Lecture 6:

Variational Bayes, Expectation Maximization

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Bayesian Data Analysis and Machine Learning for Physical Sciences

Course Map

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Variational Bayes, Expectation Maximization

Outline

The Problem

K-means

Actual EM

Variational Bayes

Outline

The Problem

K-means

Actual EM

Variational Bayes

Variational Bayes, Expectation Maximization:

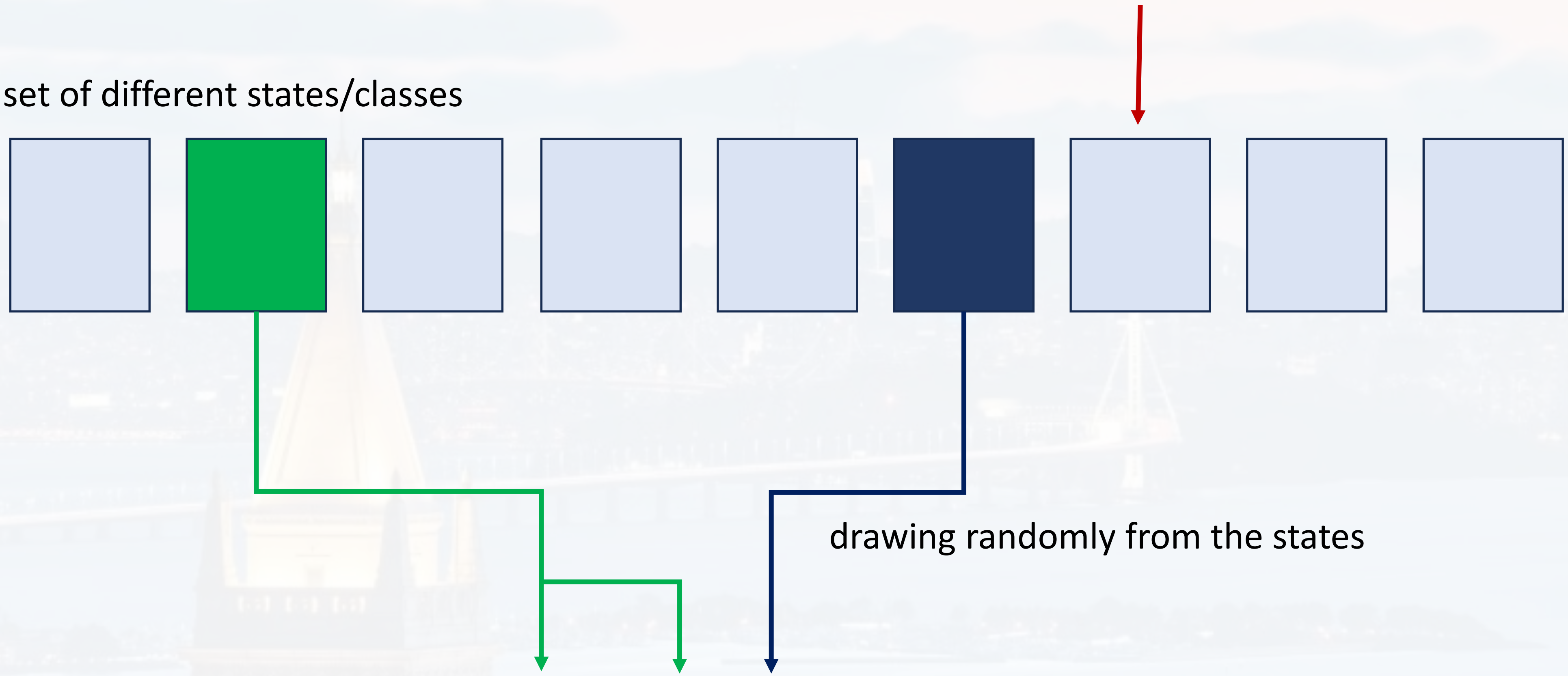
set of different states/classes

a set of N observations $\{x_1, x_2, x_3, \dots x_n, \dots x_N\}$

Variational Bayes, Expectation Maximization:

each state k draws x_n from a distribution $L_k\left(x_{n,k}\middle|\theta\right)$ with parameters θ

set of different states/classes



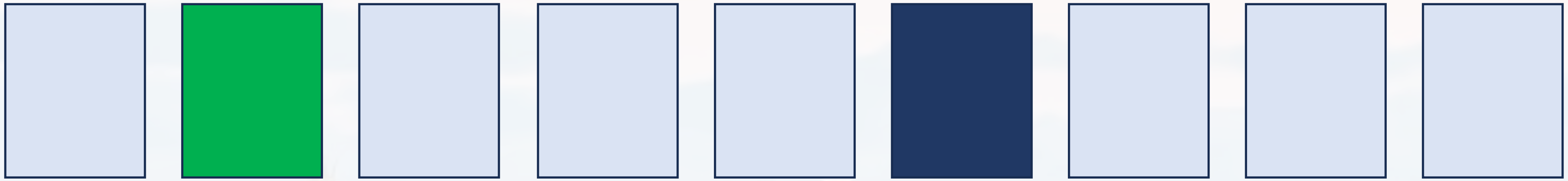
drawing randomly from the states

a set of *N* observations $\{x_1, x_2, x_3, \dots x_n, \dots x_N\}$

Variational Bayes, Expectation Maximization:

The Problem

set of different states/classes $L_k(x_{n,k}|\theta)$



a set of *N* observations $\{x_1, x_2, x_3, \ldots x_n, \ldots x_N\}$

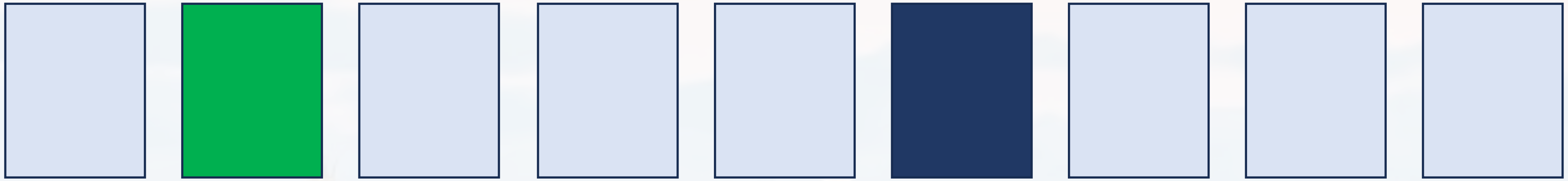
problem: - we have a model of $L_k(x_{n,k}|\theta)$, but
- we don't know θ and
- we don't know from which class/state *k* x_n has been generated

goal: - find an estimator for θ and find the class/state *k* of each x_n

Variational Bayes, Expectation Maximization:

set of different states/classes

$L_k(x_{n,k}|\theta)$



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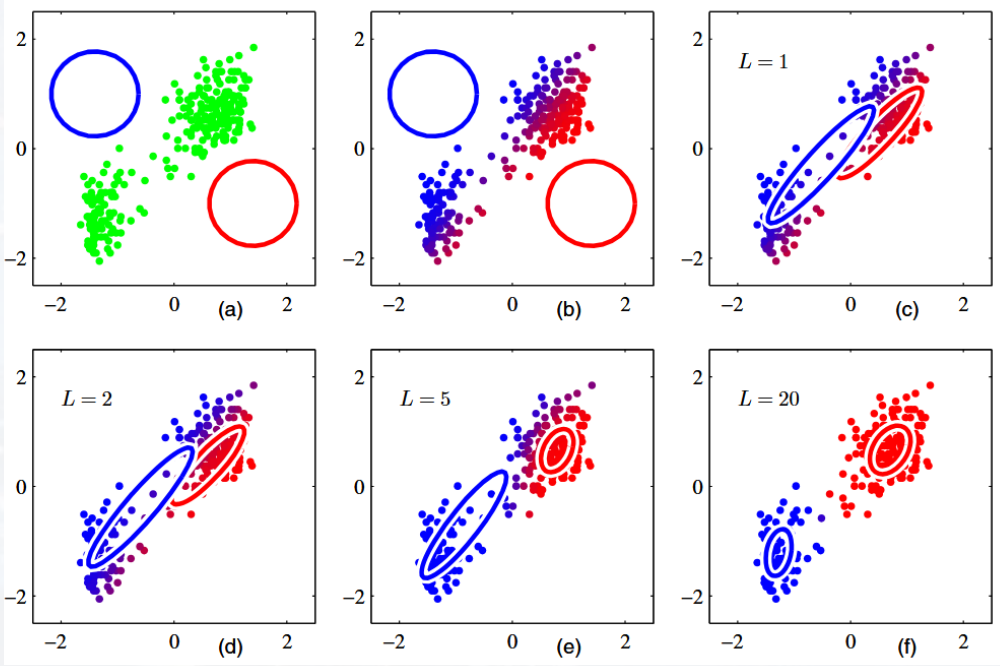


Bishop: “Pattern Recognition”

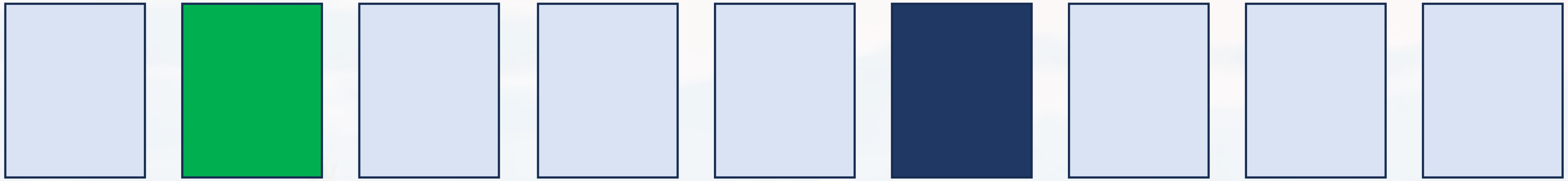
- Gaussian Mixture Models (GMM)
- K-means (clustering, image segmentation)
- HMM

→ unsupervised

Variational Bayes, Expectation Maximization:

set of different states/classes

$L_k(x_{n,k}|\theta)$



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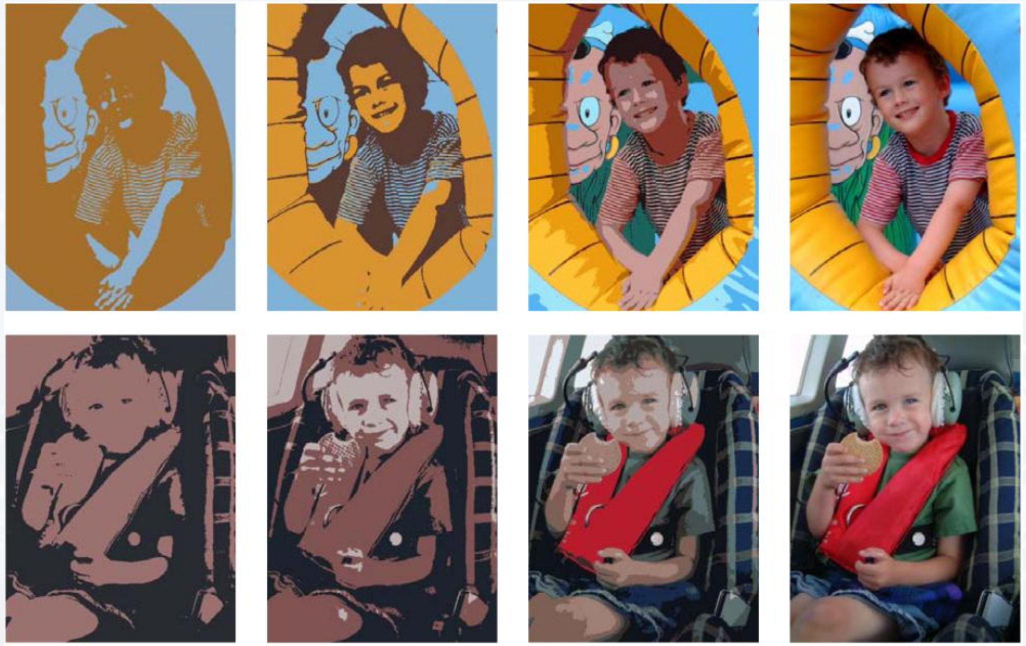

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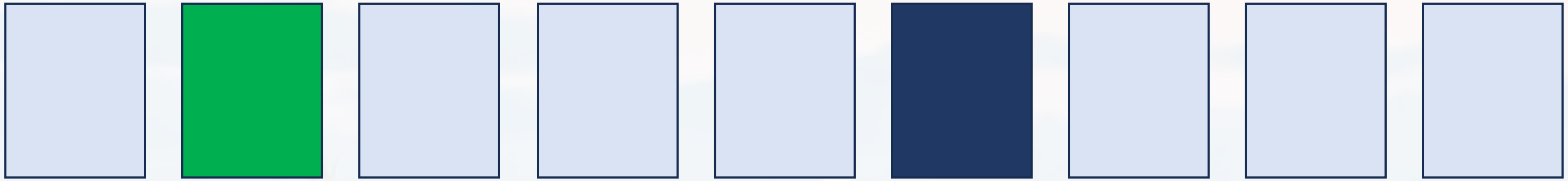
Variational Bayes, Expectation Maximization:

set of different states/classes

$L_k\left(x_{n,k} \middle| \theta\right)$



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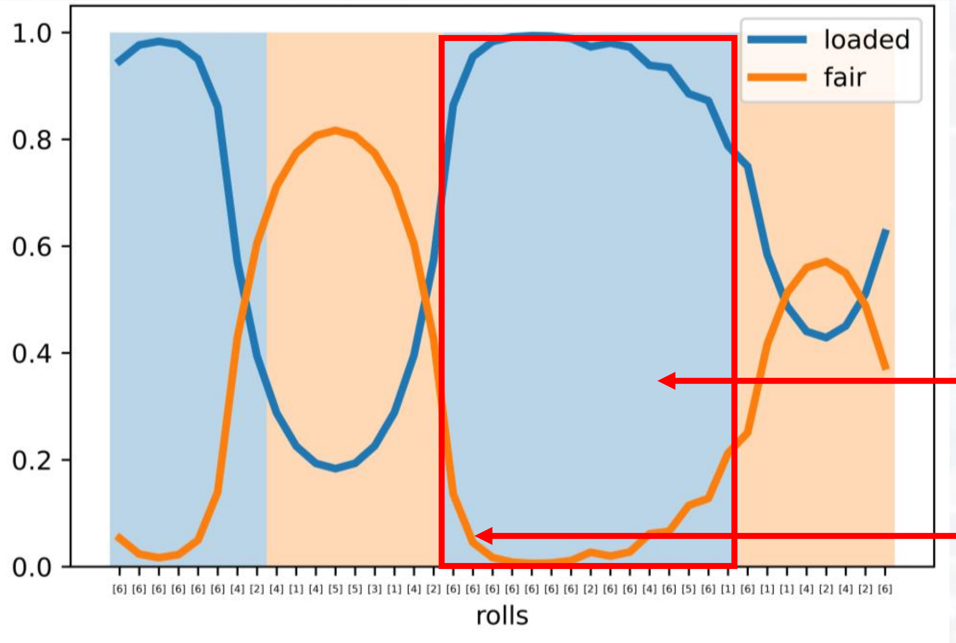

- Gaussian Mixture Models (GMM)
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- HMM

→ unsupervised

actual (unknown, aka hidden state)

predicted state, given observation

Outline

The Problem

K-means

Actual EM

Variational Bayes

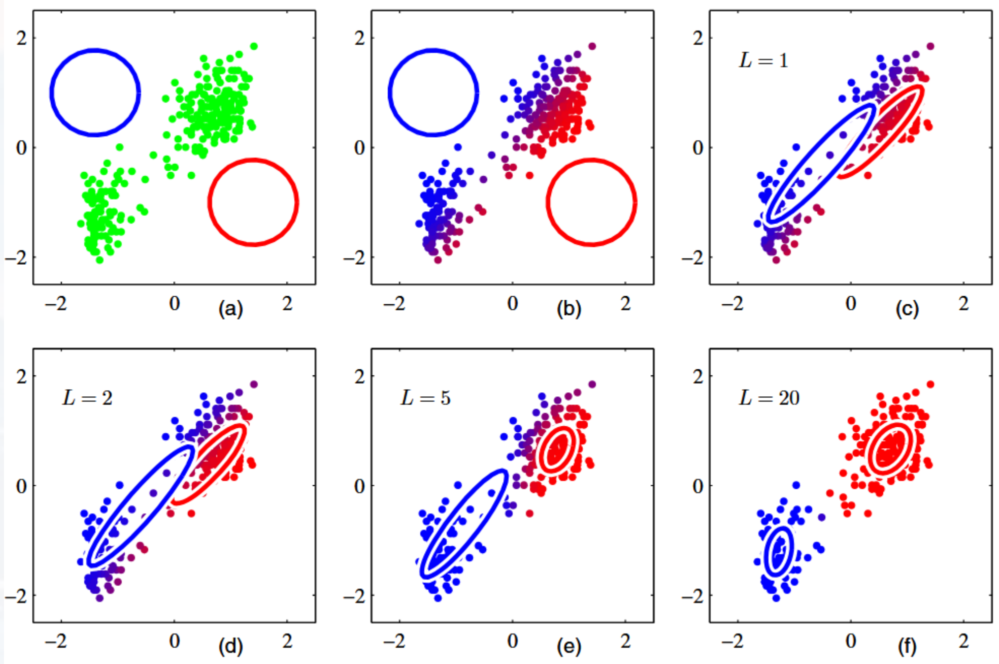
Variational Bayes, Expectation Maximization:

K	: number of cluster
N	: number of observations

indicator function $r_{n,k} \epsilon \{0, 1\}$

goal: minimizing

$$J = \sum_{n=1}^{N} \sum_{k=1}^{K} r_{n,k} \; \|x_n - \mu_k\|^2$$



μ_k: barycenter of cluster k

assigning x_n to its closed mean

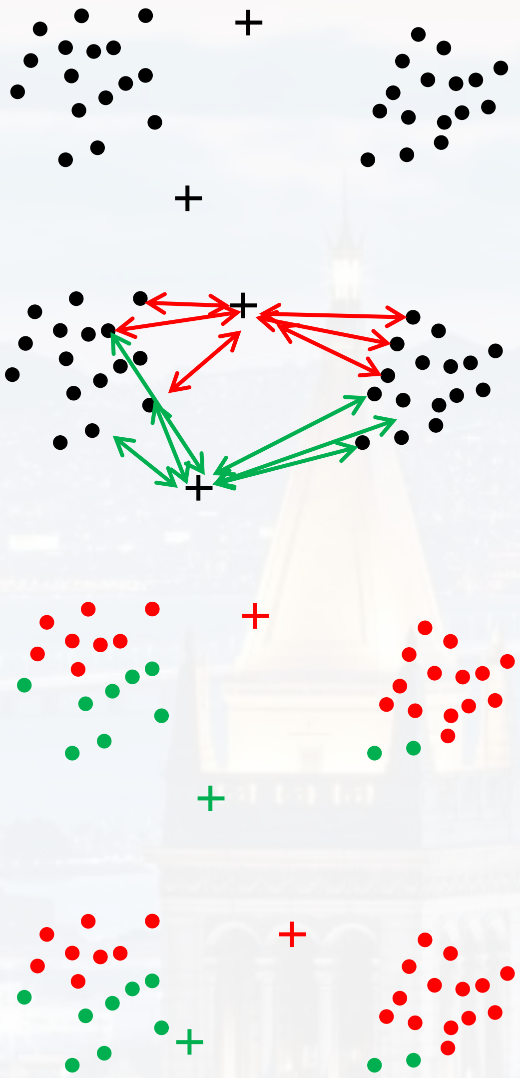
$$r_{n,k} = \begin{cases} 1, & if \; k = \; argmin_j \|x_n - \mu_j\|^2 \\ 0, & else \end{cases}$$

$$\frac{\partial J}{\partial \mu_k} = 0 \qquad \text{MLE} \longrightarrow \qquad \mu_k = \frac{\sum_{n=1}^{N} r_{n,k} x_n}{\sum_{n=1}^{N} r_{n,k}}$$

K – means is an iterative process

Variational Bayes, Expectation Maximization:

K – means is an iterative process



a) assign k means randomly

b) calculate *distance* from each point to each mean

c) assign each point to its closest mean

d) update the means accordingly

K	: number of cluster
N	: number of observations
μ_k	: barycenter of the cluster

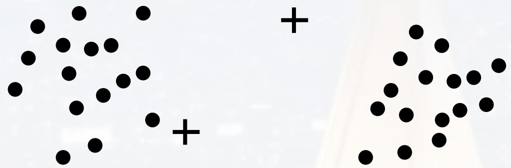
$$r_{n,k} = \begin{cases} 1, & \text{if } k = \operatorname{argmin}_j \|x_n - \mu_j\|^2 \\ 0, & \text{else} \end{cases}$$

$$\boxed{J = \sum_{n=1}^{N} \sum_{k=1}^{K} r_{n,k} \, \|x_n - \mu_k\|^2}$$

Variational Bayes, Expectation Maximization:

K – means is an iterative process

d) update the means accordingly



e) go back to b)

K : number of cluster

N : number of observations

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$$r_{n,k} = \begin{cases} 1, & \text{if } k = \operatorname{argmin}_j \|x_n - \mu_j\|^2 \\ 0, & \text{else} \end{cases}$$

$$\boxed{J = \sum_{n=1}^{N} \sum_{k=1}^{K} r_{n,k} \, \|x_n - \mu_k\|^2}$$

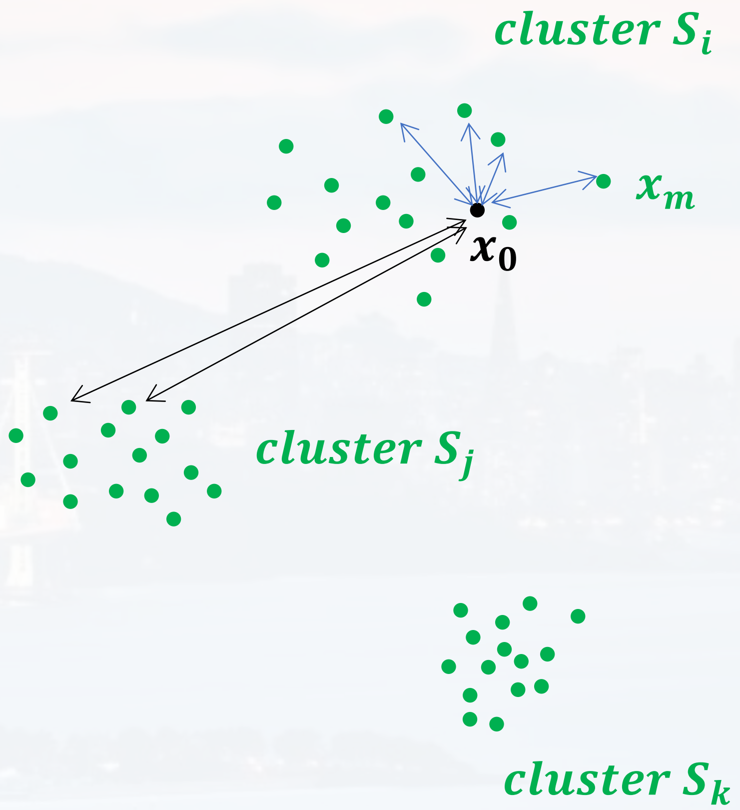
Variational Bayes, Expectation Maximization:

problem: ***K = number of cluster,*** is a hyperparameter. How do I know the correct value for ***K***?

→ silhouette Ψ

- distance $\boldsymbol{d_1}$ of a data point x_0 to ***its assigned cluster*** $\boldsymbol{S_i}$
  vs distance $\boldsymbol{d_2}$ to ***closest cluster (here*** $\boldsymbol{S_j}$***)***

$$\boxed{\Psi(x_0) = \begin{cases} 0 & if\ d_1 = 0 \\ \dfrac{d_2 - \boldsymbol{d_1}}{max[\boldsymbol{d_1}; d_2]} & \end{cases}}$$



- average over all points → ψ_{tot}

if

$\psi_{tot} = 0.75\ \ldots 1.00$ → well clustered
$\psi_{tot} = 0.50\ \ldots 0.75$ → medium clustered
$\psi_{tot} = 0.25\ \ldots 0.50$ → poorly clustered
$\psi_{tot} < 0.25$ → data has no structure

Variational Bayes, Expectation Maximization:

problem: ***K* = number of cluster,** is a hyperparameter. How do I know the correct value for ***K***?

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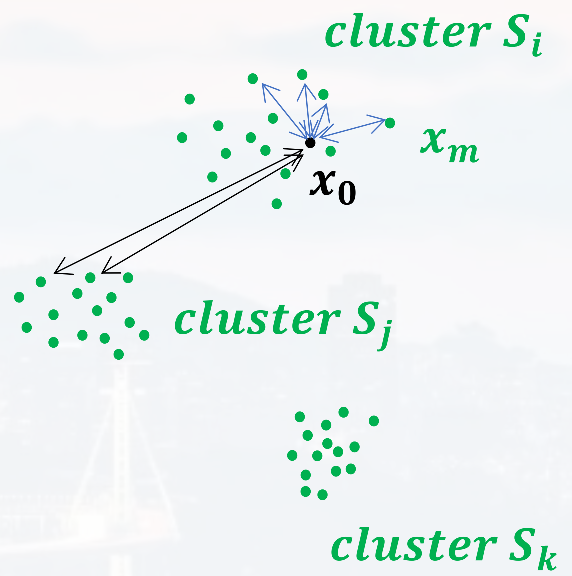

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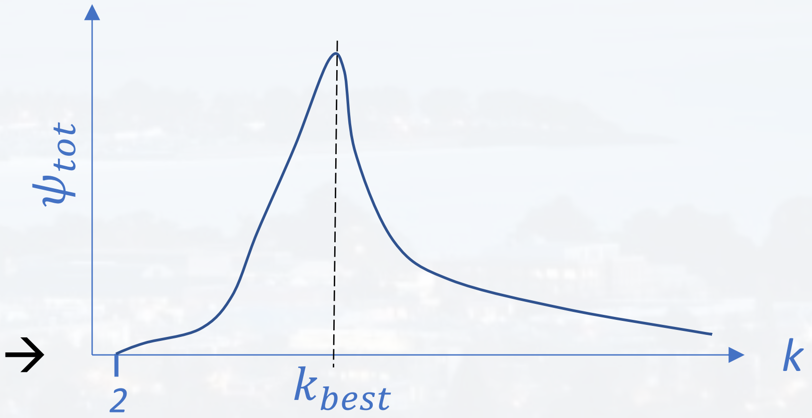
- average over all points → ψ_{tot}

if	$\psi_{tot} = 0.75\ \ldots 1.00$	→ well clustered
	$\psi_{tot} = 0.50\ \ldots 0.75$	→ medium clustered
	$\psi_{tot} = 0.25\ \ldots 0.50$	→ poorly clustered
	$\psi_{tot} < 0.25$	→ data has no structure

see `Walk_Through_Kmeans.ipynb`

ideal world →



Variational Bayes, Expectation Maximization:

$\boldsymbol{K}$	: number of cluster
$\boldsymbol{N}$	: number of observations
$\boldsymbol{\mu_k}$	: barycenter of the cluster

$$\boldsymbol{r_{n,k}} = \begin{cases} \boldsymbol{1}, & \boldsymbol{if\ k = argmin_j \|x_n - \mu_j\|^2} \\ \boldsymbol{0}, & \boldsymbol{else} \end{cases}$$

$$\boxed{J = \sum_{n=1}^{N} \sum_{k=1}^{K} r_{n,k} \, \|x_n - \mu_k\|^2}$$

the actual problem:

observation x_n has been drawn from any of the cluster C_k

$$P(x_n) = \sum_{k=1}^{K} P(x_n|C_k)\, P(C_k) \qquad \sum_{k=1}^{K} P(C_k) = 1$$

$P(x_n|C_k)$ likelihood function

$P(C_k)$ mixing coefficient

general:

$$P(x_n) = \sum_{z} P(x_n|z)\, P(z)$$

z: latent variable **(i. e. not observable, but can be inferred from $\{x_n\}$)**

Outline

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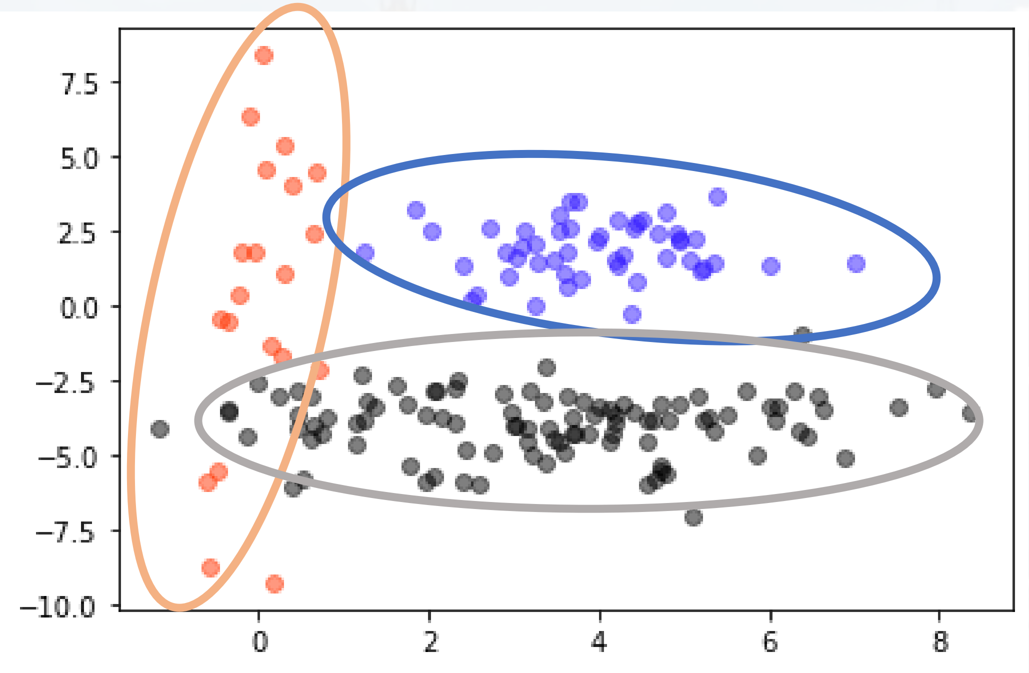
Variational Bayes, Expectation Maximization:

example: **G**aussian **M**ixture **M**odels

K	**: number of cluster**
π_k	**: mixing coefficient**

f features

$$\mathcal{N}_k(x|\mu_k, \Sigma_k) = \frac{1}{(2\pi)^{f/2}\det(\Sigma_k)^{1/2}}\, exp\left[-\frac{1}{2}(x - \mu_k)^T {\Sigma_k}^{-1}(x - \mu_k)\right]$$



two features, $K = 3$ components

$$P(x) = \sum_z P(x|z)\, P(z)$$

$$= \sum_{k=1}^{K} \mathcal{N}_k(x|\mu_k, \Sigma_k)\, \pi_k$$

Variational Bayes, Expectation Maximization:

N	**: number of observations**
K	**: number of cluster**
$\boldsymbol{\pi_k}$	**: mixing coefficient**

example: **G**aussian **M**ixture **M**odels

$$P(x) = \sum_z P(x|z)\, P(z) = \sum_{k=1}^{K} \mathcal{N}_k(x|\mu_k, \Sigma_k)\, \pi_k$$

indicator variable $z_k \, \epsilon \, \{0, 1\}$

goal: $$P(z_k = 1|x) = \frac{P(z_k = 1)\, P(x|z_k = 1)}{P(x)} = \frac{\pi_k\, \mathcal{N}_k(x|\mu_k, \Sigma_k)}{\sum_{j=1}^{K} \mathcal{N}_j\left(x|\mu_j, \Sigma_j\right) \pi_j}$$

via maximizing likelihood by finding best θ

$$L = ln[P(x|\pi, \mu, \Sigma)] = \sum_{n=1}^{N} ln\left\{\sum_{k=1}^{K} \pi_k\, \mathcal{N}_k(x_n|\mu_k, \Sigma_k)\right\}$$

model parameter $\theta = \{\pi, \mu, \Sigma\}$

Variational Bayes, Expectation Maximization:

N	**: number of observations**
K	**: number of cluster**
π_k	**: mixing coefficient**

example: **G**aussian **M**ixture **M**odels

1) initialize $\theta = \{\pi, \mu, \Sigma\}$ and $P(z_k = 1|x_n)$

2) **E**xpectation step *t*:

$$P(z_k = 1|x_n) = \frac{\pi_k\, \mathcal{N}_k(x_n|\mu_k, \Sigma_k)}{\sum_{j=1}^{K} \mathcal{N}_j\left(x_n|\mu_j, \Sigma_j\right) \pi_j}$$

i.e. evaluate $P(z|x, \theta)$

3) **M**aximization step *t* (for example MLE):

$$\mu_k^t = \frac{1}{N_k} \sum_{n=1}^{N} P(z_k = 1|x_n)\, x_n$$

$$\Sigma_k^t = \frac{1}{N_k} \sum_{n=1}^{N} P(z_k = 1|x_n)\, (x_n - \mu_k^t)\, (x_n - \mu_k^t)^T$$

$$N_k^t = \sum_{n=1}^{N} P(z_k = 1|x_n)$$

$$\pi_k^t = \frac{N_k}{N}$$

$$\theta^t = \underset{\theta}{argmax} \{L(\theta^{t-1})\}$$

Variational Bayes, Expectation Maximization:

example: **G**aussian **M**ixture **M**odels

N	**: number of observations**
K	**: number of cluster**
$\boldsymbol{\pi_k}$	**: mixing coefficient**

1) initialize $\theta = \{\pi, \mu, \Sigma\}$ and $P(z_k = 1|x_n)$

2) **E**xpectation step *t*:

$$P(z_k = 1|x_n) = \frac{\pi_k\, \mathcal{N}_k(x_n|\mu_k, \Sigma_k)}{\sum_{j=1}^{K} \mathcal{N}_j\left(x_n|\mu_j, \Sigma_j\right) \pi_j}$$

i.e. evaluate $P(z|x, \theta)$

3) **M**aximization step *t* (for example MLE):

$$\mu_k^t = \frac{1}{N_k} \sum_{n=1}^{N} P(z_k = 1|x_n)\, x_n \qquad \Sigma_k^t = \frac{1}{N_k} \sum_{n=1}^{N} P(z_k = 1|x_n)\, (x_n - \mu_k^t)\, (x_n - \mu_k^t)^T \qquad N_k = \sum_{n=1}^{N} P(z_k = 1|x_n) \qquad \pi_k^t = \frac{N_k}{N}$$

$$\theta^t = \underset{\theta}{argmax} \left\{ \sum_z P(z|x, \theta^{t-1})\ ln[P(x, z|\theta^{t-1})] \right\}$$

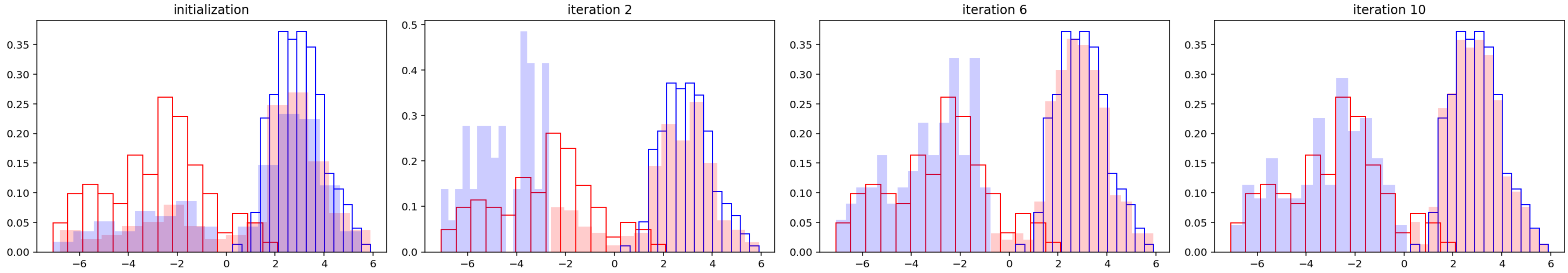
4) evaluate log likelihood

$$ln[P(x|\pi, \mu, \Sigma)] = \sum_{n=1}^{N} ln \left\{ \sum_{k=1}^{K} \pi_k\, \mathcal{N}_k(x_n|\mu_k, \Sigma_k) \right\}$$

Variational Bayes, Expectation Maximization:

note:

- one can show that EM indeed converges and maximizes the likelihood function (Bishop, Sec 9.4)
- no guarantee to find a **global** minimum
- applications: HMM, unsupervised clustering, image segmentation (before CNNs)
- iterative process: $\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + \Delta\theta$ → connection to gradient descent (see later)
- see also `EM__Example.py`



blue and red swapped
→ unsupervised learning

Variational Bayes, Expectation Maximization

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Variational Bayes, Expectation Maximization:

EM: - likelihood function given

- parameter via MLE (point estimate)

problem: - integrals can be complicated, calculation takes too long etc ("intractable")

- need **pdf of desired parameter** (see BPE) without ad-hoc constrain

only two assumptions: - 1) maximum entropy

- 2) pdf factorizes wrt its parameters (→ mean field approximation)

D : data set
$Z = \{Z_1, \dots Z_n\}$: set of (latent) parameter

goal: - find an approximation $Q(Z)$ for the posterior $P(Z|D)$ via **max ent**
- the more data, $Q(Z) \rightarrow P(Z|D)$ ("learning")

idea: - $\mathrm{Q}(Z) = \prod_{i=1}^{n} q_i(Z_i|D)$ mean field approximation

Variational Bayes, Expectation Maximization:

find an approximation $Q(Z)$ for the posterior $P(Z|D)$

D	**: data set**
Z	**: set of n (latent) parameter**

KL divergence: tells us how much our information is "off" if we work with the approximation $Q(Z)$

$$KL(P||Q) = -\int P(x)\log\left[\frac{Q(x)}{P(x)}\right]dx \qquad \text{(module 1)}$$

goal: find the $Q(x)$ that **minimizes $\boldsymbol{KL(P||Q)}$** → variational calculus (Euler-Lagrange)

same principle: **maximizing evidence** $P(D)$ → variational calculus (Euler−Lagrange)

we run the 2nd idea → find an equation we know from stat TD
see also reading Paisley, Blei & Jordan

$$ln[P(D)] = ln\left[\int P(D,Z)\,dZ\right] = ln\left[\int P(D,Z)\frac{Q(Z)}{Q(Z)}dZ\right] \geq \int Q(Z)\ln\left[\frac{P(D,Z)}{Q(Z)}\right]dZ$$

"Jensen inequality" → evidence lower bound ("ELBO")

Variational Bayes, Expectation Maximization:

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maximizing evidence $P(D)$

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"Jensen inequality"

$$ln[P(D)] \geq \int Q(Z)\ ln[P(D,Z)]\ dZ\ - \boxed{\int Q(Z)\ ln[Q(Z)]\ dZ}$$

entropy of $Q(Z)$

$$ln[P(D)] \geq \int \boxed{\prod_{i=1}^{n} q_i(Z_i|D)}\ ln[P(D,Z)]\ dZ\ - \underbrace{\int \boxed{\prod_{i=1}^{n} q_i(Z_i|D)}\ ln\left[\boxed{\prod_{i=1}^{n} q_i(Z_i|D)}\right]\ dZ}_{S_q}$$

assumption 2

Variational Bayes, Expectation Maximization:

D	: data set
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find an approximation $Q(Z)$ for the posterior $P(Z|D)$

maximizing evidence $P(D)$

$$S_q = \int \prod_{i=1}^{n} q_i(Z_i|D)\, ln\left[\prod_{i=1}^{n} q_i(Z_i|D)\right] dZ$$

one n-dimensional problem (hard)

$$= \int \{q_1(Z_1|D)\, q_2(Z_2|D) \ldots\, q_N(Z_N|D)\}\, ln[q_1(Z_1|D)]\, dZ_1 dZ_2 \ldots dZ_n +$$

$$\int \{q_1(Z_1|D)\, q_2(Z_2|D) \ldots\, q_N(Z_N|D)\}\, ln[q_2(Z_2|D)]\, dZ_1 dZ_2 \ldots dZ_n +$$

$$\ldots +$$

$$\int \{q_1(Z_1|D)\, q_2(Z_2|D) \ldots\, q_N(Z_N|D)\}\, ln[q_n(Z_n|D)]\, dZ_1 dZ_2 \ldots dZ_n$$

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$$\int \{q_1(Z_1|D)\, q_2(Z_2|D) \ldots\, q_N(Z_N|D)\}\, ln[q_n(Z_n|D)]\, dZ_1 dZ_2 \ldots dZ_n$$

$$= \int q_1(Z_1|D)\, ln[q_1(Z_1|D)]\, dZ_1 \int q_2(Z_2|D)\, dZ_2 \ldots \int q_n(Z_n|D)\, dZ_n +$$

$$\int q_2(Z_2|D)\, ln[q_2(Z_2|D)]\, dZ_2 \int q_1(Z_1|D)\, dZ_1 \ldots \int q_n(Z_n|D)\, dZ_n + \ldots +$$

Variational Bayes, Expectation Maximization:

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$$\int q_2(Z_2|D)\, ln[q_2(Z_2|D)]\, dZ_2 \int q_1(Z_1|D)\, dZ_1 \ldots \int q_n(Z_n|D)\, dZ_n + \cdots +$$

$$= \sum_{i=1}^{n} \int q_i(Z_i|D)\, ln[q_i(Z_i|D)]\, dZ_i \left\{ \prod_{j\neq i}^{n-1} \int q_j(Z_j|D)\, dZ_j \right\}$$

$$= \sum_{i=1}^{n} \langle ln[q_i(Z_i|D)] \rangle \left\{ \prod_{j\neq i}^{n-1} \int q_j(Z_j|D)\, dZ_j \right\}$$

= 1 (the q_j are all a pdf of Z_j)

$$S_q = \sum_{i=1}^{n} \langle ln[q_i(Z_i|D)] \rangle = \sum_{i=1}^{n} \int q_i(Z_i|D)\, ln[q_i(Z_i|D)]\, dZ_i$$

n one-dimensional problems (not so hard)

Variational Bayes, Expectation Maximization:

D	: data set
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find an approximation $Q(Z)$ for the posterior $P(Z|D)$

maximizing evidence $P(D)$

$$ln[P(D)] \geq \int \prod_{i=1}^{n} q_i(Z_i|D)\ ln[P(D|Z)]\, dZ \ - \underbrace{\int \prod_{i=1}^{n} q_i(Z_i|D)\, ln\left[\prod_{i=1}^{n} q_i(Z_i|D)\right] dZ}_{S_q}$$

$$ln[P(D)] \geq \int \prod_{i=1}^{n} q_i(Z_i|D)\ ln[P(D,Z)]\, dZ \ - \sum_{i=1}^{n} \int q_i(Z_i|D)\ ln[q_i(Z_i|D)]\, dZ_i$$

$$E(D,Z) := ln[P(D,Z)]$$

$$ln[P(D)] \geq \int \prod_{i=1}^{n} q_i(Z_i|D)\, E(D,Z)\, dZ \ - \sum_{i=1}^{n} \int q_i(Z_i|D)\ ln[q_i(Z_i|D)]\, dZ_i$$

Variational Bayes, Expectation Maximization:

D	: data set
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find an approximation $Q(Z)$ for the posterior $P(Z|D)$

maximizing evidence $P(D)$

$$ln[P(D)] \geq \int \prod_{i=1}^{n} q_i(Z_i|D)\, E(D,Z)\, dZ \; - \sum_{i=1}^{n} \int q_i(Z_i|D)\, ln[q_i(Z_i|D)]\, dZ_i$$

has the structure of $F = U - TS$, but with $-\mathrm{U} + TS$ term (aka negative variational free energy)

solution:

$$q_i(Z_i|D) = \frac{1}{\mathcal{Z}} \exp\left(\langle E(Z_i, \{Z_{j\neq i}\}, D)\rangle_{\{j\neq i\}}\right)$$

$$\mathcal{Z} = \int \exp\left(\langle E(Z_i, \{Z_{j\neq i}\}, D)\rangle_{\{j\neq i\}}\right) dZ_{\{j\neq i\}}$$

Variational Bayes, Expectation Maximization:

D	: data set
Z	: set of n (latent) parameter
σ^2	: variance
μ	: mean
$\frac{1}{\sigma^2} = \tau$	: precision

example: we can measure $\boldsymbol{\mu}$ and $\frac{1}{\sigma^2} = \boldsymbol{\tau}$ from D

goal: find $q_\mu(\mu|D)$ and $q_\tau(\tau|D)$

$$q_i(Z_i|D) = \frac{1}{\mathcal{Z}} \exp\left(\langle E(Z_i, \{Z_{j\neq i}\}, D)\rangle_{\{j\neq i\}}\right)$$

$$ln\left[q_\mu(\mu|D)\right] = \langle E(\mu, \tau, D)\rangle_\tau - \ln(\mathcal{Z})$$

$$= \langle \ln[P(D|\mu,\tau)P(\mu|\tau)P(\tau)]\rangle_\tau - \ln(\mathcal{Z})$$

$$= \langle \ln[P(D|\mu,\tau)]\rangle_\tau + \langle \ln[P(\mu|\tau)]\rangle_\tau + \langle \ln[P(\tau)]\rangle_\tau - \ln(\mathcal{Z})$$

Variational Bayes, Expectation Maximization:

D	: data set of size K
Z	: set of n (latent) parameter
σ^2	: variance
μ	: mean
$\frac{1}{\sigma^2} = \tau$	: precision

example: we can measure $\boldsymbol{\mu}$ and $\frac{1}{\sigma^2} = \boldsymbol{\tau}$ from D

goal: find $q_\mu(\mu|D)$ and $q_\tau(\tau|D)$

$$ln\left[q_\mu(\mu|D)\right] = \langle \ln[P(D|\mu,\tau)]\rangle_\tau + \langle \ln[P(\mu|\tau)]\rangle_\tau + \langle \ln[P(\tau)]\rangle_\tau - \ln(\mathcal{Z})$$

if no constrain: gaussian

$$P(D|\mu,\tau) = \prod_{k=1}^{K} \mathcal{N}(x_k|\mu,\tau)$$

support is $[0,+\infty)$ → max ent

$$P(\tau) = \Gamma(\tau|a,b) = \frac{b^a\, \tau^{a-1}\, e^{-b\tau}}{\int_0^\infty t^{a-1}\, e^{-t}\, dt}$$

with yet unknown parameter a and b

if μ has support (-∞, + ∞) it is drawn from a gaussian of yet unknown μ_0 and precision τ_0 (= small pos number, max ent)

$$P(\mu|\tau) = \mathcal{N}(\mu|\mu_0,\tau_0)$$

Variational Bayes, Expectation Maximization:

D	: data set of size K
Z	: set of n (latent) parameter
σ^2	: variance
μ	: mean
$\frac{1}{\sigma^2} = \tau$	: precision

example: we can measure $\boldsymbol{\mu}$ and $\frac{1}{\sigma^2} = \boldsymbol{\tau}$ from D

goal: find $q_\mu(\mu|D)$ and $q_\tau(\tau|D)$

$$ln\left[q_\mu(\mu|D)\right] = \langle \ln[P(D|\mu,\tau)] \rangle_\tau + \langle \ln[P(\mu|\tau)] \rangle_\tau + \langle \ln[P(\tau)] \rangle_\tau - \ln(\mathcal{Z})$$

$$P(D|\mu,\tau) = \prod_{k=1}^{K} \mathcal{N}(x_k|\mu,\tau) \qquad P(\mu|\tau) = \mathcal{N}(\mu|\mu_0,\tau_0) \qquad P(\tau) = \Gamma(\tau|a,b) = \frac{b^a\, \tau^{a-1}\, e^{-b\tau}}{\int_0^\infty t^{a-1}\, e^{-t}\, dt}$$

after some (lengthy algebra):

$$ln\left[q_\mu(\mu|D)\right] = -\frac{\langle \tau \rangle_\tau}{2} \left\{ \sum_k (x_k - \mu)^2 + \tau_0 (\mu - \mu_0)^2 \right\} + constant\ terms$$

$$q_\mu(\mu|D) \sim \mathcal{N}(\mu|\mu_K, \lambda_K^{-1})$$

where

$$\mu_K = \frac{\tau_0\, \mu_0 + K\, \bar{x}}{\tau_0 + K}$$

$$\lambda_K = (\tau_0 + K)\, \langle \tau \rangle_\tau$$

$$\bar{x} = \frac{1}{K} \sum_{k=1}^{K} x_k$$

Variational Bayes, Expectation Maximization:

D	: data set of size K
Z	: set of n (latent) parameter
σ^2	: variance
μ	: mean
$\frac{1}{\sigma^2} = \tau$	: precision

example: we can measure $\boldsymbol{\mu}$ and $\frac{1}{\sigma^2} = \boldsymbol{\tau}$ from D

goal: find $q_\mu(\mu|D)$ and $q_\tau(\tau|D)$

$$ln\left[q_\mu(\mu|D)\right] = \langle \ln[P(D|\mu,\tau)]\rangle_\tau + \langle \ln[P(\mu|\tau)]\rangle_\tau + \langle \ln[P(\tau)]\rangle_\tau - \ln(\mathcal{Z})$$

$$P(D|\mu,\tau) = \prod_{k=1}^{K} \mathcal{N}(x_k|\mu,\tau) \qquad P(\mu|\tau) = \mathcal{N}(\mu|\mu_0,\tau_0) \qquad P(\tau) = \Gamma(\tau|a,b) = \frac{b^a\,\tau^{a-1}\,e^{-b\tau}}{\int_0^\infty t^{a-1}\,e^{-t}\,dt}$$

same for $q_\tau(\tau|D)$ (lengthy algebra)

$$q_\tau(\tau|D) \sim \Gamma(\tau|a_K, b_K)$$

where
$$a_K = a + \frac{K+1}{2}$$
$$b_K = b + \frac{1}{2}\left\langle \textstyle\sum_k (x_k-\mu)^2 + \tau_0(\mu-\mu_0)^2\right\rangle_\mu$$

Variational Bayes, Expectation Maximization:

$\boldsymbol{D}$	: data set of size K
$\boldsymbol{Z}$	: set of n (latent) parameter
$\boldsymbol{\sigma^2}$	: variance
$\boldsymbol{\mu}$	: mean
$\frac{1}{\sigma^2} = \tau$	: precision

$$q_\mu(\mu|D) \sim \mathcal{N}(\mu|\mu_K, \lambda_K^{-1})$$

where

$$\mu_K = \frac{\tau_0\, \mu_0 + K\, \bar{x}}{\tau_0 + K}$$
$$\lambda_K = (\tau_0 + K)\, \boxed{\langle\tau\rangle_\tau}$$
$$\bar{x} = \frac{1}{K}\sum_{k=1}^{K} x_k$$

$$q_\tau(\tau|D) \sim \Gamma(\tau|a_K, b_K)$$

where

$$a_K = a + \frac{K+1}{2}$$
$$b_K = b + \frac{1}{2}\langle \textstyle\sum_k (x_k - \mu)^2 + \tau_0(\mu - \mu_0)^2 \rangle_\mu$$

$$\langle\tau\rangle_\tau = \int \tau\, q_\tau(\tau|D)\, d\tau = \frac{a_K}{b_K}$$

We start with setting τ_0, μ_0, a and b to **small positive values** (largest ignorance, broad peaks) and use the circular dependencies (like actual EM)

- from $\langle\tau\rangle_\tau$ (here integral over the gamma distribution) we can calculate $q_\mu(\mu|D)$
- from that we can calculate b_K and $\langle\mu\rangle_\mu = \mu_K$ and $\langle\mu^2\rangle_\mu = \frac{1}{\lambda_K} + \mu_K^2$ etc

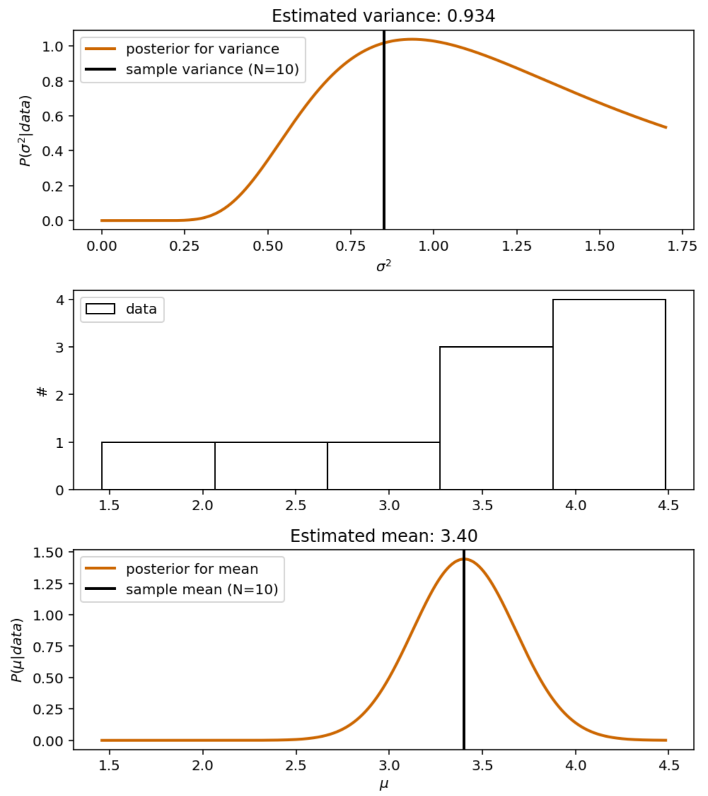
Variational Bayes, Expectation Maximization:

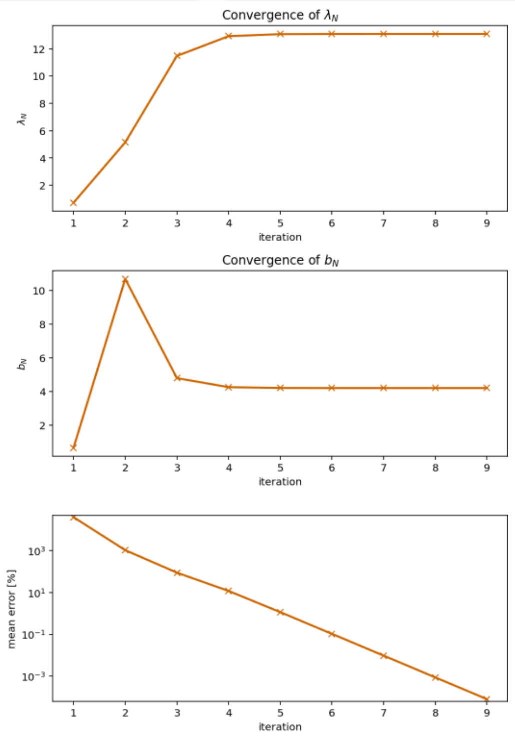
note:

- calculations for $q_i(Z_i|D)$ will lead to an **iterative procedure like for actual EM**
- instead of point estimates for Z_i (MLE) as before, **we get the actual posterior $q_i(Z_i|D)$**
- these distributions get more accurate the larger D → **learning** (see BPE)
- we only use maximum entropy

see `Var_Bayes_Example.py`

```
data = np.random.normal(3, 1, (10,))
Var_Bayes_Example(data)
```





Thank you very much for your attention!