Discussion: Chain Rule and Product Rule

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chain rule

$$\frac{d}{dx} f[g(x)] = \underbrace{\frac{df(x)}{dg(x)}}_{\text{outer derivative}} \underbrace{\frac{d}{dx} g(x)}_{\text{inner derivative}}$$

derivatives *dx, dy* were derived from finite differences Δx and Δy

$$\frac{df}{dx} = \underbrace{\frac{df}{dg}}_{\text{outer derivative}} \underbrace{\frac{dg}{dx}}_{\text{inner derivative}}$$

$$\frac{df}{dx} = \frac{\frac{df}{dg}}{\frac{dx}{dg}}$$

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chain rule

$$\frac{d}{dx} f[g(x)] = \frac{df(x)}{dg(x)} \frac{d}{dx} g(x)$$

outer derivative inner derivative

example:

$$f(t) = e^{x(t)} \qquad x(t) = \cos(\omega t + \varphi)$$

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} = e^{x(t)}\,[-\sin(\omega t + \varphi)]\,\omega$$

product rule

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

$$f(x)g(x) = \int g(x)\frac{d}{dx}f(x)\,dx + \int f(x)\frac{d}{dx}g(x)\,dx \qquad fg = \int gf' + \int fg'$$

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product rule

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

$$fg = \int gf' + \int fg'$$

$$f(x)g(x) = \int g(x)\frac{d}{dx}f(x)\,dx + \int f(x)\frac{d}{dx}g(x)\,dx$$

$$\int gf' = fg - \int fg'$$

$$\int_{\tau=0}^{\tau=\infty} \tau\,\vartheta e^{-\vartheta\tau} d\tau = \int_{\tau=0}^{\tau=\infty} \tau\,\vartheta e^{-\vartheta\tau} d\tau = \vartheta\left[\int_{\tau=0}^{\tau=\infty} \tau\, e^{-\vartheta\tau} d\tau\right] = \vartheta\left[\tau\frac{e^{-\vartheta\tau}}{-\vartheta}\Big|_{\tau=0}^{\tau=\infty} - \int_{\tau=0}^{\tau=\infty}\frac{e^{-\vartheta\tau}}{-\vartheta}d\tau\right]$$

$$= \vartheta\left[\tau\frac{e^{-\vartheta\tau}}{-\vartheta}\Big|_{\tau=0}^{\tau=\infty} - \frac{e^{-\vartheta\tau}}{\vartheta^2}\Big|_{\tau=0}^{\tau=\infty}\right]$$

$$= \vartheta\left[\infty\frac{e^{-\vartheta\infty}}{-\vartheta} - 0\frac{e^{-\vartheta 0}}{-\vartheta} - \frac{e^{-\vartheta\infty}}{\vartheta^2} + \frac{e^{-\vartheta 0}}{\vartheta^2}\right]$$

$$= \vartheta\left[0 - 0 + \frac{1}{\vartheta^2}\right] = \frac{1}{\vartheta}$$

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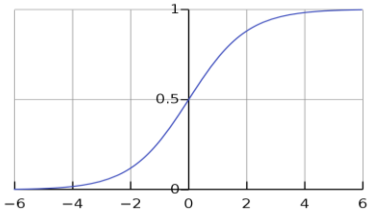
sigmoid:



$$(-\infty;\ +\infty) \rightarrow (0; 1)$$

$$\sigma(x) = \frac{1}{1+\ e^{-x}} = \frac{e^{x}}{1+\ e^{x}}$$

$$= \frac{e^{x}+1-1}{1+\ e^{x}}$$

$$= \frac{e^{x}+1}{1+\ e^{x}} - \frac{1}{1+\ e^{x}}$$

$$= 1 - \frac{1}{1+\ e^{x}} \qquad \boldsymbol{= 1 - \sigma(-x)}$$

$$\sigma(x) = \boldsymbol{1 - \sigma(-x)}$$

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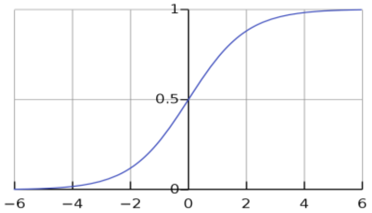
sigmoid:



$$(-\infty;\ +\infty) \rightarrow (0; 1) \qquad \sigma(x) = \frac{1}{1 + e^{-x}}$$

$$y \coloneqq 1 + e^{-x} \qquad \frac{d\sigma(x)}{dx} = \frac{d\sigma(y[x])}{dx} = \frac{d\sigma}{dy}\frac{dy}{dx} = \frac{-1}{y^2}(-e^{-x}) = \frac{e^{-x}}{(1 + e^{-x})^2}$$

$$= \frac{e^{-x}}{(1 + e^{-x})(1 + e^{-x})} = \frac{e^{-x}}{(1 + e^{-x})}\frac{1}{(1 + e^{-x})}$$

$$= \frac{e^{x} + 1 - 1}{1 + e^{x}}\frac{1}{(1 + e^{-x})}$$

$$= \left[1 - \frac{1}{1 + e^{x}}\right]\frac{1}{(1 + e^{-x})}$$

$$= [1 - \sigma(x)]\sigma(x)$$

$$\boldsymbol{\frac{d\sigma(x)}{dx} = [1 - \sigma(x)]\sigma(x)}$$

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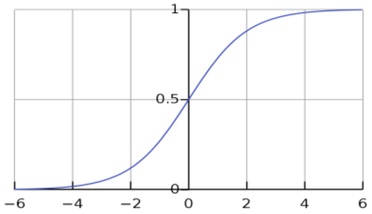
sigmoid:



$$(-\infty;\ +\infty) \rightarrow (0; 1) \qquad \sigma(x) = \frac{1}{1+\ e^{-x}} \qquad \boldsymbol{\frac{d\sigma(x)}{dx} = [1 - \sigma(x)]\sigma(x)}$$

$$\mathcal{L} = \frac{1}{2}(y - \hat{y})^2 \qquad \hat{y} = \sigma(x) \qquad x = \sum_i I_i\, w_i + b$$

$$\frac{d\mathcal{L}}{dw_i} = \frac{d\mathcal{L}}{d\hat{y}}\,\frac{d\hat{y}}{dx}\,\frac{dx}{dw_i} = -(y - \hat{y})\,[1 - \sigma(x)]\sigma(x)\, I_i$$

$$\frac{d\mathcal{L}}{db} = \frac{d\mathcal{L}}{d\hat{y}}\,\frac{d\hat{y}}{dx}\,\frac{dx}{db} = -(y - \hat{y})\,[1 - \sigma(x)]\sigma(x)\, 1$$

$$\frac{d\mathcal{L}}{dI_i} = \frac{d\mathcal{L}}{d\hat{y}}\,\frac{d\hat{y}}{dx}\,\frac{dx}{dI_i} = -(y - \hat{y})[1 - \sigma(x)]\sigma(x)\, w_i$$

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