

# Aarhus University

# Beyond Ballmer's Peak U Frank

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T	Contest	1		6.1 Permutations	11	<pre>using pii = pair<int, int="">;</int,></pre>
	template.cpp	1		6.2 Inversions	11	<pre>using vi = vector<int>;</int></pre>
	template.py	1		6.3 Partitions and subsets	11	int main() {
	.bashrc	1		6.4 General purpose numbers	12	<pre>cin.tie(0)-&gt;sync_with_stdio(0);</pre>
	.vimrc	1				cin.exceptions(cin.failbit);
	hash.sh	1	7	Graph	12	}
	troubleshoot.txt	1		7.1 Fundamentals	12	template.py
				7.2 Network flow	13	<b>Description:</b> Fast I/O. Improves performance when reading many ( $\geq 10^4$ )
<b>2</b>	Mathematics	2		7.3 Matching	14	lines. Returns an empty string instead of raising EOFError!  4 lines
	2.1 Equations	2		7.4 DFS algorithms	15	import sys
	2.2 Recurrences	2		7.5 Coloring	16	<pre>input = lambda: sys.stdin.readline().rstrip('\r\n') # Many calls to print is also slow, consider:</pre>
	2.3 Trigonometry	2		7.6 Heuristics	16	<pre>print('\n'.join(buffered_output))</pre>
	2.4 Geometry	2		7.7 Trees	17	
	2.5 Derivatives/Integrals	2		7.8 Tree diameter	18	.bashrc 2 lines
	2.6 Sums	2		7.9 Math	18	alias c='q++ -Wall -Wconversion -Wfatal-errors -q -std=c++14 \
	2.7 Series	3				-fsanitize=undefined,address'
	2.8 Probability theory	3	8	Geometry	18	
	2.9 Markov chains	3		8.1 Geometric primitives	18	.Vimrc 6 lines
				8.2 Circles	19	set cin aw ai is ts=4 sw=4 tm=50 nu noeb bg=dark ru cul
3	Data structures	3		8.3 Polygons	20	sy on   im jk <esc>   im kj <esc>   no;: "Select region and then type :Hash to hash your selection.</esc></esc>
	OrderStatisticTree.h	3		8.4 Misc. Point Set Problems	21	" Useful for verifying that there aren't mistypes.
	HashMap.h	3		8.5 3D	22	ca Hash w !cpp -dD -P -fpreprocessed \  tr -d '[:space:]' \
	SegmentTree.h	4				\  md5sum \  cut -c-6
	LazySegmentTree.h	4	9	Strings	23	hash.sh
	UnionFindRollback.h	4		Regex.h	23	3 lines
	SubMatrix.h	4		KMP.h	23	# Hashes a file, ignoring all whitespace and comments. Use for # verifying that code was correctly typed.
	Matrix.h	4		Zfunc.h	23	cpp -dD -P -fpreprocessed   tr -d '[:space:]'   md5sum  cut -c-6
	LineContainer.h	4		Manacher.h	23	
	Treap.h	5		MinRotation.h	23	troubleshoot.txt 52 lines
	FenwickTree.h	5		SuffixArray.h	23	Pre-submit:
	FenwickTree2d.h	5		SuffixTree.h	23	Write a few simple test cases if sample is not enough.  Are time limits close? If so, generate max cases.
	RMQ.h	5		Hashing.h	24	Is the memory usage fine?
	MoQueries.h	5		AhoCorasick.h		Could anything overflow?
						Make sure to submit the right file.
4	Numerical	5	10	Various	${\bf 24}$	Wrong answer:
	4.1 Polynomials and recurrences	5		10.1 Hashing	24	Print your solution! Print debug output, as well.
	4.2 Optimization	6		10.2 Intervals	24	Are you clearing all data structures between test cases?  Can your algorithm handle the whole range of input?
	4.3 Matrices	7		10.3 Misc. algorithms	25	Read the full problem statement again.
	4.4 Fourier transforms	8		10.4 Dynamic programming	25	Do you handle all corner cases correctly? Have you understood the problem correctly?
				10.5 Debugging tricks	25	Any uninitialized variables?
<b>5</b>	Number theory	9		10.6 Optimization tricks	25	Any overflows?
	5.1 Modular arithmetic	9				Confusing N and M, i and j, etc.?  Are you sure your algorithm works?
	5.2 Primality	9	C	$\underline{\text{bontest}}$ (1)		What special cases have you not thought of?
	5.3 Divisibility	10		,		Are you sure the STL functions you use work as you think?
	5.4 Fractions	10	te	$\operatorname{mplate.cpp}$	5 lines	Add some assertions, maybe resubmit.  Create some testcases to run your algorithm on.
	5.5 Pythagorean Triples	11		nclude <bits stdc++.h=""></bits>	_	Go through the algorithm for a simple case.
	5.6 Primes	11	us	ing namespace std;		Go through this list again.
	5.7 Estimates	11	#de	<b>efine</b> rep(i, a, b) <b>for</b> ( <b>int</b> i = a; i < (b); ++i)		Explain your algorithm to a teammate.  Ask the teammate to look at your code.
	5.8 Mobius Function	11	#de	efine all(x) begin(x), end(x)		Go for a small walk, e.g. to the toilet.
				efine sz(x) (int)(x).size() efine endl "\n"		Is your output format correct? (including whitespace) Rewrite your solution from the start or let a teammate do it.
G	Combinatorial	11		ing 11 = long long;		New Itee your solution from the Start of fet a teammate do it.

## template template .bashrc .vimrc hash troubleshoot

Runtime error: Have you tested all corner cases locally? Any uninitialized variables? Are you reading or writing outside the range of any vector? Any assertions that might fail? Any possible division by 0? (mod 0 for example) Any possible infinite recursion?

Invalidated pointers or iterators?

Are you using too much memory?

Debug with resubmits (e.g. remapped signals, see Various).

Time limit exceeded:

Do you have any possible infinite loops? What is the complexity of your algorithm? Are you copying a lot of unnecessary data? (References) How big is the input and output? (consider scanf) Avoid vector, map. (use arrays/unordered\_map) What do your teammates think about your algorithm?

Memory limit exceeded:

What is the max amount of memory your algorithm should need? Are you clearing all data structures between test cases?

# Mathematics (2)

## 2.1 Equations

$$ax^{2} + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^{2} - 4ac}}{2a}$$

The extremum is given by x = -b/2a.

$$ax + by = e$$

$$cx + dy = f \Rightarrow x = \frac{ed - bf}{ad - bc}$$

$$y = \frac{af - ec}{ad - bc}$$

In general, given an equation Ax = b, the solution to a variable  $x_i$  is given by

$$x_i = \frac{\det A_i'}{\det A}$$

where  $A'_i$  is A with the *i*'th column replaced by b.

## 2.2 Recurrences

If  $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$ , and  $r_1, \ldots, r_k$  are distinct roots of  $x^k + c_1 x^{k-1} + \cdots + c_k$ , there are  $d_1, \ldots, d_k$  s.t.

$$a_n = d_1 r_1^n + \dots + d_k r_k^n.$$

Non-distinct roots r become polynomial factors, e.g.  $a_n = (d_1 n + d_2)r^n.$ 

## 2.3 Trigonometry

$$\sin(v+w) = \sin v \cos w + \cos v \sin w$$
$$\cos(v+w) = \cos v \cos w - \sin v \sin w$$

$$\tan(v+w) = \frac{\tan v + \tan w}{1 - \tan v \tan w}$$

$$\sin v + \sin w = 2\sin\frac{v+w}{2}\cos\frac{v-w}{2}$$

$$\cos v + \cos w = 2\cos\frac{v+w}{2}\cos\frac{v-w}{2}$$

 $(V+W)\tan(v-w)/2 = (V-W)\tan(v+w)/2$ where V, W are lengths of sides opposite angles v, w.

$$a\cos x + b\sin x = r\cos(x - \phi)$$

$$a\sin x + b\cos x = r\sin(x+\phi)$$

where  $r = \sqrt{a^2 + b^2}$ ,  $\phi = \operatorname{atan2}(b, a)$ .

## 2.4 Geometry

## 2.4.1 Triangles

Given a triangle with side lengths a, b, c and opposite angles  $\alpha, \beta, \gamma$ , we can find the unknown sides and angles by repeatedly using the first usable equation from the following list:

$$\alpha = \pi - \beta - \gamma$$

$$\alpha = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$\alpha = \sin^{-1}\left(\frac{a\sin\beta}{b}\right)$$

$$a = \sqrt{b^2 + c^2 - 2bc \times \cos\alpha}$$

$$a = \frac{b\sin\alpha}{\sin\beta}$$

Semiperimeter:  $p = \frac{a+b+c}{2}$ 

Area:  $A = \sqrt{p(p-a)(p-b)(p-c)}$ 

Circumradius:  $R = \frac{abc}{4A}$ 

Inradius:  $r = \frac{A}{r}$ 

Length of median (divides triangle into two equal-area triangles):  $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$ 

Length of bisector (divides angles in two):

$$s_a = \sqrt{bc \left[ 1 - \left( \frac{a}{b+c} \right)^2 \right]}$$

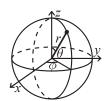
## 2.4.2 Quadrilaterals

With side lengths a, b, c, d, diagonals e, f, diagonals angle  $\theta$ , area A and magic flux  $F = b^2 + d^2 - a^2 - c^2$ :

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180°, ef = ac + bd, and  $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$ 

## 2.4.3 Spherical coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z/\sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \operatorname{atan2}(y, x) \end{aligned}$$

Distance between to points along the surface:

$$2r\sin^{-1}\left(\sqrt{\frac{1-\cos\theta_1\cos\theta_2\cos(\phi_1-\phi_2)-\sin\theta_1\sin\theta_2}{2}}\right)$$

## Derivatives/Integrals

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\int \tan ax = -\frac{\ln|\cos ax|}{a} \qquad \int x\sin ax = \frac{\sin ax - ax\cos ax}{a^2}$$

$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) \qquad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

Integration by parts:

$$\int_{a}^{b} f(x)g(x)dx = [F(x)g(x)]_{a}^{b} - \int_{a}^{b} F(x)g'(x)dx$$

#### 2.6 Sums

$$c^{a} + c^{a+1} + \dots + c^{b} = \frac{c^{b+1} - c^{a}}{c-1}, c \neq 1$$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$1^{2} + 2^{2} + 3^{2} + \dots + n^{2} = \frac{n(2n+1)(n+1)}{6}$$

$$1^{3} + 2^{3} + 3^{3} + \dots + n^{3} = \frac{n^{2}(n+1)^{2}}{4}$$

$$1^{4} + 2^{4} + 3^{4} + \dots + n^{4} = \frac{n(n+1)(2n+1)(3n^{2} + 3n - 1)}{20}$$

#### OrderStatisticTree HashMap

#### Series 2.7

$$e^{x} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \dots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^{2}}{2} + \frac{x^{3}}{3} - \frac{x^{4}}{4} + \dots, (-1 < x \le 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^{2}}{8} + \frac{2x^{3}}{32} - \frac{5x^{4}}{128} + \dots, (-1 \le x \le 1)$$

$$\sin x = x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} - \frac{x^{7}}{7!} + \dots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^{2}}{2!} + \frac{x^{4}}{4!} - \frac{x^{6}}{6!} + \dots, (-\infty < x < \infty)$$

## Probability theory

Let X be a discrete random variable with probability  $p_X(x)$  of assuming the value x. It will then have an expected value (mean)  $\mu = \mathbb{E}(X) = \sum_{x} x p_X(x)$  and variance  $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$  where  $\sigma$ is the standard deviation. If X is instead continuous it will have a probability density function  $f_X(x)$  and the sums above will instead be integrals with  $p_X(x)$  replaced by  $f_X(x)$ .

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent X and Y,

$$V(aX + bY) = a^2V(X) + b^2V(Y).$$

## 2.8.1 Discrete distributions Binomial distribution

The number of successes in n independent yes/no experiments, each which yields success with probability p is Bin(n, p), n = 1, 2, ..., 0

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np, \ \sigma^2 = np(1-p)$$

Bin(n, p) is approximately Po(np) for small p.

#### First success distribution

The number of trials needed to get the first success in independent yes/no experiments, each wich yields success with probability p is Fs(p), 0 .

$$p(k) = p(1-p)^{k-1}, k = 1, 2, \dots$$
  
$$\mu = \frac{1}{p}, \sigma^2 = \frac{1-p}{p^2}$$

#### Poisson distribution

The number of events occurring in a fixed period of time t if these events occur with a known average rate  $\kappa$  and independently of the time since the last event is  $Po(\lambda)$ ,  $\lambda = t\kappa$ .

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, 2, \dots$$
$$\mu = \lambda, \sigma^2 = \lambda$$

## 2.8.2 Continuous distributions Uniform distribution

If the probability density function is constant between a and band 0 elsewhere it is U(a, b), a < b.

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2}, \, \sigma^2 = \frac{(b-a)^2}{12}$$

#### Exponential distribution

The time between events in a Poisson process is  $\operatorname{Exp}(\lambda), \lambda > 0.$ 

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0\\ 0 & x < 0 \end{cases}$$
$$\mu = \frac{1}{\lambda}, \, \sigma^2 = \frac{1}{\lambda^2}$$

#### Normal distribution

Most real random values with mean  $\mu$  and variance  $\sigma^2$  are well described by  $\mathcal{N}(\mu, \sigma^2)$ ,  $\sigma > 0$ .

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If  $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$  and  $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$  then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

## 2.9 Markov chains

A Markov chain is a discrete random process with the property that the next state depends only on the current state. Let  $X_1, X_2, \dots$  be a sequence of random variables generated by the Markov process. Then there is a transition matrix  $\mathbf{P} = (p_{ij})$ , with  $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$ , and  $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$  is the probability distribution for  $X_n$  (i.e.,  $p_i^{(n)} = \Pr(X_n = i)$ ), where  $\mathbf{p}^{(0)}$  is the initial distribution.

 $\pi$  is a stationary distribution if  $\pi = \pi \mathbf{P}$ . If the Markov chain is irreducible (it is possible to get to any state from any state), then  $\pi_i = \frac{1}{\mathbb{E}(T_i)}$  where  $\mathbb{E}(T_i)$  is the expected time between two visits in state i.  $\pi_i/\pi_i$  is the expected number of visits in state j between two visits in state i.

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors,  $\pi_i$  is proportional to node i's degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and aperiodic (i.e., the gcd of cycle lengths is 1).  $\lim_{k\to\infty} \mathbf{P}^k = \mathbf{1}\pi$ .

A Markov chain is an A-chain if the states can be partitioned into two sets A and G, such that all states in A are absorbing  $(p_{ii}=1)$ , and all states in **G** leads to an absorbing state in **A**. The probability for absorption in state  $i \in \mathbf{A}$ , when the initial state is j, is  $a_{ij} = p_{ij} + \sum_{k \in \mathbf{G}} a_{ik} p_{kj}$ . The expected time until absorption, when the initial state is i, is  $t_i = 1 + \sum_{k \in \mathbf{G}} p_{ki} t_k$ .

# Data structures (3)

#### OrderStatisticTree.h

Description: A set (not multiset!) with support for finding the n'th element, and finding the index of an element. To get a map, change null\_type. Time:  $\mathcal{O}(\log N)$ 

782797, 16 lines

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
template<class T>
using Tree = tree<T, null_type, less<T>, rb_tree_tag,
    tree order statistics node update>;
void example() {
 Tree<int> t, t2; t.insert(8);
 auto it = t.insert(10).first;
 assert(it == t.lower bound(9));
 assert(t.order_of_key(10) == 1);
 assert(t.order_of_key(11) == 2);
 assert(*t.find_by_order(0) == 8);
 t.join(t2); // assuming T < T2 or T > T2, merge t2 into t
```

#### HashMap.h

Description: Hash map with mostly the same API as unordered\_map, but ~3x faster. Uses 1.5x memory. Initial capacity must be a power of 2 (if provided).

```
#include <bits/extc++.h>
// To use most bits rather than just the lowest ones:
struct chash { // large odd number for C
  const uint64_t C = 11(4e18 * acos(0)) | 71;
  11 operator()(11 x) const { return __builtin_bswap64(x*C); }
__gnu_pbds::gp_hash_table<11, int, chash> h({}, {}, {}, {}, {1<<16});
```

#### SegmentTree.h

Description: Zero-indexed max-tree. Bounds are inclusive to the left and exclusive to the right. Can be changed by modifying T, f and unit. If n is a power of two or f is commutative s[1] holds the result for the full range.

Time:  $\mathcal{O}(\log N)$ 

```
struct Tree {
  typedef int T;
  static constexpr T unit = INT_MIN;
  T f (T a, T b) { return max(a, b); } // (any associative fn)
  Tree(int n = 0, T def = unit) : s(2*n, def), n(n) {}
  void update(int pos, T val) {
    for (s[pos += n] = val; pos /= 2;)
     s[pos] = f(s[pos * 2], s[pos * 2 + 1]);
  T query (int b, int e) { // query (b, e)
   T ra = unit, rb = unit;
    for (b += n, e += n; b < e; b /= 2, e /= 2) {
     if (b % 2) ra = f(ra, s[b++]);
     if (e % 2) rb = f(s[--e], rb);
    return f(ra, rb);
};
```

#### LazySegmentTree.h

else {

Description: Segment tree with ability to add or set values of large intervals, and compute max of intervals. Can be changed to other things. Use with a bump allocator for better performance, and SmallPtr or implicit indices to save memory.

```
Usage: Node* tr = new Node(v, 0, sz(v));
```

```
Time: \mathcal{O}(\log N).
"../various/BumpAllocator.h"
                                                         34ecf5, 50 lines
const int inf = 1e9;
struct Node {
 Node *1 = 0, *r = 0;
 int lo, hi, mset = inf, madd = 0, val = -inf;
 Node (int lo, int hi):lo(lo), hi(hi) {} // Large interval of -inf
  Node(vi& v, int lo, int hi) : lo(lo), hi(hi) {
   if (lo + 1 < hi) {
      int mid = lo + (hi - lo)/2;
     1 = new Node(v, lo, mid); r = new Node(v, mid, hi);
     val = max(1->val, r->val);
    else val = v[lo];
  int query(int L, int R) {
   if (R <= lo || hi <= L) return -inf;</pre>
   if (L <= lo && hi <= R) return val;</pre>
   push();
   return max(1->query(L, R), r->query(L, R));
  void set(int L, int R, int x) {
   if (R <= lo || hi <= L) return;</pre>
    if (L <= lo && hi <= R) mset = val = x, madd = 0;</pre>
      push(), l\rightarrow set(L, R, x), r\rightarrow set(L, R, x);
      val = max(1->val, r->val);
  void add(int L, int R, int x) {
   if (R <= lo || hi <= L) return;</pre>
    if (L <= lo && hi <= R) {
      if (mset != inf) mset += x;
      else madd += x;
      val += x;
```

```
push(), l->add(L, R, x), r->add(L, R, x);
     val = max(1->val, r->val);
 void push() {
   if (!1) {
     int mid = lo + (hi - lo)/2;
     1 = new Node(lo, mid); r = new Node(mid, hi);
    if (mset != inf)
     l->set(lo,hi,mset), r->set(lo,hi,mset), mset = inf;
    else if (madd)
     1- add (lo, hi, madd), r- add (lo, hi, madd), madd = 0;
};
```

#### UnionFindRollback.h.

struct RollbackUF {

**Description:** Disjoint-set data structure with undo. If undo is not needed, skip st, time() and rollback().

Usage: int t = uf.time(); ...; uf.rollback(t); Time:  $\mathcal{O}(\log(N))$ 

de4ad0, 21 lines

```
vi e; vector<pii> st;
 RollbackUF(int n) : e(n, -1) {}
 int size(int x) { return -e[find(x)]; }
 int find(int x) { return e[x] < 0 ? x : find(e[x]); }</pre>
 int time() { return sz(st); }
 void rollback(int t) {
    for (int i = time(); i --> t;)
     e[st[i].first] = st[i].second;
    st.resize(t);
 bool join(int a, int b) {
   a = find(a), b = find(b);
   if (a == b) return false;
   if (e[a] > e[b]) swap(a, b);
   st.push_back({a, e[a]});
   st.push back({b, e[b]});
   e[a] += e[b]; e[b] = a;
   return true;
};
```

#### SubMatrix.h

Description: Calculate submatrix sums quickly, given upper-left and lowerright corners (half-open).

```
Usage: SubMatrix<int> m(matrix);
m.sum(0, 0, 2, 2); // top left 4 elements
Time: \mathcal{O}(N^2+Q)
```

c59ada 13 lines

```
template<class T>
struct SubMatrix {
 vector<vector<T>> p;
 SubMatrix(vector<vector<T>>& v) {
   int R = sz(v), C = sz(v[0]);
   p.assign(R+1, vector<T>(C+1));
   rep(r, 0, R) rep(c, 0, C)
     p[r+1][c+1] = v[r][c] + p[r][c+1] + p[r+1][c] - p[r][c];
 T sum(int u, int 1, int d, int r) {
   return p[d][r] - p[d][l] - p[u][r] + p[u][l];
};
```

## Matrix.h

**Description:** Basic operations on square matrices.

```
Usage: Matrix<int, 3> A;
A.d = \{\{\{1,2,3\}\}, \{\{4,5,6\}\}, \{\{7,8,9\}\}\}\};
vector < int > vec = \{1, 2, 3\};
vec = (A^N) * vec;
                                                        c43c7d, 26 lines
template<class T, int N> struct Matrix {
 typedef Matrix M;
  array<array<T, N>, N> d{};
 M operator*(const M& m) const {
    rep(i,0,N) rep(j,0,N)
      rep(k, 0, N) \ a.d[i][j] += d[i][k]*m.d[k][j];
  vector<T> operator*(const vector<T>& vec) const {
    vector<T> ret(N);
    rep(i, 0, N) rep(j, 0, N) ret[i] += d[i][j] * vec[j];
    return ret:
 M operator^(ll p) const {
    assert (p >= 0);
    M a, b(*this);
    rep(i, 0, N) \ a.d[i][i] = 1;
    while (p) {
      if (p&1) a = a*b;
      b = b*b;
      p >>= 1;
    return a:
};
```

#### LineContainer.h

Description: Container where you can add lines of the form kx+m, and query maximum values at points x. Useful for dynamic programming ("convex hull trick").

```
Time: \mathcal{O}(\log N)
                                                                                    8ec1c7, 30 lines
```

```
struct Line {
 mutable 11 k, m, p;
 bool operator<(const Line& o) const { return k < o.k; }</pre>
 bool operator<(ll x) const { return p < x; }</pre>
struct LineContainer : multiset<Line, less<>>> {
  // (for doubles, use inf = 1/.0, div(a,b) = a/b)
  static const ll inf = LLONG MAX;
  ll div(ll a, ll b) { // floored division
    return a / b - ((a ^ b) < 0 && a % b); }
  bool isect(iterator x, iterator y) {
    if (y == end()) return x \rightarrow p = inf, 0;
    if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
    else x->p = div(y->m - x->m, x->k - y->k);
    return x->p >= y->p;
 void add(ll k, ll m) {
    auto z = insert(\{k, m, 0\}), y = z++, x = y;
    while (isect(y, z)) z = erase(z);
    if (x != begin() \&\& isect(--x, y)) isect(x, y = erase(y));
    while ((y = x) != begin() && (--x)->p >= y->p)
      isect(x, erase(v));
 ll query(ll x) {
    assert(!empty());
    auto 1 = *lower_bound(x);
    return l.k * x + l.m;
};
```

```
Treap.h
```

**Description:** A short self-balancing tree. It acts as a sequential container with log-time splits/joins, and is easy to augment with additional data.

```
Time: \mathcal{O}(\log N)
```

```
struct Node {
  Node *1 = 0, *r = 0;
  int val, y, c = 1;
 Node (int val) : val(val), y(rand()) {}
  void recalc();
int cnt(Node* n) { return n ? n->c : 0; }
void Node::recalc() { c = cnt(1) + cnt(r) + 1; }
template < class F > void each (Node * n, F f) {
 if (n) { each (n->1, f); f(n->val); each (n->r, f); }
pair<Node*, Node*> split(Node* n, int k) {
  if (!n) return {};
  if (cnt(n->1) >= k) { // "n->val>= k" for lower_bound(k)}
    auto pa = split(n->1, k);
   n->1 = pa.second;
   n->recalc();
    return {pa.first, n};
    auto pa = split(n->r, k - cnt(n->l) - 1); // and just "k"
   n->r = pa.first;
   n->recalc();
    return {n, pa.second};
Node* merge(Node* 1, Node* r) {
 if (!1) return r;
  if (!r) return 1;
  if (1->y > r->y) {
   1->r = merge(1->r, r);
   1->recalc();
   return 1;
  } else {
    r->1 = merge(1, r->1);
    r->recalc();
   return r:
Node* ins(Node* t, Node* n, int pos) {
  auto pa = split(t, pos);
  return merge(merge(pa.first, n), pa.second);
// Example application: move the range (l, r) to index k
void move(Node*& t, int 1, int r, int k) {
 Node *a, *b, *c;
  tie(a,b) = split(t, 1); tie(b,c) = split(b, r - 1);
  if (k \le 1) t = merge(ins(a, b, k), c);
  else t = merge(a, ins(c, b, k - r));
```

#### FenwickTree.h

**Description:** Computes partial sums a[0] + a[1] + ... + a[pos - 1], and updates single elements a[i], taking the difference between the old and new

**Time:** Both operations are  $\mathcal{O}(\log N)$ .

e62fac, 22 lines

```
struct FT {
 vector<ll> s;
 FT(int n) : s(n) {}
```

```
void update(int pos, ll dif) { // a[pos] += dif
    for (; pos < sz(s); pos |= pos + 1) s[pos] += dif;</pre>
 11 query (int pos) { // sum of values in [0, pos)
   11 \text{ res} = 0;
    for (; pos > 0; pos &= pos - 1) res += s[pos-1];
    return res;
 int lower_bound(ll sum) \{// min \ pos \ st \ sum \ of \ [0, \ pos] >= sum
    // Returns n if no sum is >= sum, or -1 if empty sum is.
    if (sum \le 0) return -1;
    int pos = 0;
    for (int pw = 1 << 25; pw; pw >>= 1) {
     if (pos + pw <= sz(s) && s[pos + pw-1] < sum)</pre>
        pos += pw, sum -= s[pos-1];
    return pos;
};
FenwickTree2d.h
Description: Computes sums a[i,j] for all i<I, j<J, and increases single ele-
ments a[i,j]. Requires that the elements to be updated are known in advance
(call fakeUpdate() before init()).
Time: \mathcal{O}(\log^2 N). (Use persistent segment trees for \mathcal{O}(\log N).)
"FenwickTree.h"
struct FT2 {
 vector<vi> ys; vector<FT> ft;
 FT2(int limx) : vs(limx) {}
 void fakeUpdate(int x, int y) {
    for (; x < sz(ys); x |= x + 1) ys[x].push_back(y);
 void init() {
    for (vi& v : ys) sort(all(v)), ft.emplace back(sz(v));
 int ind(int x, int y) {
    return (int) (lower_bound(all(ys[x]), y) - ys[x].begin()); }
 void update(int x, int y, ll dif) {
    for (; x < sz(ys); x | = x + 1)
      ft[x].update(ind(x, y), dif);
 11 query(int x, int y) {
   11 \text{ sum} = 0;
   for (; x; x &= x - 1)
      sum += ft[x-1].query(ind(x-1, y));
    return sum;
};
RMQ.h
Description: Range Minimum Queries on an array. Returns min(V[a], V[a
+1], ... V[b - 1]) in constant time.
Usage: RMO rmg(values);
rmq.query(inclusive, exclusive);
Time: \mathcal{O}(|V|\log|V|+Q)
                                                        510c32, 16 lines
template<class T>
struct RMO {
 vector<vector<T>> imp;
 RMQ(const vector<T>& V) : jmp(1, V) {
    for (int pw = 1, k = 1; pw * 2 <= sz(V); pw *= 2, ++k) {
```

 $jmp.emplace_back(sz(V) - pw * 2 + 1);$ 

assert (a < b); // or return inf if a == b

int dep = 31 - \_\_builtin\_clz(b - a);

rep(j, 0, sz(jmp[k]))

T query(int a, int b) {

# Numerical (4)

I[i++] = b, b = par[b];

## 4.1 Polynomials and recurrences

return min(jmp[dep][a], jmp[dep][b - (1 << dep)]);</pre>

Description: Answer interval or tree path queries by finding an approxi-

mate TSP through the queries, and moving from one query to the next by adding/removing points at the ends. If values are on tree edges, change step

to add/remove the edge (a, c) and remove the initial add call (but keep in).

void add(int ind, int end) { ... } // add a[ind] (end = 0 or 1)

**#define** K(x) pii(x.first/blk, x.second ^ -(x.first/blk & 1))

sort(all(s), [&](int s, int t){ return  $K(Q[s]) < K(Q[t]); });$ 

vi moTree(vector<array<int, 2>> Q, vector<vi>& ed, int root=0) {

**else** { add(c, end); in[c] = 1; } a = c; }

int N = sz(ed), pos[2] = {}, blk = 350;  $// \sim N/sqrt(Q)$ 

vi s(sz(Q)), res = s, I(N), L(N), R(N), in(N), par(N);

auto dfs = [&] (int x, int p, int dep, auto& f) -> void {

void del(int ind, int end) { ... } // remove a[ind]

int calc() { ... } // compute current answer

int L = 0, R = 0, blk = 350;  $// \sim N/sqrt(Q)$ 

while (L > q.first) add(--L, 0);

while (R < q.second) add(R++, 1);</pre>

while (L < q.first) del(L++, 0);

while (R > q.second) del(--R, 1);

## Polynomial.h

c9b7b0, 17 lines

```
struct Poly {
```

```
L[x] = N;
                                                              if (dep) I[x] = N++;
                                                              for (int y : ed[x]) if (y != p) f(y, x, !dep, f);
                                                              if (!dep) I[x] = N++;
                                                              R[x] = N;
                                                            dfs(root, -1, 0, dfs);
                                                          #define K(x) pii(I[x[0]] / blk, I[x[1]] ^ -(I[x[0]] / blk & 1))
                                                            iota(all(s), 0);
                                                            sort(all(s), [\&](int s, int t) { return K(O[s]) < K(O[t]); });
                                                            for (int qi : s) rep(end, 0, 2) {
                                                              int &a = pos[end], b = Q[qi][end], i = 0;
                                                          #define step(c) { if (in[c]) { del(a, end); in[a] = 0; }
                                                               while (!(L[b] <= L[a] && R[a] <= R[b]))</pre>
                                                              while (a != b) step(par[a]);
                                                              while (i--) step(I[i]);
                                                              if (end) res[qi] = calc();
                                                            return res;
jmp[k][j] = min(jmp[k - 1][j], jmp[k - 1][j + pw]);
```

};

MoQueries.h

Time:  $\mathcal{O}(N\sqrt{Q})$ 

vi mo(vector<pii> Q) {

iota(all(s), 0);

for (int qi : s) {

pii q = Q[qi];

res[qi] = calc();

add(0, 0), in[0] = 1;

par[x] = p;

return res;

vi s(sz(0)), res = s;

```
vector<double> a;
  double operator()(double x) const {
    double val = 0;
    for (int i = sz(a); i--;) (val *= x) += a[i];
   return val:
  void diff() {
    rep(i, 1, sz(a)) a[i-1] = i*a[i];
   a.pop_back();
  void divroot(double x0) {
    double b = a.back(), c; a.back() = 0;
    for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i+1]*x0+b, b=c;
   a.pop_back();
};
```

#### PolyRoots.h

**Description:** Finds the real roots to a polynomial.

**Usage:** polyRoots( $\{\{2, -3, 1\}\}, -1e9, 1e9\}$ ) // solve  $x^2-3x+2=0$ Time:  $\mathcal{O}\left(n^2\log(1/\epsilon)\right)$ 

"Polynomial.h" b00bfe, 23 lines vector<double> polyRoots(Poly p, double xmin, double xmax) { if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; } vector<double> ret; Poly der = p; der.diff(); auto dr = polyRoots(der, xmin, xmax); dr.push\_back(xmin-1); dr.push\_back(xmax+1); sort(all(dr)); rep(i, 0, sz(dr) -1) { double l = dr[i], h = dr[i+1]; **bool** sign = p(1) > 0; **if** (sign  $^{(p(h) > 0)}$ ) { rep(it, 0, 60) { // while (h - l > 1e-8)**double** m = (1 + h) / 2, f = p(m); **if**  $((f \le 0) ^ sign) 1 = m;$ else h = m; ret.push back((1 + h) / 2); return ret;

#### PolyInterpolate.h

**Description:** Given n points (x[i], y[i]), computes an n-1-degree polynomial p that passes through them:  $p(x) = a[0] * x^0 + ... + a[n-1] * x^{n-1}$ . For numerical precision, pick  $x[k] = c * \cos(k/(n-1) * \pi), k = 0 \dots n-1.$ Time:  $\mathcal{O}\left(n^2\right)$ 

```
typedef vector<double> vd;
vd interpolate(vd x, vd y, int n) {
  vd res(n), temp(n);
  rep(k, 0, n-1) rep(i, k+1, n)
   y[i] = (y[i] - y[k]) / (x[i] - x[k]);
  double last = 0; temp[0] = 1;
  rep(k,0,n) rep(i,0,n) {
   res[i] += y[k] * temp[i];
   swap(last, temp[i]);
   temp[i] -= last * x[k];
  return res;
```

#### BerlekampMassev.h

**Description:** Recovers any n-order linear recurrence relation from the first 2n terms of the recurrence. Useful for guessing linear recurrences after bruteforcing the first terms. Should work on any field, but numerical stability for floats is not guaranteed. Output will have size  $\leq n$ .

Usage: berlekampMassey({0, 1, 1, 3, 5, 11}) // {1, 2} Time:  $\mathcal{O}(N^2)$ 

```
96548<u>b</u>, 20 lines
"../number-theory/ModPow.h"
vector<11> berlekampMassey(vector<11> s) {
 int n = sz(s), L = 0, m = 0;
 vector<ll> C(n), B(n), T;
 C[0] = B[0] = 1;
 11 b = 1;
 rep(i,0,n) { ++m;
   11 d = s[i] % mod;
   rep(j,1,L+1) d = (d + C[j] * s[i - j]) % mod;
   if (!d) continue;
   T = C; 11 coef = d * modpow(b, mod-2) % mod;
   rep(j, m, n) C[j] = (C[j] - coef * B[j - m]) % mod;
   if (2 * L > i) continue;
   L = i + 1 - L; B = T; b = d; m = 0;
 C.resize(L + 1); C.erase(C.begin());
 for (11& x : C) x = (mod - x) % mod;
 return C;
```

#### LinearRecurrence.h

**Description:** Generates the k'th term of an n-order linear recurrence  $S[i] = \sum_{i} S[i-j-1]tr[j]$ , given  $S[0... \ge n-1]$  and tr[0...n-1]. Faster than matrix multiplication. Useful together with Berlekamp-Massey. Usage: linearRec( $\{0, 1\}, \{1, 1\}, k$ ) // k'th Fibonacci number Time:  $\mathcal{O}\left(n^2 \log k\right)$ 

```
f4e444, 26 lines
typedef vector<ll> Poly;
11 linearRec(Poly S, Poly tr, 11 k) {
 int n = sz(tr);
  auto combine = [&](Poly a, Poly b) {
   Poly res(n \star 2 + 1);
   rep(i,0,n+1) rep(j,0,n+1)
     res[i + j] = (res[i + j] + a[i] * b[j]) % mod;
    for (int i = 2 * n; i > n; --i) rep(j,0,n)
     res[i - 1 - j] = (res[i - 1 - j] + res[i] * tr[j]) % mod;
    res.resize(n + 1);
   return res;
  Poly pol(n + 1), e(pol);
  pol[0] = e[1] = 1;
  for (++k; k; k /= 2) {
   if (k % 2) pol = combine(pol, e);
    e = combine(e, e);
 11 \text{ res} = 0;
 rep(i,0,n) res = (res + pol[i + 1] * S[i]) % mod;
 return res;
```

## Optimization

GoldenSectionSearch.h

**Description:** Finds the argument minimizing the function f in the interval [a, b] assuming f is unimodal on the interval, i.e. has only one local minimum. The maximum error in the result is eps. Works equally well for maximization with a small change in the code. See TernarySearch.h in the Various chapter for a discrete version. Usage: double xmin = gss(-1000, 1000, [](double x){

```
Time: \mathcal{O}(\log((b-a)/\epsilon))
                                                       1f3e0a, 15 lines
template <class F>
double gss(double a, double b, F f) {
  double r = (sgrt(5)-1)/2, eps = 1e-7;
  double x1 = b - r*(b-a), x2 = a + r*(b-a);
  double f1 = f(x1), f2 = f(x2);
  while (b-a > eps)
    if (f1 < f2 || f1 == INFINITY) { //negate to find maximum
      b = x2; x2 = x1; f2 = f1;
      x1 = b - r*(b-a); f1 = f(x1);
      a = x1; x1 = x2; f1 = f2;
      x2 = a + r*(b-a); f2 = f(x2);
  return a;
```

#### HillClimbing.h

return  $4 + x + .3*x*x; });$ 

Description: Poor man's optimization for unimodal functions<sub>8eeeaf, 14 lines</sub>

```
typedef array<double, 2> P;
template<class F> pair<double, P> hillClimb(P start, F f) {
 pair<double, P> cur(f(start), start);
  for (double jmp = 1e9; jmp > 1e-20; jmp /= 2) {
    rep(j, 0, 100) rep(dx, -1, 2) rep(dy, -1, 2) {
      P p = cur.second;
      q = 0 + dx * dy;
     p[1] += dy * jmp;
      cur = min(cur, make_pair(f(p), p));
 return cur;
```

#### Integrate.h

template <class F>

dc = (a + b) / 2;

d rec(F& f, da, db, deps, dS) {

Description: Simple integration of a function over an interval using Simpson's rule. The error should be proportional to  $h^4$ , although in practice you will want to verify that the result is stable to desired precision when epsilon

```
template<class F>
double quad(double a, double b, F f, const int n = 1000) {
  double h = (b - a) / 2 / n, v = f(a) + f(b);
  rep(i.1.n*2)
  v += f(a + i*h) * (i&1 ? 4 : 2);
  return v * h / 3;
IntegrateAdaptive.h
Description: Fast integration using an adaptive Simpson's rule.
Usage: double sphereVolume = quad(-1, 1, [](double x) {
return quad(-1, 1, [&](double y)
return quad(-1, 1, [&] (double z) {
return x*x + y*y + z*z < 1; {);});});
                                                     92dd79, 15 lines
typedef double d;
#define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (b-a) / 6
```

```
d S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
if (abs(T - S) <= 15 * eps || b - a < 1e-10)
return T + (T - S) / 15;
return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2, S2);
}
template<class F>
d quad(d a, d b, F f, d eps = 1e-8) {
return rec(f, a, b, eps, S(a, b));
}
```

## Simplex.h

**Description:** Solves a general linear maximization problem: maximize  $c^Tx$  subject to  $Ax \leq b$ ,  $x \geq 0$ . Returns -inf if there is no solution, inf if there are arbitrarily good solutions, or the maximum value of  $c^Tx$  otherwise. The input vector is set to an optimal x (or in the unbounded case, an arbitrary solution fulfilling the constraints). Numerical stability is not guaranteed. For better performance, define variables such that x = 0 is viable.

```
Usage: vvd A = \{\{1,-1\}, \{-1,1\}, \{-1,-2\}\}; vd b = \{1,1,-4\}, c = \{-1,-1\}, x; T val = LPSolver(A, b, c).solve(x);
```

**Time:**  $\mathcal{O}(NM*\#pivots)$ , where a pivot may be e.g. an edge relaxation.  $\mathcal{O}(2^n)$  in the general case.

aa8530, 68

```
typedef double T; // long double, Rational, double + mod < P > \dots typedef vector<T> vd; typedef vector<vd> vvd;
```

```
const T eps = 1e-8, inf = 1/.0;
#define MP make pair
#define ltj(X) if (s == -1 \mid | MP(X[j], N[j]) < MP(X[s], N[s])) s=j
struct LPSolver {
 int m, n;
 vi N, B;
  vvd D:
  LPSolver (const vvd& A, const vd& b, const vd& c) :
   m(sz(b)), n(sz(c)), N(n+1), B(m), D(m+2), vd(n+2)) {
     rep(i, 0, m) rep(j, 0, n) D[i][j] = A[i][j];
     rep(i, 0, m) \{ B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[i]; \}
     rep(j,0,n) \{ N[j] = j; D[m][j] = -c[j]; \}
     N[n] = -1; D[m+1][n] = 1;
  void pivot(int r, int s) {
   T *a = D[r].data(), inv = 1 / a[s];
    rep(i, 0, m+2) if (i != r \&\& abs(D[i][s]) > eps) {
     T *b = D[i].data(), inv2 = b[s] * inv;
     rep(j, 0, n+2) b[j] -= a[j] * inv2;
     b[s] = a[s] * inv2;
    rep(j,0,n+2) if (j != s) D[r][j] *= inv;
   rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
   D[r][s] = inv;
   swap(B[r], N[s]);
  bool simplex(int phase) {
    int x = m + phase - 1;
    for (;;) {
     int s = -1;
     rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
     if (D[x][s] >= -eps) return true;
     int r = -1;
      rep(i,0,m) {
       if (D[i][s] <= eps) continue;</pre>
       if (r == -1 || MP(D[i][n+1] / D[i][s], B[i])
```

< MP(D[r][n+1] / D[r][s], B[r])) r = i;

```
if (r == -1) return false;
    pivot(r, s);
}

T solve(vd &x) {
    int r = 0;
    rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
    if (D[r][n+1] < -eps) {
        pivot(r, n);
        if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
        rep(i,0,m) if (B[i] == -1) {
            int s = 0;
               rep(j,1,n+1) ltj(D[i]);
               pivot(i, s);
        }
}
bool ok = simplex(1); x = vd(n);
    rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
    return ok ? D[m][n+1] : inf;
}</pre>
```

#### 4.3 Matrices

#### Determinant.h

};

**Description:** Calculates determinant of a matrix. Destroys the matrix. Time:  $\mathcal{O}(N^3)$ 

```
double det(vector<vector<double>>& a) {
   int n = sz(a); double res = 1;
   rep(i,0,n) {
     int b = i;
     rep(j,i+1,n) if (fabs(a[j][i]) > fabs(a[b][i])) b = j;
     if (i != b) swap(a[i], a[b]), res *= -1;
     res *= a[i][i];
     if (res == 0) return 0;
     rep(j,i+1,n) {
        double v = a[j][i] / a[i][i];
        if (v != 0) rep(k,i+1,n) a[j][k] -= v * a[i][k];
     }
   }
   return res;
}
```

#### IntDeterminant.h

**Description:** Calculates determinant using modular arithmetics. Modulos can also be removed to get a pure-integer version.

```
Time: \mathcal{O}(N^3)
                                                       3313dc, 18 lines
const 11 mod = 12345;
11 det(vector<vector<11>>& a) {
 int n = sz(a); ll ans = 1;
 rep(i,0,n) {
    rep(j,i+1,n) {
      while (a[j][i] != 0) { // gcd step
        11 t = a[i][i] / a[j][i];
        if (t) rep(k,i,n)
          a[i][k] = (a[i][k] - a[j][k] * t) % mod;
        swap(a[i], a[j]);
        ans \star = -1;
    ans = ans * a[i][i] % mod;
    if (!ans) return 0;
 return (ans + mod) % mod;
```

#### SolveLinear.h

typedef vector<double> vd;

**Description:** Solves A \* x = b. If there are multiple solutions, an arbitrary one is returned. Returns rank, or -1 if no solutions. Data in A and b is lost. **Time:**  $\mathcal{O}(n^2m)$ 

```
const double eps = 1e-12;
int solveLinear(vector<vd>& A, vd& b, vd& x) {
 int n = sz(A), m = sz(x), rank = 0, br, bc;
 if (n) assert(sz(A[0]) == m);
 vi col(m); iota(all(col), 0);
 rep(i,0,n) {
    double v, bv = 0;
    rep(r,i,n) rep(c,i,m)
     if ((v = fabs(A[r][c])) > bv)
       br = r, bc = c, bv = v;
    if (bv <= eps) {
      rep(j,i,n) if (fabs(b[j]) > eps) return -1;
     break;
    swap(A[i], A[br]);
    swap(b[i], b[br]);
   swap(col[i], col[bc]);
   rep(j,0,n) swap(A[j][i], A[j][bc]);
   bv = 1/A[i][i];
   rep(j,i+1,n) {
     double fac = A[j][i] * bv;
     b[j] -= fac * b[i];
     rep(k,i+1,m) A[j][k] = fac*A[i][k];
   rank++;
 x.assign(m. 0):
 for (int i = rank; i--;) {
   b[i] /= A[i][i];
   x[col[i]] = b[i];
   rep(j,0,i) b[j] -= A[j][i] * b[i];
 return rank; // (multiple solutions if rank < m)
```

#### SolveLinear2.h

**Description:** To get all uniquely determined values of x back from Solve-Linear, make the following changes:

```
"SolveLinear.h"
rep(j,0,n) if (j != i) // instead of rep(j,i+1,n)
// ... then at the end:
x.assign(m, undefined);
rep(i,0,rank) {
   rep(j,rank,m) if (fabs(A[i][j]) > eps) goto fail;
   x[col[i]] = b[i] / A[i][i];
fail:; }
```

#### SolveLinearBinary.h

**Description:** Solves Ax = b over  $\mathbb{F}_2$ . If there are multiple solutions, one is returned arbitrarily. Returns rank, or -1 if no solutions. Destroys A and b. **Time:**  $\mathcal{O}\left(n^2m\right)$ 

```
typedef bitset<1000> bs;

int solveLinear(vector<bs>& A, vi& b, bs& x, int m) {
  int n = sz(A), rank = 0, br;
  assert(m <= sz(x));
  vi col(m); iota(all(col), 0);
  rep(i,0,n) {
    for (br=i; br<n; ++br) if (A[br].any()) break;</pre>
```

```
if (br == n) {
    rep(j,i,n) if(b[j]) return -1;
   break;
  int bc = (int)A[br]._Find_next(i-1);
  swap(A[i], A[br]);
  swap(b[i], b[br]);
  swap(col[i], col[bc]);
  rep(j, 0, n) if (A[j][i] != A[j][bc]) {
   A[j].flip(i); A[j].flip(bc);
  rep(j,i+1,n) if (A[j][i]) {
   b[j] ^= b[i];
   A[j] ^= A[i];
 rank++;
x = bs();
for (int i = rank; i--;) {
 if (!b[i]) continue;
 x[col[i]] = 1;
 rep(j,0,i) b[j] ^= A[j][i];
return rank; // (multiple solutions if rank < m)
```

#### MatrixInverse.h

**Description:** Invert matrix A. Returns rank; result is stored in A unless singular (rank < n). Can easily be extended to prime moduli; for prime powers, repeatedly set  $A^{-1} = A^{-1}(2I - AA^{-1}) \pmod{p^k}$  where  $A^{-1}$  starts as the inverse of A mod p, and k is doubled in each step.

```
Time: \mathcal{O}\left(n^3\right)
                                                        ebfff6, 35 lines
int matInv(vector<vector<double>>& A) {
  int n = sz(A); vi col(n);
  vector<vector<double>> tmp(n, vector<double>(n));
  rep(i, 0, n) tmp[i][i] = 1, col[i] = i;
  rep(i,0,n) {
   int r = i, c = i;
    rep(j,i,n) rep(k,i,n)
     if (fabs(A[j][k]) > fabs(A[r][c]))
        r = j, c = k;
    if (fabs(A[r][c]) < 1e-12) return i;</pre>
   A[i].swap(A[r]); tmp[i].swap(tmp[r]);
    rep(j,0,n)
     swap(A[j][i], A[j][c]), swap(tmp[j][i], tmp[j][c]);
    swap(col[i], col[c]);
    double v = A[i][i];
    rep(j,i+1,n) {
     double f = A[j][i] / v;
     A[j][i] = 0;
     rep(k,i+1,n) A[j][k] = f*A[i][k];
     rep(k,0,n) tmp[j][k] -= f*tmp[i][k];
    rep(j, i+1, n) A[i][j] /= v;
    rep(j,0,n) tmp[i][j] /= v;
    A[i][i] = 1;
  for (int i = n-1; i > 0; --i) rep(j, 0, i) {
   double v = A[j][i];
    rep(k,0,n) tmp[j][k] \rightarrow v*tmp[i][k];
  rep(i,0,n) rep(j,0,n) A[col[i]][col[j]] = tmp[i][j];
 return n;
```

```
Tridiagonal.h
```

**Description:** x = tridiagonal(d, p, q, b) solves the equation system

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & 0 & q_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

This is useful for solving problems on the type

$$a_i = b_i a_{i-1} + c_i a_{i+1} + d_i, \ 1 \le i \le n,$$

where  $a_0, a_{n+1}, b_i, c_i$  and  $d_i$  are known. a can then be obtained from

$$\{a_i\} = \operatorname{tridiagonal}(\{1,-1,-1,\ldots,-1,1\},\{0,c_1,c_2,\ldots,c_n\},\\ \{b_1,b_2,\ldots,b_n,0\},\{a_0,d_1,d_2,\ldots,d_n,a_{n+1}\}).$$

Fails if the solution is not unique.

If  $|d_i| > |p_i| + |q_{i-1}|$  for all i, or  $|d_i| > |p_{i-1}| + |q_i|$ , or the matrix is positive definite, the algorithm is numerically stable and neither tr nor the check for diag[i] == 0 is needed.

Time:  $\mathcal{O}(N)$ 8f9fa8, 26 lines

```
typedef double T;
vector<T> tridiagonal(vector<T> diag, const vector<T>& super,
   const vector<T>& sub, vector<T> b) {
 int n = sz(b); vi tr(n);
 rep(i, 0, n-1) {
    if (abs(diag[i]) < 1e-9 * abs(super[i])) { // diag[i] == 0
     b[i+1] -= b[i] * diag[i+1] / super[i];
     if (i+2 < n) b[i+2] -= b[i] * sub[i+1] / super[i];</pre>
     diag[i+1] = sub[i]; tr[++i] = 1;
     diag[i+1] -= super[i]*sub[i]/diag[i];
     b[i+1] -= b[i]*sub[i]/diag[i];
 for (int i = n; i--;) {
   if (tr[i]) {
     swap(b[i], b[i-1]);
     diag[i-1] = diag[i];
     b[i] /= super[i-1];
    } else {
     b[i] /= diag[i];
     if (i) b[i-1] -= b[i]*super[i-1];
 return b;
```

## 4.4 Fourier transforms

FastFourierTransform.h

**Description:** fft(a) computes  $\hat{f}(k) = \sum_{x} a[x] \exp(2\pi i \cdot kx/N)$  for all k. N must be a power of 2. Useful for convolution: conv(a, b) = c, where  $c[x] = \sum a[i]b[x-i]$ . For convolution of complex numbers or more than two vectors: FFT, multiply pointwise, divide by n, reverse(start+1, end), FFT back. Rounding is safe if  $(\sum a_i^2 + \sum b_i^2) \log_2 N < 9 \cdot 10^{14}$  (in practice  $10^{16}$ ; higher for random inputs). Otherwise, use NTT/FFTMod. **Time:**  $O(N \log N)$  with  $N = |A| + |B| (\sim 1s \text{ for } N = 2^{22})$ 

```
typedef complex<double> C;
typedef vector<double> vd;
void fft(vector<C>& a) {
 int n = sz(a), L = 31 - builtin clz(n);
 static vector<complex<long double>> R(2, 1);
 static vector<C> rt(2, 1); // (^ 10% faster if double)
 for (static int k = 2; k < n; k \neq 2) {
   R.resize(n); rt.resize(n);
    auto x = polar(1.0L, acos(-1.0L) / k);
```

```
rep(i,k,2*k) rt[i] = R[i] = i&1 ? R[i/2] * x : R[i/2];
 vi rev(n);
  rep(i,0,n) \ rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
  rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
  for (int k = 1; k < n; k *= 2)
    for (int i = 0; i < n; i += 2 * k) rep(j, 0, k) {
     Cz = rt[j+k] * a[i+j+k]; // (25\% faster if hand-rolled)
      a[i + j + k] = a[i + j] - z;
     a[i + j] += z;
vd conv(const vd& a, const vd& b) {
 if (a.empty() || b.empty()) return {};
 vd res(sz(a) + sz(b) - 1);
  int L = 32 - __builtin_clz(sz(res)), n = 1 << L;</pre>
 vector<C> in(n), out(n);
  copy(all(a), begin(in));
  rep(i,0,sz(b)) in[i].imag(b[i]);
  fft(in);
  for (C& x : in) x *= x;
  rep(i, 0, n) out[i] = in[-i & (n - 1)] - conj(in[i]);
  rep(i, 0, sz(res)) res[i] = imag(out[i]) / (4 * n);
  return res;
```

#### FastFourierTransformMod.h

Description: Higher precision FFT, can be used for convolutions modulo arbitrary integers as long as  $N \log_2 N \cdot \text{mod} < 8.6 \cdot 10^{14}$  (in practice  $10^{16}$  or higher). Inputs must be in [0, mod).

**Time:**  $\mathcal{O}(N \log N)$ , where N = |A| + |B| (twice as slow as NTT or FFT) "FastFourierTransform.h" b82773, 22 lines

```
typedef vector<11> v1;
template<int M> vl convMod(const vl &a, const vl &b) {
  if (a.empty() || b.empty()) return {};
  vl res(sz(a) + sz(b) - 1);
  int B=32-__builtin_clz(sz(res)), n=1<<B, cut=int(sqrt(M));</pre>
  vector<C> L(n), R(n), outs(n), outl(n);
  rep(i,0,sz(a)) L[i] = C((int)a[i] / cut, (int)a[i] % cut);
  rep(i, 0, sz(b)) R[i] = C((int)b[i] / cut, (int)b[i] % cut);
  fft(L), fft(R);
  rep(i,0,n) {
    int j = -i \& (n - 1);
    outl[j] = (L[i] + conj(L[j])) * R[i] / (2.0 * n);
    outs[j] = (L[i] - conj(L[j])) * R[i] / (2.0 * n) / 1i;
  fft (outl), fft (outs);
  rep(i, 0, sz(res)) {
    11 \text{ av} = 11(\text{real}(\text{outl}[i]) + .5), \text{ cv} = 11(\text{imag}(\text{outs}[i]) + .5);
    11 \text{ bv} = 11(\text{imag}(\text{outl}[i]) + .5) + 11(\text{real}(\text{outs}[i]) + .5);
    res[i] = ((av % M * cut + bv) % M * cut + cv) % M;
  return res;
```

#### NumberTheoreticTransform.h

**Description:** ntt(a) computes  $\hat{f}(k) = \sum_{x} a[x]g^{xk}$  for all k, where  $g = \sum_{x} a[x]g^{xk}$  $root^{(mod-1)/N}$ . N must be a power of 2. Useful for convolution modulo specific nice primes of the form  $2^a b + 1$ , where the convolution result has size at most  $2^a$ . For arbitrary modulo, see FFTMod. conv(a, b) = c, where  $c[x] = \sum a[i]b[x-i]$ . For manual convolution: NTT the inputs, multiply pointwise, divide by n, reverse(start+1, end), NTT back. Inputs must be in [0, mod).

```
Time: \mathcal{O}(N \log N)
```

00ced6, 35 lines

```
"../number-theory/ModPow.h"
```

**const** 11 mod = (119 << 23) + 1, root = 62; // = 998244353// For p < 2^30 there is also e.g. 5 << 25, 7 << 26, 479 << 21

```
// and 483 \ll 21 (same root). The last two are > 10^9.
typedef vector<ll> v1;
void ntt(vl &a) {
  int n = sz(a), L = 31 - __builtin_clz(n);
  static v1 rt(2, 1);
  for (static int k = 2, s = 2; k < n; k \neq 2, s++) {
   rt.resize(n);
   11 z[] = \{1, modpow(root, mod >> s)\};
   rep(i,k,2*k) rt[i] = rt[i / 2] * z[i & 1] % mod;
  vi rev(n);
  rep(i,0,n) \ rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
  rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
  for (int k = 1; k < n; k *= 2)
    for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {
     11 z = rt[j + k] * a[i + j + k] % mod, &ai = a[i + j];
     a[i + j + k] = ai - z + (z > ai ? mod : 0);
     ai += (ai + z >= mod ? z - mod : z);
vl conv(const vl &a, const vl &b) {
 if (a.empty() || b.empty()) return {};
  int s = sz(a) + sz(b) - 1, B = 32 - builtin clz(s), n = 1
      << B:
  int inv = modpow(n, mod - 2);
  vl L(a), R(b), out(n);
  L.resize(n), R.resize(n);
  ntt(L), ntt(R);
  rep(i, 0, n) out[-i \& (n - 1)] = (11)L[i] * R[i] % mod * inv %
  return {out.begin(), out.begin() + s};
```

#### FastSubsetTransform.h

**Description:** Transform to a basis with fast convolutions of the form  $c[z] = \sum_{z=x \oplus y} a[x] \cdot b[y]$ , where  $\oplus$  is one of AND, OR, XOR. The size of a must be a power of two.

Time:  $\mathcal{O}(N \log N)$  464cf3, 16 lines

# Number theory (5)

## 5.1 Modular arithmetic

#### Modular Arithmetic.h

**Description:** Operators for modular arithmetic. You need to set mod to some number first and then you can use the structure.

```
euclid.h" 35bfea, 18 lines
```

```
\begin{tabular}{ll} \beg
```

```
11 x;
Mod(11 xx) : x(xx) {}
Mod operator+(Mod b) { return Mod((x + b.x) % mod); }
Mod operator-(Mod b) { return Mod((x - b.x + mod) % mod); }
Mod operator*(Mod b) { return Mod((x * b.x) % mod); }
Mod operator/(Mod b) { return *this * invert(b); }
Mod invert(Mod a) {
    11 x, y, g = euclid(a.x, mod, x, y);
    assert(g == 1); return Mod((x + mod) % mod);
}
Mod operator^(11 e) {
    if (!e) return Mod(1);
    Mod r = *this ^ (e / 2); r = r * r;
    return e&1 ? *this * r : r;
}
};
```

#### ModInverse.h

**Description:** Pre-computation of modular inverses. Assumes LIM  $\leq$  mod and that mod is a prime. If the base is a prime then  $a^{-1} \equiv a^{p-2}_{-6f684f,~3~{\rm lines}}$ 

```
const 11 mod = 1000000007, LIM = 200000;
l1* inv = new l1[LIM] - 1; inv[1] = 1;
rep(i,2,LIM) inv[i] = mod - (mod / i) * inv[mod % i] % mod;
```

#### ModPow.h

**Description:** If e is really large and the base m is small use  $a^{\varphi(m)} \equiv 1 \mod m$  for gcd(a,m)=1. So  $a^e \equiv a^{e \mod \varphi(m)} \mod m$ . For  $gcd(a,m) \neq 1$  set  $g=gcd(a,m), x=e \mod \varphi(m)$ . If e=0 return 1. Otherwise  $a^e \equiv (g^x \mod m)((a/g)^x \mod m) \mod m$ .

```
Otherwise a = (g mod m)((a/g) mod m) mod m.

b83e45, 8 lines

const 11 mod = 1000000007; // faster if const

11 modpow(11 b, 11 e) {
    11 ans = 1;
    for (; e; b = b * b % mod, e /= 2)
        if (e & 1) ans = ans * b % mod;
    return ans;
}
```

#### ModLog.h

**Description:** Returns the smallest x > 0 s.t.  $a^x = b \pmod{m}$ , or -1 if no such x exists.  $\operatorname{modLog}(a,1,m)$  can be used to calculate the order of a.

Time:  $\mathcal{O}(\sqrt{m})$ 

```
Imme: O(\sqrt{m})

co40b8, 11 lines

ll modLog(ll a, ll b, ll m) {
    ll n = (ll) sqrt(m) + 1, e = 1, f = 1, j = 1;
    unordered_map*ll, ll> A;

while (j <= n && (e = f = e * a % m) != b % m)
    A[e * b % m] = j++;
    if (e == b % m) return j;
    if (__gcd(m, e) == __gcd(m, b))
        rep(i,2,n+2) if (A.count(e = e * f % m))
        return n * i - A[e];
    return -1;
}</pre>
```

#### ModSum.h

**Description:** Sums of mod'ed arithmetic progressions. modsum(to, c, k, m) =  $\sum_{i=0}^{\text{to}-1} (ki+c)\%m$ . divsum is similar but for floored division.

**Time:**  $\log(m)$ , with a large constant.

```
typedef unsigned long long ull;
ull sumsq(ull to) { return to / 2 * ((to-1) | 1); }
ull divsum(ull to, ull c, ull k, ull m) {
  ull res = k / m * sumsq(to) + c / m * to;
  k %= m; c %= m;
```

```
if (!k) return res;
ull to2 = (to * k + c) / m;
return res + (to - 1) * to2 - divsum(to2, m-1 - c, m, k);
}

ll modsum(ull to, ll c, ll k, ll m) {
  c = ((c % m) + m) % m;
  k = ((k % m) + m) % m;
  return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
}
```

#### ModMulLL.h

**Description:** Calculate  $a \cdot b \mod c$  (or  $a^b \mod c$ ) for  $0 \le a, b \le c \le 7.2 \cdot 10^{18}$ . **Time:**  $\mathcal{O}(1)$  for modmul,  $\mathcal{O}(\log b)$  for modpow

```
typedef unsigned long long ull;
ull modmul(ull a, ull b, ull M) {
    ll ret = a * b - M * ull(1.L / M * a * b);
    return ret + M * (ret < 0) - M * (ret >= (ll)M);
}
ull modpow(ull b, ull e, ull mod) {
    ull ans = 1;
    for (; e; b = modmul(b, b, mod), e /= 2)
        if (e & 1) ans = modmul(ans, b, mod);
    return ans;
}
```

#### ModSqrt.h

**Description:** Tonelli-Shanks algorithm for modular square roots. Finds x s.t.  $x^2 = a \pmod{p}$  (-x gives the other solution).

**Time:**  $\mathcal{O}(\log^2 p)$  worst case,  $\mathcal{O}(\log p)$  for most p

```
"ModPow.h"
                                                      19a793, 24 lines
ll sgrt(ll a, ll p) {
 a \% = p; if (a < 0) a += p;
 if (a == 0) return 0;
 assert (modpow(a, (p-1)/2, p) == 1); // else no solution
 if (p % 4 == 3) return modpow(a, (p+1)/4, p);
  // a^{(n+3)/8} \text{ or } 2^{(n+3)/8} * 2^{(n-1)/4} \text{ works if } p \% 8 == 5
  11 s = p - 1, n = 2;
 int r = 0, m;
  while (s % 2 == 0)
   ++r, s /= 2;
  while (modpow(n, (p-1) / 2, p) != p-1) ++n;
  11 x = modpow(a, (s + 1) / 2, p);
  ll b = modpow(a, s, p), g = modpow(n, s, p);
  for (;; r = m) {
   11 t = b:
    for (m = 0; m < r && t != 1; ++m)
     t = t * t % p;
    if (m == 0) return x;
    11 qs = modpow(q, 1LL \ll (r - m - 1), p);
    q = qs * qs % p;
    x = x * gs % p;
   b = b * g % p;
```

## 5.2 Primality

FastEratosthenes.h

**Description:** Prime sieve for generating all primes smaller than LIM.

Time: LIM=1e9  $\approx 1.5s$ 

5c5bc5, 16 lines

6b2912, 20 lines

```
const int LIM = 1e6;
bitset<LIM> isPrime;
vi eratosthenes() {
  const int S = (int)round(sqrt(LIM)), R = LIM / 2;
  vi pr = {2}, sieve(S+1); pr.reserve(int(LIM/log(LIM)*1.1));
  vector<pii> cp;
```

```
for (int i = 3; i <= S; i += 2) if (!sieve[i]) {</pre>
  cp.push_back(\{i, i * i / 2\});
  for (int j = i * i; j <= S; j += 2 * i) sieve[j] = 1;</pre>
for (int L = 1; L <= R; L += S) {
  array<bool, S> block{};
  for (auto &[p, idx] : cp)
    for (int i=idx; i < S+L; idx = (i+=p)) block[i-L] = 1;</pre>
  rep(i, 0, min(S, R - L))
    if (!block[i]) pr.push_back((L + i) * 2 + 1);
for (int i : pr) isPrime[i] = 1;
return pr;
```

#### MillerRabin.h

Description: Deterministic Miller-Rabin primality test. Guaranteed to work for numbers up to  $7 \cdot 10^{18}$ ; for larger numbers, use Python and extend A randomly.

**Time:** 7 times the complexity of  $a^b \mod c$ .

```
"ModMulLL.h"
                                                        60dcd1, 12 lines
bool isPrime(ull n) {
  if (n < 2 | | n % 6 % 4 != 1) return (n | 1) == 3;</pre>
  ull A[] = \{2, 325, 9375, 28178, 450775, 9780504, 1795265022\},
      s = \underline{\quad} builtin_ctzll(n-1), d = n >> s;
  for (ull a : A) { // ^ count trailing zeroes
    ull p = modpow(a%n, d, n), i = s;
    while (p != 1 && p != n - 1 && a % n && i--)
     p = modmul(p, p, n);
    if (p != n-1 && i != s) return 0;
  return 1;
```

#### Factor.h

Description: Pollard-rho randomized factorization algorithm. Returns prime factors of a number, in arbitrary order (e.g. 2299 -> {11, 19, 11}).

**Time:**  $\mathcal{O}\left(n^{1/4}\right)$ , less for numbers with small factors.

```
"ModMulLL.h", "MillerRabin.h"
                                                      a33cf6, 18 lines
ull pollard(ull n) {
  auto f = [n](ull x) { return modmul(x, x, n) + 1; };
  ull x = 0, y = 0, t = 30, prd = 2, i = 1, q;
  while (t++ % 40 || __gcd(prd, n) == 1) {
   if (x == y) x = ++i, y = f(x);
   if ((q = modmul(prd, max(x,y) - min(x,y), n))) prd = q;
   x = f(x), y = f(f(y));
 return __gcd(prd, n);
vector<ull> factor(ull n) {
 if (n == 1) return {};
 if (isPrime(n)) return {n};
  ull x = pollard(n);
  auto 1 = factor(x), r = factor(n / x);
 1.insert(l.end(), all(r));
 return 1:
```

#### Divisibility 5.3

**Description:** Finds two integers x and y, such that  $ax + by = \gcd(a, b)$ . If you just need gcd, use the built in \_\_gcd instead. If a and b are coprime, then x is the inverse of  $a \pmod{b}$ . b2b420, 8 lines

```
11 gcd(ll a, ll b) { return __gcd(a, b); }
11 lcm(11 a, 11 b) { return abs(a) / gcd(a, b) * abs(b); }
```

```
ll euclid(ll a, ll b, ll &x, ll &v) {
 if (!b) return x = 1, y = 0, a;
 11 d = euclid(b, a % b, v, x);
 return v -= a/b * x, d;
CRT.h
Description: Chinese Remainder Theorem.
crt(a, m, b, n) computes x such that x \equiv a \pmod{m}, x \equiv b \pmod{n}. If
|a| < m and |b| < n, x will obey 0 < x < \operatorname{lcm}(m, n). Assumes mn < 2^{62}.
Time: \log(n)
"euclid.h"
11 crt(ll a, ll m, ll b, ll n) {
 if (n > m) swap(a, b), swap(m, n);
 ll x, y, g = euclid(m, n, x, y);
  assert((a - b) % g == 0); // else no solution
 x = (b - a) % n * x % n / q * m + a;
 return x < 0 ? x + m*n/q : x;
```

#### CRT.pv

**Description:** Chinese Remainder Theorem. crt (r, m) computes x such that  $x \equiv r_i \pmod{m_i}$  for all i and  $0 \le x < \text{lcm}(m)$ .

Time:  $\sum_{i} \log(m_i)$ 18 lines def euclid(a, b): if not b: return a, 1, 0

d, y, x = euclid(b, a % b)return d, x, y - a // b \* x **def** crt2(a, b, m, n): **if** n > m: a, b, m, n = b, a, n, m

q, x, y = euclid(m, n)assert (a - b) % q == 0x = (b - a) \* x % n // q \* m + areturn x, a

def crt(rem, mod): a, m = 0, 1for b, n in zip(rem, mod): a, q = crt2(a, b, m, n)m = m // g \* nreturn a

## 5.3.1 Bézout's identity

For  $a \neq b \neq 0$ , then d = gcd(a, b) is the smallest positive integer for which there are integer solutions to

$$ax + by = d$$

If (x, y) is one solution, then all solutions are given by

Fermat's little thm:  $p \text{ prime } \Rightarrow a^{p-1} \equiv 1 \pmod{p} \ \forall a.$ 

$$\left(x + \frac{kb}{\gcd(a,b)}, y - \frac{ka}{\gcd(a,b)}\right), \quad k \in \mathbb{Z}$$

#### phiFunction.h

**Description:** Euler's  $\phi$  function is defined as  $\phi(n) := \#$  of positive integers  $\leq n$  that are coprime with n.  $\phi(1) = 1$ , p prime  $\Rightarrow \phi(p^k) = (p-1)p^{k-1}$ ,  $m, n \text{ coprime } \Rightarrow \phi(mn) = \phi(m)\phi(n).$  If  $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$  then  $\phi(n) =$  $(p_1-1)p_1^{k_1-1}...(p_r-1)p_r^{k_r-1}.$   $\phi(n)=n\cdot\prod_{p\mid n}(1-1/p).$  $\sum_{d|n} \phi(d) = n, \sum_{1 \le k \le n, \gcd(k,n)=1} k = n\phi(n)/2, n > 1$ Euler's thm:  $a, n \text{ coprime } \Rightarrow a^{\phi(n)} \equiv 1 \pmod{n}$ .

cf7d6d, 8 lines

```
const int LIM = 5000000;
int phi[LIM];
void calculatePhi() {
 rep(i,0,LIM) phi[i] = i&1 ? i : i/2;
 for (int i = 3; i < LIM; i += 2) if(phi[i] == i)</pre>
    for (int j = i; j < LIM; j += i) phi[j] -= phi[j] / i;</pre>
```

#### 5.4 Fractions

## Rational.h "euclid.h"

04d93a, 7 lines

Description: Class for rational arithmetic

ad9060, 26 lines

```
template <typename T>
struct Rational {
 using R = Rational<T>;
 T n, d;
  Rational(T n): n(n), d(1) {}
  Rational(T _n, T _d): n(_n), d(_d) {
   T q = gcd(n, d);
    n /= q, d /= q;
    if (d < 0) n *= -1, d *= -1;
    assert (d != 0);
 R operator+(R o) const {
   T l = lcm(d, o.d);
    return {n * (1 / d) + o.n * (1 / o.d), 1};
 R operator-() const { return {-n, d}; }
  R operator-(R o) const { return *this + (-o); }
  R operator*(R o) const { return {n * o.n, d * o.d}; }
  R operator/(R o) const { return *this * R(o.d, o.n); }
 bool operator == (R o) const {
    return tie(n, d) == tie(o.n, o.d);
 bool operator<(R o) const {</pre>
    return (*this - o).n < 0;</pre>
};
```

#### ContinuedFractions.h

**Description:** Given N and a real number x > 0, finds the closest rational approximation p/q with p, q < N. It will obey |p/q - x| < 1/qN.

For consecutive convergents,  $p_{k+1}q_k - q_{k+1}p_k = (-1)^k$ .  $(p_k/q_k$  alternates between > x and < x.) If x is rational, y eventually becomes  $\infty$ ; if x is the root of a degree 2 polynomial the a's eventually become cyclic.

Time:  $\mathcal{O}(\log N)$ 

dd6c5e, 21 lines

```
typedef double d; // for N \sim 1e7; long double for N \sim 1e9
pair<11, 11> approximate(d x, 11 N) {
 11 LP = 0, LQ = 1, P = 1, Q = 0, inf = LLONG_MAX; d y = x;
 for (;;) {
    ll lim = min(P ? (N-LP) / P : inf, Q ? (N-LQ) / Q : inf),
       a = (11) floor(v), b = min(a, lim),
       NP = b*P + LP, NQ = b*Q + LQ;
    if (a > b) {
      // If b > a/2, we have a semi-convergent that gives us a
      // better approximation; if b = a/2, we *may* have one.
      // Return {P, Q} here for a more canonical approximation.
      return (abs(x - (d)NP / (d)NQ) < abs(x - (d)P / (d)Q)) ?
        make_pair(NP, NQ) : make_pair(P, Q);
    if (abs(y = 1/(y - (d)a)) > 3*N) {
      return {NP, NQ};
    LP = P; P = NP;
    LQ = Q; Q = NQ;
```

## FracBinarySearch IntPerm inversions multinomial

FracBinarySearch.h

**Description:** Given f and N, finds the smallest fraction  $p/q \in [0,1]$  such that f(p/q) is true, and p, q < N. You may want to throw an exception from f if it finds an exact solution, in which case N can be removed. Usage: fracBS([](Frac f) { return f.p>=3\*f.q; }, 10); // {1,3}

Time:  $\mathcal{O}(\log(N))$ struct Frac { ll p, q; };

```
template<class F>
Frac fracBS(F f, 11 N) {
 bool dir = 1, A = 1, B = 1;
  Frac lo{0, 1}, hi{1, 1}; // Set hi to 1/0 to search (0, N]
  if (f(lo)) return lo;
  assert (f(hi));
  while (A | | B) {
   11 adv = 0, step = 1; // move hi if dir, else lo
    for (int si = 0; step; (step *= 2) >>= si) {
     Frac mid{lo.p * adv + hi.p, lo.q * adv + hi.q};
     if (abs(mid.p) > N || mid.q > N || dir == !f(mid)) {
       adv -= step; si = 2;
   hi.p += lo.p * adv;
   hi.q += lo.q * adv;
    dir = !dir;
    swap(lo, hi);
    A = B; B = !!adv;
  return dir ? hi : lo;
```

## Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), b = k \cdot (2mn), c = k \cdot (m^2 + n^2),$$

with m > n > 0, k > 0,  $m \perp n$ , and either m or n even.

## 5.6 Primes

p = 962592769 is such that  $2^{21} \mid p - 1$ , which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than  $1\,000\,000.$ 

Primitive roots exist modulo any prime power  $p^a$ , except for p=2, a>2, and there are  $\phi(\phi(p^a))$  many. For p=2, a>2, the group  $\mathbb{Z}_{2^a}^{\times}$  is instead isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$ .

## 5.7 Estimates

$$\sum_{d|n} d = O(n \log \log n).$$

The number of divisors of n is at most around 100 for n < 5e4. 500 for n < 1e7, 2000 for n < 1e10, 200 000 for n < 1e19.

## Mobius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Mobius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d)g(n/d)$$

Other useful formulas/forms:

$$\sum_{d|n} \mu(d) = [n = 1] \text{ (very useful)}$$

$$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n)g(d)$$

$$g(n) = \sum_{1 \le m \le n} f(\left|\frac{n}{m}\right|) \Leftrightarrow f(n) = \sum_{1 \le m \le n} \mu(m)g(\left|\frac{n}{m}\right|)$$

## Combinatorial (6)

## 6.1 Permutations

#### 6.1.1 Factorial

IntPerm.h

Description: Permutation -> integer conversion. (Not order preserving.) Integer -> permutation can use a lookup table. Time:  $\mathcal{O}(n)$ 

int permToInt(vi& v) { int use = 0, i = 0, r = 0; for (int x:v)  $r = r * ++i + \underline{\quad}$  builtin\_popcount (use & -(1<<x)), // (note: minus, not  $\sim$ !) use |= 1 << x;return r;

## Inversions

Let  $\sigma$  be a permutation with disjoint cycles of length  $r_1, \ldots, r_n$ . Then  $\operatorname{sgn}(\sigma) = \prod_{i=1}^n (-1)^{r_i-1}$ , where  $\operatorname{sgn}(\sigma) = 1$  if the permutation has an even number of inversions else -1.

inversions.h

Description: Counts the number of inversions in an array.

Usage: ll c = invs(arr, 0, sz(arr));Time:  $\mathcal{O}(n \log n)$ 

e6eb7e, 15 lines template <class T> 11 invs(vector<T> & arr, int lo, int hi) {

```
if (hi - lo <= 1) return 0;
int m = (lo + hi) / 2;
11 r = invs(arr, lo, m) + invs(arr, m, hi);
vector<T> res(hi - lo);
auto it = res.begin();
for (int i = lo, j = m; i < m || j < hi; it++) {</pre>
  if (i >= m || (j < hi && arr[i] > arr[j]))
       *it = arr[j], j++, r += m - i;
  else *it = arr[i], i++;
copy(all(res), arr.begin() + lo);
```

return r;

## **6.2.1** Cycles

Let  $q_S(n)$  be the number of n-permutations whose cycle lengths all belong to the set S. Then

$$\sum_{n=0}^{\infty} g_S(n) \frac{x^n}{n!} = \exp\left(\sum_{n \in S} \frac{x^n}{n}\right)$$

## 6.2.2 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1) + D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor \frac{n!}{e} \right\rfloor$$

#### 6.2.3 Burnside's lemma

Given a group G of symmetries and a set X, the number of elements of X up to symmetry equals

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where  $X^g$  are the elements fixed by q (q.x = x).

If f(n) counts "configurations" (of some sort) of length n, we can ignore rotational symmetry using  $G = \mathbb{Z}_n$  to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n,k)) = \frac{1}{n} \sum_{k|n} f(k)\phi(n/k).$$

## Partitions and subsets

#### 6.3.1 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \ p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k - 1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

#### 6.3.2 Lucas' Theorem

Let n, m be non-negative integers and p a prime. Write  $n = n_k p^k + ... + n_1 p + n_0$  and  $m = m_k p^k + ... + m_1 p + m_0$ . Then  $\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod{p}.$ 

## BellmanFord FloydWarshall TopoSort TSP

#### 6.3.3 Binomials

multinomial.h

Description: Computes 
$$\binom{k_1+\cdots+k_n}{k_1,k_2,\ldots,k_n} = \frac{(\sum k_i)!}{k_1!k_2!\ldots k_n!}$$
.

11 multinomial (vi& v) {
 11 c = 1, m = v.empty() ? 1 : v[0];
 rep(i,1,sz(v)) rep(j,0,v[i])
 c = c \* ++m / (j+1);
 return c;

## 6.4 General purpose numbers

#### 6.4.1 Bernoulli numbers

EGF of Bernoulli numbers is  $B(t)=\frac{t}{e^t-1}$  (FFT-able).  $B[0,\ldots]=[1,-\frac12,\frac16,0,-\frac1{30},0,\frac1{42},\ldots]$ 

Sums of powers:

$$\sum_{i=1}^{n} n^{m} = \frac{1}{m+1} \sum_{k=0}^{m} {m+1 \choose k} B_{k} \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty} f(i) = \int_{m}^{\infty} f(x)dx - \sum_{k=1}^{\infty} \frac{B_{k}}{k!} f^{(k-1)}(m)$$

$$\approx \int_{m}^{\infty} f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

## 6.4.2 Stirling numbers of the first kind

Number of permutations on n items with k cycles.

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k), \ c(0,0) = 1$$
$$\sum_{k=0}^{n} c(n,k)x^{k} = x(x+1)\dots(x+n-1)$$

 $c(8,k) = 8,0,5040,13068,13132,6769,1960,322,28,1 \\ c(n,2) = 0,0,1,3,11,50,274,1764,13068,109584,\dots$ 

#### 6.4.3 Eulerian numbers

Number of permutations  $\pi \in S_n$  in which exactly k elements are greater than the previous element. k j:s s.t.  $\pi(j) > \pi(j+1)$ , k+1 j:s s.t.  $\pi(j) \geq j$ , k j:s s.t.  $\pi(j) > j$ .

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$

$$E(n,0) = E(n,n-1) = 1$$

$$E(n,k) = \sum_{i=0}^{k} (-1)^{i} \binom{n+1}{i} (k+1-j)^{n}$$

## 6.4.4 Stirling numbers of the second kind

Partitions of n distinct elements into exactly k groups.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

$$S(n,1) = S(n,n) = 1$$

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^{n}$$

#### 6.4.5 Bell numbers

Total number of partitions of n distinct elements. B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, .... For <math>p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

#### 6.4.6 Labeled unrooted trees

```
# on n vertices: n^{n-2}
# on k existing trees of size n_i: n_1 n_2 \cdots n_k n^{k-2}
# with degrees d_i: (n-2)!/((d_1-1)!\cdots(d_n-1)!)
```

#### 6.4.7 Catalan numbers

$$C_n = \frac{1}{n+1} {2n \choose n} = {2n \choose n} - {2n \choose n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, \ C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \ C_{n+1} = \sum_{n=1}^{\infty} C_n C_{n-n}$$

 $C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$ 

- sub-diagonal monotone paths in an  $n \times n$  grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with n+1 leaves (0 or 2 children).
- ordered trees with n+1 vertices.
- ways a convex polygon with n + 2 sides can be cut into triangles by connecting vertices with straight lines.
- $\bullet$  permutations of [n] with no 3-term increasing subseq.

# Graph (7)

#### 7.1 Fundamentals

BellmanFord.h

**Description:** Calculates shortest paths from s in a graph that might have negative edge weights. Unreachable nodes get dist = inf; nodes reachable through negative-weight cycles get dist = -inf. Assumes  $V^2 \max |w_i| < \sim 2^{63}$ . **Time:**  $\mathcal{O}(VE)$ 

```
const ll inf = LLONG MAX;
struct Ed { int a, b, w, s() { return a < b ? a : -a; }};</pre>
struct Node { ll dist = inf; int prev = -1; };
void bellmanFord(vector<Node>& nodes, vector<Ed>& eds, int s) {
 nodes[s].dist = 0;
 sort(all(eds), [](Ed a, Ed b) { return a.s() < b.s(); });</pre>
 int lim = sz(nodes) / 2 + 2; // /3+100 with shuffled vertices
 rep(i,0,lim) for (Ed ed : eds) {
   Node cur = nodes[ed.a], &dest = nodes[ed.b];
   if (abs(cur.dist) == inf) continue;
   11 d = cur.dist + ed.w;
   if (d < dest.dist) {</pre>
      dest.prev = ed.a;
     dest.dist = (i < lim-1 ? d : -inf);
 rep(i,0,lim) for (Ed e : eds) {
   if (nodes[e.a].dist == -inf)
     nodes[e.b].dist = -inf;
```

#### FlovdWarshall.h

Time:  $\mathcal{O}(N^3)$ 

**Description:** Calculates all-pairs shortest path in a directed graph that might have negative edge weights. Input is an distance matrix m, where  $m[i][j] = \inf$  if i and j are not adjacent. As output, m[i][j] is set to the shortest distance between i and j, inf if no path, or -inf if the path goes through a negative-weight cycle.

```
const ll inf = 1LL << 62;
void floydWarshall(vector<vector<1l>>& m) {
   int n = sz(m);
   rep(i,0,n) m[i][i] = min(m[i][i], 0LL);
   rep(k,0,n) rep(i,0,n) rep(j,0,n)
   if (m[i][k] != inf && m[k][j] != inf) {
      auto newDist = max(m[i][k] + m[k][j], -inf);
      m[i][j] = min(m[i][j], newDist);
   }
   rep(k,0,n) if (m[k][k] < 0) rep(i,0,n) rep(j,0,n)
   if (m[i][k] != inf && m[k][j] != inf) m[i][j] = -inf;
}</pre>
```

#### TopoSort.h

**Description:** Topological sorting. Given is an oriented graph. Output is an ordering of vertices, such that there are edges only from left to right. If there are cycles, the returned list will have size smaller than n – nodes reachable from cycles will not be returned.

Time:  $\mathcal{O}\left(|V|+|E|\right)$ 

66a137, 14 lines

```
vi topoSort(const vector<vi>& gr) {
  vi indeg(sz(gr)), ret;
  for (auto& li : gr) for (int x : li) indeg[x]++;
  queue<int> q; // use priority_queue for lexic. largest ans.
  rep(i,0,sz(gr)) if (indeg[i] == 0) q.push(i);
  while (!q.empty()) {
    int i = q.front(); // top() for priority queue
    ret.push_back(i);
    q.pop();
    for (int x : gr[i])
        if (--indeg[x] == 0) q.push(x);
  }
  return ret;
}
```

#### TSP.h

return DP[M-1];

**Description:** Given a distance matrix, computes for all i < S = N - 1, the min. length of all paths visiting all nodes starting at S and ending at i. To compute the cyclic TSP-tour length use min{res $[i] + C[i][S] \mid i < S$ }.

rep(bs, 3, M) if(\_\_builtin\_popcount(bs) > 1) repbits(j, bs) {

```
Time: \mathcal{O}\left(2^{N}N^{2}\right). N=22\approx0.5s.

const int MAXS = 21; // 1 less than MAXN
using T = 11; // can also use double
const T INF = 1e12;
#define repbits(x, bs) for(int x, y = bs; \
    y && ((x = _builtin_ctz(y)) || y); y ^= y&-y)
T DP[1 << MAXS] [MAXS];
T * TSP(const vector<vector<T>>& C) {
    int S = sz(C)-1, M = 1 << S;
    rep(i, 0, S) DP[1 << i][i] = C[S][i];
```

int nbs = bs ^ (1 << j); T& r = DP[bs][j] = INF;</pre>

repbits(i, nbs) r = min(r, DP[nbs][i] + C[i][j]);

#### PushRelabel MinCostMaxFlow Dinics EdmondsKarp

## 7.2 Network flow

#### PushRelabel.h

**Description:** Push-relabel using the highest label selection rule and the gap heuristic. Quite fast in practice. To obtain the actual flow, look at positive values only.

```
Time: \mathcal{O}\left(V^2\sqrt{E}\right)
```

0ae1d4, 48 lines

```
struct PushRelabel {
  struct Edge {
   int dest, back;
   11 f, c;
  vector<vector<Edge>> g;
  vector<11> ec:
  vector<Edge*> cur;
  vector<vi> hs; vi H;
  PushRelabel(int n) : g(n), ec(n), cur(n), hs(2*n), H(n) {}
  void addEdge(int s, int t, ll cap, ll rcap=0) {
   if (s == t) return;
   g[s].push_back({t, sz(g[t]), 0, cap});
   g[t].push_back({s, sz(g[s])-1, 0, rcap});
  void addFlow(Edge& e, ll f) {
   Edge &back = g[e.dest][e.back];
   if (!ec[e.dest] && f) hs[H[e.dest]].push_back(e.dest);
   e.f += f; e.c -= f; ec[e.dest] += f;
   back.f -= f; back.c += f; ec[back.dest] -= f;
  11 calc(int s, int t) {
   int v = sz(q); H[s] = v; ec[t] = 1;
   vi co(2*v); co[0] = v-1;
    rep(i,0,v) cur[i] = g[i].data();
   for (Edge& e : g[s]) addFlow(e, e.c);
    for (int hi = 0;;) {
      while (hs[hi].empty()) if (!hi--) return -ec[s];
     int u = hs[hi].back(); hs[hi].pop_back();
     while (ec[u] > 0) // discharge u
       if (cur[u] == g[u].data() + sz(g[u])) {
         H[u] = 1e9;
          for (Edge& e : g[u]) if (e.c && H[u] > H[e.dest]+1)
           H[u] = H[e.dest]+1, cur[u] = &e;
          if (++co[H[u]], !--co[hi] && hi < v)</pre>
            rep(i, 0, v) if (hi < H[i] && H[i] < v)
              --co[H[i]], H[i] = v + 1;
        } else if (cur[u]->c && H[u] == H[cur[u]->dest]+1)
         addFlow(*cur[u], min(ec[u], cur[u]->c));
        else ++cur[u];
 bool leftOfMinCut(int a) { return H[a] >= sz(g); }
```

#### MinCostMaxFlow.h

**Description:** Min-cost max-flow. cap[i][j] != cap[j][i] is allowed; double edges are not. If costs can be negative, call setpi before maxflow, but note that negative cost cycles are not supported. To obtain the actual flow, look at positive values only.

Time: Approximately  $\mathcal{O}(E^2)$ 

fe85cc, 81 lines

```
#include <bits/extc++.h>
const ll INF = numeric_limits<ll>::max() / 4;
typedef vector<ll> VL;
struct MCMF {
```

```
int N;
vector<vi> ed, red;
vector<VL> cap, flow, cost;
vi seen;
VL dist, pi;
vector<pii> par;
MCMF (int N) .
  N(N), ed(N), red(N), cap(N, VL(N)), flow(cap), cost(cap),
  seen(N), dist(N), pi(N), par(N) {}
void addEdge(int from, int to, ll cap, ll cost) {
  this->cap[from][to] = cap;
  this->cost[from][to] = cost;
  ed[from].push_back(to);
  red[to].push_back(from);
void path(int s) {
  fill(all(seen), 0);
  fill(all(dist), INF);
  dist[s] = 0; ll di;
  __gnu_pbds::priority_queue<pair<ll, int>> q;
  vector<decltype(q)::point_iterator> its(N);
  q.push({0, s});
  auto relax = [&](int i, ll cap, ll cost, int dir) {
    11 val = di - pi[i] + cost;
    if (cap && val < dist[i]) {
      dist[i] = val;
      par[i] = \{s, dir\};
      if (its[i] == q.end()) its[i] = q.push({-dist[i], i});
      else q.modify(its[i], {-dist[i], i});
  };
  while (!q.empty()) {
    s = q.top().second; q.pop();
    seen[s] = 1; di = dist[s] + pi[s];
    for (int i : ed[s]) if (!seen[i])
      relax(i, cap[s][i] - flow[s][i], cost[s][i], 1);
    for (int i : red[s]) if (!seen[i])
      relax(i, flow[i][s], -cost[i][s], 0);
  rep(i, 0, N) pi[i] = min(pi[i] + dist[i], INF);
pair<11, 11> maxflow(int s, int t) {
  11 totflow = 0, totcost = 0;
  while (path(s), seen[t]) {
    11 fl = INF;
    for (int p,r,x = t; tie(p,r) = par[x], x != s; x = p)
      fl = min(fl, r ? cap[p][x] - flow[p][x] : flow[x][p]);
    totflow += fl:
    for (int p,r,x = t; tie(p,r) = par[x], x != s; x = p)
      if (r) flow[p][x] += fl;
      else flow[x][p] -= fl;
  rep(i, 0, N) rep(j, 0, N) totcost += cost[i][j] * flow[i][j];
  return {totflow, totcost};
// If some costs can be negative, call this before maxflow:
void setpi(int s) { // (otherwise, leave this out)
  fill(all(pi), INF); pi[s] = 0;
  int it = N, ch = 1; 11 v;
  while (ch-- && it--)
    rep(i,0,N) if (pi[i] != INF)
```

```
for (int to : ed[i]) if (cap[i][to])
    if ((v = pi[i] + cost[i][to]) < pi[to])
        pi[to] = v, ch = 1;
    assert(it >= 0); // negative cost cycle
}
};
```

#### Dinics.py

**Description:** Maximum flow. Very fast in practice. To obtain edge flows, compare input and output capacities and discard negative values.

```
Time: \mathcal{O}(V^2E). \mathcal{O}(\sqrt{V}E) for bipartite matching.
                                                             33 lines
import sys
from collections import defaultdict
sys.setrecursionlimit(1<<30)
def dinics(graph, s, t):
    assert s != t
    N = len(graph)
    cap = [defaultdict(int, adj) for adj in graph]
    def augment(i, of):
        if i == t: return of
        used = 0
        for j, c in cap[i].items():
            if level[j] != level[i] + 1 or not c: continue
            r = augment(j, min(of - used, c))
            used += r
            cap[i][j] -= r
            cap[j][i] += r
            if of == used: break
        return used
    flow = 0
    while True:
        level = [-1] * N
        level[s], Q = 0, [s]
        for i in Q:
            if i == t: break
            for j, c in cap[i].items():
                if level[j] == -1 and c:
                     level[j] = level[i] + 1
                     O.append(j)
```

#### EdmondsKarp.h

else: return flow, cap

flow += augment(s, float('inf'))

**Description:** Flow algorithm with guaranteed complexity  $O(VE^2)$ . To get edge flow values, compare capacities before and after, and take the positive values only.

482fe0, 35 lines

```
template < class T > T edmondsKarp(vector < unordered_map < int, T >> &
    graph, int source, int sink) {
  assert (source != sink);
 T flow = 0;
 vi par(sz(graph)), q = par;
  for (;;) {
    fill(all(par), -1);
    par[source] = 0;
    int ptr = 1;
    q[0] = source;
    rep(i,0,ptr) {
      int x = q[i];
      for (auto e : graph[x]) {
        if (par[e.first] == -1 && e.second > 0) {
          par[e.first] = x;
          q[ptr++] = e.first;
```

```
if (e.first == sink) goto out;
    }
}
return flow;
out:
    T inc = numeric_limits<T>::max();
    for (int y = sink; y != source; y = par[y])
        inc = min(inc, graph[par[y]][y]);

flow += inc;
    for (int y = sink; y != source; y = par[y]) {
        int p = par[y];
        if ((graph[p][y] -= inc) <= 0) graph[p].erase(y);
        graph[y][p] += inc;
    }
}</pre>
```

#### MinCut.h

**Description:** After running max-flow, the left side of a min-cut from s to t is given by all vertices reachable from s, only traversing edges with positive residual capacity.

#### GlobalMinCut.h

**Description:** Find a global minimum cut in an undirected graph, as represented by an adjacency matrix.

Time:  $\mathcal{O}\left(V^3\right)$  8b0e19, 21 lines

```
pair<int, vi> globalMinCut(vector<vi> mat) {
  pair<int, vi> best = {INT_MAX, {}};
  int n = sz(mat);
  vector<vi> co(n);
  rep(i, 0, n) co[i] = {i};
  rep(ph,1,n) {
   vi w = mat[0];
   size_t s = 0, t = 0;
    rep(it,0,n-ph) { //O(V^2) \rightarrow O(E log V) with prio. queue
      w[t] = INT_MIN;
      s = t, t = max\_element(all(w)) - w.begin();
      rep(i, 0, n) w[i] += mat[t][i];
   best = min(best, \{w[t] - mat[t][t], co[t]\});
   co[s].insert(co[s].end(), all(co[t]));
   rep(i, 0, n) mat[s][i] += mat[t][i];
   rep(i, 0, n) mat[i][s] = mat[s][i];
   mat[0][t] = INT_MIN;
 return best;
```

#### GomorvHu.h

**Description:** Given a list of edges representing an undirected flow graph, returns edges of the Gomory-Hu tree. The max flow between any pair of vertices is given by minimum edge weight along the Gomory-Hu tree path.

**Time:**  $\mathcal{O}(V)$  Flow Computations

```
return tree;
```

## 7.3 Matching

hopcroftKarp.h

**Description:** Fast bipartite matching algorithm. Graph g should be a list of neighbors of the left partition, and btoa should be a vector full of -1's of the same size as the right partition. Returns the size of the matching. btoa[i] will be the match for vertex i on the right side, or -1 if it's not matched. Usage: vi btoa(m, -1); hoperoftKarp(q, btoa);

```
Time: \mathcal{O}\left(\sqrt{V}E\right)
                                                      f612e4, 42 lines
bool dfs(int a, int L, vector<vi>& q, vi& btoa, vi& A, vi& B) {
 if (A[a] != L) return 0;
 A[a] = -1;
 for (int b : g[a]) if (B[b] == L + 1) {
   B[b] = 0;
    if (btoa[b] == -1 || dfs(btoa[b], L + 1, g, btoa, A, B))
      return btoa[b] = a, 1;
 return 0;
int hopcroftKarp(vector<vi>& g, vi& btoa) {
 int res = 0;
 vi A(g.size()), B(btoa.size()), cur, next;
 for (;;) {
   fill(all(A), 0);
   fill(all(B), 0);
    cur.clear();
    for (int a : btoa) if (a !=-1) A[a] = -1;
   rep(a, 0, sz(g)) if(A[a] == 0) cur.push_back(a);
    for (int lay = 1;; lay++) {
     bool islast = 0;
      next.clear();
      for (int a : cur) for (int b : g[a]) {
       if (btoa[b] == -1) {
          B[b] = lay;
          islast = 1;
        else if (btoa[b] != a && !B[b]) {
          B[b] = lav;
          next.push_back(btoa[b]);
      if (islast) break;
     if (next.empty()) return res;
      for (int a : next) A[a] = lay;
      cur.swap(next);
   rep(a, 0, sz(q))
      res += dfs(a, 0, g, btoa, A, B);
```

#### DFSMatching.h

**Description:** Simple bipartite matching algorithm. Graph g should be a list of neighbors of the left partition, and btoa should be a vector full of -1's of the same size as the right partition. Returns the size of the matching. btoa[i] will be the match for vertex i on the right side, or -1 if it's not matched.

```
Usage: vi btoa(m, -1); dfsMatching(g, btoa); Time: \mathcal{O}(VE)
```

```
bool find(int j, vector<vi>& g, vi& btoa, vi& vis) {
   if (btoa[j] == -1) return 1;
   vis[j] = 1; int di = btoa[j];
   for (int e : g[di])
```

```
if (!vis[e] && find(e, g, btoa, vis)) {
    btoa[e] = di;
    return 1;
}
return 0;
}
int dfsMatching(vector<vi>& g, vi& btoa) {
    vi vis;
    rep(i,0,sz(g)) {
       vis.assign(sz(btoa), 0);
       for (int j : g[i])
        if (find(j, g, btoa, vis)) {
          btoa[j] = i;
          break;
       }
}
return sz(btoa) - (int)count(all(btoa), -1);
}
```

#### MinimumVertexCover.h

**Description:** Finds a minimum vertex cover in a bipartite graph. The size is the same as the size of a maximum matching, and the complement is a maximum independent set.

```
"DFSMatching.h"
vi cover(vector<vi>& g, int n, int m) {
 vi match (m, -1);
 int res = dfsMatching(q, match);
  vector<bool> lfound(n, true), seen(m);
  for (int it : match) if (it != -1) lfound[it] = false;
  rep(i,0,n) if (lfound[i]) q.push_back(i);
  while (!q.empty()) {
    int i = q.back(); q.pop_back();
    lfound[i] = 1;
    for (int e : g[i]) if (!seen[e] && match[e] != -1) {
      seen[e] = true;
      q.push_back(match[e]);
 rep(i,0,n) if (!lfound[i]) cover.push_back(i);
 rep(i,0,m) if (seen[i]) cover.push_back(n+i);
 assert(sz(cover) == res);
 return cover;
```

#### WeightedMatching.h

**Description:** Given a weighted bipartite graph, matches every node on the left with a node on the right such that no nodes are in two matchings and the sum of the edge weights is minimal. Takes cost[N][M], where cost[i][j] = cost for L[i] to be matched with R[j] and returns (min cost, match), where L[i] is matched with R[match[i]]. Negate costs for max cost.

```
ep(1,1,n) {
  p[0] = i;
  int j0 = 0; // add "dummy" worker 0
  vi dist(m, INT_MAX), pre(m, -1);
  vector<bool> done(m + 1);
  do { // dijkstra
      done[j0] = true;
  int i0 = p[j0], j1, delta = INT_MAX;
  rep(j,1,m) if (!done[j]) {
      auto cur = a[i0 - 1][j - 1] - u[i0] - v[j];
      if (cur < dist[j]) dist[j] = cur, pre[j] = j0;
      if (dist[j] < delta) delta = dist[j], j1 = j;</pre>
```

2965e5, 33 lines

```
}
rep(j,0,m) {
    if (done[j]) u[p[j]] += delta, v[j] -= delta;
    else dist[j] -= delta;
}
j0 = j1;
} while (p[j0]);
while (j0) { // update alternating path
    int j1 = pre[j0];
    p[j0] = p[j1], j0 = j1;
}
rep(j,1,m) if (p[j]) ans[p[j] - 1] = j - 1;
return {-v[0], ans}; // min cost
```

#### GeneralMatching.h

**Description:** Matching for general graphs. Fails with probability N/mod. Time:  $\mathcal{O}(N^3)$ 

```
"../numerical/MatrixInverse-mod.h"
vector<pii> generalMatching(int N, vector<pii>& ed) {
  vector<vector<ll>> mat(N, vector<ll>(N)), A;
  for (pii pa : ed) {
   int a = pa.first, b = pa.second, r = rand() % mod;
   mat[a][b] = r, mat[b][a] = (mod - r) % mod;
  int r = matInv(A = mat), M = 2*N - r, fi, fj;
  assert (r % 2 == 0);
  if (M != N) do {
   mat.resize(M, vector<ll>(M));
   rep(i,0,N) {
     mat[i].resize(M);
     rep(j,N,M) {
       int r = rand() % mod;
       mat[i][j] = r, mat[j][i] = (mod - r) % mod;
  } while (matInv(A = mat) != M);
  vi has (M, 1); vector<pii> ret;
  rep(it,0,M/2) {
   rep(i,0,M) if (has[i])
     rep(j,i+1,M) if (A[i][j] && mat[i][j]) {
        fi = i; fj = j; goto done;
    } assert(0); done:
   if (fj < N) ret.emplace_back(fi, fj);</pre>
   has[fi] = has[fj] = 0;
    rep(sw, 0, 2) {
     11 a = modpow(A[fi][fj], mod-2);
     rep(i,0,M) if (has[i] && A[i][fj]) {
       ll b = A[i][fj] * a % mod;
        rep(j, 0, M) A[i][j] = (A[i][j] - A[fi][j] * b) % mod;
     swap(fi,fj);
  return ret;
```

#### StableMarriage.h

**Description:** Stable marriage, with A and B being  $N \times N$  matrices and A[a] being the ordered list of B's according to a's preference and similarly for B[b]. b and B-to\_A[b] are matched up.

```
Time: \mathcal{O}\left(N^2\right)
```

bfc6f2, 45 lines

```
using vvi = vector<vi>;
vvi prefer_rank(const vvi& B) {
```

```
int N = sz(B); vvi res(N, vi(N));
  rep(b, 0, N) rep(i, 0, N) res[b][B[b][i]] = i;
  return res;
vi stable_marriage(const vvi& A, const vvi& B) {
 int N = sz(A);
  queue<int> unmatched;
  vi B_{to}A(N, -1);
  auto prefer_B = prefer_rank(B);
  rep(a, 0, N) unmatched.push(a);
 while (!unmatched.empty()) {
    int a = unmatched.front();
    unmatched.pop();
    for(int b : A[a]) {
      int a2 = B_to_A[b];
      if (a2 == -1) {
        B_to_A[b] = a;
       break;
      if (prefer_B[b][a] < prefer_B[b][a2]) {
        unmatched.push(a2);
        B_to_A[b] = a;
        break;
  return B_to_A;
bool is_stable(const vvi& A, const vvi& B, const vi& B_to_A) {
 int N = sz(A);
  auto prefer_A = prefer_rank(A);
  auto prefer_B = prefer_rank(B);
  rep(b, 0, N) {
    int a = B_to_A[b];
    for(int b2 : A[a]) {
     if (b2 == b) break;
      if (prefer_A[a][b2] < prefer_A[a][b] &&</pre>
          prefer_B[b2][a] < prefer_B[b2][B_to_A[b2]])</pre>
        return false;
 return true;
```

## 7.4 DFS algorithms

#### SCC.h

**Description:** Finds strongly connected components in a directed graph. If vertices u,v belong to the same component, we can reach u from v and vice versa.

**Usage:**  $sc(graph, [\&](vi\& v) \{ ... \})$  visits all components in reverse topological order. comp[i] holds the component index of a node (a component only has edges to components with lower index). ncomps will contain the number of components.

```
ncomps++;
}
return val[j] = low;
}
template<class G, class F> void scc(G& g, F f) {
  int n = sz(g);
  val.assign(n, 0); comp.assign(n, -1);
  Time = ncomps = 0;
  rep(i,0,n) if (comp[i] < 0) dfs(i, g, f);
}</pre>
```

#### BiconnectedComponents.h

for each edge (a,b) {

ed[a].emplace\_back(b, eid);

Usage: int eid = 0; ed.resize(N);

**Description:** Finds all biconnected components in an undirected graph, and runs a callback for the edges in each. In a biconnected component there are at least two distinct paths between any two nodes. Note that a node can be in several components. An edge which is not in a component is a bridge, i.e., not part of any cycle.

```
ed[b].emplace_back(a, eid++); }
bicomps([&](const vi& edgelist) {...});
Time: \mathcal{O}\left(E+V\right)
vi num, st;
vector<vector<pii>> ed;
int Time:
template < class F>
int dfs(int at, int par, F& f) {
  int me = num[at] = ++Time, e, y, top = me;
  for (auto pa : ed[at]) if (pa.second != par) {
    tie(y, e) = pa;
    if (num[y]) {
      top = min(top, num[y]);
      if (num[y] < me)
        st.push back(e);
    } else {
      int si = sz(st);
      int up = dfs(y, e, f);
      top = min(top, up);
      if (up == me) {
        st.push_back(e);
        f(vi(st.begin() + si, st.end()));
        st.resize(si);
      else if (up < me) st.push_back(e);</pre>
      else { /* e is a bridge */ }
  return top;
template<class F>
void bicomps(F f) {
  num.assign(sz(ed), 0);
  rep(i,0,sz(ed)) if (!num[i]) dfs(i, -1, f);
```

#### 2sat.h

**Description:** Calculates a valid assignment to boolean variables a, b, c,... to a 2-SAT problem, so that an expression of the type (a|||b)&&(!a|||c)&&(d|||!b)&&... becomes true, or reports that it is unsatisfiable. Negated variables are represented by bit-inversions ( $\sim$ x).

```
Usage: TwoSat ts(number of boolean variables); ts.either(0, ~3); // Var 0 is true or var 3 is false ts.setValue(2); // Var 2 is true ts.atMostOne(\{0, \sim 1, 2\}); // <= 1 of vars 0, \sim 1 and 2 are true ts.solve(); // Returns true iff it is solvable ts.values[0..N-1] holds the assigned values to the vars
```

**Time:**  $\mathcal{O}(N+E)$ , where N is the number of boolean variables, and E is the number of clauses.

```
struct TwoSat {
 int N;
  vector<vi> gr;
  vi values; // 0 = false, 1 = true
  TwoSat(int n = 0) : N(n), gr(2*n) {}
  int addVar() { // (optional)
   gr.emplace back();
   gr.emplace_back();
    return N++;
  void either(int f, int j) {
   f = \max(2 * f, -1 - 2 * f);
   j = \max(2 * j, -1 - 2 * j);
   gr[f].push_back(j^1);
   gr[j].push_back(f^1);
  void setValue(int x) { either(x, x); }
  void atMostOne(const vi& li) { // (optional)
   if (sz(li) <= 1) return;</pre>
   int cur = ~li[0];
    rep(i,2,sz(li)) {
     int next = addVar();
     either(cur, ~li[i]);
     either(cur, next);
     either(~li[i], next);
     cur = ~next;
    either(cur, ~li[1]);
  vi val, comp, z; int time = 0;
  int dfs(int i) {
   int low = val[i] = ++time, x; z.push_back(i);
   for(int e : gr[i]) if (!comp[e])
     low = min(low, val[e] ?: dfs(e));
   if (low == val[i]) do {
     x = z.back(); z.pop_back();
     comp[x] = low;
     if (values[x>>1] == -1)
       values[x>>1] = x&1;
    } while (x != i);
   return val[i] = low;
  bool solve() {
   values.assign(N, -1);
   val.assign(2*N, 0); comp = val;
   rep(i,0,2*N) if (!comp[i]) dfs(i);
   rep(i,0,N) if (comp[2*i] == comp[2*i+1]) return 0;
    return 1;
```

#### EulerWalk.h

};

Description: Eulerian undirected/directed path/cycle algorithm. Input should be a vector of (dest, global edge index), where for undirected graphs, forward/backward edges have the same index. Returns a list of nodes in the Eulerian path/cycle with src at both start and end, or empty list if no cycle/path exists. To get edge indices back, add .second to s and ret.

Time:  $\mathcal{O}(V+E)$ 780b64, 15 lines

```
vi eulerWalk(vector<vector<pii>>& gr, int nedges, int src=0) {
  int n = sz(qr);
```

```
vi D(n), its(n), eu(nedges), ret, s = {src};
D[src]++; // to allow Euler paths, not just cycles
while (!s.empty()) {
  int x = s.back(), y, e, &it = its[x], end = sz(gr[x]);
  if (it == end) { ret.push_back(x); s.pop_back(); continue; }
  tie(y, e) = gr[x][it++];
  if (!eu[e]) {
   D[x] --, D[y] ++;
    eu[e] = 1; s.push_back(y);
for (int x : D) if (x < 0 \mid | sz(ret) != nedges+1) return \{\};
return {ret.rbegin(), ret.rend()};
```

## 7.5 Coloring

EdgeColoring.h

**Description:** Given a simple, undirected graph with max degree D, computes a (D+1)-coloring of the edges such that no neighboring edges share a color. (D-coloring is NP-hard, but can be done for bipartite graphs by repeated matchings of max-degree nodes.)

```
Time: \mathcal{O}(NM)
                                                     e210e2, 31 lines
vi edgeColoring(int N, vector<pii> eds) {
 vi cc(N + 1), ret(sz(eds)), fan(N), free(N), loc;
 for (pii e : eds) ++cc[e.first], ++cc[e.second];
 int u, v, ncols = *max_element(all(cc)) + 1;
 vector<vi> adj(N, vi(ncols, -1));
 for (pii e : eds) {
   tie(u, v) = e;
   fan[0] = v;
   loc.assign(ncols, 0);
    int at = u, end = u, d, c = free[u], ind = 0, i = 0;
    while (d = free[v], !loc[d] && (v = adj[u][d]) != -1)
     loc[d] = ++ind, cc[ind] = d, fan[ind] = v;
    cc[loc[d]] = c;
    for (int cd = d; at != -1; cd ^= c ^ d, at = adj[at][cd])
     swap(adj[at][cd], adj[end = at][cd ^ c ^ d]);
    while (adj[fan[i]][d] != -1) {
     int left = fan[i], right = fan[++i], e = cc[i];
     adj[u][e] = left;
     adj[left][e] = u;
     adj[right][e] = -1;
     free[right] = e;
   adj[u][d] = fan[i];
   adj[fan[i]][d] = u;
   for (int y : {fan[0], u, end})
     for (int& z = free[y] = 0; adj[y][z] != -1; z++);
 rep(i, 0, sz(eds))
   for (tie(u, v) = eds[i]; adj[u][ret[i]] != v;) ++ret[i];
 return ret;
```

## VertexColoring.h

**Description:** Calculates the chromatic number of an undirected graph. Time:  $\mathcal{O}(2.4422^n)$ 

```
"MaximalCliques.h"
                                                        011734, 17 lines
const int MAXN = 15;
int memo[1<<MAXN] = {0};</pre>
int aux(vector<B>& eds, B nodes) {
 auto k = nodes.to_ulong();
 if (!nodes.any()) return 0;
 if (memo[k]) return memo[k];
 int r = MAXN;
 cliques(eds, [&](B x) {
    r = min(r, 1 + aux(eds, nodes & \sim x));
 }, nodes);
```

```
return memo[k] = r;
int chromaticNumber(vector<B>& eds) {
 vector<B> comp;
 rep(i, 0, sz(eds)) comp.push_back((~eds[i]).reset(i));
 return aux(comp, (1 << sz(comp)) - 1);</pre>
```

#### 7.6 Heuristics

MaximalCliques.h

**Description:** Runs a callback for all maximal cliques in a graph (given as a symmetric bitset matrix; self-edges not allowed). Callback is given a bitset representing the maximal clique. For a subgraph provide vertices in P.

**Time:**  $\mathcal{O}\left(3^{n/3}\right)$ , much faster for sparse graphs

d7d308, 13 lines

```
typedef bitset<128> B;
template < class F>
void cliques (vector B \ge a eds, F f, B P = B \ge a (), B X={}, B R={}) {
  if (!P.any()) { if (!X.any()) f(R); return; }
  auto q = (P | X)._Find_first();
  auto cands = P & ~eds[q];
  for (int i = cands. Find first(); i < sz(eds);</pre>
   i = cands._Find_next(i)) {
   R[i] = 1;
    cliques(eds, f, P & eds[i], X & eds[i], R);
    R[i] = P[i] = 0; X[i] = 1;
```

#### MaximumClique.h

Description: Quickly finds a maximum clique of a graph (given as symmetric bitset matrix; self-edges not allowed). Can be used to find a maximum independent set by finding a clique of the complement graph.

Time: Runs in about 1s for n=155 and worst case random graphs (p=.90).

```
Runs faster for sparse graphs.
                                                     f7c0bc, 49 lines
typedef vector<bitset<200>> vb;
struct Maxclique {
  double limit=0.025, pk=0;
 struct Vertex { int i, d=0; };
 typedef vector<Vertex> vv;
  vb e:
 vv V;
  vector<vi> C;
 vi qmax, q, S, old;
  void init(vv& r) {
    for (auto& v : r) v.d = 0;
    for (auto& v : r) for (auto j : r) v.d += e[v.i][j.i];
    sort(all(r), [](auto a, auto b) { return a.d > b.d; });
    int mxD = r[0].d;
    rep(i, 0, sz(r)) r[i].d = min(i, mxD) + 1;
 void expand(vv& R, int lev = 1) {
    S[lev] += S[lev - 1] - old[lev];
    old[lev] = S[lev - 1];
    while (sz(R)) {
      if (sz(q) + R.back().d <= sz(qmax)) return;</pre>
      g.push back(R.back().i);
      vv T:
      for(auto v:R) if (e[R.back().i][v.i]) T.push_back({v.i});
      if (sz(T)) {
        if (S[lev]++ / ++pk < limit) init(T);
        int j = 0, mxk = 1, mnk = max(sz(qmax) - sz(q) + 1, 1);
        C[1].clear(), C[2].clear();
        for (auto v : T) {
          int k = 1;
          auto f = [&](int i) { return e[v.i][i]; };
```

while (any\_of(all(C[k]), f)) k++;

```
if (k > mxk) mxk = k, C[mxk + 1].clear();
    if (k < mnk) T[j++].i = v.i;
        C[k].push_back(v.i);
    }
    if (j > 0) T[j - 1].d = 0;
        rep(k,mnk,mxk + 1) for (int i : C[k])
        T[j].i = i, T[j++].d = k;
        expand(T, lev + 1);
    } else if (sz(q) > sz(qmax)) qmax = q;
    q.pop_back(), R.pop_back();
}
vi maxClique() { init(V), expand(V); return qmax; }
Maxclique(vb conn) : e(conn), C(sz(e)+1), S(sz(C)), old(S) {
        rep(i,0,sz(e)) V.push_back({i});
}
};
```

#### MaximumIndependentSet.h

**Description:** To obtain a maximum independent set of a graph, find a max clique of the complement. If the graph is bipartite, see MinimumVertex-Cover

#### 7.7 Trees

#### BinaryLifting.h

**Description:** Calculate power of two jumps in a tree, to support fast upward jumps and LCAs. Assumes the root node points to itself.

**Time:** construction  $\mathcal{O}(N \log N)$ , queries  $\mathcal{O}(\log N)$ 

bfce85, 25 lines

```
vector<vi> treeJump(vi& P){
  int on = 1, d = 1;
  while (on < sz(P)) on *= 2, d++;
  vector<vi> jmp(d, P);
  rep(i,1,d) rep(j,0,sz(P))
    jmp[i][j] = jmp[i-1][jmp[i-1][j]];
  return jmp;
int jmp(vector<vi>& tbl, int nod, int steps) {
  rep(i, 0, sz(tbl))
    if(steps&(1<<i)) nod = tbl[i][nod];
  return nod;
int lca(vector<vi>& tbl, vi& depth, int a, int b) {
  if (depth[a] < depth[b]) swap(a, b);</pre>
  a = jmp(tbl, a, depth[a] - depth[b]);
  if (a == b) return a;
  for (int i = sz(tbl); i--;) {
   int c = tbl[i][a], d = tbl[i][b];
    if (c != d) a = c, b = d;
  return tbl[0][a];
```

#### LCA.h

**Description:** Data structure for computing lowest common ancestors in a tree (with 0 as root). C should be an adjacency list of the tree, either directed or undirected.

Time:  $\mathcal{O}(N \log N + Q)$ 

```
void dfs(vector<vi>& C, int v, int par) {
   time[v] = T++;
   for (int y : C[v]) if (y != par) {
      path.push_back(v), ret.push_back(time[v]);
      dfs(C, y, v);
   }
}

int lca(int a, int b) {
   if (a == b) return a;
   tie(a, b) = minmax(time[a], time[b]);
   return path[rmq.query(a, b)];
}

//dist(a,b){return depth[a] + depth[b] - 2*depth[lca(a,b)];}
};
```

#### CompressTree.h

**Description:** Given a rooted tree and a subset S of nodes, compute the minimal subtree that contains all the nodes by adding all (at most |S|-1) pairwise LCA's and compressing edges. Returns a list of (par, orig\_index) representing a tree rooted at 0. The root points to itself.

Time:  $\mathcal{O}\left(|S|\log|S|\right)$ 

```
"LCA.h"
                                                     9775a0, 21 lines
typedef vector<pair<int, int>> vpi;
vpi compressTree(LCA& lca, const vi& subset) {
 static vi rev; rev.resize(sz(lca.time));
 vi li = subset, &T = lca.time;
 auto cmp = [&](int a, int b) { return T[a] < T[b]; };</pre>
 sort(all(li), cmp);
 int m = sz(1i)-1;
 rep(i,0,m) {
   int a = li[i], b = li[i+1];
   li.push_back(lca.lca(a, b));
 sort(all(li), cmp);
 li.erase(unique(all(li)), li.end());
 rep(i, 0, sz(li)) rev[li[i]] = i;
 vpi ret = {pii(0, li[0])};
 rep(i, 0, sz(li) - 1)  {
   int a = li[i], b = li[i+1];
   ret.emplace_back(rev[lca.lca(a, b)], b);
 return ret;
```

#### HLD.h

**Description:** Decomposes a tree into vertex disjoint heavy paths and light edges such that the path from any leaf to the root contains at most  $\log(n)$  light edges. Code does additive modifications and max queries, but can support commutative segtree modifications/queries on paths and subtrees. Takes as input the full adjacency list. VALS\_EDGES being true means that values are stored in the edges, as opposed to the nodes. All values initialized to the segtree default. Root must be 0.

```
Time: \mathcal{O}\left((\log N)^2\right)
```

dfsSz(u);

"../data-structures/LazySegmentTree.h"

```
template <bool VALS_EDGES> struct HLD {
   int N, tim = 0;
   vector<vi> adj;
   vi par, siz, depth, rt, pos;
   Node *tree;
   HLD(vector<vi> adj_)
   : N(sz(adj_)), adj(adj_), par(N, -1), siz(N, 1), depth(N),
        rt(N),pos(N),tree(new Node(0, N)) { dfsSz(0); dfsHld(0); }
   void dfsSz(int v) {
        if (par[v] != -1) adj[v].erase(find(all(adj[v]), par[v]));
        for (int& u : adj[v]) {
            par[u] = v, depth[u] = depth[v] + 1;
        }
}
```

```
if (siz[u] > siz[adj[v][0]]) swap(u, adj[v][0]);
 void dfsHld(int v) {
   pos[v] = tim++;
    for (int u : adj[v]) {
     rt[u] = (u == adj[v][0] ? rt[v] : u);
     dfsHld(u);
 template <class B> void process(int u, int v, B op) {
    for (; rt[u] != rt[v]; v = par[rt[v]]) {
     if (depth[rt[u]] > depth[rt[v]]) swap(u, v);
     op(pos[rt[v]], pos[v] + 1);
   if (depth[u] > depth[v]) swap(u, v);
    op(pos[u] + VALS_EDGES, pos[v] + 1);
 void modifyPath(int u, int v, int val) {
   process(u, v, [&](int l, int r) { tree->add(l, r, val); });
 int queryPath(int u, int v) { // Modify depending on problem
   int res = -1e9;
   process(u, v, [&](int l, int r) {
       res = max(res, tree->query(1, r));
    return res;
 int querySubtree(int v) { // modifySubtree is similar
    return tree->query(pos[v] + VALS_EDGES, pos[v] + siz[v]);
};
```

#### LinkCutTree.h

6f34db, 46 lines

**Description:** Represents a forest of unrooted trees. You can add and remove edges (as long as the result is still a forest), and check whether two nodes are in the same tree.

**Time:** All operations take amortized  $\mathcal{O}(\log N)$ .

5000e2 00 lines

```
struct Node { // Splay tree. Root's pp contains tree's parent.
 Node *p = 0, *pp = 0, *c[2];
 bool flip = 0;
 Node() { c[0] = c[1] = 0; fix(); }
  void fix() {
   if (c[0]) c[0]->p = this;
    if (c[1]) c[1]->p = this;
    // (+ update sum of subtree elements etc. if wanted)
  void pushFlip() {
    if (!flip) return;
    flip = 0; swap(c[0], c[1]);
    if (c[0]) c[0]->flip ^= 1;
    if (c[1]) c[1]->flip ^= 1;
  int up() { return p ? p->c[1] == this : -1; }
  void rot(int i, int b) {
    int h = i ^ b;
    Node *x = c[i], *y = b == 2 ? x : x -> c[h], *z = b ? y : x;
    if ((y->p = p)) p->c[up()] = y;
    c[i] = z -> c[i ^ 1];
    if (b < 2) {
      x->c[h] = y->c[h ^ 1];
      z \rightarrow c[h ^1] = b ? x : this;
    y - c[i ^1] = b ? this : x;
    fix(); x->fix(); y->fix();
    if (p) p->fix();
    swap(pp, y->pp);
```

```
void splay() {
    for (pushFlip(); p; ) {
      if (p->p) p->p->pushFlip();
     p->pushFlip(); pushFlip();
      int c1 = up(), c2 = p->up();
     if (c2 == -1) p->rot(c1, 2);
     else p->p->rot(c2, c1 != c2);
  Node* first() {
   pushFlip();
    return c[0] ? c[0]->first() : (splay(), this);
};
struct LinkCut {
  vector<Node> node;
  LinkCut(int N) : node(N) {}
  void link(int u, int v) { // add an edge (u, v)
    assert(!connected(u, v));
   makeRoot(&node[u]);
   node[u].pp = &node[v];
  void cut(int u, int v) { // remove an edge (u, v)
   Node *x = &node[u], *top = &node[v];
   makeRoot(top); x->splay();
    assert(top == (x->pp ?: x->c[0]));
    if (x->pp) x->pp = 0;
     x->c[0] = top->p = 0;
     x \rightarrow fix();
  bool connected (int u, int v) { // are u, v in the same tree?
   Node* nu = access(&node[u])->first();
    return nu == access(&node[v])->first();
  void makeRoot (Node* u) {
    access(u);
    u->splav();
    if(u->c[0]) {
     u - > c[0] - > p = 0;
     u - c[0] - flip ^= 1;
     u - c[0] - pp = u;
     u - > c[0] = 0;
     u->fix();
  Node* access(Node* u) {
   u->splav();
    while (Node* pp = u->pp) {
     pp->splay(); u->pp = 0;
     if (pp->c[1]) {
       pp - c[1] - p = 0; pp - c[1] - pp = pp; 
     pp->c[1] = u; pp->fix(); u = pp;
    return u:
};
```

## DirectedMST.h

**Description:** Finds a minimum spanning tree/arborescence of a directed graph, given a root node. If no MST exists, returns -1.

```
Time: \mathcal{O}\left(E\log V\right)
```

"../data-structures/UnionFindRollback.h"

39e620, 60 lines

```
struct Edge { int a, b; ll w; };
struct Node {
```

```
Edge kev;
  Node *1, *r;
 11 delta;
 void prop() {
   kev.w += delta;
   if (1) 1->delta += delta;
   if (r) r->delta += delta;
   delta = 0;
 Edge top() { prop(); return key; }
Node *merge(Node *a, Node *b) {
 if (!a || !b) return a ?: b;
 a->prop(), b->prop();
 if (a->key.w > b->key.w) swap(a, b);
 swap(a->1, (a->r = merge(b, a->r)));
 return a;
void pop(Node*\& a) { a->prop(); a = merge(a->1, a->r); }
pair<ll, vi> dmst(int n, int r, vector<Edge>& g) {
 RollbackUF uf(n);
 vector<Node*> heap(n);
  for (Edge e : q) heap[e.b] = merge(heap[e.b], new Node{e});
 11 \text{ res} = 0;
 vi seen(n, -1), path(n), par(n);
  seen[r] = r;
  vector<Edge> Q(n), in(n, \{-1,-1\}), comp;
  deque<tuple<int, int, vector<Edge>>> cycs;
  rep(s, 0, n) {
    int u = s, qi = 0, w;
    while (seen[u] < 0) {</pre>
     if (!heap[u]) return {-1,{}};
      Edge e = heap[u]->top();
      heap[u]->delta -= e.w, pop(heap[u]);
      Q[qi] = e, path[qi++] = u, seen[u] = s;
      res += e.w, u = uf.find(e.a);
      if (seen[u] == s) {
       Node \star cvc = 0;
       int end = qi, time = uf.time();
        do cyc = merge(cyc, heap[w = path[--qi]]);
        while (uf.join(u, w));
       u = uf.find(u), heap[u] = cyc, seen[u] = -1;
        cycs.push front({u, time, {&O[qi], &O[end]}});
    rep(i, 0, qi) in[uf.find(0[i].b)] = 0[i];
  for (auto& [u,t,comp] : cycs) { // restore sol (optional)
   uf.rollback(t);
    Edge inEdge = in[u];
    for (auto& e : comp) in[uf.find(e.b)] = e;
    in[uf.find(inEdge.b)] = inEdge;
 rep(i,0,n) par[i] = in[i].a;
  return {res, par};
```

## 7.8 Tree diameter

Given an unrooted tree the longest path between two nodes in the tree is the diameter. To find it do BFS from an arbitrary node and pick a node farthest away. Do BFS from this node and the size of the longest path is the diameter.

## 7.9 Math

## 7.9.1 Number of Spanning Trees

Create an  $N \times N$  matrix mat, and for each edge  $a \to b \in G$ , do mat[a][b]--, mat[b][b]++ (and mat[b][a]--, mat[a][a]++ if G is undirected). Remove the ith row and column and take the determinant; this yields the number of directed pranting transprotected at i (if G is undirected, remove any row column) with node degrees  $d_1 \ge \cdots \ge d_n$  exists iff  $d_1 + \cdots + d_n$  is even and for every  $k = 1 \dots n$ .

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k).$$

# Geometry (8)

## 8.1 Geometric primitives

#### Point.h

**Description:** Class to handle points in the plane. T can be e.g. double or long long. (Avoid int.)

```
template \langle class T \rangle int sgn(T x) \{ return (x > 0) - (x < 0); \}
template<class T>
struct Point {
 typedef Point P;
  explicit Point (T x=0, T y=0) : x(x), y(y) {}
 bool operator<(P p) const { return tie(x,y) < tie(p.x,p.y); }</pre>
 bool operator==(P p) const { return tie(x,y)==tie(p.x,p.y); }
 P operator+(P p) const { return P(x+p.x, y+p.y); }
  P operator-(P p) const { return P(x-p.x, y-p.y); }
 P operator*(T d) const { return P(x*d, y*d); }
  P operator/(T d) const { return P(x/d, y/d); }
 T dot(P p) const { return x*p.x + y*p.y; }
 T cross(P p) const { return x*p.y - y*p.x; }
 T cross(P a, P b) const { return (a-*this).cross(b-*this);
 T dist2() const { return x*x + y*y; }
  double dist() const { return sqrt((double)dist2()); }
  // angle to x-axis in interval [-pi, pi]
  double angle() const { return atan2(y, x); }
 P unit() const { return *this/dist(); } // makes dist()=1
 P perp() const { return P(-y, x); } // rotates +90 degrees
 P normal() const { return perp().unit(); }
  // returns point rotated 'a' radians ccw around the origin
  P rotate (double a) const {
    return P(x*cos(a)-v*sin(a),x*sin(a)+v*cos(a)); }
 friend ostream& operator<<(ostream& os, P p) {</pre>
    return os << "(" << p.x << "," << p.y << ")"; }
};
```

lineDistance.h

#### Description:

Returns the signed distance between point p and the line containing points a and b. Positive value on left side and negative on right as seen from a towards b. a==b gives nan. P is supposed to be Point<T> or Point3D<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long. Using Point3D will always give a non-negative distance. For Point3D, call .dist on the result of the cross product.



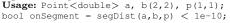
f6bf6b, 4 lines

```
template<class P>
double lineDist(const P& a, const P& b, const P& p) {
  return (double) (b-a).cross(p-a)/(b-a).dist();
}
```

### SegmentDistance.h

#### Description:

Returns the shortest distance between point p and the line segment from point s to e.



nt.h" 5c88f4, 6 lines

```
typedef Point<double> P;
double segDist(P& s, P& e, P& p) {
   if (s==e) return (p-s).dist();
   auto d = (e-s).dist2(), t = min(d,max(.0,(p-s).dot(e-s)));
   return ((p-s)*d-(e-s)*t).dist()/d;
}
```

#### SegmentIntersection.h

#### Description:

If a unique intersection point between the line segments going from s1 to e1 and from s2 to e2 exists then it is returned. If no intersection point exists an empty vector is returned. If infinitely many exist a vector with 2 elements is returned, containing the endpoints of the common line segment. The wrong position will be returned if P is Point<1|> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or long long.



```
Usage: vector<P> inter = segInter(s1,e1,s2,e2);
if (sz(inter)==1)
cout << "segments intersect at " << inter[0] << endl;
"Point.h", "OnSegment.h"

9d57f
```

```
template < class P > vector < P > segInter (P a, P b, P c, P d) {
   auto oa = c.cross(d, a), ob = c.cross(d, b),
        oc = a.cross(b, c), od = a.cross(b, d);
   // Checks if intersection is single non-endpoint point.
   if (sgn(oa) * sgn(ob) < 0 && sgn(oc) * sgn(od) < 0)
        return {(a * ob - b * oa) / (ob - oa)};
        set < P > s;
   if (onSegment(c, d, a)) s.insert(a);
   if (onSegment(a, b, c)) s.insert(c);
   if (onSegment(a, b, d)) s.insert(d);
   return {all(s)};
```

#### SegmentSegmentDistance.h

**Description:** Returns the shortest distance between two segments.

```
"SegmentDistance.h", "SegmentIntersection.h" 119f3e, 7 lines
double segsegDist(P s1, P e1, P s2, P e2) {
   if (!segInter(s1, e1, s2, e2).empty()) return 0;
   return min(segDist(s1, e1, s2),
        min(segDist(s1, e1, e2),
        min(segDist(s2, e2, s1),
        segDist(s2, e2, e1))));
}
```

#### lineIntersection.h

#### Description:

If a unique intersection point of the lines going through s1,e1 and s2,e2 exists {1, point} is returned. If no intersection point exists {0, (0,0)} is returned and if infinitely many exists {-1, (0,0)} is returned. The wrong position will be returned if P is Point<||scale="1">| Point<||scale="1">| Point<||scale="1">| Point<||scale="1">| S1 mediate steps so watch out for overflow if using int or ll.

Usage: auto res = lineInter(s1,e1,s2,e2);



#### sideOf.h

**Description:** Returns where p is as seen from s towards e.  $1/0/-1 \Leftrightarrow \text{left/on}$  line/right. If the optional argument eps is given 0 is returned if p is within distance eps from the line. P is supposed to be Point<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long.

#### OnSegment.h

**Description:** Returns true iff p lies on the line segment from s to e. Use (seqDist(s,e,p)<=epsilon) instead when using Point<double>.

```
"Point.h" c597e8, 3 lines
template<class P> bool onSegment(P s, P e, P p) {
   return p.cross(s, e) == 0 && (s - p).dot(e - p) <= 0;
}</pre>
```

# linearTransformation.h Description:

Apply the linear transformation (translation, rotation and scaling) which takes line p0-p1 to line q0-q1 to point r.



```
typedef Point<double> P;
P linearTransformation(const P& p0, const P& p1,
    const P& q0, const P& q1, const P& r) {
    P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));
    return q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.dist2();
}
```

#### Angle.h

"Point.h"

**Description:** A class for ordering angles (as represented by int points and a number of rotations around the origin). Useful for rotational sweeping. Sometimes also represents points or vectors.

```
Usage: vector<Angle> v = \{w[0], w[0].t360() ...\}; // sorted
int j = 0; rep(i,0,n) { while (v[j] < v[i].t180()) ++j; }
// sweeps j such that (j-i) represents the number of positively
oriented triangles with vertices at 0 and i
struct Angle {
 int x, v;
 int t:
  Angle(int x, int y, int t=0) : x(x), y(y), t(t) {}
 Angle operator-(Angle b) const { return {x-b.x, y-b.y, t}; }
  int half() const {
    assert(x || v);
    return y < 0 || (y == 0 && x < 0);
 Angle t90() const { return \{-y, x, t + (half() \&\& x >= 0)\}; \}
  Angle t180() const { return {-x, -y, t + half()}; }
 Angle t360() const { return {x, y, t + 1}; }
bool operator < (Angle a, Angle b) {
  // add a. dist2() and b. dist2() to also compare distances
 return make_tuple(a.t, a.half(), a.y * (11)b.x) <</pre>
         make tuple(b.t, b.half(), a.x \star (ll)b.v);
// Given two points, this calculates the smallest angle between
// them, i.e., the angle that covers the defined line segment.
pair<Angle, Angle> segmentAngles(Angle a, Angle b) {
 if (b < a) swap(a, b);
  return (b < a.t180() ?
          make_pair(a, b) : make_pair(b, a.t360()));
Angle operator+(Angle a, Angle b) { // point \ a + vector \ b
 Angle r(a.x + b.x, a.y + b.y, a.t);
 if (a.t180() < r) r.t--;
 return r.t180() < a ? r.t360() : r;
Angle angleDiff(Angle a, Angle b) { // angle b - angle a}
 int tu = b.t - a.t; a.t = b.t;
 return {a.x*b.x + a.y*b.y, a.x*b.y - a.y*b.x, tu - (b < a)};</pre>
```

#### 8.2 Circles

#### CircleIntersection.h

**Description:** Computes the pair of points at which two circles intersect. Returns false in case of no intersection.

#### CircleTangents.h

**Description:** Finds the external tangents of two circles, or internal if r2 is negated. Can return 0, 1, or 2 tangents – 0 if one circle contains the other (or overlaps it, in the internal case, or if the circles are the same); 1 if the circles are tangent to each other (in which case .first = .second and the tangent line is perpendicular to the line between the centers). .first and .second give the tangency points at circle 1 and 2 respectively. To find the tangents of a circle with a point set r2 to 0.

"Point.h" b0153d, 13 lines

template<class P>

```
vector<pair<P, P>> tangents(P c1, double r1, P c2, double r2) {
  P d = c2 - c1;
  double dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr * dr;
  if (d2 == 0 || h2 < 0) return {};
  vector<pair<P, P>> out;
  for (double sign : {-1, 1}) {
    P v = (d * dr + d.perp() * sqrt(h2) * sign) / d2;
    out.push_back({c1 + v * r1, c2 + v * r2});
  }
  if (h2 == 0) out.pop_back();
  return out;
}
```

#### CirclePolygonIntersection.h

**Description:** Returns the area of the intersection of a circle with a ccw polygon.

Time:  $\mathcal{O}\left(n\right)$ 

```
"../../content/geometry/Point.h"
                                                      a1ee63, 19 lines
typedef Point < double > P;
#define arg(p, q) atan2(p.cross(q), p.dot(q))
double circlePoly(P c, double r, vector<P> ps) {
  auto tri = [&] (P p, P q) {
   auto r2 = r * r / 2;
   P d = q - p;
   auto a = d.dot(p)/d.dist2(), b = (p.dist2()-r*r)/d.dist2();
   auto det = a * a - b;
   if (det <= 0) return arg(p, q) * r2;</pre>
   auto s = max(0., -a-sqrt(det)), t = min(1., -a+sqrt(det));
   if (t < 0 || 1 <= s) return arg(p, q) * r2;</pre>
   Pu = p + d * s, v = p + d * t;
   return arg(p,u) * r2 + u.cross(v)/2 + arg(v,q) * r2;
  auto sum = 0.0;
  rep(i, 0, sz(ps))
   sum += tri(ps[i] - c, ps[(i + 1) % sz(ps)] - c);
  return sum;
```

#### circumcircle.h

#### Description:

The circumcirle of a triangle is the circle intersecting all three vertices. ccRadius returns the radius of the circle going through points A, B and C and ccCenter returns the center of the same circle.

"Point.h"



```
typedef Point<double> P;
double ccRadius(const P& A, const P& B, const P& C) {
  return (B-A).dist()*(C-B).dist()*(A-C).dist()/
      abs((B-A).cross(C-A))/2;
}
P ccCenter(const P& A, const P& B, const P& C) {
  P b = C-A, c = B-A;
  return A + (b*c.dist2()-c*b.dist2()).perp()/b.cross(c)/2;
```

#### MinimumEnclosingCircle.h

**Description:** Computes the minimum circle that encloses a set of points. **Time:** expected  $\mathcal{O}(n)$ 

```
r = (o - ps[i]).dist();
rep(k,0,j) if ((o - ps[k]).dist() > r * EPS) {
    o = ccCenter(ps[i], ps[j], ps[k]);
    r = (o - ps[i]).dist();
}
}
return {o, r};
```

## 8.3 Polygons

#### InsidePolygon.h

**Description:** Returns true if p lies within the polygon. If strict is true, it returns false for points on the boundary. The algorithm uses products in intermediate steps so watch out for overflow.

```
Usage: vector < P > v = \{P\{4,4\}, P\{1,2\}, P\{2,1\}\}; bool in = inPolygon(v, P{3, 3}, false);

Time: \mathcal{O}(n)
```

"Point.h", "OnSegment.h", "SegmentDistance.h" 2bf504, 11 lines

```
template < class P >
bool inPolygon (vector < P > &p, P a, bool strict = true) {
  int cnt = 0, n = sz(p);
  rep(i,0,n) {
    P q = p[(i + 1) % n];
    if (onSegment(p[i], q, a)) return !strict;
    //or: if (segDist(p[i], q, a) <= eps) return !strict;
    cnt ^= ((a.y < p[i].y) - (a.y < q.y)) * a.cross(p[i], q) > 0;
  }
  return cnt;
```

#### PolygonArea.h

**Description:** Returns twice the signed area of a polygon. Clockwise enumeration gives negative area. Watch out for overflow if using int as T! For a regular n-gon with side length s use  $ns^2/(4\tan(\pi/n))$ .

#### PolygonCenter.h

**Description:** Returns the center of mass for a polygon.

#### Time: $\mathcal{O}(n)$

## PolygonCut.h

#### Description:

Returns a vector with the vertices of a polygon with everything to the left of the line going from s to e cut away.

```
Usage: vector<P> p = ...;
p = polygonCut(p, P(0,0), P(1,0));
"Point.h", "lineIntersection.h"
```

f2b7d4, 13 lines

```
typedef Point<double> P;
vector<P> polygonCut(const vector<P>& poly, P s, P e) {
```

```
vector<P> res;
rep(i,0,sz(poly)) {
  P cur = poly[i], prev = i ? poly[i-1] : poly.back();
  bool side = s.cross(e, cur) < 0;
  if (side != (s.cross(e, prev) < 0))
    res.push_back(lineInter(s, e, cur, prev).second);
  if (side)
    res.push_back(cur);
}
return res;
}</pre>
```

#### ConvexHull.h

#### Description:

"Point.h"

Returns a vector of the points of the convex hull in counterclockwise order. Points on the edge of the hull between two other points are not considered part of the hull. Time:  $\mathcal{O}(n \log n)$ 



71446b, 14 lines

310954, 13 lines

```
typedef Point<11> P;
vector<P> convexHull(vector<P> pts) {
    if (sz(pts) <= 1) return pts;
    sort(all(pts));
    vector<P> h(sz(pts)+1);
    int s = 0, t = 0;
    for (int it = 2; it--; s = --t, reverse(all(pts)))
        for (P p: pts) {
        while (t >= s + 2 && h[t-2].cross(h[t-1], p) <= 0) t--;
        h[t++] = p;
    }
    return {h.begin(), h.begin() + t - (t == 2 && h[0] == h[1])};
}</pre>
```

#### HullDiameter.h

**Description:** Returns the two points with max distance on a convex hull (ccw, no duplicate/collinear points).

"Point.h" c571b8, 12 lines

```
typedef Point<11> P;
array<P, 2> hullDiameter(vector<P> S) {
  int n = sz(S), j = n < 2 ? 0 : 1;
  pair<1l, array<P, 2>> res({0, {S[0], S[0]}});
  rep(i,0,j)
  for (;; j = (j + 1) % n) {
    res = max(res, {(S[i] - S[j]).dist2(), {S[i], S[j]}});
    if ((S[(j + 1) % n] - S[j]).cross(S[i + 1] - S[i]) >= 0)
        break;
  }
  return res.second;
```

#### PointInsideHull.h

**Description:** Determine whether a point t lies inside a convex hull (CCW order, with no collinear points). Returns true if point lies within the hull. If strict is true, points on the boundary aren't included.

#### Time: $\mathcal{O}(\log N)$

"Point.h", "sideOf.h", "OnSegment.h"

```
typedef Point<1l> P;

bool inHull(const vector<P>& 1, P p, bool strict = true) {
  int a = 1, b = sz(1) - 1, r = !strict;
  if (sz(1) < 3) return r && onSegment(1[0], 1.back(), p);
  if (sideOf(1[0], 1[a], 1[b]) > 0) swap(a, b);
  if (sideOf(1[0], 1[a], p) >= r || sideOf(1[0], 1[b], p) <= -r)
    return false;
  while (abs(a - b) > 1) {
    int c = (a + b) / 2;
    (sideOf(1[0], 1[c], p) > 0 ? b : a) = c;
}
```

```
return sgn(1[a].cross(1[b], p)) < r;
}</pre>
```

#### LineHullIntersection.h

**Description:** Line-convex polygon intersection. The polygon must be ccw and have no collinear points. lineHull(line, poly) returns a pair describing the intersection of a line with the polygon:  $\bullet$  (-1,-1) if no collision,  $\bullet$  (i,-1) if touching the corner  $i, \bullet$  (i,i) if along side  $(i,i+1), \bullet$  (i,j) if crossing sides (i,i+1) and (j,j+1). In the last case, if a corner i is crossed, this is treated as happening on side (i,i+1). The points are returned in the same order as the line hits the polygon. extrVertex returns the point of a hull with the max projection onto a line.

Time:  $\mathcal{O}(\log n)$ 

```
"Point.h"
                                                     7cf45b, 39 lines
#define cmp(i,j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
template <class P> int extrVertex(vector<P>& poly, P dir) {
 int n = sz(poly), lo = 0, hi = n;
  if (extr(0)) return 0;
  while (10 + 1 < hi) {
   int m = (lo + hi) / 2;
   if (extr(m)) return m;
   int 1s = cmp(1o + 1, 1o), ms = cmp(m + 1, m);
    (1s < ms \mid | (1s == ms \&\& 1s == cmp(1o, m)) ? hi : 1o) = m;
 return lo:
#define cmpL(i) sqn(a.cross(poly[i], b))
template <class P>
array<int, 2> lineHull(P a, P b, vector<P>& poly) {
 int endA = extrVertex(poly, (a - b).perp());
  int endB = extrVertex(poly, (b - a).perp());
  if (cmpL(endA) < 0 \mid \mid cmpL(endB) > 0)
   return {-1, -1};
  array<int, 2> res;
  rep(i, 0, 2) {
   int lo = endB, hi = endA, n = sz(poly);
   while ((lo + 1) % n != hi) {
     int m = ((lo + hi + (lo < hi ? 0 : n)) / 2) % n;
      (cmpL(m) == cmpL(endB) ? lo : hi) = m;
   res[i] = (lo + !cmpL(hi)) % n;
   swap (endA, endB);
  if (res[0] == res[1]) return {res[0], -1};
  if (!cmpL(res[0]) && !cmpL(res[1]))
   switch ((res[0] - res[1] + sz(poly) + 1) % sz(poly)) {
     case 0: return {res[0], res[0]};
     case 2: return {res[1], res[1]};
  return res;
```

## 8.4 Misc. Point Set Problems

#### ClosestPair.h

Description: Finds the closest pair of points.

Time:  $\mathcal{O}(n \log n)$ 

```
while (v[j].y <= p.y - d.x) S.erase(v[j++]);
  auto lo = S.lower_bound(p - d), hi = S.upper_bound(p + d);
  for (; lo != hi; ++lo)
    ret = min(ret, {(*lo - p).dist2(), {*lo, p}});
  S.insert(p);
}
return ret.second;
</pre>
```

#### FarthestPair.h

**Description:** Given an ordered convex hull, computes the farthest pair of points.

```
Time: \mathcal{O}\left(n\right)
```

#### kdTree.h

using T = long long;

rep(i, 0, DIM)

Description: KD-tree (any dimension)

1f90e6, 73 lines

e71847, 17 lines

```
constexpr int DIM = 2;
using P = array<T, DIM>;
const T INF = numeric_limits<T>::max();
struct Node {
 P pt; // if this is a leaf, the single point in it
 P lo_bound, hi_bound;
 Node *first = 0, *second = 0;
 T distance (const P& p) { // min squared distance to a point
   T r = 0;
   rep(i, 0, DIM) {
     T d = p[i] - max(lo_bound[i], min(hi_bound[i], p[i]));
     r += d * d:
   return r;
 Node (vector<P>&& vp) : pt(vp[0]) {
   rep(i, 0, DIM) {
     lo_bound[i] = INF; hi_bound[i] = -INF;
   for (const P & p : vp) {
     rep(i, 0, DIM) {
       lo_bound[i] = min(lo_bound[i], p[i]);
       hi_bound[i] = max(hi_bound[i], p[i]);
   if (sz(vp) > 1) {
     // split on x if width >= height (not ideal...)
     pair<T, int> biggest = \{-1, -1\};
```

```
biggest = max(biggest, {hi_bound[i] - lo_bound[i], i});
      int i = biggest.second;
      sort(all(vp), [&](const P& a, const P& b) { return a[i] <</pre>
          b[i]; });
      // divide by taking half the array for each child (not
      // best performance with many duplicates in the middle)
      int half = sz(vp)/2;
      first = new Node({vp.begin(), vp.begin() + half});
      second = new Node({vp.begin() + half, vp.end()});
struct KDTree {
  Node* root;
  KDTree(const vector<P>& vp) : root(new Node({all(vp)})) {}
 pair<T, P> search(Node *node, const P& p) {
    if (!node->first) {
      // uncomment if we should not find the point itself:
      // if (p = node \rightarrow pt) return \{INF, P()\};
      return {node->distance(p), node->pt};
    Node *f = node->first, *s = node->second;
    T bfirst = f->distance(p), bsec = s->distance(p);
    if (bfirst > bsec) swap(bsec, bfirst), swap(f, s);
    // search closest side first, other side if needed
    auto best = search(f, p);
    if (bsec < best.first)</pre>
     best = min(best, search(s, p));
    return best;
  // find nearest point to a point, and its squared distance
  // (requires an arbitrary operator< for Point)
 pair<T, P> nearest (const P& p) {
    return search (root, p);
```

#### FastDelaunav.h

**Description:** Fast Delaunay triangulation. Each circumcircle contains none of the input points. There must be no duplicate points. If all points are on a line, no triangles will be returned. Should work for doubles as well, though there may be precision issues in 'circ'. Returns triangles in order  $\{t[0][0], t[0][1], t[0][2], t[1][0], \dots\}$ , all counter-clockwise.

Time:  $\mathcal{O}(n \log n)$ 

```
"Point.h"
                                                     bf87ec, 88 lines
typedef Point<11> P:
typedef struct Quad* Q;
typedef __int128_t lll; // (can be ll if coords are < 2e4)
P arb(LLONG_MAX, LLONG_MAX); // not equal to any other point
struct Quad {
  bool mark; Q o, rot; P p;
  P F() { return r()->p; }
  O r() { return rot->rot; }
  Q prev() { return rot->o->rot; }
  Q next() { return r()->prev(); }
bool circ(P p, P a, P b, P c) { // is p in the circumcircle?
 111 p2 = p.dist2(), A = a.dist2()-p2,
      B = b.dist2()-p2, C = c.dist2()-p2;
  return p.cross(a,b) *C + p.cross(b,c) *A + p.cross(c,a) *B > 0;
Q makeEdge(P orig, P dest) {
```

6fd268, 21 lines

```
Q q[] = \{new Quad\{0,0,0,orig\}, new Quad\{0,0,0,arb\},
           new Quad{0,0,0,dest}, new Quad{0,0,0,arb}};
  rep(i, 0, 4)
   q[i] -> o = q[-i \& 3], q[i] -> rot = q[(i+1) \& 3];
  return *q;
void splice(Q a, Q b) {
  swap(a->o->rot->o, b->o->rot->o); swap(a->o, b->o);
Q connect(Q a, Q b) {
  Q = makeEdge(a->F(), b->p);
  splice(q, a->next());
  splice(q->r(), b);
  return q;
pair<Q,Q> rec(const vector<P>& s) {
  if (sz(s) <= 3) {
    Q = \text{makeEdge}(s[0], s[1]), b = \text{makeEdge}(s[1], s.back());
    if (sz(s) == 2) return { a, a->r() };
    splice(a->r(), b);
    auto side = s[0].cross(s[1], s[2]);
   0 c = side ? connect(b, a) : 0;
    return {side < 0 ? c->r() : a, side < 0 ? c : b->r() };
#define H(e) e->F(), e->p
#define valid(e) (e->F().cross(H(base)) > 0)
 Q A, B, ra, rb;
  int half = sz(s) / 2;
  tie(ra, A) = rec({all(s) - half});
  tie(B, rb) = rec({sz(s) - half + all(s)});
  while ((B->p.cross(H(A)) < 0 && (A = A->next())) | |
         (A->p.cross(H(B)) > 0 && (B = B->r()->o)));
  Q base = connect(B->r(), A);
  if (A->p == ra->p) ra = base->r();
  if (B->p == rb->p) rb = base;
#define DEL(e, init, dir) O e = init->dir; if (valid(e)) \
    while (circ(e->dir->F(), H(base), e->F())) { \
     Q t = e->dir; \
      splice(e, e->prev()); \
      splice(e->r(), e->r()->prev()); \
      e = t; \
  for (;;) {
   DEL(LC, base->r(), o); DEL(RC, base, prev());
    if (!valid(LC) && !valid(RC)) break;
    if (!valid(LC) || (valid(RC) && circ(H(RC), H(LC))))
     base = connect(RC, base->r());
     base = connect(base->r(), LC->r());
  return { ra, rb };
vector<P> triangulate(vector<P> pts) {
  sort(all(pts)); assert(unique(all(pts)) == pts.end());
  if (sz(pts) < 2) return {};</pre>
  O e = rec(pts).first;
  vector < Q > q = \{e\};
  int qi = 0;
  while (e->o->F().cross(e->F(), e->p) < 0) e = e->o;
#define ADD { Q c = e; do { c->mark = 1; pts.push_back(c->p); \
  q.push_back(c->r()); c = c->next(); } while (c != e); }
  ADD; pts.clear();
  while (qi < sz(q)) if (!(e = q[qi++])->mark) ADD;
  return pts;
```

#### 8.5 3D

#### PolyhedronVolume.h

Description: Magic formula for the volume of a polyhedron. Faces should point outwards.

3058c3, 6 lines

```
template<class V, class L>
double signedPolyVolume(const V& p, const L& trilist) {
  double v = 0;
  for (auto i : trilist) v += p[i.a].cross(p[i.b]).dot(p[i.c]);
  return v / 6;
}
```

#### Point3D.h

**Description:** Class to handle points in 3D space. T can be e.g. double or long long.

```
template < class T > struct Point3D {
 typedef Point3D P;
 typedef const P& R;
 T x, y, z;
 explicit Point3D(T x=0, T y=0, T z=0) : x(x), y(y), z(z) {}
 bool operator<(R p) const {</pre>
   return tie(x, y, z) < tie(p.x, p.y, p.z); }
 bool operator==(R p) const {
   return tie(x, y, z) == tie(p.x, p.y, p.z); }
 P operator+(R p) const { return P(x+p.x, y+p.y, z+p.z); }
 P operator-(R p) const { return P(x-p.x, y-p.y, z-p.z); }
 P operator*(T d) const { return P(x*d, y*d, z*d); }
 P operator/(T d) const { return P(x/d, y/d, z/d); }
 T dot(R p) const { return x*p.x + y*p.y + z*p.z; }
 P cross(R p) const {
   return P(y*p.z - z*p.y, z*p.x - x*p.z, x*p.y - y*p.x);
 T dist2() const { return x*x + y*y + z*z; }
 double dist() const { return sgrt((double)dist2()); }
 //Azimuthal angle (longitude) to x-axis in interval [-pi, pi]
 double phi() const { return atan2(y, x); }
 //Zenith angle (latitude) to the z-axis in interval [0, pi]
 double theta() const { return atan2(sgrt(x*x+y*y),z); }
 P unit() const { return *this/(T) dist(); } //makes dist()=1
 //returns unit vector normal to *this and p
 P normal(P p) const { return cross(p).unit(); }
 //returns point rotated 'angle' radians ccw around axis
 P rotate (double angle, P axis) const {
   double s = sin(angle), c = cos(angle); P u = axis.unit();
   return u*dot(u)*(1-c) + (*this)*c - cross(u)*s;
};
```

#### 3dHull.h

assert (sz(A) >= 4);

**Description:** Computes all faces of the 3-dimension hull of a point set. \*No four points must be coplanar\*, or else random results will be returned. All faces will point outwards.

```
vector<vector<PR>>> E(sz(A), vector<PR>(sz(A), {-1, -1}));
#define E(x,y) E[f.x][f.y]
 vector<F> FS;
  auto mf = [&](int i, int j, int k, int l) {
   P3 q = (A[j] - A[i]).cross((A[k] - A[i]));
   if (q.dot(A[1]) > q.dot(A[i]))
     q = q * -1;
   F f{q, i, j, k};
   E(a,b).ins(k); E(a,c).ins(j); E(b,c).ins(i);
   FS.push_back(f);
 rep(i, 0, 4) rep(j, i+1, 4) rep(k, j+1, 4)
   mf(i, j, k, 6 - i - j - k);
 rep(i,4,sz(A)) {
    rep(j,0,sz(FS)) {
     F f = FS[j];
     if(f.q.dot(A[i]) > f.q.dot(A[f.a])) {
       E(a,b).rem(f.c);
       E(a,c).rem(f.b);
       E(b,c).rem(f.a);
       swap(FS[j--], FS.back());
       FS.pop back();
    int nw = sz(FS);
    rep(j,0,nw) {
     F f = FS[i];
#define C(a, b, c) if (E(a,b).cnt() != 2) mf(f.a, f.b, i, f.c);
     C(a, b, c); C(a, c, b); C(b, c, a);
 for (F& it : FS) if ((A[it.b] - A[it.a]).cross(
   A[it.c] - A[it.a]).dot(it.q) \ll 0) swap(it.c, it.b);
 return FS:
```

#### sphericalDistance.h

Description: Returns the shortest distance on the sphere with radius radius between the points with azimuthal angles (longitude) f1  $(\phi_1)$  and f2  $(\phi_2)$  from x axis and zenith angles (latitude) t1  $(\theta_1)$  and t2  $(\theta_2)$  from z axis (0 = 1) north pole). All angles measured in radians. The algorithm starts by converting the spherical coordinates to cartesian coordinates so if that is what you have you can use only the two last rows. dx\*radius is then the difference between the two points in the x direction and d\*radius is the total distance between the points.

```
double sphericalDistance(double f1, double t1,
    double f2, double t2, double radius) {
    double dx = sin(t2)*cos(f2) - sin(t1)*cos(f1);
    double dy = sin(t2)*sin(f2) - sin(t1)*sin(f1);
    double dz = cos(t2) - cos(t1);
    double d = sqrt(dx*dx + dy*dy + dz*dz);
    return radius*2*asin(d/2);
```

#### PolyhedronVolume6.h

"Point3D.h"

**Description:** Returns six times the volume of a polyhedron. Points in faces can be given in either cw or ccw order.

```
template<class T>
T polyhedronVolume6(const vector<vector<Point3D<T>>> & faces) {
  T volume = 0;
  Point3D<T> center;
  set<Point3D<T>> vertices;
  for(auto& face : faces) for(auto& p : face) {
    if (!vertices.count(p)) {
      center = center + p;
      vertices.insert(p);
    }
}
```

```
}
}
center = center / sz(vertices);
for(auto& f : faces) {
    rep(i, 1, sz(f) - 1) {
        auto normal = (f[i] - f[0]).cross(f[i+1] - f[0]);
        T s = normal.dot(center - f[0]) < 0? 1: -1;
        volume += f[0].dot(normal) * s;
    }
}
return volume;</pre>
```

#### LinePlaneIntersection.h

**Description:** Given a line with point 10 and direction 1 and a plane with point p0 and normal vector n returns the intersection and the distance.

"Point3D.h" 8f1e37, 13 lines

#### LineSphereIntersection.h

**Description:** Given a sphere with center s and radius r and a line with point l0 and direction l returns whether they intersect.

```
"LinePlaneIntersection.h" 8229ef, 7 lines
bool lsIntersects(P 10, P 1, P s, double r) {
  P ip; double d;
  if (!!pIntersection(10, 1, s, 1, ip, d))
    return false;
  if (d < 0) return false;
  return (s - ip).dist() <= r;
}</pre>
```

# Strings (9)

#### Regex.h

```
#include <regex.h>
struct Regex {
  regex_t re;
  Regex(const string & p) {
    assert(regcomp(&re, p.data(), REG_EXTENDED) == 0);
  }
  ~Regex() { regfree(&re); }
  bool search(const string & s, pii & res) {
    regmatch_t m;
    if (regexec(&re, s.data(), 1, &m, 0)) return false;
    res.first = m.rm_so; res.second = m.rm_eo;
    return true;
  }
};
```

#### KMP.h

**Description:** pi[x] computes the length of the longest prefix of s that ends at x, other than s[0...x] itself (abacaba -> 0010123). Can be used to find all occurrences of a string. Time:  $\mathcal{O}(n)$ 

```
vi pi(const string& s) {
  vi p(sz(s));
  rep(i,1,sz(s)) {
    int g = p[i-1];
    while (g && s[i] != s[g]) g = p[g-1];
    p[i] = g + (s[i] == s[g]);
  }
  return p;
}

vi match(const string& s, const string& pat) {
  vi p = pi(pat + '\0' + s), res;
  rep(i,sz(p)-sz(s),sz(p))
    if (p[i] == sz(pat)) res.push_back(i - 2 * sz(pat));
  return res;
}
```

#### Zfunc.h

Time:  $\mathcal{O}(n)$ 

**Description:** z[x] computes the length of the longest common prefix of s[i:] and s, except z[0] = 0. (abacaba -> 0010301)

```
vi Z(string S) {
vi z(sz(s));
int l = -1, r = -1;
rep(i,1,sz(s)) {
    z[i] = i >= r ? 0 : min(r - i, z[i - 1]);
    while (i + z[i] < sz(s) && S[i + z[i]] == S[z[i]])
        z[i]++;
    if (i + z[i] > r)
        l = i, r = i + z[i];
}
return z;
}
```

#### Manacher.h

**Description:** For each position in a string, computes p[0][i] = half length of longest even palindrome around pos i, <math>p[1][i] = longest odd (half rounded down).

Time:  $\mathcal{O}\left(N\right)$  e7ad79, 13 lines

```
array<vi, 2> manacher(const string& s) {
  int n = sz(s);
  array<vi,2> p = {vi(n+1), vi(n)};
  rep(z,0,2) for (int i=0,l=0,r=0; i < n; i++) {
    int t = r-i+!z;
    if (i<r) p[z][i] = min(t, p[z][l+t]);
    int L = i-p[z][i], R = i+p[z][i]-!z;
    while (L>=1 && R+1<n && s[L-1] == s[R+1])
        p[z][i]++, L--, R++;
    if (R>r) l=L, r=R;
  }
  return p;
}
```

#### MinRotation.h

**Description:** Finds the lexicographically smallest rotation of a string. **Usage:** rotate(v.begin(), v.begin()+minRotation(v), v.end()); **Time:**  $\mathcal{O}(N)$ 

```
int minRotation(string s) {
  int a=0, N=sz(s); s += s;
  rep(b,0,N) rep(k,0,N) {
   if (a+k == b || s[a+k] < s[b+k]) {b += max(0, k-1); break;}</pre>
```

```
if (s[a+k] > s[b+k]) { a = b; break; }
}
return a;
```

### SuffixArray.h

**Description:** Builds suffix array for a string. sa[i] is the starting index of the suffix which is i'th in the sorted suffix array. The returned vector is of size n+1, and sa[0] = n. The 1cp array contains longest common prefixes for neighbouring strings in the suffix array: lcp[i] = lcp(sa[i], sa[i-1]), lcp[0] = 0. The input string must not contain any zero bytes. **Time:**  $O(n \log n)$ 

```
struct SuffixArray {
  vi sa, lcp;
  SuffixArray(string& s, int lim=256) { // or basic_string<int>
    int n = sz(s) + 1, k = 0, a, b;
    vi x(all(s)+1), y(n), ws(max(n, lim)), rank(n);
    sa = lcp = y, iota(all(sa), 0);
    for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim = p) {
      p = j, iota(all(y), n - j);
      rep(i,0,n) if (sa[i] >= j) y[p++] = sa[i] - j;
      fill(all(ws), 0);
      rep(i, 0, n) ws[x[i]] ++;
      rep(i,1,lim) ws[i] += ws[i - 1];
      for (int i = n; i--;) sa[--ws[x[v[i]]]] = v[i];
      swap(x, y), p = 1, x[sa[0]] = 0;
      rep(i,1,n) = sa[i-1], b = sa[i], x[b] =
        (y[a] == y[b] \&\& y[a + j] == y[b + j]) ? p - 1 : p++;
    rep(i,1,n) rank[sa[i]] = i;
    for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)</pre>
      for (k \&\& k--, j = sa[rank[i] - 1];
          s[i + k] == s[j + k]; k++);
};
```

#### SuffixTree.h

**Description:** Ukkonen's algorithm for online suffix tree construction. Each node contains indices [l, r) into the string, and a list of child nodes. Suffixes are given by traversals of this tree, joining [l, r) substrings. The root is 0 (has l=-1, r=0), non-existent children are -1. To get a complete tree, append a dummy symbol – otherwise it may contain an incomplete path (still useful for substring matching, though).

```
Time: \mathcal{O}(26N)
                                                     aae0b8, 50 lines
struct SuffixTree {
  enum { N = 200010, ALPHA = 26 }; // N \sim 2*maxlen+10
 int toi(char c) { return c - 'a'; }
  string a; //v = cur \ node, q = cur \ position
  int t[N][ALPHA], 1[N], r[N], p[N], s[N], v=0, q=0, m=2;
 void ukkadd(int i, int c) { suff:
    if (r[v] <=q) {
      if (t[v][c]==-1) { t[v][c]=m; l[m]=i;
        p[m++]=v; v=s[v]; q=r[v]; goto suff; }
      v=t[v][c]; q=l[v];
    if (q==-1 || c==toi(a[q])) q++; else {
     l[m+1]=i; p[m+1]=m; l[m]=l[v]; r[m]=q;
      p[m]=p[v]; t[m][c]=m+1; t[m][toi(a[q])]=v;
      l[v]=q; p[v]=m; t[p[m]][toi(a[l[m]])]=m;
      v=s[p[m]]; q=l[m];
      while (q<r[m]) { v=t[v][toi(a[q])]; q+=r[v]-l[v]; }</pre>
      if (q==r[m]) s[m]=v; else s[m]=m+2;
      q=r[v]-(q-r[m]); m+=2; goto suff;
```

## Hashing AhoCorasick Hashable IntervalContainer

```
SuffixTree(string a) : a(a) {
    fill(r,r+N,sz(a));
    memset(s, 0, sizeof s);
    memset(t, -1, sizeof t);
    fill(t[1],t[1]+ALPHA,0);
    s[0] = 1; 1[0] = 1[1] = -1; r[0] = r[1] = p[0] = p[1] = 0;
    rep(i,0,sz(a)) ukkadd(i, toi(a[i]));
  // example: find longest common substring (uses ALPHA = 28)
  pii best;
  int lcs(int node, int i1, int i2, int olen) {
    if (1[node] <= i1 && i1 < r[node]) return 1;</pre>
    if (1[node] <= i2 && i2 < r[node]) return 2;</pre>
    int mask = 0, len = node ? olen + (r[node] - 1[node]) : 0;
    rep(c, 0, ALPHA) if (t[node][c] != -1)
     mask |= lcs(t[node][c], i1, i2, len);
    if (mask == 3)
     best = max(best, {len, r[node] - len});
    return mask:
  static pii LCS(string s, string t) {
    SuffixTree st(s + (char) ('z' + 1) + t + (char) ('z' + 2));
    st.lcs(0, sz(s), sz(s) + 1 + sz(t), 0);
    return st.best;
};
Hashing.h
Description: Self-explanatory methods for string hashing.
// Arithmetic mod 2^64-1. 2x slower than mod 2^64 and more
// code, but works on evil test data (e.g. Thue-Morse, where
// ABBA... and BAAB... of length 2^10 hash the same mod 2^64).
// "typedef ull H;" instead if you think test data is random,
// or work mod 10^9+7 if the Birthday paradox is not a problem.
struct H {
  typedef uint64_t ull;
  ull x; H(ull x=0) : x(x) {}
#define OP(O,A,B) H operator O(H o) { ull r = x; asm \
  (A "addg %%rdx, %0\n adcg $0,%0" : "+a"(r) : B); return r; }
  OP(+,,"d"(o.x)) OP(*,"mul %1\n", "r"(o.x) : "rdx")
  H operator-(H o) { return *this + ~o.x; }
  ull get() const { return x + !~x; }
  bool operator==(H o) const { return get() == o.get(); }
  bool operator<(H o) const { return get() < o.get(); }</pre>
static const H C = (11)1e11+3; // (order \sim 3e9; random also ok)
struct HashInterval {
  vector<H> ha, pw;
  HashInterval(string& str) : ha(sz(str)+1), pw(ha) {
    pw[0] = 1;
    rep(i, 0, sz(str))
     ha[i+1] = ha[i] * C + str[i],
     pw[i+1] = pw[i] * C;
  H hashInterval(int a, int b) { // hash [a, b)
    return ha[b] - ha[a] * pw[b - a];
};
vector<H> getHashes(string& str, int length) {
  if (sz(str) < length) return {};</pre>
  H h = 0, pw = 1;
  rep(i,0,length)
   h = h * C + str[i], pw = pw * C;
  vector<H> ret = {h};
  rep(i,length,sz(str)) {
```

```
ret.push_back(h = h * C + str[i] - pw * str[i-length]);
 return ret;
H hashString(string& s){H h{}; for(char c:s) h=h*C+c;return h;}
```

#### AhoCorasick.h

**Description:** Aho-Corasick automaton, used for multiple pattern matching. Initialize with AhoCorasick ac(patterns); the automaton start node will be at index 0. find(word) returns for each position the index of the longest word that ends there, or -1 if none. findAll(-, word) finds all words (up to  $N\sqrt{N}$ many if no duplicate patterns) that start at each position (shortest first). Duplicate patterns are allowed: empty patterns are not. To find the longest words that start at each position, reverse all input. For large alphabets, split each symbol into chunks, with sentinel bits for symbol boundaries.

**Time:** construction takes  $\mathcal{O}(26N)$ , where N = sum of length of patterns. find(x) is  $\mathcal{O}(N)$ , where N = length of x. findAll is  $\mathcal{O}(NM)$ .

```
struct AhoCorasick {
 enum {alpha = 26, first = 'A'}; // change this!
 struct Node {
    // (nmatches is optional)
   int back, next[alpha], start = -1, end = -1, nmatches = 0;
   Node (int v) { memset (next, v, sizeof (next)); }
 };
 vector<Node> N;
 vi backp;
 void insert(string& s, int j) {
   assert(!s.empty());
   int n = 0;
    for (char c : s) {
     int& m = N[n].next[c - first];
     if (m == -1) { n = m = sz(N); N.emplace back(-1); }
     else n = m;
    if (N[n].end == -1) N[n].start = j;
   backp.push_back(N[n].end);
   N[n].end = i;
   N[n].nmatches++;
 AhoCorasick(vector<string>& pat) : N(1, -1) {
   rep(i,0,sz(pat)) insert(pat[i], i);
   N[0].back = sz(N);
   N.emplace_back(0);
   queue<int> q;
    for (q.push(0); !q.empty(); q.pop()) {
     int n = q.front(), prev = N[n].back;
     rep(i,0,alpha) {
       int &ed = N[n].next[i], y = N[prev].next[i];
       if (ed == -1) ed = y;
       else {
         N[ed].back = y;
         (N[ed].end == -1 ? N[ed].end : backp[N[ed].start])
           = N[y].end;
         N[ed].nmatches += N[y].nmatches;
         q.push(ed);
 vi find(string word) {
   int n = 0;
   vi res; // ll count = 0;
   for (char c : word) {
     n = N[n].next[c - first];
     res.push_back(N[n].end);
     // count \neq= N[n].nmatches;
```

```
return res:
vector<vi> findAll(vector<string>& pat, string word) {
 vi r = find(word);
  vector<vi> res(sz(word));
  rep(i,0,sz(word)) {
   int ind = r[i];
    while (ind !=-1) {
     res[i - sz(pat[ind]) + 1].push_back(ind);
      ind = backp[ind];
  return res;
```

# Various (10)

## 10.1 Hashing

Hashable.h

```
Description: Makes a POD-like type hashable
Usage: struct P { int x, y; }; MAKE_HASHABLE(P, t.x, to y); 12 lines
```

```
size_t hash_combine(size_t s) { return s; }
template <typename T, typename... Ts>
size_t hash_combine(size_t s, const T & v, Ts... vs) {
    return hash_combine(hash<T>{}(v) + 0x9e3779b9 +
           (s << 6) + (s >> 2), vs...);
#define MAKE HASHABLE(T, ...) \
    namespace std {\
        template<> struct hash<T> {\
            size_t operator()(const T &t) const {\
                return hash_combine(0, __VA_ARGS__);\
```

## 10.2 Intervals

IntervalContainer.h

Description: Add and remove intervals from a set of disjoint intervals. Will merge the added interval with any overlapping intervals in the set when adding. Intervals are [inclusive, exclusive).

Time:  $\mathcal{O}(\log N)$ 

```
edce47, 23 lines
set<pii>::iterator addInterval(set<pii>& is, int L, int R) {
 if (L == R) return is.end();
 auto it = is.lower_bound({L, R}), before = it;
  while (it != is.end() && it->first <= R) {</pre>
    R = max(R, it->second);
    before = it = is.erase(it);
 if (it != is.begin() && (--it)->second >= L) {
   L = min(L, it->first);
    R = max(R, it->second);
    is.erase(it);
  return is.insert(before, {L,R});
void removeInterval(set<pii>& is, int L, int R) {
 if (L == R) return;
 auto it = addInterval(is, L, R);
 auto r2 = it->second;
 if (it->first == L) is.erase(it);
  else (int&)it->second = L;
 if (R != r2) is.emplace(R, r2);
```

d38d2b, 18 lines

#### IntervalCover.h

**Description:** Compute indices of smallest set of intervals covering another interval. Intervals should be [inclusive, exclusive). To support [inclusive, inclusive], change (A) to add || R.empty(). Returns empty set on failure (or if G is empty).

Time:  $\mathcal{O}(N \log N)$  9e9d8d, 19 lines

```
template < class T > vi cover(pair < T, T > G, vector < pair < T, T > I) {
    vi S(sz(I)), R;
    iota(all(S), 0);
    sort(all(S), [&](int a, int b) { return I[a] < I[b]; });
    T cur = G.first;
    int at = 0;
    while (cur < G.second) { // (A)
        pair < T, int > mx = make_pair(cur, -1);
        while (at < sz(I) && I[S[at]].first <= cur) {
            mx = max(mx, make_pair(I[S[at]].second, S[at]));
            at++;
        }
        if (mx.second == -1) return {};
        cur = mx.first;
        R.push_back(mx.second);
    }
    return R;
}</pre>
```

#### ConstantIntervals.h

**Description:** Split a monotone function on [from, to) into a minimal set of half-open intervals on which it has the same value. Runs a callback g for each such interval.

```
Usage: constantIntervals(0, sz(v), [&] (int x) {return v[x];}, [&] (int lo, int hi, T val) \{\ldots\}); 
Time: \mathcal{O}\left(k\log\frac{n}{L}\right)
```

```
template < class F, class G, class T>
void rec(int from, int to, F& f, G& g, int& i, T& p, T q) {
 if (p == q) return;
  if (from == to) {
   q(i, to, p);
    i = to; p = q;
  } else {
    int mid = (from + to) >> 1;
   rec(from, mid, f, q, i, p, f(mid));
   rec(mid+1, to, f, g, i, p, q);
template<class F, class G>
void constantIntervals(int from, int to, F f, G g) {
 if (to <= from) return;</pre>
  int i = from; auto p = f(i), q = f(to-1);
 rec(from, to-1, f, g, i, p, q);
 g(i, to, q);
```

## 10.3 Misc. algorithms

#### TernarySearch.h

**Description:** Find the smallest i in [a,b] that maximizes f(i), assuming that  $f(a) < \ldots < f(i) \ge \cdots \ge f(b)$ . To reverse which of the sides allows non-strict inequalities, change the < marked with (A) to <=, and reverse the loop at (B). To minimize f, change it to >, also at (B).

```
Usage: int ind = ternSearch(0,n-1,[&](int i){return a[i];}); 
 Time: \mathcal{O}(\log(b-a)) 9155b4, 11 lines
```

```
template < class F >
int ternSearch(int a, int b, F f) {
  assert(a <= b);
  while (b - a >= 5) {
   int mid = (a + b) / 2;
}
```

```
if (f(mid) < f(mid+1)) a = mid; // (A)
else b = mid+1;
}
rep(i,a+1,b+1) if (f(a) < f(i)) a = i; // (B)
return a;</pre>
```

#### LIS.h

**Description:** Compute indices for the longest increasing subsequence. **Time:**  $\mathcal{O}\left(N\log N\right)$ 

```
template < class I > vi lis (const vector < I > & S) {
   if (S.empty()) return {};
    vi prev(sz(S));
   typedef pair < I, int > p;
   vector  res;
   rep(i,0,sz(S)) {
        // change 0 -> i for longest non-decreasing subsequence
        auto it = lower_bound(all(res), p{S[i], 0});
        if (it == res.end()) res.emplace_back(), it = res.end()-1;
        *it = {S[i], i};
        prev[i] = it == res.begin() ? 0 : (it-1)->second;
   }
   int L = sz(res), cur = res.back().second;
   vi ans(L);
   while (L--) ans[L] = cur, cur = prev[cur];
   return ans;
}
```

#### LIS.pv

**Description:** Compute indices for the longest increasing subsequence. **Time:**  $\mathcal{O}\left(N\log N\right)$ 

```
from bisect import bisect_left
def lis(S):
 prev, res = [], []
 for i, v in enumerate(S):
   el = (v, i)
   \# v + 1 for non-decreasing
    j = bisect_left(res, (v, 0))
   if i == len(res):
     res.append(el)
     res[j] = el
   prev.append(0 if j == 0 else res[j - 1][1])
 ans = []
 cur = res[-1][1]
 for _ in range(len(res)):
   ans.append(cur)
   cur = prev[cur]
 return ans[::-1]
```

#### LCS.h

**Description:** Finds the longest common subsequence.

Memory:  $\mathcal{O}(nm)$ .

Time:  $\mathcal{O}(nm)$  where n and m are the lengths of the sequences absolute  $a_{b1021,\ 14\ \mathrm{lines}}$ 

```
template < class T > T lcs (const T &X, const T &Y) {
   int a = sz(X), b = sz(Y);
   vector < vi> dp (a+1, vi (b+1));
   rep(i,1,a+1) rep(j,1,b+1)
   dp[i][j] = X[i-1]==Y[j-1] ? dp[i-1][j-1]+1 :
        max (dp[i][j-1], dp[i-1][j]);
   int len = dp[a][b];
   T ans (len,0);
   while (a && b)
        if (X[a-1]==Y[b-1]) ans [--len] = X[--a], --b;
        else if (dp[a][b-1]>dp[a-1][b]) --b;
```

```
else --ā;
return ans;
```

## 10.4 Dynamic programming

#### KnuthDP.h

**Description:** When doing DP on intervals:  $a[i][j] = \min_{i < k < j} (a[i][k] + a[k][j]) + f(i,j)$ , where the (minimal) optimal k increases with both i and j, one can solve intervals in increasing order of length, and search k = p[i][j] for a[i][j] only between p[i][j-1] and p[i+1][j]. This is known as Knuth DP. Sufficient criteria for this are if  $f(b,c) \le f(a,d)$  and  $f(a,c) + f(b,d) \le f(a,d) + f(b,c)$  for all  $a \le b \le c \le d$ . Consider also: LineContainer (ch. Data structures), monotone queues, ternary search. **Time:**  $\mathcal{O}\left(N^2\right)$ 

## DivideAndConquerDP.h

**Description:** Given  $a[i] = \min_{lo(i) \le k < hi(i)} (f(i, k))$  where the (minimal) optimal k increases with i, computes  $\overline{a[i]}$  for i = L..R - 1.

Time:  $\mathcal{O}\left(\left(N+(hi-lo)\right)\log N\right)$ 

```
struct DP { // Modify at will:
   int lo(int ind) { return 0; }
   int hi(int ind) { return ind; }
   ll f(int ind, int k) { return dp[ind][k]; }
   void store(int ind, int k, ll v) { res[ind] = pii(k, v); }

   void rec(int L, int R, int LO, int HI) {
      if (L >= R) return;
      int mid = (L + R) >> 1;
      pair<ll, int> best(LLONG_MAX, LO);
      rep(k, max(LO,lo(mid)), min(HI,hi(mid)))
      best = min(best, make_pair(f(mid, k), k));
      store(mid, best.second, best.first);
```

## 10.5 Debugging tricks

rec(L, mid, LO, best.second+1);

rec(mid+1, R, best.second, HI);

• signal (SIGSEGV, [] (int) { \_Exit(0); }); converts segfaults into Wrong Answers. Similarly one can catch SIGABRT (assertion failures) and SIGFPE (zero divisions). \_GLIBCXX\_DEBUG failures generate SIGABRT (or SIGSEGV on gcc 5.4.0 apparently).

void solve(int L, int R) { rec(L, R, INT\_MIN, INT\_MAX); }

• feenableexcept (29); kills the program on NaNs (1), 0-divs (4), infinities (8) and denormals (16).

## 10.6 Optimization tricks

\_\_builtin\_ia32\_ldmxcsr(40896); disables denormals (which make floats 20x slower near their minimum value).

#### 10.6.1 Bit backs

- x & -x is the least bit in x.
- for (int x = m; x; ) { --x &= m; ... } loops over all subset masks of m (except m itself).
- c = x&-x, r = x+c; (((r^x) >> 2)/c) | r is the next number after x with the same number of bits set.

AU

• rep(b,0,K) rep(i,0,(1 << K))
if (i & 1 << b) D[i] += D[i^(1 << b)];

## 10.6.2 Pragmas

- #pragma GCC optimize ("Ofast") will make GCC auto-vectorize loops and optimizes floating points better.
- #pragma GCC target ("avx2") can double performance of vectorized code, but causes crashes on old machines.
- #pragma GCC optimize ("trapv") kills the program on integer overflows (but is really slow).

## BumpAllocator.h

```
// Either globally or in a single class:
static char buf[450 << 20];
void* operator new(size_t s) {
    static size_t i = sizeof buf;
    assert(s < i);
    return (void*)&buf[i -= s];
}
void operator delete(void*) {}</pre>
```

computes all sums of subsets.

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