Computational Physics Assessment

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Chapter 1

Quadratic Equations

1.1 Basics

In algebra, a **quadratic equation** is any equation that can be rearranged in standard form as:

$$ax^2 + bx + c = 0 \tag{1.1}$$

where x represents an unknown, and a, b, and c represent known numbers, where $a \neq 0$. If $a = 0$, then the equation is linear, not quadratic, as there is no ax^2 term. The numbers a, b, and c are the coefficients of the equation and may be distinguished by calling them, respectively, the *quadratic coefficient*, the *linear coefficient* and the *constant* or *free term.*

The values of x that satisfy the equation are called *solutions* of the equation, and *roots* or *zeros* of the expression on its left-hand side. A quadratic equation has at most two solutions. If there is **no real solution**, there are two *comp'lex solutions.* If there is **only one solution**, I call it a *double root.*

A quadratic equation can be factored into an equivalent equation:

$$ax^2 + bx + c = a(x - r)(x - s) = 0 \tag{1.2}$$

where r and s are the solutions for x. Completing the square on a quadratic equation in standard form results in the quadratic formula, which expresses the solutions in terms of a, b, and c.We now look at solving $ax^2 + bx + c = 0$.

The equation $ax^2 + bx + c = 0$ with $a \neq 0$ has the solutions:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

1.2 Deriving the solution

We use the method of completing the square to rewrite $ax^2 + bx + c$.

$$\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x +\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 +\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.
\end{aligned}$$

Therefore $ax^2 + bx + c = 0$ can be rewritten as

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} = 0,$$

which can in turn be rearranged as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking square roots gives

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

which implies

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

as required.

1.3 Reduced Quadratic Equation

It is sometimes convenient to reduce a quadratic equation so that its leading coefficient is one. This is done by dividing both sides by a, which is always possible since a is non-zero. This produces the reduced quadratic equation:

$$x^2 + px + q = 0 \tag{1.3}$$

where $p = b/a$ and $q = c/a$. This monic equation has the same solutions as the original.

The quadratic formula for the solutions of the reduced quadratic equation, written in terms of its coefficients, is:

$$x = \frac{1}{2}\left(-p \pm \sqrt{p^2 - 4q}\right) \tag{1.4}$$

or equivalently:

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \tag{1.5}$$

1.4 Discriminant

In the quadratic formula, the expression underneath the square root sign is called the discriminant of the quadratic equation, and is often represented using an upper case D or an upper case Greek delta:[9]

$$\Delta = b^2 - 4ac. \tag{1.6}$$

A quadratic equation with real coefficients can have either one or two distinct real roots, or two distinct complex roots. In this case the discriminant determines the number and nature of the roots. There are three cases:

- If the discriminant is positive, then there are two distinct roots

$$\frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{\Delta}}{2a} \tag{1.7}$$

- If the discriminant is 0, then there is exactly one real root

$$\frac{-b}{2a}, \tag{1.8}$$

- If the discriminant is negative, then there are no real roots. Rather, there are two distinct (non-real) complex roots

$$\frac{b}{2a} + i\frac{\sqrt{-\Delta}}{2a} \quad \text{and} \quad -\frac{b}{2a} - i\frac{\sqrt{-\Delta}}{2a} \tag{1.9}$$

  which are complex conjugates of each other. In these expressions i is the imaginary unit.

Chapter 2

Fourier series

2.1 History

The Fourier series is named in honour of **Jean-Baptiste Joseph Fourier (1768–1830)**, who made important contributions to the study of trigonometric series, after preliminary investigations by *Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli.*

Fourier introduced the series for the purpose of solving the heat equation in a metal plate, publishing his initial results in his 1807 *Mémoire sur la propagation de la chaleur dans les corps solides (Treatise on the propagation of heat in solid bodies)*, and publishing his *Théorie analytique de la chaleur (Analytical theory of heat)* in 1822. The Mémoire introduced Fourier analysis, specifically Fourier series.

Through Fourier's research the fact was established that an arbitrary (at first, continuous and later generalized to any piecewise-smooth function can be represented by a trigonometric series. The first announcement of this great discovery was made by Fourier in 1807, before the French Academy. Early ideas of decomposing a periodic function into the sum of simple oscillating functions date back to the 3rd century B.C, when ancient astronomers proposed an empiric model of planetary motions, based on deferents and epicycles.

2.1.1 How the empire expanded

The heat equation is a partial differential equation. Prior to Fourier's work, no solution to the heat equation was known in the general case, although particular solutions were known if the heat source behaved in a simple way, in particular, if the heat source was a sine or cosine wave. These simple solutions are now sometimes called eigensolutions.

Fourier's idea was to model a complicated heat source as a superposition (or linear combination) of simple sine and cosine waves, and to write the solution as a superposition of the corresponding eigensolutions. This superposition or linear combination is called the Fourier series.

2.1.2 From the shoulder of giants

From a modern point of view, Fourier's results are somewhat informal, due to the lack of a precise notion of function and integral in the early nineteenth century. Later, *Peter Gustav*, *Lejeune Dirichlet* and *Bernhard Riemann* expressed Fourier's results with greater precision and formality.

2.1.3 Defeating Eigensolutions

Although the original motivation was to solve the heat equation, it later became obvious that the same techniques could be applied to a wide array of mathematical and physical problems, and especially those involving linear differential equations with constant coefficients, for which the eigensolutions are sinusoids. The Fourier series has many such applications in **electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, econometrics, shell theory, etc**.

2.2 Basic Definition

The Fourier series is the trigonometric series representing functions that are periodic. If a function $f(x)$ is defined over the interval $-L < x < L$, under certain conditions, the Fourier series is a periodic approximation to this function in that interval and it is defined as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

where a_0, a_n and b_n are the Fourier coefficients defined as:

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(x)\, dx \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \end{aligned}$$

2.2.1 A special case

In case when the period of the function $f(x)$ is $2L = 2\pi$, the Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \end{aligned}$$

2.3 A fundamental example

Problem: A periodic function is given:

$$f(x) = \begin{cases} -1 & -\pi \leq x \leq 0 \\ 1 & 0 < x \leq \pi \end{cases} \text{ and } f(x + 2\pi) = f(x).$$

Find the Fourier series for this function.

Solution:

Period of $f(x) = 2\pi$

$$\therefore, f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$a_0 = \frac{1}{\pi} \int_{-\pi}^{0} -1dx + \frac{1}{\pi} \int_{0}^{\pi} 1dx = 0,$

$a_n = \frac{1}{\pi} \int_{-\pi}^{0} -\cos nxdx + \frac{1}{\pi} \int_{0}^{\pi} \cos nxdx = 0,$

$b_n = \frac{1}{\pi} \int_{-\pi}^{0} -\sin nxdx + \frac{1}{\pi} \int_{0}^{\pi} \sin nxdx = -\frac{2}{\pi} \frac{-1+\cos n\pi}{n} = \begin{cases} \frac{4}{n\pi} & n = 1, 3, 5, ... \\ 0 & n = 2, 4, 6, ... \end{cases},$

We see that a_0 and a_n are 0, hence we move forward to the next best thing, b_n. All even n returns 0, and odd n iteratory values confining to our loop.

$$b \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$b \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{4}{\pi} \\ \frac{4}{3\pi} \\ \frac{4}{5\pi} \end{bmatrix}$$

Substituting for b_n, we arrive at the final solution:

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x$$

2.4 Kinds of Fourier series

2.4.1 Half-range Fourier series

Suppose a function $f(x)$ is not periodic but is only defined over a finite interval , $0 < x < L$. As the region of interest is only that between $0 < x < L$, we may choose to define the function arbitrarily outside the interval. In particular, we can make our choice so that the resulting function is periodic, with period $2L$.

For example, we can repeat the function periodically so that the resulting function is a *periodic even function* and that $f(x)$ can be represented by a *Fourier cosine series*. Alternatively, we can repeat the function so that the resulting function is a *periodic odd function* and that $f(x)$ can be represented by a *Fourier sine series*.

Nothing alters within the interval of interest. Whichever extension we choose, the resulting Fourier series only give a representation of the original function in the interval $0 < x < L$ and as such is termed a half range Fourier series.

Half range sine series:

$$\begin{aligned} a_0 &= a_n = 0 \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \end{aligned}$$

and

$$f(x) = \sum_{n=1}^{\infty} b(n) \sin \frac{n\pi x}{L}$$

Half range cosine series:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{1}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \\ b_n &= 0 \end{aligned}$$

and

$$f(x) = \frac{a(0)}{2} + \sum_{n=1}^{\infty} a(n)\cos\frac{n\pi x}{L}$$

2.5 Convergence

In engineering applications, the Fourier series is generally presumed to converge almost everywhere (the exceptions being at discrete discontinuities) since the functions encountered in engineering are better-behaved than the functions that mathematicians can provide as counter-examples to this presumption.

In particular, if s is continuous and the derivative of $s(x)$ (which may not exist everywhere) is square integrable, then the Fourier series of s converges absolutely and uniformly to $s(x)$. If a function is square-integrable on the interval $[x_0, x_0 + P]$, then the Fourier series converges to the function at almost every point.

Convergence of Fourier series also depends on the finite number of maxima and minima in a function which is popularly known as one of the Dirichlet's condition for Fourier series. See Convergence of Fourier series. It is possible to define Fourier coefficients for more general functions or distributions, in such cases convergence in norm or weak convergence is usually of interest.

2.6 Divergence

Since Fourier series have such good convergence properties, many are often surprised by some of the negative results. For example, the Fourier series of a continuous T-periodic function need not converge pointwise.The uniform boundedness principle yields a simple non-constructive proof of this fact.

In 1922, *Andrey Kolmogorov* published an article titled *Une série de Fourier-Lebesgue divergente presque partout* in which he gave an example of a Lebesgue-integrable function whose Fourier series diverges almost everywhere. He later constructed an example of an integrable function whose Fourier series diverges everywhere (Katznelson 1976).

2.7 Birth of harmonic analysis

Since Fourier's time, many different approaches to defining and understanding the concept of Fourier series have been discovered, all of which are consistent with one another, but each of which emphasizes different aspects of the topic. Some of the more powerful and elegant approaches are based on mathematical ideas and tools that were not available at the time Fourier completed his original work. Fourier originally defined the Fourier series for real-valued functions of real arguments, and using the sine and cosine functions as the basis set for the decomposition.

Many other Fourier-related transforms have since been defined, extending the initial idea to other applications. This general area of inquiry is now sometimes called harmonic analysis. A Fourier series, however, can be used only for periodic functions, or for functions on a bounded (compact) interval.

Chapter 3

Series

3.1 Introduction to series

In common parlance the words series and sequence are essentially synonomous, however, in mathematics the distinction between the two is that a series is the sum of the terms of a sequence.

3.1.1 Definition

Let $\{a_n\}$ be a sequence and define a new sequence $\{s_n\}$ by the recursion relation $s_1 = a_1$, and $s_{n+1} = s_n + a_{n+1}$. The sequence $\{s_n\}$ is called the sequence of partial sums of $\{a_n\}$.

Another way to think about $\{s_n\}$ is that it is given by the sum of the first n terms of the sequence $\{a_n\}$, namely:

$$\{s_n\} = a_1 + a_2 + \ldots + a_n. \tag{3.1}$$

A shorthand form of writing this sum is by using the sigma notation:

$$\{s_n\} = \sum_{j=1}^{n} a_j \tag{3.2}$$

3.1.2 Example

Using sigma notation, the sum $1 + 2 + 3 + 4 + 5$ can be written as $\sum_{j=1}^{5} j$.

It can also be denoted as $\sum_{n=1}^{5} n$ or $\sum_{n=0}^{4}(n+1)$. Similarly, the sum:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \frac{1}{n} \tag{3.3}$$

can be written as $\sum_{n=1}^{j} \frac{1}{j}$ or $\sum_{n=1}^{k} \frac{1}{k}$ etc.

3.1.3 Definition

Let $\{a_n\}$ be a sequence and let $\{s_n\}$ be the sequence of partial sums of $\{a_n\}$. If $\{s_n\}$ converges we say that $\{a_n\}$ is summable. In this case, we denote the $lim_{n\to\infty}\{s_n\}$ by

$$\sum_{j=1}^{n} a_j \tag{3.4}$$

3.1.4 Another definition

The expression $\sum_{j=1}^{\infty} a_j$ is called an infinite series (whether or not the sequence $\{a_n\}$ is summable). The sequence $\{a_n\}$ is called the sequence of terms. If the sequence of terms is summable, the infinite series is said to be *convergent*. If not, it is said to *diverge*.

3.1.5 Example

Consider the sequence of terms given by

$$\{a_n\} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \tag{3.5}$$

Then

$$\begin{aligned}
s_1 &= a_1 = 1 - \frac{1}{2} = \frac{1}{2} \\
s_2 &= a_1 + a_2 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) \\
&= 1 + (\frac{1}{2} - \frac{1}{2}) - \frac{1}{3} \\
&= \frac{2}{3},
\end{aligned}$$

etc. Continuing this regrouping, we see that:

$$\begin{aligned} s_n &= a_1 + a_2 + \ldots + a_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{n} - \frac{1}{n+1}) \\ &= 1 + (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{3}) + \ldots + (\frac{1}{n} - \frac{1}{n}) - \frac{1}{n+1} \\ &= \frac{n}{n+1} \end{aligned}$$

$\therefore$ We see that $lim_{n\to\infty}\{s_n\} = 1$ and so $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

3.2 Power series

Many functions can be represented efficiently by means of infinite series. Examples we have seen in calculus include the exponential function

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n, \tag{3.6}$$

and the trigonometric functions,

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!}x^{2k}$$

and

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!}x^{2k+1}.$$

An infinite series of this type is called a power series. To be precise, a *power series* about x_0 is an infinite sum of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

where the a_n's are constants.

In order for a power series to be useful, the infinite sum must actually converge to a finite number, at least for some values of x. Let s_N be the sum of the first $(N + 1)$ terms,

$$s_N = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_N(x - x_0)^N = \sum_{n=0}^{N} a_n(x - x_0)^n.$$

We say that the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges if the sum s_N approaches a finite limit as $N \to \infty$.

Take, for example, the *geometric series*

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

In this case we have

$$s_N = 1 + x + x^2 + x^3 + \cdots + x^N, \qquad x s_N = x + x^2 + x^3 + x^4 \cdots + x^{N+1},$$

$$s_N - x s_N = 1 - x^{N+1}, \qquad s_N = \frac{1 - x^{N+1}}{1 - x}.$$

If $|x| < 1$, then x^{N+1} gets smaller and smaller as N approaches infinity, and hence

$$\lim_{N \to \infty} x^{N+1} = 0.$$

Substituting into the expression for s_N, we find that

$$\lim_{N \to \infty} s_N = \frac{1}{1 - x}.$$

Thus if $|x| < 1$, we say that the geometric series converges, and write

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

On the other hand, if $|x| > 1$, then x^{N+1} gets larger and larger as N approaches infinity, so $\lim_{N\to\infty} x^{N+1}$ does not exist as a finite number, and neither does $\lim_{N\to\infty} s_N$. In this case, we say that the geometric series *diverges*. In summary, the geometric series

$$\sum_{n=0}^{\infty} x^n \qquad \text{converges to} \quad \frac{1}{1-x} \qquad \text{when} \quad |x| < 1,$$

and diverges when $|x| > 1$.

Theorem. *For any power series*

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

there exists R, which is a nonnegative real number or ∞, such that

1. *the power series converges when $|x - x_0| < R$,*
2. *and the power series diverges when $|x - x_0| > R$.*

We call R the *radius of convergence*.

We have seen that the geometric series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

has radius of convergence $R = 1$. More generally, if b is a positive constant, the power series

$$1 + \frac{x}{b} + \left(\frac{x}{b}\right)^2 + \left(\frac{x}{b}\right)^3 + \cdots = \sum_{n=0}^{\infty} \left(\frac{x}{b}\right)^n \tag{3.7}$$

has radius of convergence b. To see this, we make the substitution $y = x/b$, and the power series becomes $\sum_{n=0}^{\infty} y^n$, which we already know converges for $|y| < 1$ and diverges for $|y| > 1$. But

$$|y| < 1 \quad \Leftrightarrow \quad \left|\frac{x}{b}\right| < 1 \quad \Leftrightarrow \quad |x| < b,$$

$$|y| > 1 \quad \Leftrightarrow \quad \left|\frac{x}{b}\right| > 1 \quad \Leftrightarrow \quad |x| > b.$$

Thus for $|x| < b$ the power series (3.7) converges to

$$\frac{1}{1-y} = \frac{1}{1-(x/b)} = \frac{b}{b-x},$$

while for $|x| > b$, it diverges.

There is a simple criterion that often enables one to determine the radius of convergence of a power series.

Ratio Test. The radius of convergence of the power series

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is given by the formula

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

so long as this limit exists.

Let us check that the ratio test gives the right answer for the radius of convergence of the power series (3.7). In this case, we have

$$a_n = \frac{1}{b^n}, \quad \text{so} \quad \frac{|a_n|}{|a_{n+1}|} = \frac{1/b^n}{1/b^{n+1}} = \frac{b^{n+1}}{b^n} = b,$$

and the formula from the ratio test tells us that the radius of convergence is $R = b$, in agreement with our earlier determination.

In the case of the power series for e^x,

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

in which $a_n = 1/n!$, we have

$$\frac{|a_n|}{|a_{n+1}|} = \frac{1/n!}{1/(n+1)!} = \frac{(n+1)!}{n!} = n + 1,$$

and hence

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} (n+1) = \infty,$$

so the radius of convergence is infinity. In this case the power series converges for all x. In fact, we could use the power series expansion for e^x to calculate e^x for any choice of x.

On the other hand, in the case of the power series

$$\sum_{n=0}^{\infty} n! x^n,$$

in which $a_n = n!$, we have

$$\frac{|a_n|}{|a_{n+1}|} = \frac{n!}{(n+1)!} = \frac{1}{n+1}, \qquad R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0.$$

In this case, the radius of convergence is zero, and the power series does not converge for any nonzero x.

The ratio test doesn't always work because the limit may not exist, but sometimes one can use it in conjunction with the

Comparison Test. Suppose that the power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n, \qquad \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

have radius of convergence R_1 and R_2 respectively. If $|a_n| \leq |b_n|$ for all n, then $R_1 \geq R_2$. If $|a_n| \geq |b_n|$ for all n, then $R_1 \leq R_2$.

In short, power series with smaller coefficients have larger radius of convergence.

Consider for example the power series expansion for $\cos x$,

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k}.$$

In this case the coefficient a_n is zero when n is odd, while $|a_n| = 1/n!$ when n is even. In either case, we have $|a_n| \leq 1/n!$. But we have seen that the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

has infinite radius of convergence. It follows from the comparison test that the radius of convergence of

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k}$$

must be at least that large, hence must also be infinite.

Power series with positive radius of convergence are so important that we have a special term for describing functions which can be represented by such power series. A function $f(x)$ is said to be *real analytic* at x_0 if there is a power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

about x_0 with positive radius of convergence R such that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n, \qquad \text{for} \qquad |x-x_0| < R.$$

Chapter 4

Matrices

4.1 Introduction

Imagine a matrix. What comes to your mind. Boxes of numbers arranged like little cupcakes? What does the box, or the cupcakes, or let's say *the rectangular array of real numbers*, represent? They're not simply fun number carts to poke and play opertions with, they're actually **linear functions of a certain kind**. Put differently, **matrices are a result of organizing linear functions in a certain way**. Not too shabby for number boxes!

4.2 Representation of matrices

Now let's try and style it. If it's twisting cartesian planes and manipulating linear functions, let's groom it for the field by heavy-duty notations. It's obviously effortlessly easy.

An $m \times n$ matrix having m rows and n columns is represented as:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

If $m = n$, we say A is square matrix of degree n. The set of all $m \times n$ matrices with real entries will be denoted by $\mathbb{R}^{m \times n}$

4.2.1 Dimensions of a Matrix

From the example before, simply observing what comes out to be m and n notations in the matrix after we define a $m \times n$ would've given you an idea of what they are. Essentially, dimensions. Another example:

$$IfA = \begin{bmatrix} \gamma & \beta & \alpha \\ a & b & c \\ \alpha & \beta & \gamma \end{bmatrix}$$

It has 3 rows and columns, it's dimensions are 3×3. The pattern can't be clearer. If m is the number of rows and n, number of columns, the dimensions are: $m \times n$

4.3 Back to Kindergarten-(now with matrices)

4.3.1 Addition

Unlike two regular numbers who seem to have no bounds whatsoever when it comes to addition, matrices in their place need to have equal dimensions.

Formal definition

The matrix sum (or simply the sum)$A + B$ of two $m \times n$ matrices A and B is defined to be the $m \times n$ matrix C such that $c_{ij} = a_{ij} + b_{ij}$ for all pairs of indices (i, j). The scalar multiple αA of A by a real number α is the matrix obtained by multiplying each entry of A by α.

For

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ \rho & \nabla & \delta \end{bmatrix}$$

C=A+B
or

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \alpha+\rho & \beta+\nabla & \gamma+\delta \end{bmatrix}$$

4.3.2 Subtraction

When adding, invert the sign:

$A - B = A + (-B)$. Dimensional rule apply.
For

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ \alpha & \beta & \gamma \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \rho & \nabla & \delta \end{bmatrix}$$

A-B is given by:C=A-B
or

$$C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ \alpha-\rho & \beta-\nabla & \gamma-\delta \end{bmatrix}$$

4.3.3 Multiplication- maybe skip a grade or 10

It's a little less straight-forward that the other operations. But the fundamental ideas will emerge slowly.

If A is $m \times p$ and B is $p \times n$, we can form $C = AB$, which is $m \times n$

$$C_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \qquad i = 1, \ldots, n \;\; j = 1, \ldots, m \tag{4.1}$$

- to find i, j entry of the product $C = AB$, you need the ith row of A and the jth column of B
- form product of corresponding entries, e.g., third component of ith row of A and third component of jth column of B

- add up all the products

Let's understand with examples:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{bmatrix} \; and \; B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ \rho & \nabla & \delta \end{bmatrix}$$

For C=A*B
we have:

$$C = \begin{bmatrix} 0*1+0*0+0*\rho & 0*1+0*0+0*\nabla & 0*1+0*0+0*\delta \\ 1*1+1*0+1*\rho & 1*1+1*0+1*\nabla & 1*1+1*0+1*\delta \\ \alpha*1+\beta*0+\gamma*\rho & \alpha*1+\beta*0+\gamma*\nabla & \alpha*1+\beta*0+\gamma*\delta \end{bmatrix}$$

If $C = AB$, then

$$C_{n,m} = \sum_{i=1}^{\infty}(A_{n,i}B_{i,m}) \tag{4.2}$$

where, $C_{n,m}$ denotes the m^{th} element of n^{th} row of matrix C.

Properties of Matric multiplication

- $0A = 0, A0 = 0$
- $AI = A, IA = A$
- $A(BC) = (AB)C$
- $(\alpha A)B = \alpha(AB)$, where α is a scalar.
- $A(B + C) = AB + AC$
- $(AB)^T = B^T A^T$

4.4 FORTRAN program to multiply matrices

```
 program matrix_mult
implicit none

real,dimension(1000,1000) :: A,B,C
```

```
integer::p,q,r,s,i,j,k

print*, "Enter the dimensions of first matrix"
read(*,*) p
read(*,*) q

print*, "Enter the dimensions of second matrix"
read(*,*) r
read(*,*) s

!checking if multiplicable
if (q.eq.r) then
        print*, "Matrix can be multiplicated"
else
    print*,"Not multiplicable"
end if

!multiplying
if (q.eq.r) then
!inputting
    print*, "enter the elements for 1"
    do i=1,p,1
        do j=1,q,1
            read (*,*) A(i,j)
        end do
    end do
    print*," "
     print*, "A"
    do i=1,p
        print*,(A(i,j),j=1,q)
    end do

    print*, "enter the elements for 2"
    do i=1,r
        do j=1,s
            read (*,*) B(i,j)
        end do
    end do
```

```
    print*," "
     print*, "B"
    do i=1,r
            print*, (B(i,j),j=1,s)
    end do



    do i=1,p
        do j=1,s
            C(i,j)=0
            do k=1,r
                C(i,j)=C(i,j)+(A(i,k)*B(k,j))
            end do
        end do
    end do
    print*," "
    print*,"product"
    do i=1,p
            print*,(C(i,j),j=1,s)
    end do

end if

end program matrix_mult
```

Chapter 5

Projectile Motion

5.1 What is it?

Projectile motion is a **special case of two-dimensional motion**. A particle moving in a vertical plane with an initial velocity and experiencing a free-fall (downward) acceleration, displays projectile motion. Some examples of projectile motion are the motion of a ball after being hit/thrown, the motion of a bullet after being fired and the motion of a person jumping off a diving board. For now, we will assume that the air, or any other fluid through which the object is moving, does not have any effect on the motion. In reality, depending on the object, air can play a very significant role. For example, by taking advantage of air resistance, a parachute can allow a person to land safely after jumping off an airplane.

5.1.1 An analysis

Before proceeding, the following section provides a reminder of the three main equations of motion for constant acceleration. These equations are used to develop the equations for projectile motion.

Equations of motion for constant acceleration

The following equations are three commonly used equations of motion for an object moving with a constant acceleration.

$$v = v_0 + at$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

Here, a is the acceleration, v is the speed, v_0 is the initial speed, x is the position, x_0 is the initial position and t is the time.

Taking an arbitary point (x, y) in the trajectory of a projectile attained after 't' time, we see

$$x = v\cos\theta * t \quad and \quad y = v\sin\theta * t - \frac{1}{2}gt^2$$

by the eq.of motion, where 'g' is accelaration due to gravity. since,$x = v\cos\theta * t$ putting $t = x/v\cos\theta$ in equation for y

$$y = v\sin\theta * \frac{x}{v\cos\theta} - \frac{1}{2}g(\frac{x}{v\cos\theta})^2$$

$$\boxed{y = \tan\theta x - \tfrac{1}{2}(\tfrac{g}{(v\cos\theta)^2}x^2}$$

Hence, the equation of a projectile is a parabola as we can easily infer by looking at the path of anything we throw(throw something right now).

5.2 Projectile Motion-Attributes

5.2.1 Maximum Height

Let the maximum hieght attained by a projectile be H, then vertical component of velocity $v_y = 0$, So, by using the 3rd equation of motion:$2as = v^2 - u^2$, we get

$$2(-g)H = 0^2 - (v\sin\theta)^2$$

$$\implies H = \frac{v^2 \sin^2\theta}{2g} \tag{5.1}$$

5.2.2 Range

Putting y=0 in the equation of trajectory for the projectile,we get 2 solutions one of which will be 0 and other- the **maximum horizontal displacement** of a projectile which is it's **range**. Hence,

$$0 = x(\tan\theta - \frac{1}{2}(\frac{g}{(v\cos\theta)^2}x)$$

$$\implies x = 0, \quad \tan\theta - \frac{1}{2}(\frac{g}{(v\cos\theta)^2}x = 0$$

solving the 2nd equation

$$\tan\theta = \frac{1}{2}(\frac{g}{(v\cos\theta)^2}x$$

$$x = \frac{2v^2\sin\theta\cos\theta}{g}$$

Hence, we get

$$R = \frac{v^2\sin 2\theta}{g} \tag{5.2}$$

where R is the Range of the projectile.

5.2.3 Time period

The time for which the projectile travels is called the time period. Since $x = v\cos\theta * t$

$$\implies R = v\cos\theta * T$$

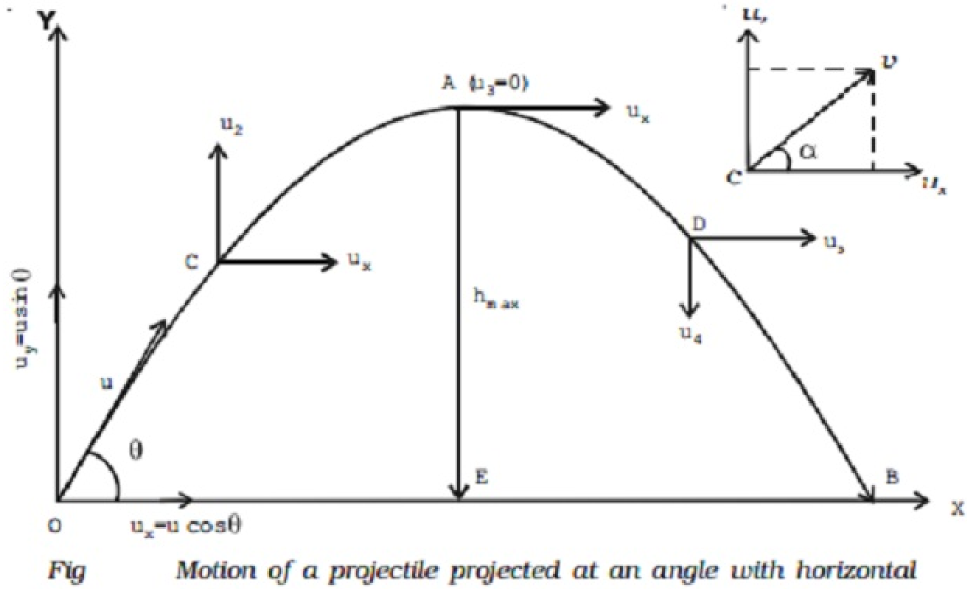

Figure 5.1: Projectilr motion

$$\frac{2v^2\sin\theta\cos\theta}{g} = v\cos\theta * T$$
$$\implies T = \frac{2v\sin\theta\theta}{g}$$
$$or \quad T = \frac{2v\sin\theta}{g} \tag{5.3}$$

5.3 Significance

- The principle of projectile motion is that, any object having a velcity making some angle $\frac{\pi}{2} - \theta$ with respect to the gravitational force from a celestial body will take a parabolic path if no other force is being applied on it. This can even be used to set the trajectory of a satellite around any celestial body by only using some little thrusts and taking the help of the gravitational field.

- Similarly we can form an analogy to every object moving with an angle ϕ to a force, will perform projectile motion,i.e., move in a parabolic path.

Chapter 6

Ordinary Differential Equations

6.1 Definitions and Basic Concepts

6.1.1 Ordinary Differential Equation (ODE)

An equation involving the derivatives of an unknown function y of a single variable x over an interval $x \in (I)$.

6.1.2 Solution

Any function $y = f(x)$ which satisfies this equation over the interval (I) is called a solution of the ODE.

For example, $y = e^{2x}$ is a solution of the ODE

$$y' = 2y$$

and $y = \sin(x^2)$ is a solution of the ODE

$$xy'' - y' + 4x^3 y = 0.$$

6.1.3 Order n of the DE

An ODE is said to be order n, if $y^{(n)}$ is the highest order derivative occurring in the equation. The simplest first order ODE is $y' = g(x)$.

The most general form of an n-th order ODE is

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

with F a function of $n+2$ variables $x, u_0, u_1, \ldots, u_n$. The equations

$$xy'' + y = x^3, \quad y' + y^2 = 0, \quad y''' + 2y' + y = 0$$

are examples of ODE's of second order, first order and third order respectively with respectively

$$F(x, u_0, u_1, u_2) = xu_2 + u_0 - x^3, \;\; F(x, u_0, u_1) = u_1 + u_0^2, \;\; F(x, u_0, u_1, u_2, u_3) = u_3 +$$

6.1.4 Linear Equation:

If the function F is linear in the variables $u_0, u_1, \ldots, u_n$ the ODE is said to be **linear**. If, in addition, F is homogeneous then the ODE is said to be homogeneous. The first of the above examples above is linear are linear, the second is non-linear and the third is linear and homogeneous. The general n-th order linear ODE can be written

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

6.1.5 Homogeneous Linear Equation:

The linear DE is homogeneous, if and only if $b(x) \equiv 0$. Linear homogeneous equations have the important property that linear combinations of solutions are also solutions. In other words, if $y_1, y_2, \ldots, y_m$ are solutions and $c_1, c_2, \ldots, c_m$ are constants then

$$c_1 y_1 + c_2 y_2 + \cdots + c_m y_m$$

is also a solution.

6.1.6 Partial Differential Equation (PDE)

An equation involving the partial derivatives of a function of more than one variable is called PED. The concepts of linearity and homogeneity can be extended to PDE's. The general second order linear PDE in two variables x, y is

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+b(x,y)\frac{\partial^2 u}{\partial x \partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}+d(x,y)\frac{\partial u}{\partial x}+e(x,y)\frac{\partial u}{\partial y}+f(x,y)u = g(x,y).$$

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a linear, homogeneous PDE of order 2. The functions $u = \log(x^2 + y^2)$, $u = xy$, $u = x^2 - y^2$ are examples of solutions of Laplace's equation. We will not study PDE's systematically in this course.

6.1.7 General Solution of a Linear Differential Equation

It represents the set of all solutions, i.e., the set of all functions which satisfy the equation in the interval (I). For example, the general solution of the differential equation $y' = 3x^2$ is $y = x^3 + C$ where C is an arbitrary constant. The constant C is the value of y at $x = 0$. This **initial condition** completely determines the solution. More generally, one easily shows that given a, b there is a unique solution y of the differential equation with $y(a) = b$. Geometrically, this means that the one-parameter family of curves $y = x^2 + C$ do not intersect one another and they fill up the plane $\mathbb{R}^2$.

6.1.8 A System of ODE's

An n-th order ODE of the form $y^{(n)} = G(x, y, y', \ldots, y^{n-1})$ can be transformed in the form of the system of first order DE's. If we introduce dependant variables $y_1 = y, y_2 = y', \ldots, y_n = y^{n-1}$ we obtain the equivalent system of first order equations

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ &\vdots \\ y_n' &= G(x, y_1, y_2, \ldots, y_n). \end{aligned}$$

For example, the ODE $y'' = y$ is equivalent to the system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_1. \end{aligned}$$

In this way the study of n-th order equations can be reduced to the study of systems of first order equations. Some times, one called the latter as the **normal form** of the n-th order ODE. Systems of equations arise in the study of the motion of particles. For example, if $P(x, y)$ is the position of a particle of mass m at time t, moving in a plane under the action of the force field

$(f(x,y), g(x,y)$, we have

$$m\frac{d^2x}{dt^2} = f(x,y),$$
$$m\frac{d^2y}{dt^2} = g(x,y).$$

This is a second order system.

The general first order ODE in normal form is

$$y' = F(x,y).$$

If F and $\frac{\partial F}{\partial y}$ are continuous one can show that, given a, b, there is a unique solution with $y(a) = b$. Describing this solution is not an easy task and there are a variety of ways to do this. The dependence of the solution on initial conditions is also an important question as the initial values may be only known approximately.

The non-linear ODE $yy' = 4x$ is not in normal form but can be brought to normal form

$$y' = \frac{4x}{y}.$$

by dividing both sides by y.

6.2 The Approaches of Finding Solutions of ODE

6.2.1 Analytical Approaches

- Analytical solution methods: finding the exact form of solutions;
- Geometrical methods: finding the qualitative behavior of solutions;

- Asymptotic methods: finding the asymptotic form of the solution, which gives good approximation of the exact solution.

6.2.2 Numerical Approaches

- Numerical algorithms — numerical methods;
- Symbolic manipulators — Maple, MATHEMATICA, MacSyma.

This course mainly discuss the analytical approaches and mainly on analytical solution methods.

For example, the functions e^x is real analytic at any $x_0 \in R$. To see this, we write $e^x = e^{x_0}e^{x-x_0}$ and utilize (3.6) with x replaced by $x - x_0$:

$$e^x = e^{x_0}\sum_{n=0}^{\infty}\frac{1}{n!}(x-x_0)^n = \sum_{n=0}^{\infty}a_n(x-x_0)^n, \qquad \text{where} \qquad a_n = \frac{e^{x_0}}{n!}.$$

This is a power series expansion of e^x about x_0 with infinite radius of convergence. Similarly, the monomial function $f(x) = x^n$ is real analytic at x_0 because

$$x^n = (x - x_0 + x_0)^n = \sum_{i=0}^{n}\frac{n!}{i!(n-i)!}x_0^{n-i}(x - x_0)^i$$

by the binomial theorem, a power series about x_0 in which all but finitely many of the coefficients are zero.

In more advanced courses, one studies criteria under which functions are real analytic. For the purposes of the present course, it is sufficient to be aware of the following facts: The sum and product of real analytic functions is real analytic. Thus any polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

is analytic at any $x_0 \in R$. The quotient of two polynomials with no common factors, $P(x)/Q(x)$, is analytic at x_0 if and only if x_0 is not a zero of the denominator $Q(x)$. Thus for example, $1/(x-1)$ is analytic whenever $x_0 \neq 1$, but fails to be analytic at $x_0 = 1$.

6.3 Solving differential equations by power series

Our main goal in this chapter is to study how to determine solutions to differential equations by means of power series. One of the simplest examples is our old friend, the equation of simple harmonic motion

$$\frac{d^2y}{dx^2} + y = 0, \tag{6.1}$$

which we have already learned how to solve by other methods. Suppose for the moment that we don't know the general solution and want to find it by means of power series. We could start by assuming that

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

Assuming that the standard technique for differentiating polynomials also works for power series, we would expect that

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

(Note that the last summation only goes from 1 to ∞.) Differentiating again would yield

$$\frac{d^2y}{dx^2} = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

We can replace n by $m+2$ in the last summation so that

$$\frac{d^2y}{dx^2} = \sum_{m+2=2}^{\infty}(m+2)(m+2-1)a_{m+2}x^{m+2-2} = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m.$$

The index m is a “dummy variable" in the summation and can be replaced by any other letter. We replace m by n and obtain the formula

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Substitution into equation(6.1) yields

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_nx^n = 0,$$

or

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + a_n]x^n = 0.$$

Now a polynomial is zero only if all its coefficients are zero. By analogy, we expect that a power series can be zero only if all of its coefficients are zero. Thus we must have

$$(n+2)(n+1)a_{n+2} + a_n = 0,$$

or

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}. \tag{6.2}$$

This is called a *recursion formula* for the coefficients a_n.

The first two coefficients a_0 and a_1 in the power series can be determined from the initial conditions,

$$y(0) = a_0, \qquad \frac{dy}{dx}(0) = a_1.$$

Then the recursion formula can be used to determine the remaining coefficients in the power series by the process of induction. Indeed it follows from (6.2) with $n = 0$ that

$$a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{1}{2}a_0.$$

Similarly, it follows from (6.2) with $n = 1$ that

$$a_3 = -\frac{a_1}{3 \cdot 2} = -\frac{1}{3!}a_1,$$

and with $n = 2$ that

$$a_4 = -\frac{a_2}{4 \cdot 3} = \frac{1}{4 \cdot 3}\frac{1}{2}a_0 = \frac{1}{4!}a_0.$$

Continuing in this manner, we find that

$$\begin{aligned} y &= a_0 + a_1 x - \frac{1}{2!}a_0 x^2 - -\frac{1}{3!}a_1 x^3 + \frac{1}{4!}a_0 x^4 + \cdots \\ &= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right). \end{aligned}$$

We recognize that the expressions within parentheses are power series expansions of the functions $\sin x$ and $\cos x$, and hence we obtain the familiar expression for the solution to the equation of simple harmonic motion,

$$y = a_0 \sin x + a_1 \cos x.$$

We want to extend the approach we have used here to more general second-order homogeneous linear differential equations. Recall from Math 5A, that if $P(x)$ and $Q(x)$ are well-behaved functions, then the solutions to the homogeneous linear differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

can be organized into a two-parameter family

$$y = a_0 y_0(x) + a_1 y_1(x),$$

called the *general solution*. Here $y_0(x)$ and $y_1(x)$ are any two linearly independent solutions and a_0 and a_1 can be arbitrary constants, and we say that $y_0(x)$ and $y_1(x)$ form a *basis* for the space of solutions. In the special case where the functions $P(x)$ and $Q(x)$ are real analytic, the solutions $y_0(x)$ and $y_1(x)$ will also be real analytic. This is the content of the following theorem, which is proven in more advanced books on differential equations:

Theorem. *If the functions $P(x)$ and $Q(x)$ can be represented by power series*

$$P(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \qquad Q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$

with positive radii of convergence R_1 and R_2 respectively, then any solution $y(x)$ to the linear differential equation

$$\frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

can be represented by a power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

whose radius of convergence is $\geq$ the minimum of R_1 and R_2.

This theorem is used to justify the solution of many well-known differential equations by means of the power series method.

Example. *Hermite's differential equation* is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2py = 0, \tag{6.3}$$

where p is a parameter. This equation is very useful for treating the simple harmonic oscillator in quantum mechanics, but for the moment, we can regard it as merely an example of an equation to which the previous theorem applies. Indeed,

$$P(x) = -2x, \qquad Q(x) = 2p,$$

both functions being polynomials, hence power series about $x_0 = 0$ with infinite radius of convergence.

As in the case of the equation of simple harmonic motion, we write

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating term by term as before, and replacing

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Once again, we can replace n by $m+2$ in the last summation so that

$$\frac{d^2y}{dx^2} = \sum_{m+2=2}^{\infty} (m+2)(m+2-1)a_{m+2}x^{m+2-2} = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m,$$

and then replace m by n once again, so that

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n. \tag{6.4}$$

Note that

$$-2x\frac{dy}{dx} = \sum_{n=0}^{\infty} -2na_n x^n, \tag{6.5}$$

while

$$2py = \sum_{n=0}^{\infty} 2pa_n x^n. \tag{6.6}$$

Adding together (6.4), (6.5) and (6.6), we obtain

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2py = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(-2n+2p)a_n x^n.$$

If y satisfies Hermite's equation, we must have

$$0 = \left[\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2} + \sum_{n=0}^{\infty}(-2n+2p)a_n\right] x^n.$$

Since the right-hand side is zero for all choices of x, each coefficient must be zero, so

$$(n+2)(n+1)a_{n+2} + (-2n+2p)a_n = 0,$$

and we obtain the *recursion formula* for the coefficients of the power series:

$$a_{n+2} = \frac{2n-2p}{(n+2)(n+1)}a_n. \tag{6.7}$$

Just as in the case of the equation of simple harmonic motion, the first two coefficients a_0 and a_1 in the power series can be determined from the initial conditions,

$$y(0) = a_0, \qquad \frac{dy}{dx}(0) = a_1.$$

The recursion formula can be used to determine the remaining coefficients in the power series. Indeed it follows from (6.7) with $n = 0$ that

$$a_2 = -\frac{2p}{2 \cdot 1} a_0.$$

Similarly, it follows from (6.7) with $n = 1$ that

$$a_3 = \frac{2 - 2p}{3 \cdot 2} a_1 = -\frac{2(p-1)}{3!} a_1,$$

and with $n = 2$ that

$$a_4 = -\frac{4 - 2p}{4 \cdot 3} a_2 = \frac{2(2-p)}{4 \cdot 3} \frac{-2p}{2} a_0 = \frac{2^2 p(p-2)}{4!} a_0.$$

Continuing in this manner, we find that

$$a_5 = \frac{6 - 2p}{5 \cdot 4} a_3 = \frac{2(3-p)}{5 \cdot 4} \frac{2(1-p)}{3!} a_1 = \frac{2^2 (p-1)(p-3)}{5!} a_1,$$

$$a_6 = \frac{8 - 2p}{6 \cdot 5 \cdot 2} a_4 = \frac{2(3-p)}{6 \cdot 5} \frac{2^2 (p-2)p}{4!} a_0 = -\frac{2^3 p(p-2)(p-4)}{6!} a_0,$$

and so forth. Thus we find that

$$y = a_0 \left[1 - \frac{2p}{2!} x^2 + \frac{2^2 p(p-2)}{4!} x^4 - \frac{2^3 p(p-2)(p-4)}{6!} x^6 + \cdots \right]$$

$$+ a_1 \left[x - \frac{2(p-1)}{3!} x^3 + \frac{2^2 (p-1)(p-3)}{5!} x^5 \right.$$

$$\left. - \frac{2^3 (p-1)(p-3)(p-5)}{7!} x^7 + \cdots \right].$$

We can now write the general solution to Hermite's equation in the form

$$y = a_0 y_0(x) + a_1 y_1(x),$$

where

$$y_0(x) = 1 - \frac{2p}{2!}x^2 + \frac{2^2p(p-2)}{4!}x^4 - \frac{2^3p(p-2)(p-4)}{6!}x^6 + \cdots$$

and

$$y_1(x) = x - \frac{2(p-1)}{3!}x^3 + \frac{2^2(p-1)(p-3)}{5!}x^5 - \frac{2^3(p-1)(p-3)(p-5)}{7!}x^7 + \cdots .$$

For a given choice of the parameter p, we could use the power series to construct tables of values for the functions $y_0(x)$ and $y_1(x)$. In the language of linear algebra, we say that $y_0(x)$ and $y_1(x)$ form a basis for the space of solutions to Hermite's equation.

When p is a positive integer, one of the two power series will collapse, yielding a polynomial solution to Hermite's equation. These polynomial solutions are known as *Hermite polynomials*.

Another Example. *Legendre's differential equation* is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0, \tag{6.8}$$

where p is a parameter. This equation is very useful for treating spherically symmetric potentials, in the theories of Newtonian gravitation and in electricity and magnetism.

To apply our theorem, we need to divide by $1 - x^2$ to obtain

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{p(p+1)}{1-x^2}y = 0.$$

Thus we have

$$P(x) = -\frac{2x}{1-x^2}, \qquad Q(x) = \frac{p(p+1)}{1-x^2}.$$

Now from the preceding section, we know that the power series

$$1 + u + u^2 + u^3 + \cdots \qquad \text{converges to} \qquad \frac{1}{1-u}$$

for $|u| < 1$. If we substitute $u = x^2$, we can conclude that

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots ,$$

the power series converging when $|x| < 1$. It follows quickly that

$$P(x) = -\frac{2x}{1-x^2} = -2x - 2x^3 - 2x^5 - \cdots$$

and

$$Q(x) = \frac{p(p+1)}{1-x^2} = p(p+1) + p(p+1)x^2 + p(p+1)x^4 + \cdots .$$

Both of these functions have power series expansions about $x_0 = 0$ which converge for $|x| < 1$. Hence our theorem implies that any solution to Legendre's equation will be expressible as a power series about $x_0 = 0$ which converges for $|x| < 1$. However, we might expect the solutions to Legendre's equation to exhibit some unpleasant behaviour near $x = \pm 1$.

Indeed, it can be shown that when p is an integer, Legendre's differential equation has a nonzero polynomial solution which is well-behaved for all x, but solutions which are not constant multiples of these *Legendre polynomials* blow up as $x \to \pm 1$.

6.4 Euler's Method

Look at the initial value problem:

$$\frac{dy}{dx} = f(x,y) \text{ where } y(a) = y_o \text{ and } x \in \text{[a,b]}$$

To solve this equation he took n steps in [a,b] such that h is the step-size between two consecutive values of x:

$$h = x_i - x_{i-1}$$

or

$$h = \frac{b-a}{n}$$

Hence, we can approximate the value of first order derivative of y by

$$y'(x) \approx \frac{y(x+h) - y(x)}{h} \text{if h is sufficiently small}$$

or

$$y(x+h) - y(x) \approx hy'(x)$$
$$\implies y(x+h) \approx y(x) + hf(x, y(x))$$

or

$$y_{i+1} = y_i + hf(x_i, y_i) \tag{6.9}$$

Hence, we can find the next value of y approximately by Euler's method if we have the value of the preceding y and x. **Note:**The larger the value of n the more accurate the solution.

6.5 Modified Euler Method

If we write the first approximation to y_{i+1} as y^*_{i+1} ,i.e.,

$$y^*_{i+1} = y_i + hf(x_i, y_i)$$

Then, a better approximation for y_{i+1} can be written as

$$y_{i+1} = y_i + \frac{h}{2}(f(x_i, y_i) + f(x_i, y^*_{i+1})$$

This is known as the **Modified or Improved Euler Method**.

Chapter 7

Integration

7.1 Introduction

In mathematics, an integral assigns numbers to functions in a way that describes displacement, area, volume, and other concepts that arise by combining infinitesimal data. The process of finding integrals is called integration. Along with differentiation, integration is a fundamental, essential operation of calculus, and serves as a tool to solve problems in mathematics and physics involving the area of an arbitrary shape, the length of a curve, and the volume of a solid, among others.

The integrals enumerated here are those termed **definite integrals**, which can be interpreted formally as the signed area of the region in the plane that is bounded by the graph of a given function between two points in the real line. Conventionally, areas above the horizontal axis of the plane are positive while areas below are negative.

Integrals also refer to the concept of an *antiderivative*, a function whose derivative is the given function. In this case, they are called **indefinite integrals**. The fundamental theorem of calculus relates definite integrals with differentiation and provides a method to compute the definite integral of a function when its antiderivative is known.

Although methods of calculating areas and volumes dated from ancient Greek mathematics, the principles of integration were formulated independently by **Isaac Newton** and **Gottfried Wilhelm Leibniz** in the late 17th century, who thought of the area under a curve as an infinite sum of rectangles of infinitesimal width. **Bernhard Riemann** later gave a rigorous definition of integrals, which is based on a limiting procedure that approximates the area of a curvilinear region by breaking the region into thin vertical slabs.

Integrals may be generalized depending on the type of the function as well as the domain over which the integration is performed. For example, a line integral is defined for functions of two or more variables, and the interval of integration is replaced by a curve connecting the two endpoints of the interval. In a surface integral, the curve is replaced by a piece of a surface in three-dimensional space.

7.2 Terminology and Notation

In general, the integral of a real-valued function $f(x)$ with respect to a real variable x on an interval $[a, b]$ is written as:

$$\int_a^b f(x)dx \tag{7.1}$$

The integral sign $\int$ represents integration. The symbol dx, called the differential of the variable x, indicates that the variable of

Figure 7.1: Newton and Leibniz, thinking about integration.

integration is x. The function $f(x)$ is called the integrand, the points a and b are called the limits of integration, and the integral is said to be over the interval $[a, b]$, called the interval of integration. A function is said to be integrable if its integral over its domain is **finite**, and when limits are specified, the integral is called a definite integral.

When the limits are omitted, as in:

$$\int f(x)dx \tag{7.2}$$

the integral is called an indefinite integral, which represents a class of functions (the antiderivative) whose derivative is the integrand. The fundamental theorem of calculus relates the evaluation of definite integrals to indefinite integrals.

7.3 Extensions

7.3.1 Multiple Integration

Just as the definite integral of a positive function of one variable represents the area of the region between the graph of the function and the x-axis, the double integral of a positive function of two variables represents the volume of the region between the surface defined by the function and the plane that contains its domain.

For example, a function in two dimensions depends on two real variables, x and y, and the integral of a function f over the rectangle R given as the Cartesian product of two intervals $\mathbb{R} = [a, b] \times [c, d]$ can be written:

$$\int_{\mathbb{R}} f(x, y)dA \tag{7.3}$$

where the differential dA indicates that integration is taken with respect to area.

7.4 Finding the Area

As we know to get a finite area the figure must have definite boundaries,hence here we will integrate a function between two

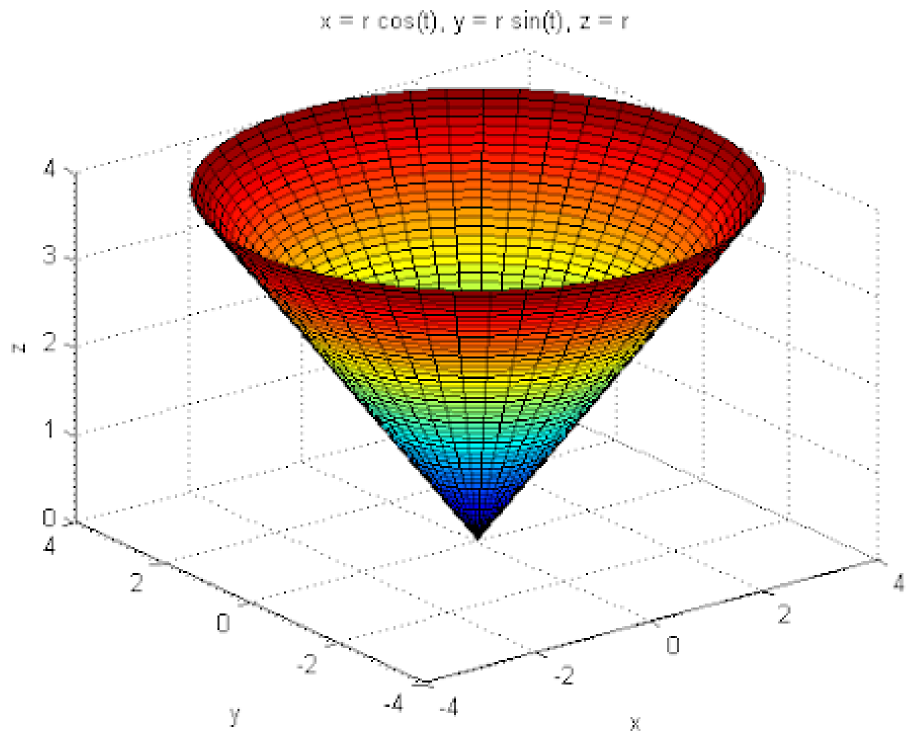

Figure 7.2: Double integral computes volume under a surface $z = f(x, y)$

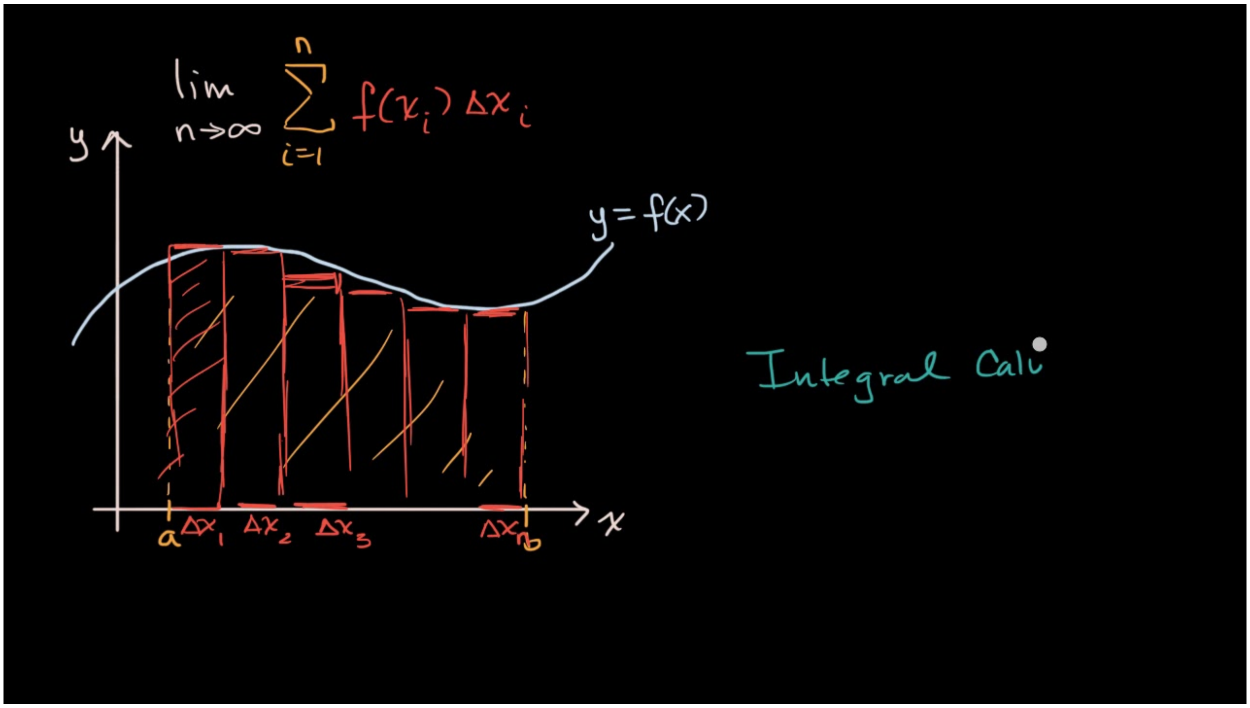

Figure 7.3: Trapezoidal method

limits. To integrate a curve, we will use two specific methods :
i)Trapezoidal Method and ii)Simpson's $\frac{1}{3}$ method

7.5 Trapezoidal Method

In the trapezoidal method, we take the interval of [a,b] ,i.e., the limits and create $n-1$ points in it making b the n^{th} value of x and a the 0_{th}.Then, we join the points $f(x_i)$ on the curve with x_i on the x-axis. This gives us n trapezoids if we take the interval between two consecutive values of x sufficiently small ,i.e., $f(x)$ acts as a straight line between its two consecutive values as it can be seen in the figure.

Let P_1, P_2 be the parallel sides of the trapezoid, and d be the distance between them. Then, Area of a trapezoid:

$$\frac{1}{2}(P_1 + P_2) \times d \tag{7.4}$$

Let the distance between two consecutive values of x be h

Area of i^{th} trapezoid$=\frac{1}{2}(f(x_i) + f(x_{i-1})(h)$

Hence,

$$\int_a^b f(x)dx = \sum_{i=1}^{n} \frac{1}{2}[f(x_i) + f(x_{i-1}](h)$$

$$\implies \int_a^b f(x)dx = \frac{h}{2}\sum_{i=1}^{n}[f(x_i) + f(x_{i-1})]$$

or

$$\int_a^b f(x)dx = \frac{h}{2}[f(a) + f(b)] + h\sum_{i=1}^{n}[f(x_i) + f(x_{i-1})] \tag{7.5}$$

7.5.1 Trapezoidal in FORTRAN

Input:

```
program trap
implicit none
real,external::f            !the function is external
real::a,b,h,s,sum
integer::n,i
print*,"Enter the lower limit "
read*,a
print*,"Enter the upper limit "
```

```
read*,b
print*,"Enter the no. of intervals"
read*,n
h = (b-a)/n
s = (h/2)*(f(a) + f(b))
sum= 0
do i=1,n-1
sum= sum +h*f(a + i*h)
end do
sum = s + sum
print*,"The Integration of x^2 from",a," to ",b," is approximately
end program trap

real function f(x)      !function to be integrated
implicit none
real::x
f = x**2
end
```

Bibliography

[1] Mathematical physics -**Arfken and Weber**

[2] Principles of Analysis-**Sherbert**

[3] https://www.overleaf.com

[4] Mathematical Physics by **HK Dass**