Problem (Exercise 1.4)

Let X be a stochastic variable on the probability field $(\Omega, \mathcal{F}, P)$, and consider the characteristic function φ_{X}.

(a) Assume, that X is binomially distributed with parameters $n \in \mathbb{N}$ and probability parameter $p \in [0,1]$. Show, that

$$\varphi_{\mathsf{X}}(t) = (1 - p + pe^{\mathrm{i}\,t})^n \quad (t \in \mathbb{R})$$

(b) Assume, that X is poisson distributed with parameters $\ell \in (0, \infty)$. Show, that

$$\varphi_{\mathsf{X}}(t) = \exp(\ell(e^{\mathrm{i}\,t} - 1)) \quad (t \in \mathbb{R})$$

Solution

In general we use example 13.2.7 and proposition 13.2.9 from [M&I] to define:

$$\psi_{\mathsf{X}}(t) = e^{\mathrm{i}\,t\mathsf{X}}$$

and recall that the characteristic function for a one-dimensional stochastic vector is defined as

$$\varphi_{\mathsf{X}} = \mathbb{E}\left[e^{\mathrm{i}\,t\mathsf{X}}\right]$$

and then use 13.2.9:

$$\mathbb{E}[\psi(\mathrm{X})] = \sum_{i \in I} \psi\left(x_i\right) p_{\mathrm{X}}\left(x_i\right).$$

(a)

$$\begin{aligned}
\varphi_{\mathsf{X}} = \mathbb{E}[e^{\mathrm{i}\,t\mathsf{X}}] = \mathbb{E}[\psi_{\mathsf{X}}] &\overset{13.2.9+13.2.7(a)}{=} \sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{\mathrm{i}\,tk} \\
&= \sum_{k=1}^{n} \binom{n}{k} (pe^{\mathrm{i}\,t})^k (1-p)^{n-k} \\
&\overset{13.2.7(A)}{=} (pe^{\mathrm{i}\,t} + (1-p))^n
\end{aligned}$$

(b)

$$\begin{aligned}
\varphi_{\mathsf{X}} = \mathbb{E}[e^{\mathrm{i}\,t\mathsf{X}}] = \mathbb{E}[\psi_{\mathsf{X}}] &\overset{13.2.9+13.2.7(b)}{=} \sum_{k=0}^{\infty} \frac{e^{i \cdot t \cdot k} \ell^k e^{-\ell}}{k!} \\
&= e^{-\ell} \sum_{k=0}^{\infty} \frac{\left(e^{it}\ell\right)^k}{k!} \\
&= e^{-\ell} \exp\left(\ell e^{\mathrm{it}}\right) \\
&= \exp\left(\ell\left(e^{\mathrm{it}} - 1\right)\right)
\end{aligned}$$

Which are the desired results.