We use $x \cdot y$ as the dot product of two vectors x and y.

We show the existence for the SDEs of the spread in the log-normal state space, meaning we will show the three conditions for the SDE of $s_t = log(S_t)$:

$$ds_t^i = \left(\tilde{\mu}_i\left(t, \exp(s_t^i)\right) - \frac{\sum_{k=1}^{m^d} \tilde{\lambda}_i^k\left(t, \exp(s_t^i)\right)^2}{2}\right) dt + \tilde{\lambda}_i\left(t, \exp(s_t)\right) \cdot dW_t$$

Let $L(t) := (L_0(t), ..., L_{N-1}(t))$ and $S(t) := (S_0(t), ..., S_{N-1}(t))$.
We assume, that the factor loadings of the non-defaultable model λ_t^{ik} are chosen such that they are Lipschitz and have sub-linear growth and prevent $L_i(t) < c_i$, where c_i is a constant with $c_i \Delta T_i > -1$ for all $i \in \{0, ..., N-1\}$.
We therefore define the box $A = \prod_{i=0}^{N-1} [c_i, \infty[$. According to our assumptions, it holds that $L(t) \in A$ for all $t \in \left[0, \tilde{T}\right]$.
Furthermore, we assume that the free parameters $f^{i,k}(t, L(t), S(t)) := f_t^{i,k}$ are continuous and bounded by $K_1 > 0$ (i.e. $|f^{ik}(t, x, s)| \leq K$) on $\left[0, \tilde{T}\right] \times A \times \mathbb{R}_{>0}^N$ (this yields Lipschitz-continuity and sub-linear growth).
We derive $\tilde{\lambda}_i^k$:

$$\tilde{\lambda}_i^k(t, x, s) = \begin{cases} \dfrac{\lambda_t^{ik} \Delta T_i}{1 + x_i \Delta T_i} & \text{for } k \in \{1, ..., m\} \\ f^{ik}(t, x, s) & \text{for } k \in \{m+1, ..., m^d\} \end{cases}$$

note that we immediately get the desired conditions for $\tilde{\lambda}_i^k$ for $(t, x, s) \in \left[0, \tilde{T}\right] \times A \times \mathbb{R}_{>0}^N$, which is sufficient, because x are given in A and are not influenced by the SDEs of s_t.

Deriving $\tilde{\mu}_i$ yields:

$$\tilde{\mu}_i(t, x, s) = \underbrace{\frac{\mu_t^i \Delta T_i}{1 + x_i \Delta T_i}}_{\text{(A)}} + \underbrace{\sum_{k=m+1}^{m^d} f^{ik}(t, x, s) \sum_{j=m(t)+1}^{i} \frac{s_j f^{jk}(t, x, s) \Delta T_j}{1 + x_j \Delta T_j + s_j \Delta T_j}}_{\text{(B)}}$$

By the same arguments as before (A) is Lipschitz and has sub-linear growth. (B) is bounded by a constant, because

$$\text{(B)} \leq \sum_{k=m+1}^{m^d} K_1 \sum_{j=m(t)+1}^{i} \frac{s_j K_1 \Delta T_j}{1 + c_j \Delta T_j + s_j \Delta T_j} \leq m^d i K_1 = K_2. \qquad \text{(C)}$$

We get

$$
\begin{aligned}
\tilde{\mu}_t^i - \frac{1}{2}\tilde{\lambda}_i \cdot \tilde{\lambda}_i = \frac{\mu_t^i \Delta T_i}{1 + x_i \Delta T_i} - \frac{\lambda_t^i \cdot \lambda_t^i (\Delta T_i)^2}{2(1 + x_i \Delta T_i)^2} \qquad &\text{(D)} \\
+ \sum_{j=m(t)+1}^{i} \frac{f^i \cdot f^j s_j \Delta T_j}{1 + x_j \Delta T_j + s_j \Delta T_j} - \frac{f^i \cdot f^i}{2} \qquad &\text{(E)}
\end{aligned}
$$

Because (E) is bounded by $(K_2^2 + \frac{K_1^2}{2})$ we only need to show that (D) is also Lipschitz and has sub linear growth:

$$
\begin{aligned}
\frac{\mu_t^i \Delta T_i}{1 + x_i \Delta T_i} - \frac{\lambda_t^i \cdot \lambda_t^i (\Delta T_i)^2}{2(1 + x_i \Delta T_i)^2} &= \frac{\Delta T_i}{1 + x_i \Delta T_i} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^i \cdot \lambda_t^j \Delta T_j}{1 + x_j \Delta T_j} - \frac{\lambda_t^i \cdot \lambda_t^i (\Delta T_i)^2}{2(1 + x_i \Delta T_i)^2} \\
&= \frac{\Delta T_i}{1 + x_i \Delta T_i} \sum_{j=m(t)+1}^{i-1} \frac{\lambda_t^i \cdot \lambda_t^j \Delta T_j}{1 + x_j \Delta T_j} + \frac{\lambda_t^i \cdot \lambda_t^i (\Delta T_i)^2}{2(1 + x_i \Delta T_i)^2}
\end{aligned}
$$