

Dialogue:

Friend: Connect each house to each utility without crossing lines.

You: It's impossible.

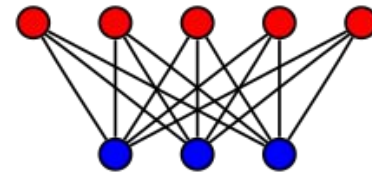
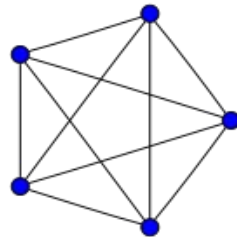
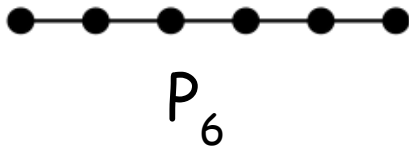
Friend: Nuh-uh.

You: It literally is. Try it.

Friend: Nuh-uh.

You: I shall prove it!

Whenever objects have a notion of connection, you have a graph. We label these objects as 'vertices' and their connections as 'edges'.

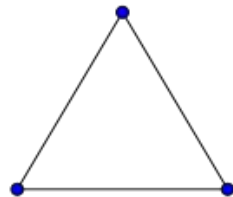


K_5

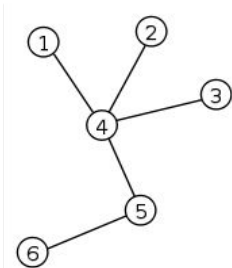
$K_{5,3}$



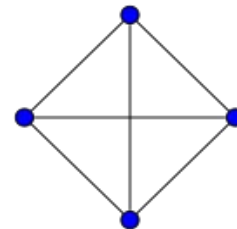
2-regular graph



K_3



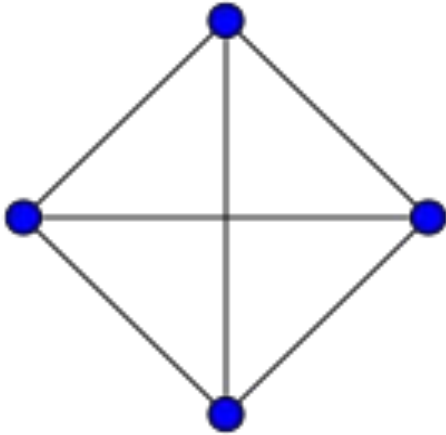
Tree



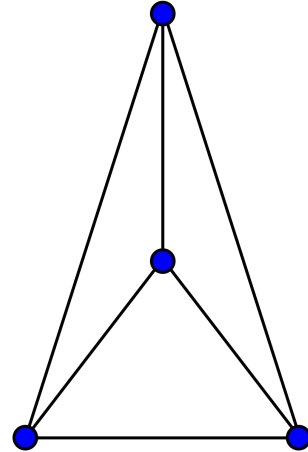
K_4

A planar graph is a graph that can be drawn on the plane in such a way that no edges cross each other.

Planar?



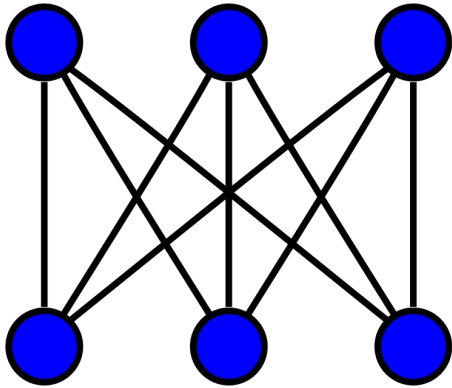
Yes!



Since there exists a planar representation, we say K_4 is a planar graph.

Recognize that the proving the unsolvability of three utilities problem is the same as proving that is $K_{3,3}$ nonplanar.

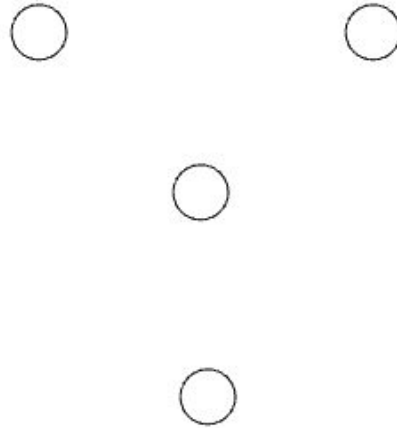
Planar?



Hopefully the answer is "no," or our Friend has made us look like fools.

Euler's Formula for Planar Graphs shown heuristically by constructing K_4 (or any planar graph in general):

Plot a "shell" of the nodes of the graph.



Observe that a new edge results in a new vertex or a new region.

(vertices + regions = edges + 2 (accounting for the initial vertex and outside region))

Properties of our utility graph $K_{3,3}$

- 6 vertices.
- 9 edges.
- If a planar representation exists it must have 5 regions, by Euler's formula (vertices - edges + regions = 2).
- Each region's boundary contains at least 4 edges.
- Since each of the 5 regions has at least 4 edges, we count a minimum of 20 edges. However each edge touches two regions, leading to double counting. Thus we have at least 10 edges in a planar representation.
- However, this contradicts the fact that the graph has only 9 edges, therefore a planar representation does not exist.



Dialogue:

You: See, Friend, it's absolutely, positively, unequivocally impossible!

Friend: Nuh-uh.

You: Are you kidding me? I just demonstrated beyond a shadow of a doubt that a planar representation of $K_{3,3}$ is utterly unattainable!

Friend: And?

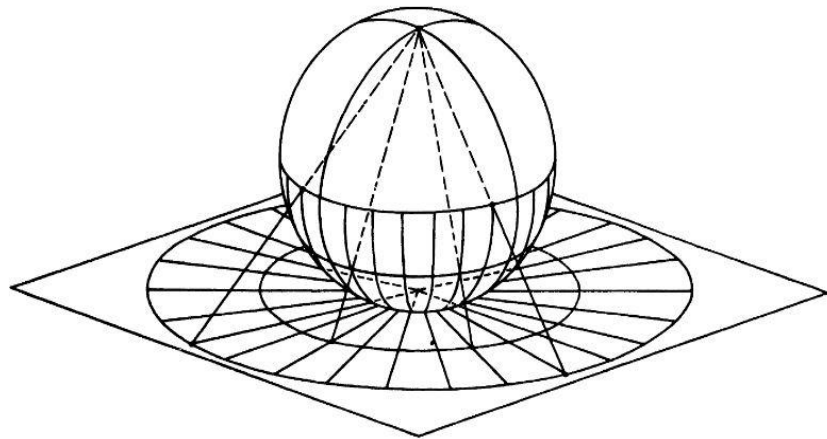
You: That's the same as our problem!

Friend: No.

You: Are you seriously suggesting that it's because we reside on a spherical surface? A stenographic projection absolutely destroys that argument. If it's possible on a sphere, it's possible on a plane, which it is not. Thus, it is undeniably impossible on a sphere!

Friend: Mug.

You: What?



Dialogue:

Don: Pourquoi, Monsieur, vous ne pouvez pas aller à l'école?

Alfred: Non, non.

Don: Pourquoi ne pouvez-vous pas aller à l'école? Vous n'avez pas de maison? Vous n'avez pas de feu? Vous n'avez pas d'électricité? Vous n'avez pas d'eau?

Don: Pourquoi ne pouvez-vous pas aller à l'école?

Alfred: Non.

Don: Pourquoi ne pouvez-vous pas aller à l'école? Vous n'avez pas de maison? Vous n'avez pas de feu? Vous n'avez pas d'électricité? Vous n'avez pas d'eau?

Alfred: Non.

Don: Pourquoi?

