Lecture 4:

Bayesian Signal Detection

Markus Hohle

University California, Berkeley

Bayesian Data Analysis and Machine Learning for Physical Sciences

Course Map

Outline

- Discrete Signals
- The Model
- The Priors
- Occam Factors
- A Code Example


WHEN YOU ESTIMATE A BAYESIAN MODEL
This little maneuver is gonna cost us 51 years
imgflip.com

Outline

literature:

A NEW METHOD FOR THE DETECTION OF A PERIODIC SIGNAL OF UNKNOWN SHAPE AND PERIOD

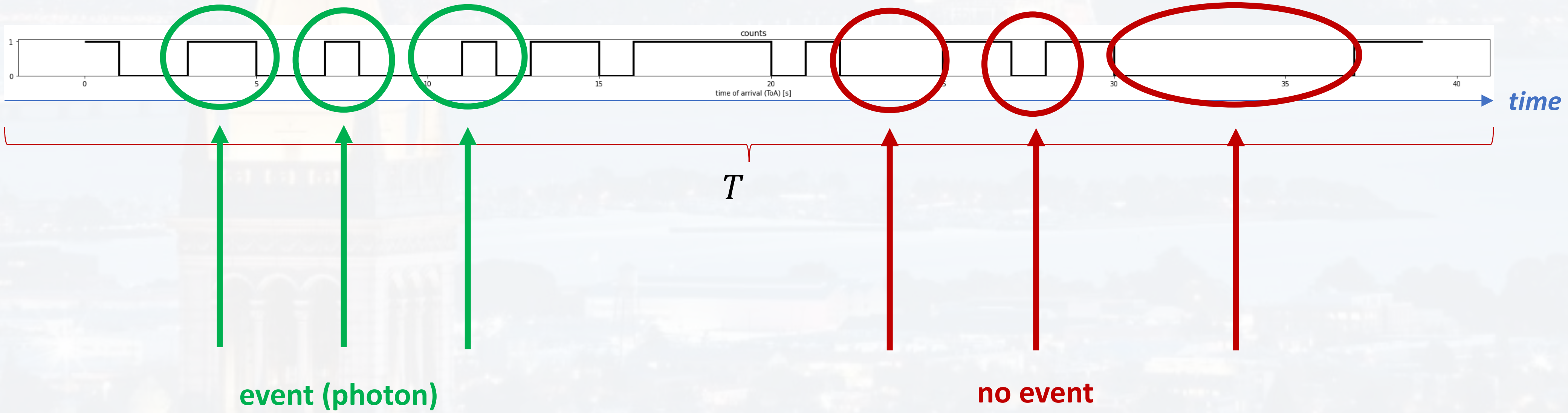
P. C. GREGORY
Department of Physics, University of British Columbia, 6224 Agricultural Road, Vancouver, British Columbia, Canada V6T 1Z1

AND

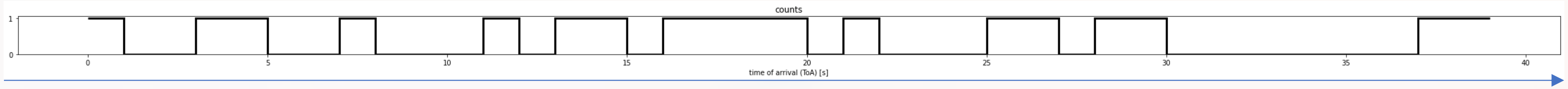
THOMAS J. LOREDO
Department of Astronomy, Space Sciences Building, Cornell University, Ithaca, NY 14853

Received 1992 January 6; accepted 1992 April 20

one application: pulsar timing (X-ray pulsars), Hambaryan et al., 20XX



Bayesian Signal Detection:



time

T

	0
0	0
1	1
2	2
3	3
4	5
5	7
6	8
7	10
8	11

T

if we *had* the frequency ω

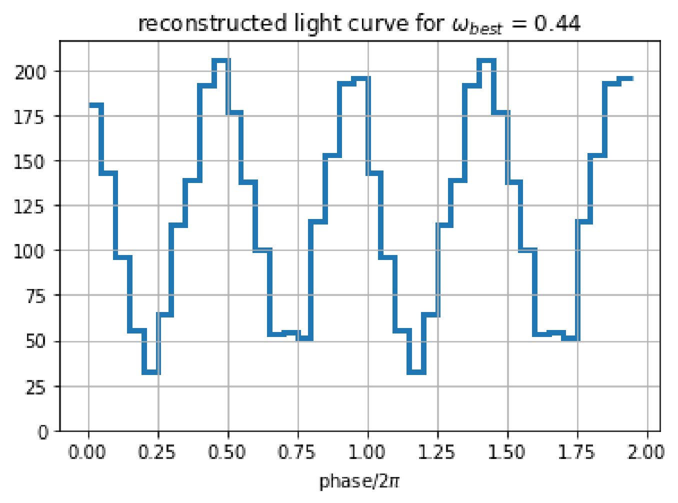
→ calculating the phase $\varphi_i(t_i) = \omega t_i + \varphi_0$

→ $\varphi(norm)_i(t_i) = \varphi_i(t_i)/2\pi$

→ using $\varphi(norm)_i(t_i) = \varphi(norm)_i(t_i) + 1$

→ histogram of all $\varphi(norm)_i(t_i)$

→ **phase binned light curve**



actual data: t_i **Time of Arrival (ToA)**

Bayesian Signal Detection:

if we *had* the frequency ω

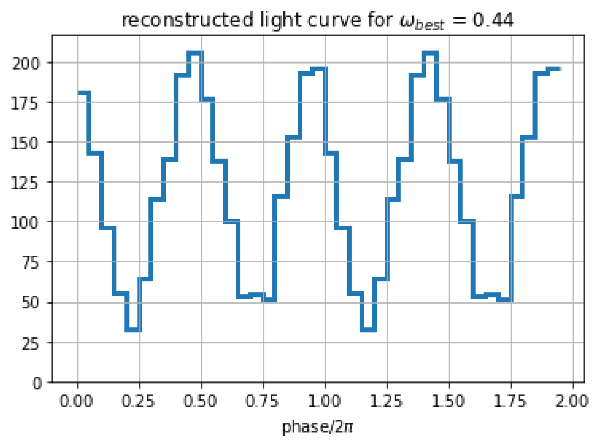
→ calculating the phase $\varphi_i(t_i) = \omega t_i + \varphi_0$

→ $\varphi(norm)_i(t_i) = \varphi_i(t_i)/2\pi$

→ using $\varphi(norm)_i(t_i) = \varphi(norm)_i(t_i) + 1$

→ histogram of all $\varphi(norm)_i(t_i)$

→ **phase binned light curve**



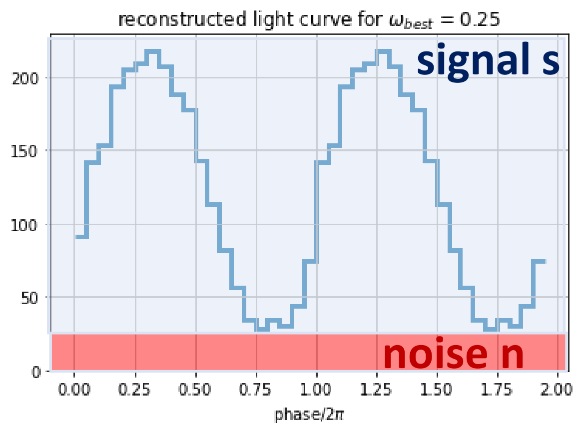
no signal:

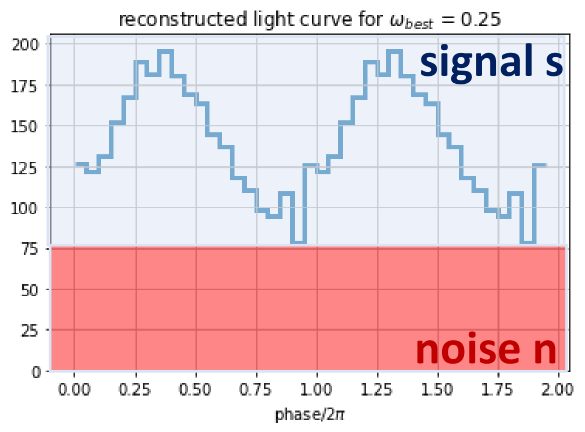
- constant rate $r(t_i) = const$
- events are evenly distributed over phase bins
- light curve is flat

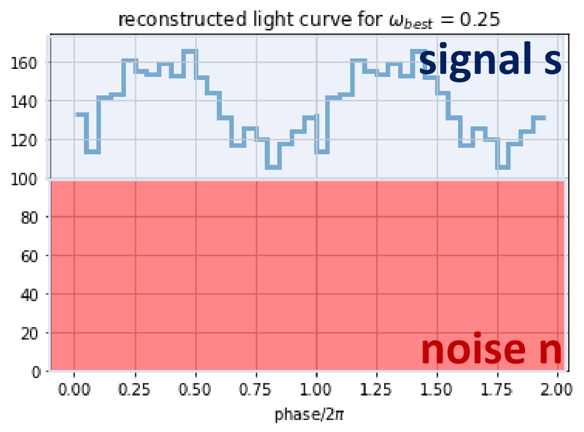
signal:

- $r(t_i)$ is a function of time
- light curve of any shape

increasing noise level







Bayesian Signal Detection:

D	: data
I	: information
M_i	: model i
M_j	: model j
$r(t_i)$	: rate
t_i	: ToA

$$P(M_i|D,I) = \frac{P(D|M_i,I)}{P(D|I)} P(M_i|I)$$

Bayes Theorem

$$\rho_{i,j} = \frac{P(M_i|D,I)}{P(M_j|D,I)} = \frac{P(D|M_i,I)}{P(D|M_j,I)} \frac{P(M_i|I)}{P(M_j|I)}$$

odds ratio

"Bayes Factor"

We want to compare the **constant rate model** M_1 with $r(t_i) = const$ to a set of **periodic models**

for N_{mod} (disjunct!) models, we can normalize and find:

$$\sum_{j=1}^{N_{mod}} P(M_j|D,I) = 1 = \sum_{j=1}^{N_{mod}} \frac{P(D|M_j,I)}{P(D|M_i,I)} \frac{P(M_j|I)}{P(M_i|I)} P(M_i|D,I)$$

$$= \frac{P(M_i|D,I)}{P(D|M_i,I)\, P(M_i|I)} \sum_{j=1}^{N_{mod}} P(D|M_j,I) P(M_j|I)$$

Bayesian Signal Detection:

D	**: data**
I	**: information**
$\boldsymbol{M_i}$	**: model i**
$\boldsymbol{M_j}$	**: model j**
$\boldsymbol{r(t_i)}$	**: rate**
$\boldsymbol{t_i}$	**: ToA**

$$P(M_i|D,I) = \frac{P(D|M_i,I)}{P(D|I)} P(M_i|I)$$

Bayes Theorem

$$\rho_{i,j} = \frac{P(M_i|D,I)}{P(M_j|D,I)} = \frac{P(D|M_i,I)}{P(D|M_j,I)} \frac{P(M_i|I)}{P(M_j|I)}$$

odds ratio

"Bayes Factor"

We want to compare the **constant rate model** M_1 with $r(t_i) = const$ to a set of **periodic models**

for N_{mod} (disjunct!) models, we can normalize and find:

$$\sum_{j=1}^{N_{mod}} P(M_j|D,I) = 1 = \frac{\boldsymbol{P(M_i|D,I)}}{P(D|M_i,I)\,P(M_i|I)} \sum_{j=1}^{N_{mod}} P(D|M_j,I)P(M_j|I)$$

$$\boldsymbol{P(M_i|D,I)} = \frac{P(D|M_i,I)\,P(M_i|I)}{\sum_{j=1}^{N_{mod}} P(D|M_j,I)P(M_j|I)}$$

$$P(M_i|D,I) = \frac{\frac{P(D|M_i,I)}{\boldsymbol{P(D|M_1,I)}} \frac{P(M_i|I)}{\boldsymbol{P(M_1|I)}}}{\sum_{j=1}^{N_{mod}} \frac{P(D|M_j,I)}{\boldsymbol{P(D|M_1,I)}} \frac{P(M_j|I)}{\boldsymbol{P(M_1|I)}}}$$

Bayesian Signal Detection:

D	: data
I	: information
M_i	: model i
M_j	: model j
$r(t_i)$	: rate
t_i	: ToA

$$P(M_i|D,I) = \frac{P(D|M_i,I)}{P(D|I)} P(M_i|I)$$

Bayes Theorem

$$\rho_{i,j} = \frac{P(M_i|D,I)}{P(M_j|D,I)} = \frac{P(D|M_i,I)}{P(D|M_j,I)} \frac{P(M_i|I)}{P(M_j|I)}$$

odds ratio

"Bayes Factor"

We want to compare the **constant rate model** M_1 with $r(t_i) = const$ to a set of **periodic models**

for N_{mod} (disjunct!) models, we can normalize and find:

$$P(M_i|D,I) = \frac{\frac{P(D|M_i,I)}{P(D|M_1,I)} \frac{P(M_i|I)}{P(M_1|I)}}{\sum_{j=1}^{N_{mod}} \frac{P(D|M_j,I)}{P(D|M_1,I)} \frac{P(M_j|I)}{P(M_1|I)}} = \frac{\rho_{i,1}}{\sum_{j=1}^{N_{mod}} \rho_{j,1}} = \frac{\rho_{i,1}}{\sum_{j=2}^{N_{mod}} \rho_{j,1} + 1}$$

Bayesian Signal Detection:

D	**: data**
I	**: information**
$\boldsymbol{M_i}$	**: model i**
$\boldsymbol{M_j}$	**: model j**
$\boldsymbol{r(t_i)}$	**: rate**
$\boldsymbol{t_i}$	**: ToA**

$$P(M_i|D,I) = \frac{P(D|M_i,I)}{P(D|I)} P(M_i|I)$$

Bayes Theorem

$$\rho_{i,j} = \frac{P(M_i|D,I)}{P(M_j|D,I)} = \frac{P(D|M_i,I)}{P(D|M_j,I)} \frac{P(M_i|I)}{P(M_j|I)}$$

odds ratio

"Bayes Factor"

We want to compare the **constant rate model** M_1 with $r(t_i) = const$ to a set of **periodic models**

for N_{mod} (disjunct!) models, we can normalize and find:

$$\boxed{P(M_i|D,I) = \frac{\rho_{i,1}}{\sum_{j=2}^{N_{mod}} \rho_{j,1} + 1}}$$


WHEN YOU ESTIMATE A BAYESIAN MODEL
This little maneuver is gonna cost us 51 years
imgflip.com

Outline

Bayesian Signal Detection:

The Model

$$\rho_{i,j} = \frac{P(M_i|D,I)}{P(M_j|D,I)} = \frac{P(D|M_i,I)}{P(D|M_j,I)} \frac{P(M_i|I)}{P(M_j|I)}$$

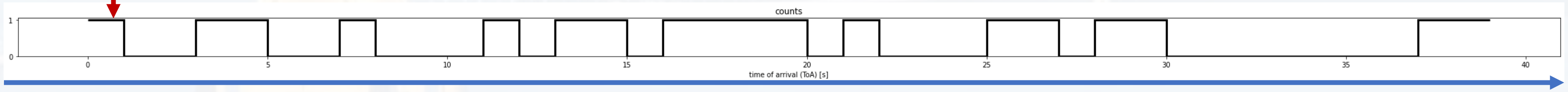
D	: data
I	: information
M_i	: model i
M_j	: model j
$r(t)$	: rate
t	: ToA
$\{\alpha\}_i$	: set of parameters of model M_i
Δt	: time resolution
n	: number of events
T	: obs. time span

marginalization

$$P(D|M_i) = \int P(D|\{\alpha\}_i\, M_i)\, P(\{\alpha\}_i|\, M_i)\; d\Omega_{\{\alpha\}_i}$$

$$P(n|r(t)) = \frac{(r(t)\cdot\Delta t)^n}{n!} e^{-r(t)\cdot\Delta t}$$

Poisson distribution



time

T

actual data (ToA)

N	intervals with $n = 1$
Q	intervals with $n = 0$

$$(N + Q)\Delta t = T$$

Bayesian Signal Detection:

$$\rho_{i,j} = \frac{P(M_i|D,I)}{P(M_j|D,I)} = \frac{P(D|M_i,I)}{P(D|M_j,I)} \frac{P(M_i|I)}{P(M_j|I)}$$

$$P(n|r(t)) = \frac{(r(t)\cdot\Delta t)^n}{n!} e^{-r(t)\cdot\Delta t}$$

Poisson distribution

N intervals with $n = 1$

Q intervals with $n = 0$

$$(N+Q)\Delta t = T$$

$\boldsymbol{D}$	**: data**
$\boldsymbol{I}$	**: information**
$\boldsymbol{M_i}$	**: model i**
$\boldsymbol{M_j}$	**: model j**
$\boldsymbol{r(t)}$	**: rate**
$\boldsymbol{t}$	**: ToA**
$\{\alpha\}_i$	**: set of parameters of model $\boldsymbol{M_i}$**
$\Delta\boldsymbol{t}$	**: time resolution**
$\boldsymbol{n}$	**: number of events**
$\boldsymbol{T}$	**: obs. time span**

joint likelihood

$$P(D|\,r(t),t) = \prod_{i=1}^{N} r(t_i)\cdot\Delta t\; e^{-r(t_i)\cdot\Delta t} \cdot \prod_{i=1}^{Q} e^{-r(t_i)\cdot\Delta t}$$

$$\boldsymbol{P(D|M_i) = \int P(D|\{\alpha\}_i\, M_i)\, P(\{\alpha\}_i|\, M_i)\; d\Omega_{\{\alpha\}_i}}$$

$$\boxed{P(D|\,r(t),t) = (\Delta t)^N \cdot \prod_{i=1}^{N} r(t_i)\cdot\; exp\left[-\sum_{i=1}^{Q+N} r(t_i)\cdot\Delta t\right]}$$

Bayesian Signal Detection:

D	: data
I	: information
M_i	: model i
M_j	: model j
$r(t)$	: rate
t	: ToA
$\{\alpha\}_i$	: set of parameters of model M_i
Δt	: time resolution
n	: number of events
T	: obs. time span

$$P(n|r(t)) = \frac{(r(t)\cdot\Delta t)^n}{n!} e^{-r(t)\cdot\Delta t}$$

Poisson distribution

N intervals with $n = 1$

Q intervals with $n = 0$

$$(N+Q)\Delta t = T$$

$$P(D|\, r(t), t) = (\Delta t)^N \cdot \prod_{i=1}^{N} r(t_i) \cdot exp\left[\underbrace{- \sum_{i=1}^{Q+N} r(t_i)\cdot \Delta t}_{= \int_0^T r(t)\, dt} \right]$$

$$\boxed{P(D|\, r(t), t) = (\Delta t)^N \cdot \prod_{i=1}^{N} r(t_i) \cdot \exp\left(- \int_0^T r(t)\, dt \right)}$$

any periodic model

Bayesian Signal Detection:

any periodic model

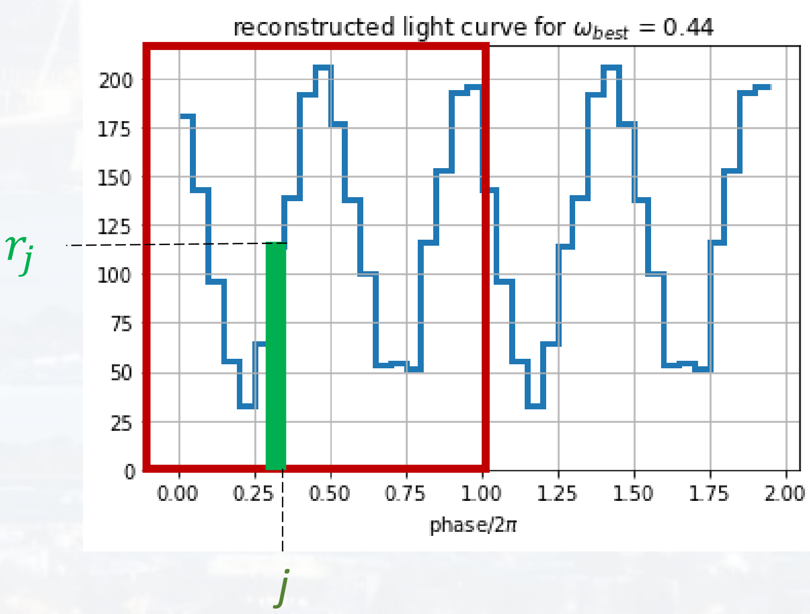
$$\boxed{P(D|\, r(t), t) = (\Delta t)^N \cdot \prod_{i=1}^{N} r(t_i) \cdot \exp\left(- \int_0^T r(t)\, dt \right)}$$

if we had the frequency already:

m phase bins:

r_j	rate in each phase bin *j*
$A = \frac{1}{m} \sum_{j=1}^{m} r_j$	average rate
$f_j = \frac{r_j}{\sum_{j=1}^{m} r_j} = \frac{r_j}{A\,m}$	fraction of total rate in each phase bin *j*

$\boldsymbol{r(t)}$	**: rate**
$\boldsymbol{\Delta t}$	**: time resolution**
$\boldsymbol{n}$	**: number of events**
$\boldsymbol{T}$	**: obs. time span**
$\boldsymbol{D}$	**: data set**
$\boldsymbol{N}$	**: number of intervals with n=1**
$\boldsymbol{r_j}$	**: rate in each phase bin *j***
$\boldsymbol{A}$	**: average rate**
$\boldsymbol{m}$	**: number of phase bins**
$\boldsymbol{f_j}$	**: fraction of total rate in *j***



Each light curve of any shape is being fully described by f_j

Bayesian Signal Detection:

any periodic model

$$\boxed{P(D|\,r(t),t) = (\Delta t)^N \cdot \prod_{i=1}^{N} r(t_i) \cdot \exp\left(-\int_0^T r(t)\,dt\right)}$$

m phase bins:

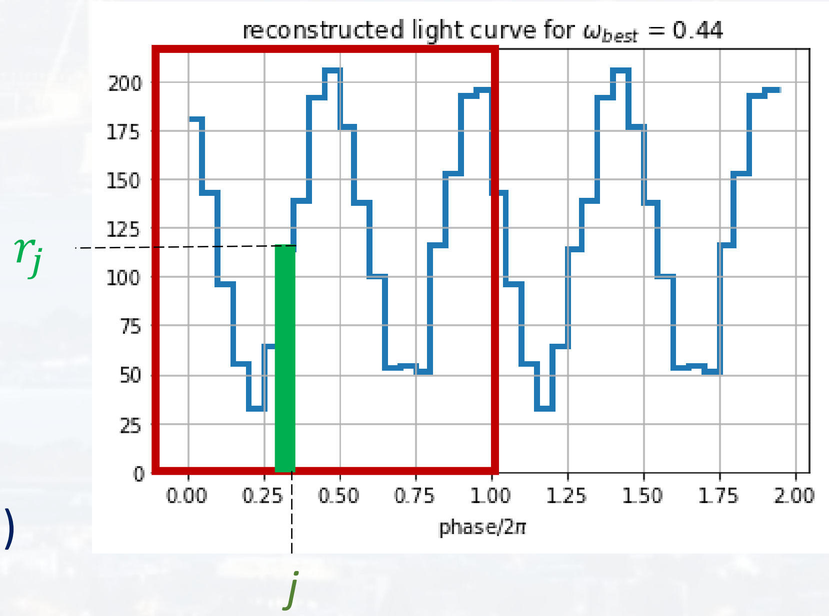
r_j	rate in each phase bin *j*
$A = \frac{1}{m}\sum_{j=1}^{m} r_j$	average rate
$f_j = \frac{r_j}{\sum_{j=1}^{m} r_j} = \frac{r_j}{A\,m}$	fraction of total rate in each phase bin *j*

constant model: r_j = const ∀ *j*

→ $r_j = A$

$$\boxed{P(D|\,r(t),t) = (\Delta t)^N \cdot A^N \cdot e^{-AT}}$$ **constant model** (poiss noise)

$r(t)$	: rate
Δt	: time resolution
n	: number of events
T	: obs. time span
D	: data set
N	: number of intervals with n=1
r_j	: rate in each phase bin *j*
A	: average rate
m	: number of phase bins
f_j	: fraction of total rate in *j*



Bayesian Signal Detection:

any periodic model

$$\boxed{P(D|\, r(t), t) = (\Delta t)^N \cdot \prod_{i=1}^{N} r(t_i) \cdot \exp\left(-\int_0^T r(t)\, dt\right)}$$

$$\boxed{P(D|\, r(t), t) = (\Delta t)^N \cdot A^N \cdot e^{-AT}}$$ **constant model** (poiss noise)

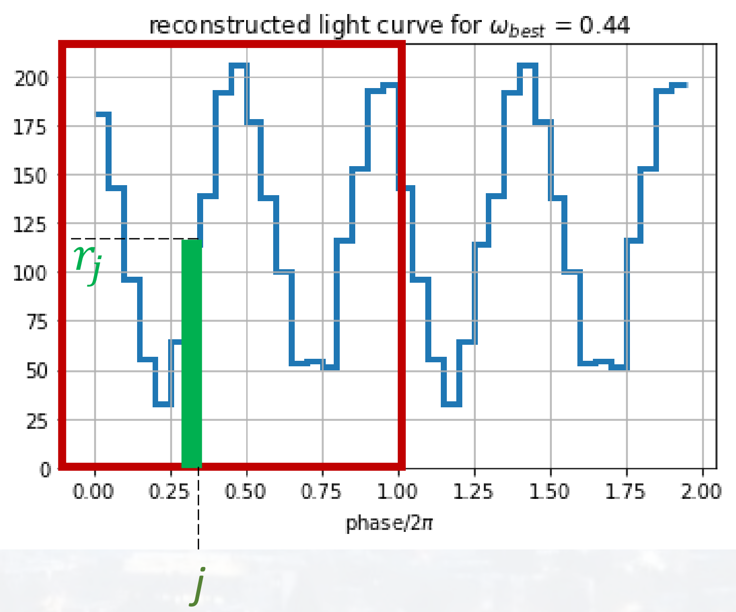
$r(t)$	: rate
Δt	: time resolution
n	: number of events
T	: obs. time span
D	: data set
N	: number of intervals with n=1
r_j	: rate in each phase bin j
A	: average rate
m	: number of phase bins
f_j	: fraction of total rate in j

actual signal

- amplitude
- phase
- frequency
- offset

detection → - t_i *analysis* →

- phase
- frequency
- $f_j(A, m)$



note:
- we assume a poissonian process
- sampling rate >> $r(t)$ troughout the observation
- $r(t)$ does not change within Δt
- T is large enough to capture at least $\approx$ 10 periods of the signal
- m is large: we capture finer details of light curve, but increase fluctuations within the bins and vice verse

Bayesian Signal Detection:

The Model

rewriting the likelihood function in terms of the **light curve parameter**:

$$\boxed{P(D|\,A, M_1) = \;(\Delta t)^N \cdot A^N \;\cdot\; e^{-AT}}$$ **constant model** (poiss noise)

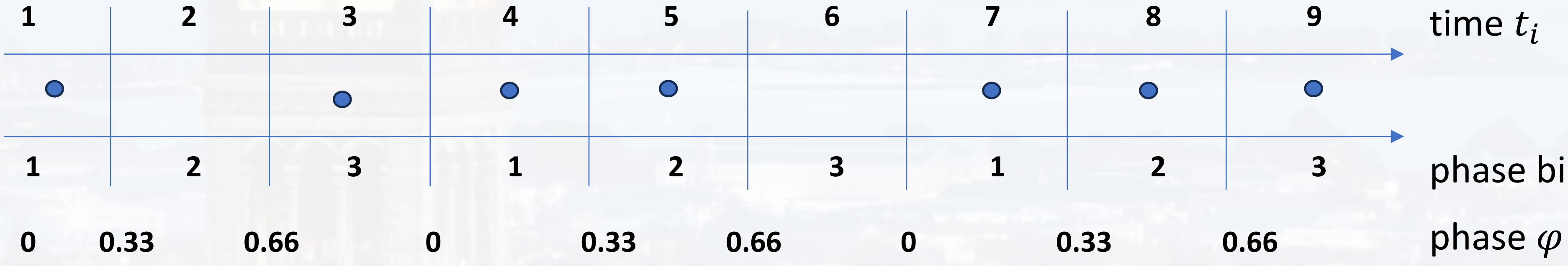
any periodic model:

$$P(D|\,r(t), t) = \;(\Delta t)^N \cdot \prod_{i=1}^{N} r(t_i) \cdot \exp\left(-\int_0^T r(t)\,dt\right)$$

$$P(D|\boldsymbol{\omega}, \boldsymbol{\varphi}, \boldsymbol{A}, \boldsymbol{f}, \boldsymbol{M_m}) = ?$$

$$f_j = \frac{r_j}{\sum_{j=1}^{m} r_j} = \frac{r_j}{A\,m}$$

$\boldsymbol{r(t)}$	**: rate**
$\boldsymbol{\Delta t}$	**: time resolution**
$\boldsymbol{T}$	**: obs. time span**
$\boldsymbol{D}$	**: data set**
$\boldsymbol{N}$	**: number of intervals with n=1**
$\boldsymbol{r_j}$	**: rate in each phase bin *j***
$\boldsymbol{A}$	**: average rate**
$\boldsymbol{m}$	**: number of phase bins**
$\boldsymbol{f_j}$	**: fraction of total rate in *j***
$\boldsymbol{f}$	**: full set of f_j**
$\boldsymbol{M_1}$	**: constant model**
$\boldsymbol{M_m}$	**: periodic model m bins**
$\boldsymbol{\omega}$	**: frequency**
$\boldsymbol{\varphi}$	**: phase**

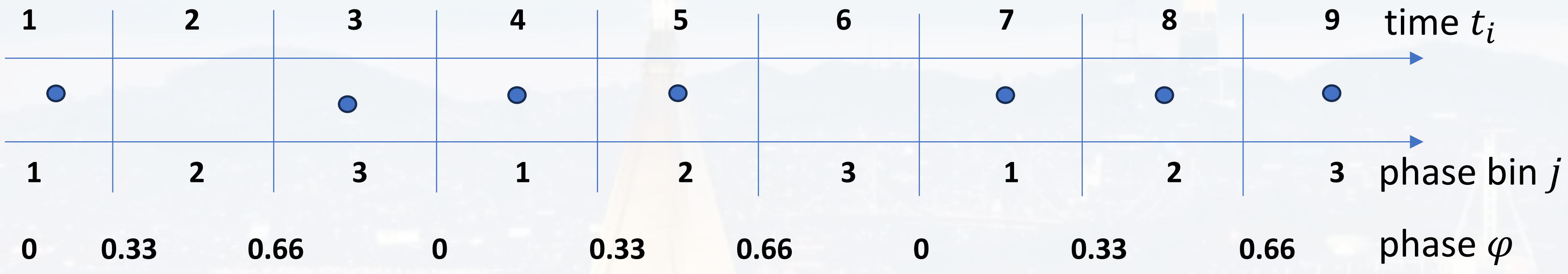


Bayesian Signal Detection:

rewriting the likelihood function in terms of the **light curve parameter**:

$$P(D|\, r(t), t) = (\Delta t)^N \cdot \boxed{\prod_{i=1}^{N} r(t_i)} \cdot \exp\left(-\int_0^T r(t)\, dt\right)$$

$$P(D|\boldsymbol{\omega}, \boldsymbol{\varphi}, \boldsymbol{A}, \boldsymbol{f}, \boldsymbol{M_m}) = ?$$



$\boldsymbol{r(t)}$	: rate
$\boldsymbol{\Delta t}$	: time resolution
$\boldsymbol{T}$	: obs. time span
$\boldsymbol{D}$	: data set
$\boldsymbol{N}$	: number of intervals with n=1
$\boldsymbol{r_j}$	: rate in each phase bin $\boldsymbol{j}$
$\boldsymbol{A}$	: average rate
$\boldsymbol{m}$	: number of phase bins
$\boldsymbol{f_j}$	: fraction of total rate in $\boldsymbol{j}$
$\boldsymbol{f}$	: full set of $\boldsymbol{f_j}$
$\boldsymbol{M_1}$	: constant model
$\boldsymbol{M_m}$	: periodic model m bins
$\boldsymbol{\omega}$	: frequency
$\boldsymbol{\varphi}$	: phase

$$f_j = \frac{r_j}{\sum_{j=1}^{m} r_j} = \frac{r_j}{A\, m}$$

$\boldsymbol{f_1 = 3/7}$
$\boldsymbol{f_2 = 2/7}$
$\boldsymbol{f_3 = 2/7}$

for $T \gg 1/\omega$ these will be average values

$n_j = n_j(\omega, \varphi)$: number of events that fall into phase bin j **(here $\boldsymbol{n_1 = 3, n_2 = 3}$ etc)**

$$\boxed{\prod_{i=1}^{N} r(t_i)} = \prod_{i=1}^{N} A\, m\ f_{j(t_i)} = (A\, m)^N f_1 f_3 f_1 f_2 f_1 f_2 f_3 = (A\, m)^N \prod_{j=1}^{m} f_j^{n_j}$$

Bayesian Signal Detection:

rewriting the likelihood function in terms of the **light curve parameter**:

$$P(D|\,r(t),t) = (\Delta t)^N \cdot (\boldsymbol{A}\,\boldsymbol{m})^N \left(\prod_{j=1}^{m} \boldsymbol{f}_j^{n_j}\right) \exp\left(-\int_0^T r(t)\,dt\right)$$

$$P(D|\boldsymbol{\omega},\boldsymbol{\varphi},\boldsymbol{A},\boldsymbol{f},\boldsymbol{M_m}) = ?$$

$n_j = n_j(\omega,\varphi)$: number of events that fall into phase bin j

r_j = events/time $= n_j/\tau_j$

τ_j: total time integrated per phase bin j $\quad \tau_j \approx \dfrac{T}{m}$

$\boldsymbol{r(t)}$	: rate
$\boldsymbol{\Delta t}$	: time resolution
$\boldsymbol{T}$	: obs. time span
$\boldsymbol{D}$	: data set
$\boldsymbol{N}$	: number of intervals with n=1
$\boldsymbol{r_j}$	: rate in each phase bin j
$\boldsymbol{A}$	: average rate
$\boldsymbol{m}$	: number of phase bins
$\boldsymbol{f_j}$	: fraction of total rate in j
$\boldsymbol{f}$	: full set of $\boldsymbol{f_j}$
$\boldsymbol{M_1}$	: constant model
$\boldsymbol{M_m}$	: periodic model m bins
$\boldsymbol{\omega}$	: frequency
$\boldsymbol{\varphi}$	: phase

$$f_j = \frac{r_j}{\sum_{j=1}^{m} r_j} = \frac{r_j}{A\,m}$$

$$\int_0^T r(t)\,dt = \sum_{i=1}^{Q+N} r(t_i)\cdot\Delta t = \sum_{i=1}^{Q+N} A\,m\,f_{j(t_i)}\,\Delta t$$

$$= A\,m \sum_{i=1}^{Q+N} (f_1 + f_3 + f_1 + f_2 + f_1 + f_2 + f_3)\,\Delta t = A\,m \sum_{j=1}^{m} f_j\,n_j\,\Delta t = A\,m \sum_{j=1}^{m} f_j\,\tau_j$$

Bayesian Signal Detection:

rewriting the likelihood function in terms of the **light curve parameter**:

any periodic model:

$$\boxed{P(D|\omega,\varphi,A,f,M_m) = (\Delta t)^N \cdot (A\,m)^N \left(\prod_{j=1}^{m} f_j^{n_j(\omega,\varphi)}\right) \exp\left(-A\,m \sum_{j=1}^{m} f_j\,\tau_j(\omega,\varphi)\right)}$$

constant model (poiss noise)

$$\boxed{P(D|A,M_1) = (\Delta t)^N \cdot A^N \cdot e^{-AT}}$$

note: - for $T \gg 1/\omega$ and $\tau_j \approx \frac{T}{m}$ on can show that

$$P(D|\omega,\varphi,A,f,M_m) \approx (\Delta t)^N \cdot (A\,m)^N \left(\prod_{j=1}^{m} f_j^{n_j(\omega,\varphi)}\right) \exp(-A\,T)$$

see paper page 152/153

$r(t)$	: rate
Δt	: time resolution
T	: obs. time span
D	: data set
N	: number of intervals with n=1
r_j	: rate in each phase bin j
A	: average rate
m	: number of phase bins
f_j	: fraction of total rate in j
f	: full set of f_j
M_1	: constant model
M_m	: periodic model m bins
ω	: frequency
φ	: phase


WHEN YOU ESTIMATE A BAYESIAN MODEL
This little maneuver is gonna cost us 51 years
imgflip.com

Outline

Bayesian Signal Detection:

any periodic model:

$$\boxed{P(D|\omega,\varphi,A,f,M_m) = (\Delta t)^N \cdot (A\,m)^N \left(\prod_{j=1}^{m} f_j^{\,n_j(\omega,\varphi)}\right) \exp\left(-A\,m \sum_{j=1}^{m} f_j\,\tau_j(\omega,\varphi)\right)}$$

constant model (poiss noise)

$$\boxed{P(D|A,M_1) = (\Delta t)^N \cdot A^N \cdot e^{-AT}}$$

$$\boldsymbol{P(D|M_i) = \int P(D|\{\alpha\}_i\, M_i)\; P(\{\alpha\}_i|\, M_i)\; d\Omega_{\{\alpha\}_i}}$$

$$P(\omega,\varphi,A,f|M_i) = P(\omega|M_i)P(\varphi|M_i)P(A|M_i)P(f|M_i)$$

$r(t)$	: rate
Δt	: time resolution
T	: obs. time span
D	: data set
N	: number of intervals with n=1
r_j	: rate in each phase bin j
A	: average rate
m	: number of phase bins
f_j	: fraction of total rate in j
f	: full set of f_j
M_1	: constant model
M_m	: periodic model m bins
ω	: frequency
φ	: phase

max entropy: $P(\omega|M_i) = \dfrac{1}{\omega \ln(\omega_{max}/\omega_{min})}$ $\qquad \omega_{max} = \dfrac{2\pi\, N}{T} \qquad \omega_{min} = \dfrac{2\pi}{T}$

in practice: $\omega_{min} = 10\,\dfrac{2\pi}{T}$

$$P(\varphi|M_i) = \frac{1}{2\pi}$$

$$P(f|M_i) = (m-1)!\;\delta\left(1 - \sum_{j=1}^{m} f_j\right)$$

delta function δ, see also page 155

$$P(A|M_i) = \frac{1}{A_{max}}$$

Bayesian Signal Detection:

a few notes on pulsar timing:

- rotating dipole → spin down $\dot{\omega} \approx -\, const\ \omega^n$

$$\varphi(t) = \omega\,(t - t_0) + \frac{1}{2}\dot{\omega}(t - t_0)^2$$

fitting $\dot{\omega}$ such that phases are consistent: phase coherent timing


WHEN YOU ESTIMATE A BAYESIAN MODEL
This little maneuver is gonna cost us 51 years
imgflip.com

Outline

Bayesian Signal Detection:

$$P(D|M_i) = \int P(D|\{\alpha\}_i\, M_i)\, P(\{\alpha\}_i|\, M_i)\;\; d\Omega_{\{\alpha\}_i}$$

for $T \gg 1/\omega$ and $\tau_j \approx \frac{T}{m}$:

$$P(D|M_m) \approx \frac{1}{A_{max}} \frac{1}{2\pi} \frac{(m-1)!}{\ln\left(\frac{\omega_{max}}{\omega_{min}}\right)} \int \frac{1}{\omega} (\Delta t)^N \cdot (A\, m)^N \left(\prod_{j=1}^{m} f_j^{\, n_j(\omega,\varphi)} \delta\left(1 - \sum_{j=1}^{m} f_j\right)\right) e^{-AT}\;\; d\omega\, df\, dA\, d\varphi$$

we need **(estimation of frequency)**:

$$P(\omega|D, M_m) = P(\omega|M_m) \frac{P(D|\omega, M_m)}{P(D|M_m)} \;\; = \frac{1}{\omega \ln(\omega_{max}/\omega_{min})} \frac{P(D|\omega, M_m)}{P(D|M_m)}$$

$$P(D|\omega, M_m) = \frac{\Delta t^N (m-1)!}{2\pi\, A_{max}} \int (A\, m)^N \left(\prod_{j=1}^{m} f_j^{\, n_j(\omega,\varphi)} \delta\left(1 - \sum_{j=1}^{m} f_j\right)\right) e^{-AT}\;\; df\, dA\, d\varphi$$

note: we can also marginalize over m, if we are **not interested in the shape** of the light curve

Bayesian Signal Detection:

$$P(D|\omega, M_m) = \frac{\Delta t^N (m-1)!}{2\pi\, A_{max}} \int (A\, m)^N \left(\prod_{j=1}^{m} f_j^{n_j(\omega,\varphi)} \delta\left(1 - \sum_{j=1}^{m} f_j\right)\right) e^{-AT}\; df\; dA\; d\varphi$$

$$P(D|\omega, M_m) = \frac{\Delta t^N (m-1)!}{2\pi\, A_{max}} \underbrace{\int (A\, m)^N e^{-AT}\; dA}_{} \int \prod_{j=1}^{m} f_j^{n_j(\omega,\varphi)} \delta\left(1 - \sum_{j=1}^{m} f_j\right) df\, d\varphi$$

$$\Gamma(a,b) = \int_a^b y^{a-1} e^{-y} dy$$

$$\int (A\, m)^N e^{-AT}\; dA = \Gamma(N+1, A_{max}\, T)\frac{1}{T^{N+1}}$$

$$\int \prod_{j=1}^{m} f_j^{n_j(\omega,\varphi)} \delta\left(1 - \sum_{j=1}^{m} f_j\right) df\, d\varphi$$ generalized β integral with constrain $\sum_{j=1}^{m} f_j = 1$

$$\int \prod_{j=1}^{m} f_j^{n_j(\omega,\varphi)} \delta\left(1 - \sum_{j=1}^{m} f_j\right) df\, d\varphi = \frac{m^N}{(N+m-1)!}\int_0^{2\pi} \prod_{j=1}^{m} n_j(\omega,\varphi)!\; d\varphi$$

Bayesian Signal Detection:

$$P(D|\omega, M_m) = \frac{\Delta t^N (m-1)!}{2\pi\, A_{max}} \Gamma(N+1, A_{max}\, T) \frac{1}{T^{N+1}} \frac{m^N}{(N+m-1)!} \int_0^{2\pi} \prod_{j=1}^{m} n_j(\omega, \varphi)!\; d\varphi$$

$$P(D|\omega, M_m) = \frac{\Delta t^N (m-1)!}{2\pi\, A_{max}} \Gamma(N+1, A_{max}\, T) \frac{1}{T^{N+1}} \frac{m^N}{(N+m-1)!}\, N! \int_0^{2\pi} \frac{1}{N!} \prod_{j=1}^{m} n_j(\omega, \varphi)!\; d\varphi$$

multiplicity: $\Omega_m(\omega, \varphi) = \frac{N!}{\prod_{j=1}^{m} n_j(\omega,\varphi)!}$

$$P(D|\omega, M_m) = \frac{\Delta t^N (m-1)!}{2\pi\, A_{max}} \Gamma(N+1, A_{max}\, T) \frac{1}{T^{N+1}} \frac{m^N}{(N+m-1)!}\, N! \int_0^{2\pi} \frac{1}{\Omega_m(\omega, \varphi)}\, d\varphi$$

“Occam Factors” penalize complex models (large values for number of bins *m*)

Bayesian Signal Detection:

$$P(D|\omega, M_m) = \frac{\Delta t^N (m-1)!}{2\pi\, A_{max}} \Gamma(N+1, A_{max}\, T) \frac{1}{T^{N+1}} \frac{m^N}{(N+m-1)!} \int_0^{2\pi} \prod_{j=1}^{m} n_j(\omega,\varphi)!\; d\varphi$$

we still need $P(D|M_m)$, but it is the same steps as for $P(D|\omega, M_m)$ just with a $\int \frac{d\omega}{\omega}$ term

finally:

using: $\Omega_m(\omega,\varphi) = \frac{N!}{\prod_{j=1}^{m} n_j(\omega,\varphi)!}$

$$P(\omega|D, M_m) = \frac{1}{\omega} \frac{1}{\int \frac{d\omega}{\omega} \frac{d\varphi}{\Omega_m(\omega,\varphi)}} \int \frac{d\varphi}{\Omega_m(\omega,\varphi)}$$

note:
- the multiplicity Ω_m of the binned ToA contains all the information about ω (and φ)
- neither A_{max} nor $\ln\left(\frac{\omega_{max}}{\omega_{min}}\right)$ appear in the above equation → rel. insensitive to these priors
- more complex models (large *m*) are penalized (more detail: paper, Sect. 5.2)


WHEN YOU ESTIMATE A BAYESIAN MODEL
This little maneuver is gonna cost us 51 years
imgflip.com

Outline

Bayesian Signal Detection: A Code Example

python package `BayesianSignalDetect.py`

```python
from BayesianSignalDetect import *
```

- a *decorator* that measures runtime of a function → my_timer
- *function* creates a test dataset for illustration → CreateSignal
- *FFT* for comparison → FFT
- plot routine for *periodogram* → PlotPvsW
- calculates $P(D|\omega, \varphi, A, m)$ → CalculateProbs
- calls *CalculateProbs* and calculates $P(D|M_i)$ → RunAnalysis
- calls main part of the code – runs *RunAnalysis* on multiple cpus in parallel → FindFrequency

BayesianSignalDetect.py
- F my_timer
- F CreateSignal
- F FFT
- F PlotPvsW
- F CalculateProbs
- C SignalDetect
  - m __init__
  - m RunAnalysis
  - m FindFrequency

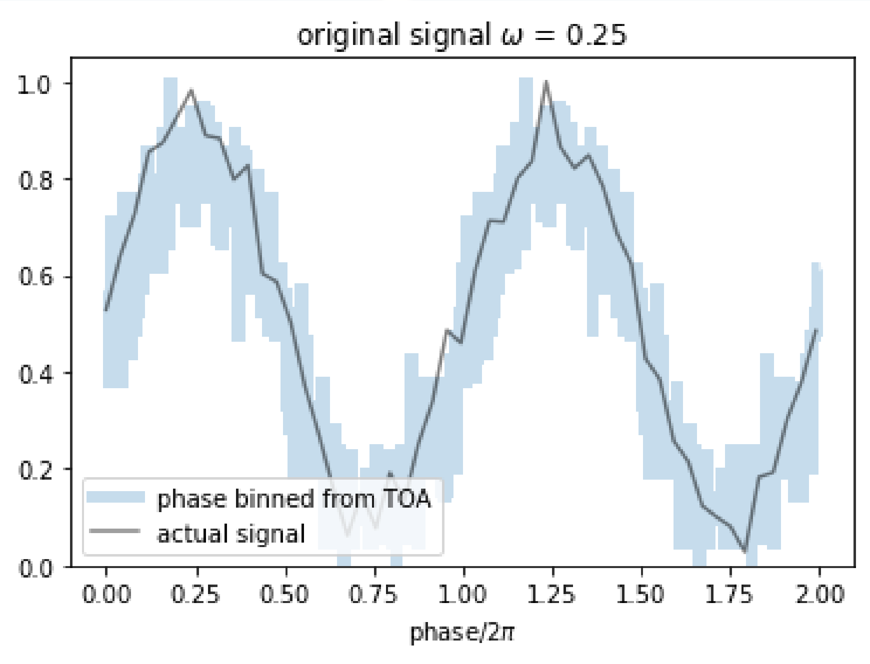
Bayesian Signal Detection:

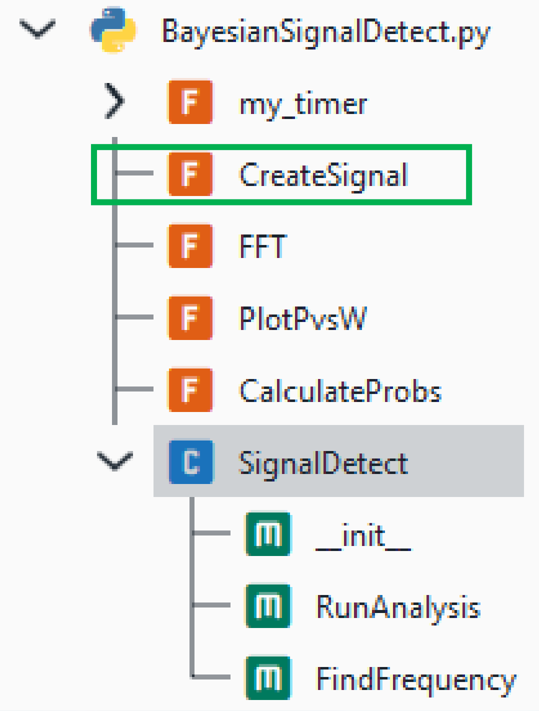
python package `BayesianSignalDetect.py`

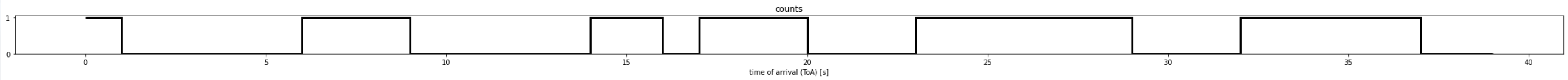
```python
from BayesianSignalDetect import *
```

N + Q

```python
T = CreateSignal(5000, 0.25, 0.1)
```







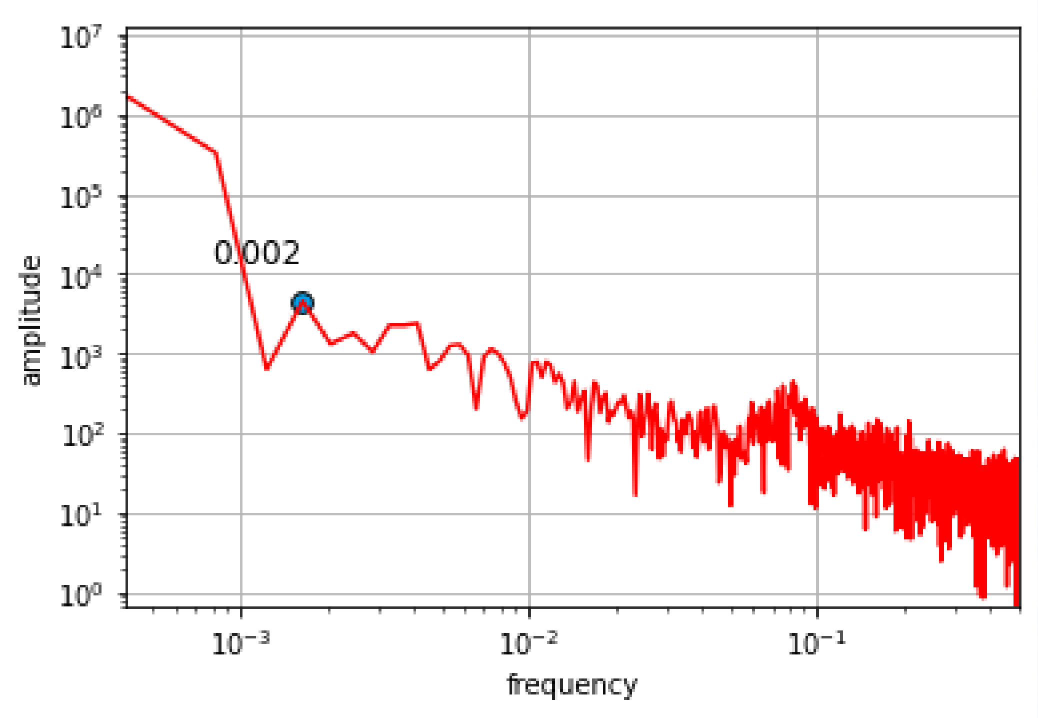
Bayesian Signal Detection:

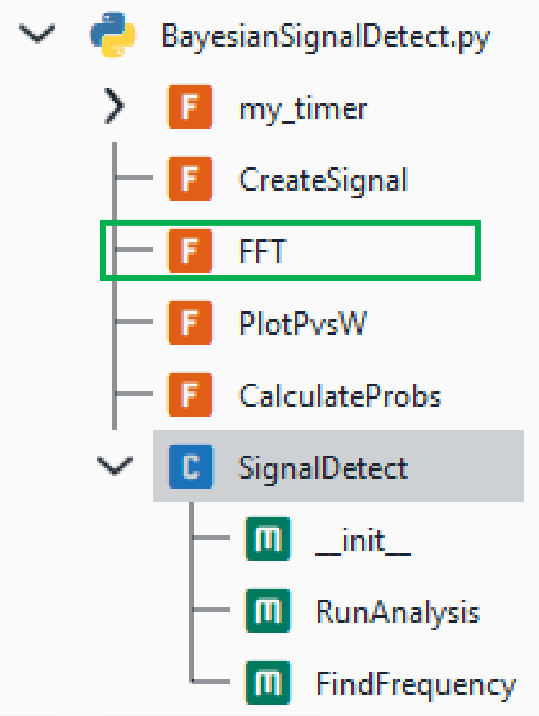
python package BayesianSignalDetect.py

```
from BayesianSignalDetect import *

T = CreateSignal(5000, 0.25, 0.1)

FFT(T)
```





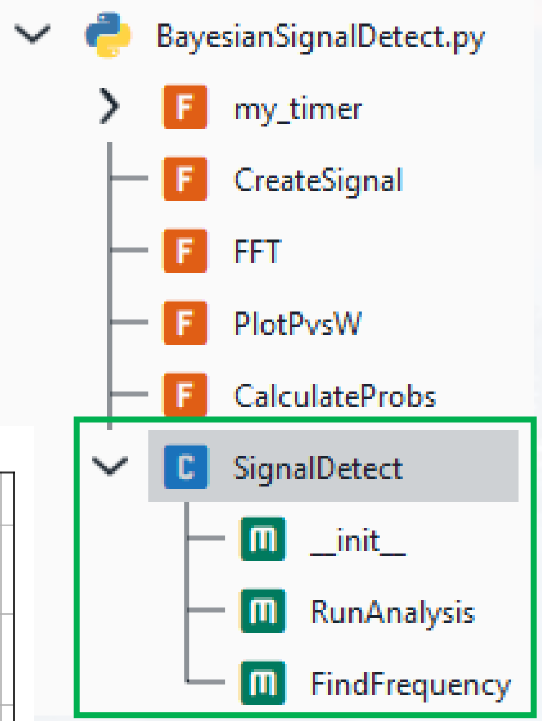
Bayesian Signal Detection:

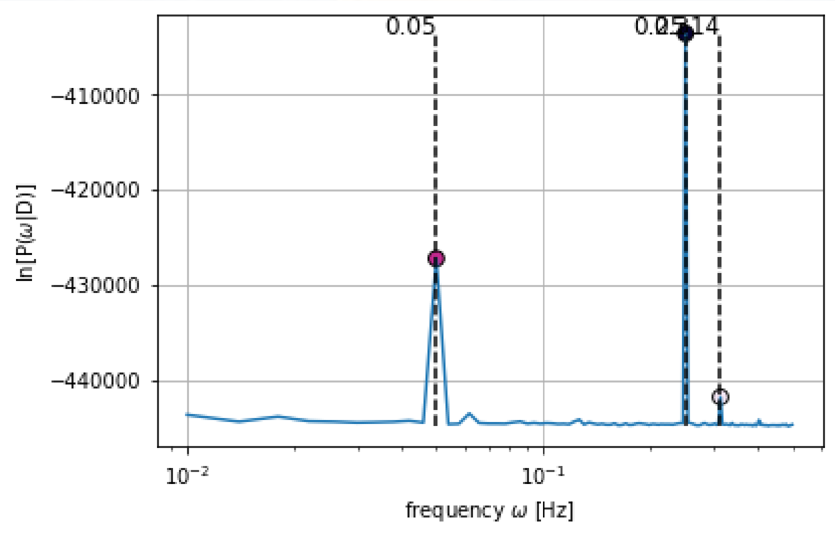
python package BayesianSignalDetect.py

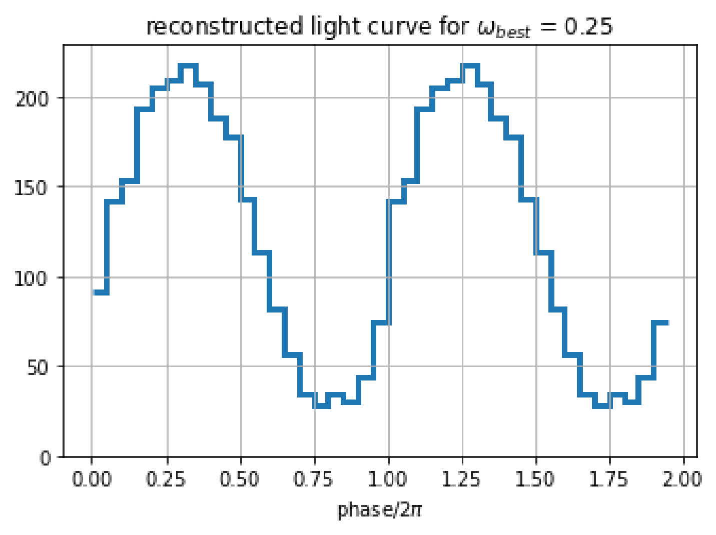
```
from BayesianSignalDetect import *


T = CreateSignal(5000, 0.25, 0.1)


S = SignalDetect(T, w_end = 0.5, w_start = 0.01)
[Omega, P] = S.FindFrequency()
```

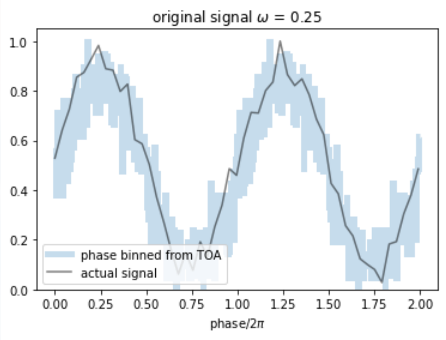


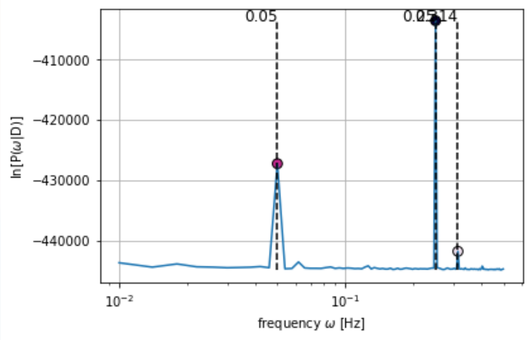


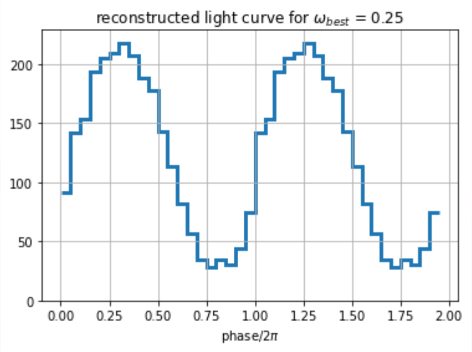


Bayesian Signal Detection:

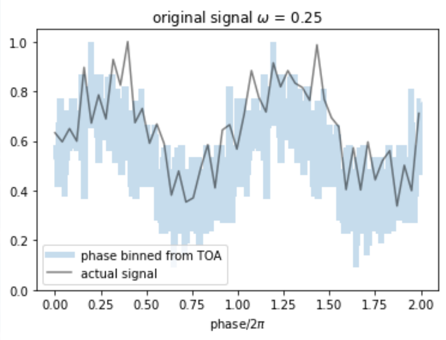

```
T = CreateSignal(5000, 0.25, 0.1)
```

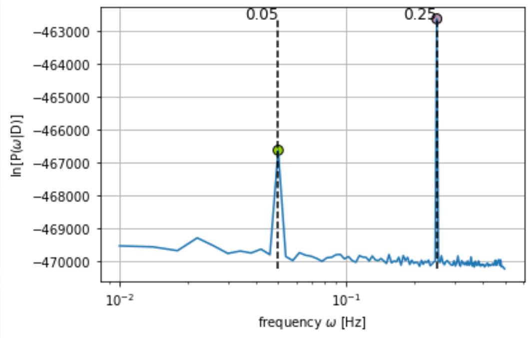


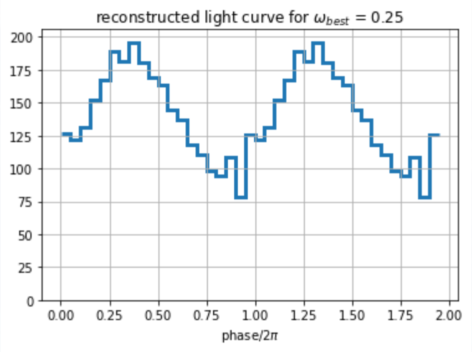




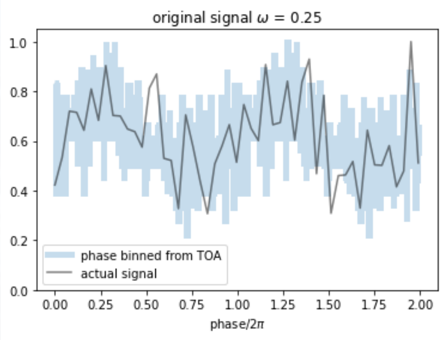

```
T = CreateSignal(5000, 0.25, 0.5)
```

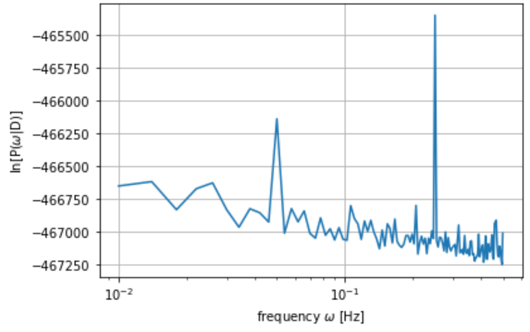


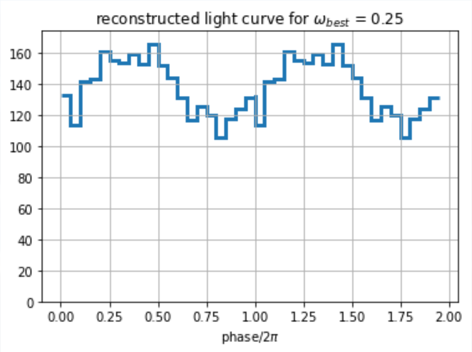





```
T = CreateSignal(5000, 0.25, 1)
```







Thank you very much for your attention!