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Master's Thesis

Default Forward Rate Models for the Valuation of Loans including Behavioral Aspects

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1 Introduction

1.1 Motivation

1.2 Aim of the thesis

1.3 Preliminaries

The thesis is aimed at an audience that has a deep analytical background and a basic knowledge in probability theory and financial mathematics.

If one has knowledge in the following areas, it is a good start:

- Brownian Motions and martingales,
- stochastic integration,
- Itô stochastic processes,
- definition of and theorems on arbitrage freeness.

Furthermore we assume a fundamental understanding of what these mathematical concepts imply on the real world and the other way around: How are the mathematical concepts motivated by the economical world?

2 Fundamentals

In this section we provide some fundamentals for the thesis. Many of the results mentioned here can also be found in different versions in other scientific papers. We stick to source [Add source] for more theoretical and source [Add source] for more application-oriented results.

2.1 Probability Theory

In our whole thesis we assume a filtered probability space

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$$

where

- Ω is the set of all states,
- $\mathcal{F}$ is a σ-algebra,
- $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\tilde{T}]}$ is a filtration with $\mathcal{F}_t \subset \mathcal{F} \quad \forall t \in [0, \tilde{T}]$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and
- $\mathbb{Q}$ is a probability measure on $\mathcal{F}$.

Let us cover some notations.

Notation 2.1. *Let $X = (X_t)_{t\in[0,\tilde{T}]}$ and $Y = (Y_t)_{t\in[0,\tilde{T}]}$ be Itô stochatic processes and W be a Brownian Motion with:*

$$X_t = X_0 + \int_0^t \mu_s^X ds + \int_0^t \phi_s dW_s,$$
$$Y_t = Y_0 + \int_0^t \mu_s^Y ds + \int_0^t \psi_s dW_s.$$

The sharp bracket *or* quadratic variation *of X is*

$$\langle X \rangle_t = \int_0^t \phi_s^2 ds.$$

. The quadratic covariation *of X and Y is*

$$\langle X, Y \rangle_t = \int_0^t \phi_s \psi_s ds.$$

For a n-dimensional Brownian Motion $W = (W^i)_{i\in\{1,\dots,n\}}$ and therefore n-dimensional diffusion processes $\phi = (\phi^i)_{i\in\{1,\dots,n\}}$, $\psi = (\psi^i)_{i\in\{1,\dots,n\}}$ the quadratic covariation is

$$\langle X, Y\rangle_t = \sum_{i=1}^{n}\int_0^t \phi_s^i \psi_s^i ds$$

For convenience we state a formula for stochastic integration by parts and an extended version of Itô's theorem:

Lemma 2.2. *Let X and Y be two Itô stochastic processes. Then*

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t \quad \text{for } t \in \left[0, \tilde{T}\right]$$

Proof. See [7], page 51. □

Theorem 2.3. *Let X be an Itô stochastic process, $f : \mathbb{R}^n \times [0, \tilde{T}] \to \mathbb{R}$, $(x, t) \rightarrowtail f(x, t)$ be two times differentiable in x and differentiable in t. Then*

$$\begin{aligned} f(X_t, t) = f(X_0, 0) &+ \int_0^t (\partial_t f)(X_s, s)ds + \sum_{i=1}^{n}\int_0^t (\partial_{x^i} f)(X_s, s)dX_s^i \\ &+ \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t (\partial^2_{x^i x^j} f)(X_s, s)d\left\langle X^i, X^j\right\rangle_s \end{aligned}$$

Proof. See [7], page 52. □

Through Itô's formula we can prove the following statement.

Lemma 2.4. *Let $W = (W^i)_{i\in\{1,\dots,d\}}$ be a d-dimensional Brownian Motion, μ be a 1-dimensional and $\sigma = (\sigma^i)_{i\in\{1,\dots,d\}}$ a d-dimensional stochastic process. Let following stochastic differential equation (SDE) be given:*

$$\begin{aligned} dY_t &= Y_t\mu_t dt + Y_t\sigma_t \cdot dW_t, \\ Y_0 &= y, \end{aligned} \tag{1}$$

where $y > 0$.

Then the solution of equation (1) is

$$\begin{aligned} Y_t &= \exp(X_t), \\ X_t &= X_0 + \int_0^t \mu_s - \frac{1}{2}\sum_{i=1}^{d}(\sigma_s^i)^2 ds + \int_0^t \sigma_s \cdot dW_s, \\ X_0 &= \log(y). \end{aligned} \tag{2}$$

Proof. We use Itô's formula on equation (2) with $f(x) = \exp(x)$, hence

$$(\partial_x f)(x) = \exp(x), \quad (\partial^2_{xx})(x) = \exp(x).$$

We get (in SDE format):

$$\begin{aligned} dY_t = d\exp(X_t) &= \exp(X_t)dX_t + \frac{1}{2}\exp(X_t)d\langle X\rangle_t \\ &= Y_t\left(\left(\mu_t - \frac{1}{2}\sum_{i=1}^d(\sigma_t^i)^2\right)dt + \sigma_t\cdot dW_t\right) + \frac{1}{2}Y_t\sum_{i=1}^d(\sigma_t^i)^2dt \\ &= Y_t\mu_t dt + Y_t\sigma_t\cdot dW_t, \end{aligned}$$

which yields equation (1). □

Remark 2.5. *It is easy to see that because of the relation*

$$Y_t = \exp(X_t),$$

it holds that $Y_t > 0$ for all $t \in [0, \tilde{T}]$.

2.2 Financial Mathematics

While the reader should have a deep understanding of what arbitrage is and how to avoid it when pricing financial products, we formulate some fundamental results as a reminder.

Theorem 2.6. *Following statements are equivalent:*

- *The market is arbitrage-free and complete.*
- *There exists exactly one probability measure $\mathbb{Q}^B$ w.r.t. a numeraire B, such that the price process of every traded asset discounted by the numeraire $\frac{X}{B}$ is a martingale w.r.t. the filtration $\mathbb{F}$.*

Proof. See [7], page 92 and 93. □

This directly gives us a notation for the price of any product in an arbitrage-free and complete market.

Lemma 2.7. *Let X be the payoff of a T-claim. Then the arbitrage-free price of the claim at any time $t \in [0, T]$ is*

$$\Pi_t^X = B(t)\mathbb{E}^{\mathbb{Q}^B}\left[\left.\frac{X}{B(T)}\right|\mathcal{F}_t\right].$$

Proof. See [7], page 89, 90. □

Elaborate!!!.

2.3 Factor Loadings

The problem we face in this section is that of correlated Brownian Motions. Assume we have two SDEs for stochastic processes X and Y:

$$\begin{aligned} dX_t &= \mu_t^X dt + \sigma_t^X dW_t^{\mathbb{Q},1}, \\ dY_t &= \mu_t^Y dt + \sigma_t^Y dW_t^{\mathbb{Q},2}, \end{aligned}$$

where $W^{\mathbb{Q},1}$ and $W^{\mathbb{Q},2}$ are instantaneously correlated Brownian Motions under the same measure $\mathbb{Q}$. This correlation ρ is expressed by a quadratic covariation of the Brownian Motions [1]:

$$\left\langle W^{\mathbb{Q},1}, W^{\mathbb{Q},2}\right\rangle_t = \int_0^t \rho_s ds$$

or as SDE:

$$d\left\langle W^{\mathbb{Q},1}, W^{\mathbb{Q},2}\right\rangle_t = \rho_t dt.$$

However, implementation-wise we need to simulate SDEs using only independent Brownian Motions.

For our computations we stick to a constant instantaneous correlation ($\rho_t \equiv \rho \in \mathbb{R}$ for all $t \in \left[0, \tilde{T}\right]$), so lets formulate this assumption first:

Assumption 2.1. *For all 1-dim. Brownian Motions under the martingale measure $\mathbb{Q}^B$ the instantaneous correlation is assumed to be constant. That is for any $\mathbb{Q}^B$-Brownian Motions W^1 and W^2 the following holds for some $\rho \in \mathbb{R}$:*

$$d\left\langle W^1, W^2\right\rangle_t = \rho dt$$

Let us now look at how to simulate correlated Brownian Motions using only independent ones by taking advantage of its properties.
Recall following lemmas:

Lemma 2.8. *$W = (W^i)_{i\in\{1,...,d\}}$ is a d-dimensional Brownian Motion if and only if $W^1, ..., W^d$ are independent 1-dimensional Brownian Motions*

Proof. See [7], page 6. $\square$

Lemma 2.9. *For any d-dimensional standard Brownian Motion $U = (U^i)_{i\in\{1,...,d\}}$ and weights $(a_i)_{i\in\{1,...d\}}$ with $a_1^2 + ... + a_d^2 = 1$ it holds that the process W given by:*

$$W_t = \sum_{i=1}^{d} a_i U_t^i$$

is a 1*-dimensional standard Brownian Motion.*

Proof. We proof that W fulfills the properties of a Brownian Motion:
Property $W_0 = 0$ a.s.:

$$W_0 = \sum_{i=1}^{d} a_i U_0^i = 0 \text{ a.s.}$$

Property $(W_t - W_s) \sim \mathcal{N}(0, t-s)$ for $s < t$:

$$\begin{aligned} W_t - W_s = \sum_{i=1}^{d} a_i U_t^i - \sum_{i=1}^{d} a_i U_s^i &\sim \mathcal{N}\left(0, \sum_{i=1}^{d} a_i^2 (t-s)\right) \\ &\sim \mathcal{N}\left(0, (t-s)\right) \end{aligned}$$

Property W has stationary independent increments:
Stationarity is given by the last property. Let $0 \leq t_0 < t_1 < ... < t_k \leq \tilde{T}$. Then the stacked vector of the increments of U: $\mathcal{U} = (U_{t_1}^1 - U_{t_0}^1, ..., U_{t_1}^d - U_{t_0}^d, ..., U_{t_k}^1 - U_{t_{k-1}}^1, ..., U_{t_k}^d - U_{t_{k-1}}^d)^T$ is a dk-dim. Gaussian vector, with the identity matrix $\mathcal{I}^{dk}$ as correlation (because of the independence). We can define a $k \times dk$-dim. matrix A, such that $A\mathcal{U} = (W_{t_1} - W_{t_0}, ..., W_{t_k} - W_{t_k-1})^T$ which is a linear transformation of $\mathcal{U}$ and therefore still a multivariate Gaussian vector with $\mathcal{I}^k$ as correlation matrix. A multivariate Gaussian vector, with identity matrix as

correlation is independent, hence W has independent increments.

Property W has continuous sample paths a.s.:

Let $U^i(\omega)$ be continuous for $\omega \in A_i \subset \Omega$ where A_i^c is a null set. Then $W(\omega) = \sum_{i=1}^d a_i U^i(\omega)$ is continuous for $\omega \in \bigcap_{i=1}^d A_i$. Because $\left(\bigcap_{i=1}^d A_i\right)^c = \bigcup_{i=1}^d A_i^c$ is still a null set, we have that also W has continuous sample paths almost surely. $\square$

We can now create different linear combinations of the same d-dimensional Brownian Motion U and look at their correlation:

Lemma 2.10. *Let $U = (U^i)_{i\in\{1,\dots,d\}}$ be a d-dimensional Brownian Motion, let $(a_i)_{i\in\{1,\dots d\}}$ and $(b_i)_{i\in\{1,\dots d\}}$ be weights with $\sum_{i=1}^d a_i^2 = 1$, $\sum_{i=1}^d b_i^2 = 1$, respectively. Then W^1 and W^2 given by*

$$W_t^1 = \sum_{i=1}^d a_i U^i$$
$$W_t^2 = \sum_{i=1}^d b_i U^i$$

are 1-dimensional Brownian Motions with:

$$d\left\langle W^1, W^2\right\rangle_t = \left(\sum_{i=1}^d a_i b_i\right) dt \tag{3}$$

Proof. By lemma 2.9 we have that W^1 and W^2 are Brownian Motions. Taking the quadratic covariation directly yields equation (3). $\square$

With this lemma we now have a way to construct correlated Brownian Motions $W^1, ..., W^d$. We still need to find the linear combinations, given a certain correlation matrix $R = (\rho_{i,j})_{i,j\in\{1,\dots,d\}}$, however. A way to do that is to use principal component analysis (PCA). It is beyond the scope of this thesis to give a detailed description of PCA, but the key idea is to take the eigenvectors of R and use them as linear combination. PCA also gives a way to reduce the number of factors, by only considering the Eigenvectors with the highest Eigenvalues [1]. Let us summarize this procedure in a lemma:

Lemma 2.11. *Let $U = (U^i)_{i\in\{1,\dots d\}}$ be a d-dim. Brownian Motion, let $R = (\rho_{i,j})_{i,j\in\{1,\dots,d\}}$ be a $\mathbb{R}^{d\times d}$ correlation matrix (positive-definite, symmetric, entries* *in* $[-1,1]$ *and* 1 *on the diagonal). Let* $(\lambda^{i,j})_{i,j\in\{1,...,d\}}$ *be a matrix constructed by PCA from R.*

For $i = 1, ..., d$ *let* W^i *be the 1-dim. Brownian Motion given by:*

$$W_t^i = \sum_{j=1}^{d} \lambda^{i,j} U_t^j.$$

Then for all $i, k \in \{1, ..., d\}$

$$\sum_{j=1}^{d} \lambda^{i,j} \lambda^{k,j} = \rho_{i,k}$$

$$d\left\langle W^i, W^k \right\rangle_t = \rho_{i,k} dt$$

Proof. This is a direct implication from the previous results together with the definition of PCA and the proof that it works in [Add source]. □

We now have a way of constructing correlated Brownian Motions. As mentioned we can perform a factor reduction with PCA, meaning we can construct d correlated Brownian Motions with a m-dim. Brownian Motion, where $m < d$. This comes at the cost of "losing" a bit of the independence of some factors. But because in most cases the advantages of a factor reduction (less computational cost) outweigh the disadvantages, we from here on assume that the correlation- and factor loading matrix are not of the same size.

3 LIBOR Market Model

While the actual LIBOR, short for "London Inter-Bank Offered Rate" [Add source] lost most of its influence after the banking crash of 2008 [Add source] the LIBOR market model – or discrete forward rate model – is still a very popular mathematical model for simulation and valuation of financial products on fixed income markets.

The idea of the model Elaborate!!!.

The basic assumption of the LIBOR Market Model is that we are in an arbitrage free and complete market.

3.1 Fixed Income Markets Terminology

We start with the definition of some fixed income market terms:

Definition 3.1. *A* zero coupon bond *with maturity $T \in [0, \tilde{T}]$ (short: T-bond) is a product that pays* 1 *at maturity. Its price process is denoted:*

$$P(t;T) := P(\omega, t; T)$$

Note: by construction $P(T;T) = 1$ and $P(\cdot, T)$ discounted with the numeraire must be a martingale under the corresponding martingale-measure $\mathbb{Q}^B$.

While the zero coupon bond does not yield any payoff (or coupons) between buying- and maturity time – hence the name – one can also find coupon paying bonds:

Definition 3.2. *Let $T_1 < ... < T_N$ be a tenor with $T_i \in [0, \tilde{T}]$ for all $i \in \{1, ..., N\}$.*

A (fixed) coupon bond *with nominal $\mathcal{N} \in \mathbb{R}$ and coupons $c_i \in \mathbb{R}$ for $i \in \{1, ..., N\}$ on the given tenor is a product that pays c_i at each time point T_i and additionally the nominal $\mathcal{N}$ at maturity T_N.*

Remark 3.3. *A variation of this definition is that c_i are defined as coupon rates and the actual coupon payment is then $c_i \mathcal{N}$ at each time step T_i. Another popular definition includes the terminal payment of the nominal $\mathcal{N}$ in the last coupon c_N.*

Additionally to this "normal" coupon bond one can also find amortizing *coupon bonds in the market that distribute the nominal N in the coupons over all periods instead of paying it all at once at the maturity time.*

Lemma 3.4. *Let T_i and c_i be as in definition 3.2. The price of a fixed coupon paying bond is:*

$$\Pi(t) = \sum_{i=1}^{N} c_i P(t;T_i) + \mathcal{N} P(t;T_N) \tag{4}$$

for $t \in [0, T_1[$.

Proof. The coupon bond can be replicated by buying c_i T_i-zero-coupon-bonds for each $i \in \{1, ..., N\}$ and $\mathcal{N}$ T_N-bonds.
Such a portfolio of zero coupon bonds has a price as given in equation (4). As we are in a complete market, the two products must have the same value. □

Definition 3.5. *We define the* simple forward rate $L(t;S,T)$ *with fixing time S and payment time T at evaluation time t to be a relation of S- and T-bonds:*

$$1 + L(t;S,T)(T-S) = \frac{P(t;S)}{P(t;T)} \tag{5}$$

We can define different products that are strictly positive as numeraires as alternatives to the money market account. We then use a change of measure which gives us a different martingale measure corresponding to the numeraire, i.e. under this new measure all price processes of traded assets discounted with the numeraire are martingales as well.
A simple example is the terminal measure:

Definition 3.6. *The* terminal measure *is the martingale measure $\mathbb{Q}^B$ gained by using the terminal bond as numeraire, i.e.*

$$B(t) = P(t;\tilde{T}) \tag{6}$$

A more complex example is the spot measure.

Definition 3.7. *Let* $0 = T_0 < T_1 < ... < T_N = \tilde{T}$ *be a tenor on the time set* $[0, \tilde{T}]$.

The spot measure *is the martingale measure* $\mathbb{Q}^B$ *gained by using the numeraire:*

$$B(t) = P(t; T_{m(t)+1}) \prod_{i=0}^{m(t)} \frac{1}{P(T_i; T_{i+1})}, \tag{7}$$

where $m(t) = \max\{i \in \{0, ..., N-1\} \mid T_i \leq t\}$.

Remark 3.8. *Note that equation* (7) *can be rewritten to:*

$$B(t) = P(t; T_{m(t)+1}) \prod_{i=0}^{m(t)} \left(1 + L\left(T_{i-1}; T_{i-1}, T_i\right)\left(T_{i+1} - T_i\right)\right) \tag{8}$$

The numeraire in the spot measure can be explained as follows:

At $T_0 = 0$ we invest 1 into T_1-bonds. Once these expire (at T_1) we reinvest the money gained from them into T_2-bonds and so on. This product is generally known as *rolling bond*.

Elaborate!!!.

3.2 Model specification

Through the LIBOR model we construct the stochastic differential equations of simple forward rates for a consecutive set of time periods.

We start with a fixed time tenor of $N+1$ ($N \in \mathbb{N}$) points, that splits our time horizon $[0, \tilde{T}]$:

$$0 = T_0 < T_1 < ... < T_N = \tilde{T}.$$

The main objective is to simulate the one step simple forward rates for this tenor:

Assumption 3.1. *The one step simple forward rate (called LIBOR rate)*

$$L_i(t) := L(t; T_i, T_{i+1}) \quad \forall i \in \{0, ..., N\},\ t \in [0, T_i],$$

where $L(t; S, T)$ *is defined as in definition 3.5, follows an Itô stochastic process satisfying*

$$dL_i = \mu_t^i dt + \sigma_t^i dW_t^{\mathbb{Q}^B, L_i}, \tag{9}$$

$$L_i(0)(T_{i+1} - T_i) = \frac{P(0; T_i)}{P(0; T_{i+1})} - 1, \tag{10}$$

where $(W^{\mathbb{Q}^B,L_i})_{i\in\{0,\dots,N\}}$ *are possibly instantaneously correlated Brownian Motions.*

In the LIBOR model, all other variables are then derived from these interest rates, most importantly the T_i-bond prices. The attentive reader will have noticed, however, that there is a hole in this derivation: the so-called short-period bond $P(t;T_{m(t)+1})$ cannot be calculated by $(L_i(t))_{i\in\{0,\dots,N\}}$ alone, which is why we need the following assumption:

Assumption 3.2. *The short-period bond*

$$P(t;T_{m(t)+1})$$

is $\mathcal{F}_{m(t)}$-measurable. This means that all T_i-bond prices are predictable in $[T_{i-1},T_i]$. A specification that satisfies this assumption is:

$$P(t;T_{m(t)+1}) = \left(1 + L_{m(t)}(t)\left(T_{m(t)+1} - t\right)\right)^{-1}$$

Let us fix the spot measure as our valuation measure $\mathbb{Q}^B$. Given a variance-structure for the rates $(\sigma^i)_{i\in\{0,\dots,N\}}$, a correlation-structure $(\rho^{i,j})_{i,j\in\{0,\dots,N\}}$ for the Brownian Motions and we can specify a drift for the SDEs of the LIBORs:

Lemma 3.9. *Let $\mathbb{Q}^B$ be the spot measure. Let L_i satisfy assumptions 3.1 and 3.2. Let $(\sigma^i)_{i\in\{0,\dots,N\}}$ and $(\rho^{i,j})_{i,j\in\{0,\dots,N\}}$ be given, where*

$$d\left\langle W^{\mathbb{Q}^B,L_i}, W^{\mathbb{Q}^B,L_j}\right\rangle_t = \rho_t^{i,j}dt.$$

Then for each $i\in\{1,\dots,N\}$:

$$\mu_t^i = \sigma_t^i \sum_{j=m(t)+1}^{i} \frac{\rho_t^{ij}\sigma_t^j \Delta T_j}{1+\Delta T_j L_j(t)},$$

Proof. The proof is beyond the scope of this thesis, but can be found in a variety of literature. We reference [1] (pages 301 - 303) and [6], (pages 81 - 86). One can also take a look at the next section at the proof of theorem 4.5, which is very similar and carries its' idea and outline. □

Note that in its original form, the LIBOR model is a log-normal model (see [1]), hence the original model assumed $\sigma_t^i = L_i(t)\tilde{\sigma}_t$. However all proofs also work on a non log-normal model as well. The only restriction we need to apply is

$$L_i(t) > -1 \quad \forall i \in \{0, ..., N\}$$

to prevent negative or no zero coupon bond prices.

As mentioned in the previous chapter, having a model in terms of correlated Brownian Motions is not ideal for numerical replication. We therefore move to another notation, which uses factor loadings instead:

Lemma 3.10. *There exist m-dimensional stochastic processes $\lambda^i = (\lambda^{ik})_{k\in\{1,...m\}}$ and a m-dimensional $\mathbb{Q}^B$-Brownian Motion $U = (U^k)_{k\in\{1,...,m\}}$ such that*

$$dL_i(t) = \mu_t^i dt + \sigma_t^i dW_t^{\mathbb{Q}^B, L_i} \iff dL_i(t) = \mu_t^i dt + \lambda_t^i \cdot dU_t$$

Furthermore

$$\sum_{k=1}^{m} \lambda_t^{ik} \lambda_t^{ik} = (\sigma_t^i)^2$$

as well as

$$\sum_{k=1}^{m} \lambda_t^{ik} \lambda_t^{jk} = \sigma_t^i \sigma_t^j \rho_t^{i,j}$$

Proof. Follows directly from section 2.3. □

With this notation we can also rewrite our drift term of the LIBOR rates:

Remark 3.11. *For the spot measure the drifts μ^i in equation* (9) *can be rewritten in terms of λ:*

$$\mu_t^i = \sum_{k=1}^{m} \lambda_t^{ik} \sum_{j=m(t)}^{i} \frac{\lambda_t^{jk} \Delta T_j}{1 + L_j(t)\Delta T_j}$$

Proof. By lemmas 3.9 and 3.10 we have:

$$\begin{aligned}
\mu_t^i = \sigma_t^i \sum_{j=m(t)+1}^{i} \frac{\rho_t^{ij} \sigma_t^j \Delta T_j}{1 + \Delta T_j L_j(t)} &= \sum_{j=m(t)+1}^{i} \frac{\rho_t^{ij,d} \sigma_t^{j,d} \sigma_t^i \Delta T_j}{1 + \Delta T_j L_j(t)} \\
= \sum_{j=m(t)+1}^{i} \frac{\left(\sum_{k=1}^{m} \lambda_t^{ik} \lambda_t^{jk}\right) \Delta T_j}{1 + \Delta T_j L_j(t)} &= \sum_{k=1}^{m^d} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{ik} \lambda_t^{jk} \Delta T_j}{1 + \Delta T_j L_j(t)} \\
= \sum_{k=1}^{m} \lambda_t^{ik} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk} \Delta T_j}{1 + \Delta T_j L_j(t)}
\end{aligned}$$

□

4 Defaultable LIBOR Market Models

In this section we introduce defaultable LIBOR market models that we can use to value credits and credit options.
Our main source for this section is the article "Defaultable Discrete Forward Rate Model with Covariance Structure guaranteeing Positive Credit Spreads" authored by Christian Fries [2].

4.1 The Defaultable Forward Rate

We remain in the same setting as in the non-defaultable model, where we have a LIBOR tenor discretization $(T_i)_{i\in\{0,1,\ldots,N\}}$ and a set of (non-defaultable) zero coupon bonds $(P(t;T_i))_{i\in\{0,1,\ldots,N\}}$. Hence we can define the same products and apply the same valuation formulas.
We extend the model by defining an additional set of zero coupon bonds which are defaultable: $(P^d(t;T_i))_{i\in\{0,\ldots,N\}}$.
Note here that by construction we must still use the riskless bonds for the calculation of the Numeraire, as one can not use a risky asset as numeraire. Furthermore we do not consider recovery rates, i.e. we assume that a party is either able to pay all or nothing. We now introduce the concept of default and defaultable zero coupon bonds.

Definition 4.1. *The* default time *is a stopping time $\tau(\omega)$ on the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$.*
The default indicator $J(t)$ *is the indicator process over the default time:*

$$J(t) := \mathbf{1}_{\{\tau(\omega)\leq t\}}$$

Definition 4.2. *The* Defaultable Zero Coupon Bond *with price process*

$$P^d(t;T_i)$$

at time $t \in [0,T]$ is a traded asset that pays $1-J(T_i)$ at maturity $T_i \in \{T_0,\ldots,T_N\}$. Hence it pays 1 if the default has not happened until maturity.

It is easy to see that if default occurs, the price of a defaultable zero coupon bond jumps to zero. This means that the price process can be discontinuous at default events. This gives notion to the definition of a zero coupon bond conditional on pre-default.

Definition 4.3. *The* Defaultable Zero Coupon Bond conditional pre-default *is a continuous Itô-stochastic process $P^{d,*}(t;T_i)$ at time $t \in [0,T_i]$ with maturity T_i ($i \in \{0,...,N\}$) such that*

$$P^d(t;T_i) = P^{d,*}(t;T_i)(1 - J(t))$$

Definition 4.4. *The* simple Defaultable Forward Rate *is the rate gained from $P^{d,*}(t;T)$ by the same concept as in a non-defaultable model:*

$$L_i^d(t) := L^d(t;T_i,T_{i+1}) := \left(\frac{P^{d,*}(t;T_i)}{P^{d,*}(t;T_{i+1})} - 1\right)\Delta T_i, \tag{11}$$

where $T_i \in \{T_0, ...T_N\}$.

The simple Defaultable Forward Rate is the rate at which one can lend money to a defaultable party (for the time period T_i to T_{i+1}) at the risk of default, if the defaultable party is not in default at the evaluation time t [Add source].

Assumption 4.1. *As in the non defaultable model we assume that the defaultable short period bond conditional pre-default*

$$P^{d,*}(t;T_{m(t)+1})$$

has no diffusion. This means that the only stochasticity on the defaultable short rate bond is the default time.
I.e. we specify the defaultable short period bond as:

$$P^{d,*}(t;T_{m(t)+1}) = (1 + L_{m(t)}^d(t)(T_{m(t)+1} - t))^{-1}$$

Theorem 4.5. *Let L_i^d be defined as in equation (11). Let $B(t)$ be the numeraire under the spot measure (i.e. $B(t)$ is given by equation (7)) and $W^{\mathbb{Q}^B}$ a Brownian*

Motion w.r.t. the spot measure. Let $\sigma_t^{i,d} := \sigma^{i,d}(t,\omega)$ be a progressive stochastic process. Let

$$dL_i^d(t) = \mu_t^{i,d}dt + \sigma_t^{i,d}dW_t^{\mathbb{Q}^B,L_i^d} \tag{12}$$

be the stochastic differential of L_i^d.
Then

$$\mu_t^{i,d} = \sigma_t^{i,d}\sum_{j=m(t)+1}^{i}\frac{\rho_t^{ij,d}\sigma_t^{j,d}\Delta T_j}{1+\Delta T_j L_j^d(t)}, \tag{13}$$

where $\rho_t^{ij,d} = d\left\langle W_t^{\mathbb{Q}^B,L_i^d}, W_t^{\mathbb{Q}^B,L_j^d}\right\rangle$.

Proof. By construction the defaultable zero coupon bond $P^d(t;T_i)$ is a traded asset. We get

$$L_i^d(t)P^d(t;T_{i+1}) = (P^d(t;T_i) - P^d(t;T_{i+1}))\Delta T_i$$

is also a traded asset, because it is a portfolio of defaultable zero coupon bonds. Hence both processes discounted with the numeraire $B(t)$ are martingales.
By Itô we have:

$$\begin{aligned} d\left(L_i^d(t)\frac{P^d(t;T_{i+1})}{B(t)}\right) &= L_i^d(t)d\left(\frac{P^d(t;T_{i+1})}{B(t)}\right) + \frac{P^d(t;T_{i+1})}{B(t)}dL_i^d(t) \\ &\quad + d\left\langle L_i^d(t), \frac{P^d(t;T_{i+1})}{B(t)}\right\rangle \end{aligned}$$

We analyze the drift terms on each diffusion:

$$d\left(L_i^d(t)\frac{P^d(t;T_{i+1})}{B(t)}\right) \quad \text{and} \quad d\left(\frac{P^d(t;T_{i+1})}{B(t)}\right)$$

are martingale diffusions and hence have no drift. Therefore the drift terms of the two remaining differentials must cancel each other out. I.e.:

$$\frac{P^d(t;T_{i+1})}{B(t)}\mu_t^{i,d}dt \stackrel{!}{=} -d\left\langle L_i^d(t), \frac{P^d(t;T_{i+1})}{B(t)}\right\rangle \tag{14}$$

To calculate the quadratic variation we need the diffusion of the discounted defaultable zero coupon bond:

$$\begin{aligned}
d\left(\frac{P^d(t;T_{i+1})}{B(t)}\right) &= d\left(\frac{P^d(t;T_{m(t)+1})}{B(t)}\prod_{j=m(t)+1}^{i}(1+\Delta T_j L_j^d(t))^{-1}\right)\\
&= (\dots)dt - \frac{P^d(t;T_{m(t)+1})}{B(t)}\sum_{j=m(t)+1}^{i}\frac{\sigma_t^{j,d}\Delta T_j}{1+\Delta T_j L_j^d(t)}dW_t^{\mathbb{Q}^B,L_j^d}\\
&\quad+\prod_{j=m(t)+1}^{i}(1+\Delta T_j L_j^d(t))^{-1} d\left(\frac{P^d(t;T_{m(t)+1})}{B(t)}\right)\\
&\quad+d\left\langle \frac{P^d(t;T_{m(t)+1})}{B(t)}, \prod_{j=m(t)+1}^{i}(1+\Delta T_j L_j^d(t))^{-1}\right\rangle.
\end{aligned}$$

With assumptions 3.2 and 4.1 we get

$$d\left(\frac{P^d(t;T_{m(t)+1})}{P(t;T_{m(t)+1})}\right) = (\dots)dt - \frac{P^{d,*}(t;T_{m(t)+1})}{P(t;T_{m(t)+1})}dJ(t).$$

Furthermore we have

$$d\left(\frac{P^d(t;T_{m(t)+1})}{B(t)}\right) = \prod_{j=0}^{m(t)}(1+\Delta T_j L_j(T_j))^{-1} d\left(\frac{P^d(t;T_{m(t)+1})}{P(t;T_{m(t)+1})}\right)$$

and for any Itô-process X: $d\langle X, J\rangle = 0$ [Add source].

This yields

$$\begin{aligned}
d\left(\frac{P^d(t;T_{i+1})}{B(t)}\right) &= (\dots)dt - \frac{P^d(t;T_{m(t)+1})}{B(t)}\sum_{j=m(t)+1}^{i}\frac{\sigma_t^{j,d}\Delta T_j}{1+\Delta T_j L_j^d(t)}dW_t^{\mathbb{Q}^B,L_j^d}\\
&\quad-\frac{P^{d,*}(t;T_{i+1})}{P(t;T_{m(t)+1})}dJ(t).
\end{aligned}$$

We get

$$d\left\langle L_i^d(t), \frac{P^d(t;T_{i+1})}{B(t)}\right\rangle = \sigma_t^{i,d}\frac{P^d(t;T_{m(t)+1})}{B(t)}\sum_{j=m(t)+1}^{i}\frac{\sigma_t^{j,d}\Delta T_j}{1+\Delta T_j L_j^d(t)}\rho_t^{ij,d}dt.$$

Inserting into equation (14) yields our statement (13). □

Just as we did in the last section, we now move from the "covariance" process model to a "factor loading" model as described in section 2.3. Note that we also need to include the non defaultable covariance structure for our new model.

Lemma 4.6. *There exist* m^d*-dimensional stochastic processes* $\lambda^{i,d} = (\lambda^{ik,d})_{k\in\{1,\dots,m^d\}}$, $\lambda^i = ((\lambda^{ik})_{k\in\{1,\dots m\}}, (0)_{k\in\{m+1,\dots m^d\}})$ *and a* m^d*-dimensional* $\mathbb{Q}^B$*-Brownian Motion* $U = (U^k)_{k\in\{1,\dots,m^d\}}$ *such that*

$$\left.\begin{aligned} dL_i(t) &= \mu_t^i dt + \sigma_t^i dW_t^{\mathbb{Q}^B, L_i} \\ dL_i^d(t) &= \mu_t^{i,d} dt + \sigma_t^{i,d} dW_t^{\mathbb{Q}^B, L_i^d} \end{aligned}\right\} \iff \begin{cases} dL_i(t) = \mu_t^i dt + \lambda_t^i \cdot dU_t \\ dL_i^d(t) = \mu_t^{i,d} dt + \lambda_t^{i,d} \cdot dU_t \end{cases}$$

Furthermore

$$\sum_{k=1}^{m^d} \lambda_t^{ik,d}\lambda_t^{ik,d} = (\sigma_t^i)^2 \quad \textit{and}$$
$$\sum_{k=1}^{m^d} \lambda_t^{ik,d}\lambda_t^{jk,d} = \sigma_t^i\sigma_t^j\rho_t^{i,j}.$$

Proof. Follows directly from section 2.3. □

Hence from here on we use this computation friendly version:

$$dL_i^d(t) = \mu_t^{i,d} dt + \lambda_t^{i,d} \cdot dU_t. \tag{15}$$

Remark 4.7. *For the spot measure the drifts* $\mu^{i,d}$ *in equation* (15) *can be rewritten in terms of* λ^d*:*

$$\mu_t^{i,d} = \sum_{k=1}^{m^d} \lambda_t^{ik,d} \sum_{j=m(t)}^{i} \frac{\lambda_t^{jk,d}\Delta T_j}{1 + L_j^d(t)\Delta T_j}$$

Proof. By equation (13) and lemma 4.6 we have:

$$\begin{aligned} \mu_t^{i,d} &= \sigma_t^{i,d} \sum_{j=m(t)+1}^{i} \frac{\rho_t^{ij,d}\sigma_t^{j,d}\Delta T_j}{1 + \Delta T_j L_j^d(t)} = \sum_{j=m(t)+1}^{i} \frac{\rho_t^{ij,d}\sigma_t^{j,d}\sigma_t^{i,d}\Delta T_j}{1 + \Delta T_j L_j^d(t)} \\ &= \sum_{j=m(t)+1}^{i} \frac{\left(\sum_{k=1}^{m^d} \lambda_t^{ik,d}\lambda_t^{jk,d}\right)\Delta T_j}{1 + \Delta T_j L_j^d(t)} = \sum_{k=1}^{m^d} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{ik,d}\lambda_t^{jk,d}\Delta T_j}{1 + \Delta T_j L_j^d(t)} \\ &= \sum_{k=1}^{m^d} \lambda_t^{ik,d} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk,d}\Delta T_j}{1 + \Delta T_j L_j^d(t)} \end{aligned}$$

□

4.2 Covariance Structures Guaranteeing positive spreads

We now investigate how we can generate positive spreads from the defaultable LIBOR market model.

For this purpose let us define a spread:

Definition 4.8. *Let L^d and L be defined as before. The* spread S_i *for the LIBOR period* $[T_i; T_{i+1}]$*, where* $i \in \{0, ..., N\}$ *is defined as:*

$$S_i(t) := L_i^d(t) - L_i(t) \quad \textit{for } t \in \left[0, \tilde{T}\right]$$

Remark 4.9. *A negative spread would mean $L_i^d(t) < L_i(t)$. This constitutes an arbitrage possibility, which is why the model needs to be specified, such that this case is "impossible" (i.e. has probability 0).*

The spreads dynamics are given by

$$dS_i(t) = \mu_t^{i,S} dt + \lambda_t^{i,S} \cdot dU_t$$

where

$$\mu_t^{i,S} = \mu_t^{i,d} - \mu_t^i, \quad \text{and} \quad \lambda_t^{i,S} = \lambda_t^{i,d} - \lambda_t^i.$$

Note that we "extended" the vectors $\lambda_t^i = (\lambda_t^{i1}, ..., \lambda_t^{im}, 0, ..., 0)^T$.
Given a numeraire and the factor loadings of the non-defaultable LIBOR rates λ^i, the goal is, to find a specification for the defaultable factor loadings $\lambda^{i,d}$ such that S_i is always positive.
The most common dynamics that guarantee positivity are log-normal dynamics as discussed in lemma 2.4. So one idea is to find restrictions on $\lambda_t^{i,d}$ such that

$$\mu_t^{i,d} - \mu_t^i = S_i(t)\tilde{\mu}_t^{i,S}, \quad \text{and} \quad \lambda_t^{i,d} - \lambda_t^i = S_i(t)\tilde{\lambda}_t^{i,S}$$

for two processes $\tilde{\mu}^{i,S}$ and $\tilde{\lambda}^{i,S}$.

Lemma 4.10. *Let $\mathbb{Q}^B$ be the spot measure (i.e. $B(t)$ is given by equation (7)). Let L^d be defined as before with*

$$\begin{aligned} \lambda_t^{ik,d} &= \frac{1 + L_i^d \Delta T_i}{1 + L_i \Delta T_i} \lambda_t^{ik} \qquad && \textit{for } k = 1, ..., m \\ \lambda_t^{ik,d} &= \left(L_i^d(t) - L_i(t)\right) f_t^{ik} && \textit{for } k = m+1, ...m^d \end{aligned} \tag{16}$$

where f_t^{ik} are (possibly stochastic) processes for $i = 1, ..., N$ and $k = m+1, ...m^d$. Then $S = (S_i)_{i \in \{0,...,N\}}$ satisfies

$$dS_i = S_i(t)\tilde{\mu}_t^{i,S} dt + S_i(t)\tilde{\lambda}_t^{i,S} \cdot dU_t.$$

for some process $\tilde{\mu}^{i,S}$ and $\tilde{\lambda}^{i,S}$.

Proof. By remark 4.7 we have:

$$
\begin{aligned}
dS_i &= dL_i^d - dL_i \\
&= \sum_{k=1}^{m^d} \lambda_t^{ik,d} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk,d}\Delta T_j}{1+L_j^d(t)\Delta T_j} dt + \sum_{k=1}^{m^d} \lambda_t^{ik,d} dU_t^k \\
&\quad - \sum_{k=1}^{m} \lambda_t^{ik} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk}\Delta T_j}{1+L_j(t)\Delta T_j} dt - \sum_{k=1}^{m} \lambda_t^{ik} dU_t^k \\
&= \sum_{k=1}^{m} \left(\lambda_t^{ik,d} - \lambda_t^{ik}\right) \left(\sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk}\Delta T_j}{1+L_j(t)\Delta T_j} dt + dU_t^k \right) && \text{(A)} \\
&\quad + \left(L_i^d(t) - L_i(t)\right) \left(\sum_{k=m+1}^{m^d} f_t^{ik} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk,d}\Delta T_j}{1+L_j^d(t)\Delta T_j} dt + dU_t^k \right) \\
&= S_i \sum_{k=1}^{m} \frac{\lambda_t^{ik}\Delta T_i}{1+L_i(t)\Delta T_i} \left(\sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk}\Delta T_j}{1+L_j(t)\Delta T_j} dt + dU_t^k \right) && \text{(B)} \\
&\quad + S_i \left(\sum_{k=m+1}^{m^d} f_t^{ik} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk,d}\Delta T_j}{1+L_j^d(t)\Delta T_j} dt + dU_t^k \right)
\end{aligned}
$$

where (A) comes from the relation:

$$
\frac{\lambda_t^{jk,d}\Delta T_j}{1+L_j^d(t)\Delta T_j} = \frac{\lambda_t^{jk}\Delta T_j}{1+L_j(t)\Delta T_j} \quad \text{for } k \in \{1,...,m\}
$$

and (B) comes from

$$
\lambda_t^{ik,d} - \lambda_t^{ik} = \left(\frac{1+L_i^d(t)\Delta T_i}{1+L_i(t)\Delta T_i} - \frac{1+L_i(t)\Delta T_i}{1+L_i(t)\Delta T_i} \right) \lambda_t^{ik} = S_i \frac{\lambda_t^{ik}\Delta T_i}{1+L_i(t)\Delta T_i}
$$

for $k \in \{1,...,m\}$. □

Elaborate!!!.

4.3 The Survival Probability

Until now we assumed every stochastic process to be adapted to the filtration $\mathbb{F}$. This is what is commonly done in mathematical finance to find theoretical prices and it simulates the availability of information at each time t. This includes the indicator over the default time $J(t)$. However the products we are interested in, are rarely dependent on the (future) default state, but rather its probability, as we see in the next chapter. So it is more efficient (and less dependent on extra assumptions) to calculate the survival probability instead of simulating the default time.

It is important to note here that for pricing we are only interested in the behavior of the rates under the martingale measure $\mathbb{Q}^B$ and so we also only focus on the survival probability under this measure. This means the probability we derive, is not to be confused with the real world probability of survival.

As we work quite a lot with conditional probability and expectation we denote for a σ-algebra $\mathcal{G}$:

$$\mathbb{E}_{\mathcal{G}}^{\mathbb{Q}^B}\left[\,\cdot\,\right] = \mathbb{E}^{\mathbb{Q}^B}\left[\,\cdot \mid \mathcal{G}\,\right]$$
$$\mathbb{Q}_{\mathcal{G}}^{B}\left(\,\cdot\,\right) = \mathbb{Q}^{B}\left(\,\cdot \mid \mathcal{G}\right).$$

With the assumption that the short period bond is deterministic w.r.t. the previous LIBOR time, we can derive the price of a zero-coupon bond that only lives inside a LIBOR period:

Lemma 4.11. *For $T_i \leq s < t \leq T_{i+1}$ it holds that*

- $P(s;T_{i+1}) = P(s;t)P(t;T_{i+1})$ *or* $P(s;t) = \frac{P(s;T_{i+1})}{P(t;T_{i+1})}$ *and*
- $P^{d,*}(s;T_{i+1}) = P^{d,*}(s;t)P^{d,*}(t;T_{i+1})$ *or* $P^{d,*}(s;t) = \frac{P^{d,*}(s;T_{i+1})}{P^{d,*}(t;T_{i+1})}$ *for* $\tau > s$.

Proof. Let $T_i \leq s < t \leq T_{i+1}$. With assumption 3.2 we have:

$$\begin{aligned} P(s;T_{i+1}) &= B(s)\mathbb{E}_{\mathcal{F}_s}^{\mathbb{Q}^B}\left[\frac{P(t;T_{i+1})}{B(t)}\right] = P(t;T_{i+1})B(s)\mathbb{E}_{\mathcal{F}_s}^{\mathbb{Q}^B}\left[\frac{P(t;t)}{B(t)}\right] \\ &= P(t;T_{i+1})P(s;t) \end{aligned}$$

We can do a very similar derivation for $P^{d,*}(s;t)$ with assumption 4.1:

$$\begin{aligned} P^{d,*}(s;T_{i+1})\mathbf{1}_{\{\tau>s\}} &= B(s)\mathbb{E}_{\mathcal{F}_s}^{\mathbb{Q}^B}\left[\frac{P^d(t;T_{i+1})}{B(t)}\right] \\ &= P^{d,*}(t;T_{i+1})B(s)\mathbb{E}_{\mathcal{F}_s}^{\mathbb{Q}^B}\left[\frac{\mathbf{1}_{\{\tau>t\}}}{B(t)}\right] \\ &= P^{d,*}(t;T_{i+1})B(s)\mathbb{E}_{\mathcal{F}_s}^{\mathbb{Q}^B}\left[\frac{P^d(t;t)}{B(t)}\right] \\ &= P^{d,*}(t;T_{i+1})P^{d,*}(s;t)\mathbf{1}_{\{\tau>s\}} \end{aligned}$$

$\square$

With this property we can express the probability for the survival inside a LIBOR period w.r.t. the spot measure:

Lemma 4.12. *Let $\mathbb{Q}^B$ be the spot measure. For $T_i \leq s < t \leq T_{i+1}$ it holds that:*

$$\mathbb{Q}^B_{\mathcal{F}_s}\left(\{\tau > t\} \middle| \{\tau > s\}\right) = \frac{P^{d,*}(s;t)}{P(s;t)}$$

Proof. Let $T_i \leq s < t \leq T_{i+1}$. From the martingale property of $\frac{P^d(s;t)}{B(s)}$ we have:

$$\begin{aligned}
\frac{P^{d,*}(s;t)}{P(s;t)} &= \mathbb{E}^{\mathbb{Q}^B}_{\mathcal{F}_s}\left[\frac{B(s)}{B(t)}\right]^{-1} B(s)\mathbb{E}^{\mathbb{Q}^B}_{\mathcal{F}_s}\left[\frac{\mathbf{1}_{\{\tau>t\}}}{B(t)}\middle|\{\tau > s\}\right] \\
&= \left(\frac{B(s)}{B(t)}\right)^{-1}\frac{B(s)}{B(t)}\mathbb{E}^{\mathbb{Q}^B}_{\mathcal{F}_s}\left[\mathbf{1}_{\{\tau>t\}}\middle|\{\tau > s\}\right] \\
&= \mathbb{Q}^B_{\mathcal{F}_s}\left(\{\tau > t\} \middle| \{\tau > s\}\right)
\end{aligned}$$

$\square$

Setting $s = T_i$ and $t = T_{i+1}$ gives us the one step survival probability:

Lemma 4.13. *Let B be the numeraire under the spot measure, $i \in \{1, ..., N\}$. The one-step survival probability until T_{i+1} w.r.t. $\mathcal{F}_{T_i}$ given pre-default is*

$$q(T_{i+1}, \omega) := \mathbb{Q}^B_{\mathcal{F}_{T_i}}\left(\{\tau(\omega) > T_{i+1}\} \mid \{\tau(\omega) > T_i\}\right) = \frac{P^{d,*}(T_i;T_{i+1})}{P(T_i;T_{i+1})}. \tag{17}$$

Proof. Can directly be derived from the lemma above with $s = T_i$ and $t = T_{i+1}$. $\square$

Note that while we calculate a probability here, it is a *pathwise* probability: at each start of a LIBOR period T_i – if default has not happened yet – we can evaluate how probable it is that P^d survives the coming period.
Let us define a new value:

Definition 4.14. *Let $i \in \{1, ..., N\}$. We define the random variable Q as*

$$Q(T_i, \omega) = \prod_{j=1}^{i} q(T_j, \omega)$$

While Q in theory is not a probability, we can view it as the pathwise survival probability until T_i, if we have no information regarding the actual default state even at time T_i. Note that this is an *interpretation* and the actual proof for this statement needs extra assumptions [Attach proof for this?].

What we actually proof in the next chapter is that the total survival probability until T_i is given by the expectation of Q:

$$\mathbb{Q}^B(\{\tau > T_i\}) = \mathbb{E}^{\mathbb{Q}^B}[Q(T_i;\omega)]$$

Let us rewrite the before derived/defined values in terms of L and L^d:

Remark 4.15. *Note that*

$$q(T_{i+1},\omega) = \frac{1+\Delta T_i L_i(T_i)(\omega)}{1+\Delta T_i L_i^d(T_i)(\omega)}.$$

Furthermore, with the definition of $q(T_i,\omega)$ one can rewrite Q in terms of the numeraire $B(T_i)$ (corresponding to the Spot measure):

$$Q(T_{i+1},\omega) = \frac{B(T_i)}{\prod_{k=0}^{i}(1+\Delta T_k L_k^d(T_k)(\omega))} =: \frac{B(T_i)}{B^d(T_i)}.$$

Note that $Q(T_i,\omega)$ is $\mathcal{F}_{T_{i-1}}$-measurable.

The denominator in this equation can be seen as a "defaultable numeraire", which we define next:

Definition 4.16. *We call B^d given by:*

$$B^d(t) = P^{d,*}(t;T_{m(t)+1})\prod_{k=1}^{m(t)}\frac{1}{P^{d,*}(T_k;T_{k+1})} = P^{d,*}(t;T_{m(t)+1})\prod_{k=1}^{m(t)}(1+\Delta T_k L_k^d(T_k))$$

the defaultable numeraire*.*

Due to B^d not being a traded product, it is not a real numeraire. Hence there is not an equivalent martingale measure w.r.t. B^d that we can use to price other products. And yet, we will see that the pricing formula for defaultable products is similar to a change of numeraires with B^d. Therefore, let us look at pricing in more detail.

5 Loans and Credit Option Pricing

A model is useless without a practical application for it. Hence we take a look at how to apply our results to pricing loans and credit options.
Note here that the model is for use in numerical option pricing, so deriving prices in terms of an expectation over model primitives (directly simulated values or values that can be derived from these) is sufficient. We can then use Monte Carlo Methods to approximate the expectation.
For a start let us define our simulation filtration:

Definition 5.1. *The filtration* $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,\tilde{T}]}$ *is given by*

$$\mathcal{G}_t := \sigma(U_t^k | k = 1, ..., m^d) \subset \mathcal{F}_t \quad \forall t \in [0, \tilde{T}].$$

Note that all model primitives are $\mathbb{G}$-adapted, as $\mathbb{G}$ is the filtration over the only simulated stochastic processes (in our case the Brownian Motion U). As mentioned before we are interested in pricing without simulating the default time τ, which might be driven by other stochastic values. This means that τ is not necessarily a stopping time w.r.t. $\mathbb{G}$.

5.1 General Application

In this subsection, we derive a general pricing methodology before moving to specific products.
A problem is that our model does not capture default. This is illustrated in figure 1.
While the model simulates only paths of $P^{d,*}(t;T_i)$ (in the figure $P^{d,*}(t;4.0)$), there are indefinitely many paths of $P^d(t;T_i)$ (in the figure $P^d(t;4.0)(\omega_j)$ for $j \in \{1,2,3\}$) which satisfy

$$P^d(t;T_i) = P^{d,*}(t;T_i)(1 - J(t)),$$

but are not equal.
We therefore use a trick and replace the default state $1 - J(t)$ in all our pricing

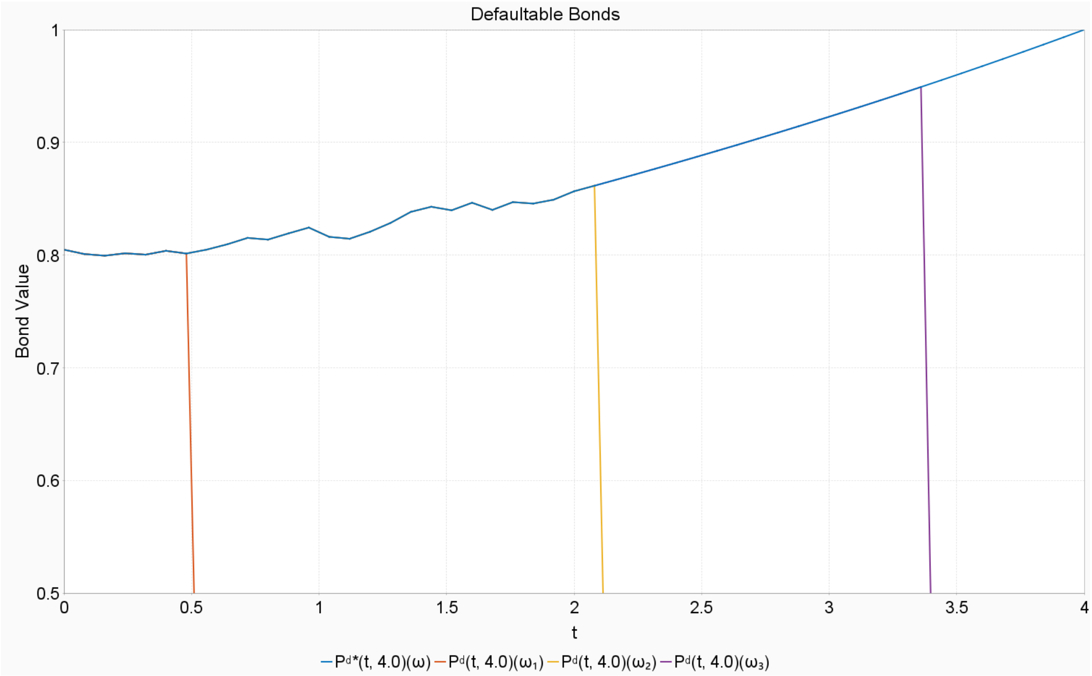

Figure 1: Paths of P^d

formulas with the random variable Q that we defined in the previous chapter. To proof this works, assume a T_i-claim X that has a payoff that is dependent on survival of the defaultable bonds. We can then use the pricing methodology described in te following lemma.

Lemma 5.2. *Let X be a payoff at $T_i \in \{T_1, ..., T_N\}$, with*

$$X = X^S \mathbf{1}_{\{\tau(\omega) > T_i\}},$$

where X^S is the payoff conditional pre-default. Let X^S be $\mathcal{G}_{T_i}$-measurable. Then the price at time $t = 0$ is

$$\Pi^X = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^S}{B(T_i)} Q\left(T_i, \omega\right)\right].$$

Note that by construction $Q(T_i, \omega)$ is $\mathcal{G}_{T_i}$-measurable, which means, we have a price in terms of model primitives. Let us now proof this statement.

Proof. For our proof we use following property of the conditional expectation extensively:

$$\mathbb{E}^{\mathbb{Q}^B}\left[Y\right] = \mathbb{E}^{\mathbb{Q}^B}\left[Y|A\right]\mathbb{Q}^B(A) + \mathbb{E}^{\mathbb{Q}^B}\left[Y|A^c\right]\mathbb{Q}^B(A^c) \tag{A}$$

for any set $A \in \mathcal{F}$.

For ease of notation we write $q(T_i) := q(T_i, \omega)$ and $Q(T_i) := Q(T_i, \omega)$. Note $B(T_0) = 1$. We have:

$$
\begin{aligned}
\Pi^X &= \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^S}{B(T_i)}\mathbf{1}_{\{\tau(\omega)>T_i\}}\right] \\
&= \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^S}{B(T_i)}\mathbf{1}_{\{\tau(\omega)>T_i\}}\middle| \{\tau > T_1\}\right] q(T_1) && \text{(B)} \\
&= \mathbb{E}^{\mathbb{Q}^B}\left[\mathbb{E}^{\mathbb{Q}^B}_{\mathcal{F}_{T_1}}\left[q(T_1)\frac{X^S}{B(T_i)}\mathbf{1}_{\{\tau(\omega)>T_i\}}\middle| \{\tau > T_1\}\right]\right] && \text{(C)} \\
&= \mathbb{E}^{\mathbb{Q}^B}\left[\mathbb{E}^{\mathbb{Q}^B}_{\mathcal{F}_{T_1}}\left[q(T_1)\frac{X^S}{B(T_i)}\mathbf{1}_{\{\tau(\omega)>T_i\}}\middle| \{\tau > T_2\}\right] q(T_2)\right] \\
&= \mathbb{E}^{\mathbb{Q}^B}\left[\mathbb{E}^{\mathbb{Q}^B}_{\mathcal{F}_{T_2}}\left[Q(T_2)\frac{X^S}{B(T_i)}\mathbf{1}_{\{\tau(\omega)>T_i\}}\middle| \{\tau > T_2\}\right]\right] \\
&= \dots \\
&= \mathbb{E}^{\mathbb{Q}^B}\left[\mathbb{E}^{\mathbb{Q}^B}_{\mathcal{F}_{T_i}}\left[Q(T_i)\frac{X^S}{B(T_i)}\mathbf{1}_{\{\tau(\omega)>T_i\}}\middle| \{\tau > T_i\}\right]\right] && \text{(D)} \\
&= \mathbb{E}^{\mathbb{Q}^B}\left[Q(T_i)\frac{X^S}{B(T_i)}\right],
\end{aligned}
$$

where (B) comes from (A) and (C) comes from the tower property and the fact that $q(T_1)$ is a constant. Then we repeat these steps until eq. (D) always keeping in mind $q(T_j)$ is $\mathcal{F}_{T_i}$-measurable for all $j \leq i$. □

This methodology can easily be extended to products where only parts of the payoff are dependent on survival and even where other parts are dependent on default:

Lemma 5.3. *Let X be a payoff at $T_i \in \{T_1, ..., T_N\}$, with*

$$X = X^O + X^S \mathbf{1}_{\{\tau(\omega)>T_i\}} + X^D \mathbf{1}_{\{\tau(\omega)\leq T_i\}},$$

where the payoff X^O is unconditional, X^S is conditional pre-default and X^D is conditional past-default and all are $\mathcal{G}_{T_i}$-measurable.

Then the price at time $t = 0$ is

$$\Pi^X = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^O}{B(T_i)}\right] + \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^S}{B(T_i)}Q(T_i)\right] + \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^D}{B(T_i)}\left(1 - Q(T_i)\right)\right].$$

Proof. For the parts X^O and X^S the statement clear. For X^D we have:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^D}{B(T_i)}J(T_i)\right] &= \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^D}{B(T_i)}\right] - \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^D}{B(T_i)}\mathbf{1}_{\{\tau(\omega)>T_i\}}\right] \\ &= \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^D}{B(T_i)}\left(1-Q(T_i)\right)\right]\end{aligned}$$

with the same arguments as in the lemma before. □

Lets re-evaluate the above equation with our expression for the Q. One might notice that pricing defaultable products is similar to a change of numeraire with the defaultable numeraire:

$$\mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^S}{B(T_i)}Q(T_i)\right] = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^S}{B^d(T_i)}\right]. \tag{18}$$

While this cannot hold up as a proof, it can be used as an intuitive explanation: we exchange the theoretical time value of future money with one that also accounts for default possibilities.

With this expression it is also easy to derive an alternative numerical price for the defaultable zero-coupon bonds, by setting $X^S = 1$:

$$P^d(0;T_i) = P^{d,*}(0;T_i) = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{1}{B^d(T_i)}\right]$$

While we normally have the analytic value of $P^d(0,T_i)$ as an input value to our model (or at least calibrated from other input values), the numerical price can act as an error measurement for the model we create. Furthermore, we can later use it as a control variate in pricing, as it is done for the non defaultable model (see [1]), which is beyond the scope of this thesis.

We can also get a numerical expression for the total survival probability, because setting $X^S = B(T_i)$ and pricing the claim yields:

$$\begin{aligned}\Pi^X &= \mathbb{E}^{\mathbb{Q}^B}\left[\frac{X^S}{B(T_i)}\mathbf{1}_{\{\tau(\omega)>T_i\}}\right] = \mathbb{E}^{\mathbb{Q}^B}\left[\mathbf{1}_{\{\tau(\omega)>T_i\}}\right] = \mathbb{Q}^B(\{\tau > T_i\}) \\ &= \mathbb{E}^{\mathbb{Q}^B}\left[Q(T_i)\right]\end{aligned}$$

With these formulas we now have an expression for claim prices in terms of an expectation over model primitives. We can therefore move to more specific products now.

5.2 General Loan Pricing

Let us first take a look what a loan is in general. "A loan is a sum of money that one or more individuals or companies borrow from banks or other financial institutions [...]. In doing so, the borrower incurs a debt, which he has to pay back with interest and within a given period of time."[4].

Hence a loan is nothing else than a fixed coupon bond, where the nominal $\mathcal{N}$ represents the debt and the coupons c_i represent the interest. In the non defaultable case, "pricing" the loan is therefore nothing else than setting the coupons such that (we assume the same coupon tenor as LIBOR tenor):

$$\mathcal{N} \stackrel{!}{=} \sum_{i=1}^{N} c_i P(0;T_i) + \mathcal{N} P(0;T_N) \tag{19}$$

Remark 5.4. *Setting*

$$c_i = \mathcal{N} \Delta T_i L_i(0)$$

satisfies equation (19) *and therefore yields a valid loan.*

Proof. We have

$$\Delta T_i L_{i-1}(0) = \frac{P(0;T_{i-1})}{P(0;T_i)} - 1$$

Inserting in equation (19) yields (note $P(0;T_0) = 1$):

$$\begin{aligned} &\sum_{i=1}^{N} \mathcal{N} \Delta T_i L_i(0) P(0;T_i) + \mathcal{N} P(0;T_N) \\ &= \sum_{i=1}^{N} \mathcal{N} \left(\frac{P(0;T_{i-1})}{P(0;T_i)} - 1\right) P(0;T_i) + \mathcal{N} P(0;T_N) \\ &= \sum_{i=1}^{N} \mathcal{N} (P(0;T_{i-1}) - P(0;T_i)) + \mathcal{N} P(0;T_N) = \mathcal{N} \end{aligned}$$

□

Let us look at the defaultable case:

Lemma 5.5. *The price of a coupon paying bond which has defaultable cashflows is:*

$$\mathcal{N} = \sum_{i=1}^{N} c_i P^{d,*}(0;T_i) + \mathcal{N} P^{d,*}(0;T_N)$$

Proof. Let $c \in \mathbb{R}$ be constant. Then for $i \in \{1, ..., N\}$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^B}\left[\frac{c\,\mathbf{1}_{\{\tau(\omega)>T_i\}}}{B(T_i)}\right] &= c\,\mathbb{E}^{\mathbb{Q}^B}\left[\frac{P^d(T_i;T_i)}{B(T_i)}\right] \\
&= c\,P^d(0;T_i) = c\,P^{d,*}(0;T_i).
\end{aligned}$$

Replacing c with c_i, $\mathcal{N}$ repectively, and taking the sum yields the statement. □

A loan, where the debtor is defaultable hence must satisfy:

$$\mathcal{N} \overset{!}{=} \sum_{i=1}^{N} c_i P^{d,*}(0;T_i) + \mathcal{N} P^{d,*}(0;T_N)$$

We again have a natural solution to this equation. With the same arguments as for remark 5.4 we find that setting

$$c_i = \mathcal{N} \Delta T_i L_i^d(0)$$

yields a valid loan.

Note that for a simple loan it does not matter if the creditor is defaultable, because they have no debt w.r.t. the loan after $t = 0$ and in case of their default the pending claim would still be collected.

Let us take a look at a future on defaultable loans, or rather a future on a defaultable coupon paying bond. Remember that in contrast to an option a future brings the obligation to enter into an agreement. Hence from the debtors point of view we have a payoff function

$$\Psi(T_s) = \left(\mathcal{N} - \sum_{i=s+1}^{N} c_i P^{d,*}(T_s;T_i)\right) \mathbf{1}_{\{\tau>T_s\}} \tag{20}$$

where T_s is the start time of the bond and for ease of notation the terminal coupon c_N includes the terminal payment of the nominal $\mathcal{N}$.

This payoff function gives us a direct approach to price the product:

Lemma 5.6. *A T_s claim with a payoff function as in equation* (20) *has the price*

$$\begin{aligned}
\Pi^{\Psi} &= \mathcal{N} P^d(0;T_s) - \sum_{i=s+1}^{N} c_i P^d(0;T_i) \\
&= \mathcal{N} P^{d,*}(0;T_s) - \sum_{i=s+1}^{N} c_i P^{d,*}(0;T_i)
\end{aligned}$$

at $t = 0$.

Proof. We have:

$$\begin{aligned}
\Pi^{\Psi} &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\left(\mathcal{N}-\sum_{i=s+1}^{N} c_i P^{d,*}(T_s;T_i)\right)\mathbf{1}_{\{\tau>T_s\}}}{B(T_s)}\right] \\
&= \mathcal{N}\mathbb{E}^{\mathbb{Q}}\left[\frac{P^d(T_s;T_s)}{B(T_s)}\right]-\sum_{i=s+1}^{N} c_i\mathbb{E}^{\mathbb{Q}}\left[\frac{P^d(T_s;T_i)}{B(T_s)}\right] \\
&= \mathcal{N}P^d(0;T_s)-\sum_{i=s+1}^{N} c_i P^d(0;T_i)
\end{aligned}$$

□

To gain an initial price of zero one can set the coupons as done before:

$$c_i = \mathcal{N}\Delta T_i L_i^d(0)$$

In the past example default of the creditor did not matter, but in this case actually the future coupon bond could not be bought. Hence creditor default possibility does indeed change the payoff and hence the price of the product:

$$\Psi(T_s) = \left(\mathcal{N}-\sum_{i=s+1}^{N} c_i P^{d^1,*}(T_s;T_i)\right)\mathbf{1}_{\{\tau^1>T_s\}}\mathbf{1}_{\{\tau^2>T_s\}} \tag{21}$$

where Elaborate i.e. on default of the issuer!!!.

5.3 Credit Options

In the last subsection we considered pricing loans and loan forwards. Essentially cash flow that is set in stone (with the exception of default). Now let us consider more intriguing products which bring an optionality with them: We start with a simple put option on a coupon bond (note that we work a lot with coupon bonds as to their equivalence with a loan).

Remark 5.7. *A put option on a coupon paying bond, with maturity T_s and strike price $\mathcal{N}$ where the debtor/payer is defaultable has the payoff function*

$$\Psi(T_s) = \left(\mathcal{N}-\sum_{i=s+1}^{N} c_i P^{d,*}(T_s;T_i)\right)^{+}\mathbf{1}_{\{\tau>T_s\}} \tag{22}$$

where $(\cdot)^+ := \max(0,\cdot)$.

We can numerically compute the price.

Lemma 5.8. *The price of a claim with a payoff function as in equation* (22) *has the price:*

$$\Pi^{\Psi} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{\left(\mathcal{N} - \sum_{i=s+1}^{N} c_i P^{d,*}(T_s;T_i)\right)^+}{B^d(T_s)}\right]$$

Proof. Directly follows from lemma 5.2. □

Also this product can be extended to both counter parties being defaultable: the debtor and the creditor.

The adjusted formulas are

$$\Psi(T_s) = \left(\mathcal{N} - \sum_{i=s+1}^{N} c_i P^{d^1,*}(T_s;T_i)\right)^+ \mathbf{1}_{\{\tau^1>T_s\}}\mathbf{1}_{\{\tau^2>T_s\}}$$

for the payoff and

$$\Pi^{\Psi} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{\left(\mathcal{N} - \sum_{i=s+1}^{N} c_i P^{d^1,*}(T_s;T_i)\right)^+}{B^{d^1}(T_s)} Q_2(T_s)\right]$$

for pricing. A proof is omitted as it is trivial.

A more advanced product is a cancellable loan. We start with the case, where one can cancel the loan at a single time point (that is part of the LIBOR tenor). Consider the cash flow of such a product from the debtors point of view: We get the nominal $\mathcal{N}$ at the start of the loan. Then we pay coupons each tenor until we can decide to cancel the loan. If we cancel the loan we have to pay the redemption $\tilde{R}$, otherwise we keep paying coupons until the end of the loan. This is visualized in figure 2.

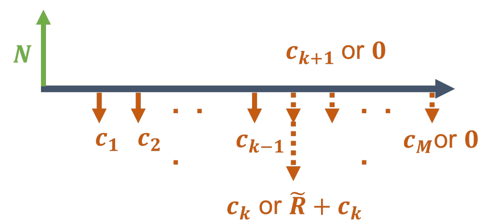

Figure 2: Cashflow of a cancellable loan

5.4 Introducing Behavioral Aspects

6 Implementation Details

The implementation of the described concepts are a crucial part of this thesis. We use Java, which is a purely object oriented programming language, for having a good code readability while still performing very well. We assume a basic knowledge in Java. As build system we use Maven.
Before we can jump into any code, though, we need to describe ways to discretize a stochastic process.

6.1 Numerical Schemes for Stochastic Processes

In this chapter we introduce three ways to approach the simulation of stochastic processes described by SDEs. All of these numerical schemes can also be found in [5], which is also the main source of this section.
As basis for all our schemes we have a known SDE for a n-dimensional stochastic process $X = (X_t)_{t\in[0,\tilde{T}]}$ that we want to simulate:

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t) \cdot dU_t, \\ X_0 &= x, \end{aligned} \tag{23}$$

where U is a d-dimensional standard Brownian Motion, $\mu : [0, \tilde{T}] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, \tilde{T}] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are functions and $x \in \mathbb{R}^n$.
Let us start with the well-known Euler or Euler-Maruyama scheme:

Definition 6.1. *Let $X = (X_t)_{t\in[0,\tilde{T}]}$ be as above in (23).*
Furthermore let $m \in \mathbb{N}$, $(t_i)_{i\in\{0,...,m\}}$, where $0 = t_0 < t_1 < ... < t_m = \tilde{T}$.
Then we call the discrete process $\hat{X} = (\hat{X}_{t_i})_{i\in\{0,...,m\}}$, given by

$$\begin{aligned} \hat{X}_{t_0} &= x, \\ \hat{X}_{t_{i+1}} &= \hat{X}_{t_i} + \mu(t_i, \hat{X}_{t_i})\Delta t_i + \sigma(t_i, \hat{X}_{t_i}) \cdot \Delta U_{t_i}, \end{aligned}$$

where $i \in \{0, ..., m-1\}$, an Euler-Maruyama scheme *of X.*

The Euler-Maruyama scheme takes the concept of approaching deterministic integrals and applies it on stochastic integrals. While this works quite well for constant and deterministic factor loadings $\sigma(t_i, x) = \sigma(t_i)$ [Add source], it has a weakness for stochastic factor loadings.

An improvement is the Milstein scheme:

Definition 6.2. *Let again* $X = (X_t)_{t\in[0,\tilde{T}]}$ *be as above in (23) and* $m \in \mathbb{N}$, $(t_i)_{i\in\{0,...,m\}}$, *where* $0 = t_0 < t_1 < ... < t_m = \tilde{T}$.
Then we call the discrete process $\hat{X} = (\hat{X}_{t_i})_{i\in\{0,...,m\}}$, *given by*

$$\begin{aligned}\hat{X}_{t_0} &= x,\\ \hat{X}^l_{t_{i+1}} &= \hat{X}^l_{t_i} + \mu(t_i, \hat{X}_{t_i})\Delta t_i + \sigma(t_i, \hat{X}_{t_i}) \cdot \Delta U_{t_i}\\ &\qquad + \frac{1}{2}\sum_{k=1}^{d} \sigma_k(t_i, \hat{X}_{t_i})(\partial_{x^l}\sigma_k)(t_i, \hat{X}_{t_i})((\Delta U^k_{t_i})^2 - \Delta t_i),\end{aligned}$$

where $i \in \{0, ..., m-1\}$ *and* $l \in \{1, ..., n\}$, *a* Milstein scheme of X.

This gives an improvement due to the correction term at the end. We can also easily see that if σ is independent of X_t the Milstein is equal to the Euler-Maruyama scheme.

Another numerical scheme for the approximation of stochastic processes is the functional Euler Scheme:

Definition 6.3. *Let* $X = (X_t)_{t\in[0,\tilde{T}]}$ *be as above in (23).*
Let $Y = (Y_t)_{t\in[0,\tilde{T}]}$ *be defined as* $Y_t = f(t, X_t) \quad \forall t \in [0, \tilde{T}]$ *for a 2-times differentiable function* $f : [0, \tilde{T}] \times \mathbb{R}^n \to \mathbb{R}^k$ *and* $k \in \mathbb{N}$.
Let $m \in \mathbb{N}$, $(t_i)_{i\in\{0,...,m\}}$, *where* $0 = t_0 < t_1 < ... < t_m = \tilde{T}$. *Let* $\hat{X} = (\hat{X}_{t_i})_{i\in\{0,...,m\}}$ *be an Euler-Maruyama scheme of* X.
Then we call the discrete process $\hat{Y} = (\hat{Y}_{t_i})_{i\in\{0,...,m\}}$, *given by*

$$\hat{Y}_{t_i} = f(t, \hat{X}_{t_i}),$$

where $i \in \{0, ..., m\}$, *a* functional Euler scheme of Y.

This scheme performs well in situations, where the factor loadings of X are less dependent on X, than the factor loadings of Y are on Y.

It is also very useful, if f preserves a property of Y which otherwise might be lost through the numerical error of the other schemes (such as staying in a certain domain).

Considering this, the functional Euler scheme is great for the approximation of log-normal processes, as it cancels a part of the numerical error due to the dependency of the factor loadings on the approximated process and at the same time it preserves its' positivity.

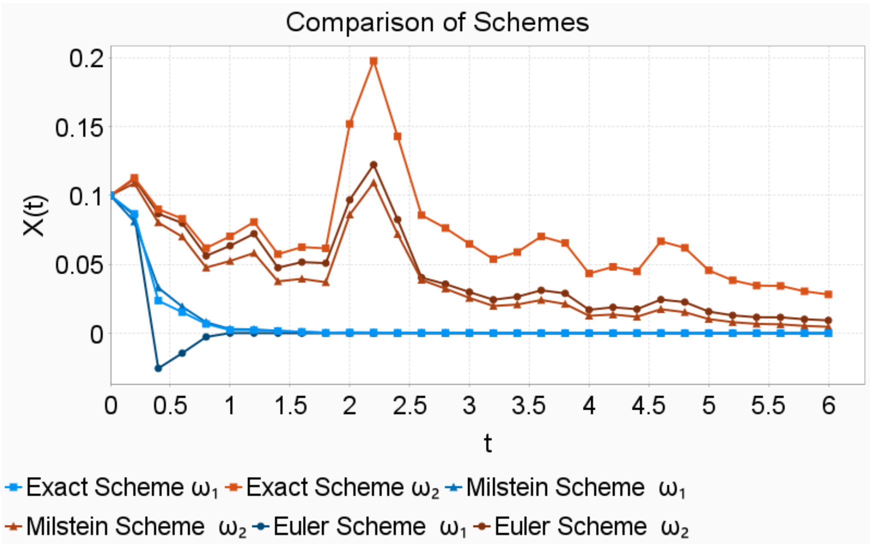

Figure 3: Comparison of Numerical Schemes for a Geometric Brownian Motion

Elaborate (i.e. error estimates and reference for proofs that these work)!!!.

6.2 The finMath Library

We can now start to look at the actual implementation.

As a starting point for the code base we use the finMath Library of Prof. Christian Fries.

The finMath library is written in Java and provides interfaces and classes for the use in stochastics and financial mathematics.

In this section we take a look at the ones that are most frequently used by our code. For a more thorough description one can use the website of the finMath library [3], which is also our main source for this section.

RandomVariable

As one can tell from the name, `RandomVariable` is an interface for working with stochastic values, hence it is the basis for Monte Carlo methods. Operator overloading is not possible in Java so the workaround is to have methods that represent these operations, which is what `RandomVariable` declares. Taking the expectation (the mean) and the variance is also supported. A nice feature is a `RandomVariableFactory`, which handles all creations of a `RandomVariable`.

ProcessModel

The `ProcessModel` interface is the finMath equivalent of an SDE. It specifies that any implementation has methods for getting the initial state, the drift and the factor loadings.

```java
public interface ProcessModel {
        int getNumberOfComponents();
        RandomVariable[] getInitialState(MonteCarloProcess process);
        RandomVariable[] getDrift(MonteCarloProcess process, int timeIndex, RandomVariable[] realizationAtTimeIndex, RandomVariable[] realizationPredictor);
        RandomVariable[] getFactorLoading(MonteCarloProcess process, int timeIndex, int componentIndex, RandomVariable[] realizationAtTimeIndex);
}
```

Listing 1: The `ProcessModel` interface (some methods were ommitted)

We can use a `ProcessModel` as a plug in to numerical schemes simulating an SDE such as `EulerSchemeFromProcessModel`.

MonteCarloProcess

The `MonteCarloProcess` is an interface for the simulation of an SDE that was before specified as `ProcessModel`.

LIBORCovarianceModel

6.3 Implementation of the Defaultable LIBOR Model

To use the full flexibility of the finMath Library, the implementation of our models is structured in the same way: we have an Interface for the defaultable LIBOR market model which extends the `ProcessModel`:

```
public interface DefaultableLIBORMarketModel extends LIBORMarketModel,
    ↪ ProcessModel {
```

Listing 2: Declaration of `DefaultableLIBORMarketModel`

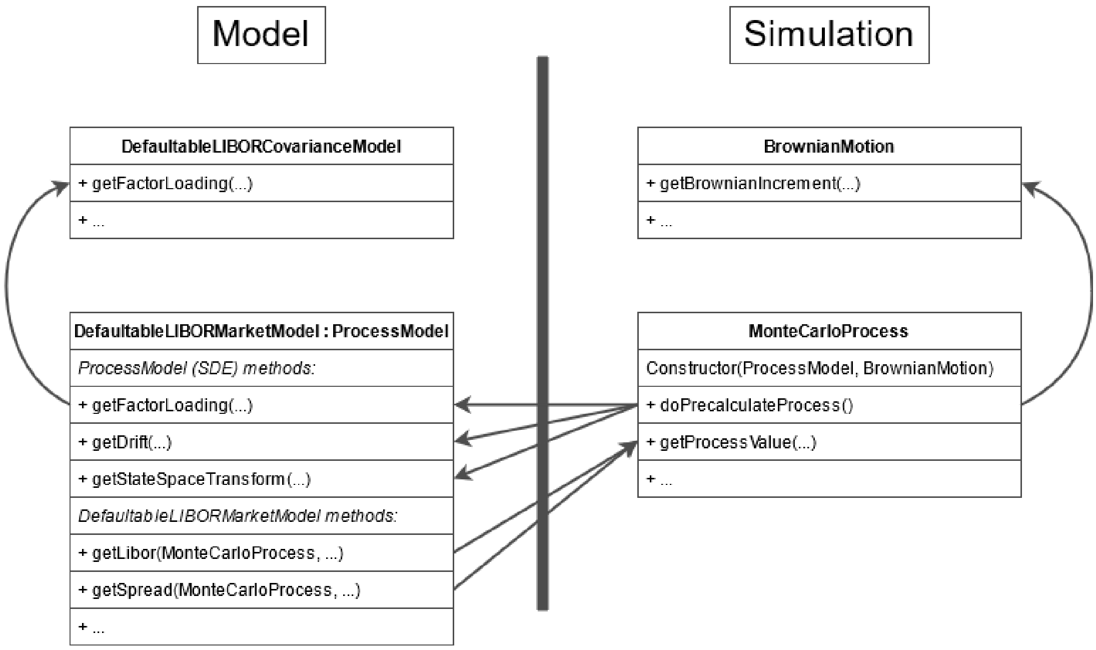

Figure 4: Using `DefaultableLIBORMarketModel` as Plug-In for `MonteCarloProcess`

Let us take a look at the cooperation between our model and the numerical scheme simulating the Process in figure 4.

The `DefaultableLIBORMarketModel` can be used as a plug in for a class implementing the `MonteCarloProcess` interface, which calls SDE-methods of the `ProcessModel` to pre-calculate the process. Calls to `getFactorLoading(...)` should be delegated to the `DefaultableLIBORCovarianceModel`, as we can then reuse the code of `DefaultableLIBORMarketModel`, even though creating different kinds of covariance (factor-loading) models.

Note here that we also extend the `LIBORMarketModel`. This however does not necessarily mean that the SDE specified by the `ProcessModel` simulates the LIBOR rates. In fact the process values approximating the SDE are accessed by our model through input parameters and can be processed further. We therefore have a clean separation of model specifications and simulation.
A way to

6.4 Valuation of Financial Products

7 List of Symbols

Annotation	Meaning
SDE	stochastic differential equation
w.r.t.	with respect to
$\mathbb{Q}^B$	martingale measure w.r.t. the numeraire $B(t)$
$m(t)$	For a tenor $T_0 < ... < T_N$, $m(t) := \max\{i \in \{0, ..., N-1\} \mid T_i \leq t\}$
$(\partial_x f)(x)$	Same as $\frac{\partial}{\partial x} f(x)$.
$(\partial_{xy} f)(x, y)$	Same as $\frac{\partial^2}{\partial x \partial y} f(x, y)$.
$\mathbb{E}_{\mathcal{G}}^{\mathbb{Q}^B}[\,\cdot\,]$	Same as $\mathbb{E}^{\mathbb{Q}^B}[\,\cdot \mid \mathcal{G}\,]$. for a σ-algebra $\mathcal{G}$

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Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbständig angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht.

Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

Ort, Datum Unterschrift