

Lecture 04:

Linear and Non-Linear Regression



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Machine Learning Algorithms

MSSE 277B, 3 Units

Spring 2025



Outline

Linear Regression

- Mathematical Notation
- What is Linear?
- Some Statistics
- a Python example

Logistic Regression





Outline

Linear Regression

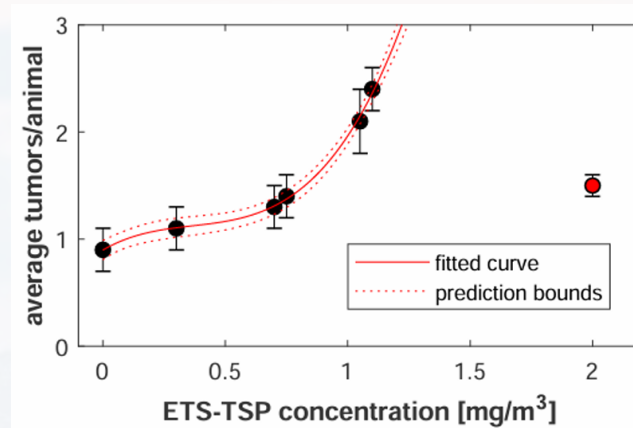
- Mathematical Notation
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- Some Statistics
- a Python example

Logistic Regression



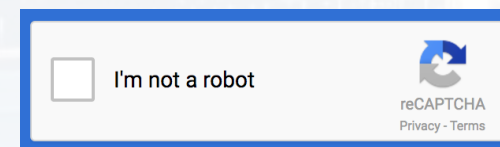
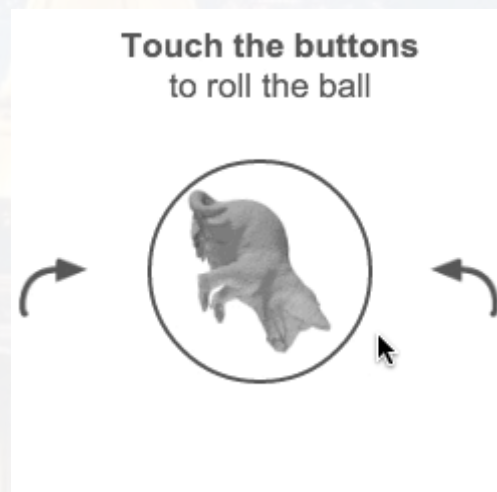
Regression vs Classification

regression



curve fit: finding model parameters by **minimizing** χ^2

$$\chi^2 = \sum_k \frac{(\bar{y}_k - y_k)^2}{\sigma_k^2}$$

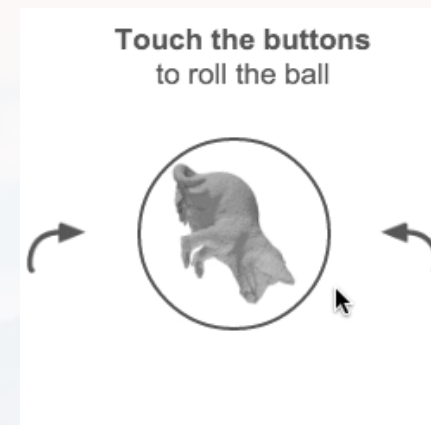
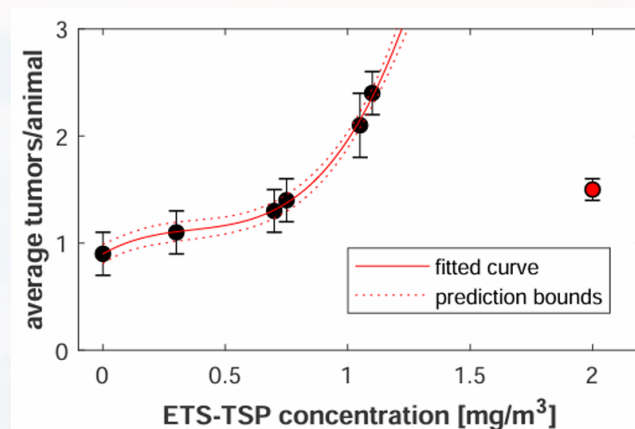


turning an image the right way:

- **maximizing** autocorrelation function
- training an AI

Regression vs Classification

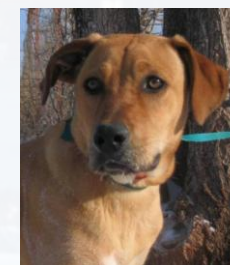
regression



classification



cat

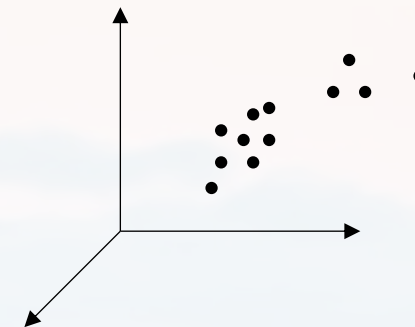


dog

note: we can use (non-linear) regression for classification!

idea: data point y_k in N dimensional space

$$\rightarrow y_k = f(x_1, \dots, x_n, \dots, x_N) + \epsilon \quad \text{for each data point } k$$

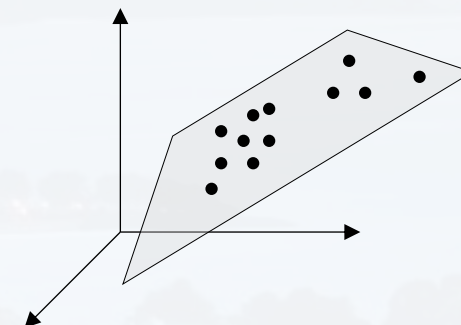


ansatz:

$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$

linear combination

y : response
 x : regressors (assumed to be independent)
 β : factors (how a regressor contributes to the response)
 β_0 : intercept
 ϵ : error (stochasticity of the data, assumed to be normally dist.)





Outline

Linear Regression

- Mathematical Notation
- What is Linear?
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Logistic Regression



linear \neq not curved

$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n^n + \epsilon \quad \dots \text{is still linear}$$

just define: $\bar{x}_n := x_n^n$

$$y_k = \beta_1 x_n^{\beta_2} \quad \dots \text{is still linear}$$

$$\begin{aligned} \text{just use log: } \bar{y}_k &= \log(y_k) = \log(\beta_1) + \beta_2 \log(x_n) \\ &= \bar{\beta}_1 + \beta_2 \bar{x}_n \end{aligned}$$

As long as we can recover the linear structure by any transformation \rightarrow it is linear

in part. log scaling is quite common examples:

- log fold change (DESeq/RNASeq)
- log odds ratio (comparing models, HMM)
- sound \rightarrow dB is a log unit
- log incidence rates (medical studies)
- percentiles (medical studies)
-

y:	response
x:	regressors
β :	factors
β_0 :	intercept
ϵ :	error

...what is **not** linear?

$$y_k = \beta_0 + \beta_1 x_n^{\beta_2} \quad \text{log trick does not work here}$$

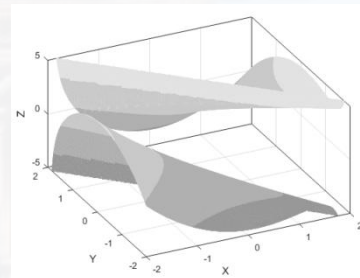
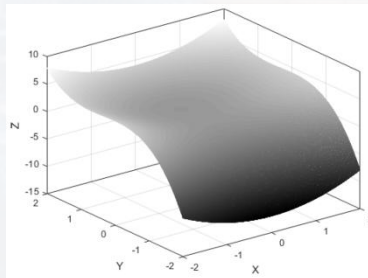
general: linear refers to the **factors**

y :	response
x :	regressors
β :	factors
β_0 :	intercept
ϵ :	error

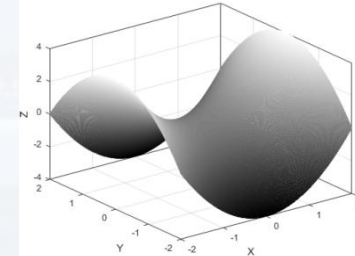
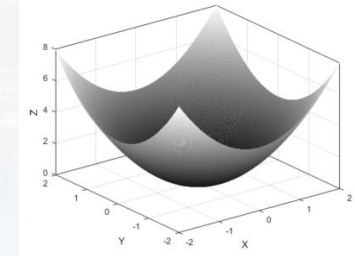
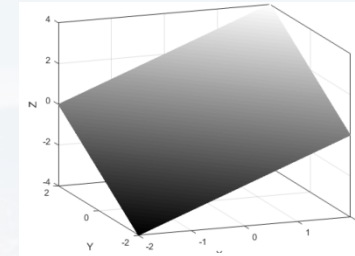
$$y_k = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \quad \text{2D plane in 3D space}$$

$$y_k = \beta_0 + \beta_1 x_1^2 + \beta_2 x_2^2 \quad \text{2D parabolic}$$

$$y_k = \beta_0 + \beta_1 x_1^2 - \beta_2 x_2^2 \quad \text{2D hyperbolic}$$



...and many more...



all linear

$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$



Outline

Linear Regression

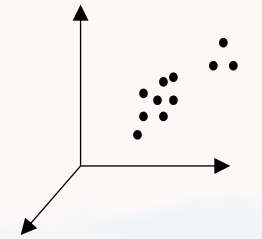
- Mathematical Notation
- What is Linear?
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- a Python example

Logistic Regression



for K data points in N dimensional space

$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$



y :	response
x :	regressors
β :	factors
β_0 :	intercept
ϵ :	error

$$\underbrace{\begin{pmatrix} y_1 \\ \dots \\ y_k \\ \dots \\ y_K \end{pmatrix}}_Y = \underbrace{\begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1n} & \dots & x_{1N} \\ \dots & \dots & \dots & & \dots & & \dots \\ 1 & x_{k1} & & & x_{kn} & & \\ \dots & \dots & & & \dots & & \dots \\ 1 & \dots & & & \dots & & \dots \\ 1 & x_{K1} & x_{K2} & \dots & x_{Kn} & \dots & x_{KN} \end{pmatrix}}_X \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \dots \\ \beta_n \\ \dots \\ \beta_N \end{pmatrix}}_{\beta} + \underbrace{\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \dots \\ \epsilon_k \\ \dots \\ \epsilon_K \end{pmatrix}}_{\epsilon}$$

$$Y = X\beta + \epsilon$$

fitting: finding the best β in terms of minimizing the errors

$$(Y - X\beta)^T (Y - X\beta) = \sum_k \epsilon_k^2$$

the model

$$\frac{\partial}{\partial \beta} \sum_k \epsilon_k^2 = 0 \longrightarrow \beta_{best} = \hat{\beta} = (X^T X)^{-1} X^T Y \longrightarrow \hat{Y} = X\hat{\beta} = X(X^T X)^{-1} X^T Y$$

X and Y are all
observables

$$Y = X\beta + \varepsilon$$

fitting: finding the best β in by minimizing the errors

$$(Y - X\beta)^T(Y - X\beta) = \sum_k \varepsilon_k^2$$

$$\frac{\partial}{\partial \beta} \sum_k \varepsilon_k^2 = 0 \longrightarrow \beta_{best} = \hat{\beta} = (X^T X)^{-1} X^T Y \longrightarrow \text{the model } \hat{Y} = X\hat{\beta} = \underbrace{X(X^T X)^{-1} X^T}_{\text{hat matrix } H} Y$$

some properties of the hat matrix:

- $H = H^T$ (symmetry)
- $HH = H \rightarrow H^n = H$ (idempotency)

evaluating the fit:

$$\hat{\varepsilon} = Y - X\hat{\beta} = Y - \hat{Y} = (I - H)Y \quad \rightarrow \quad \hat{\varepsilon}^T \hat{\varepsilon} = [(I - H)Y]^T (I - H)Y = Y^T (I - H)^T (I - H)Y = Y^T (I - H)Y$$

$$\text{variance} \rightarrow \hat{\sigma}^2 = \frac{\hat{\varepsilon}^T \hat{\varepsilon}}{K - N} \quad \leftarrow \text{degrees of freedom}$$

y:	response
x:	regressors
β :	factors
β_0 :	intercept
ε :	error

summary:

y:	response
x:	regressors
β:	factors
β_0:	intercept
ε:	error

the model: $Y = X\beta + \varepsilon$

the fit: $\hat{Y} = X\hat{\beta} = X(X^T X)^{-1} X^T Y$

residual sum of squares:
(after the fit) $\hat{\varepsilon}^T \hat{\varepsilon} = Y^T (I - H) Y$

variance:
(after the fit) $\hat{\sigma}^2 = \frac{\hat{\varepsilon}^T \hat{\varepsilon}}{K - N}$

often fit quality is judged by $R^2 := 1 - \frac{\sum_k (\hat{y}_k - y_k)^2}{\sum_k (y_k - \langle y \rangle)^2}$

or adjusted R^2 $\bar{R}^2 := R^2 - (1 - R^2) \frac{N}{K - N - 1}$

and it is said that the fit is good if R^2 is close to one....

...but that is not true...

$$\chi_{red}^2 = \frac{1}{df} \sum_{i=1}^N \left(\frac{y_i - \hat{y}_i}{\sigma_i} \right)^2 \quad df = N - p - 1$$

y_i :	measured value of data point
σ_i :	statistical error of y_i (often aka e_{y_i})
\hat{y}_i :	prediction by the model <i>after the fit</i>
N :	number of data points
p :	number of fit parameter

def:

\bar{y} : mean of the data point values

$$R^2 = 1 - \frac{\sum_{i=1}^N (y_i - \hat{y}_i)^2}{\sum_{i=1}^N (y_i - \bar{y})^2}$$

← variance data vs model
(aka residual sum of squares)

← variance of the data
(aka total sum of squares)

Note: do not confuse R^2 with Pearsons coefficient: $\rho = \frac{cov(x,y)}{\sqrt{var(x)var(y)}}$

$$\chi^2_{red} = \frac{1}{df} \sum_{i=1}^N \left(\frac{y_i - \hat{y}_i}{\sigma_i} \right)^2$$

$$df = N - p - 1$$

- scales difference between model and data to the error bars
- can be directly translated to a p-value via the Students distribution

H0: the fitted model has in fact generated the data

$$R^2 = 1 - \frac{\sum_{i=1}^N (y_i - \hat{y}_i)^2}{\sum_{i=1}^N (y_i - \bar{y})^2}$$

variance data vs model
(aka residual sum of squares)

variance of the data
(aka total sum of squares)

\bar{y} : mean of the data point values



data variance can be huge
(i. e. exponential functions)
→ R^2 could be around 1.0
even if fit is completely off!

$$\chi^2_{red} = \frac{1}{df} \sum_{i=1}^N \left(\frac{y_i - \hat{y}_i}{\sigma_i} \right)^2$$

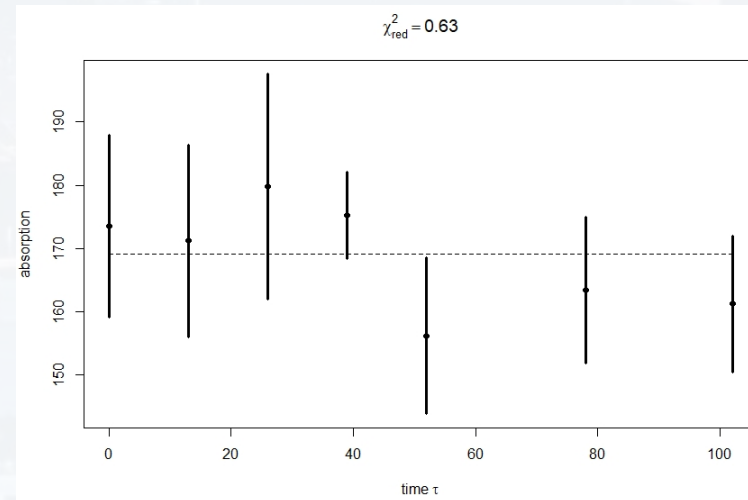
$$df = N - p - 1$$

- scales difference between model and data to the error bars
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H0: the fitted model has in fact generated the data

$$R^2 = 1 - \frac{\sum_{i=1}^N (y_i - \hat{y}_i)^2}{\sum_{i=1}^N (y_i - \bar{y})^2} \quad \frac{\text{variance data vs model (aka residual sum of squares)}}{\text{variance of the data (aka total sum of squares)}}$$

\bar{y} : mean of the data point values



$$\frac{\text{variance data vs model (aka residual sum of squares)}}{\text{variance of the data (aka total sum of squares)}} \approx 1 \rightarrow R^2 = 0$$

→ although the fit is good

$$\chi_{red}^2 = \frac{1}{df} \sum_{i=1}^N \left(\frac{y_i - \hat{y}_i}{\sigma_i} \right)^2$$

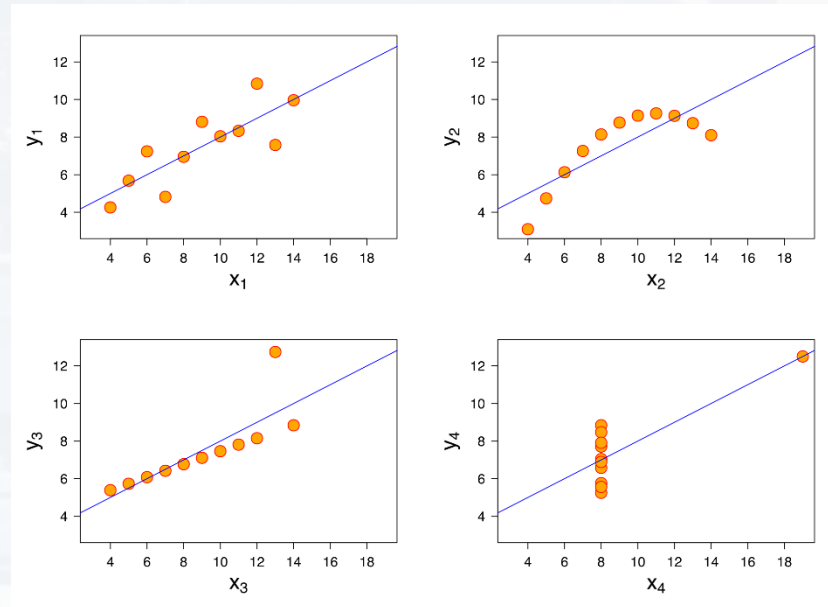
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\bar{y} : mean of the data point values



all plots: same R^2 , but different χ_{red}^2

$$\chi^2_{red} = \frac{1}{df} \sum_{i=1}^N \left(\frac{y_i - \hat{y}_i}{\sigma_i} \right)^2$$

$$df = N - p - 1$$

- scales difference between model and data to the error bars
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H0: the fitted model has in fact generated the data

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\bar{y} : mean of the data point values

conclusion:

- R^2 is not a measure of the fit quality (but χ^2 is)
- error bars are important
- **given a good fit**, R^2 tells how strong the dependent variable responds to the independent variable

Also, Wiki is full of examples...

...and warnings (see “caveats” therein)



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Logistic Regression





```
import numpy as np
```

```
import pandas as pd
```

```
import matplotlib.pyplot as plt
```

```
import seaborn as sns
```

```
import pylab
```

```
import scipy.stats as stats
```

```
import statsmodels.api as sm
```

```
from statsmodels.formula.api import ols
```

```
from sklearn.preprocessing import MinMaxScaler
```

reading .xlsx
.csv
.txt
...

standard plots

fancy plots:
here a pair-
plot

Q-Q plot

the actual
super tool for
superb data
analysis

scaling and normalizing



```
Train = pd.read_csv("molecular_train_gbc.csv")  
Test  = pd.read_csv("molecular_test_gbc.csv")
```

- 1) loading data
- 2) plotting data
- 3) scaling data
- 4) fitting model
- 5) evaluating model

	x_1	x_2	x_3	x_4	x_5	y_k
Index	molecular_weight	electronegativity	bond_lengths	num_hydrogen_bonds	logP	toxicity_score
0	341.704	2.65585	3.09407	2	9.11147	80.9281
1	335.951	3.22262	2.89039	7	8.92848	83.4911
2	235.203	2.44115	2.48203	1	6.49731	61.8406
3	246.505	2.76656	2.71547	7	7.45089	57.0538
4	437.939	3.4801	3.59569	3	10.9156	131.326

$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$

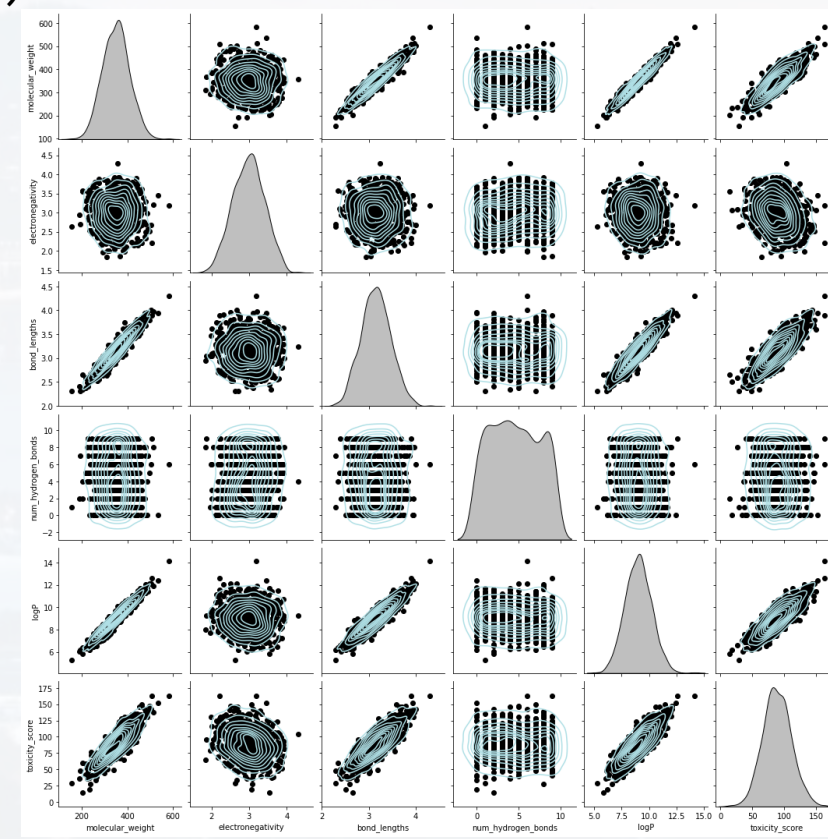
y : toxicity_score
 x_n : molecular_weight, electronegativity,
bond_lengths, num_hydrogen_bonds, logP



```
Train = pd.read_csv("molecular_train_gbc.csv")  
Test  = pd.read_csv("molecular_test_gbc.csv")
```

- 1) loading data
- 2) plotting data
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- 5) evaluating model

```
out = sns.pairplot(Train, kind = "kde", \  
                  plot_kws = {'color': [176/255, 224/255, 230/255]}, \  
                  diag_kws = {'color': 'black'})  
out.map_offdiag(plt.scatter, color = 'black')
```





- 1) loading data
- 2) plotting data
- 3) scaling data
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- 5) evaluating model

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Train = pd.read_csv("molecular_train_gbc.csv")
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```
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                    diag_kws = {'color': 'black'})
out.map_offdiag(plt.scatter, color = 'black')
```

```
scaler      = MinMaxScaler(feature_range = (0, 1))
All         = pd.concat((Train, Test), axis = 0)
(rows, _)   = Train.shape
AllS        = scaler.fit_transform(All)
```

the scaler returns an `np.array`
→ convert back to data frame

```
TrainS = pd.DataFrame(AllS[:rows,:], columns = Train.columns)
TestS  = pd.DataFrame(AllS[rows:,:], columns = Train.columns)
```




```
TrainS = pd.DataFrame(TrainS, columns = Train.columns)
TestS   = pd.DataFrame(TestS,  columns = Train.columns)
```

```
equation = 'toxicity_score ~ ' + '+'.join(Train.columns[:-1])
print(equation)
```

- 1) loading data
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$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$

```
toxicity_score ~      molecular_weight + electronegativity +
                      bond_lengths + num_hydrogen_bonds + logP
```

```
my_model = ols(equation, data = TrainS).fit()
my_model.summary()
```

OLS (ordinary least squares)



```
my_model.summary()
```

- 1) loading data
- 2) plotting data
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- 5) evaluating model

number of data points
is much larger than
the number of regressors
→ degree of freedom
approx. no of obs

OLS Regression Results

not the fit quality!

Dep. Variable:	toxicity_score	R-squared:	0.790
Model:	OLS	Adj. R-squared:	0.789
Method:	Least Squares	F-statistic:	597.5
Date:	Fri, 13 Sep 2024	Prob (F-statistic):	3.34e-266
Time:	20:57:10	Log-Likelihood:	1013.0
No. Observations:	800	AIC:	-2014.
Df Residuals:	794	BIC:	-1986.
Df Model:	5		
Covariance Type:	nonrobust		

	coef	std err	t	P> t	[0.025
Intercept	0.1494	0.012	12.533	0.000	0.126
molecular_weight	0.7961	0.089	8.982	0.000	0.622
electronegativity	-0.1682	0.015	-11.591	0.000	-0.197
bond_lengths	0.0204	0.049	0.417	0.677	-0.076
num_hydrogen_bonds	0.0035	0.008	0.458	0.647	-0.011
logP	0.1246	0.072	1.723	0.085	-0.017

Omnibus:	2.249	Durbin-Watson:	1.984
Prob(Omnibus):	0.325	Jarque-Bera (JB):	2.240
Skew:	-0.129	Prob(JB):	0.326
Kurtosis:	2.980	Cond. No.	65.6

p-value for
constant model

p-values for
factors

2σ conf range of
factors

$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$

Notes:

[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

more accurate: determining **the p-values for the factors using ANOVA** for the corresponding residuals

```
table = sm.stats.anova_lm(my_model, typ = 1)
print(table)
```

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$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$

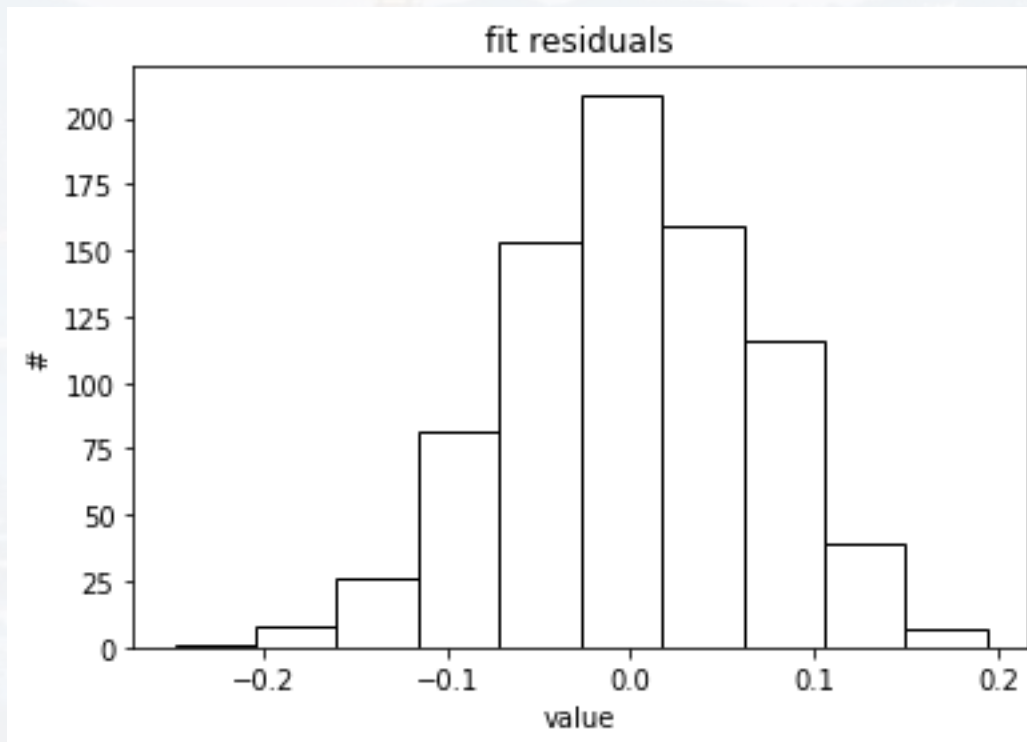
	df	sum_sq	mean_sq	F	PR(>F)	vs from t-test
molecular_weight	1.0	13.346285	13.346285	2847.525516	8.024085e-265	0.0000
electronegativity	1.0	0.640388	0.640388	136.631363	3.085962e-29	0.0000
bond_lengths	1.0	0.000684	0.000684	0.145954	7.025342e-01	0.6766
num_hydrogen_bonds	1.0	0.000703	0.000703	0.150055	6.985866e-01	0.6473
logP	1.0	0.013917	0.013917	2.969353	8.524510e-02	0.0852
Residual	794.0	3.721459	0.004687	NaN	NaN	


```
residuals = my_model.resid
```

```
plt.hist(residuals, color = 'w', edgecolor = 'black')  
plt.title('fit residuals')  
plt.ylabel('#')  
plt.xlabel('value')  
plt.show()
```

- 1) loading data
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$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$



residuals approx.
normally distributed
around $\mu = 0$



```
residuals = my_model.resid
```

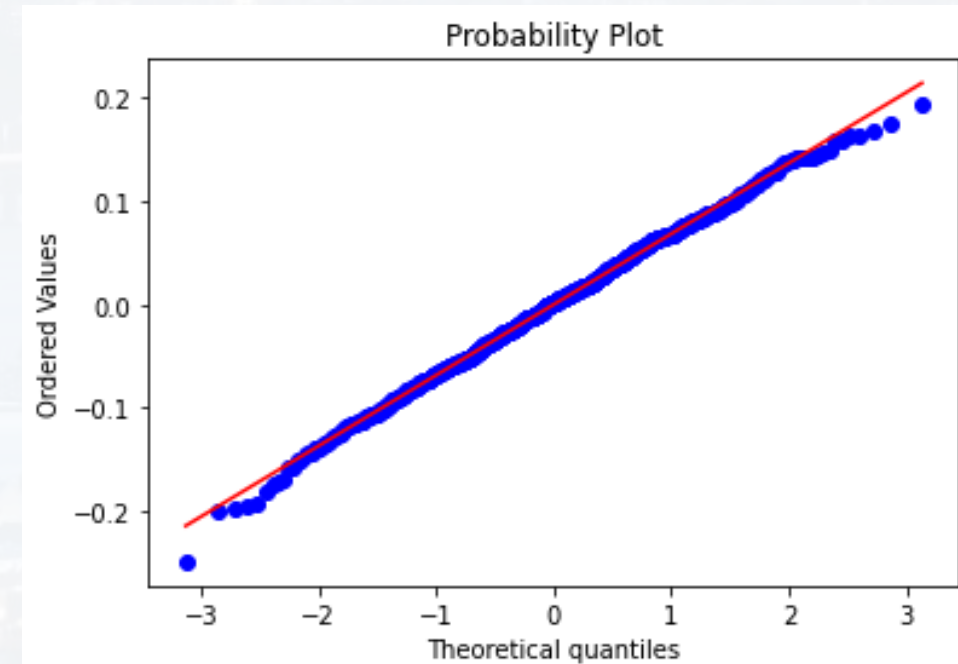
```
plt.hist(residuals, color = 'w', edgecolor = 'black')  
plt.title('fit residuals')  
plt.ylabel('#')  
plt.xlabel('value')  
plt.show()
```

```
stats.probplot(residuals, dist = "norm", plot = pylab)  
pylab.show()
```

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residuals approx.
normally distributed
around $\mu = 0$

$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$





```
Ypred = my_model.predict(TestS)
```

```
higher = np.max([Ypred, TestS.toxicity_score])
```

```
lower = np.min([Ypred, TestS.toxicity_score])
```

```
plt.plot([lower, higher], [lower, higher], c = [0, 0, 0, 0.2],\n         linewidth = 4)
```

```
plt.scatter(TestS.toxicity_score, Ypred, marker = '.', c = 'k')
```

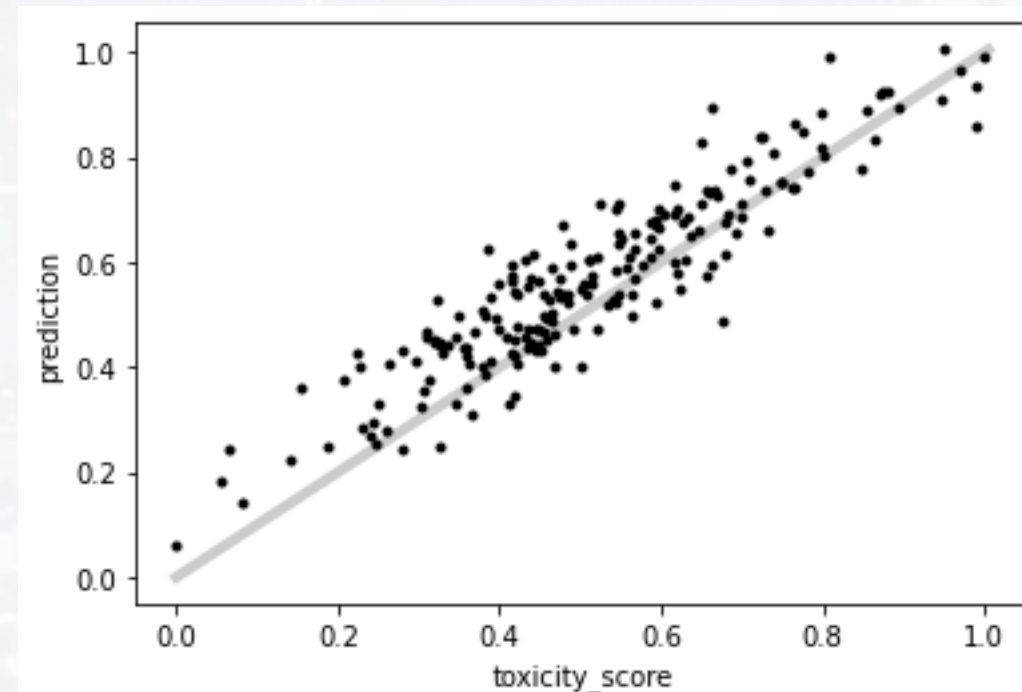
```
plt.ylabel('prediction')
```

```
plt.xlabel('toxicity score')
```

```
plt.show()
```

- 1) loading data
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$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$





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```

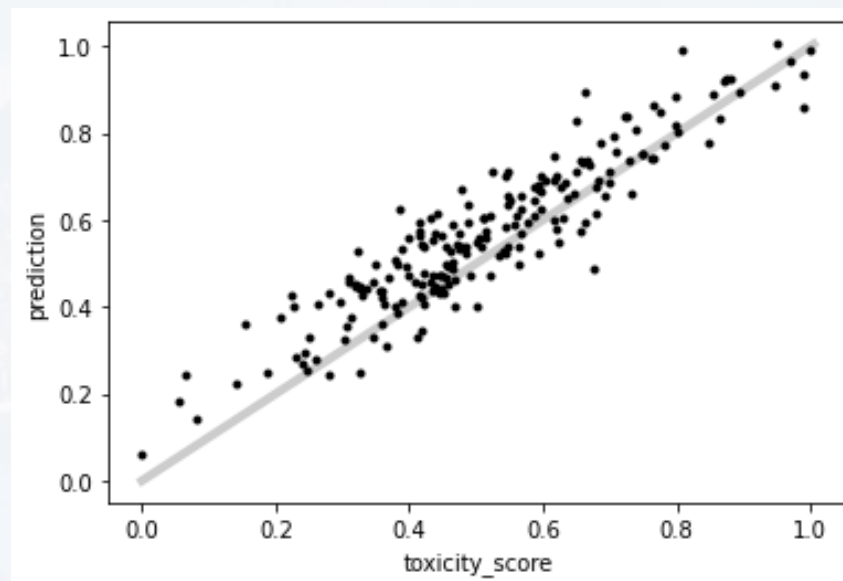
```
plt.plot([lower, higher], [lower, higher], c = [0, 0, 0, 0.2], linewidth = 4)
```

```
plt.scatter(TestS.toxicity_score, Ypred, marker = '.', c = 'k')
```

```
plt.ylabel('prediction')
```

```
plt.xlabel('toxicity score')
```

```
plt.show()
```



```
mean_dev = np.sum( abs(TestS.toxicity_score - Ypred) )/len(Ypred)  
print(mean_dev)
```

5%

- 1) loading data
- 2) plotting data
- 3) scaling data
- 4) fitting model
- 5) evaluating model

$$y_k = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$



Outline

Linear Regression

- Mathematical Notation
- What is Linear?
- Some Statistics
- a Python example

Logistic Regression



linear model: regressors are continuous or categorical,
response is continuous

logistic model: response is **categorical**

y :	response
x :	regressors (assumed to be independent)
β :	factors
β_0 :	intercept
ϵ :	error (stochasticity of the data, assumed to be normally dist.)

Index	molecular_weight	electronegativity	bond_lengths	num_hydrogen_bonds	logP	label
0	341.704	2.65585	3.09407	2	9.11147	Toxic
1	335.951	3.22262	2.89039	7	8.92848	Toxic
2	235.203	2.44115	2.48203	1	6.49731	Non-Toxic
3	246.505	2.76656	2.71547	7	7.45089	Non-Toxic
4	437.939	3.4801	3.59569	3	10.9156	Non-Toxic

linear model: regressors are continuous or categorical,
response is continuous

logistic model: response is **categorical**

y:	response
x:	regressors (assumed to be independent)
β :	factors
β_0 :	intercept
ϵ :	error (stochasticity of the data, assumed to be normally dist.)

dichotomic model: **probability** to be in state A) $\rightarrow p$

probability to be in state B) $\rightarrow 1 - p$

label
Toxic
Toxic
Non-Toxic
Non-Toxic
Non-Toxic

ansatz:

$$\log \left(\frac{p}{1-p} \right) = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$

log odds ratio: linear model

dichotomic model:

label
Toxic
Toxic
Non-Toxic
Non-Toxic
Non-Toxic

probability to be in state A) $\rightarrow p$

probability to be in state B) $\rightarrow 1 - p$

y:	response
x:	regressors (assumed to be independent)
β :	factors
β_0 :	intercept
ϵ :	error (stochasticity of the data, assumed to be normally dist.)

ansatz:

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$

log odds ratio: linear model

\rightarrow probability for being in a certain state

$$p = \frac{e^{\beta_0 + \beta_1 x_1 + \dots}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots}}$$

often:

$$\text{logit}(p) = \log\left(\frac{p}{1-p}\right)$$

examples:

- probability that a gene has been mutated
- probability of being diseased (cancer, alzheimer etc) as function of age, environmental influence etc ...
- Verhulst equation: $N(t) = N_0 \frac{e^{rt}}{C + e^{rt}}$
- activation functions in ANNs

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \sum_{n=1}^N \beta_n x_n + \epsilon$$

Note: one can derive the logit function from max. entropy too!

y:	response
x:	regressors (assumed to be independent)
β :	factors
β_0 :	intercept
ϵ :	error (stochasticity of the data, assumed to be normally dist.)

$$p = \frac{e^{\beta_0 + \beta_1 x_1 + \dots}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots}} = \frac{1}{1 + e^{-\beta_0 - \beta_1 x_1 - \dots}}$$

onset of Alzheimer's disease (AD) is a function of **age** and years spent in **education**
(and other risk factors we ignore here for the sake of simplicity)

education: $d = x_1$ [yrs]

age: $a = x_2$ [yrs]

model:
$$p_{AD} = \frac{1}{1 + e^{-\beta_0 - \beta_1 d - \beta_2 a}}$$

+ data set + fit \rightarrow

$$\begin{aligned}\beta_0 &= +0.1 \\ \beta_1 &= -1.5 \\ \beta_2 &= +0.12\end{aligned}$$

- positive value \rightarrow increasing p
- negative value \rightarrow decreasing p
- intercept: "background" prevalence, not related to environmental/internal conditions

model:
$$p_{AD} = \frac{1}{1 + e^{-\beta_0 - \beta_1 d - \beta_2 a}}$$

education: $d = x_1$ [yrs] $\beta_0 = +0.1$
 age: $a = x_2$ [yrs] $\beta_1 = -1.5$
 $\beta_2 = +0.12$

y:	response
x:	regressors (assumed to be independent)
β :	factors
β_0 :	intercept
ε :	error (stochasticity of the data, assumed to be normally dist.)

example: **65yrs** old person, **8yrs** spent in education
 $\rightarrow p_{AD} = 1.6\%$

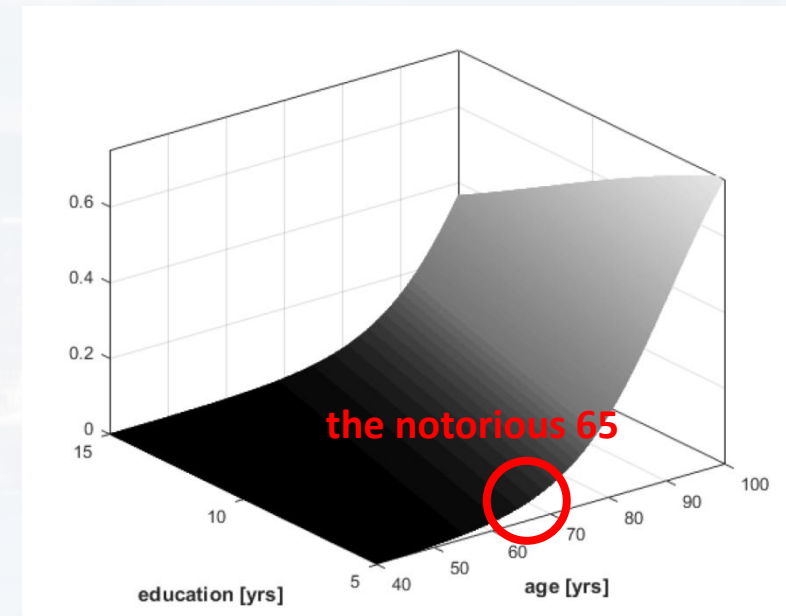
65yrs old person, **13yrs** spent in education
 $\rightarrow p_{AD} = 0.001\%$

How does education compensate aging?

$$p_{AD}(d + \bar{d}, a + \bar{a}) = p_{AD}(d, a)$$

$$\rightarrow \bar{a} = 12.5 \bar{d}$$

hence, one more year prolonged education compensates
12.5 years of aging
 (warning: don't confuse correlation with causation here!)



model:
$$p_{AD} = \frac{1}{1 + e^{-\beta_0 - \beta_1 d - \beta_2 a}}$$

education: $d = x_1$ [yrs]

age: $a = x_2$ [yrs]

$$\beta_0 = +0.1$$

$$\beta_1 = -1.5$$

$$\beta_2 = +0.12$$

y:	response
x:	regressors (assumed to be independent)
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How does the risk of onset changes *per year*?

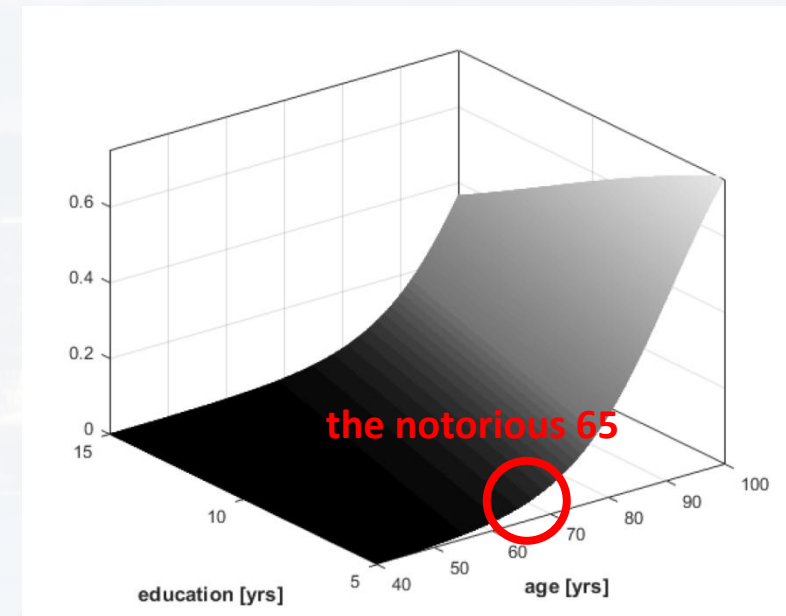
relative change:

$$\frac{p_{AD}(a+1) - p_{AD}(a)}{p_{AD}(a)} \approx e^{\beta_2} - 1 \approx 12.7\%$$

$p_{AD} \ll 1$ (hence, for small Δa and “young” ages, i. e. below ≈ 80 yrs)

the risk of getting AD increases by 12.7% every year

(warning: does not mean that it increases by 127% in ten yrs – we made an approximation!)



model:
$$p_{AD} = \frac{1}{1 + e^{-\beta_0 - \beta_1 d - \beta_2 a}}$$

education: $d = x_1$ [yrs] $\beta_0 = +0.1$
 age: $a = x_2$ [yrs] $\beta_1 = -1.5$
 $\beta_2 = +0.12$

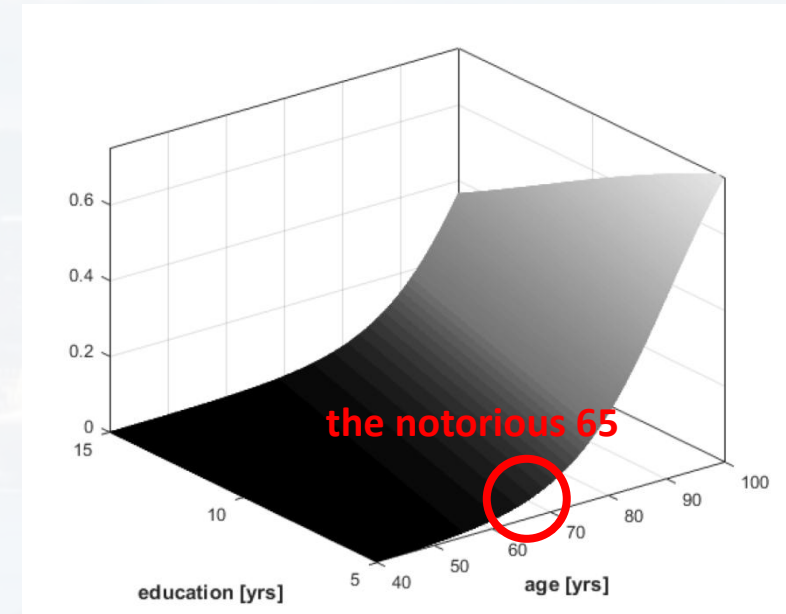
y:	response
x:	regressors (assumed to be independent)
β :	factors
β_0 :	intercept
ε :	error (stochasticity of the data, assumed to be normally dist.)

How does the risk of onset changes *per year*?

more precise: relative change of the odds ratio

$$\frac{\frac{\partial}{\partial x_i} \left(\frac{p_{AD}}{1 - p_{AD}} \right)}{\frac{p_{AD}}{1 - p_{AD}}} = \beta_i$$

x_i is the desired regressor,
for example, age again (x_2)



the factors β_i indicate how strong (and in which direction) p changes wrt a regressor x_i

let us return to the molecule data set:

```
Train = pd.read_csv("molecular_train_gbc_cat.csv")  
Test  = pd.read_csv("molecular_test_gbc_cat.csv")
```

- 1) loading data
- 2) plotting data
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Index	molecular_weight	electronegativity	bond_lengths	num_hydrogen_bonds	logP	label
0	341.704	2.65585	3.09407	2	9.11147	Toxic
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3	246.505	2.76656	2.71547	7	7.45089	Non-Toxic
4	437.939	3.4801	3.59569	3	10.9156	Non-Toxic

it is the same data set → plotting and scaling is as before

```
X = sm.add_constant(TrainS)
```

adding the
intercept

```
Y = pd.get_dummies(Train['Label'])
```

Python needs
True/False
as categorical

- 1) loading data
- 2) plotting data
- 3) scaling data
- 4) fitting model
- 5) evaluating model

```
In [48]: print(Y)
      Non-Toxic  Toxic
0         False   True
1         False   True
2          True  False
3          True  False
4          True  False
```

$$p = \frac{e^{\beta_0 + \beta_1 x_1 + \dots}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots}}$$

we have two
states: toxic /
non-toxic

```
my_model = sm.GLM(Y, X, family = sm.families.Binomial()).fit()
```

```
my_model.summary()
```

GLM: general
linear model

it is the same data set → plotting and scaling is as before

```
X = sm.add_constant(TrainS)
Y = pd.get_dummies(Train['Label'])
```

```
my_model = sm.GLM(Y, X, family = sm.families.Binomial()).fit()
my_model.summary()
```

- 1) loading data
- 2) plotting data
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Generalized Linear Model Regression Results						
=====						
Dep. Variable:	['Non-Toxic', 'Toxic']		No. Observations:	800		
Model:	GLM		Df Residuals:	794		
Model Family:	Binomial		Df Model:	5		
Link Function:	Logit		Scale:	1.0000		
Method:	IRLS		Log-Likelihood:	-332.82		
Date:	Sat, 14 Sep 2024		Deviance:	665.64		
Time:	20:59:18		Pearson chi2:	1.14e+03		
No. Iterations:	6		Pseudo R-squ. (CS):	0.4243		
Covariance Type:	nonrobust		p-values for factors			
=====						
	coef	std err	z	P> z	[0.025	0.975]

const	6.1641	0.585	10.536	0.000	5.017	7.311
molecular_weight	-10.4920	3.626	-2.893	0.004	-17.599	-3.385
electronegativity	3.2874	0.599	5.492	0.000	2.114	4.461
bond_lengths	0.6736	1.913	0.352	0.725	-3.075	4.422
num_hydrogen_bonds	-0.3082	0.303	-1.018	0.309	-0.902	0.285
logP	-7.6090	2.978	-2.555	0.011	-13.447	-1.771
=====						

$$p = \frac{e^{\beta_0 + \beta_1 x_1 + \dots}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots}}$$

p-value for
constant model

2σ conf range of
factors

accuracy: How **often** did the model make the correct prediction.
cross-entropy: How **certain** was the model when making the prediction.

- 1) loading data
- 2) plotting data
- 3) scaling data
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accuracy: How **often** did the model make the correct prediction.

cross-entropy: How **certain** was the model when making the prediction.

- 1) loading data
- 2) plotting data
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```
predProbs = my_model.predict(sm.add_constant(TestS))
```

```
Pred = np.round(predProbs).astype(int)
```

```
predictions = ['Non-Toxic' if i==1 else 'Toxic' for i in Pred]
```

```
Dep. Variable:      ['Non-Toxic', 'Toxic']
```

```
In [51]: predictions
Out[51]:
['Toxic',
 'Toxic',
 'Non-Toxic',
 'Non-Toxic',
 'Toxic',
 'Toxic',
 'Toxic']
```

```
TestY = Test['label']
```

```
accuracy = 100*(TestY == predictions).sum()/len(predictions)
```

```
print(f'accuracy = {accuracy: .2f}%')
```

```
accuracy = 80.50%
```

accuracy: How **often** did the model make the correct prediction.

cross-entropy: How **certain** was the model when making the prediction.

accuracy is $\approx 80\%$ But does it depend on the class? \rightarrow **confusion matrix**

- 1) loading data
- 2) plotting data
- 3) scaling data
- 4) fitting model
- 5) evaluating model

ideal world:

true label	non-toxic	toxic
	100%	0%
non-toxic	100%	0%
toxic	0%	100%
predicted label		

```
from sklearn.metrics import confusion_matrix, ConfusionMatrixDisplay
```


accuracy: How **often** did the model make the correct prediction.

cross-entropy: How **certain** was the model when making the prediction.

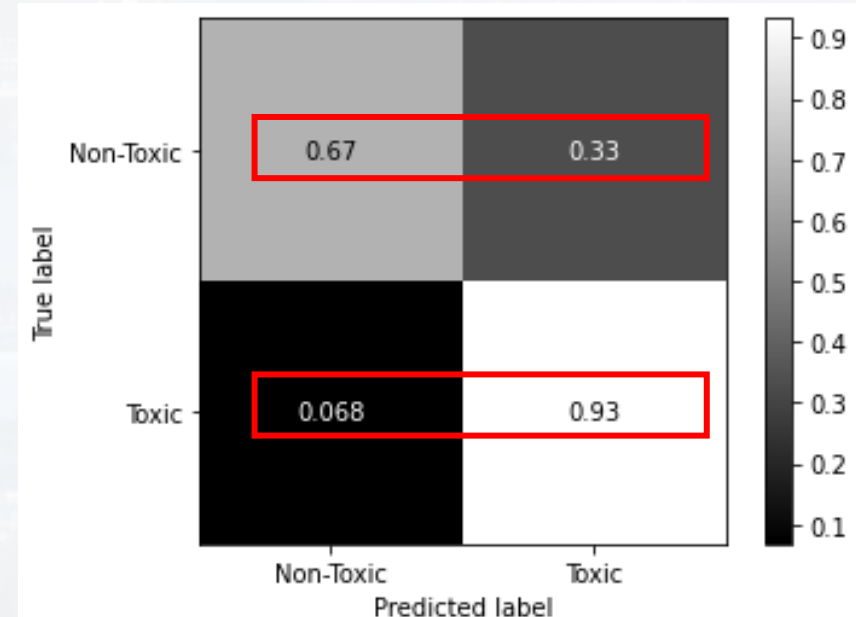
- 1) loading data
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- 5) evaluating model

accuracy is $\approx 80\%$ But does it depend on the class? \rightarrow **confusion matrix**

`L = ['Non-Toxic', 'Toxic']`

two labels

```
cm = confusion_matrix(TestY, predictions, labels = L, normalize = 'true')  
disp = ConfusionMatrixDisplay(confusion_matrix = cm, display_labels = L)  
disp.plot(cmap = 'gray')  
plt.show()
```

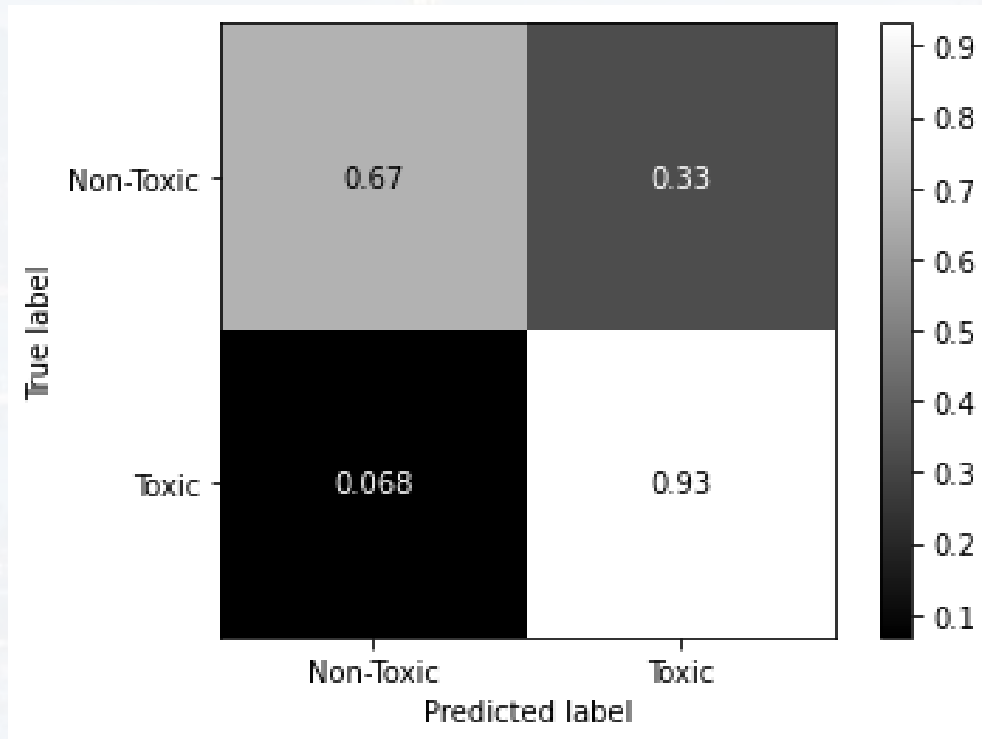


accuracy: How **often** did the model make the correct prediction.

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- 1) loading data
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accuracy is $\approx 80\%$ But does it depend on the class? \rightarrow **confusion matrix**



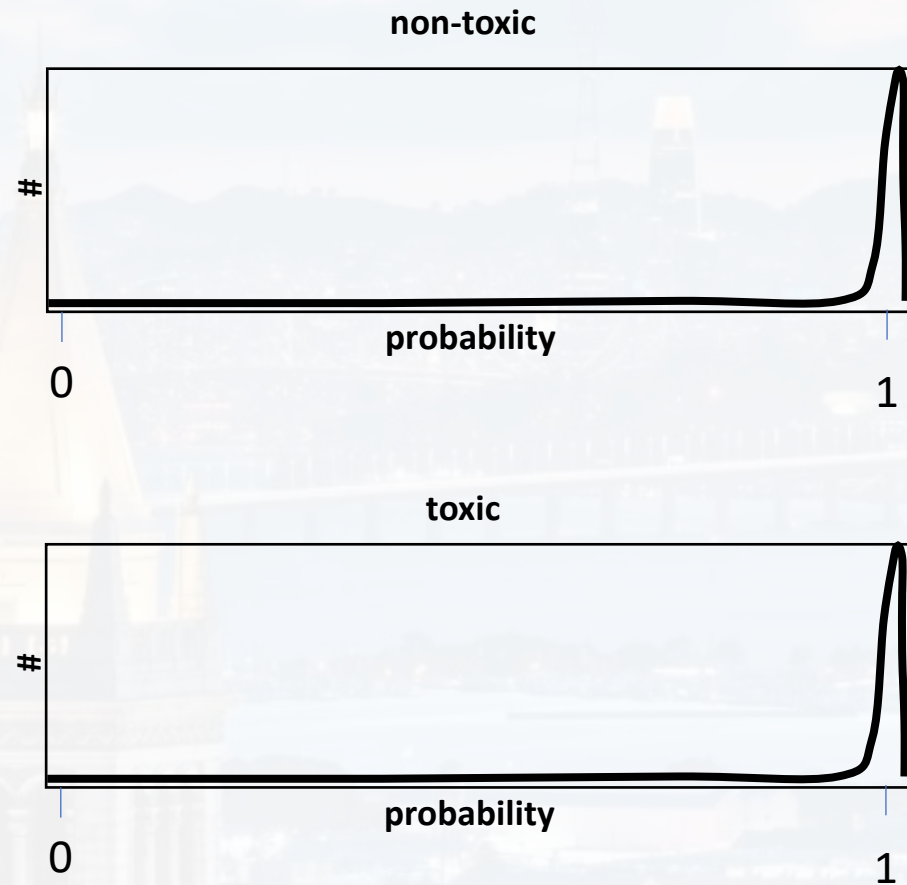
true label	non-toxic	100%	0%
	toxic	0%	100%
		non-toxic	toxic
		predicted label	

accuracy: How *often* did the model make the correct prediction.

cross-entropy: How *certain* was the model when making the prediction.

- 1) loading data
- 2) plotting data
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ideal world:



accuracy: How *often* did the model make the correct prediction.

cross-entropy: How *certain* was the model when making the prediction.

```
PredProbs = np.vstack((predProbs, 1 - predProbs))
```

```
fig, ax = plt.subplots(len(L), 1, sharex = True)
fig.set_figheight(6)
fig.subplots_adjust(hspace = 0.5)
fig.suptitle('entropy')
for i, l in enumerate(L):
    idx = [k for k, y in enumerate(TestY) if y == 1]
    idx = np.array(idx)
    (value, where) = np.histogram(PredProbs[i,idx],\
                                bins = np.arange(0, 1, 0.01),\
                                density = True)

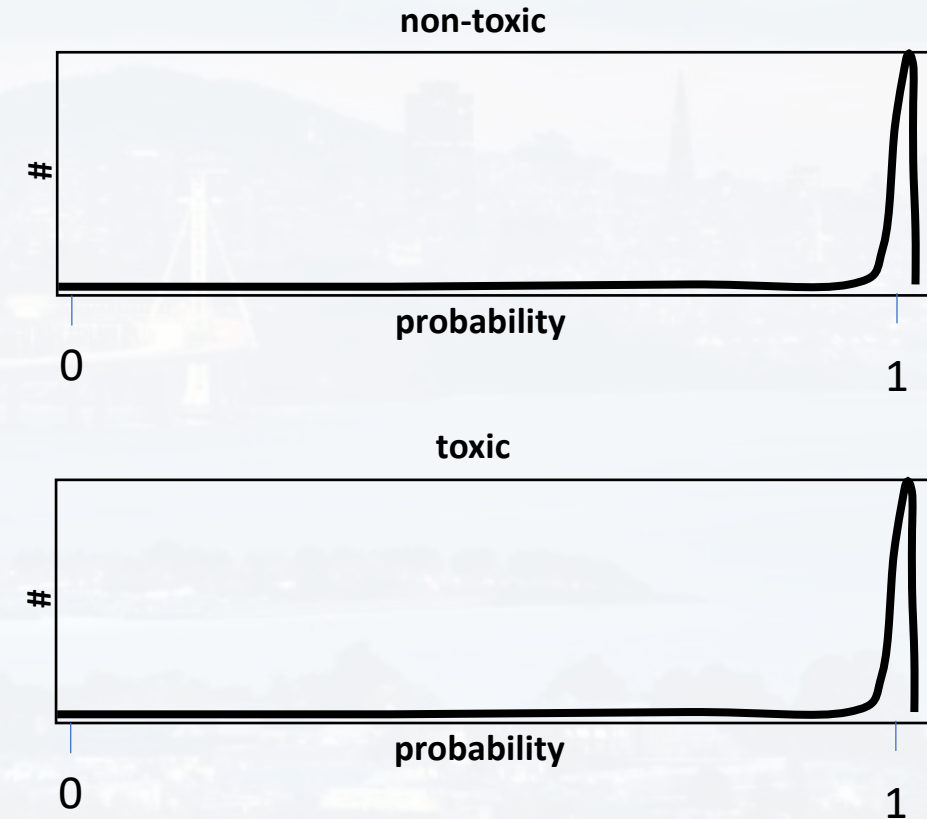
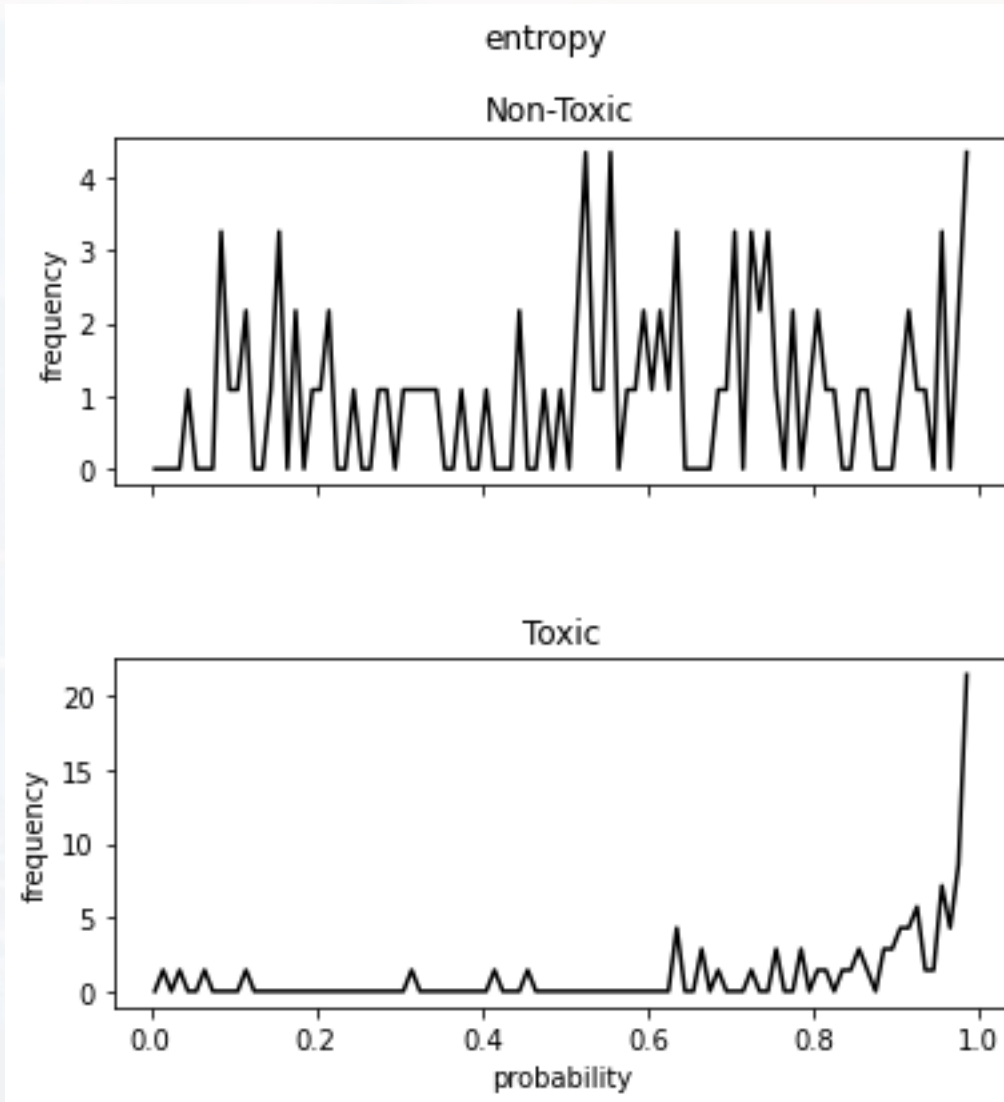
    w = 0.5*(where[1:] + where[:-1])
    ax[i].plot(w, value, 'k-')
    ax[i].set_ylabel('frequency')
    ax[i].set_title(l)
ax[len(L)-1].set_xlabel('probability')
plt.show()
```

- 1) loading data
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accuracy: How *often* did the model make the correct prediction.

cross-entropy: How *certain* was the model when making the prediction.

- 1) loading data
- 2) plotting data
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Thank you very much for your attention!