Defaultable LIBOR Market Models for Loan Valuation and their Implementation

Master's Thesis

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Table of Contents

Framework

During the whole talk we make a few assumptions:

- The market is arbitrage free and complete.
- We work in the setting of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{Q}^B)$.
- $B(t)$ is a numeraire that is $\mathcal{F}_t$-measurable. We mainly consider the Spot measure.
- $\mathbb{Q}^B$ is the martingale measure corresponding to the numeraire.
- This means that all traded assets discounted by $B(t)$ are martingales w.r.t. $\mathbb{Q}^B$ (see [6]).
- We have a time tenor $0 = T_0 < ... < T_N = T$ that splits $[0, T]$.

General Setting

We are in the setting of a general LIBOR market model:

- We have N zero coupon bonds with price $P(t; T_i)$, where $P(T_i; T_i) = 1$.
- These are traded assets $\Rightarrow$ discounted price processes are martingales.
- We derive simple forward rates for the tenor:

$$L_i(t) := L(t; T_i, T_{i+1}) = \frac{1}{T_{i+1} - T_i} \left(\frac{P(t; T_i)}{P(t; T_{i+1})} - 1 \right)$$

- ... and their SDE:

$$dL_i(t) = \mu_i dt + \sum_{k=0}^{M} \lambda_{i,k} dW_t^k,$$

where $\lambda_{i,k}$ are factor loadings (we assume $M + 1$ factors), W_k are $M + 1$ independent brownian motions and $L = (L_0, ..., L_{N-1})^T$.

The Numeraire: Spot measure

The choice of numeraire influences the SDE, i.e. the drift μ_k:

- Under the **Spot** measure we specify the numeraire:

$$B(t) := P(t; T_{m(t)+1}) \prod_{j=0}^{m(t)} (1 + L_j(T_j)\Delta T_j)$$

- The drift is then:

$$\mu_i(t, L(t)) = \sum_{j=m(t)+1}^{i} \lambda_i \lambda_j^T \frac{\Delta T_j}{1 + \Delta T_j \, L_j(t)}$$

where $m(t) = \max(i \in \{0, ..., N-1\} | T_i \leq t)$. Note that $\lambda_i \lambda_j^T = |\sigma_i||\sigma_j|\rho_{i,j}$ is the "covariance" between the i-th and j-th LIBOR rate.

The Numeraire: Terminal measure

- Under the **Terminal** measure we have the numeraire:

$$B(t) := P(t; T_N)$$

- The drift is then:

$$\mu_i(t, L(t)) = -\sum_{j=i+1}^{N-1} \lambda_i \lambda_j^T \frac{\Delta T_j}{1 + \Delta T_j \, L_j(t)}$$

For a proof of these results see [1] and [5].

Defaultable LIBOR Market Model

Defaultable Bonds

Let us extend this model:

- We now assume an additional set of **defaultable** zero coupon bonds $P^d(t; T_i)$ with the same maturities.
- Their payoff is:
$$P^d(T_i; T_i) = \mathbf{1}_{\{\tau > T_i\}},$$
- $\tau(\omega)$ denotes the (stochastic) default time.
- Lets split the defaultable bond into a continuous and discontinuous part:
$$P^d(t; T_i) = P^{d,*}(t; T_i)(1 - J(t)) \quad J(t) := \mathbf{1}_{\{\tau \le t\}}$$
- where $P^{d,*}$ denotes the defaultable zero coupon bond conditional on predefault (see also [2]).

Defaultable Libor rates

- We can also define LIBOR rates on the defaultable bonds conditional on predefault:

$$L_i^d(t) := L^d(t; T_i, T_{i+1}) = \frac{1}{\Delta T_i}\left(\frac{P^{d,*}(t; T_i)}{P^{d,*}(t; T_{i+1})} - 1\right).$$

- Assume L^d is driven by $M^d - M$ extra factors
- With factor loadings $\lambda_i^d = (\lambda_{i\,0}^d, \ldots, \lambda_{i\,M^d}^d)$ for component i the defaultable LIBOR rates then follow the dynamic:

$$dL_i^d(t) = \mu_i^d dt + \sum_{k=0}^{M^d} \lambda_{i\,k}^d dW_t^k,$$

where we again need to derive $\mu^d(t, L^d(t))$.

μ^d under the Spot measure

It turns out the drift term of the defaultable LIBORs under the Spot measure looks very similar to those of the non defaultable ones:

$$\mu_i^d(t, L^d) = \sum_{j=m(t)+1}^{i} \lambda_i^d \lambda_j^{d\,T} \frac{\Delta T_j}{1 + \Delta T_j \, L_j^d(t)}.$$

Let us take a peek at an outline of the proof.

Outline of the proof

- We first realize $P^d(t;T_i)$ and hence also $L_i^d(t)P^d(t;T_{i+1})$ are traded assets and thus discounted with $B(t)$ must be martingales w.r.t $\mathbb{Q}^B$.
- We use the product rule to see:

$$d(L_i^d(t)\frac{P^d(t;T_{i+1})}{B(t)}) = \frac{P^d(t;T_{i+1})}{B(t)} dL_i^d(t) + L_i^d(t) d\frac{P^d(t;T_{i+1})}{B(t)} + dL_i^d(t) d\frac{P^d(t;T_{i+1})}{B(t)}$$

- Comparing drifts we find:

$$\frac{P^d(t;T_{i+1})}{B(t)} \mu^d dt = -dL_i^d(t) d\frac{P^d(t;T_{i+1})}{B(t)}$$

Outline of the proof

- We have $P^d(t;T_{i+1}) = P^d(t;T_{m(t)+1})\prod_{j=m(t)+1}^{i}(1+\Delta T_j L_j^d)^{-1}$
- We assume $P^{(d)}(t;T_{m(t)+1})$ have no diffusion, hence:

$$d\frac{P^d(t;T_{m(t)+1})}{P(t;T_{m(t)+1})} = (\ldots)dt - \frac{P^{d,*}(t;T_{m(t)+1})}{P(t;T_{m(t)+1})}dJ(t),$$

- We get (note $dXdJ = 0$ for all Itô processes X):

$$d\frac{P^d(t;T_{i+1})}{B(t)} = (\ldots)dt - \frac{P^d(t;T_{i+1})}{B(t)}\sum_{k=0}^{M^d}\sum_{j=m(t)+1}^{i}\frac{\Delta T_j \lambda_{j\,k}}{1+\Delta T_j L_j^d}dW_t^k$$
$$-\frac{P^{d,*}(t;T_{i+1})}{B(t)}dJ(t).$$

- Which concludes the proof.

Spreads

A spread is the difference between defaultable and non defaultable forward rates (i.e. $S_i = L_i^d - L_i$).

- The general dynamic for the spread is then:

$$dS_i = (\mu^d(t, L_i^d) - \mu(t, L_i))dt + \sum_{k=0}^{M^d}(\lambda_{i\,k}^d - \lambda_{i\,k}\mathbf{1}_{k<M})dW_t^k$$

- This does not imply positive spread values.
- A negative spread would constitute an arbitrage possibility.
- Hence through the choice of λ^d we need to generate a dynamic for S that **guarantees positivity**.

Diffusions generating Log-Normal Spreads

Let us choose as factor loadings

$$\lambda^d_{i\,k} = \frac{1 + L^d_i(t)\Delta T_i}{1 + L_i(t)\Delta T_i}\lambda_{i\,k} \qquad \text{if } k \leq M \text{, and}$$

$$\lambda^d_{i\,k} = (L^d_i(t) - L_i(t))f_{i\,k} \qquad \text{if } M < k \leq M^d,$$

where $f_{i\,k}$ are free parameters.
This will yield a log normal spread dynamic (I challenge you to try proving this yourself ;), otherwise a proof can be found in [2]):

$$dS_i = S_i\mu^S dt + S_i \sum_{k=0}^{M^d} \lambda^S_{ik} dW^k_t$$

Survival Probability

We can now construct the survival probabilities:

- At the tenor times T_i the numeraire under the Spot measure is $\mathcal{F}_{T_{i-1}}$ measurable.
- This implies the one step survival probability:

$$\mathbb{Q}^B(\{\tau > T_{i+1} \mid \tau > T_i\})(\omega) = \frac{P^{d,*}(T_i; T_{i+1})}{P(T_i; T_{i+1})}$$

- Since we assume $\mathbb{Q}^B(\tau > T_0) = 1$, Bayes-rule implies:

$$\mathbb{Q}^B(\tau > T_i)(\omega) = \prod_{k=0}^{i-1} \frac{P^{d,*}(T_k; T_{k+1})}{P(T_k; T_{k+1})} = \frac{B(T_i)}{B^d(T_i)},$$

where $B^d(t) := P^{d,*}(t; T_{m(t)+1}) \prod_{k=0}^{m(t)} (1 + L_k^d(T_k)\Delta T_k)$

Implementation

Introduction to Finmath Library

The Finmath Library is an object oriented Java library that is open source and was built by Prof. Dr. Fries [3].
It strongly follows the concepts of **abstraction**, **high cohesion** and **single responsibility principle**.

- **Abstraction**: take in interfaces instead of classes.
- **High Cohesion**: closely related constructs should be given in the same module, object or function.
- **Single Responsibility Principle**: each part of the program should have a well defined and single responsibility.

In the diagram we can see how we use these concepts to create a flexible DefaultableLIBORMarketModel:

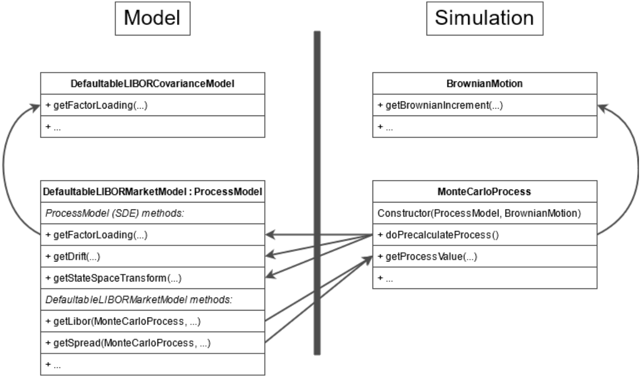

Figure: Separation of Model and Simulation

Discretization Schemes

- Euler scheme:

$$\tilde{X}_{t_{i+1}} = \mu^X(t_i, \tilde{X}_{t_i})\Delta t_i + \lambda^X(t_i, \tilde{X}_{t_i}) \cdot \Delta W_{t_i}$$

- Milstein scheme:

$$\begin{aligned}\tilde{X}_{t_{i+1}} =& \mu^X(t_i, \tilde{X}_{t_i})\Delta t_i + \lambda^X(t_i, \tilde{X}_{t_i}) \cdot \Delta W_{t_i} \\ & + \frac{1}{2}\lambda^X(t_i, \tilde{X}_{t_i})\frac{\partial \lambda^X}{\partial x}(t_i, \tilde{X}_{t_i})((\Delta W_{t_i})^2 - \Delta t_i)\end{aligned}$$

- State-space adjusted Euler Scheme for a process $X_t = f(Y_t)$:
  1. Approximate Y_t by Euler scheme $\tilde{Y}_{t_i}$
  2. Set $\tilde{X}_{t_i} = f(\tilde{Y}_{t_i})$

The schemes are all described and analyzed in [4], [1] and [3].

Problems with Schemes

```
Testing Spread for Positivity (Run 1: Modelling defaultable LIBORs):

LIBOR -index/-time | Minimum Spread      time      path
LIBOR  0     0,00  |  5.00000E-03          0         0
LIBOR  1     2,00  | -6.98142E-03        199      1932
LIBOR  2     4,00  | -2.35999E-03        399      9519
LIBOR  3     6,00  |  3.44480E-04        278      4239
LIBOR  4     8,00  |  3.98306E-04        395       257

Overall Minimum is:
LIBOR  1     2,00  | -6.98142E-03        199      1932
```

Figure: Minimum Path Values Euler Scheme

Problems with Schemes

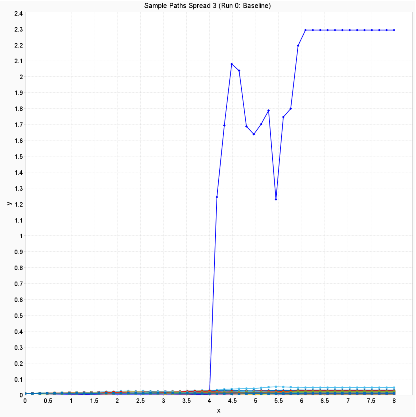

Figure: Paths Functional Euler Scheme

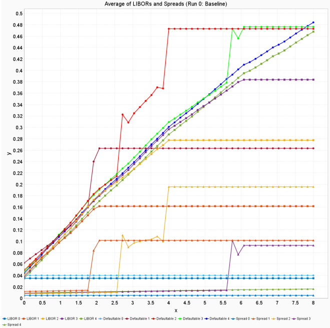

Figure: Averages Functional Euler Scheme

Results

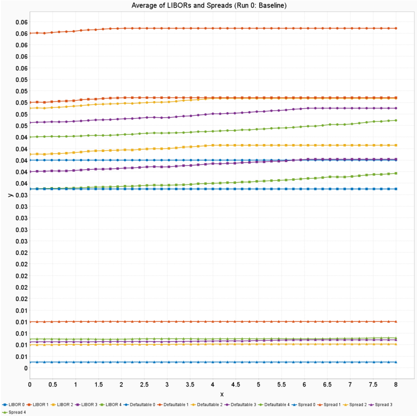

Figure: Averages of LIBORs and Spreads for $f_{ik} \in [-0.5, 0.5]$

Results

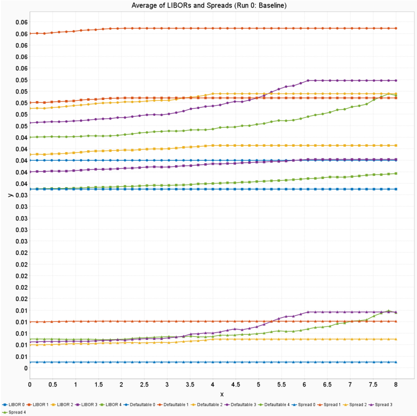

Figure: Averages of LIBORs and Spreads for $f_{ik} \in [-1.0, 1.0]$

Results

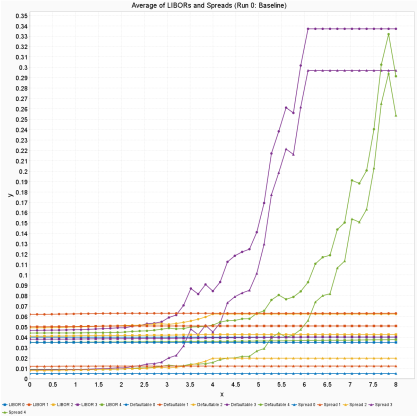

Figure: Averages of LIBORs and Spreads for $f_{ik} \in [-1.5, 1.5]$

Pricing claims

To price claims with this model we apply the default probability to those parts of the claim, that are dependent on survival.
E.g. a T_i claim which is fully conditional on the survival of the modeled entity has the price:

$$B(0)\mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^d}{B(T_i)}\right] = B(0)\mathbb{E}^{\mathbb{Q}^B}\left[\frac{X}{B(T_i)}\mathbb{Q}^B(\tau > T_i)\right],$$

where X^d is the claim, and X is the claim if the entity would have no default probability.
An application for this are loan related options.

Simple Loans

A loan is nothing else than a defaultable coupon bond issued by the debtor. The price of a defaultable coupon bond with nominal N and coupons c_i paid at $\tilde{T}_i \in \{T_1, ..., T_N\}$ is:

$$C^d(t) = \sum_{i=1}^{M} c_i P^d(t; \tilde{T}_i),$$

for $t \in [0, \tilde{T}_1]$. Pricing the loan comes down to establishing the coupon rates such that the initial price of the coupon bond is equal to the nominal:

$$N \stackrel{!}{=} \sum_{i=1}^{M} c_i P^d(0; \tilde{T}_i)$$

Note: if the loan is not an amortizing one $c_M = N + \tilde{c}_M$.

Cancellable Loans

A cancellable loan is a loan that can be canceled (payed back completely) at - in this simple example - a fixed time $T_k \in \{T_1, ..., T_{M-1}\}$:

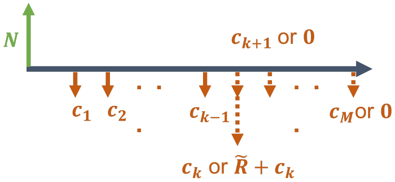

Figure: Cashflow of a cancellable loan

where $\tilde{R} = \dfrac{\sum_{i=k+1}^{M} c_i P^d(0; T_i)}{P^d(0; T_k)}$ is the redemption of the loan.

Cancellable Loans

We can derive a price formula by splitting the definite loan and the optional part:

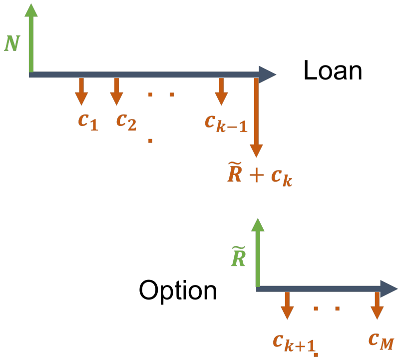

Figure: Cashflow of a cancellable loan

Cancellable Loans

Hence the option is a European put option with strike price $\tilde{R}$ on a coupon bond with coupons $c_{k+1}, \ldots, c_M$ and nominal $\tilde{R}$. In other words the debtor has the right (but not the obligation) to enter a second loan for the conditions of today.
This yields an adjustment in the derivation of the coupons:

$$N + V^P(0) \stackrel{!}{=} \sum_{i=1}^{M} c_i P^d(0; T_i)$$

where

$$V^P(t) = B(t)\mathbb{E}^{\mathbb{Q}^B}\left[\frac{(\tilde{R} - C^d(T_k))^+}{B(T_k)} \mathbb{Q}^B(\tau > T_k | \tau > t) \,\middle|\, \mathcal{F}_t \right]$$

What's next?

- Deriving more complicated products like loans that can be canceled at any tenor.
- Constructing products, where both counterparties are defaultable: the issuer and the buyer.
- Construct products, where the decision to cancel the loan is flawed.

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