Problem (1)

Consider i.i.d. real random variables $Y_1, Y_2, \dots$ with common density f given by

$$f(y) = \begin{cases} \frac{1}{2}e^{-y}, & y \geq 0 \\ e^{2y} & y < 0 \end{cases}$$

Let $S_0 = 0$ and define $(S_n)_{n\geq 1}$ as

$$S_n = Y_1 + \cdots + Y_n, \quad \text{for } n \in \mathbb{N}$$

Also define

$$\mathcal{F}_n = \begin{cases} \{\emptyset, \Omega\} & \text{for } n = 0 \\ \sigma(Y_1, \dots, Y_n) & \text{for } n \in \mathbb{N} \end{cases}$$

(a) For $n \in \mathbb{N}_0$ determine $\mathbb{E}\left[e^{\alpha S_{n+1}} \mid \mathcal{F}_n\right]$ for all $\alpha \in \mathbb{R}$.

(b) Determine $\mathbb{P}(S_n < 0 < S_{n+1} \mid \mathcal{F}_n)$ for $n \in \mathbb{N}$.

Solution

(a) Let $n \in \mathbb{N}_0$,

$$\begin{aligned}
\mathbb{E}[e^{\alpha S_{n+1}}|\mathcal{F}_n] &= \mathbb{E}[e^{\alpha(Y_1+\cdots+Y_n+Y_{n+1})}|\mathcal{F}_n] \\
&= \mathbb{E}[e^{\alpha(Y_1+\cdots+Y_n)}e^{\alpha Y_{n+1}}|\mathcal{F}_n] \\
&\overset{D.10(6)}{=} e^{\alpha(Y_1+\cdots+Y_n)}\mathbb{E}[e^{\alpha Y_{n+1}}|\mathcal{F}_n] \\
&\overset{D.10(8)}{=} e^{\alpha(Y_1+\cdots+Y_n)}\mathbb{E}[e^{\alpha Y_{n+1}}] \\
&\overset{LOTUS}{=} e^{\alpha(Y_1+\cdots+Y_n)} \lim_{n\to\infty} \int_{-n}^{n} e^{\alpha y} f(y)\ \mathrm{d}y \\
&= e^{\alpha(Y_1+\cdots+Y_n)} \left(\lim_{n\to\infty} \int_{-n}^{0} e^{\alpha y} e^{2y}\ \mathrm{d}y + \lim_{n\to\infty} \int_{0}^{n} e^{\alpha y} \frac{1}{2} e^{-y}\ \mathrm{d}y \right) \\
&= e^{\alpha S_n} \left(\lim_{n\to\infty} \int_{-n}^{0} e^{\alpha y} e^{2y}\ \mathrm{d}y + \lim_{n\to\infty} \int_{0}^{n} e^{\alpha y} \frac{1}{2} e^{-y}\ \mathrm{d}y \right)
\end{aligned}$$

For $\lim_{n\to\infty} \int_{-n}^{0} e^{\alpha y} e^{2y}$ we have:

$$\begin{aligned}
\lim_{n\to n} \int_{-\infty}^{0} e^{\alpha y} e^{2y} &= \int_{-\infty}^{0} e^{(\alpha+2)y} \\
&= \lim_{n\to\infty} \left[\frac{e^{(\alpha+2)y}}{\alpha+2} \right]_{-n}^{0} \\
&= \begin{cases} \frac{1}{\alpha+2}, & \text{if } \alpha > -2 \\ \infty, & \text{otherwise} \end{cases}
\end{aligned}$$

For $\lim_{n\to\infty}\int_0^n e^{\alpha y}\frac{1}{2}e^{-y}\ \mathrm{d}y$ we have:

$$\begin{aligned}\lim_{n\to\infty}\int_0^n e^{\alpha y}\frac{1}{2}e^{-y}\ \mathrm{d}y &= 2\int_0^\infty e^{(\alpha-1)y}\ \mathrm{d}y \\ &= \frac{1}{2}\lim_{n\to\infty}\left[\frac{e^{(\alpha-1)}y}{\alpha-1}\right]_0^n \\ &= \begin{cases}\frac{1}{2\alpha-2}, & \text{if } \alpha<1 \\ \infty, & \text{otherwise}\end{cases}\end{aligned}$$

Combining these we get:

$$\mathbb{E}\left[e^{\alpha S_{n+1}}\mid\mathcal{F}_n\right]=\begin{cases}e^{\alpha S_n}\left(\frac{1}{\alpha+2}+\frac{1}{2\alpha-2}\right), & \text{if } -2<\alpha<1 \\ \infty, & \text{otherwise}\end{cases}$$

(b) Let $n\in\mathbb{N}$:

$$\begin{aligned}\mathbb{P}(S_n<0<S_{n+1}|\mathcal{F}_n) &\overset{D.1(4)}{=} \mathbb{E}[\mathbb{1}_{\{S_n<0<S_{n+1}\}}|\mathcal{F}_n] \\ &= \mathbb{E}[\mathbb{1}_{\{S_n<0\}}\mathbb{1}_{\{0<S_{n+1}\}}|\mathcal{F}_n] \\ &= \mathbb{E}[\mathbb{1}_{\{S_n<0\}}\mathbb{1}_{\{-S_n<Y_{n+1}\}}|\mathcal{F}_n] \\ &\overset{\perp\!\!\!\perp}{=} \mathbb{1}_{\{S_n<0\}}\mathbb{E}[\mathbb{1}_{\{-S_n<Y_{n+1}\}}|\mathcal{F}_n] \\ &\overset{D.7}{=} \mathbb{1}_{\{S_n<0\}}\mathbb{E}[\mathbb{1}_{\{-s<Y_{n+1}\}}|\mathcal{F}_n]\Big|_{s=S_n} \\ &= \mathbb{1}_{\{S_n<0\}}\lim_{n\to\infty}\int_{-s}^n\frac{1}{2}e^{-y}\ \mathrm{d}y \\ &= \mathbb{1}_{\{S_n<0\}}\frac{e^s}{2} \\ &= \begin{cases}\frac{e^{S_n}}{2}, & \text{if } S_n<0 \\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

Problem (2)

Let the state space be $S = \mathbb{N}_0 = \{0, 1, \ldots\}$ and for $i \in S$ let $p_i \in [0, 1]$. Consider the Markov chain with state space S specified as follows: A particle which in state $i \in S$ moves to state $i + 1$ in one step with probability p_i and to state 0 with probability $1 - p_i$

(a) Argue that the chain is irreducible if $p_i \in (0, 1)$ for all $i \in S$. Also, indicate the transition diagram in this case.

(b) Determine p_{02}^4. For $n \in \mathbb{N}$ determine $\mathbb{P}_0(T_0 \geq n)$ and $\mathbb{P}_0(T_0 = n)$.

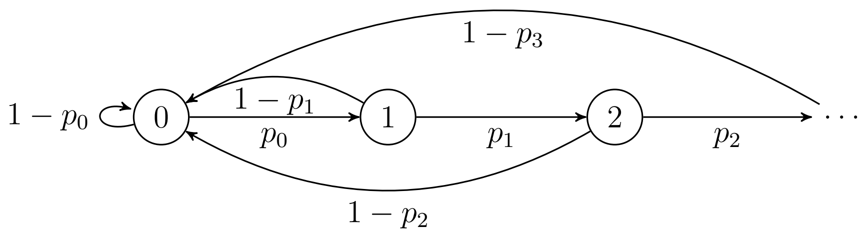
(c) Show that state 0 is recurrent if and only if $\prod_{i=0}^{\infty} p_i = 0$.

(d) Give a concrete example of probabilities $(p_i)_{i=0}^{\infty}$ such that $p_i \in (0, 1)$ for all $i \in S$ and the chain is transient.

(e) Assume that $p_i \in (0, 1)$ for all $i \in S$. Show that there exists a unique stationary distribution if and only if $\sum_{n=0}^{\infty} (p_0 p_1 \cdots p_n) < \infty$. Determine the stationary distribution when it exists.

Solution

(a) To argue that the chain is irreducible, we need to argue that any state can be reached from any other state with positive probability.
If $p_i \in (0, 1) \forall i \in S$, we have probability $p_i > 0$ for moving to state $i + 1$ and probability $1 - p_i > 0$ for moving to state 0.
For states i, j, we have that if $j > i$, then to go from i to j we can move from i to $i+1, i+2, \ldots, j$, with positive probability. If $j < i$ we can move to 0 from any state (with positive probability) and then move to state $1, 2, \ldots, i$ with positive probability. As this covers all cases, the chain is irreducible. Here is a sketch for the transition digram:



(b) We want to determine to probability to go from state 0 to state 2 in exactly 4 steps. There are only 2 combinations where this is possible as represented by the following paths:

$$0 \to 1 \to 0 \to 1 \to 2$$

with probability $p_0^2 p_1 (1 - p_1)$, or

$$0 \to 0 \to 0 \to 1 \to 2$$

with probability $(1-p_0)^2 p_0 p_1$,
So we have that

$$p_{02}^4 = p_0^2 p_1(1-p_1) + (1-p_0)^2 p_0 p_1$$

We now calculate $\mathbb{P}_0(T_0 \geq n)$ and $\mathbb{P}_0(T_0 = n)$. Let $n \in \mathbb{N}$. T_0 is the return time to state 0, meaning that if $T_0 = n$ then we move to state 0 at time $n-1$ from a state i:
For $\mathbb{P}_0(T_0 \geq n)$, we want to calculate the probability of the first return time being at n or later, if we start in state 0, i.e. not returning to 0 for the first $n-2$ steps and then no restrictions from time $n-1$ and onwards. Intuitively this is

$$\mathbb{P}_0(T_0 \leq n) = \prod_{i=0}^{n-2} p_i$$

For $\mathbb{P}_0(T_0 = n)$ we want the probability of the first return time being exactly at time n. This is the same as not returning for the first $n-2$ steps and then exactly returning at the $n-1'th$ step

$$\mathbb{P}_0(T_0 = n) = \left(\prod_{i=0}^{n-2} p_i\right)(1-p_{n-1})$$

(c) Assume state 0 is recurrent:
Then by definition,

$$\mathbb{P}_0(T_0 < \infty) = 1$$

This implies that

$$\mathbb{P}_0(T_0 \geq \infty) = \prod_{i=0}^{\infty} p_i = 0$$

On the other hand, assume that $\Pi_{i=0}^{\infty} p_i = 0$, then there is probability 0 of the first time being after $n = \infty$. I.e. that

$$\mathbb{P}_0(T_0 \geq \infty) = \prod_{i=0}^{\infty} p_i = 0 \Longrightarrow \mathbb{P}_0(T_0 < \infty) = 1$$

Which is the definition of state 0 being recurrent.

(d) To ensure that the chain is transient me must have that

$$\mathbb{P}_0(T_0 = \infty) = \prod_{i=0}^{\infty} p_i > 0 \Rightarrow \mathbb{P}_0(T_0 < \infty) < 1$$

We therefore choose $p_i = e^{-1/2^i}$, which is always in $(0,1)$ and

$$\prod_{i=0}^{\infty} p_i > 0$$

meaning that the chain is transient.

(e) Assume that there exists a unique stationary distribution, π. By Theorem 6.22 we have that $\pi_j = 1/\mathbb{E}_j[T_j]$ and that it is positive reccurent. We then have:

$$\mathbb{E}_0[T_0] < \infty \Rightarrow \sum_{n=0}^{\infty} \prod_{i=0}^{n} p_i < \infty$$

Now, assume that $\sum_{n=0}^{\infty} \prod_{i=0}^{n} p_i < \infty$:

$$\sum_{n=0}^{\infty} \prod_{i=0}^{n} p_i < \infty \Rightarrow \mathbb{E}_0 \left[\sum_{n=0}^{\infty} \prod_{i=0}^{n} p_i \right] < \infty$$

By remark 6.20, P is positive recurrent, and by 6.22(4) and 6.22(1), we then have that there exists a unique stationary distribution.
To determine the stationary distribution we know that $\pi_k = \pi_{k-1} \prod_{i=0}^{k-1} p_i$, and by extension, $\pi_k = \pi_0 \prod_{i=0}^{k-1}$. So we can conclude that

$$\pi_0 = \frac{1}{\sum_{k=0}^{\infty} \prod_{i=0}^{k-1} p_i} \quad \text{for } i = 0$$

and

$$\pi_k = \pi_0 \prod_{i=0}^{k-1} p_i = \frac{\prod_{i=0}^{k-1} p_i}{\sum_{k=0}^{\infty} \prod_{i=0}^{k-1} p_i}, \quad \text{for } i \in \mathbb{N}$$

Problem (3)

Consider Proposition 6.3 in the lecture notes.

(a) Prove (6.4).

Let $i \in S$ and μ denote a probability measure on S. Assume $\mathbb{P}_\mu(T_i < \infty) = 1$ and $\mathbb{P}_i(T_i < \infty) = 1$.

(b) Show that T_i^n is finite $\mathbb{P}_\mu$-a.s. for every $n \in \mathbb{N}$.

(c) Show that T_i^1 and $T_i^2 - T_i^1$ are independent under $\mathbb{P}_\mu$ and that the distribution of $T_i^2 - T_i^1$ under $\mathbb{P}_\mu$ is the same as the distribution of T_i^1 under $\mathbb{P}_i$.

Essentially, the above proves Proposition 6.3. The only remaining step is to extend (c) above by induction to get Proposition 6.3(2)(b) – (c).

Solution

(a) We need to prove that

$$\mathbb{E}_\mu = \mathbb{P}_\mu(T_i < \infty)(1 + \mathbb{E}_i[N_i])$$

Let $i \in S$ and μ a probability measure on S:

$$\begin{aligned}
\mathbb{E}_\mu[N_i] &\overset{6.2}{=} \mathbb{E}_\mu\left[\left((1+N_i)\circ\theta^{T_i}\right)\mathbb{1}_{\{T_i<\infty\}}\right] \\
&\overset{\text{Tower}}{=} \mathbb{E}_\mu\left[\mathbb{E}_\mu\left[\left((1+N_i)\circ\theta^{T_i}\right)\mathbb{1}_{\{T_i<\infty\}}\right]|\mathcal{F}_{T_i}^X\right] \\
&= \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i<\infty\}}\mathbb{E}_\mu\left[\left((1+N_i)\circ\theta^{T_i}\right)\right]|\mathcal{F}_{T_i}^X\right] \\
&\overset{4.25}{=} \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i<\infty\}}\mathbb{E}_{X_{T_i}}\left[(1+N_i)\right]\right] \\
&= \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i<\infty\}}\mathbb{E}_i\left[(1+N_i)\right]\right] \\
&= \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i<\infty\}}\right]\mathbb{E}_i\left[(1+N_i)\right] \\
&= \mathbb{P}_\mu(T_i<\infty)\mathbb{E}_i\left[(1+N_i)\right] \\
&= \mathbb{P}_\mu(T_i<\infty)\left(1+\mathbb{E}_i\left[N_i\right]\right)
\end{aligned}$$

Which is the desired result.

(b) We assume that $\mathbb{P}_\mu(T_i < \infty) = 1$ and $\mathbb{P}_i(T_i < \infty) = 1$. From (6.3) we know that

$$\mathbb{P}_\mu\left(T_i^{n+1} < \infty\right) = \mathbb{P}_\mu(T_i < \infty)\left[\mathbb{P}_i(T_i < \infty)\right]^n$$

Which is equivalent with

$$\mathbb{P}_\mu(T_i^n < \infty) = \mathbb{P}_\mu(T_i < \infty)\left[\mathbb{P}_i(T_i < \infty)\right]^{n-1}$$

So we have:

$$\mathbb{P}_\mu(T_i^n < \infty) = 1 \cdot 1^{n-1} = 1$$

Which means that $T_i^n < \infty$ $\mathbb{P}_\mu$ – a.s.

(c) We first show that they are equally distributed. We need to show that

$$\mathbb{P}_\mu(T_i^2 - T_i^1 \in B) = \mathbb{P}_i(T_i^1 \in B)\ \forall B \in \mathcal{B}(S).$$

Let $B \in \mathcal{B}(S)$:

$$\begin{aligned}
P_\mu(T_i^2 - T_i^1 \in B) &= \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i^2 - T_i^1 \in B\}}\right] \\
&\overset{6.1}{=} \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i \circ \theta^{T_i^1} \in B\}}\right] \\
&\overset{\text{Assumption}}{=} \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i^1 < \infty\}}\mathbb{1}_{\{T_i \circ \theta^{T_i^1} \in B\}}\right] \\
&\overset{\text{Tower}}{=} \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i^1 < \infty\}}\mathbb{E}_\mu\left[\mathbb{1}_{\{T_i \circ \theta^{T_i^1} \in B\}}|\mathcal{F}_{T_i^1}^X\right]\right] \\
&\overset{4.25}{=} \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i^1 < \infty\}}\mathbb{E}_{X_{T_i^1}}\left[\mathbb{1}_{\{T_i^1 \in B\}}\right]\right] \\
&= \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i^1 < \infty\}}\mathbb{P}_{X_{T_i^1}}\left(T_i^1 \in B\right)\right] \\
&\overset{X_{T_i^1}=i}{=} \mathbb{P}_\mu\left(T_i^1 < \infty\right)\mathbb{P}_i\left(T_i^1 \in B\right) \\
&\overset{\text{Assumption}}{=} \mathbb{P}_i\left(T_i^1 \in B\right)
\end{aligned}$$

We now show that they are independent. Let $B_1, B_2 \in \mathcal{B}(S)$:

$$\begin{aligned}
\mathbb{P}_\mu\left(T_1 \in B_1, T_2 - T_1 \in B_2\right) &= \mathbb{E}_\mu\left[\mathbb{1}_{\{T_1 \in B_1\}}\mathbb{1}_{\{T_2 - T_1 \in B_2\}}\right] \\
&\overset{\text{Tower}}{=} \mathbb{E}_\mu\left[\mathbb{E}_\mu\left[\mathbb{1}_{\{T_1 \in B_1\}}\mathbb{1}_{\{T_2 - T_1 \in B_2\}}|\mathcal{F}_{T_1}^X\right]\right] \\
&\overset{D.10(6)}{=} \mathbb{E}_\mu\left[\mathbb{1}_{\{T_1 \in B_1\}}\mathbb{E}_\mu\left[\mathbb{1}_{\{T_2 - T_1 \in B_2\}}|\mathcal{F}_{T_1}^X\right]\right] \\
&= \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i^1 \in B_1\}}\mathbb{E}_\mu\left[\mathbb{1}_{\{T_i^1 \circ \theta^{T_i^1} \in B_2\}}\mathbb{1}_{\{T_i^1 < \infty\}}|\mathcal{F}_{T_1}^X\right]\right] \\
&\overset{4.25}{=} \mathbb{E}_\mu\left[\mathbb{1}_{\{T_i^1 \in B_1\}}\mathbb{1}_{\{T_i^1 < \infty\}}\mathbb{E}_{X_{T_i}}\left[\mathbb{1}_{\{T_i^1 \in B_2\}}|\mathcal{F}_{T_1}^X\right]\right] \\
&= \mathbb{P}_\mu(T_i^1 \in B_1)\mathbb{P}_\mu(T_i^1 < \infty)\mathbb{P}_i(T_i^1 \in B_2) \\
&= \mathbb{P}_\mu(T_i^1 \in B_1)\mathbb{P}_i(T_i^1 \in B_2)
\end{aligned}$$

Which are the desired results.

Problem (4)

Let $S = \{1, 2, 3, 4, 5, 6\}$ and

$$\mathsf{P} = \begin{pmatrix} 0.1 & 0.4 & 0.1 & 0 & 0 & 0.4 \\ 0 & 0.2 & 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.2 & 0.2 & 0.1 \\ 0 & 0 & 0.8 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Argue that there exists a unique stationary distribution, determine it, and determine $\mathbb{E}_i [T_i]$ for all $i \in S$

Solution

We have a transition matrix, P. We see that the states 1,2 are transient states, since you cannot return to them if you leave them. By exercise 60 this means that $\pi_1 = \pi_2 = 0$ with $\mathbb{E}[T_1] = \mathbb{E}[T_2] = \infty$. We now look at $S = \{3, 4, 5, 6\}$. We see that this chain is irreducible, and that all states are recurrent.
We have the following transition matrix:

$$\mathsf{P}^* = \begin{pmatrix} 0.2 & 0.3 & 0.5 & 0 \\ 0.5 & 0.2 & 0.2 & 0.1 \\ 0.8 & 0 & 0.2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We determine the stationary distribution π^* for $S^* = \{3, 4, 5, 6\}$ by using that is satisfies the following:

$$\pi^* = \pi^* \mathsf{P}^*$$

We solve this by using CAS to obtain:

$$\pi^* = \begin{pmatrix} 0.4591 & 0.1722 & 0.3515 & 0.0172 \end{pmatrix}$$

To determine $\mathbb{E}_i[T_i]$ we know that

$$\pi_i = \frac{1}{\mathbb{E}_i[T_i]} \; \forall i \in S$$

We thus take each element in π to the power of -1 to obtain the following vector of expected values:

$$\mathsf{E}^* = \begin{pmatrix} 2.1782 & 5.8072 & 2.8450 & 58.1395 \end{pmatrix}$$

Since we have that P^* is irreducible and that $\mathbb{E}[T_i] < \infty \; \forall i \in S^*$, P^* is positive recurrent, and by 6.22 has a unique stationary distribution.