Problem (1)

Consider the measure space $(X, \mathcal{E}, \mu)$, and let $f, g, f_1, f_2, f_3, \ldots$ be functions from $\mathcal{M}(\mathcal{E})$. Further let

$$\xrightarrow{(1)} \quad \text{and} \quad \xrightarrow{(2)}$$

denote two (possibly different) of the three fundamental types of convergence defined in Definition 2.1.1 in the notes.

(a) Generalize Proposition 2.1.3 by proving, that for any choice of $\xrightarrow{(1)}$, $\xrightarrow{(2)}$, the following implication holds:

$$f_n \xrightarrow{(1)} f \text{ and } f_n \xrightarrow{(2)} g \Longleftrightarrow f = g \ \mu\text{-a.e.} \tag{1}$$

(b) Can we in (a) generally conclude, that $f(x) = g(x)$ for all $x \in X$?

Solution

(a) We have the following 4 cases:

(i) If $\xrightarrow{(1)}$ and $\xrightarrow{(2)}$ denote the same type of convergence, then by 2.1.3 we have that (1) is satisfied.

(ii) If $\xrightarrow{(1)}$ and $\xrightarrow{(2)}$ are convergence in μ-p-mean and convergence in μ-measure such that

$$f_n \longrightarrow f \text{ in } \mu\text{-p-mean, and } f_n \longrightarrow g \text{ in } \mu\text{-measure}$$

then by 2.1.4(i), $f_n \longrightarrow f$ in μ-p-measure, and again by 2.1.3, (1) is satisfied.

(iii) If $\xrightarrow{(1)}$ and $\xrightarrow{(2)}$ are convergence μ-a.e. and convergence in μ-p-mean such that

$$f_n \longrightarrow f \text{ in } \mu\text{-a.e., and } f_n \longrightarrow g \text{ in } \mu\text{-p-mean}$$

then, by 7.4.10 [M&I] there exists an increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that $f_{n_k} \longrightarrow g$ μ-a.e. for $k \to \infty$.
We also have that, since $f_n \longrightarrow f$ μ-a.e., this also holds for any subsequence. Specifically, we have that $f_{n_k} \longrightarrow f$ μ-a.e. for $k \to \infty$ for the specified sequence (n_k). Again by 2.1.3, (1) is satisfied.

(iv) If $\xrightarrow{(1)}$ and $\xrightarrow{(2)}$ are convergence μ-a.e. and convergence in μ-measure such that

$$f_n \longrightarrow f \text{ in } \mu\text{-a.e., and } f_n \longrightarrow g \text{ in } \mu\text{-measure}$$

then by 2.1.4(iii) there exists an increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that $f_{n_k} \longrightarrow g$ μ-a.e. for $k \to \infty$. Again, by the same argument as in (iii) we have $f_{n_k} \longrightarrow f$ μ-a.e., and again by 2.1.3, (1) is satisfied.

Therefore we have shown the desired property.

(b) If we consider the functions $f_n = \mathbb{1}_{\{0\}}$, $f = \mathbb{1}_{\{0\}}$ and $g = 0$ from $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, then we have that

$$f_n \to f$$

for all three main types of convergence, since $f_n = f \ \forall n \in \mathbb{N}$.
In the following, we show that $f_n \to g$ for all three main types of convergence:
For $f_n \to g$ convergence λ-a.e. and convergence in the λ-p-mean we have that $\mathbb{1}_{\{0\}}$ and 0 are both equal to 0 λ-a.e. and

$$\int_{\mathbb{R}} \left|\mathbb{1}_{\{0\}} - 0\right|^p d\lambda = \int_{\mathbb{R}} \mathbb{1}_{\{0\}} d\lambda = \lambda(\{0\}) = 0 \forall n \in \mathbb{N}$$

.

We now argue that $f_n \to g$ in λ-measure:
Let $\epsilon > 0$. Then $\forall n \in \mathbb{N}$ it holds that:

$$\begin{aligned}\lambda\left(\{x \in X : |f_n(x) - g(x)| > \epsilon\}\right) &= \lambda\left(\left\{x \in X : \mathbb{1}_{\{0\}}(x) > \epsilon\right\}\right) \\ &= \begin{cases} \lambda(\{0\}) = 0, & \text{for } \epsilon < 1 \\ \lambda(\emptyset) = 0, & \text{otherwise} \end{cases}\end{aligned}$$

So to use the notation from (a), we have that

$$f_n \xrightarrow{(1)} \mathbb{1}_{\{0\}} \quad \text{and} \quad f_n \xrightarrow{(2)} 0$$

for any two of the three main types of convergence. We also have that $\mathbb{1}_{\{0\}}$ and 0 are not equal on the λ-null set $\{0\}$. Therefore, we cannot generally conclude that $f(x) = g(x)$ for all $x \in X$ in (a).