Assignment_1_Solution

March 6, 2019

1 Assignment 1

1.1 Task 1

Suppose we have a two-class classification problem, where we denote the two classes with $+1$ and -1. Further assume that the joint distribution of $\mathbf{x}$ and y, $p(\mathbf{x}, y)$, is known and that the distributions of the two classes do not overlap, i.e.

$$\min\{p(\mathbf{x}|y=+1), p(\mathbf{x}|y=-1)\} = 0.$$

Determine an optimal classification function g and compute the generalization error using the zero-one loss function.

1.1.1 Solution

Since $p(\mathbf{x}, y)$, is known, we can also compute the marginals $p(\mathbf{x})$ and $p(y)$ by averaging(i.e. integrating or summing) over variables(e.g. if we are interested in $p(\mathbf{x})$, we need to average over y). Then, by definition of conditional probability, we can also compute $p(\mathbf{x}|y) = \frac{p(\mathbf{x},y)}{p(y)}$ and similarly $p(y|\mathbf{x})$. According to the slides, we can therefore find an optimal classifier, given by any function g that satisfies

$$g(\mathbf{x}) \begin{cases} > 0 & \text{for} \quad p\left(y=1 \mid \mathbf{x}\right) \ > \ p\left(y=-1 \mid \mathbf{x}\right) \\ < 0 & \text{for} \quad p\left(y=-1 \mid \mathbf{x}\right) \ > \ p\left(y=1 \mid \mathbf{x}\right) \end{cases}$$

Moreover,by the definition of conditional probability(or also known as Bayes' Theorem), we have

$$p(y|\mathbf{x}) = \frac{p(\mathbf{x}|y)p(y)}{p(\mathbf{x})}.$$

This immediately leads to the conclusion that $\min\{p(\mathbf{x}|y=+1), p(\mathbf{x}|y=-1)\} = 0$ if and only if $\min\{p(y=+1 \mid \mathbf{x}), p(y=-1 \mid \mathbf{x})\} = 0$. We also know from the slides how the minimal risk can be computed and therefore conclude:

$$R_{\min} = \int_X \min\{p\left(y=-1 \mid \mathbf{x}\right), p\left(y=1 \mid \mathbf{x}\right)\} \ p(\mathbf{x}) \ d\mathbf{x} = 0.$$

1.2 Task 2

Assume that the two classes in data set `DataSet6` are distributed according to multivariate normal distributions. Estimate the means and covariance matrices as well as $p(y=+1)$ and $p(y=-1)$ from the data (you may use all 200 samples), compute an optimal classification function (see slide 9) and visualize it graphically(a two-dimensional plot suffices).

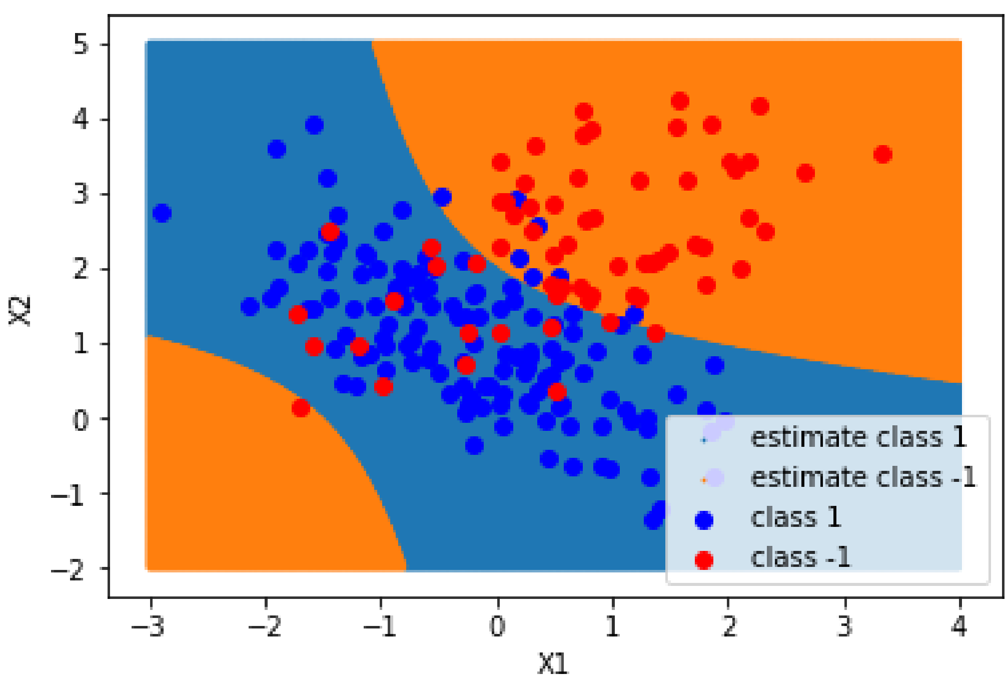
1.2.1 Solution

In [5]:
```
import sklearn
import numpy as np
import matplotlib.pyplot as plt

#read data, split into X(features) and y(labels)
Z = np.genfromtxt('DataSet6.csv', delimiter=',')
X, y = Z[:,:-1], Z[:,-1]
#further split features according to labels
Xpos=X[y==1]
Xneg=X[y==-1]
#compute mean and covariance
meanXpos=np.mean(Xpos.T,axis=1)
meanXneg=np.mean(Xneg.T,axis=1)
covXpos=np.cov(Xpos.T)
covXneg=np.cov(Xneg.T)
#compute distribution p(y=+-1)
p_ypos=Xpos.T.shape[1]/X.shape[0]
p_yneg=1-p_ypos
#print corresponding values
print("Cov of positve class=", covXpos)
print("Cov of negative class=", covXneg)
print("Mean of positve class=", meanXpos)
print("Mean of negative class=", meanXneg)
print("P(y=1)=", p_ypos)
print("P(y=-1)=", p_yneg)
#compute inverses of covariance matrices
covXpos_inv = np.linalg.inv(covXpos)
covXneg_inv = np.linalg.inv(covXneg)
#compute parameters of discrimination function g according to slide 9
A= covXpos_inv - covXneg_inv
w = np.matmul(covXpos_inv, meanXpos) - np.matmul(covXneg_inv, meanXneg)
b = (-1/2*np.matmul(np.matmul(meanXpos, covXpos_inv), meanXpos)
+ 1/2*np.matmul(np.matmul(meanXneg, covXneg_inv), meanXneg)
- 1/2 * np.log(np.linalg.det(covXpos)) + 1/2 * np.log(np.linalg.det(covXneg))
+ np.log(p_ypos) - np.log(p_yneg) )
#Print corresponding values
print("Value of A=", A)
print("Value of w=", w)
print("Value of b=", b)
```

```
#Create grid and use it to evaluate g
X1, X2 = np.mgrid[-3:4:500j, -2:5:500j]
points = np.c_[X1.ravel(), X2.ravel()]
g = -1/2*np.sum(np.dot(points, A) * points, axis=1) + np.dot(points, w) + b
#Create Plot
plt.scatter(X1,X2, g > 0, label="estimate class 1")
plt.scatter(X1,X2, g < 0, label="estimate class -1")
plt.scatter(Xpos[:,0],Xpos[:,1], color= 'blue', label="class 1")
plt.scatter(Xneg[:,0],Xneg[:,1], color='red', label="class -1")
plt.xlabel("X1")
plt.ylabel("X2")
plt.legend(loc=4)
plt.show()
```

```
Cov of positve class= [[ 0.97776086 -0.63279722]
 [-0.63279722  0.96214327]]
Cov of negative class= [[1.18686086 0.55312556]
 [0.55312556 0.98948508]]
Mean of positve class= [-0.23072318  1.0973771 ]
Mean of negative class= [0.72081505 2.3428049 ]
P(y=1)= 0.6716417910447762
P(y=-1)= 0.32835820895522383
Value of A= [[0.64132121 1.80808986]
 [1.80808986 0.44294787]]
Value of w= [ 1.54525683 -1.02711802]
Value of b= 3.083468597232569
```



1.3 Task 3

1.3.1 Part 1

Given the model class M of all exponential distributions with parameter λ. i.e. $M = \{f_\lambda(x) \mid \lambda > 0\}$ with

$$f_\lambda(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Derive a formula for the maximum likelihood estimator λ^* for the parameter λ.

Hint: Maximize the logarithm of the likelihood function instead of the likelihood function itself.

1.3.2 Solution

Assume that $\mathbf{x} = (x_1, ..., x_n)$, where the x_is are iid, according to an exponential distribution with parameter λ . Without restriction we can work with positive x_i's. The liklehood can then be computed, according to the slides, as

$$\mathcal{L}(\{\mathbf{x}\}; \lambda) = p(\{\mathbf{x}\}; \lambda\}) = \prod_{i=1}^{n} p(x_i; \lambda) = \prod_{i=1}^{n} f_\lambda(x_i) = \lambda^n \prod_{i=1}^{n} e^{-\lambda x_i}.$$

Now we take the logarithm and obtain

$$\log(\mathcal{L}(\{\mathbf{x}\}; \lambda)) = n \log(\lambda) - \lambda \sum_{i=1}^{n} x_i.$$

Taking the derivative wrt. λ yields

$$\frac{\partial \log(\mathcal{L}(\{\mathbf{x}\}; \lambda))}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i.$$

Setting the above expression to 0, finally leads us to the following result for the optimal λ^*:

$$\lambda^* = \frac{n}{\sum_{i=1}^{n} x_i}.$$

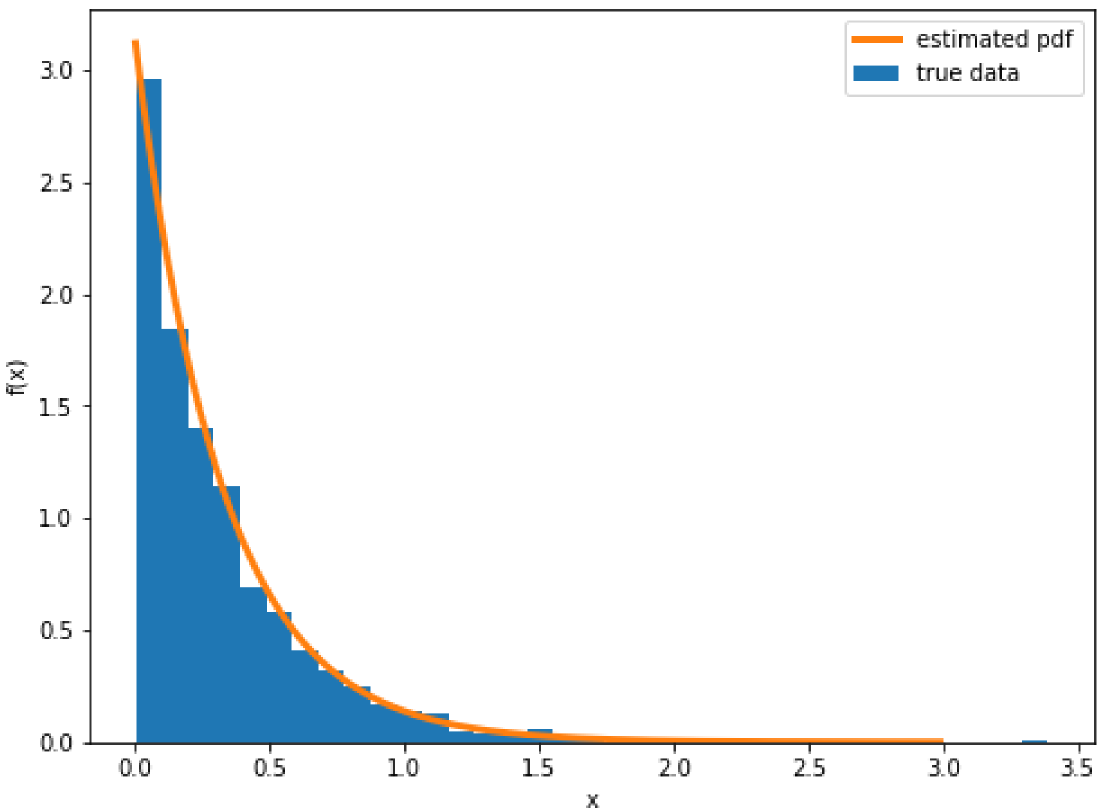
1.3.3 Part 2

Apply the formula for λ^* from the previous part to the data of `DataSet7`. Visualize the density defined by this optimal λ^* and compare it to the true data distribution (e.g. by using a histogram).

1.3.4 Solution

```
In [6]: # load the data
        data = np.genfromtxt('DataSet7.csv', delimiter=',', skip_header=1)
        # estimate the parameter according to the previously derived formula
        lambd_est = len(data)/np.sum(data)
        print('Estimated lambda = ',lambd_est)
        #Define function that computes estimated pdf
        def f(x_i, lambd):
            if x_i < 0:
                return 0
            return lambd * np.exp(-lambd * x_i)
        #Evaluate f on points with distance 0.01 in interval [0,3], store values
        x = np.arange(0, 3.0, 0.01)
        y = [f(x_i, lambd_est) for x_i in x]
        #Make plot
        plt.figure(figsize = (8,6))
        plt.hist(data, 35, density = True, label = 'true data')
        plt.plot(x, y, label = 'estimated pdf', linewidth = 3)
        plt.xlabel('x')
        plt.ylabel('f(x)')
        plt.legend()
        plt.show()
```

```
Estimated lambda =  3.120304521828994
```




```
In [ ]:
```