Problem (1.7)

Let $\mathsf{X} = (\mathsf{X}_1, \ldots, \mathsf{X}_d)$ and $\mathsf{Y} = (\mathsf{Y}_1, \ldots, \mathsf{Y}_m)$ be $d-$ and $m-$ dimensional stochastic vectors defined on the probability field $(\Omega, \mathcal{F}, P)$. Further consider the usual inner products, $\langle \cdot, \cdot \rangle_d$ and $\langle \cdot, \cdot \rangle_m$ on $\mathbb{R}^d$ and $\mathbb{R}^m$.

(a) Assume $d = m$. Show, that $\mathsf{X} \sim \mathsf{Y}$, if and only if $\langle t, \mathsf{X} \rangle_d \sim \langle t, \mathsf{Y} \rangle_d$ for all vectors $t = (t_1, \ldots, t_d)$ in $\mathbb{R}^d$

(b) Show, that X and Y are independent, if and only if the stochastic variables $\langle t, \mathsf{X} \rangle_d$ and $\langle s, \mathsf{Y} \rangle_m$ are independent for all vectors $t = (t_1, \ldots, t_d)$ in $\mathbb{R}^d$ and $s = (s_1, \ldots, s_m)$ in $\mathbb{R}^m$.

Solution

(a) We first assume $\mathsf{X} \sim \mathsf{Y}$. Then it holds that $\langle t, \mathsf{X} \rangle_d \sim \langle t, \mathsf{Y} \rangle_d$ for $t \in \mathbb{R}^d$, since this will be the result of multiplying by some scalar.

We now assume $\langle t, \mathsf{X} \rangle_d \sim \langle t, \mathsf{Y} \rangle_d$ for $t \in \mathbb{R}^d$. We calculate:

$$\varphi_{\mathsf{X}}(t) = \mathbb{E}\left[e^{i\langle t, \mathsf{X} \rangle_d}\right] = \mathbb{E}\left[e^{i\langle t, \mathsf{Y} \rangle_d}\right] = \varphi_{\mathsf{Y}}(t) \forall t \in \mathbb{R}^d$$

Since their characteristic functions are equal, by 1.2.5(i) they must be identically distributed. So in conclusion:

$$\mathsf{X} \sim \mathsf{Y} \Leftrightarrow \langle t, \mathsf{X} \rangle_d \sim \langle t, \mathsf{Y} \rangle_d \forall t \in \mathbb{R}^d$$

(b) Assume that X and Y are independent. Then it again follows, that $\langle t, \mathsf{X} \rangle_d \sim \langle s, \mathsf{Y} \rangle_m$ also are independent $\forall t \in \mathbb{R}^d, \forall s \in \mathbb{R}^m$.

We now assume $\langle t, \mathsf{X} \rangle_d \sim \langle s, \mathsf{Y} \rangle_m$ are independent $\forall t \in \mathbb{R}^d, \forall s \in \mathbb{R}^m$. We once again calculate the characteristic functions:

$$\begin{aligned}\varphi_{(\mathsf{X},\mathsf{Y})}(t, s) &= \mathbb{E}\left[e^{i\langle (t,s), (\mathsf{X},\mathsf{Y}) \rangle_{d+m}}\right] = \mathbb{E}\left[e^{i(\langle t, \mathsf{X} \rangle_d + \langle s, \mathsf{Y} \rangle_m)}\right] = \mathbb{E}\left[e^{i\langle t, \mathsf{X} \rangle_d} e^{i\langle s, \mathsf{Y} \rangle_m}\right] \\ &\overset{\perp\!\!\!\perp}{=} \mathbb{E}\left[e^{i\langle t, X \rangle_d}\right] \mathbb{E}\left[e^{i\langle s, Y \rangle_m}\right] = \phi_X(t)\phi_Y(s), \forall t \in \mathbb{R}^d, \forall s \in \mathbb{R}^m\end{aligned}$$

Which by 1.2.7 means that X and Y are independent. We have thus shown that

$$\mathsf{X} \perp\!\!\!\perp \mathsf{Y} \Leftrightarrow \langle t, \mathsf{X} \rangle_d \perp\!\!\!\perp \langle t, \mathsf{Y} \rangle_d$$