Ludwig-Maximilians-Universität München

Mathematisches Institut

Master's Thesis

Default Forward Rate Models for the Valuation of Loans including Behavioral Aspects

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1 Introduction

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The thesis is aimed at an audience that has a deep analytical background and a basic knowledge in stochasticity and financial mathematics.

If one has knowledge in the following areas, it is a good start:

- Brownian-Motions and martingales,
- stochastic integration,
- Itô stochastic processes,
- definition of and theorems on arbitrage freeness.

Furthermore we assume a fundamental understanding of what these mathematical concepts imply on the real world and the other way around: How are the mathematical concepts motivated by the economical world?

2 Fundamentals

In this section we provide some fundamentals for the thesis. Many of the results mentioned here can also be found in different versions in other scientific papers. We will stick to source [Add source] for more application-oriented and source [Add source] for more theoretical results.

2.1 Probability Theory

In our whole thesis we will assume a filtered probability space

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$$

where

- Ω is the set of all states,
- $\mathcal{F}$ is a σ-algebra,
- $\mathbb{F} = (\mathcal{F}_t)_{t \in [0;\tilde{T}]}$ is a filtration with $\mathcal{F}_t \subset \mathcal{F} \quad \forall t \in [0; \tilde{T}]$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and
- $\mathbb{Q}$ is a probability measure on $\mathcal{F}$.

Let us cover some notations.

Notation 2.1. *Let X and Y be Itô stochatic processes and W be a Brownian-Motion with:*

$$X_t = X_0 + \int_0^t \mu_s^X ds + \int_0^t \phi_s dW_s,$$
$$Y_t = Y_0 + \int_0^t \mu_s^Y ds + \int_0^t \psi_s dW_s.$$

The sharp bracket *or* quadratic variation *of X is*

$$\langle X \rangle_t = \int_0^t \phi_s^2 ds.$$

. The quadratic covariation *of X and Y is*

$$\langle X, Y \rangle_t = \int_0^t \phi_s \psi_s ds.$$

For a n-dimensional Brownian-Motion $W = (W^i)_{i\in\{1,\dots,n\}}$ and therefore n-dimensional diffusion processes $\phi = (\phi^i)_{i\in\{1,\dots,n\}}$, $\psi = (\psi^i)_{i\in\{1,\dots,n\}}$ the quadratic covariation is

$$\langle X, Y\rangle_t = \sum_{i=1}^{n} \int_0^t \phi_s^i \psi_s^i ds$$

For convenience we state a formula for stochastic integration by parts and a extended version of Itô's theorem:

Lemma 2.2. *Let X and Y be two Itô stochastic processes. Then*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t$$

Proof. See [3]. □

Theorem 2.3. *Let X be an Itô stochastic process, $f : \mathbb{R}^n \times [0; \tilde{T}] \to \mathbb{R}$, $(x,t) \rightarrowtail f(x,t)$ be two times differentiable in x and differentiable in t. Then*

$$f(X_t, t) = f(X_0, 0) + \int_0^t (\partial_t f)(X_s, s) ds + \sum_{i=1}^{n} \int_0^t (\partial_{x^i} f)(X_s, s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t (\partial^2_{x^i x^j} f)(X_s, s) d\left\langle X^i, X^j \right\rangle_s$$

Proof. See [3]. □

Through Itô's formula we can prove the following statement.

Lemma 2.4. *Let $W = (W^i)_{i\in\{1,\dots,d\}}$ be a d-dimensional Brownian Motion, μ be a 1- dimensional and $\sigma = (\sigma^i)_{i\in\{1,\dots,d\}}$ a d-dimensional stochastic process. Let following stochastic differential equation (SDE) be given:*

$$\begin{aligned} dY_t &= Y_t \mu_t dt + Y_t \sigma_t \cdot dW_t, \\ Y_0 &= y, \end{aligned} \tag{1}$$

where $y > 0$.

Then solving eq. (1) is equivalent to solving

$$\begin{aligned} Y_t &= \exp(X_t), \\ dX_t &= (\mu_t - \frac{1}{2} \sum_{i=1}^{d} (\sigma_t^i)^2) dt + \sigma_t \cdot dW_t, \\ X_0 &= \log(y). \end{aligned} \tag{2}$$

Proof. We use Itô's formula on eq. (2) with $f(x) = \exp(x)$, hence

$$(\partial_x f)(x) = \exp(x), \quad (\partial^2_{xx})(x) = \exp(x).$$

We get (in SDE format):

$$\begin{aligned} dY_t = d\exp(X_t) &= \exp(X_t)dX_t + \frac{1}{2}\exp(X_t)d\left\langle X\right\rangle_t \\ &= Y_t((\mu_t - \frac{1}{2}\sum_{i=1}^{d}(\sigma_t^i)^2)dt + \sigma_t \cdot dW_t) + \frac{1}{2}Y_t\sum_{i=1}^{d}(\sigma_t^i)^2 dt \\ &= Y_t\mu_t dt + Y_t\sigma_t \cdot dW_t, \end{aligned}$$

which yields eq. (1). □

Remark 2.5. *It is easy to see, that because of the relation*

$$Y_t = \exp(X_t),$$

it holds that $Y_t > 0 \quad \forall t \in [0; \tilde{T}]$.

2.2 Factor Loadings

The problem we face in this section is that of correlated Brownian-Motions. Assume we have two SDEs for stochastic processes X and Y:

$$\begin{aligned} dX_t &= \mu_t^X dt + \sigma_t^X dW_t^{\mathbb{Q},1}, \\ dY_t &= \mu_t^Y dt + \sigma_t^Y dW_t^{\mathbb{Q},2}, \end{aligned}$$

where $W^{\mathbb{Q},1}$ and $W^{\mathbb{Q},2}$ are instantaneously correlated Brownian-Motions under the same measure $\mathbb{Q}$. This correlation ρ is expressed by a quadratic covariation of the Brownian-Motions [1]:

$$\left\langle W^{\mathbb{Q},1}, W^{\mathbb{Q},2}\right\rangle_t = \int_0^t \rho_s ds$$

or as SDE:

$$d\left\langle W^{\mathbb{Q},1}, W^{\mathbb{Q},2}\right\rangle_t = \rho_t dt.$$

However implementation-wise we need to simulate SDEs using only independent Brownian-Motions. For this purpose we will take advantage of the properties of a Brownian-Motion. Recall following Lemmatas:

Lemma 2.6. *$W = (W^i)_{i\in\{1,...,d\}}$ is a d-dimensional Brownian-Motion if and only if $W^1, ..., W^d$ are independent 1-dimensional Brownian-Motions*

Proof. See [3]. □

Lemma 2.7. *For any d-dimensional Brownian-Motion $W = (W^i)_{i\in\{1,...,d\}}$ and weights $(a_i)_{i\in\{1,...d\}}$ with $a_1 + ... + a_d = 1$ it holds, that U given by:*

$$U_t = \sum_{i=1}^{d} a_i W_t^i$$

is also a Brownian-Motion.

Elaborate!!!.

3 LIBOR Market Model

While the actual LIBOR, short for "London Inter-Bank Offered Rate" [Add source] lost most of its influence after the banking crash of 2008 [Add source] the LIBOR market model - or discrete forward rate model - is still a very popular mathematical model for simulation and valuation of financial products on fixed income markets.

The idea of the model Elaborate!!!.

The basic assumption of the LIBOR Market Model is that we are in an arbitrage free and complete market.

3.1 Fixed Income Markets Terminology

We start with the definition of some fixed income market terms:

Definition 3.1. *A* zero coupon bond *with maturity* $T \in [0; \tilde{T}]$ *(short:* T*-bond) is a product that pays* 1 *at maturity. It's price process is denoted:*

$$P(t;T) := P(\omega, t; T)$$

Note: by construction $P(T;T) = 1$ and $P(\cdot, T)$ discounted with the numeraire must be a martingale under the corresponding martingale-measure $\mathbb{Q}^B$.

While the zero coupon bond does not yield any payoff (or coupons) between buying- and maturity time - hence the name - one can also find coupon paying bonds:

Definition 3.2. *Let* $T_1 < ... < T_M$ *be a tenor with* $T_1, T_M \in [0; \tilde{T}]$*.*
A (fixed) coupon bond *with nominal* $N \in \mathbb{R}$ *and coupons* $c_i \in \mathbb{R}$ *for* $i = 1, ..., M$ *on the given tenor is a product that pays* c_i *at each time point* T_i *and additionally the nominal* N *at maturity* T_M*.*

Remark 3.3. *A variation of this definition is that* c_i *are defined as coupon rates and the actual coupon payment is then* $c_i N$ *at each time step* T_i*. Another popular definition includes the terminal payment of the nominal* N *in the last coupon* c_M*. Additionally to this "normal" coupon bond one can also find* amortizing *coupon* *bonds in the market that distribute the nominal N in the coupons over all periods instead of paying it all at once at the maturity time.*

Lemma 3.4. *Let T_i and c_i be as in the definition above. The price of a fixed coupon paying bond is:*

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Definition 3.5. *We define the* simple forward rate $L(t;S,T)$ *with fixing time S and payment time T at evaluation time t to be a relation of S- and T-bonds:*

$$1 + L(t;S,T)(T-S) = \frac{P(t;S)}{P(t;T)} \tag{3}$$

We can define different products that are strictly positive as numeraires as alternatives to the money market account. We then use a change of measure which gives us a different martingale measure corresponding to the numeraire, i.e. under this new measure all price processes of traded assets discounted with the numeraire are martingales as well.
A simple example is the terminal measure:

Definition 3.6. *The* terminal measure *is the martingale measure $\mathbb{Q}^B$ gained by using the terminal Bond as numeraire, i.e.*

$$B(t) = P(t;\tilde{T}) \tag{4}$$

A more complex example is the spot measure.

Definition 3.7. *Let $0 = T_0 < T_1 < ... < T_N = \tilde{T}$ be a tenor on the time set $[0;\tilde{T}]$.*
The spot measure *is the martingale measure $\mathbb{Q}^B$ gained by using the numeraire:*

$$B(t) = P(t;T_{m(t)+1}) \prod_{i=0}^{m(t)} \frac{1}{P(T_i;T_{i+1})}, \tag{5}$$

where $m(t) = \max\{i \in \{0,...,N-1\} \mid T_i \leq t\}$.

Remark 3.8. *Equation (5) can be rewritten to:*

$$B(t) = P(t;T_{m(t)+1}) \prod_{i=0}^{m(t)} (1 + L(T_{i-1};T_{i-1},T_i)(T_{i+1}-T_i)) \tag{6}$$

The numeraire in the spot measure can be explained as follows:
At $T_0 = 0$ we invest 1 into T_1-bonds. Once these expire (at T_1) we reinvest the money gained from them into T_2-bonds and so on. This product is generally known as *rolling bond*.
Elaborate!!!.

3.2 Model specification

Through the LIBOR model we construct the stochastic differential equations of simple forward rates for a consecutive set of time periods.
We start with a fixed time tenor of $N+1$ ($N \in \mathbb{N}$) points, that splits our time horizon $[0; \tilde{T}]$:

$$0 = T_0 < T_1 < ... < T_N = \tilde{T}.$$

The main objective is to simulate the one step simple forward rates for this tenor:

Assumption 3.1. *The one step simple forward rate (called LIBOR rate)*

$$L_i(t) := L(t; T_i, T_{i+1}) \quad \forall i \in \{0, ..., N\},\, t \in [0; T_i],$$

where $L(t; S, T)$ is defined as in definition 3.5, follows an Itô stochastic process satisfying

$$\begin{aligned} dL_i &= \mu_t^i dt + \sigma_t^i dW_t^{\mathbb{Q}^B, L_i}, \\ L_i(0)(T_{i+1} - T_i) &= \frac{P(0; T_i)}{P(0; T_{i+1})} - 1, \end{aligned}$$

where $(W^{\mathbb{Q}^B, L_i})_{i \in \{0,...,N\}}$ are possibly instantaneously correlated Brownian-Motions.

In the LIBOR model, all other variables are then derived from these interest rates, most importantly the T_i-Bond prices. The attentive reader will have noticed, however, that there is a hole in this derivation: the so-called short-period Bond $P(t; T_{m(t)+1})$ cannot be calculated by $(L_i(t))_{i \in \{0,...,N\}}$ alone, which is why we need the following assumption:

Assumption 3.2. *The short-period Bond*

$$P(t; T_{m(t)+1})$$

is $\mathcal{F}_{m(t)}$-measurable. This means that all T_i-Bond prices are predictable in $[T_{i-1}; T_i]$. A specification that satisfies this assumption is:

$$P(t; T_{m(t)+1}) = (1 + L_{m(t)}(t)(T_{m(t)+1} - t))^{-1}$$

Given a variance-structure for the rates $(\sigma^i)_{i\in\{0,\ldots,N\}}$, a correlation-structure $(\rho^{i,j})_{i,j\in\{0,\ldots,N\}}$ for the Brownian-Motions and we can specify a drift for the SDEs of the LIBORs:

Lemma 3.9. *Let L_i satisfy assumption 3.1. Let $(\sigma^i)_{i\in\{0,\ldots,N\}}$ and $(\rho^{i,j})_{i,j\in\{0,\ldots,N\}}$ be given, where*

$$d\left\langle W^{\mathbb{Q}^B,L_i}, W^{\mathbb{Q}^B,L_j}\right\rangle_t = \rho_t^{i,j} dt.$$

Then Elaborate!!!.

4 Defaultable LIBOR Market Models

In this section we will introduce defaultable LIBOR market models that we can use to value credits and credit options.

Our main source for this section is the article "Defaultable Discrete Forward Rate Model with Covariance Structure guaranteeing Positive Credit Spreads" authored by Christian Fries [2].

4.1 The Defaultable Forward Rate

We remain in the same setting as in the non-defaultable model, where we have a LIBOR tenor discretization $(T_i)_{i\in\{0,1,\dots,N\}}$ and a set of (non-defaultable) zero coupon bonds $(P(t;T_i))_{i\in\{0,1,\dots,N\}}$. Hence we can define the same products and apply the same valuation formulas.

We extend the model by defining an additional set of zero coupon bonds which are defaultable: $(P^d(t;T_i))_{i\in\{0,\dots,N\}}$.

Note here, that by construction we must still use the riskless bonds for the calculation of the Numeraire, as one can not use a risky asset as numeraire. Furthermore we will - for now - not consider recovery rates, i.e. we assume that a party is either able to pay all or nothing. We will now introduce the concept of default and defaultable zero coupon bonds.

Definition 4.1. *The* default time *is a stopping time $\tau(\omega)$ on the Filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$.*

The default indicator *$J(t)$ is the indicator process over the default time:*

$$J(t) := \mathbf{1}_{\{\tau(\omega)<t\}}$$

Definition 4.2. *The* Defaultable Zero Coupon Bond *with price process*

$$P^d(t;T_i)$$

at time $t \in [0,T]$ is a traded asset that pays $1-J(T_i)$ at maturity $T_i \in \{T_0,\dots,T_N\}$. Hence it pays 1 if the default has not happened until maturity.

It is easy to see, that if default occurs, the price of a defaultable zero coupon bond jumps to zero. This means that the price process can be discontinuous at default events. This gives notion to the definition of a zero coupon bond conditional on pre-default.

Definition 4.3. *The* Defaultable Zero Coupon Bond conditional pre-default *is a continuous Itô-stochastic process $P^{d,*}(t;T_i)$ at time $t \in [0,T_i]$ with maturity T_i ($i \in \{0,...,N\}$) such that*

$$P^d(t;T_i) = P^{d,*}(t;T_i)(1 - J(t))$$

Definition 4.4. *The* simple Defaultable Forward Rate *is the rate gained from $P^{d,*}(t;T)$ by the same concept as in a non-defaultable model:*

$$L_i^d(t) := L^d(t;T_i,T_{i+1}) := \left(\frac{P^{d,*}(t;T_i)}{P^{d,*}(t;T_{i+1})} - 1\right)\Delta T_i, \tag{7}$$

where $T_i \in \{T_0, ...T_N\}$.

The simple Defaultable Forward Rate is the rate at which one can lend money to a defaultable party (for the time period T_i to T_{i+1}) at the risk of default, if the defaultable party is not in default at the evaluation time t [Add source].

Assumption 4.1. *As in the non defaultable model we assume, that the defaultable short period bond conditional pre-default*

$$P^{d,*}(t;T_{m(t)+1})$$

has no diffusion. This means that the only stochasticity on the defaultable short rate bond is the default time.
I.e. we specify the defaultable short period bond as:

$$P^{d,*}(t;T_{m(t)+1}) = (1 + L_{m(t)}^d(t)(T_{m(t)+1} - t))^{-1}$$

Theorem 4.5. *Let L_i^d be defined as in eq. (7). Let $B(t)$ be the numeraire under the spot measure (i.e. $B(t)$ is given by eq. (5)) and $W^{\mathbb{Q}^B}$ a Brownian Motion* *w.r.t. the spot measure. Let* $\sigma_t^{i,d} := \sigma^{i,d}(t,\omega)$ *be a progressive stochastic process. Let*

$$dL_i^d(t) = \mu_t^{i,d} dt + \sigma_t^{i,d} dW_t^{\mathbb{Q}^B, L_i^d} \tag{8}$$

be the stochastic differential of L_i^d.
Then

$$\mu_t^{i,d} = \sigma_t^{i,d} \sum_{j=m(t)+1}^{i} \frac{\rho_t^{ij,d} \sigma_t^{j,d} \Delta T_j}{1 + \Delta T_j L_j^d(t)}, \tag{9}$$

where $\rho_t^{ij,d} = d\left\langle W_t^{\mathbb{Q}^B, L_i^d}, W_t^{\mathbb{Q}^B, L_j^d} \right\rangle$.

Proof. By construction the defaultable zero coupon bond $P^d(t;T_i)$ is a traded asset. We get

$$L_i^d(t) P^d(t;T_{i+1}) = (P^d(t;T_i) - P^d(t;T_{i+1}))\Delta T_i$$

is also a traded asset, because it is a portfolio of defaultable zero coupon bonds. Hence both processes discounted with the numeraire $B(t)$ are martingales.
By Itô we have:

$$\begin{aligned} d\left(L_i^d(t)\frac{P^d(t;T_{i+1})}{B(t)}\right) &= L_i^d(t) d\left(\frac{P^d(t;T_{i+1})}{B(t)}\right) + \frac{P^d(t;T_{i+1})}{B(t)} dL_i^d(t) \\ &\quad + d\left\langle L_i^d(t), \frac{P^d(t;T_{i+1})}{B(t)} \right\rangle \end{aligned}$$

We analyze the drift terms on each diffusion:

$$d\left(L_i^d(t)\frac{P^d(t;T_{i+1})}{B(t)}\right) \quad \text{and} \quad d\left(\frac{P^d(t;T_{i+1})}{B(t)}\right)$$

are martingale diffusions and hence have no drift. Therefore the drift terms of the two remaining differentials must cancel each other out. I.e.:

$$\frac{P^d(t;T_{i+1})}{B(t)} \mu_t^{i,d} dt \stackrel{!}{=} -d\left\langle L_i^d(t), \frac{P^d(t;T_{i+1})}{B(t)} \right\rangle \tag{10}$$

To calculate the quadratic variation we need the diffusion of the discounted defaultable zero coupon bond:

$$
\begin{aligned}
d\left(\frac{P^d(t;T_{i+1})}{B(t)}\right) &= d\left(\frac{P^d(t;T_{m(t)+1})}{B(t)}\prod_{j=m(t)+1}^{i}(1+\Delta T_j L_j^d(t))^{-1}\right)\\
&= (...)dt - \frac{P^d(t;T_{m(t)+1})}{B(t)}\sum_{j=m(t)+1}^{i}\frac{\sigma_t^{j,d}\Delta T_j}{1+\Delta T_j L_j^d(t)}dW_t^{\mathbb{Q}^B,L_j^d}\\
&\quad + \prod_{j=m(t)+1}^{i}(1+\Delta T_j L_j^d(t))^{-1} d\left(\frac{P^d(t;T_{m(t)+1})}{B(t)}\right)\\
&\quad + d\left\langle \frac{P^d(t;T_{m(t)+1})}{B(t)}, \prod_{j=m(t)+1}^{i}(1+\Delta T_j L_j^d(t))^{-1}\right\rangle .
\end{aligned}
$$

With Assumption 3.2 and 4.1 we get

$$
d\left(\frac{P^d(t;T_{m(t)+1})}{P(t;T_{m(t)+1})}\right) = (...)dt - \frac{P^{d,*}(t;T_{m(t)+1})}{P(t;T_{m(t)+1})}dJ(t).
$$

Furthermore we have

$$
d\left(\frac{P^d(t;T_{m(t)+1})}{B(t)}\right) = \prod_{j=0}^{m(t)}(1+\Delta T_j L_j(T_j))^{-1} d\left(\frac{P^d(t;T_{m(t)+1})}{P(t;T_{m(t)+1})}\right)
$$

and for any Itô-process X: $d\langle X, J\rangle = 0$ [Add source].

This yields

$$
\begin{aligned}
d\left(\frac{P^d(t;T_{i+1})}{B(t)}\right) &= (...)dt - \frac{P^d(t;T_{m(t)+1})}{B(t)}\sum_{j=m(t)+1}^{i}\frac{\sigma_t^{j,d}\Delta T_j}{1+\Delta T_j L_j^d(t)}dW_t^{\mathbb{Q}^B,L_j^d}\\
&\quad - \frac{P^{d,*}(t;T_{i+1})}{P(t;T_{m(t)+1})}dJ(t).
\end{aligned}
$$

We get

$$
d\left\langle L_i^d(t), \frac{P^d(t;T_{i+1})}{B(t)}\right\rangle = \sigma_t^{i,d}\frac{P^d(t;T_{m(t)+1})}{B(t)}\sum_{j=m(t)+1}^{i}\frac{\sigma_t^{j,d}\Delta T_j}{1+\Delta T_j L_j^d(t)}\rho_t^{ij,d}dt.
$$

Inserting into (10) yields our statement (9). □

We now move from the "covariance" process model to a "factor loading" model as described in Section 2.2.

Lemma 4.6. *There exist* m^d*-dimensional stochastic processes* $\lambda^{i,d} = (\lambda^{ik,d})_{k\in\{1,\dots m^d\}}$ *and a* m^d1*-dimensional* $\mathbb{Q}^B$*-Brownian-Motion* $U = (U^k)_{k\in\{1,\dots,m^d\}}$ *such that*

$$dL_i^d(t) = \mu_t^{i,d}dt + \sigma_t^{i,d}dW_t^{\mathbb{Q}^B,L_i^d} \iff dL_i^d(t) = \mu_t^{i,d}dt + \lambda_t^{i,d}\cdot dU_t$$

Furthermore

$$\sum_{k=1}^{m^d} \lambda_t^{ik,d}\lambda_t^{ik,d} = (\sigma_t^i)^2$$

as well as

$$\sum_{k=1}^{m^d} \lambda_t^{ik,d}\lambda_t^{jk,d} = \sigma_t^i\sigma_t^j\rho_t^{i,j}$$

Proof. Follows directly from Section 2.2. □

Hence from here on we will use this computation friendly version:

$$dL_i^d(t) = \mu_t^{i,d}dt + \lambda_t^{i,d}\cdot dU_t. \tag{11}$$

Remark 4.7. *For the spot measure the drifts* $\mu^{i,d}$ *in eq.* (11) *can be rewritten in terms of* λ^d*:*

$$\mu_t^{i,d} = \sum_{k=1}^{m^d} \lambda_t^{ik,d} \sum_{j=m(t)}^{i} \frac{\lambda_t^{jk,d}\Delta T_j}{1 + L_j^d(t)\Delta T_j}$$

Proof. By Equation (9) and Lemma 4.6 we have:

$$\begin{aligned}
\mu_t^{i,d} &= \sigma_t^{i,d} \sum_{j=m(t)+1}^{i} \frac{\rho_t^{ij,d}\sigma_t^{j,d}\Delta T_j}{1+\Delta T_j L_j^d(t)} = \sum_{j=m(t)+1}^{i} \frac{\rho_t^{ij,d}\sigma_t^{j,d}\sigma_t^{i,d}\Delta T_j}{1+\Delta T_j L_j^d(t)} \\
&= \sum_{j=m(t)+1}^{i} \frac{\left(\sum_{k=1}^{m^d}\lambda_t^{ik,d}\lambda_t^{jk,d}\right)\Delta T_j}{1+\Delta T_j L_j^d(t)} = \sum_{k=1}^{m^d}\sum_{j=m(t)+1}^{i} \frac{\lambda_t^{ik,d}\lambda_t^{jk,d}\Delta T_j}{1+\Delta T_j L_j^d(t)} \\
&= \sum_{k=1}^{m^d}\lambda_t^{ik,d} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk,d}\Delta T_j}{1+\Delta T_j L_j^d(t)}
\end{aligned}$$

□

4.2 Covariance Structures Guaranteeing positive spreads

We now investigate how we can generate positive spreads from the defaultable LIBOR market model.

For this purpose let us define a spread:

Definition 4.8. *Let L^d and L be defined as before. The* spread S_i *for the LIBOR period $[T_i; T_{i+1}]$, where $i \in \{0, ..., N\}$ is defined as:*

$$S_i(t) := L_i^d(t) - L_i(t)$$

Remark 4.9. *A negative spread would mean $L_i^d(t) < L_i(t)$. This constitutes an arbitrage possibility, which is why the model needs to be specified, such that this case is "impossible" (i.e. has probability 0).*

The spreads dynamics are given by

$$dS_i(t) = \mu_t^{i,S} dt + \lambda_t^{i,S} \cdot dU_t$$

where

$$\mu_t^{i,S} = \mu_t^{i,d} - \mu_t^i, \quad \text{and} \quad \lambda_t^{i,S} = \lambda_t^{i,d} - \lambda_t^i.$$

Note that we "extended" the vectors $\lambda_t^i = (\lambda_t^{i1}, ..., \lambda_t^{im}, 0, ..., 0)^T$.

Given a numeraire and the factor loadings of the non-defaultable LIBOR rates λ^i, the goal is, to find a specification for the defaultable factor loadings $\lambda^{i,d}$ such that S_i is always positive.

The most common dynamics that guarantee positivity are log-normal dynamics as discussed in Lemma 2.4. So one idea is to find restrictions on $\lambda_t^{i,d}$ such that

$$\mu_t^{i,d} - \mu_t^i = S_i(t)\tilde{\mu}_t^{i,S}, \quad \text{and} \quad \lambda_t^{i,d} - \lambda_t^i = S_i(t)\tilde{\lambda}_t^{i,S}$$

for two processes $\tilde{\mu}^{i,S}$ and $\tilde{\lambda}^{i,S}$.

Lemma 4.10. *Let $\mathbb{Q}^B$ be the spot measure (i.e. $B(t)$ is given by eq. (5)). Let L^d be defined as before with*

$$\begin{aligned} \lambda_t^{ik,d} &= \frac{1 + L_i^d \Delta T_i}{1 + L_i \Delta T_i} \lambda_t^{ik} && \textit{for } k = 1, ..., m \\ \lambda_t^{ik,d} &= \left(L_i^d(t) - L_i(t)\right) f_t^{ik} && \textit{for } k = m+1, ...m^d \end{aligned} \tag{12}$$

where f_t^{ik} are (possibly stochastic) processes for $i = 1, ..., N$ and $k = m+1, ...m^d$. Then $S = (S_i)_{i\in\{0,...,N\}}$ satisfies

$$dS_i = S_i(t)\tilde{\mu}_t^{i,S}dt + S_i(t)\tilde{\lambda}_t^{i,S} \cdot dU_t.$$

for some process $\tilde{\mu}^{i,S}$ and $\tilde{\lambda}^{i,S}$.

Proof. By Remark 4.7 we have:

$$\begin{aligned}
dS_i &= dL_i^d - dL_i \\
&= \sum_{k=1}^{m^d} \lambda_t^{ik,d} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk,d}\Delta T_j}{1 + L_j^d(t)\Delta T_j}dt + \sum_{k=1}^{m^d} \lambda_t^{ik,d} dU_t^k && \text{(A)} \\
&- \sum_{k=1}^{m} \lambda_t^{ik} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk}\Delta T_j}{1 + L_j(t)\Delta T_j}dt - \sum_{k=1}^{m} \lambda_t^{ik} dU_t^k \\
&= \sum_{k=1}^{m} \left(\lambda_t^{ik,d} - \lambda_t^{ik}\right) \left(\sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk}\Delta T_j}{1 + L_j(t)\Delta T_j}dt + dU_t^k \right) && \text{(B)} \\
&+ \left(L_i^d(t) - L_i(t)\right) \left(\sum_{k=m+1}^{m^d} f_t^{ik} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk,d}\Delta T_j}{1 + L_j^d(t)\Delta T_j}dt + dU_t^k \right) \\
&= S_i \sum_{k=1}^{m} \frac{\lambda_t^{ik}\Delta T_i}{1 + L_i(t)\Delta T_i} \left(\sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk}\Delta T_j}{1 + L_j(t)\Delta T_j}dt + dU_t^k \right) && \text{(C)} \\
&+ S_i \left(\sum_{k=m+1}^{m^d} f_t^{ik} \sum_{j=m(t)+1}^{i} \frac{\lambda_t^{jk,d}\Delta T_j}{1 + L_j^d(t)\Delta T_j}dt + dU_t^k \right)
\end{aligned}$$

where (A) to (B) used the relation:

$$\frac{\lambda_t^{jk,d}\Delta T_j}{1 + L_j^d(t)\Delta T_j} = \frac{\lambda_t^{jk}\Delta T_j}{1 + L_j(t)\Delta T_j} \quad \text{for } k = 1, ..., m$$

and (B) to (C) used:

$$\lambda_t^{ik,d} - \lambda_t^{ik} = \left(\frac{1 + L_i^d(t)\Delta T_i}{1 + L_i(t)\Delta T_i} - \frac{1 + L_i(t)\Delta T_i}{1 + L_i(t)\Delta T_i} \right) \lambda_t^{ik} = S_i \frac{\lambda_t^{ik}\Delta T_i}{1 + L_i(t)\Delta T_i}$$

for $k = 1, ..., m$. □

Elaborate!!!.

4.3 The Survival Probability

Until now we assumed every stochastic process to be adapted to the filtration $\mathbb{F}$. This is what is commonly done in mathematical finance to find theoretical prices and it simulates the availability of information at each time t. This includes the indicator over the default time $J(t)$. However the products we are interested in, are rarely dependent on the (future) default state, but rather its' probability, as we will see in the next chapter. So it is more efficient (and less dependent on extra assumptions) to calculate the survival probability instead of simulating the default time.

It is important to note here, that for pricing we are only interested in the behavior of the rates under the martingale measure $\mathbb{Q}^B$ and so we will also only focus on the survival probability under this measure. This means the probability we will derive, is not to be confused with the real world probability of survival.

Let us start by stating an assumption that is important for the derivation:

Assumption 4.2. *The default time $\tau(\omega)$ has a stochastic driver, that is not fully dependent on the Brownian-Motion U.*
Furthermore $\{\tau(\omega) > 0\} = \Omega$.

This statement seems to be a bit vague, but this is only a requirement to find σ-Algebras $\mathcal{G}_t$ such that $L_i^d(t)$ and $L_i(t)$ are $\mathcal{G}_t$-measurable but $J(t) = \mathbf{1}_{\{\tau<t\}}$ is not $\forall t \in [0; \tilde{T}]$.
The second part of the assumption means that, independent of the filtration, the default has not happened at time $t = 0$.
Let us now define a new filtration with this property:

Definition 4.11. *The filtration $\mathbb{G} := (\mathcal{G}_t)_{t\in[0;\tilde{T}]}$ is given by*

$$\mathcal{G}_t := \sigma(U_t^k | k = 1, ..., m^d) \subset \mathcal{F}_t \quad \forall t \in [0; \tilde{T}].$$

Remark 4.12. *It is worth to note that all processes that are dependent on $J(t)$*

are not $\mathbb{G}$-adapted. This includes $P^d(t;T_i)$ for $i = 1, ..., N$ (remember $P(0;T_0) = 1$).

Lemma 4.13. *For $T_i < s < t < T_{i+1}$ it holds that*

- *$P(s;T_{i+1}) = P(s;t)P(t;T_{i+1})$ or $P(s;t) = \frac{P(s;T_{i+1})}{P(t;T_{i+1})}$ and*
- *$P^{d,*}(s;T_{i+1}) = P^{d,*}(s;t)P^{d,*}(t;T_{i+1})$ or $P^{d,*}(s;t) = \frac{P^{d,*}(s;T_{i+1})}{P^{d,*}(t;T_{i+1})}$.*

Proof. Let $T_i < s < t < T_{i+1}$. With Assumption 3.2 we have:

$$\begin{aligned} P(s;T_{i+1}) &= B(s)\mathbb{E}^{\mathbb{Q}^B}\left[\frac{P(t;T_{i+1})}{B(t)} \,\middle|\, \mathcal{F}_s\right] = P(t;T_{i+1})B(s)\mathbb{E}^{\mathbb{Q}^B}\left[\frac{P(t;t)}{B(t)} \,\middle|\, \mathcal{F}_s\right] \\ &= P(t;T_{i+1})P(s;t) \end{aligned}$$

Prove the same for $P^{d,*}$!!! ∎

Lemma 4.14. *Let B be the numeraire under the spot measure, $i = 0, ..., N$ and $j = i, ..., N$. The one-step survival probability w.r.t. $\mathcal{G}_{T_i}$ (resp. $\mathcal{G}_{T_j}$), given pre-default is*

$$\begin{aligned} q_i(T_i,\omega) &:= \mathbb{Q}^B\left((\{\tau(\omega) > T_{i+1}\} \mid \{\tau(\omega) > T_i\})| \,\mathcal{G}_{T_i}\right) = \frac{P^{d,*}(T_i;T_{i+1})}{P(T_i;T_{i+1})} \qquad (13) \\ &= \mathbb{Q}^B\left((\{\tau(\omega) > T_{i+1}\} \mid \{\tau(\omega) > T_i\})| \,\mathcal{G}_{T_j}\right) =: q_j(T_i,\omega). \qquad (14) \end{aligned}$$

The total survival probability until T_{i+1} w.r.t. $\mathcal{G}_{T_j}$ is

$$\mathbb{Q}^B\left(\{\tau(\omega) > T_{i+1}\}| \,\mathcal{G}_{T_i}\right) = \prod_{k=1}^{i+1} q_i(T_k,\omega) \qquad (15)$$

Proof. Let us start with statement 13. For ease of notation we will use for a σ-algebra $\mathcal{G}$: $\mathbb{E}^{\mathbb{Q}^B}_{\mathcal{G}}[\,\cdot\,] = \mathbb{E}^{\mathbb{Q}^B}[\,\cdot \mid \mathcal{G}\,]$. We get:

$$\begin{aligned} q_i(T_i,\omega) &= \mathbb{E}^{\mathbb{Q}^B}_{\mathcal{G}_{T_i}}\left[\mathbf{1}_{\{\tau > T_{i+1}\}} \,\middle|\, \{\tau(\omega) > T_i\}\right] && \text{(A)} \\ &= \mathbb{E}^{\mathbb{Q}^B}_{\mathcal{G}_{T_i}}\left[\frac{B(T_i)B(T_{i+1})}{B(T_i)B(T_{i+1})}\mathbb{E}^{\mathbb{Q}^B}_{\mathcal{F}_{T_i}}\left[\mathbf{1}_{\{\tau > T_{i+1}\}} \,\middle|\, \{\tau(\omega) > T_i\}\right]\right] && \text{(B)} \\ &= \mathbb{E}^{\mathbb{Q}^B}_{\mathcal{G}_{T_i}}\left[\left(\frac{B(T_i)}{B(T_{i+1})}\right)^{-1} B(T_i)\mathbb{E}^{\mathbb{Q}^B}_{\mathcal{F}_{T_i}}\left[\frac{\mathbf{1}_{\{\tau > T_{i+1}\}}}{B(T_{i+1})} \,\middle|\, \{\tau(\omega) > T_i\}\right]\right] && \text{(C)} \\ &= \mathbb{E}^{\mathbb{Q}^B}_{\mathcal{G}_{T_i}}\left[\left(B(T_i)\mathbb{E}^{\mathbb{Q}^B}_{\mathcal{G}_{T_i}}\left[\frac{1}{B(T_{i+1})}\right]\right)^{-1}\left(P^d(T_i;T_{i+1}) \mid \{\tau(\omega) > T_i\}\right)\right] && \text{(D)} \\ &= \mathbb{E}^{\mathbb{Q}^B}_{\mathcal{G}_{T_i}}\left[\frac{P^{d,*}(T_i;T_{i+1})}{P(T_i;T_{i+1})}\right] = \frac{P^{d,*}(T_i;T_{i+1})}{P(T_i;T_{i+1})} && \text{(E)} \end{aligned}$$

where from (A) to (B) we use the tower property of conditional expectation. From (B) to (C) we use the $\mathcal{G}_{T_i}$- and therefore $\mathcal{F}_{T_i}$-measurability of $B(T_{i+1})$, which we again use from (C) to (D). From (C) to (D) and from (D) to (E) we use the martingale property of $\frac{P^d(t;T_{i+1})}{B(t)}$ and $\frac{P(t;T_{i+1})}{B(t)}$ respectively.

Prove that $q_j(T_i, \omega) = q_i(T_i, \omega)$!!!

Now we get Equation (15) with Bayes-rule and statement (14). $\square$

Note, that while we calculate a probability here, it is a *pathwise* probability. At first glance this seems counter-intuitive, but consider this: $\mathbb{G}$ does not capture the default state. I.e. while we can calculate $P^{d,*}(t;T_i)$, the default might have already happened, which we would not know, even past t and T_i. This is why we can and need to calculate the probability.

Figure 1 shows this concept in a very simplified and discretized way.

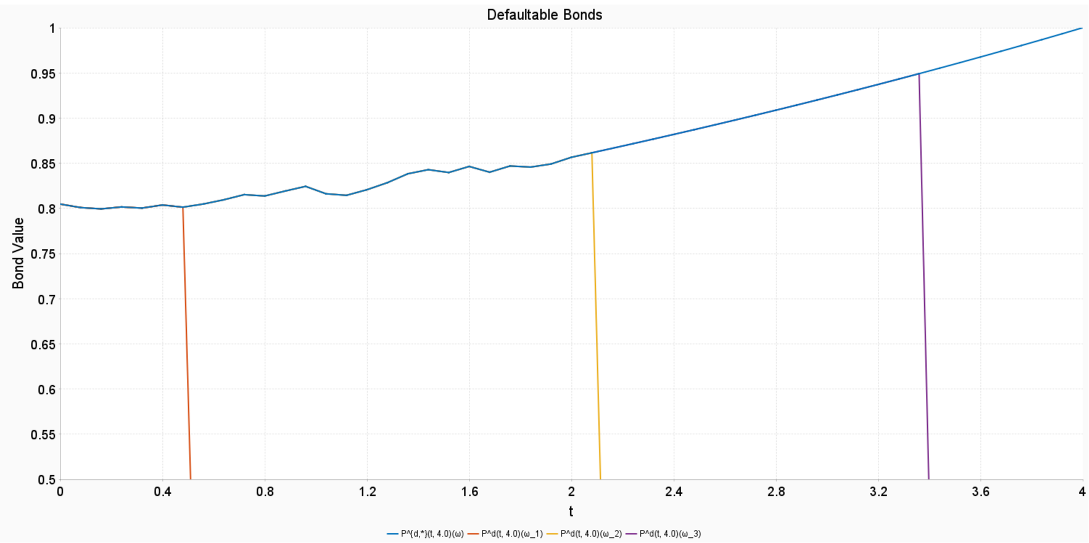

Figure 1: Paths of P^d

In this example $\mathbb{G}$ captures only the path of $P^{d,*}(t;4.0)$, which is equal for $\omega = \omega_1, \omega_2$ and ω_3, while $P^d(t;4.0)(\omega_1) \neq P^d(t;4.0)(\omega_2) \neq P^d(t;4.0)(\omega_3)$ for $t = 4.0$. With this pathwise probability we will now be able to numerically price credits and credit derivatives.

5 Loans and Credit Option Pricing

In this section we take a look at pricing loans and credit options.

5.1 General Loan Pricing

Let us first take a look what a loan is in general. "A loan is a sum of money that one or more individuals or companies borrow from banks or other financial institutions [...]. In doing so, the borrower incurs a debt, which he has to pay back with interest and within a given period of time."[4].

In this sense a loan is nothing else than a fixed coupon bond, where the nominal N represents the debt and the coupons c_i

5.2 Credit Options

6 Introducing Behavioral Aspects

6.1 General Credit Pricing

6.2 Pricing Credit Options

7 Implementation Details

8 List of Symbols

Annotation	Meaning
SDE	stochastic differential equation
w.r.t.	with respect to
$\mathbb{Q}^B$	martingale measure w.r.t. the numeraire $B(t)$
$m(t)$	For a tenor $T_0 < ... < T_N$, $m(t) := \max\{i \in \{0, ..., N-1\} \mid T_i \leq t\}$

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