Let $L_t := (L_0(t), ..., L_{N-1}(t))$ and $S_t := (S_0(t), ..., S_{N-1}(t))$.

We show the existence for the SDEs of the log spread, meaning we will show the three conditions for the SDE of $s_t = \log(S_t)$:

$$ds_t^i = \left(\tilde{\mu}^i(t, L_t, S_t) - \frac{\|\tilde{\lambda}^i(t, L_t, S_t)\|^2}{2}\right) dt + \tilde{\lambda}^i(t, L_t, S_t) \cdot dU_t$$

We assume, that the SDEs for L are specified, such that the Lipschitz-continuity and the sub-linear growth conditions are fulfilled and such that they prevent $L_i(t) < c_i$, where c_i is a constant with $c_i \Delta T_i > -1$ for all $i \in \{0, ..., N-1\}$. We also assume that the factor loadings of $\lambda^i(t, L_t)$ depend only on the LIBOR they correspond to: $\lambda^i(t, L_i(t)) := \lambda^i(t, L_t)$ (e.g. a logmormal statespace with bounded volatility is sufficient).

We define the box $A = \prod_{i=0}^{N-1} [c_i, \infty[$. According to our assumptions, it holds that $L_t \in A$ for all $t \in \left[0, \tilde{T}\right]$.

Furthermore, we assume that the free parameters are given by $\mathbb{R}$-valued functions $f^{ik}(t, x, s)$ on $\left[0, \tilde{T}\right] \times A \times \mathbb{R}^N_{>0}$ with $f^{i,k}(t, L_t, S_t) := f_t^{i,k}$ and we assume they are Lipschitz in the second and third parameter, and bounded by $K_1 > 0$, i.e. $|f^{ik}(t, x, s)| \leq K_1$), which yields sub-linear growth.

We first prove the following statement for $x \in A$: there exist a constant $K_2 > 0$ such that

$$0 < \frac{1}{1 + x_i \Delta T_i} \leq \frac{1 + |x_i|}{1 + x_i \Delta T_i} \leq K_2 \tag{A}$$

The first two inequalities are trivial, as $1 + x_i \Delta T_i > 0$. For the second equation first note that for $x_i \geq 0$ this is always true, because (assuming $x_i \geq 0$):

$$\begin{aligned} \frac{1 + |x_i|}{1 + x_i \Delta T_i} &= \underbrace{\frac{1}{1 + x_i \Delta T_i}}_{\leq 1} \underbrace{\frac{x_i}{\frac{1}{\Delta T_i} + x_i}}_{\leq 1} \frac{1}{\Delta T_i} \\ &\leq 1 + \frac{1}{\Delta T_i} \end{aligned}$$

Also note

$$\lim_{x_i \to -\frac{1}{\Delta T_i}} \frac{\overbrace{1 + |x_i|}^{\geq 1}}{1 + \underbrace{x_i \Delta T_i}_{\to -1}} = \infty$$

Since $\frac{1+|z|}{1+z\Delta T_i}$ is continuous for all $z > -\frac{1}{\Delta T_i}$, we can (for any c_i) find an ϵ, such that $-1 < \epsilon \Delta T_i < c_i \Delta T_i$ with:

$$K_2 := \frac{1+|\epsilon|}{1+\epsilon\Delta T_i} > \frac{1+|x_i|}{1+x_i\Delta T_i} \quad \text{for all } x \in A$$

This concludes the proof of statement (A).

Let us now prove that for some constant $K_3 > 0$:

$$\left| y_i \lambda^{ik}(t,x) - x_i \lambda^{ik}(t,y) \right| \leq K_3 \; |x_i - y_i| \left(1 + |x_i|\right)\left(1 + |y_i|\right) \tag{B}$$

W.l.o.g. we assume $|y_i| < |x_i|$ and get:

$$\begin{aligned}
\left| y_i \lambda^{ik}(t,x) - x_i \lambda^{ik}(t,y) \right| &= \left| y_i \lambda^{ik}(t,x) - y_i \lambda^{ik}(t,y) + y_i \lambda^{ik}(t,y) - x_i \lambda^{ik}(t,y) \right| \\
&\leq |y_i| \left| \lambda^{ik}(t,x) - \lambda^{ik}(t,y) \right| + \left| \lambda^{ik}(t,y) \right| |y_i - x_i| \\
&\leq C_1 |y_i| \, |x_i - y_i| + C_2 \left(1 + |y_i|\right) |y_i - x_i| \\
&\leq \max\left(C_1, C_2\right) \left(1 + |x_i| + |y_i|\right) |x_i - y_i| \\
&\leq \max\left(C_1, C_2\right) \left(1 + |x_i|\right)\left(1 + |y_i|\right) |x_i - y_i|
\end{aligned}$$

where C_1 and C_2 are the Lipschitz- resp. sub-linear growth constants of λ^{ik}. Setting $K_3 := \max\left(C_1, C_2\right)$ yields the result.

We now prove the conditions for the factor loadings of the spread. Note that by **??** we have:

$$\tilde{\lambda}^{ik}\left(t,x,s\right) = \begin{cases} \dfrac{\lambda^{ik}(t,x)\Delta T_i}{1 + x_i \Delta T_i} & \text{for } k \in \{1, ..., m\} \\ f^{ik}(t,x,s) & \text{for } k \in \{m+1, ..., m^d\} \end{cases}$$

We can prove the conditions for each component and factor, because they are stable under addition. Note that for $k \in \{m+1, ..., m^d\}$ they are immediately given.

We start with sub-linear growth for $k \in \{1, ..., m\}$ for all $x \in A$, and for all $s \in \mathbb{R}^N_{>0}$:

$$\left| \tilde{\lambda}^{ik}\left(t,x,s\right) \right| = \left| \frac{\lambda^{ik}(t,x)\Delta T_i}{1 + x_i \Delta T_i} \right| \leq C_2 \frac{1 + |x_i|}{1 + x_i \Delta T_i} \leq C_2 \; K_2 =: K_4 \tag{C}$$

where we used the sub-linear growth of $\lambda^{ik}(t,x)$ in the first inequality and (A) in the second. Hence we even have that λ^{ik} is bounded for all $i \in \{0, ...N-1\}$

and $k \in \{1, ...m^d\}$.

For the Lipschitz condition we have:

$$\left|\frac{\lambda^{ik}(t,x)\Delta T_i}{1+x_i\Delta T_i} - \frac{\lambda^{ik}(t,y)\Delta T_i}{1+y_i\Delta T_i}\right| = \Delta T_i \left|\frac{\lambda^{ik}(t,x)\left(1+y_i\Delta T_i\right) - \lambda^{ik}(t,y)\left(1+x_i\Delta T_i\right)}{\left(1+x_i\Delta T_i\right)\left(1+y_i\Delta T_i\right)}\right|$$

$$\leq \Delta T_i \left|\frac{\left|\lambda^{ik}(t,x) - \lambda^{ik}(t,y)\right| + \Delta T_i \left|\lambda^{ik}(t,x)y_i - \lambda^{ik}(t,y)x_i\right|}{\left(1+x_i\Delta T_i\right)\left(1+y_i\Delta T_i\right)}\right| \tag{D}$$

$$\leq \Delta T_i \left|\frac{C_1\left|x_i-y_i\right| + \Delta T_i K_3\left(1+\left|x_i\right|\right)\left(1+\left|y_i\right|\right)\left|x_i-y_i\right|}{\left(1+x_i\Delta T_i\right)\left(1+y_i\Delta T_i\right)}\right| \tag{E}$$

$$\leq \Delta T_i \ \left|x_i-y_i\right| \left|\frac{C_1 + \Delta T_i K_3\left(1+\left|x_i\right|\right)\left(1+\left|y_i\right|\right)}{\left(1+x_i\Delta T_i\right)\left(1+y_i\Delta T_i\right)}\right| \tag{F}$$

$$\leq \left|x_i-y_i\right| \underbrace{\Delta T_i \left(C_1\left(K_2\right)^2 + \Delta T_i K_3\left(K_2\right)^2\right)}_{=:K_5}, \tag{G}$$

where for (D) we used the triangle inequality, (E) comes from (B) and the Lipschitz condition of λ^{ik} and (G) from (A). This yields the Lipschitz condition for the factor loadings $\tilde{\lambda}^{ik}$.

Next we will prove the conditions also for the drift $\tilde{\mu}^i$ but first we need to pave the way.

First note that any multiplication of two factor loadings is again Lipschitz and bounded, i.e. there exist a $K_6 > 0$ and $K_7 > 0$ such that for all $i,j \in \{0,...,N-1\}$ and $k \in \{1,...,m^d\}$:

$$\left|\tilde{\lambda}^{ik}(t,x,s)\,\tilde{\lambda}^{jk}(t,x,s) - \tilde{\lambda}^{ik}(t,y,u)\,\tilde{\lambda}^{jk}(t,y,u)\right| \leq K_6 \left\|(x,s)-(y,u)\right\|$$
$$\left|\tilde{\lambda}^{ik}(t,x,s)\,\tilde{\lambda}^{jk}(t,x,s)\right| \leq K_7 \tag{H}$$

This stems from **??**, together with the fact that both $\tilde{\lambda}^{ik}$ and $\tilde{\lambda}^{jk}$ are bounded.

Now we can prove that $\tilde{\mu}^i$ is Lipschitz and has sub linear growth.

By **??** we have:

$$\tilde{\mu}^i_t = \sum_{j=m(t)+1}^{i} \frac{\sum_{k=1}^{m} \lambda^{ik}(t,L_t)\,\Delta T_i \lambda^{jk}(t,L_t)\,\Delta T_j}{(1+L_i(t)\Delta T_i)(1+L_j(t)\Delta T_j)}$$
$$+ \sum_{j=m(t)+1}^{i} \sum_{k=m+1}^{m^d} f^{ik}(t,L_t,S_t)\frac{S_j(t) f^{jk}(t,L_t,S_t)}{1+L^d_j(t)\Delta T_j}$$

hence with the relation $L_t^d = L_t + S_t$ we have:

$$
\begin{aligned}
\tilde{\mu}^i(t,x,s) = & \sum_{j=m(t)+1}^{i} \sum_{k=1}^{m} \frac{\lambda^{ik}(t,x)\,\Delta T_i \lambda^{jk}(t,x)\,\Delta T_j}{(1+x_i\Delta T_i)(1+x_j\Delta T_j)} \\
& + \sum_{j=m(t)+1}^{i} \sum_{k=m+1}^{m^d} f^{ik}(t,x,s) f^{jk}(t,x,s) \frac{s_j}{1+x_j\Delta T_j + s_j\Delta T_j} \\
= & \sum_{j=m(t)+1}^{i} \sum_{k=1}^{m} \tilde{\lambda}^{ik}(t,x,s)\,\tilde{\lambda} jk(t,x,s) && \text{(I)} \\
& + \sum_{j=m(t)+1}^{i} \sum_{k=m+1}^{m^d} \tilde{\lambda}^{ik}(t,x,s)\,\tilde{\lambda}^{jk}(t,x,s) \underbrace{\frac{s_j}{1+x_j\Delta T_j + s_j\Delta T_j}}_{=(K)} && \text{(J)}
\end{aligned}
$$

As (I) is just the sum of Lipschitz-continuous and bounded functions, it is again Lipschitz and bonded. Because $x_j \Delta T_j > c_j > -1$ and $s_j \geq 0$, (K) is naturally bounded by some constant C_3 with the same arguments as for (A). Also with an exactly analogous proof to (A) we have that $\frac{1+|y_j|+|v_j|}{1+\Delta T_j(x_j+s_j)}$ is bounded by some constant C_4. For Lipschitz continuity we have therefore:

$$
\begin{aligned}
& \left| \frac{s_j}{1+x_j\Delta T_j + s_j\Delta T_j} - \frac{v_j}{1+y_j\Delta T_j + v_j\Delta T_j} \right| \\
& = \left| \frac{(s_j - v_j) + \Delta T_j(s_j y_j - v_j x_j)}{(1+\Delta T_j(x_j+s_j))\,(1+\Delta T_j(y_j+v_j))} \right| \\
& \leq \left| \frac{\|(x_j,s_j)-(y_j,v_j)\| + \Delta T_j(|y_j|\,|s_j-v_j| + |v_j|\,|y_j-x_j|)}{(1+\Delta T_j(x_j+s_j))\,(1+\Delta T_j(y_j+v_j))} \right| \\
& \leq \|(x_j,s_j)-(y_j,v_j)\| \left| \frac{1+\Delta T_j(|y_j|+|v_j|)}{(1+\Delta T_j(x_j+s_j))\,(1+\Delta T_j(y_j+v_j))} \right| \\
& \leq \|(x_j,s_j)-(y_j,v_j)\| \left| \frac{1+\Delta T_j\,(1+|y_j|+|v_j|)\,(1+|x_j|+|s_j|)}{(1+\Delta T_j(x_j+s_j))\,(1+\Delta T_j(y_j+v_j))} \right| \\
& \leq \|(x_j,s_j)-(y_j,v_j)\| \left(C_3^2 + C_4^2 \right)
\end{aligned}
$$

Hence we have that (J) is nothing else than a sum of multiplications of bounded, implying that the drift $\tilde{\mu}^i$ is Lipschitz and bounded for all $i \in \{0, ...N-1\}$. This concludes the proof, because by (H) we already have $\|\tilde{\lambda}^i\|^2$ is Lipschitz and bounded, proving

$$
\tilde{\mu}^i(t,x,s) - \frac{\|\tilde{\lambda}^i(t,x,s)\|^2}{2}
$$

is Lipschitz and bounded.