Problem (3.6)

Let $(X, \mathcal{E}, \mu)$ be a finite measure space, and let (f_n) be a sequence of functions from $\mathcal{M}(\mathcal{E})$ such that $\{f_n | n \in \mathbb{N}\}$ is uniformly integrable. Define a new sequence (g_n) of functions by:

$$g_n := n^{-1} \sum_{j=1}^{n} f_j$$

Show then, that the set $\{g_n | n \in \mathbb{N}\}$ likewise is uniformly integrable.

Proposition 3.1.4. *A subset $\mathcal{H}$ of $\mathcal{M}(\mathcal{E})$ is uniformly integrable, if and only if it satisfies the following two conditions:*

(i) $\sup_{f \in \mathcal{H}} \int_X |f| \mathrm{d}\mu < \infty$.

(ii) $\forall \epsilon > 0 \exists \delta > 0 \forall A \in \mathcal{E} : \mu(A) \leq \delta \Longrightarrow \sup_{\delta \in \nu} \int_\Delta |f| \mathrm{d}\mu \leq \epsilon$.

Solution

We show uniform integrability by Proposition 3.1.4.
For $A \in \mathcal{E}$ we have, that

$$\begin{aligned}
\int_A |g_n| \ \mathrm{d}\mu &= n^{-1} \int_A \left| \sum_{j=1}^{n} f_j \right| \ \mathrm{d}\mu \\
&\leq n^{-1} \int_A \sum_{j=1}^{n} |f_j| \ \mathrm{d}\mu \\
&\overset{5.2.9}{=} n^{-1} \sum_{j=1}^{n} \int_A |f_j| \ \mathrm{d}\mu \\
&\leq \sup_{m \in \mathbb{N}} \int_A |f_m| \ \mathrm{d}\mu \\
&\leq k < \infty
\end{aligned}$$

for any $n \in \mathbb{N}$. Then it follows that

$$\sup_{n \in \mathbb{N}} \int_A |g_n| \ \mathrm{d}\mu \leq \sup_{m \in \mathbb{N}} \int_A |f_m| \ \mathrm{d}\mu$$

This especially holds for $A = X$, and we have that $\{g_n | n \in \mathbb{N}\}$ satisfies condition (i) in 3.1.4.
To prove the second condition, we let $\epsilon > 0$. Since $\{f_n | n \in \mathbb{N}\}$ is uniformly integrable, and thus satisfies condition (ii) in 3.1.4, we can choose $\delta > 0$ such that:

$$\forall A \in \mathcal{E} : \mu(A) \leq \delta \Longrightarrow \sup_{n \in \mathbb{N}} \int_A |f_n| \ \mathrm{d}\mu \leq \epsilon$$

From this it follows that:

$$\forall A \in \mathcal{E} : \mu(A) \leq \delta \Longrightarrow \sup_{n \in \mathbb{N}} \int_A |g_n| \ \mathrm{d}\mu \leq \epsilon$$

Which shows that $\{g_n | n \in \mathbb{N}\}$ satisfies condition (ii) in Proposition 3.1.4, and thus the proof is done.