Problem (1)

Let $(B_t)_{t\geq 0}$ denote a standard Brownian motion.

(a) Determine the distribution of $B_s + B_t$ for $0 \leq s \leq t$. Determine the distribution of B_2^2 and of $B_1^2 + (B_2 - B_1)^2$.

(b) Argue that $\mathbb{P}(0 < B_1 < B_2 < B_\pi) = \frac{1}{8}$. Determine $\mathbb{E}\left[B_1 B_2 B_\pi \mid \mathcal{F}_2^B\right]$ and $\mathbb{E}\left[B_1 B_2 B_\pi\right]$.

(c) Define $U_t = tB_{\frac{1}{t}}$ for $t > 0$. Let $0 < s < t$. Determine the distribution of $U_,$ and of $U_t - U_s$, and determine whether U_n and $U_t - U_s$ are independent. For $a \in \mathbb{R}$ let

$$T_a = \inf\{t \geq 0 : B_t = a\}$$

Later in the course we show both that $\mathbb{P}(T_a < \infty) = 1$ for all $a \in \mathbb{R}$ and that $\mathrm{P}(T_a < T_b) = b/(b-a)$ for $a < 0 < b$.

(d) Determine $\mathbb{P}(T_1 < T_{-1} < T_2)$.

(e) Define $(X_t)_{t\geq 0}$ as $X_t = \sqrt{t}B_1$ for $t \geq 0$. Does $X_t \overset{t}{=} B_t$ hold for $t \geq 0$? Is $(X_t)_{t\geq 0}$ a Brownian motion?

(f) Let $Y \sim N(0,1)$ and assume that Y is independent of $(B_t)_{t\geq 0}$. Let

$$Z_t = B_t + t(Y - B_1), \quad t \in [0,1].$$

Are $(B_t)_{t\in[0,1]}$ and $(Z_t)_{t\in[0,1]}$ versions? Are they modifications or indistinguishable (dansk: uskelnelige)?

Solution

(a)

Problem (2)

Let $(N_t)_{t\geq 0}$ denote a Poisson process with parameter $\lambda > 0$.

(a) Determine $\mathbb{E}\left[T_{12}\right]$ and $\mathrm{Var}\left(T_{12}\right)$.

(b) Determine $\mathbb{E}\left[T_{10} \mid \{N_2 = 5\} \cap \{T_3 < 2\}\right]$.

(c) Determine $\mathbb{E}\left[N_5 \mid N_2 = 7\right]$ and $\mathrm{Var}\left[N_5 \mid N_2 = 7\right]$. Also determine $\mathbb{E}\left[N_5 \mid N_2\right]$.

Solution

(a)

Problem (3 - Durrett 2.27)

Rock concert tickets are sold at a ticket counter. Females and males arrive at times of independent Poisson processes with rates 30 and 20 customers per hour.

(a) What is the probability the first three customers are female?

(b) If exactly two customers arrived in the first five minutes, what is the probability both arrived in the first three minutes.

(c) Suppose that customers regardless of sex buy one ticket with probability 1/2, two tickets with probability 2/5, and three tickets with probability 1/10. Let N_i be the number of customers that buy i tickets in the first hour. Find the joint distribution of (N_1, N_2, N_3).

Solution

(a)