Lecture 15:

Language Models and Transformer

Markus Hohle

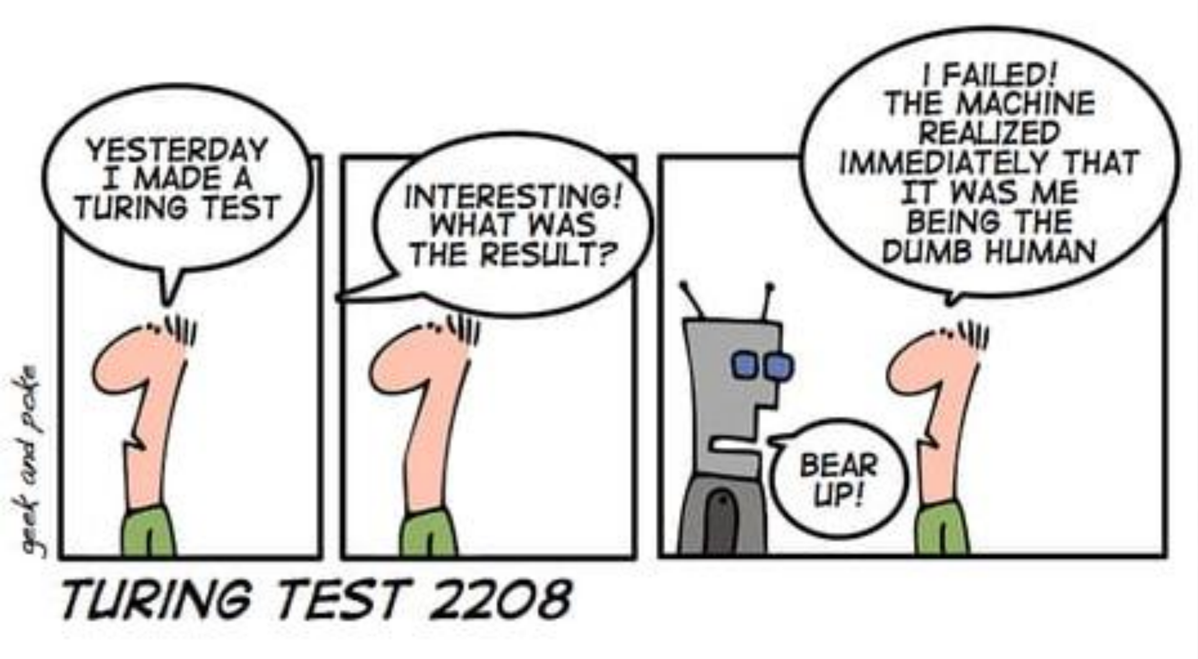
University California, Berkeley

Machine Learning Algorithms

MSSE 277B, 3 Units

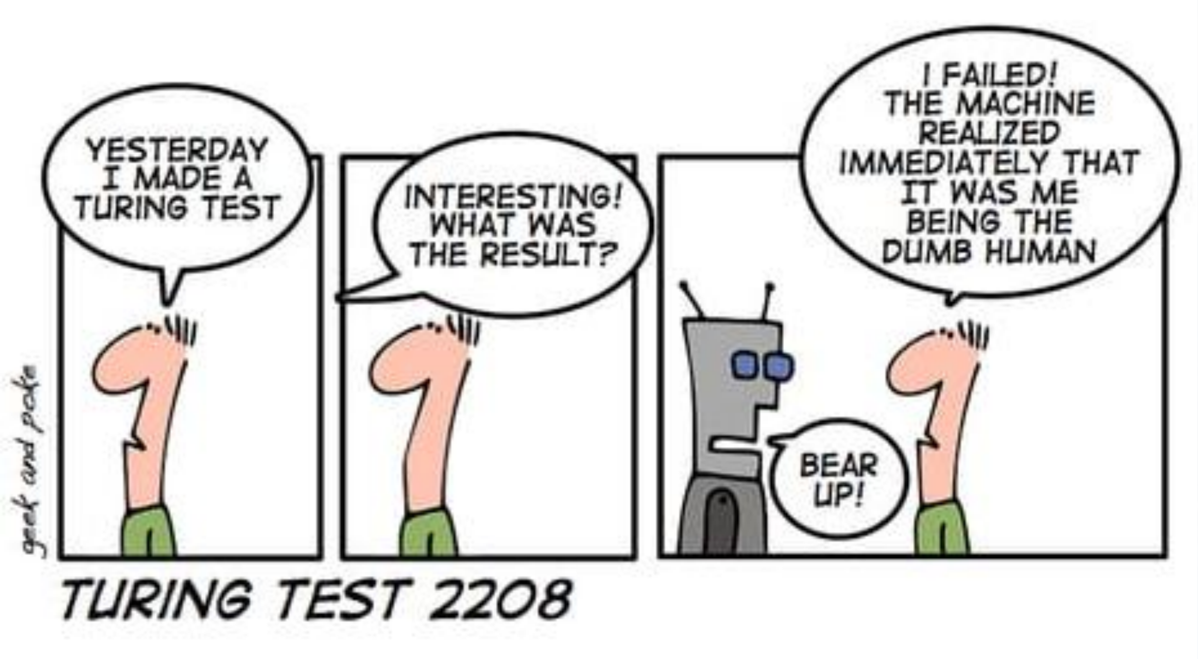
Spring 2025

Outline



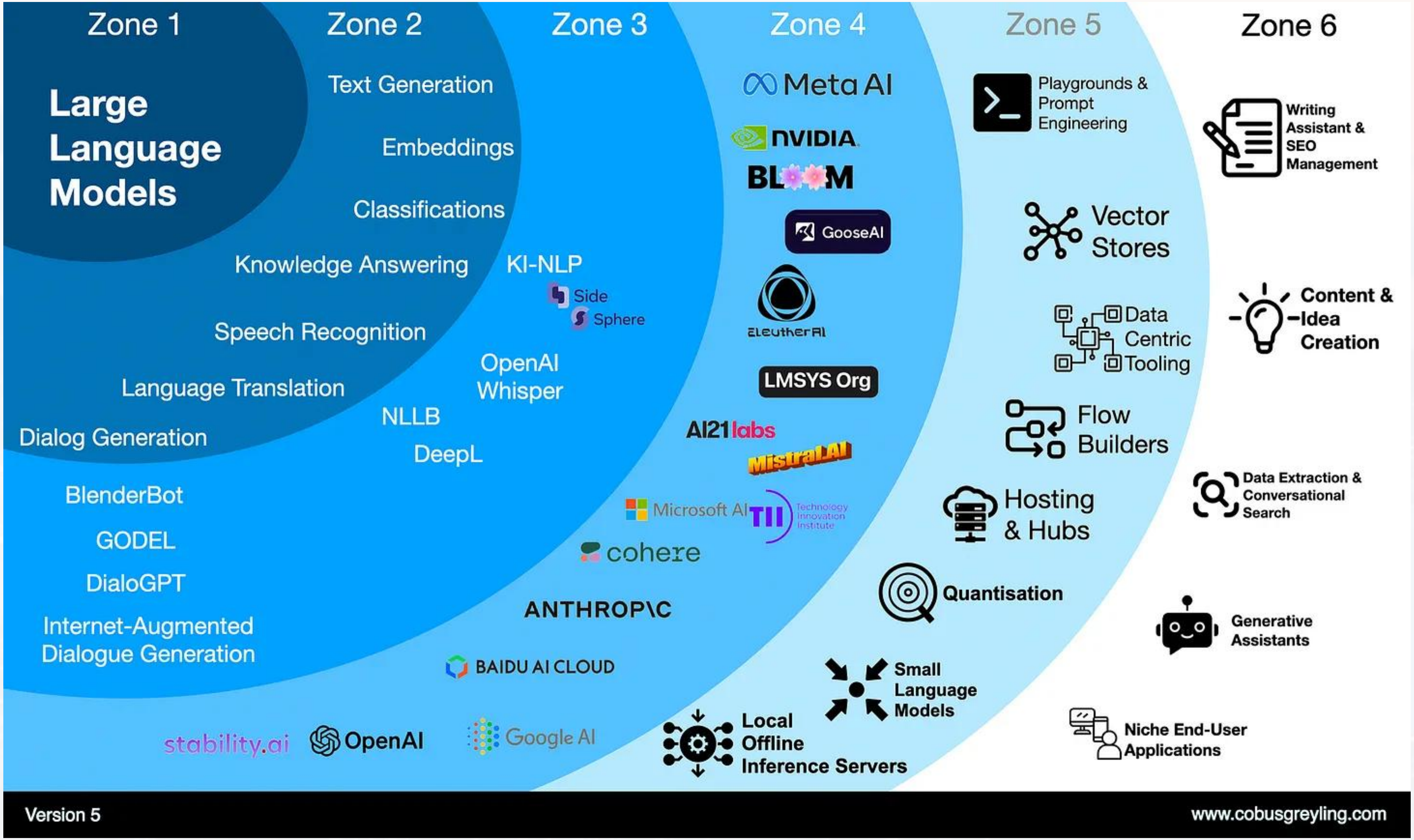
- **Introduction**
- **Bigram and MAP**
- **Positional Encoding**
- **Word Embedding**
- **Attention**
- **Transformer Architecture**

Outline



- **Introduction**
- Bigram and MAP
- Positional Encoding
- Word Embedding
- Attention
- Transformer Architecture

LLM & Transformer: Introduction



LLM & Transformer:

corpus: (large, representative) data set containing sequences of a language

token: individual, independent entity of a language

alphabet/vocabulary: set of tokens

token	**size of *alphabet***
- letters in a word	- 10^2
- words in a sentence	- $10^4 \dots 10^6$
(**upper/lower case, cases, gender, tenses, conjugations**)	
- amino acids in a protein sequence	- 21
- nucleotides in a DNA/RNA sequence	- 4
- motifs in a DNA/RNA sequence	- 10^4

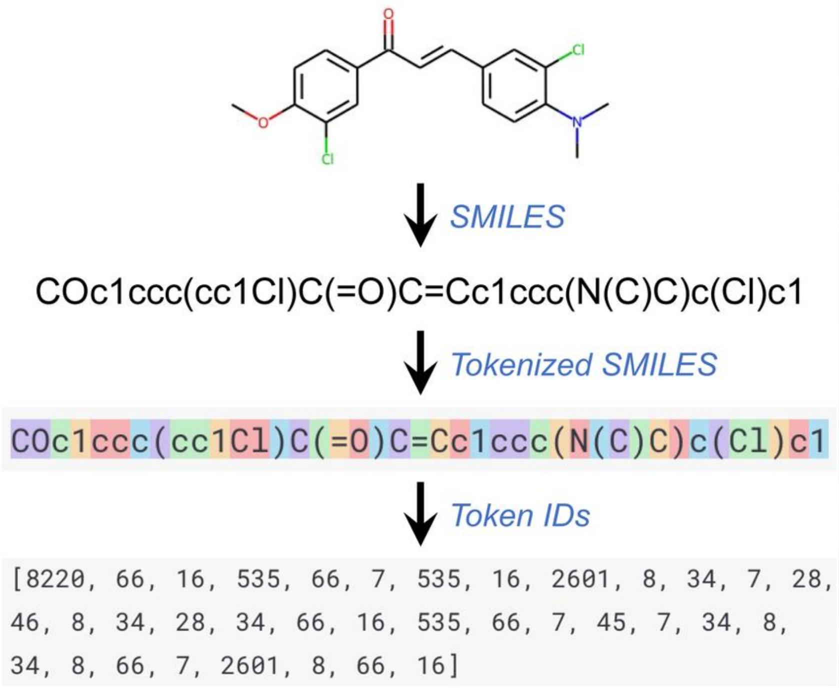
LLM & Transformer:

corpus: (large, representative) data set containing sequences of a language

token: individual, independent entity of a language

alphabet/vocabulary: set of tokens

tokenization



DOI:10.1039/D3SC04610A

token:
- single atom vs…
- …functional group

LLM & Transformer:

corpus: (large, representative) data set containing sequences of a language

token: individual, independent entity of a language

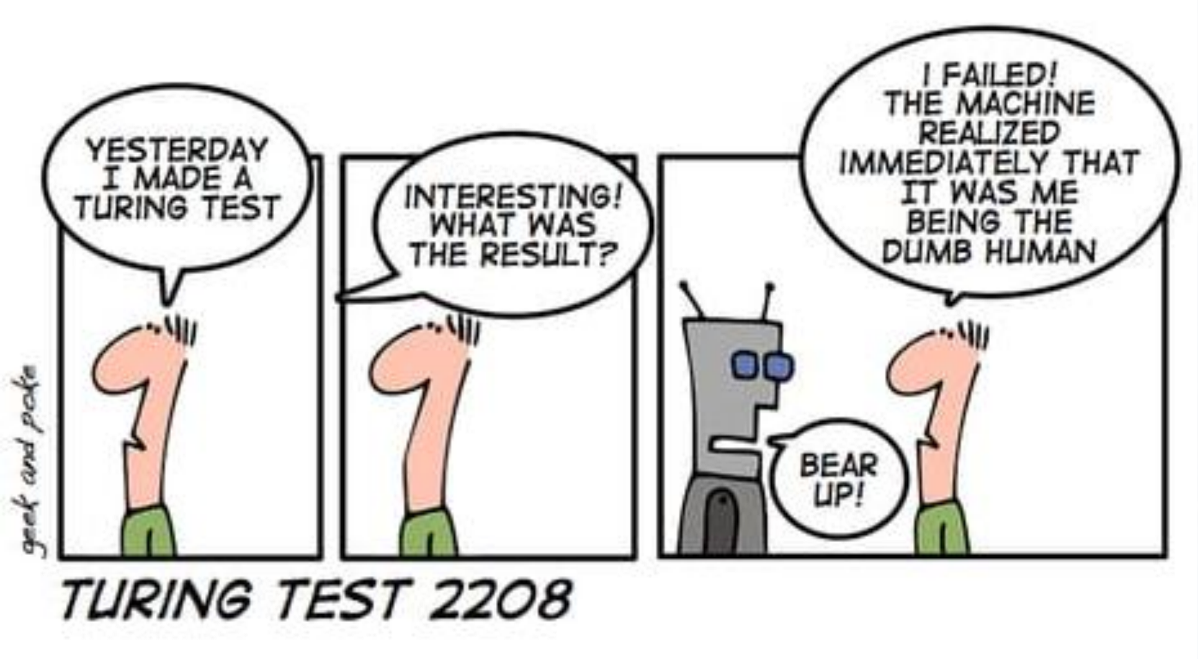
alphabet/vocabulary: set of tokens

note: language models don’t know grammar as we do, but they don’t need to anyway…

three things make context (details: see later**):**

- **word embedding** (relation between similar/different token)
- **positional encoding** (location of token in a sequence)
- **attention** (relation between token within a sequence)

Outline



- Introduction
- **Bigram and MAP**
- Positional Encoding
- Word Embedding
- Attention
- Transformer Architecture

LLM & Transformer: Bigram & MAP

$X_1\ X_2\ X_3\ X_4\ X_5\ \ldots .\ X_n$ sequence of *n* **token** *X*

actually:

$$P(X_1\ X_2\ X_3\ X_4\ X_5\ \ldots .\ X_n) = P(X_n|X_{n-1} \ldots X_1)P(X_{n-1}|X_{n-2} \ldots X_1) \ldots P(X_1)$$

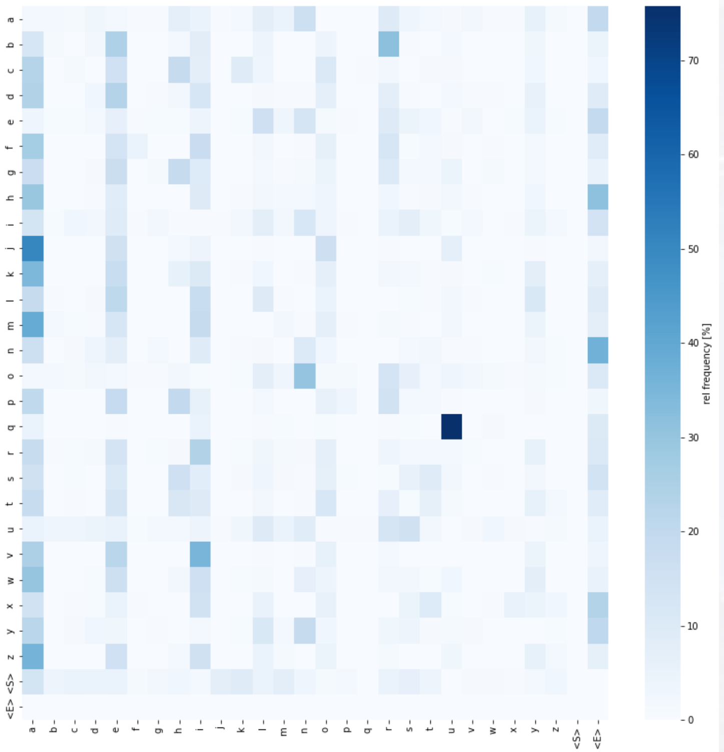
bigram (1st order Markov Chain, see e.g. first WhatsApp versions):

$$P(X_1\ X_2\ X_3\ X_4\ X_5\ \ldots .\ X_n) = P(X_n|X_{n-1})P(X_{n-1}|X_{n-2}) \ldots P(X_1)$$

P(i|j): that token ***i*** is generated after token ***j***

→ N x N transition matrix from frequencies

→ "bigram" = "two words"



frequency matrix of letters in common names

bigram (1st order Markov Chain):

let's build our own bigram model: **generate new names** based on a corpus of names

```
In [15]: words[0:12]
Out[15]:
['emma',
 'olivia',
 'ava',
 'isabella',
 'sophia',
 'charlotte',
 'mia',
 'amelia',
 'harper',
 'evelyn',
 'abigail',
 'emily']
```

see **Andrej Karpathy's GitHub** repository

$$P(X_1\, X_2\, X_3\, X_4\, X_5 \,\ldots .X_n) = \boxed{P(X_n|X_{n-1})P(X_{n-1}|X_{n-2})} \ldots \boxed{P(X_1)}$$

we only need to count **how often** a letter is followed by another

we also need to indicate when a name has **started** and **ended**

`['<S>'] + ['olivia'] + ['<E>']`

→ **alphabet:** 26 letters + the two special "letters"

let's create a dictionary first (will help for counting):

bigram (1st order Markov Chain):

let's build our own bigram model: **generate new names** based on a corpus of names

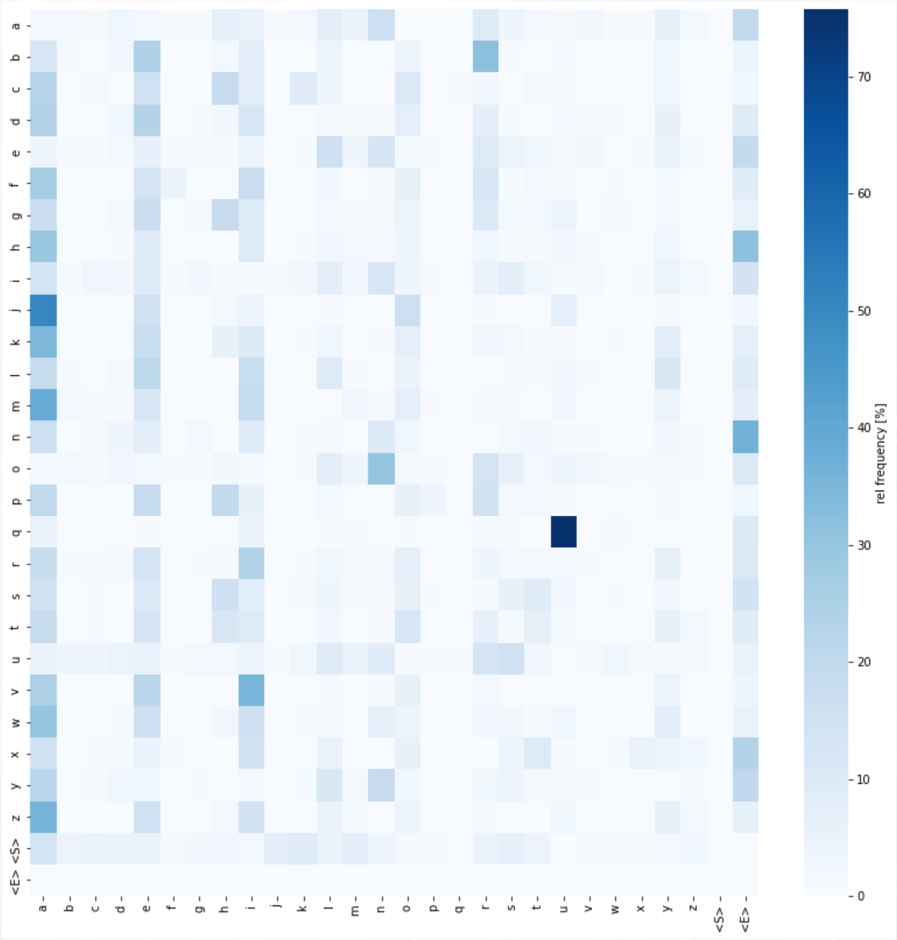
1) dictionary first (will help for counting)
2) count how often letter ***i*** is followed by letter ***j***

   → bigram matrix ***N***

3) normalize ***N*** accordingly
4) begin with a start token
5) draw a letter randomly based on ***N***, using

```
np.argmax(np.random.multinomial(1,p))
```

6) if next token is stop token → stop



bigram (1st order Markov Chain):

let's build our own bigram model: **generate new names** based on a corpus of names

check out **`Bigram.ipynb`**

`B.SampleNames(15)` vs totally random **`B.SampleNames(15, False)`**

some names are gibberish

some names sound real

some names **are** real

```
In [295]: B.SampleNames(15)
keesa
ann
ja
jon
nma
malynojana
sall
daha
drvah
lzaxi
tyunusthun
jorrwro
ja
asoow
s
```

```
In [296]: B.SampleNames(15,False)

mtkgy
yufexhviovmorhqvikbbbjxebpxwurejaqlzzuwuanxmmomhr<S>uhb
xlmusadjfdzxadaotd
ik<S>vdtydvxevtaselkykcfbamceprtvl
zyr<S>inzoerobzwuovx
eg<S>pbdvikf<S>tomcnkfsjay<S>rikatnaykizszcivpds<S>zj
kh<S>y<S>ualzugqgakakeubjbasc
bblupnibtqmyl<S>vyobf
kybs
rznjgpmlo
tnhoxuckkjjzbwmj<S>vshkycicf<S>kowskphy
rxodh
jvswmzw
jzpcfnpcg
```

Note, there is no conceptual difference between applying our model to *letters in a word* vs *words in a sentence*

<u>caveats:</u>

- the bigram model derives $P(X_n)$ from **observed** frequencies
  → essentially **MLE** (problematic if a letter hasn't appeared in the sequence yet
  → $P(X_n)$ assumed to be zero!)

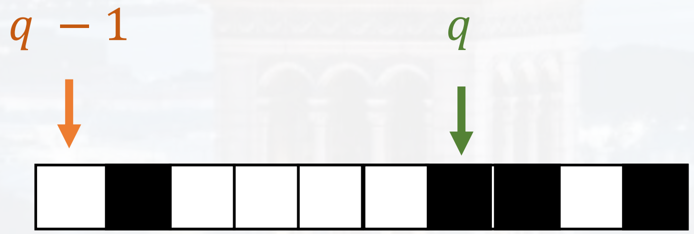
```
Nsam = N/np.sum(N+0.0001, axis = 1, keepdims = True)
S_bi += np.sum(-N[:,i]*np.log(N[:,i]+1e-16))
```

- can we implement something that is closer to:

$$P(X_1\, X_2\, X_3\, X_4\, X_5 \ldots . X_n) = P(X_n|X_{n-1} \ldots X_1)P(X_{n-1}|X_{n-2} \ldots X_1) \ldots P(X_1) \quad ?$$

binomial process

$$P(k|n,q) = \binom{n}{k} q^k (1-q)^{n-k}$$

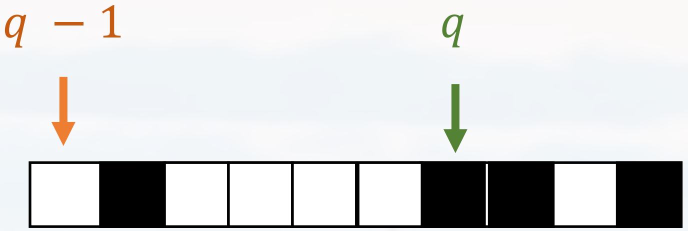


$$q = ?$$

LLM & Transformer: Bigram & MAP

Bayesian **P**arameter **E**stimation*)

$$P(X_1\, X_2\, X_3\, X_4\, X_5 \ldots .\, X_n) = P(X_n|X_{n-1} \ldots X_1)P(X_{n-1}|X_{n-2} \ldots X_1) \ldots P(X_1)$$



$$P(k|n,q) = \binom{n}{k} q^k (1-q)^{n-k} \qquad q = ?$$

likelihood function **(here: binomial)**

$$P(q|data\ set) = \frac{\boxed{P(data\ set|q)}\boxed{P(q)}}{\boxed{P(data\ set)}}$$

prior **(~$P(X_n|X_{n-1} \ldots X_1)$)**

evidence **(const wrt q)**

$$= \frac{1}{\int_0^1 P(q|data\ set)dq}(1-q)^{n-k}q^k$$

$q = const$
before 1st data point
(max entropy!)

$$= \frac{q^{k+\alpha-1}(1-q)^{n-k+\beta-1}}{\int_0^1 q^{k+\alpha-1}(1-q)^{n-k+\beta-1}\,dq}$$

q = conjugate prior
after nth data point

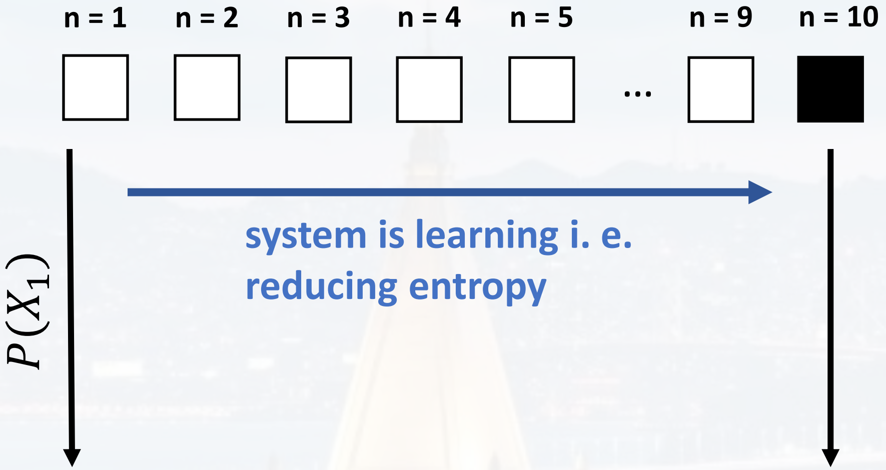
*) see lecture 2

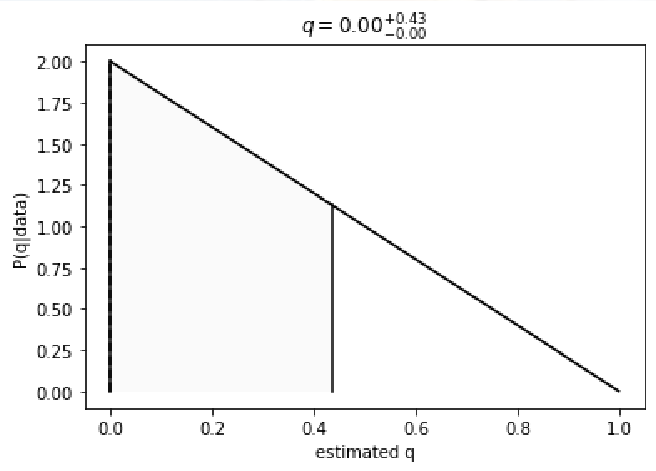
LLM & Transformer: Bigram & MAP

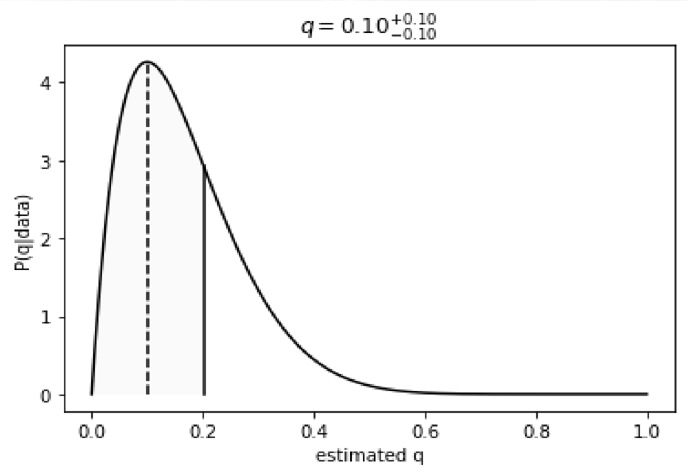
$$P(X_1\, X_2\, X_3\, X_4\, X_5 \dots . X_n) = P(X_n|X_{n-1} \dots X_1)P(X_{n-1}|X_{n-2} \dots X_1) \dots P(X_1)$$

Bayesian **P**arameter **E**stimation*)

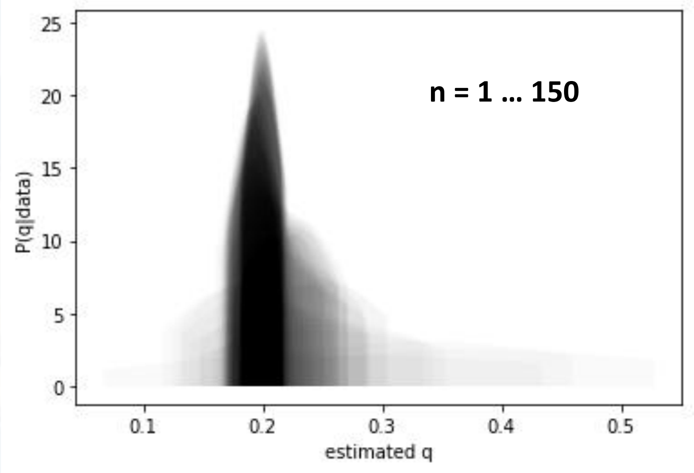
$$P(k|n,q) = \binom{n}{k} q^k (1-q)^{n-k} \qquad q = ?$$







$P(X_1\, X_2\, X_3\, X_4\, X_5 \dots . X_n)$



*) see lecture 2

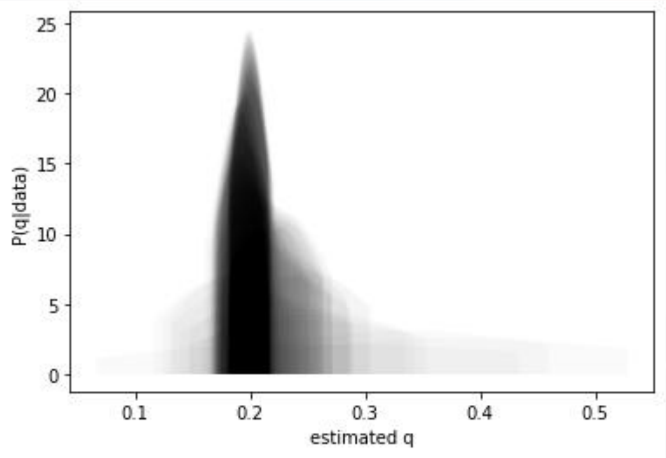
Bayesian **P**arameter **E**stimation*)

$$P(X_1\, X_2\, X_3\, X_4\, X_5 \ldots .\, X_n) = P(X_n|X_{n-1} \ldots X_1)P(X_{n-1}|X_{n-2} \ldots X_1) \ldots P(X_1)$$

$$P(k|n,q) = \binom{n}{k} q^k (1-q)^{n-k}$$

$$P(q|data\ set) = \frac{q^{k+\alpha-1}(1-q)^{n-k+\beta-1}}{\boxed{\int_0^1 q^{k+\alpha-1}(1-q)^{n-k+\beta-1}\, dq}}$$

Beta function



more general, we want to learn the probability $P_j(a)$ of letter a at position j $\quad q \;\rightarrow P_j(a)$

→ multinomial problem

→ conjugate prior is the **Dirichlet distribution**

$$P(sequence) \sim \prod_j \boxed{\prod_a P(a)_j^{\alpha(a)-1}}$$

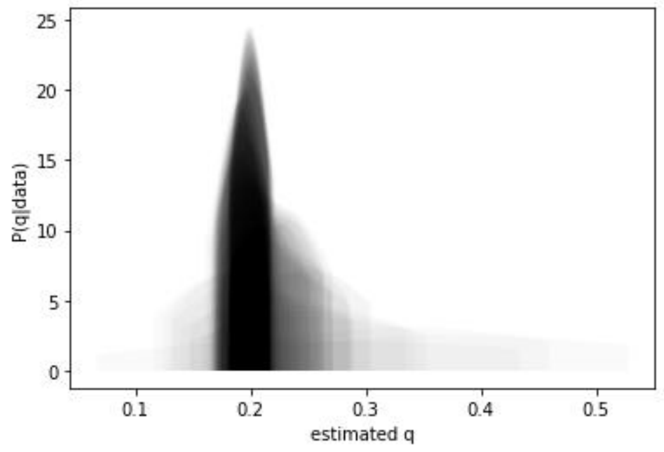
equivalent to what was $P(q|data\ set)$ earlier

***) see lecture 2**

$$P(X_1\, X_2\, X_3\, X_4\, X_5 \,\ldots .\, X_n) = P(X_n | X_{n-1} \ldots X_1) P(X_{n-1} | X_{n-2} \ldots X_1) \ldots P(X_1)$$

Dirichlet distribution

$$P(sequence) \sim \prod_j \prod_a P(a)_j^{\alpha(a)-1}$$



note: $\sum_{over\ all\ a} P(a)_j = 1$ → N – dim simplex

note:
- we don't need to extract $P(a)$ from the maximum of the pdf given by the BPE posterior

- we can directly derive the maximum of $P(a)$ from $P(sequence)$ given the constrain $\sum_{over\ all\ a} P(a)_j = 1$ **(Lagrangian multipliers)**

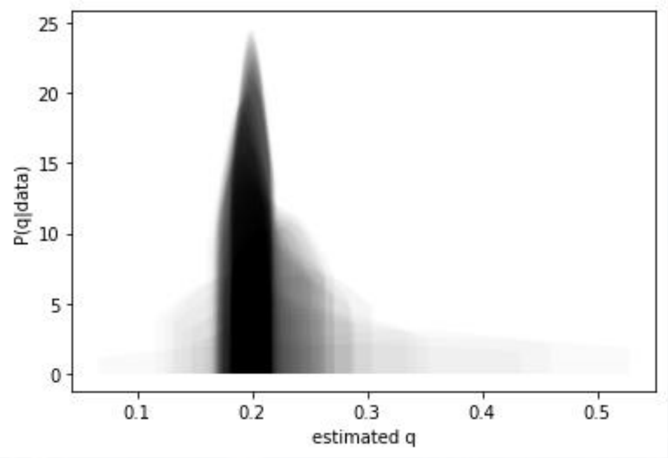
- **Maximum a-posteriori (MAP)** approach → see XXmotif (Siebert & Soeding, 2016)

LLM & Transformer:

$$P(X_1\ X_2\ X_3\ X_4\ X_5\ \ldots.X_n) = P(X_n|X_{n-1}\ldots X_1)P(X_{n-1}|X_{n-2}\ldots X_1)\ldots P(X_1)$$

Dirichlet distribution

$$P(sequence) \sim \prod_j \prod_a P(a)_j^{\alpha(a)-1}$$



note: $$\sum_{over\ all\ a} P(a)_j = 1$$ → N – dim simplex

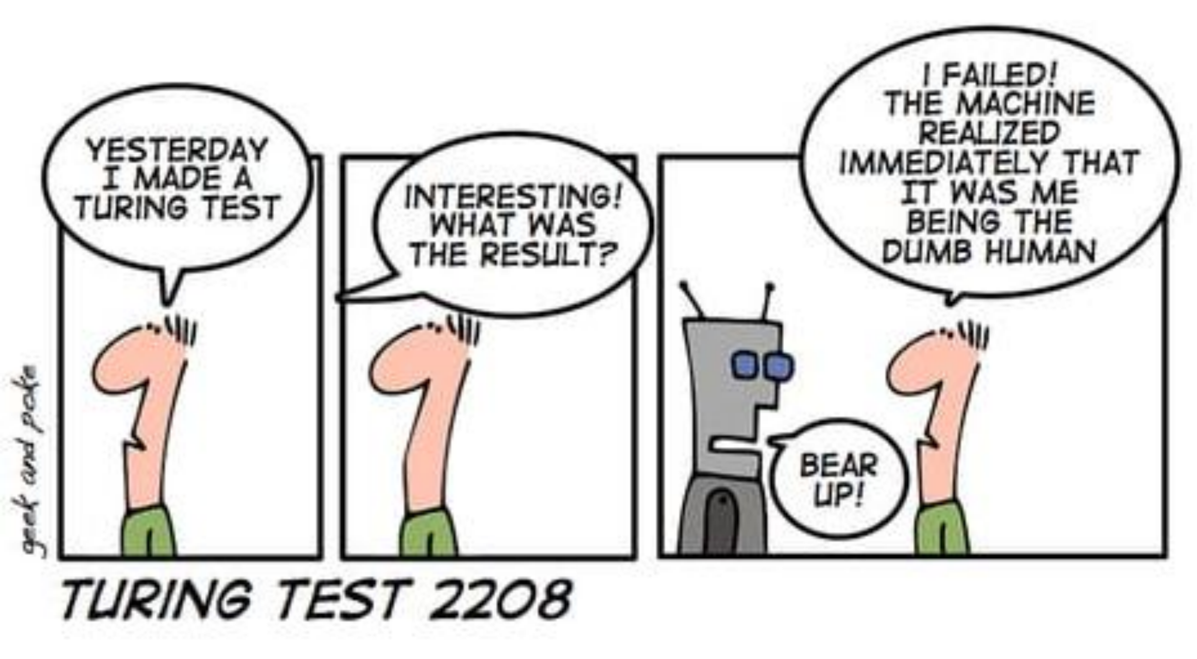
Maximum a-posteriori (MAP)

- XXmotif (Siebert & Soeding, 2016) significantly outperformed PWMs

- it struggled however with related motifs which where **physically located far apart** from each other

→ solution see later: attention
→ older solutions: LSTMs

Outline



- Introduction
- Bigram and MAP
- **Positional Encoding**
- Word Embedding
- Attention
- Transformer Architecture

three things make context:

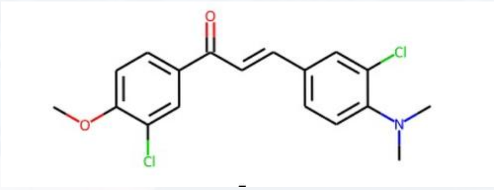
- **positional encoding** (location of token in a sequence)
- **word embedding** (relation between similar/different token)
- **attention** (relation between token within a sequence)

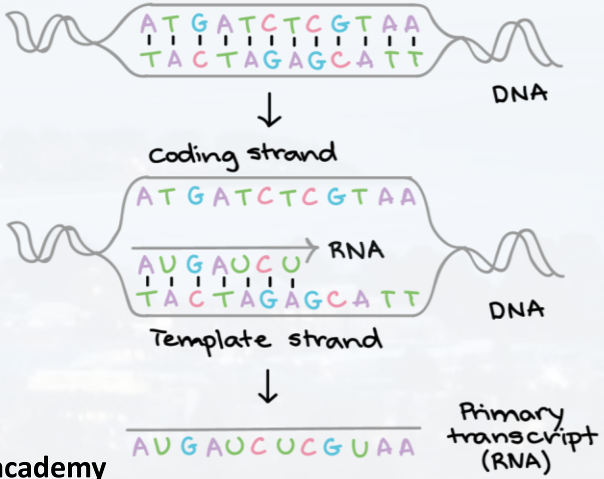
"The cat jumped on the roof."

order matters!:

1st: article
2nd: noun/subject
3rd: verb
4th : noun/object (in English)

→ **positional encoding**





Khan academy

LLM & Transformer: Positional Encoding

goal: find a positional encoding that is

- reasonably simple
- independent from the length of the sequence
- somehow normalized

one idea: n-bit binary encoding

position	code
	76543210 ← 8bit i.e. eight dimensions
1	00000001
2	00000010
3	00000011
4	00000100
5	00000101
6	00000110
7	00000111
8	00001000
9	00001001
10	00001010
11	00001011
12	00001100
13	00001101
14	00001110
15	00001111
16	00010000

depending on dimensions (bit)

→ different frequencies

bit	**frequency**
0	1/2
1	1/4
2	1/8
…	

Does that look familiar?
→ *like* Fourier Series

LLM & Transformer: Positional Encoding

even dimensions: $E(p, 2k) = \sin\left(\frac{p}{10{,}000^{\frac{2k}{d}}}\right)$

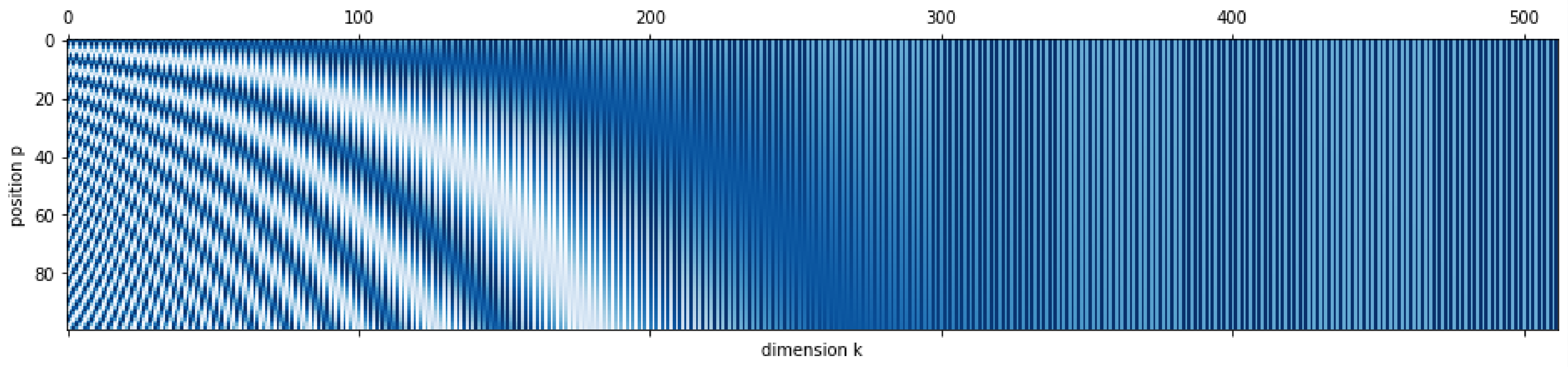
odd dimensions: $E(p, 2k+1) = \cos\left(\frac{p}{10{,}000^{2k/d}}\right)$

p: position in sequence
k: dimension index
d: number of dimensions
10,000: an arbitrary number

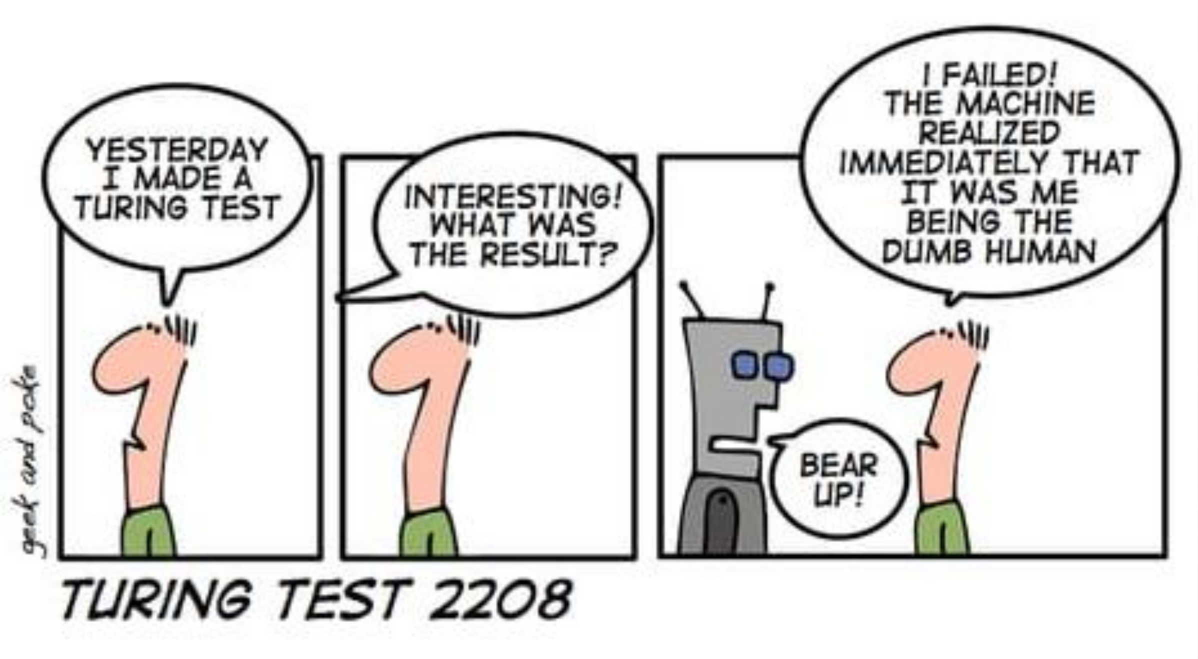
Vaswani et al., 2017

run **`PlotPositionEncoding.py`**

more info here



Outline



- Introduction
- Bigram and MAP
- Positional Encoding
- **Word Embedding**
- Attention
- Transformer Architecture

three things make context:

- **positional encoding** (location of token in a sequence)
- **word embedding** (relation between similar/different token)
- **attention** (relation between token within a sequence)

problem: turning token (words/letters) into numbers

<u>single letters:</u>

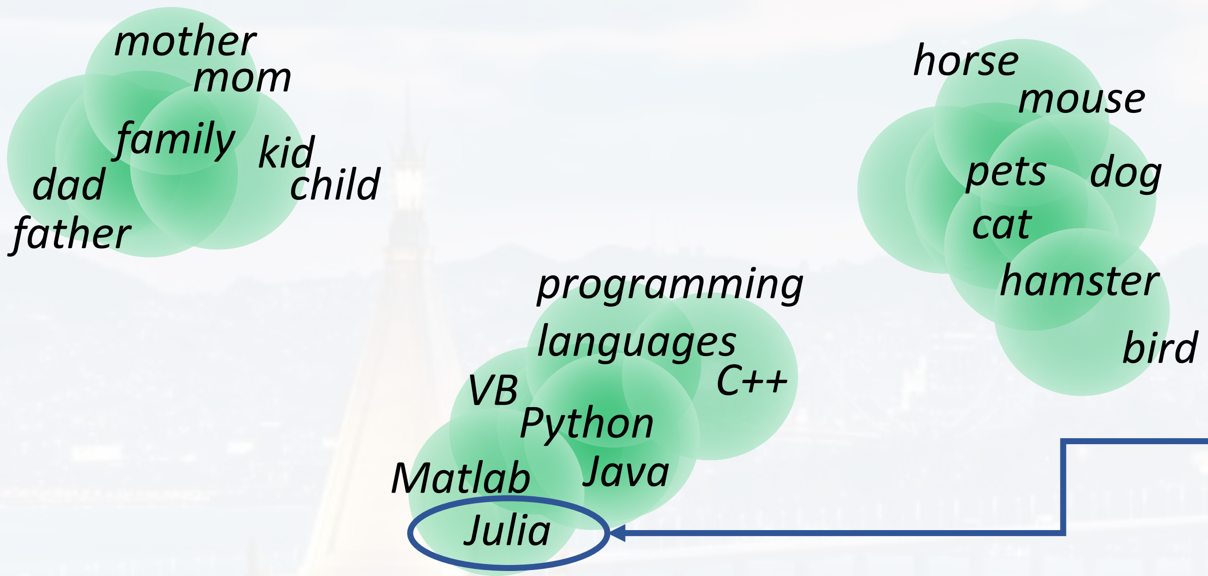
ACGT - one – hot works perfectly (four different token)
abcd… - lower/upper case, special characters (50 different token), one – hot is fine too

<u>words:</u>

actual words - 10^4 … 10^6 (upper/lower case, cases, gender, tenses, conjugations)
→ one – hot doesn't work (matrices would be too large)
→ some words have a **similar meaning,** should be **close** in parameter space (**cluster**)

LLM & Transformer:

words with a **similar meaning** should form **cluster**

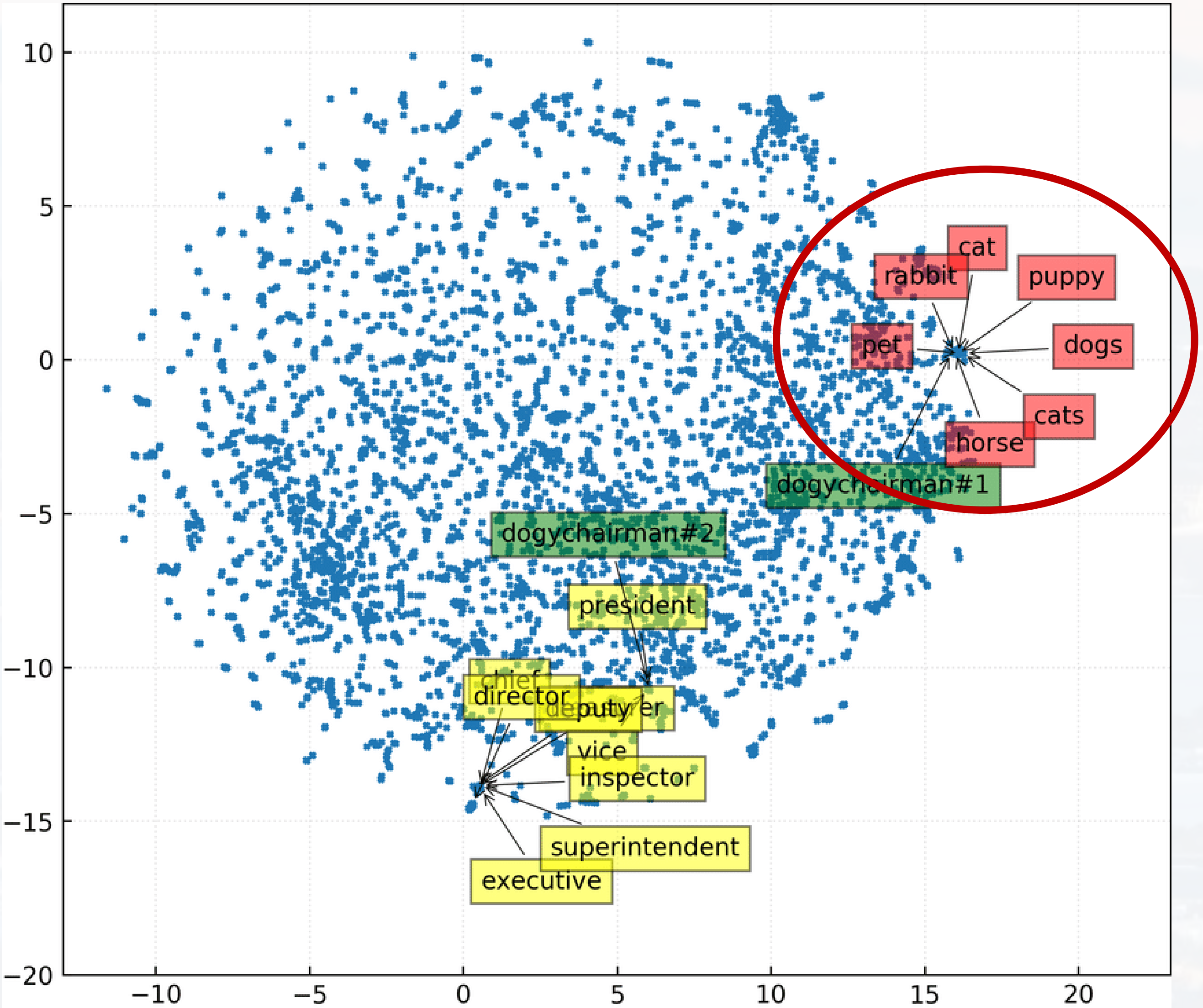


could be a name too
→ *context matters!*

- embedding, instead of one – hot encoding
- from experience: **N = 30 – 300** dim vector for each token **(which is a lot less than 10^4 … 10^6)** is sufficient
- as a result: token with **similar meaning are close** in the vector space!

LLM & Transformer:

words with a **similar meaning** should form **cluster**



"Joint Learning of Sense and Word Embeddings"
M Alsuhaibani & D Bollegala

common training set: recorded speeches from the European Parliament:

…It seems absolutely disgraceful that we pass legislation and do not adhere to it ourselves. Mrs Lynne, you are quite right and I shall check whether this has actually not been done. I shall also refer the matter to the College of Quaestors, and I am certain that they will be keen to ensure that we comply with the regulations we ourselves vote on.
Madam President, Mrs Díez González and I had tabled questions on certain opinions of the Vice-President, Mrs de Palacio, which appeared in a Spanish newspaper.
The competent services have not included them in the agenda on the grounds that they had been answered in a previous part-session.
I would ask that they reconsider, since this is not the case….

words of similar meaning should appear in similar environment

→ *target token within a window*

Two common algorithms are ***C****ontinuous* ***B****ag* ***O****f* ***W****ords* and ***skip gram***

Continuous Bag Of Words

n:	**number of unique token from corpus**
m:	**desired number of dimensions for embedding**

The competent services have not included them in the agenda on the grounds that they…

target: *have*

context window: *The competent services not included them*

one – hot encoding

one – hot encoding

target vector of length *n*

shallow ANN

input layer of shape *n x m*

hidden layer of shape *m x n*

softmax

output vector of length *n*

learning weights such that both vectors **are identical**
→ **weights*[:,i]*** of the **hidden layer** make up the embedding vector of ***token i***

Continuous Bag Of Words

n:	**number of unique token from corpus**
m:	**desired number of dimensions for embedding**

The competent services have not included them in the agenda on the grounds that they…

target — *not*

context window — *competent services have included them in*

one – hot encoding

one – hot encoding

shallow ANN

input layer of shape *n x m*

hidden layer of shape *m x n*

softmax

output vector of length *n*

target vector of length *n*

learning weights such that both vectors **are identical**

→ **weights*[:,i]*** of the **hidden layer** make up the embedding vector of ***token i***

LLM & Transformer: Word Embedding

*C*ontinuous *B*ag *O*f *W*ords

n:	**number of unique token from corpus**
m:	**desired number of dimensions for embedding**

The competent services have not included them in the agenda on the grounds that they…

target: *included*

context window: *services have not* *them in the*

one – hot encoding

shallow ANN:

- input layer of shape n x m
- hidden layer of shape m x n
- softmax

output vector of length n

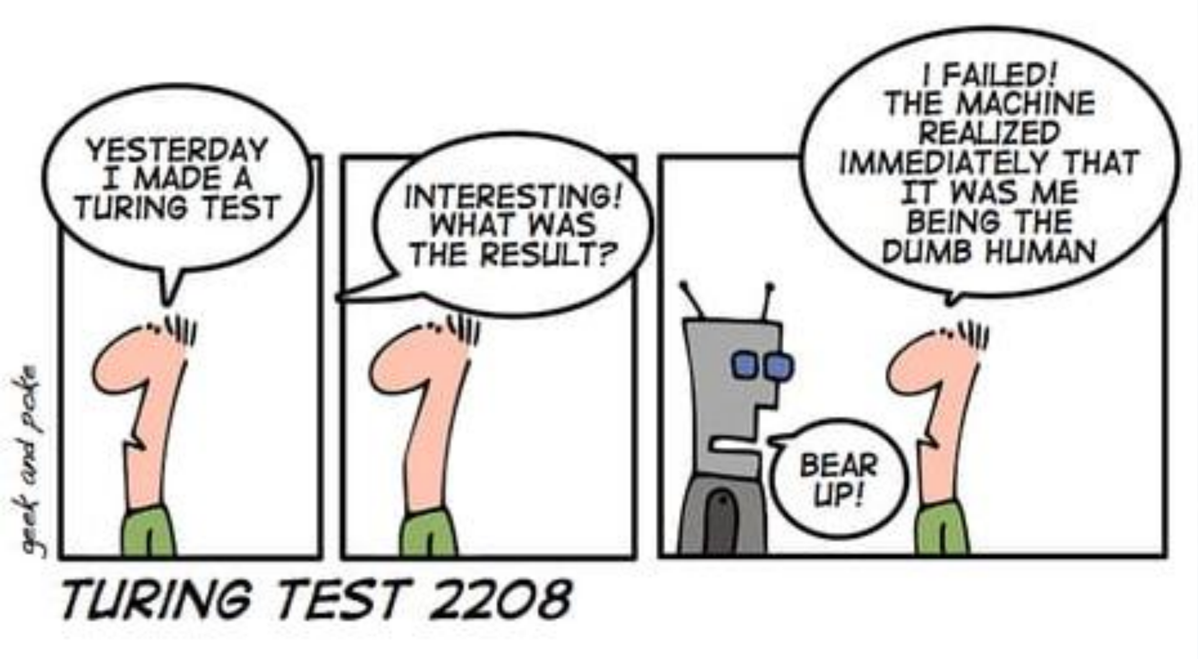
one – hot encoding

target vector of length n

learning weights such that both vectors **are identical**
→ **weights*[:,i]*** of the **hidden layer** make up the embedding vector of ***token i***

… and so on….

Outline



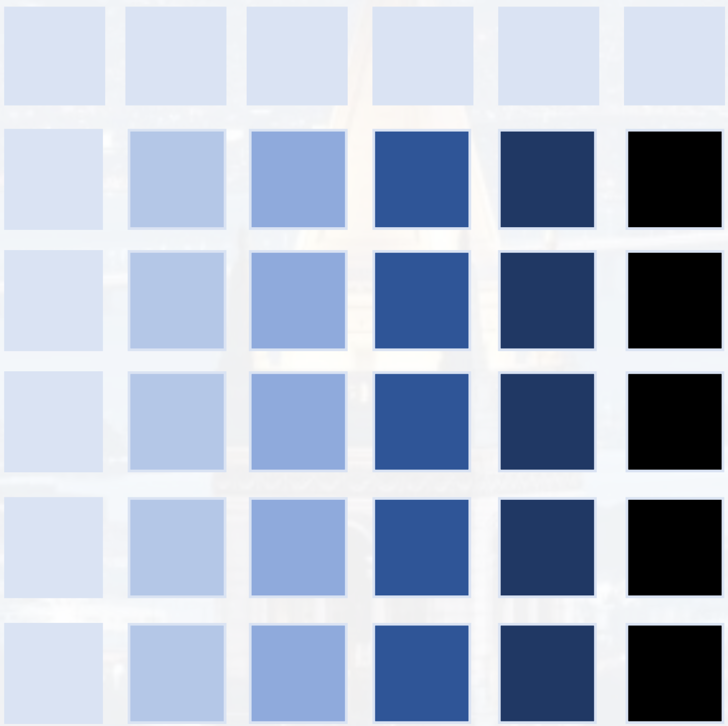
- Introduction
- Bigram and MAP
- Positional Encoding
- Word Embedding
- **Attention**
- Transformer Architecture

LLM & Transformer: Attention

three things make context:

- **positional encoding** (location of token in a sequence)
- **word embedding** (relation between similar/different token)
- **attention** (relation between token within a sequence)

"The cat jumped on the roof."

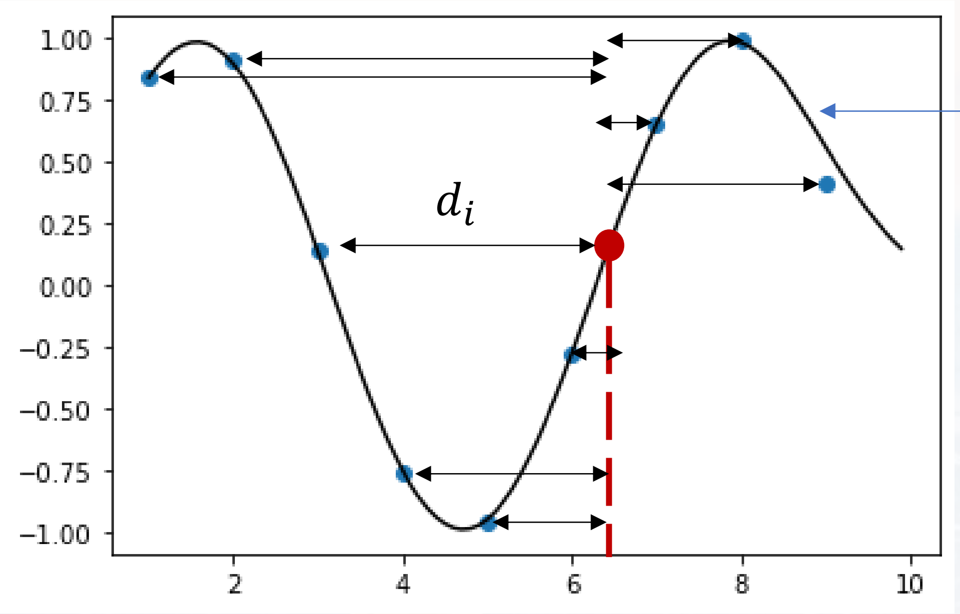


how the first token influences all other token

how the second token influences all other token

.... and so on

LLM & Transformer: Attention



We want to interpolate between the blue dots
→ generating the black line
→ **no curve fitting!**

<u>idea:</u>

- select a point for which we want the interpolation for
- calculate distance d_i to every other point
- each data point should influence the value of the interpolated point
- the closer, the stronger the influence → weighted mean

$$y_{int} \sim \sum_{i=1}^{I} w_i \, y_i \qquad w_i \sim \frac{1}{\boxed{d_i}}$$

LLM & Transformer: Attention

calculating distance


```
D = np.tile(V, (1, len(V))) -  np.tile(L.transpose(), (len(V), 1))
```

- each data point should influence the value of the interpolated point
- the closer, the stronger the influence → weighted mean

$$y_{int} \sim \sum_{i=1}^{I} w_i \, y_i$$

Gaussian kernel

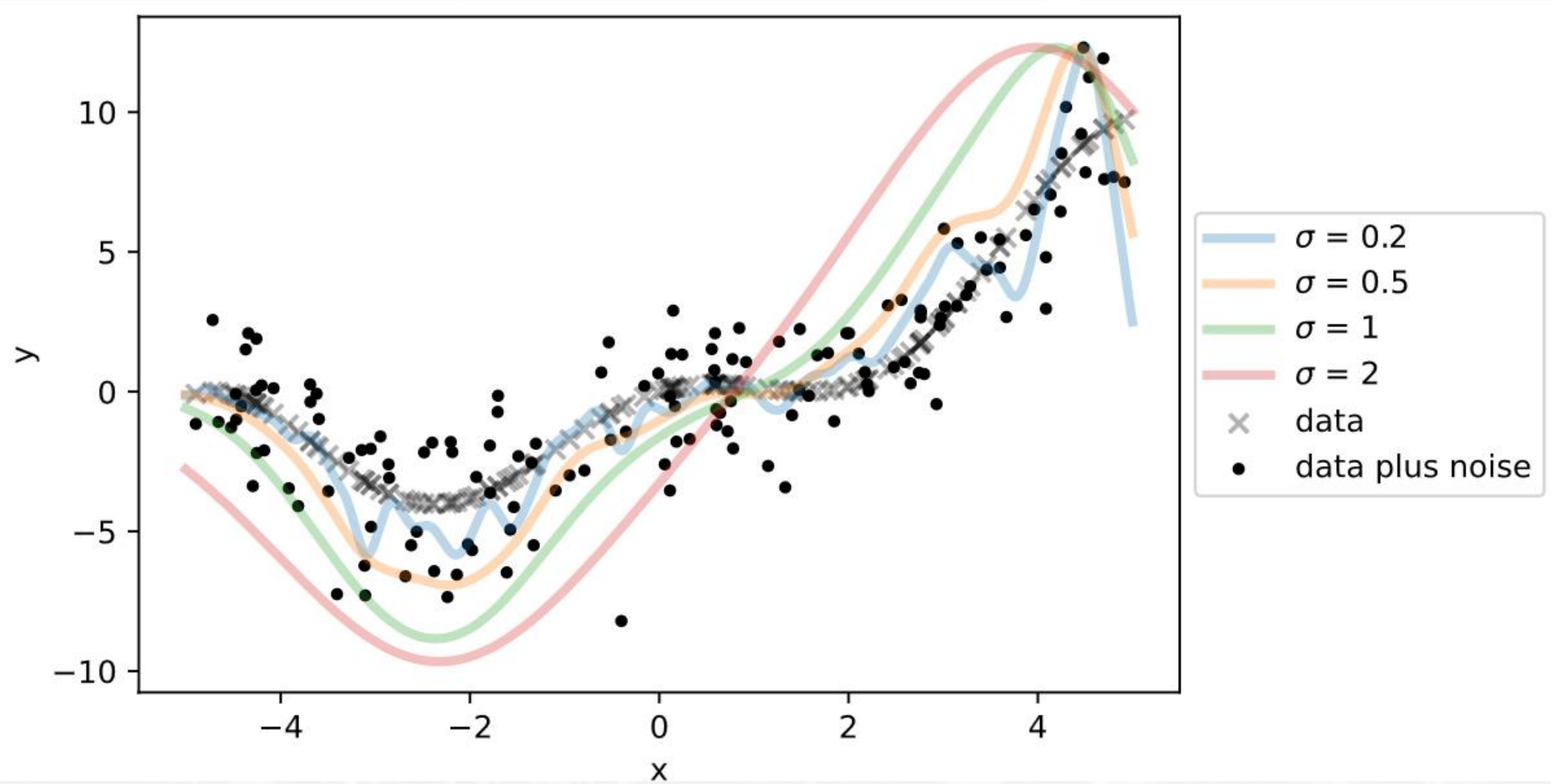
```
W      = np.exp(-(D**2)/(sigma))
W      = W/np.sum(W + 1e-16, axis = 0)
yint   = np.dot(W.transpose(), y)
```

LLM & Transformer: Attention

```
D = np.tile(V, (1, len(V))) -  np.tile(L.transpose(), (len(V), 1))
```

Gaussian kernel

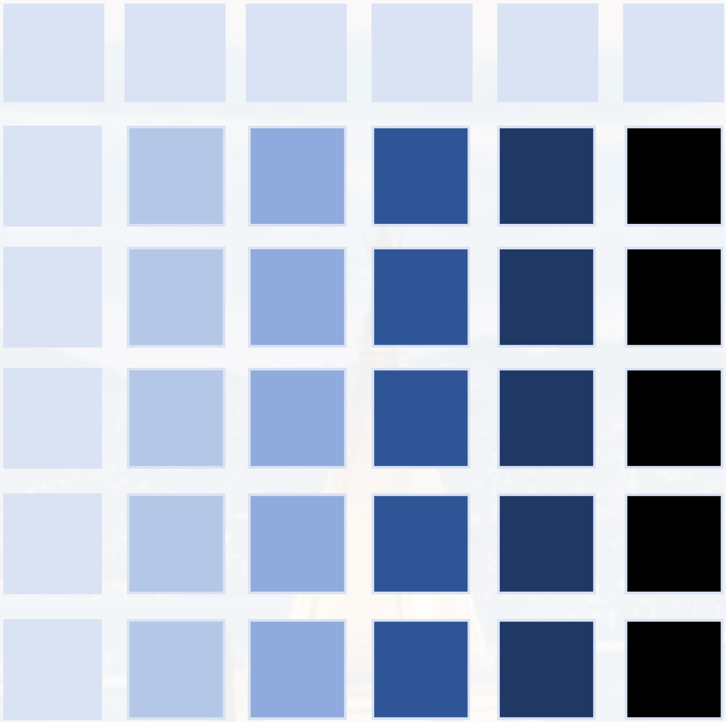
```
W       = np.exp(-(D**2)/(sigma))
W       = W/np.sum(W + 1e-16, axis = 0)
yint    = np.dot(W.transpose(), y)
```

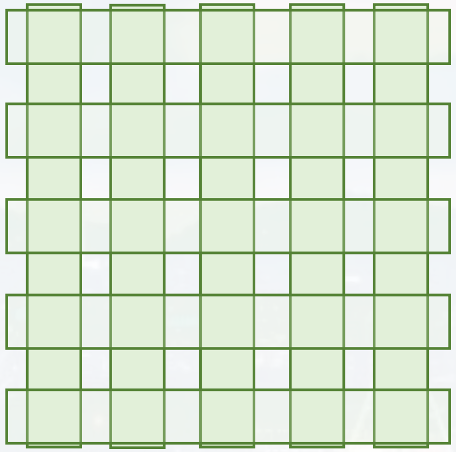


check out:

SmoothGaussKernel.py
SmoothExamples.py

“The cat jumped on the roof.”





Gaussian kernel

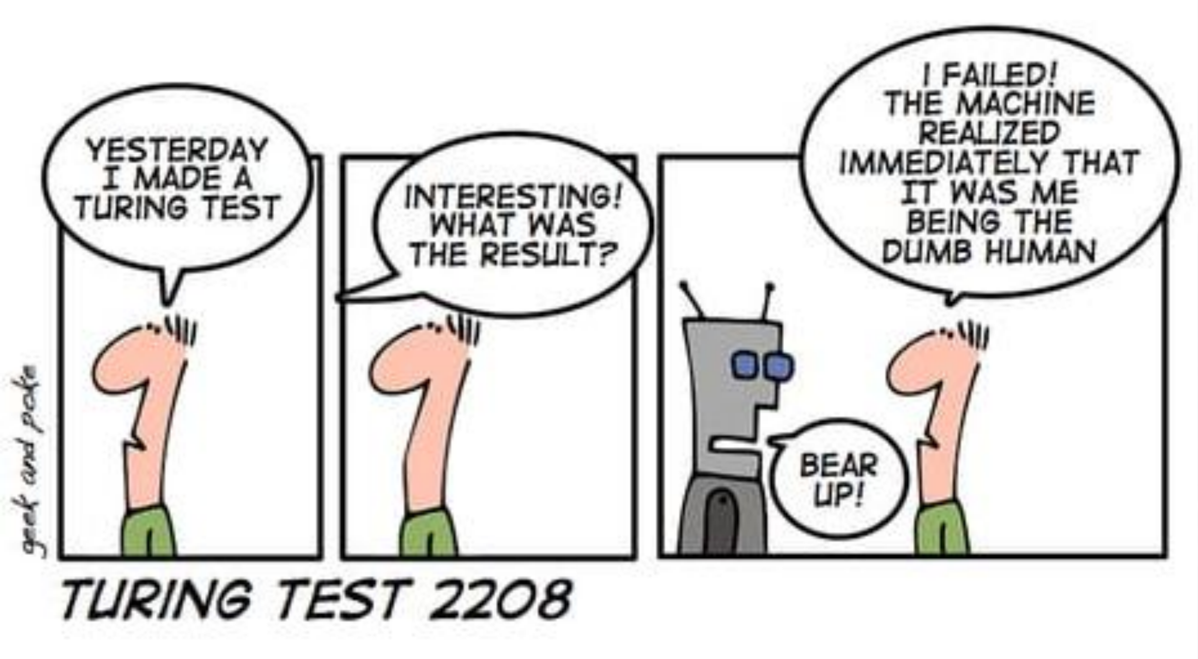
```
W     = np.exp(-(D**2)/(sigma))
W     = W/np.sum(W + 1e-16, axis = 0)
yint  = np.dot(W.transpose(), y)
```

actual attention:

these weights are learnable,

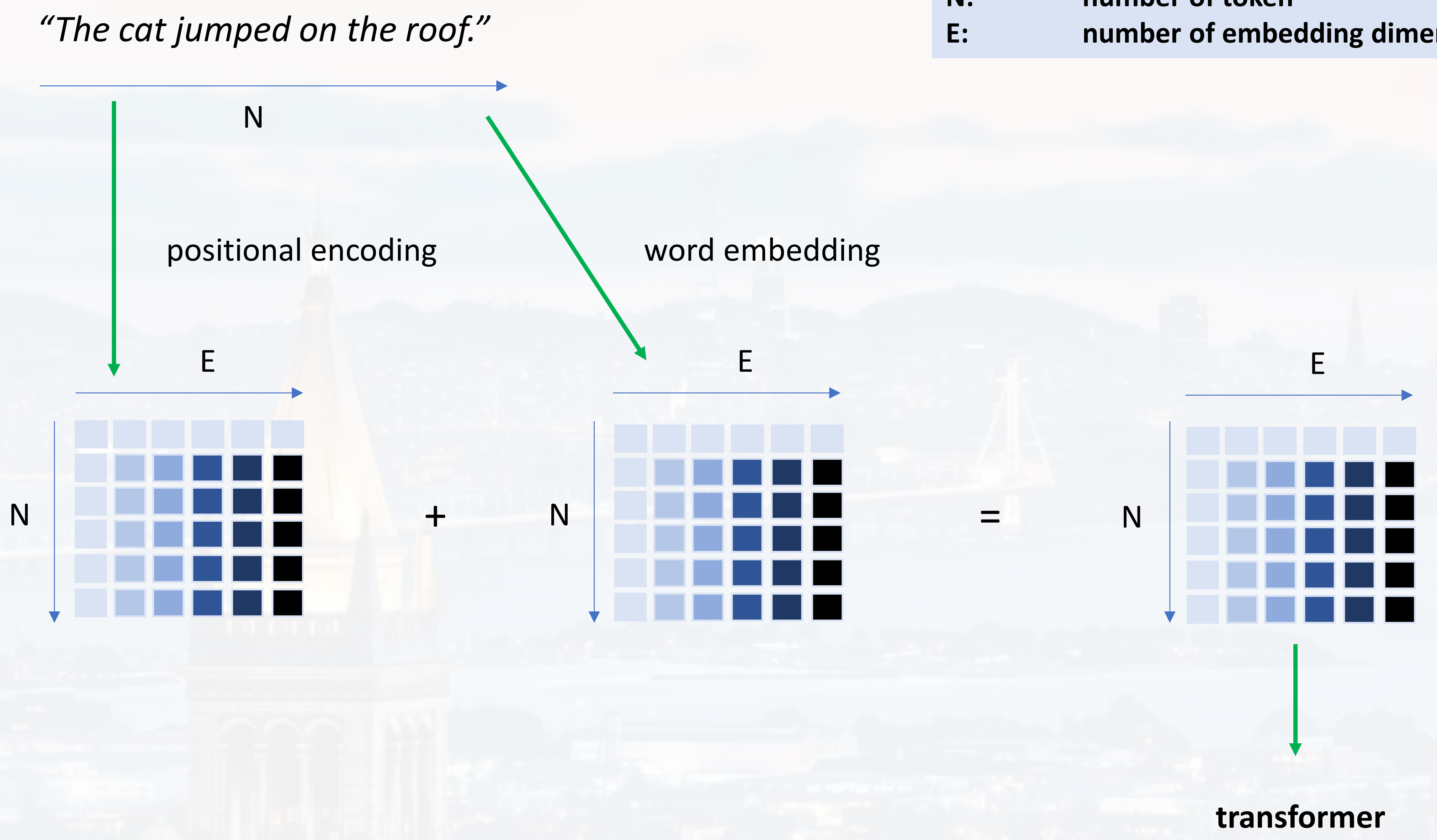
no kernel assumed!

Outline



- Introduction
- Bigram and MAP
- Positional Encoding
- Word Embedding
- Attention
- **Transformer Architecture**

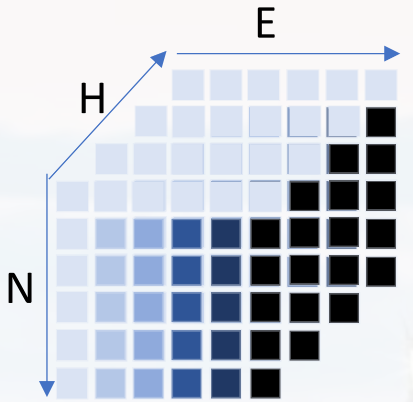
N:	**number of token**
E:	**number of embedding dimensions**

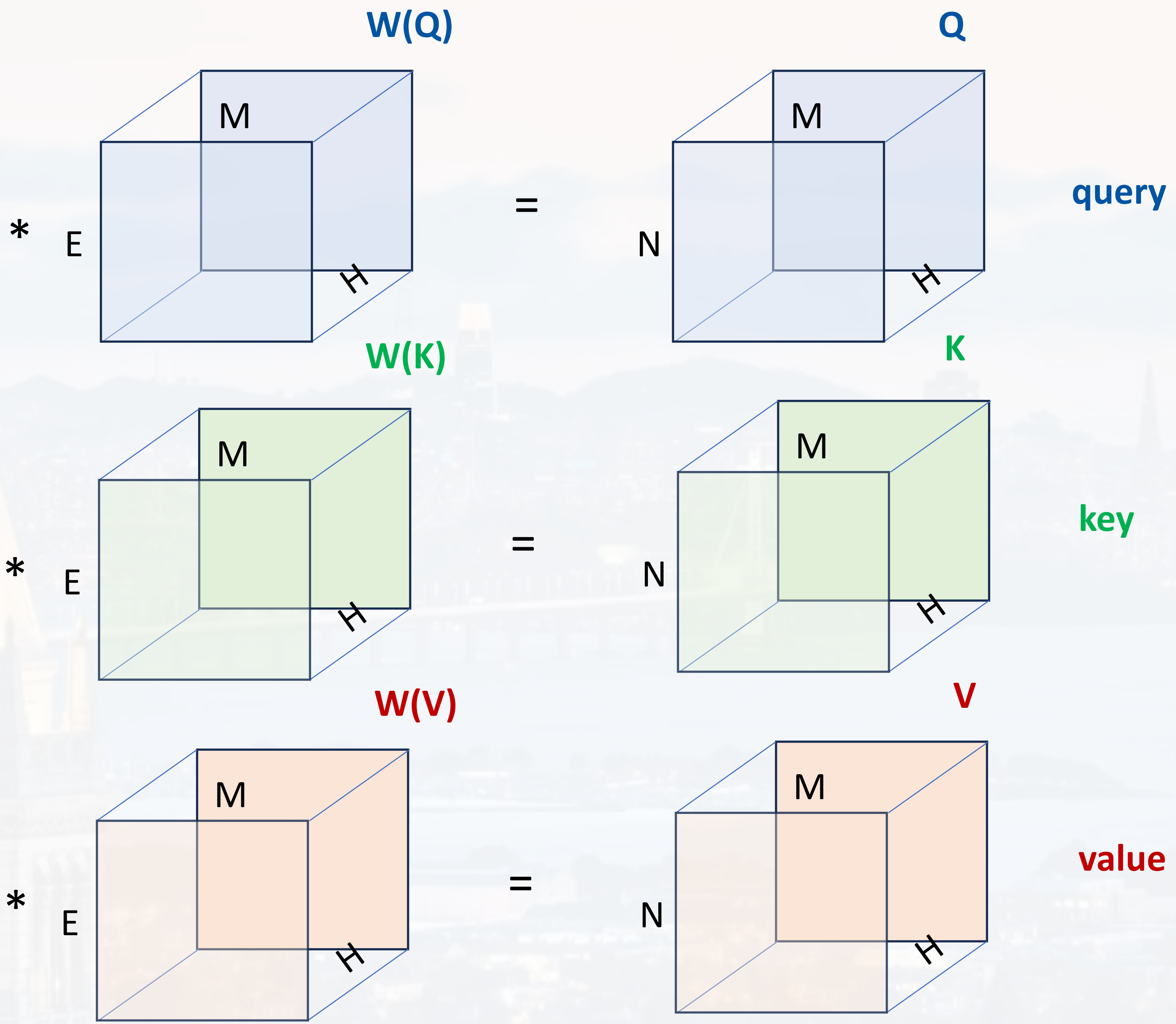


LLM & Transformer: Transformer Architecture

within the transformer:

attention:





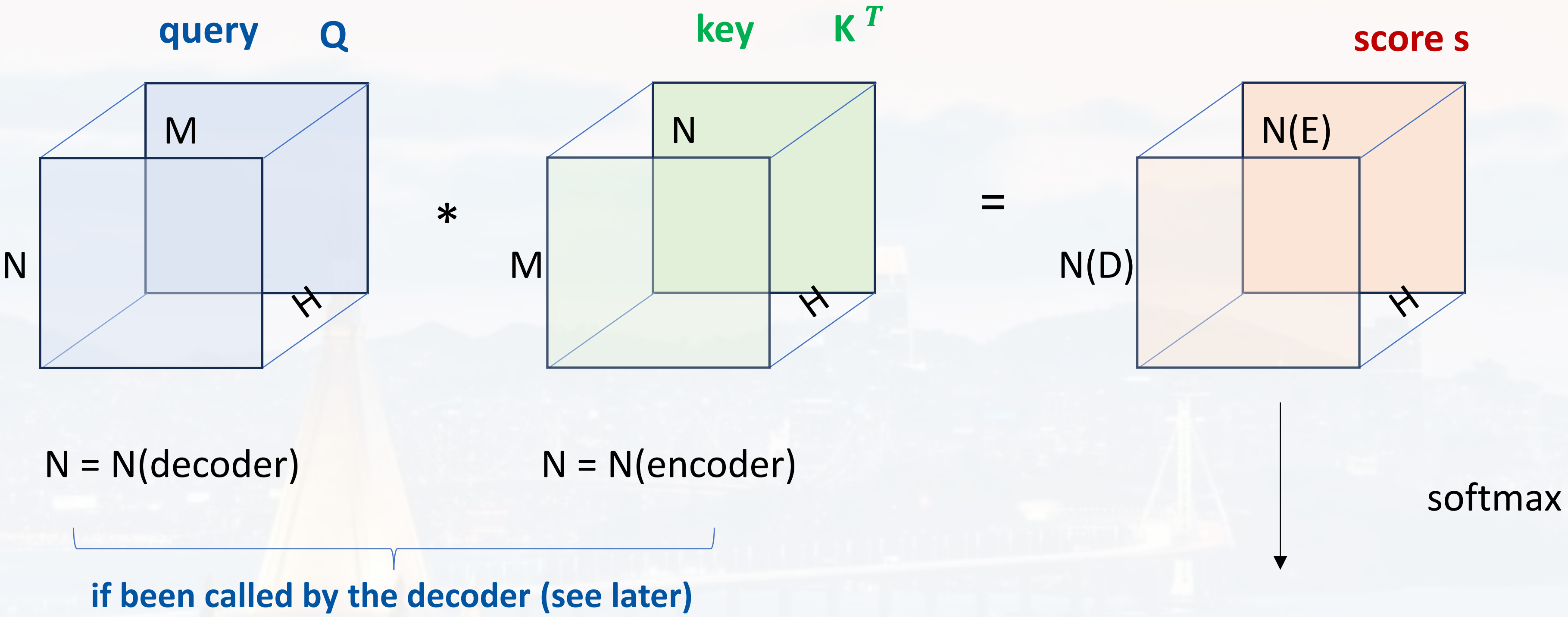
W(Q), W(K), W(V): learnable

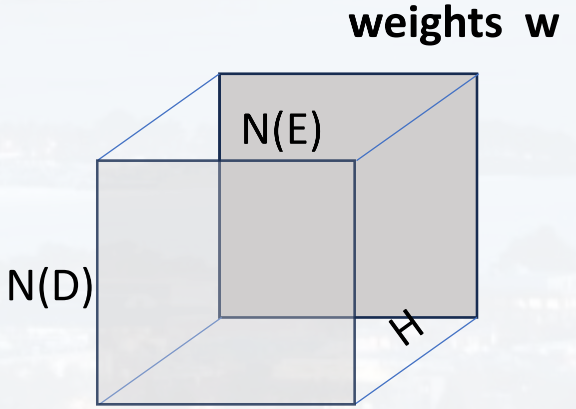
N:	number of token
E:	number of embedding dimensions
H:	number of heads (= 8)
M:	head size (= 64)

LLM & Transformer: Transformer Architecture

within the transformer:

attention:





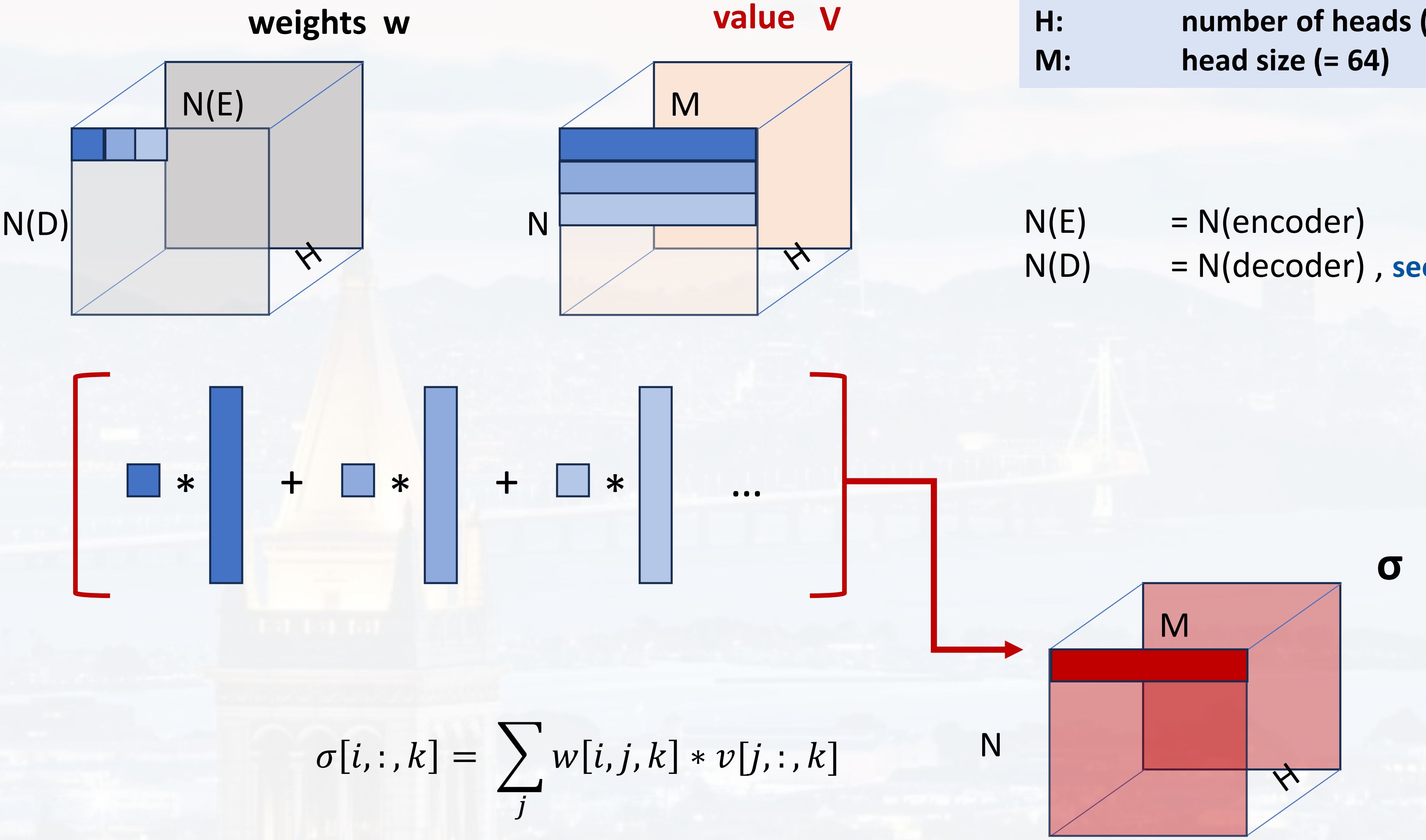
W(Q), W(K), W(V): learnable

N:	**number of token**
E:	**number of embedding dimensions**
H:	**number of heads (= 8)**
M:	**head size (= 64)**

within the transformer:

N:	**number of token**
E:	**number of embedding dimensions**
H:	**number of heads (= 8)**
M:	**head size (= 64)**

attention:



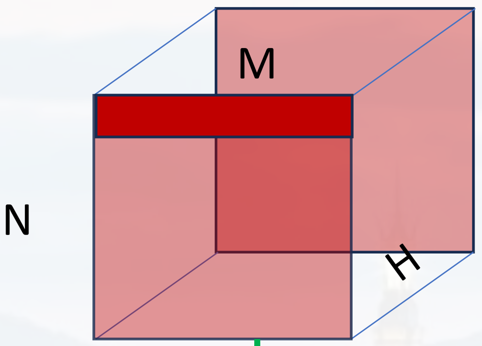
N(E) = N(encoder)
N(D) = N(decoder) , see later

$$\sigma[i,:,k] = \sum_j w[i,j,k] * v[j,:,k]$$

within the transformer:

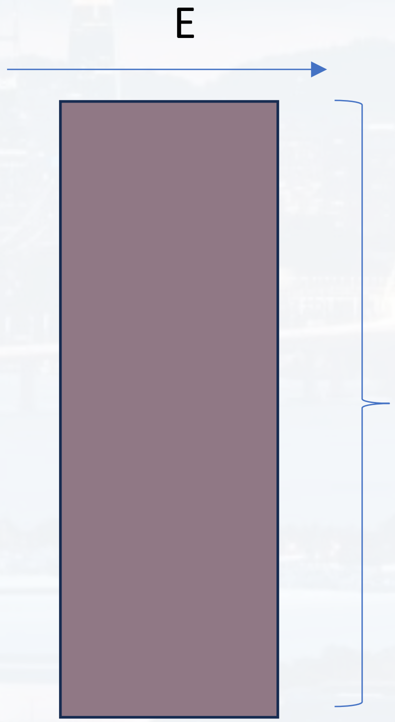
N:	**number of token**
E:	**number of embedding dimensions**
H:	**number of heads (= 8)**
M:	**head size (= 64)**

σ



concatenating σ

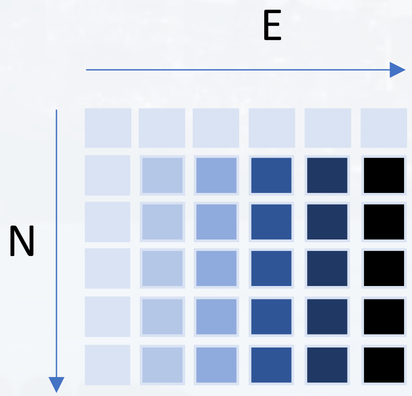
$$\sigma_{con} = \underbrace{\boxed{\sigma[:,:,0]}\boxed{\sigma[:,:,1]}\ldots\boxed{\sigma[:,:,H]}}_{\mathbf{H \cdot M}} \downarrow N \quad *$$



learnable *Ω*

=

output:

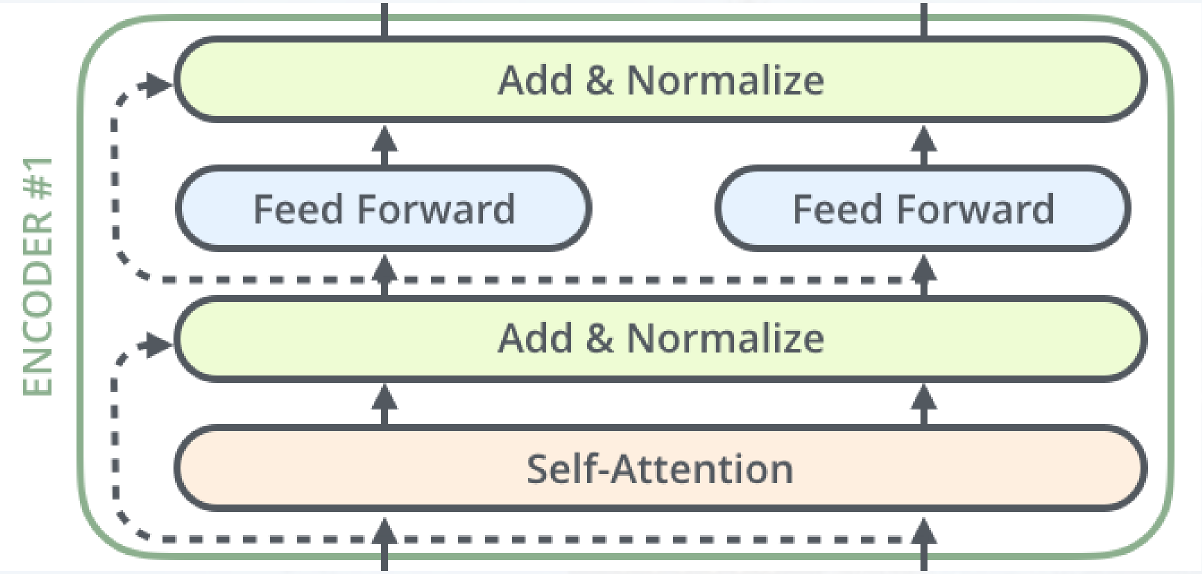


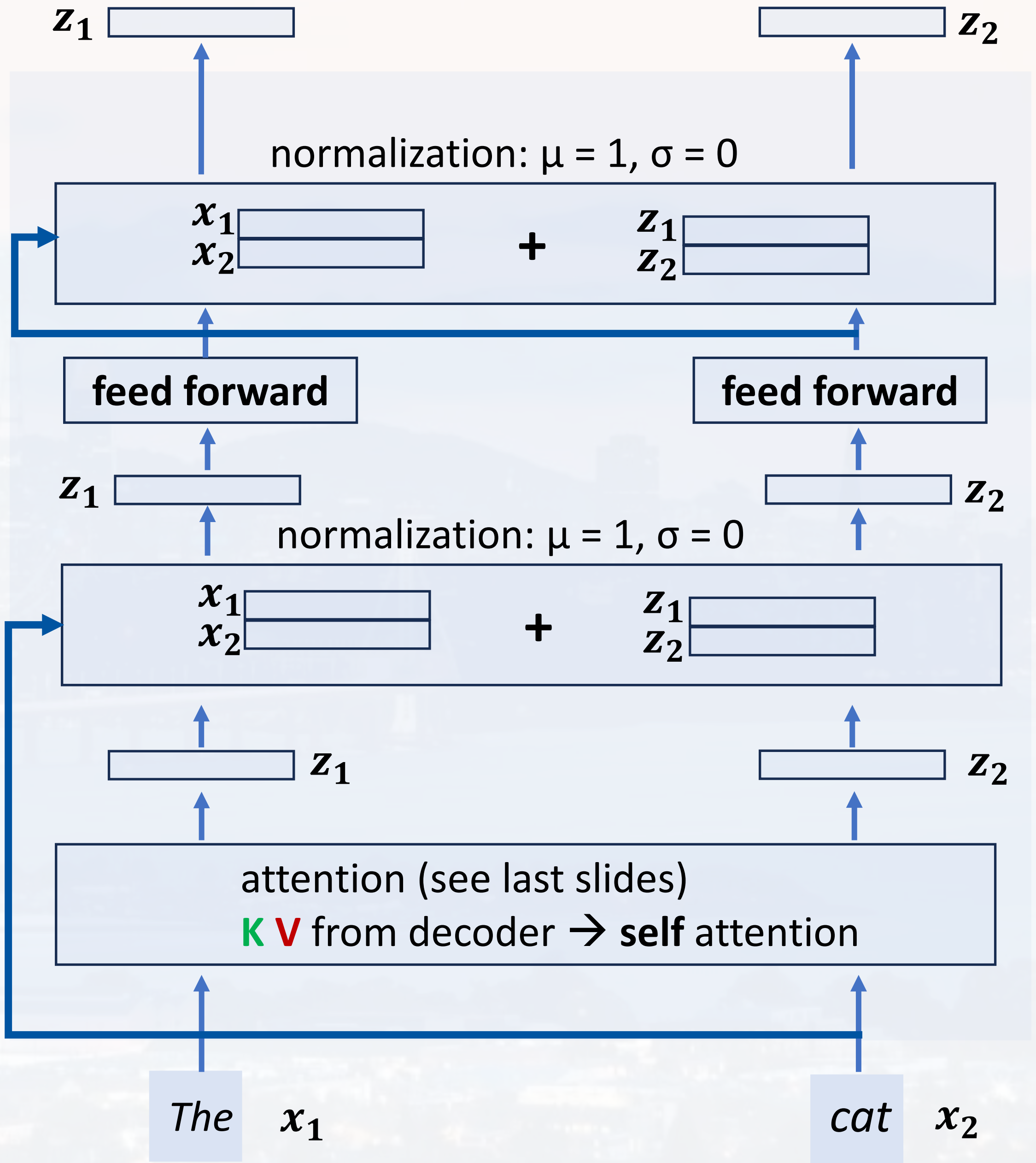
N = N(dencoder) ,
if been called by the decoder (see later)

LLM & Transformer: Transformer Architecture

that was attention → now: encoder

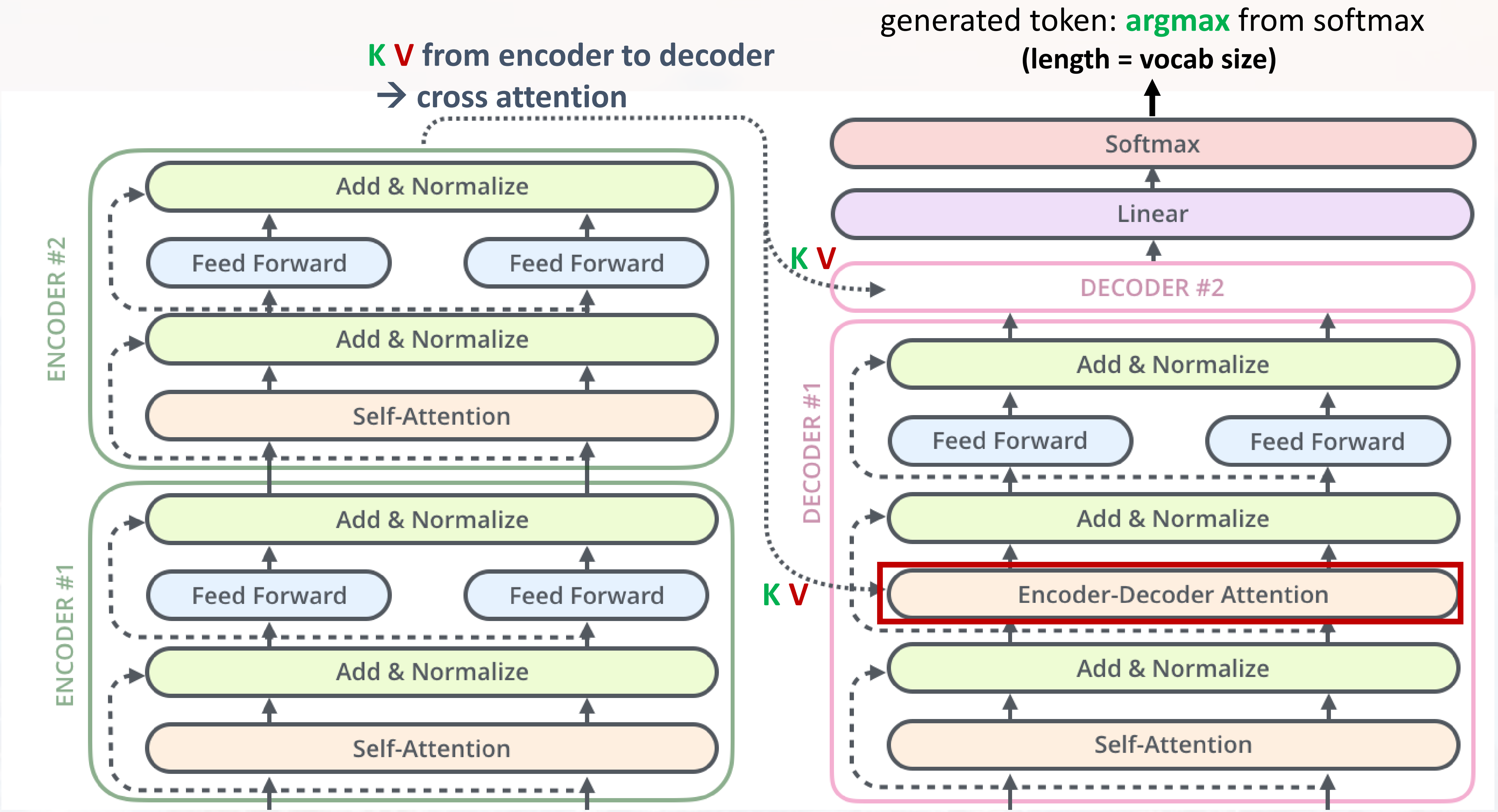
https://jalammar.github.io/illustrated-transformer/



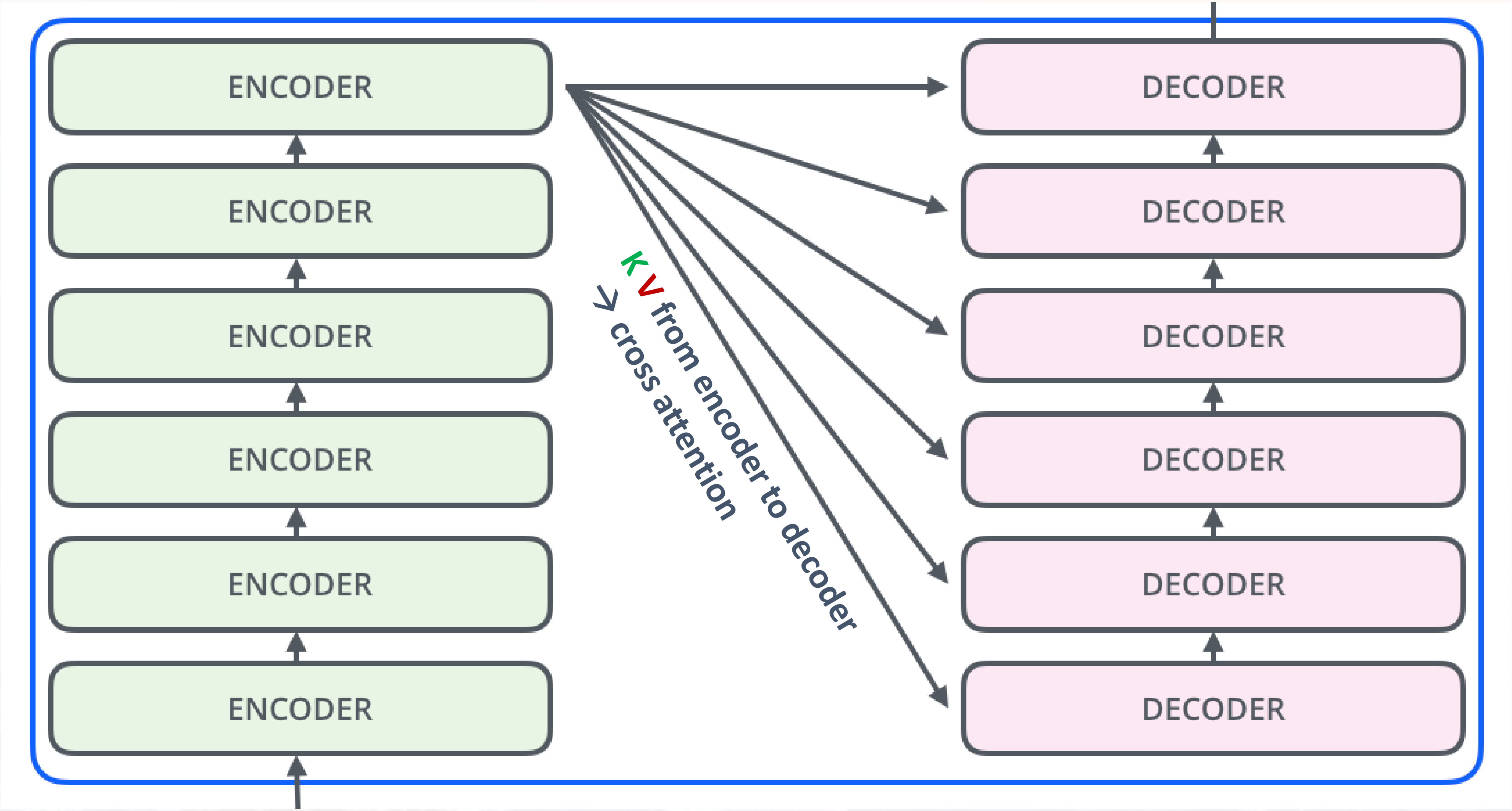


positional encoding + word embedding

LLM & Transformer:

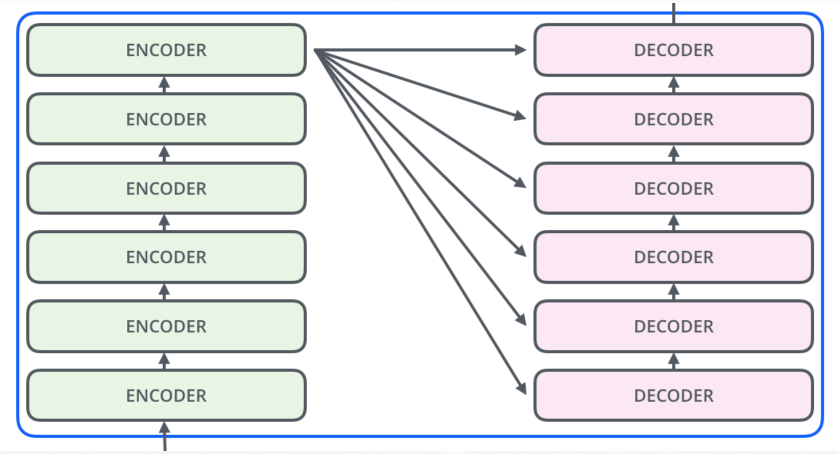


https://jalammar.github.io/illustrated-transformer/



https://jalammar.github.io/illustrated-transformer/

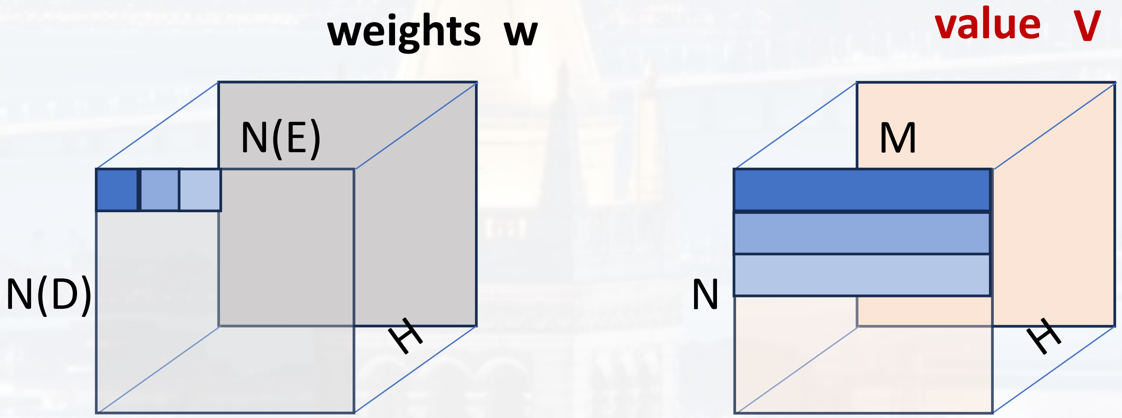
K V from encoder to decoder
→ cross attention



general: *N(E) ≠ N(D)*

English:	Let there be light.	N(E) = 4
German:	Es werde Licht.	N(D) = 3
Latin:	Fiat lux.	N(D) = 2
Hebrew:	yehi ,or	N(D) = 2

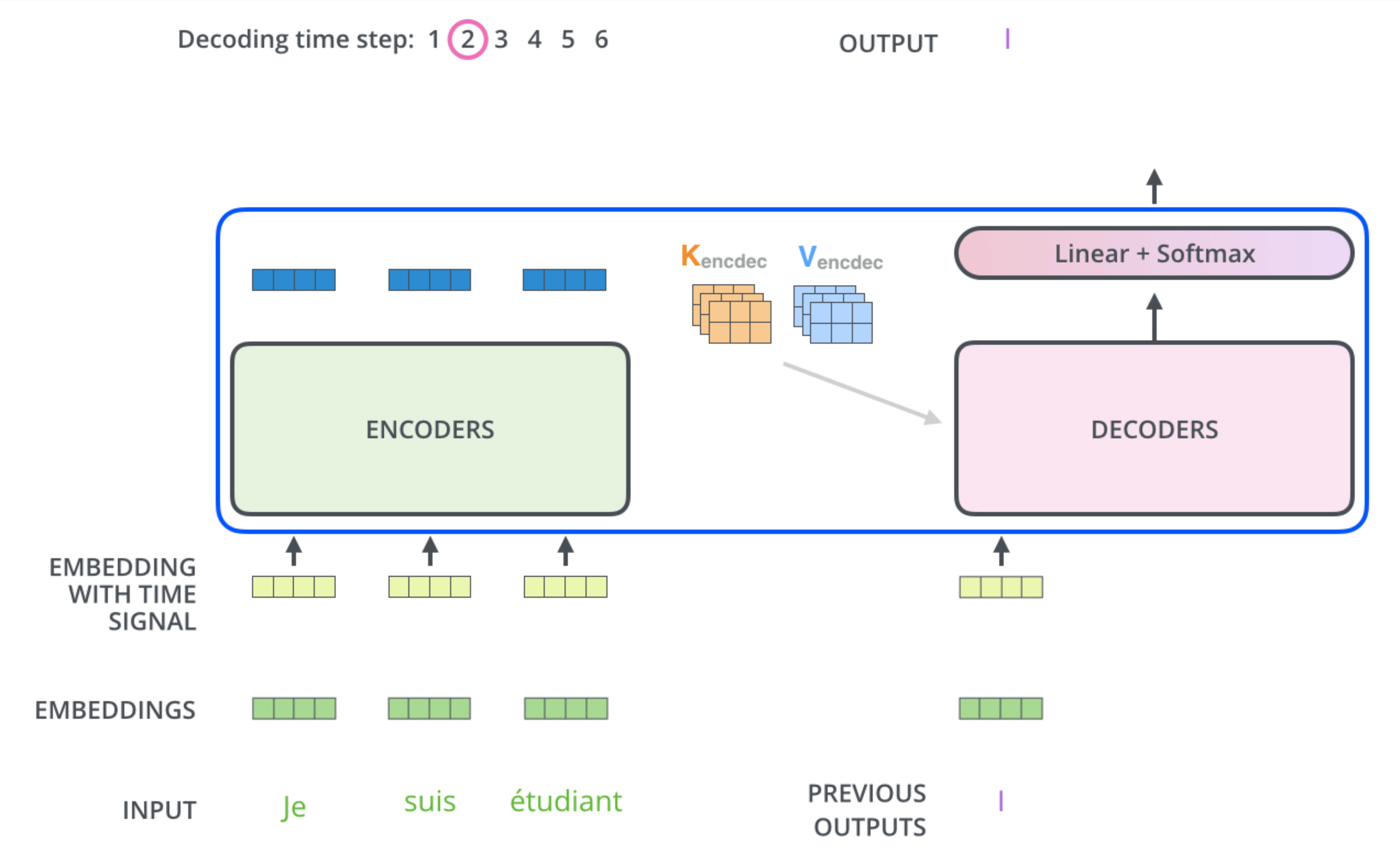
attention:

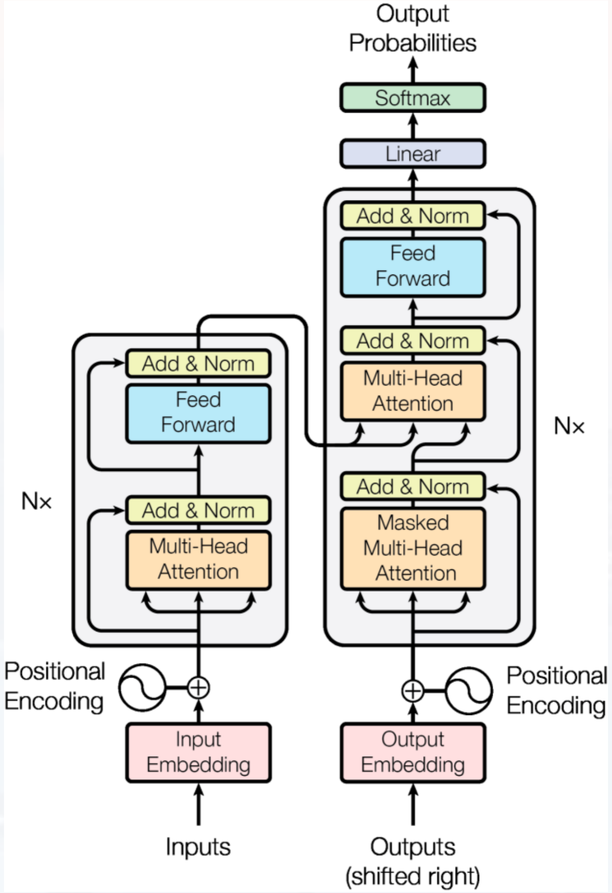


N = N(encoder) , **if been called by the decoder**

https://jalammar.github.io/illustrated-transformer/

LLM & Transformer: Transformer Architecture





https://jalammar.github.io/illustrated-transformer/

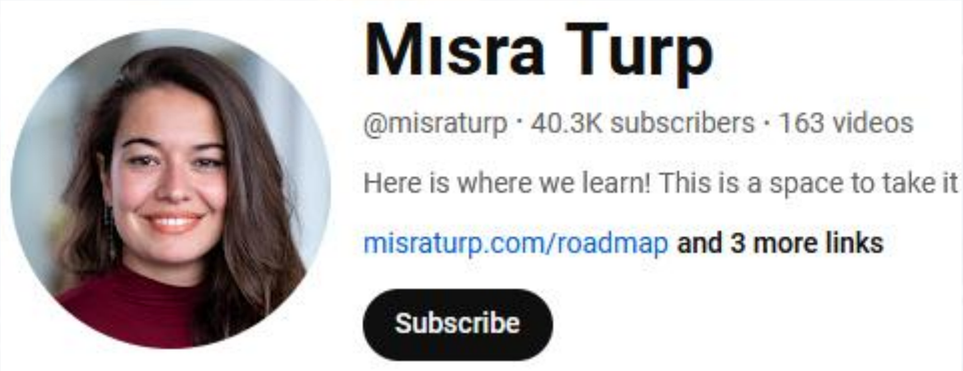
Attention Is All You Need (Vaswani et al, 2017)

more about transformers:

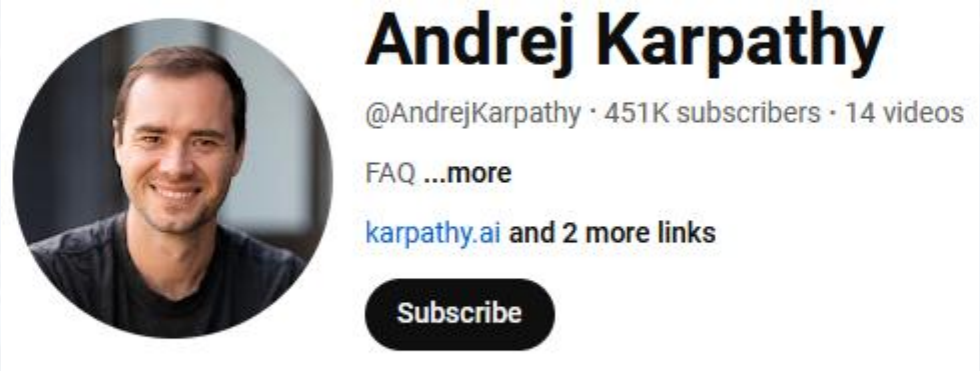
[Jay Alammar](#)

[Interactive Visualization](#)

[transformers intro](#)



[building GPT from scratch!](#)



Thank you very much for your