Fluid Flow and Heat Transfer in Open Cell Foams

towards a port-Hamiltonian approach to study thermoacoustic engines

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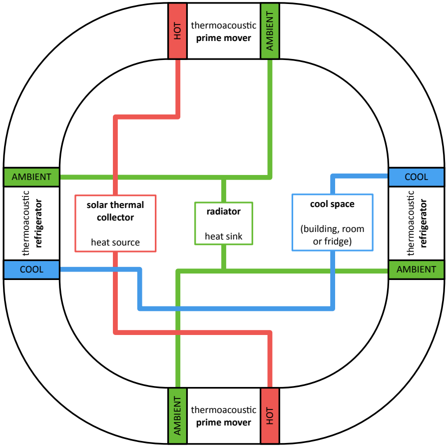
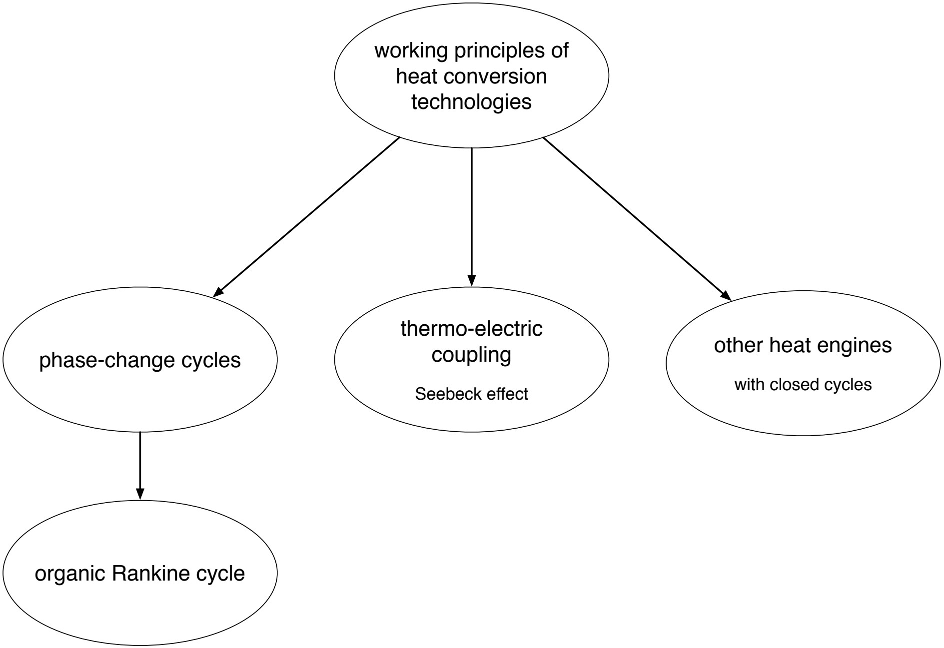


Technologies for low-temperature heat conversion



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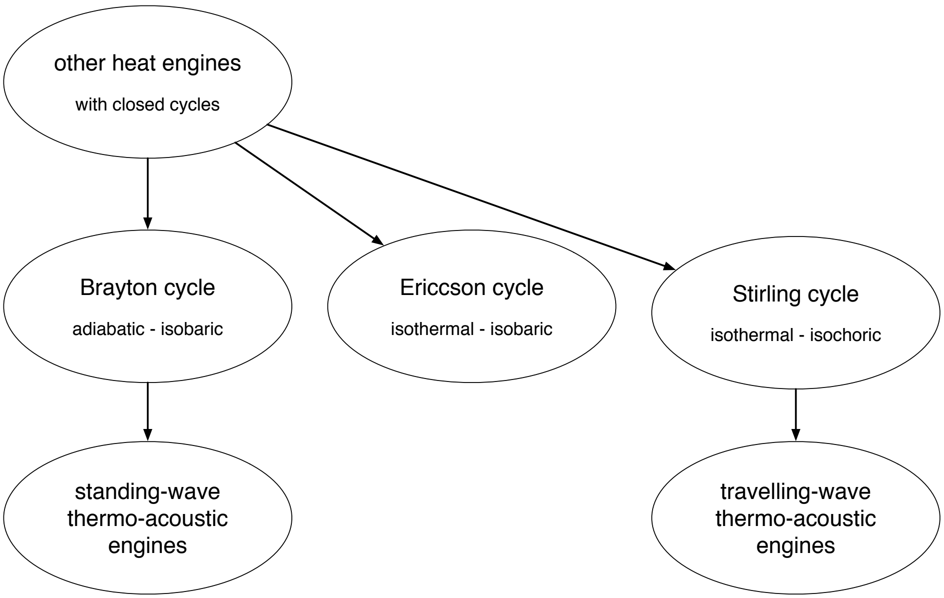
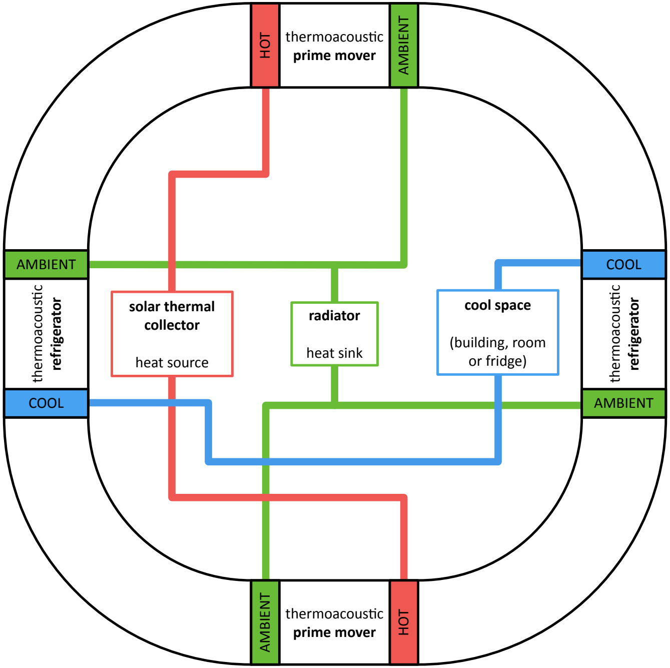
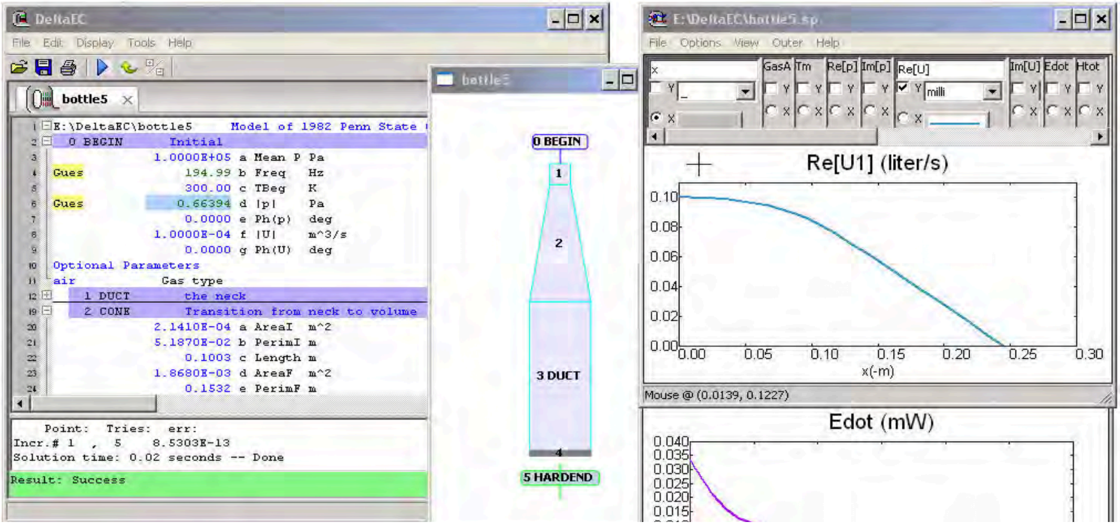


Thermoacoustic devices



How to design thermoacoustic devices?



DeltaEC

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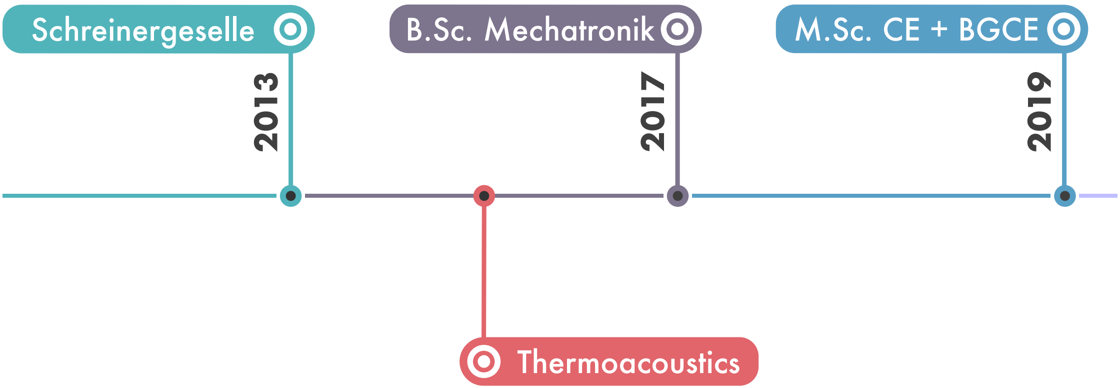
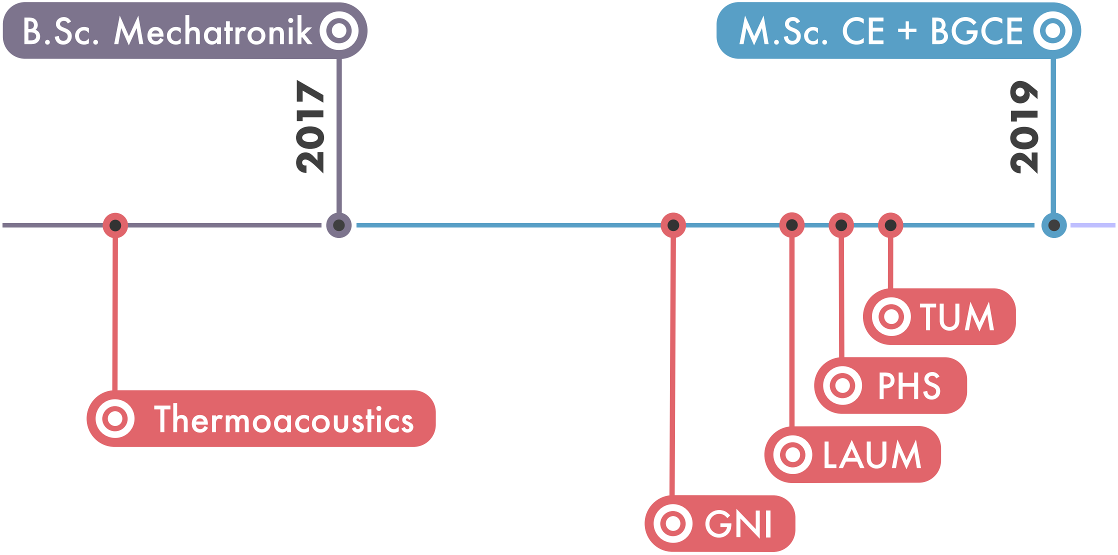
- low-amplitude acoustic approximation (linearized Navier-Stokes equations)
- frequency domain $\left(\mathrm{p}e^{\mathrm{i}\omega t}, \mathrm{u}e^{\mathrm{i}\omega t}\right)$ (Helmholtz equation)

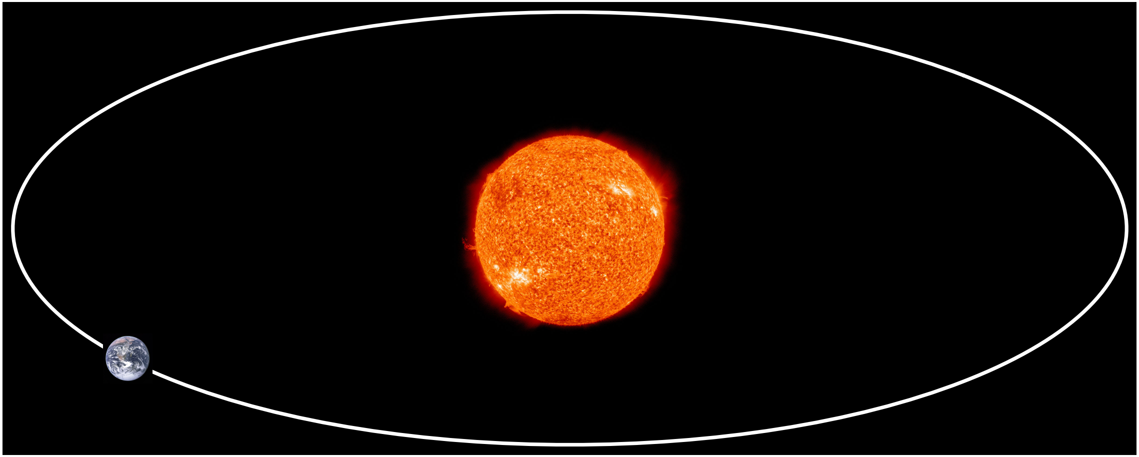
My career path



My master thesis

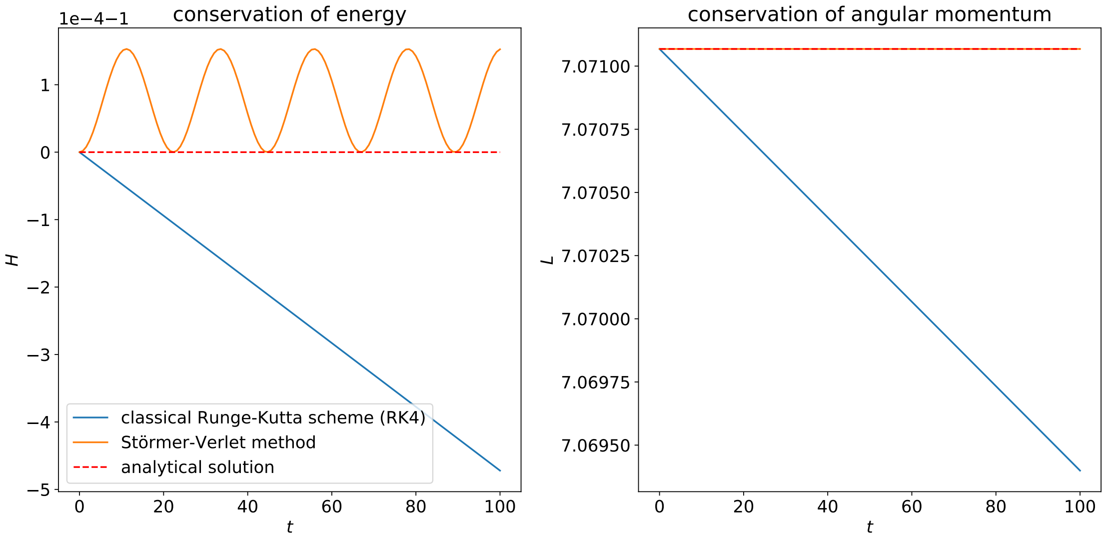


Kepler problem

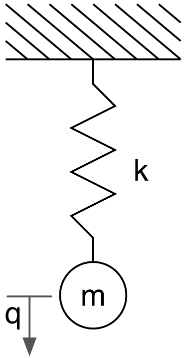


$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = -\frac{G\,M}{(x^2+y^2)^{\frac{3}{2}}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Geometric numerical integration



Mass-spring harmonic oscillator



free vibration: $\mathrm{m}\,\ddot{\mathrm{q}} + \mathrm{k}\,\mathrm{q} = 0$

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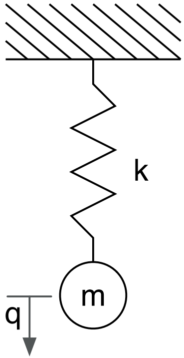

$$\begin{bmatrix} 1 & 0 \\ 0 & \mathrm{m} \end{bmatrix} \begin{bmatrix} \dot{\mathrm{q}} \\ \dot{\mathrm{v}} \end{bmatrix} = \begin{bmatrix} \mathrm{v} \\ -\mathrm{k}\,\mathrm{q} \end{bmatrix}$$

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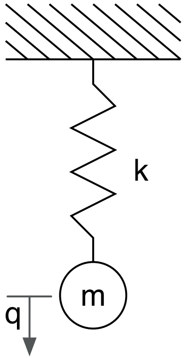

$$\begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -k\,q \end{bmatrix}$$

$$p := mv \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{1}{m}\,p \\ -k\,q \end{bmatrix}$$

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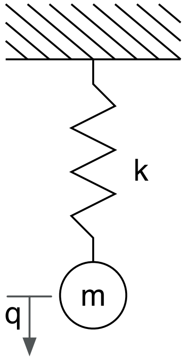

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$$H(q,p) = H_q(q) + H_p(p) = \frac{1}{2}\,q\,k\,q + \frac{1}{2}\,p\,\frac{1}{m}\,p$$

free vibration: $m\,\ddot{q} + k\,q = 0$

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{1}{m}\,p \\ -k\,q \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{bmatrix}$$

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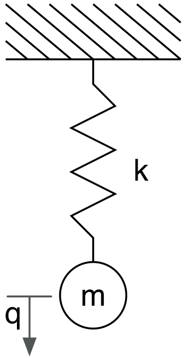

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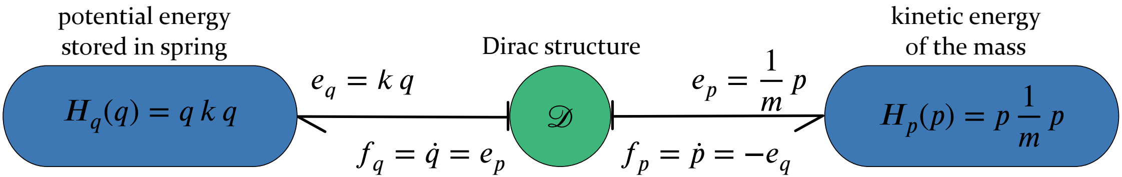
$$\mathrm{H}(\mathrm{q},\mathrm{p}) = \mathrm{H}_{\mathrm{q}}(\mathrm{q}) + \mathrm{H}_{\mathrm{p}}(\mathrm{p}) = \frac{1}{2}\,\mathrm{q}\,\mathrm{k}\,\mathrm{q} + \frac{1}{2}\,\mathrm{p}\,\frac{1}{\mathrm{m}}\,\mathrm{p}$$

$$\begin{bmatrix} \dot{\mathrm{q}} \\ \dot{\mathrm{p}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\mathrm{m}}\,\mathrm{p} \\ -\mathrm{k}\,\mathrm{q} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathrm{H}}{\partial \mathrm{p}} \\ -\frac{\partial \mathrm{H}}{\partial \mathrm{q}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathrm{H}}{\partial \mathrm{q}} \\ \frac{\partial \mathrm{H}}{\partial \mathrm{p}} \end{bmatrix}$$

Hamiltonian system shown as a bond graph



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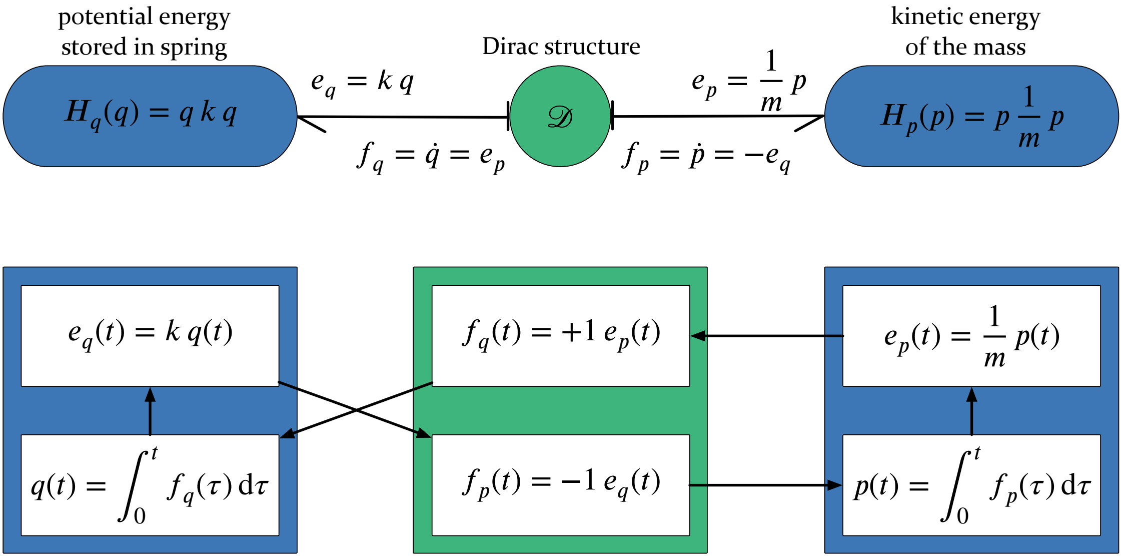

Energy balance

harmonic oscillator as a Hamiltonian system:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\text{canonical structure matrix}} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix}$$

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more general Hamiltonian system:

$$\dot{x} = J \frac{\partial H}{\partial x} \quad \text{with} \quad J^T = -J$$

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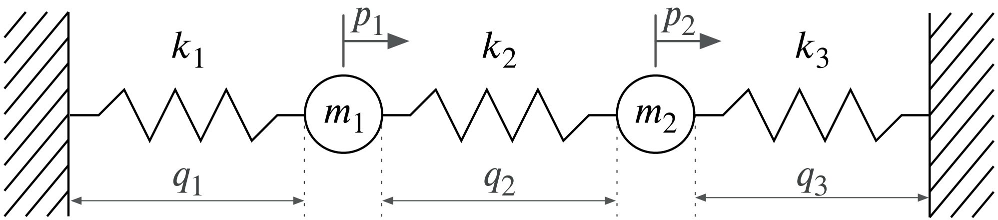
conservation of energy:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} = \frac{\partial H}{\partial x} J \frac{\partial H}{\partial x} = \frac{\partial H}{\partial x} J^T \frac{\partial H}{\partial x} = -\frac{\partial H}{\partial x} J \frac{\partial H}{\partial x} = 0$$

Spring-mass system



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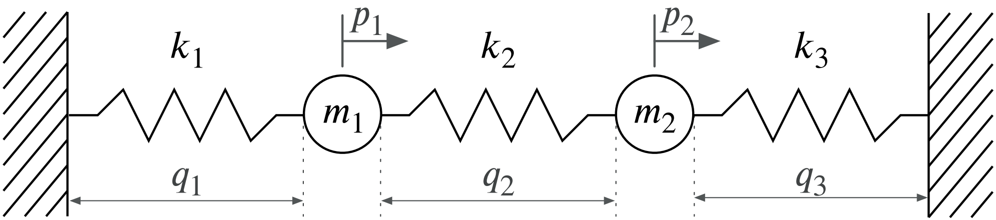
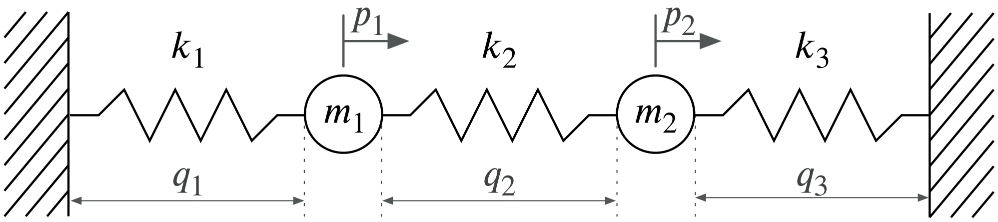


state: $\mathrm{x} = \begin{bmatrix} \mathrm{q} & \mathrm{p} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \mathrm{q}_1 & \mathrm{q}_2 & \mathrm{q}_3 & \mathrm{p}_1 & \mathrm{p}_2 \end{bmatrix}^{\mathrm{T}}$

Spring-mass system



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Hamiltonian: $\mathrm{H} = \frac{1}{2}\,\mathrm{q}^{\mathrm{T}}\,\mathsf{diag}\left(\mathrm{k}_1,\, \mathrm{k}_2,\, \mathrm{k}_3\right)\mathrm{q} + \frac{1}{2}\,\mathrm{p}^{\mathrm{T}}\,\mathsf{diag}\left(\frac{1}{\mathrm{m}_1},\, \frac{1}{\mathrm{m}_2}\right)\mathrm{p}$

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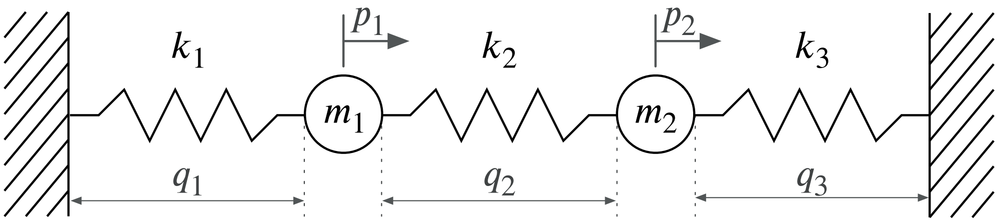

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incidence matrix: $\mathrm{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

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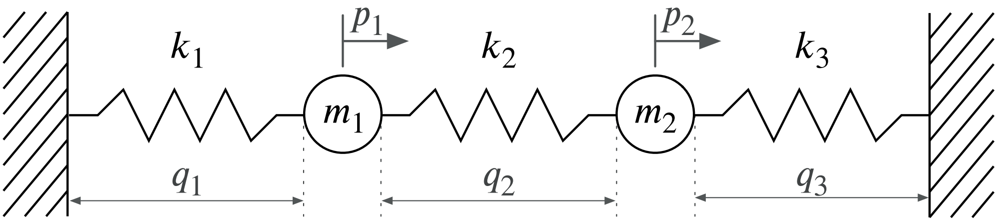

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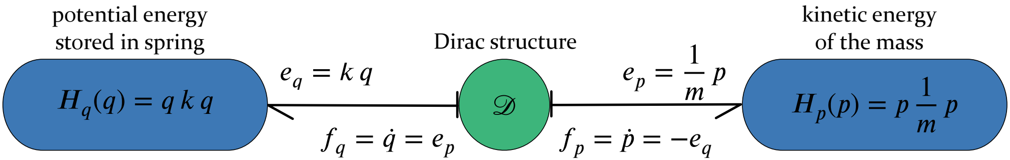
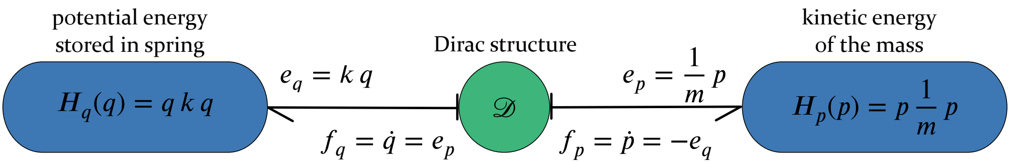
Hamiltonian system: $\dot{\mathrm{x}} = \begin{bmatrix} 0 & \mathrm{A}^{\mathrm{T}} \\ -\mathrm{A} & 0 \end{bmatrix} \frac{\partial \mathrm{H}}{\partial \mathrm{x}}$

Dirac structures – power-conserving interconnection



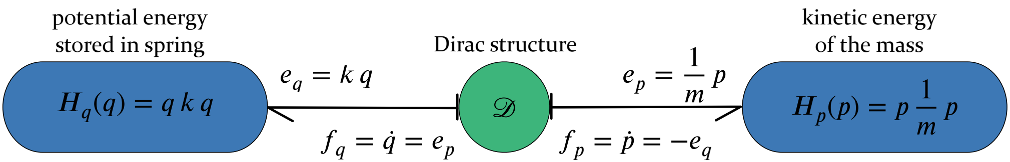
Dirac structures – power-conserving interconnection



space of flow variables: $\mathcal{F} = \mathbb{R}^2 \ni \mathrm{f} = (\mathrm{f_q},\, \mathrm{f_p})$

space of effort variables: $\mathcal{E} = \mathcal{F}^* \ni \mathrm{e} = (\mathrm{e_q},\, \mathrm{e_p})$

Dirac structures – power-conserving interconnection



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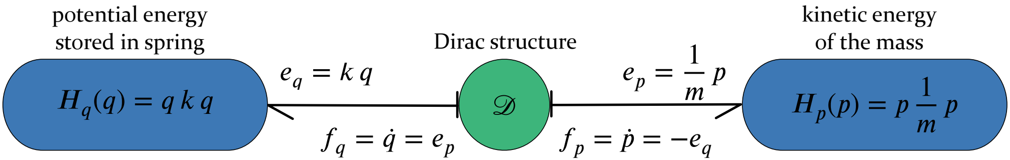
space of power variables (bond space): $\mathcal{B} = \mathcal{F} \times \mathcal{F}^* \ni (\mathrm{f}, \mathrm{e})$

Dirac structures – power-conserving interconnection



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space of power variables (bond space): $\mathcal{B} = \mathcal{F} \times \mathcal{F}^* \ni (\mathrm{f}, \mathrm{e})$

with natural pairing $\langle \cdot \mid \cdot \rangle : \mathcal{B} \xrightarrow{\sim} \mathbb{R}$

$$(\mathrm{f}, \mathrm{e}) \mapsto \langle \mathrm{e} \mid \mathrm{f} \rangle = \mathrm{e}^{\mathrm{T}}\,\mathrm{f} = \mathrm{e_q}\,\mathrm{f_q} + \mathrm{e_p}\,\mathrm{f_p}$$

Dirac structures – power-conserving interconnection

canonically defined symmetric bilinear form on $\mathcal{B}$:

$$
\begin{aligned}
\langle \cdot, \cdot \rangle : \mathcal{B} \times \mathcal{B} &\xrightarrow{\sim} \mathbb{R} \\
\Big(\big(\mathrm{f}^1, \mathrm{e}^1\big), \big(\mathrm{f}^2, \mathrm{e}^2\big) \Big) &\mapsto \langle \big(\mathrm{f}^1, \mathrm{e}^1\big), \big(\mathrm{f}^2, \mathrm{e}^2\big) \rangle = \langle \mathrm{e}^1 \mid \mathrm{f}^2 \rangle + \langle \mathrm{e}^2 \mid \mathrm{f}^1 \rangle \\
&= \mathrm{e}^1_{\mathrm{q}} \, \mathrm{f}^2_{\mathrm{q}} + \mathrm{e}^1_{\mathrm{p}} \, \mathrm{f}^2_{\mathrm{p}} + \mathrm{e}^2_{\mathrm{q}} \, \mathrm{f}^1_{\mathrm{q}} + \mathrm{e}^2_{\mathrm{p}} \, \mathrm{f}^1_{\mathrm{p}}
\end{aligned}
$$

Dirac structures – power-conserving interconnection

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Dirac structure: $\mathcal{D} \subset_{\mathrm{vec}} \mathcal{B}$ such that $\mathcal{D} = \mathcal{D}^\perp$

Dirac structures – power-conserving interconnection

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Dirac structures – power-conserving interconnection

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$$

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$$

$$
(\mathrm{f}, \mathrm{e}) \in \mathcal{D} \quad \Rightarrow \quad 0 = \langle (\mathrm{f}, \mathrm{e}), (\mathrm{f}, \mathrm{e}) \rangle = 2 \, \langle \mathrm{e} \mid \mathrm{f} \rangle
$$

Dirac structures – power-conserving interconnection

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$$(\mathrm{f}, \mathrm{e}) \in \mathcal{D} \quad \Rightarrow \quad 0 = \langle (\mathrm{f}, \mathrm{e}), (\mathrm{f}, \mathrm{e}) \rangle = 2 \langle \mathrm{e} \mid \mathrm{f} \rangle$$

$$\begin{aligned}\mathsf{dim}(\mathrm{B}) &= \mathsf{dim}(\mathrm{F} \times \mathrm{F}^*) = 2\, \mathsf{dim}(\mathrm{F}) \\ &= \mathsf{dim}(\mathrm{D}) + \mathsf{dim}(\mathrm{D}^\perp) = 2\, \mathsf{dim}(\mathrm{D})\end{aligned}$$

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\begin{aligned}
\langle\cdot,\,\cdot\rangle \,:\, \mathcal{B}\times\mathcal{B} &\xrightarrow{\sim} \mathbb{R} \\
\Big(\big(\mathrm{f}^1,\,\mathrm{e}^1\big),\,\big(\mathrm{f}^2,\,\mathrm{e}^2\big)\Big) &\mapsto \big\langle\big(\mathrm{f}^1,\,\mathrm{e}^1\big),\,\big(\mathrm{f}^2,\,\mathrm{e}^2\big)\big\rangle \;=\; \langle\mathrm{e}^1 \mid \mathrm{f}^2\rangle \;+\; \langle\mathrm{e}^2 \mid \mathrm{f}^1\rangle \\
&\qquad\qquad = \mathrm{e}^1_\mathrm{q}\,\mathrm{f}^2_\mathrm{q} + \mathrm{e}^1_\mathrm{p}\,\mathrm{f}^2_\mathrm{p} \;+\; \mathrm{e}^2_\mathrm{q}\,\mathrm{f}^1_\mathrm{q} + \mathrm{e}^2_\mathrm{p}\,\mathrm{f}^1_\mathrm{p}
\end{aligned}
$$

Dirac structure: $\mathcal{D} \subset_{\mathrm{vec}} \mathcal{B}$ such that $\mathcal{D} = \mathcal{D}^\perp$

$$
\mathcal{D} \;=\; \Big\{(\mathrm{f},\,\mathrm{e}) \in \mathcal{B} \;\mid\; \forall\;(\mathrm{f}',\,\mathrm{e}') \in \mathcal{D} :\; \langle(\mathrm{f},\,\mathrm{e})\,,\,(\mathrm{f}',\,\mathrm{e}')\rangle = 0\Big\}
$$

$$
(\mathrm{f},\,\mathrm{e}) \in \mathcal{D} \quad\Rightarrow\quad 0 \;=\; \langle(\mathrm{f},\,\mathrm{e})\,,\,(\mathrm{f},\,\mathrm{e})\rangle \;=\; 2\,\langle\mathrm{e} \mid \mathrm{f}\rangle
$$

$$
\left.
\begin{aligned}
\mathsf{dim}(\mathrm{B}) \;&=\; \mathsf{dim}(\mathrm{F}\times\mathrm{F}^*) \;=\; 2\;\mathsf{dim}(\mathrm{F}) \\
&=\; \mathsf{dim}(\mathrm{D}) + \mathsf{dim}(\mathrm{D}^\perp) \;=\; 2\;\mathsf{dim}(\mathrm{D})
\end{aligned}
\right\}
\quad\Rightarrow\quad \mathsf{dim}(\mathrm{D}) \;=\; \mathsf{dim}(\mathrm{F})
$$

Dirac structures – power-conserving interconnection

canonically defined symmetric bilinear form on $\mathcal{B}$:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \ \mathcal{B} \times \mathcal{B} &\xrightarrow{\sim} \mathbb{R} \\ \Big(\big(\mathrm{f}^1, \mathrm{e}^1\big), \big(\mathrm{f}^2, \mathrm{e}^2\big) \Big) &\mapsto \langle \big(\mathrm{f}^1, \mathrm{e}^1\big), \big(\mathrm{f}^2, \mathrm{e}^2\big) \rangle \ = \ \langle \mathrm{e}^1 \mid \mathrm{f}^2 \rangle + \langle \mathrm{e}^2 \mid \mathrm{f}^1 \rangle \\ &= \ \mathrm{e}^1_\mathrm{q}\, \mathrm{f}^2_\mathrm{q} + \mathrm{e}^1_\mathrm{p}\, \mathrm{f}^2_\mathrm{p} \ + \ \mathrm{e}^2_\mathrm{q}\, \mathrm{f}^1_\mathrm{q} + \mathrm{e}^2_\mathrm{p}\, \mathrm{f}^1_\mathrm{p} \end{aligned}$$

Dirac structure: $\mathcal{D} \subset_{\mathrm{vec}} \mathcal{B}$ such that $\mathcal{D} = \mathcal{D}^\perp$

$$\mathcal{D} \ = \ \Big\{ (\mathrm{f}, \mathrm{e}) \in \mathcal{B} \ \big| \ \forall \ (\mathrm{f}', \mathrm{e}') \in \mathcal{D} : \ \langle (\mathrm{f}, \mathrm{e}), (\mathrm{f}', \mathrm{e}') \rangle = 0 \Big\}$$

$$(\mathrm{f}, \mathrm{e}) \in \mathcal{D} \quad \Rightarrow \quad 0 \ = \ \langle (\mathrm{f}, \mathrm{e}), (\mathrm{f}, \mathrm{e}) \rangle \ = \ 2 \, \langle \mathrm{e} \mid \mathrm{f} \rangle$$

$$\left.\begin{aligned} \mathsf{dim}(\mathrm{B}) \ &= \ \mathsf{dim}(\mathrm{F} \times \mathrm{F}^*) \ = \ 2 \ \mathsf{dim}(\mathrm{F}) \\ &= \ \mathsf{dim}(\mathrm{D}) + \mathsf{dim}(\mathrm{D}^\perp) \ = \ 2 \ \mathsf{dim}(\mathrm{D}) \end{aligned}\right\} \quad \Rightarrow \quad \mathsf{dim}(\mathrm{D}) \ = \ \mathsf{dim}(\mathrm{F})$$

$$\mathcal{D} \ = \ \mathsf{graph}(\mathrm{J}) \ = \ \{ (\mathrm{f}, \mathrm{e}) \in \mathcal{B} \ | \ \mathrm{f} = \mathrm{J}\, \mathrm{e} \}$$

Dirac structures and port-Hamiltonian systems

Dirac structures ...

- can be written as the graph of a skew-symmetric map.
- generalize symplectic and Poisson structures (which exist for infinite-dimensional state spaces).

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- Enables energy-based control methods such as passivity-based control.

A more general class of port-Hamiltonian systems

$$\dot{\mathrm{x}} = \mathrm{J}(\mathrm{x})\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}} + \mathrm{B}(\mathrm{x})\,\mathrm{u}$$

A more general class of port-Hamiltonian systems

$$
\begin{aligned}
\dot{\mathrm{x}} &= \mathrm{J}(\mathrm{x})\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}} + \mathrm{B}(\mathrm{x})\,\mathrm{u} \\
\mathrm{y} &= \mathrm{B}^{\mathrm{T}}(\mathrm{x})\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}}
\end{aligned}
$$

A more general class of port-Hamiltonian systems

$$\begin{aligned} \dot{x} &= J(x)\,\frac{\partial H}{\partial x} + B(x)\,u \\ y &= B^T(x)\,\frac{\partial H}{\partial x} \end{aligned}$$

Inputs and outputs are collocated.

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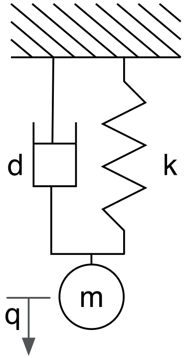
Inputs and outputs are collocated.

energy balance:

$$\frac{\mathrm{dH}}{\mathrm{dt}} = \frac{\partial \mathrm{H}}{\partial \mathrm{x}}\,\frac{\mathrm{dx}}{\mathrm{dt}} = \frac{\partial \mathrm{H}}{\partial \mathrm{x}}\,\mathrm{J}(\mathrm{x})\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}} + \frac{\partial \mathrm{H}}{\partial \mathrm{x}}\,\mathrm{B}(\mathrm{x})\,\mathrm{u} = \left(\mathrm{B}^{\mathrm{T}}(\mathrm{x})\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}}\right)^{\mathrm{T}} \mathrm{u} = \mathrm{y}^{\mathrm{T}}\,\mathrm{u}$$

Damped harmonic oscillator



free vibration: $\mathrm{m}\,\ddot{\mathrm{q}} + \mathrm{d}\,\dot{\mathrm{q}} + \mathrm{k}\,\mathrm{q} = 0$

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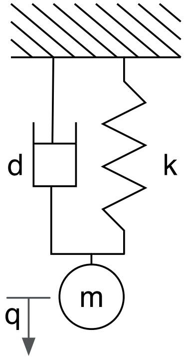


free vibration: $\mathrm{m}\,\ddot{\mathrm{q}} + \mathrm{d}\,\dot{\mathrm{q}} + \mathrm{k}\,\mathrm{q} = 0$

$$\begin{bmatrix} \dot{\mathrm{q}} \\ \dot{\mathrm{p}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{k\,x} \\ \mathrm{v} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{u}$$

$$\mathrm{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{k\,x} \\ \mathrm{v} \end{bmatrix} = \mathrm{v}$$

connect damping model to input-output pair:

$$\mathrm{u} = -\mathrm{d}\,\mathrm{y}$$

energy balance:

$$\frac{\mathrm{dH}}{\mathrm{dt}} = \mathrm{y}^{\mathrm{T}}\,\mathrm{u} = -\mathrm{vdv} < 0$$

Damped port-Hamiltonian systems

$$\dot{x} = J \frac{\partial H}{\partial x} + B u$$

$$y = B^T \frac{\partial H}{\partial x}$$

$$u = -D y$$

Damped port-Hamiltonian systems

$$\left.\begin{aligned} \dot{\mathrm{x}} &= \mathrm{J}\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}} + \mathrm{B}\,\mathrm{u} \\ \mathrm{y} &= \mathrm{B}^{\mathrm{T}}\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}} \\ \mathrm{u} &= -\mathrm{D}\,\mathrm{y} \end{aligned}\right\} \quad \Rightarrow \quad \dot{\mathrm{x}} = \mathrm{J}\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}} + \mathrm{B}\,\mathrm{u} = \mathrm{J}\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}} - \underbrace{\mathrm{B}\,\mathrm{D}\,\mathrm{B}^{\mathrm{T}}}_{=:\mathrm{R}}\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}} = (\mathrm{J} - \mathrm{R})\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}}$$

Damped port-Hamiltonian systems

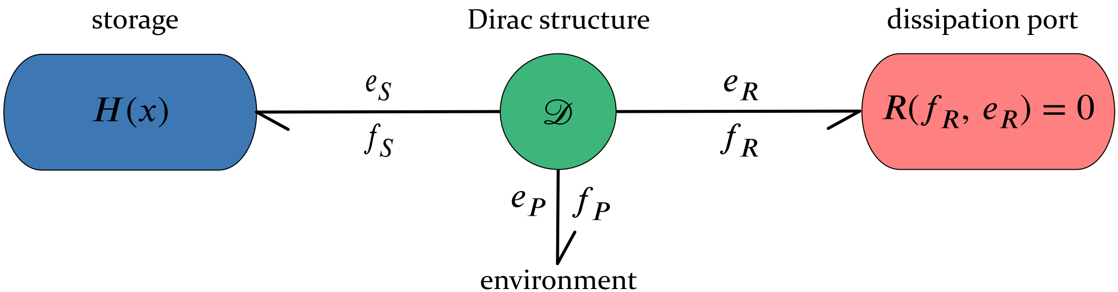
$$\left.\begin{aligned} \dot{x} &= J\,\frac{\partial H}{\partial x} + B\,u \\ y &= B^T\,\frac{\partial H}{\partial x} \\ u &= -D\,y \end{aligned}\right\} \quad \Rightarrow \quad \dot{x} = J\,\frac{\partial H}{\partial x} + B\,u = J\,\frac{\partial H}{\partial x} - \underbrace{B\,D\,B^T}_{=:R}\,\frac{\partial H}{\partial x} = (J - R)\,\frac{\partial H}{\partial x}$$

energy balance:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\,(J - R)\,\frac{\partial H}{\partial x} = -\frac{\partial H}{\partial x}\,R\,\frac{\partial H}{\partial x} < 0$$

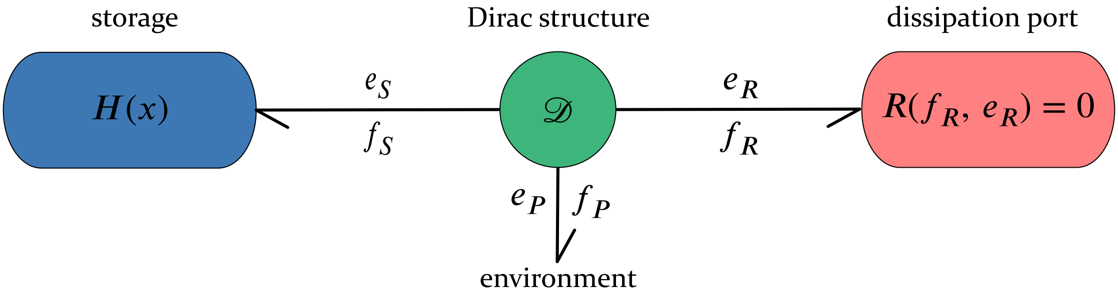
with a symmetric and positive-semidefinite dissipation matrix R

Port-Hamiltonian systems and dissipation



$$e_R^T f_R < 0$$

Port-Hamiltonian systems and dissipation



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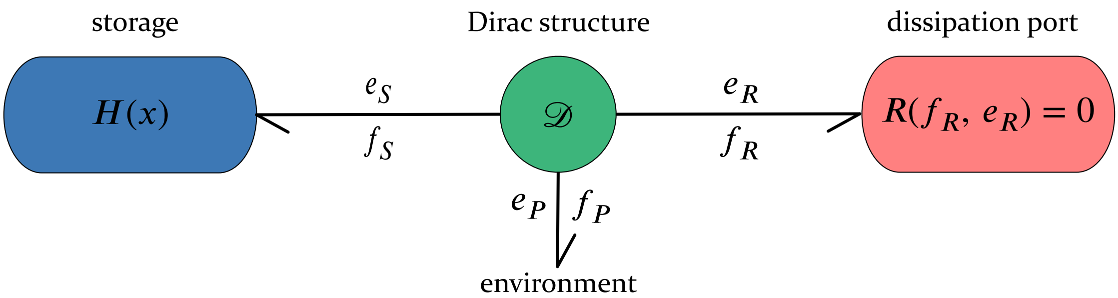
The dissipated energy disappears by leaving the system through a port.

Port-Hamiltonian systems and dissipation

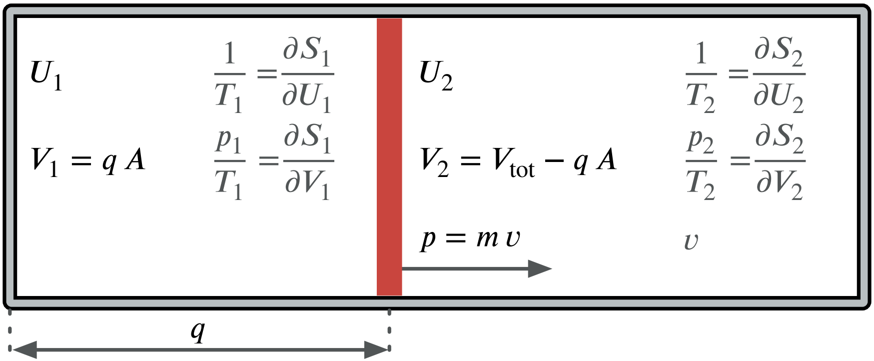


$$e_R^T f_R < 0$$

The dissipated energy disappears by leaving the system through a port.

$\Rightarrow$ H is a “free energy” or exergy.

Heat conduction through a moving piston



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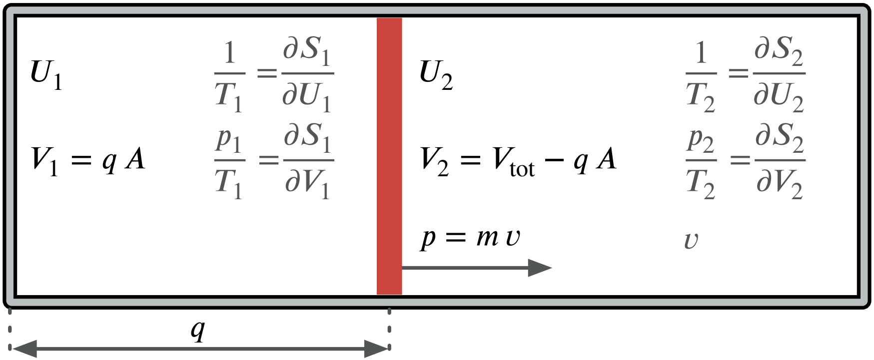

Irreversible dynamics depend on temperature!

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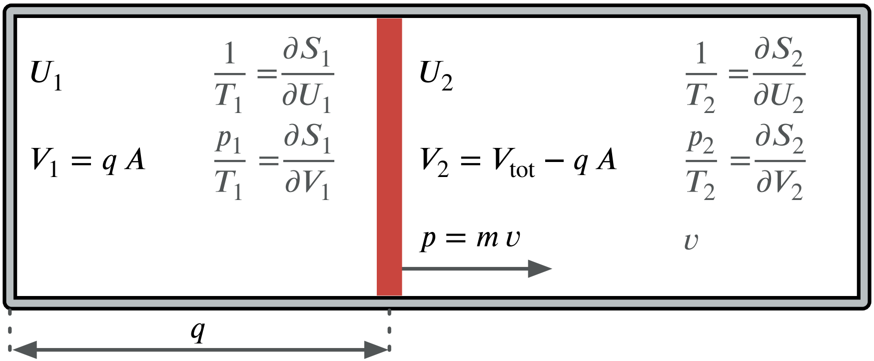

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$\Rightarrow$ H must include the internal energy of the gas inside each volume.

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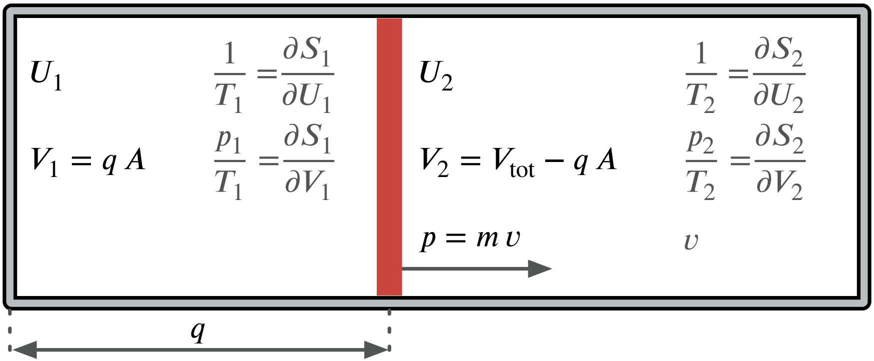

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$$\text{state:} \quad x = \begin{bmatrix} q & p & U_1 & U_2 \end{bmatrix}$$

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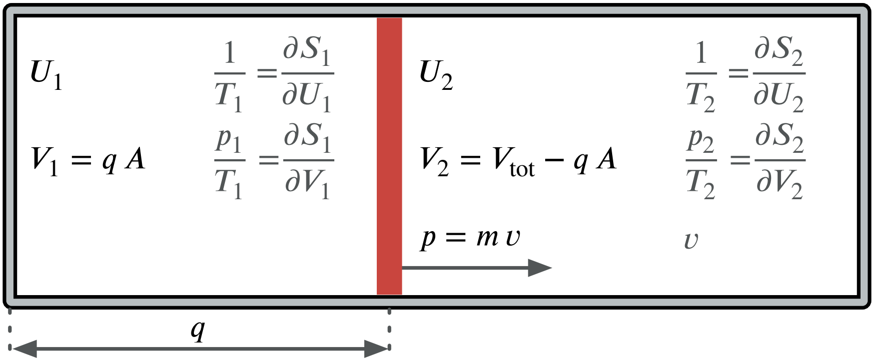

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$$\mathrm{H}(\mathrm{x}) = \frac{1}{2}\,\mathrm{p}\,\frac{1}{\mathrm{m}}\,\mathrm{p} + \mathrm{U}_1 + \mathrm{U}_2$$

GENERIC

General Equation for the Nonequilibrium Reversible-Irreversible Coupling
(Öttinger and Grmela, 1997)

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \underbrace{\mathrm{J}(\mathrm{x})\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}}}_{\dot{\mathrm{x}}_{\mathrm{rev}}} + \underbrace{\mathrm{R}(\mathrm{x})\,\frac{\partial \mathrm{S}}{\partial \mathrm{x}}}_{\dot{\mathrm{x}}_{\mathrm{irr}}}$$

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Irreversible dynamics leave the total energy invariant.

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$$\forall\, \mathrm{H} : \ \frac{\mathrm{dS}}{\mathrm{dt}} = \frac{\partial \mathrm{S}}{\partial \mathrm{x}}\,\dot{\mathrm{x}}_{\mathrm{rev}} = \frac{\partial \mathrm{S}}{\partial \mathrm{x}}\,\mathrm{J(x)}\,\frac{\partial \mathrm{H}}{\partial \mathrm{x}} = 0$$

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Reversible dynamics leave the total entropy invariant.

$$\text{energy balance:}\quad \frac{\mathrm{dH}}{\mathrm{dt}} = \frac{\partial \mathrm{H}}{\partial \mathrm{x}}\,\frac{\mathrm{dx}}{\mathrm{dt}} = 0$$

$$\text{entropy balance:}\quad \frac{\mathrm{dS}}{\mathrm{dt}} = \frac{\partial \mathrm{S}}{\partial \mathrm{x}}\,\frac{\mathrm{dx}}{\mathrm{dt}} \geq 0$$

Heat conduction through a moving piston

$$\text{state:} \quad x = \begin{bmatrix} q & p & U_1 & U_2 \end{bmatrix}$$

$$\text{total energy:} \quad H(x) = \frac{1}{2} p \frac{1}{m} p + U_1 + U_2$$

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$$\frac{\partial \mathrm{H}}{\partial \mathrm{x}} = \begin{bmatrix} 0 & \frac{1}{\mathrm{m}}\,\mathrm{p} & 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\begin{aligned} \frac{\partial \mathrm{S}}{\partial \mathrm{x}} &= \begin{bmatrix} \frac{\partial \mathrm{S_1}}{\partial \mathrm{V_1}}\frac{\partial \mathrm{V_1}}{\partial \mathrm{q}} + \frac{\partial \mathrm{S_2}}{\partial \mathrm{V_2}}\frac{\partial \mathrm{V_2}}{\partial \mathrm{q}} & 0 & \frac{1}{\mathrm{T_1}} & \frac{1}{\mathrm{T_2}} \end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} \left(\frac{\mathrm{p_1}}{\mathrm{T_1}} - \frac{\mathrm{p_2}}{\mathrm{T_2}}\right)\mathrm{A} & 0 & \frac{1}{\mathrm{T_1}} & \frac{1}{\mathrm{T_2}} \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

$$\begin{bmatrix} \dot{\mathrm{q}} \\ \dot{\mathrm{p}} \\ \dot{\mathrm{U}}_1 \\ \dot{\mathrm{U}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \mathrm{p_1\,A} & -\mathrm{p_2\,A} \\ 0 & -\mathrm{p_1\,A} & 0 & 0 \\ 0 & \mathrm{p_2\,A} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \mathrm{v} \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & -\alpha \\ 0 & 0 & -\alpha & \alpha \end{bmatrix} \begin{bmatrix} \left(\frac{\mathrm{p_1}}{\mathrm{T_1}} - \frac{\mathrm{p_2}}{\mathrm{T_2}}\right)\mathrm{A} \\ 0 \\ \frac{1}{\mathrm{T_1}} \\ \frac{1}{\mathrm{T_2}} \end{bmatrix}$$