Probing the $\rho(770)$ Resonance: Data Analysis, Modelling and Parameter Estimation

Shahnaz Abdul Hameed

27/07/2023

1 Abstract

In this study, we aim to investigate the $\rho(770)$ Resonance: Analyzing experimental data (E2 vs δ), modelling through T-matrix (from K-matrix and Breit-Wigner formula), fitting and determining the best fit parameters with Chi-square (χ^2), predicting the pole position and estimating uncertainties. The methodologies outlined in this study can be useful in building similar tools for analysing other *events* and systems.

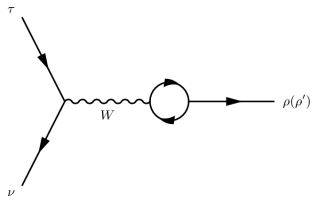


Figure 1: Feynman diagram of the decay: $\tau \to \rho(\rho')\nu_\tau$

2 $\rho(770)$ Resonance

2.1 What is it

The Rho-resonance $\rho(770)$ is known as the lightest vector-meson resonance and plays a crucial role in particle physics. It is described by a Breit-Wigner resonance formula:

$$BW(m) = \frac{1}{m^2 - m_\rho^2 + i m_\rho \Gamma_\rho}, \tag{1}$$

where m is the invariant mass of the final-state particles, m_ρ is the mass of the $\rho(770)$ resonance, and Γ_ρ is its total width. The narrowness of the $\rho(770)$ resonance ($\Gamma_\rho \ll m_\rho$) makes it well pronounced in experimental observations.

The $\rho(770)$ resonance dominates numerous processes involving vector mesons due to Vector Meson Dominance (VMD). VMD is a concept in particle physics where the exchange of vector mesons, such as ρ mesons, mediates certain interactions between other particles. It is described by the Lagrangian density:

$$\mathcal{L}_{\text{VMD}} = \frac{g_\rho}{m_\rho} \rho_\mu \left(\sum_f \bar{f} \gamma^\mu f \right), \tag{2}$$

where g_ρ is the ρ-meson coupling constant and ρ_μ is the $\rho(770)$ vector meson field.

The existence and properties of the $\rho(770)$ resonance have been extensively studied through various experimental measurements. It appears in scattering experiments, decay processes, production reactions, basically everywhere- it's a resonance that doesn't quit. Additionally, lattice data, obtained from lattice quantum chromodynamics calculations, has also contributed valuable insights into the properties and behavior of $\rho(770)$ and its interactions with other particles. The combination of experimental and lattice data has greatly improved our understanding of the $\rho(770)$ resonance and its impact on fundamental interactions in the realm of particle physics.

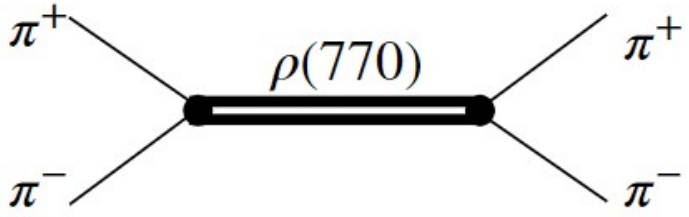


Figure 2

2.2 Experimental Data

The experimental data presenting Energy vs. Delta, where"delta" denotes the phase shift in pion-nucleus scattering, provides valuable information about the existence and properties of resonances. Resonances are characterized by sharp changes in the phase shift as a function of energy, indicating the presence of intermediate states that significantly affect the scattering process. To identify resonances from this data, we look for abrupt changes or sharp peaks in the phase shift values as the energy varies.

A widely used theoretical framework for analyzing resonance behavior in scattering processes is the Breit-Wigner formula, which describes the energy dependence of the phase shift in the vicinity of a resonance:

$$\delta(E) = \arctan\left(\frac{\Gamma_\rho/2}{E - m_\rho}\right), \tag{3}$$

where $\delta(E)$ is the phase shift at energy E, m_ρ is the resonance mass, and Γ_ρ is the total width of the resonance. In the presence of a resonance, the phase shift exhibits a characteristic behavior with a maximum or minimum value, corresponding to the resonance peak position ($E_{\text{peak}} = m_\rho$). Additionally, the width of the resonance in the data (ΔE_{FWHM}) is related to the total width by $\Delta E_{\text{FWHM}} \approx \Gamma_\rho$.

By fitting the experimental data with the Breit-Wigner formula and adjusting the resonance parameters (m_ρ and Γ_ρ), we can quantitatively determine the properties of the resonance, such as its mass and total width. This analysis allows for the identification of the Rho-resonance $\rho(770)$ and other resonant states.

2.3 Through the lens of K- Matrix

The K-matrix approach is a powerful technique used in scattering theory to describe resonant states and scattering amplitudes in a model-independent way. It provides an alternative representation of the scattering matrix (S-matrix) and is particularly useful when dealing with strong interactions and resonances.

In the K-matrix approach, the S-matrix is related to the K-matrix through the following equation:

$$S = \eta \frac{1 - iK}{1 + iK}, \tag{4}$$

where S is the usual S-matrix, and η is a diagonal matrix representing the relative phase between the different scattering channels. The K-matrix is a complex symmetric matrix, and its eigenvalues correspond to the poles of the scattering matrix, which are associated with resonant states. It can be parameterized by its eigenvalues and eigenvectors, providing a clear physical interpretation of the resonant behavior.

The K-matrix approach offers several advantages over other methods in dealing with resonances:

1. **Model Independence:** The K-matrix approach does not rely on specific theoretical models, making it more flexible and applicable in a wide range of scattering processes.

2. **Unitarity:** The K-matrix approach ensures that the resulting S-matrix is unitary, preserving the conservation of probability in scattering interactions.

3. **Pole Resonances:** The poles of the K-matrix correspond to resonant states, providing valuable information about their masses, widths, and couplings without assuming a specific Breit-Wigner resonance form.

The T-matrix form of the K-matrix is an alternative representation that simplifies the calculation of the K-matrix for certain scattering processes. The T-matrix is related to the K-matrix through the equation:

$$T = (1 + iK)\eta, \tag{5}$$

where T is the T-matrix. The T matrix allows for a straightforward description of the scattering amplitude without the need to invert complex matrices. This simplification can be particularly useful in practical calculations involving multiple scattering channels and resonance states.

In the context of partial-wave analysis, the T-matrix can be expressed in terms of the K-matrix as follows:

$$T_l(s) = \frac{\sqrt{s}}{K_l^{-1}(s) - ip_{\text{cms}}}, \tag{6}$$

where s is the center-of-mass energy squared, p_{cms} is the center-of-mass momentum. And

$$p_{cms} = \sqrt{\frac{s}{4} - m^2}, \tag{7}$$

where m is the mass of the pion.

Caution: The one in numerator comes from fixed angular momentum at 1.

In scattering theory, the function $K_l^{-1}(s)$ is the inverse of the K-matrix and is related to the scattering phase shift $\delta_l(s)$ for the partial wave l as:

$$K_l^{-1}(s) = p_{cms} \cot \delta_l(s), \tag{8}$$

where $\delta_l(s)$ is the phase shift for the l-th partial wave at the given energy s.

2.4 Using this information & parametrising the T matrix

To analyze the system, we will represent the T-matrix as a function of the center-of-mass energy squared (s) and parametrise it by $T(s, a, b, c)$ (*why, what is going on?* $\rightarrow$).

The T-matrix can be expressed (as explained earlier) in terms of the K-matrix, and we can "rely" on the K matrix for our *degrees of freedom*:

$$T_l(s) = \frac{\sqrt{s}}{K_l^{-1}(s) - ip_{\text{cms}}}, \tag{9}$$

We can expand this out by substituting for p_{cms}, hence obtaining finer grained flexibility in choosing how T changes:

$$T(s, a, b, c) = \frac{\sqrt{s}}{2\left(a + \sqrt{1 - \frac{s}{4}} + b \cdot s + c \cdot s^2\right)}, \tag{10}$$

where a, b, c, and d are the adjustable parameters that determine the behavior of the scattering amplitude.

2.5 Minimizing error, enter Chi-Square

To fit our model to the experimental data, we will utilize the chi-square (χ^2) method. This allows us to compare our theoretical predictions with the experimental data by calculating the sum of the squared differences between the two. The goal is to minimize the χ^2 value by adjusting the parameters a, b, c, and d, thereby obtaining the best-fit model that describes the scattering behavior and identifies resonant states in the system.

2.5.1 What's that now

The chi-square (χ^2) statistic is a measure used in statistical analysis to quantify the goodness-of-fit between a theoretical model and observed data. It compares the expected values from the model with the actual observed data and assesses how well the model describes the observed outcomes.

Mathematically, the χ^2 statistic is computed as the sum of the squared differences between the observed data (O_i) and the corresponding expected values from the model (E_i), normalized by the uncertainty (σ_i) in each observation:

$$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{\sigma_i^2}. \tag{11}$$

A smaller value of χ^2 indicates a better agreement between the model and the data, implying a more accurate representation of the underlying system. Conversely, a larger χ^2 suggests that the model does not fit the data well.

We will now study the parameters that best "fit" the experimental data to identify the presence of where poles occur in subsequent Energy-T complex 3-D plots.

2.6 Computation

2.6.1 Trial one: the Breit-Wigner formula

The Breit-Wigner formula, also known as the Breit-Wigner resonance formula, is a mathematical expression commonly used to describe the behavior of resonances in scattering processes. It relates the phase shift (δ) to the squared energy (s) and parameters M (resonance mass) and Γ (resonance width) as follows:

$$\delta = \arctan\left(\frac{2\Gamma\sqrt{s}}{M^2 - s}\right) + \texttt{constant}$$

where δ is the phase shift in degrees, and `constant` represents an additional constant term.

What we try to do

We fit the theoretical model represented by the Breit-Wigner formula to the experimental data for the phase shift (δ_{exp}) and energy ($E2_{\text{exp}}$). The fitting process aims to determine the best-fit values of M, Γ, and `constant` that provide the closest agreement between the theoretical model and the experimental observations.

By using the Breit-Wigner formula, the code can extract parametric information about the resonance state, such as its mass (M_{fit}), width (Γ_{fit}), and pole position.

Theoretical Model Definition:

The function `theoretical_model(E2, M, Gamma, constant)` defines a theoretical model for the phase shift (δ) as a function of the squared energy ($E2$) and three parameters (M, Γ, and `constant`). The model is given by:

$$\delta = \text{np.degrees}\left(\arctan\left(\frac{2 \cdot \Gamma \cdot E2}{M^2 - E2^2}\right)\right) + \texttt{constant}.$$

The function calculates the phase shift in degrees and returns the result.

Experimental Dataset:

The code defines experimental data for the energy ($E2$) and the corresponding phase shift (δ_{exp}). Additionally, it provides the errors associated with the phase shift measurements.

Parameter Fitting:

The `curve_fit` function from the `scipy.optimize` module is used to find the best-fit parameters that minimize the differences between the theoretical model and the experimental data. It starts from an initial guess (`initial_guess`) for the parameter values M, Γ, and `constant`.

Extracting Best-fit Parameters and Uncertainties:

The code extracts the best-fit parameters (M_{fit}, Γ_{fit}, and `constant_fit`) and their uncertainties (M_{err}, Γ_{err}, and `constant_err`) from the fitting procedure.

Calculating Chi-Square Value:

The code calculates the chi-square statistic to evaluate the goodness-of-fit between the model and the experimental data.

Calculating Pole Position:

The code calculates the pole position (`pole_position`) of the resonance state (rho(770)) using the fitted parameters M_{fit} and Γ_{fit}.

Plotting Fitted Model and Experimental Data:

The code plots the experimental data with error bars and overlays the best-fit model for the phase shift as a function of energy.

Results

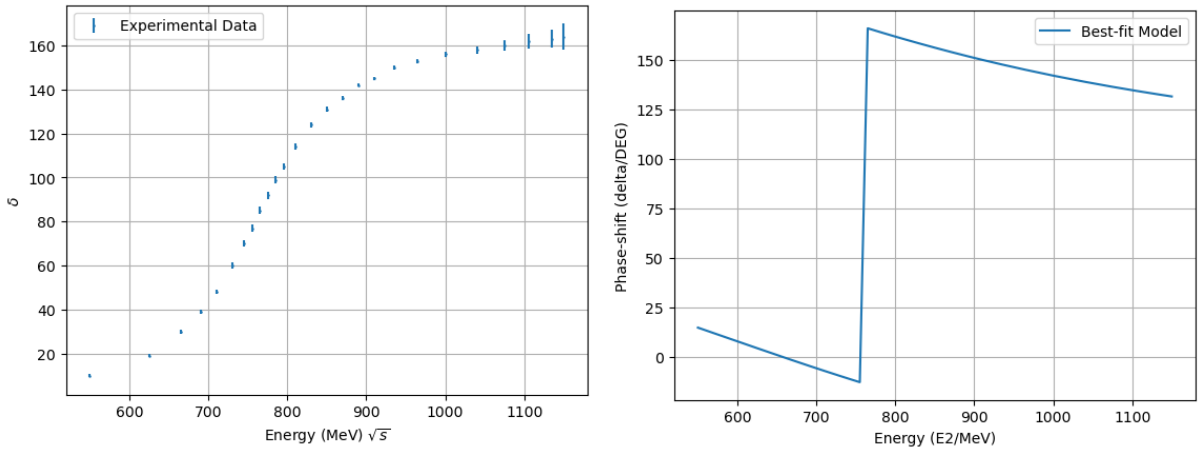
```
Best-fit Parameters:
Mass (M): 765.000 MeV
Width (Gamma): -467.735 MeV
Constant: 76.046 degrees

Parameter Uncertainties:
Mass (M) Uncertainty: 0.000 MeV
Width (Gamma) Uncertainty: 241.235 MeV
Constant Uncertainty: 7.796 degrees

Chi-Square Value: 26027.299

Pole Position of rho(770):
Real Part (Mass): 765.000 MeV
Imaginary Part (Width): -467.735 MeV
```

Plots



(a) (b)

Figure 3: Experimental data and best fit model

2.6.2 Trial 2: Parametrising T explicitly

Theoretical Model Definition:

The function `T_full(s, a, b, c, d)` defines a theoretical model for the partial-wave scattering amplitude (T) as a function of the center-of-mass energy squared (s) and several parameters (a, b, c, and d). The model is given by:

$$T(s,a,b,c,d) = \frac{\sqrt{s}}{2\left(a + \sqrt{1-\frac{s}{4}} + b\cdot s + c\cdot s^2\right)}.$$

The function then calculates the real part of the scattering amplitude (`np.real(T)`) and returns the result.

Experimental Data:

The code defines/stores experimental data for the center-of-mass energy (E) and the corresponding scattering phase shift (δ_{exp}). Additionally, it provides the errors associated with the phase shift measurements.

Chi-Square Function:

The `chi_square(params)` function computes the chi-square statistic for a given set of model parameters (`params`). It calculates the theoretical scattering amplitude (`T_model`) using the model defined earlier and compares it to the experimental phase shift (δ_{exp}). The residuals (differences between the model and the data) are normalized by the error associated with each measurement. The function returns the sum of the squared residuals.

Parameter Fitting:

The `minimize` function from the `scipy.optimize` module is then used to find the best-fit parameters that minimize the chi-square statistic. It starts from an initial guess (`initial_guess`) for the parameter values and employs the Nelder-Mead optimization method.

Visualization:

The code extracts the best-fit parameters (`best_fit_params`) and calculates the real part of the scattering momentum (`pc`) based on the fitted model. It then sorts the scattering momenta in reverse order and plots them against the center-of-mass energy (E). The plot shows the best-fit model's behavior compared to the experimental data.

Results

```
Best fit-parameters:
[5.84993668e+00
9.63922184e-02
1.06047922e-03
1.43748784e-03]
```

2.6.3 Checking the goodness of fit, ipywidget: an interactive widget-based tool

Visualization and Parameter Adjustment:

We first calculate the real part of the scattering momentum (`pc`) using the experimental data for the center-of-mass energy (E) and the scattering phase shift (δ_{exp}). We then sort the scattering momenta in reverse order.

Next, we define an initial guess for the parameters a, b, c, and d based on previous chi-square results, which will be adjusted during the interactive visualization.

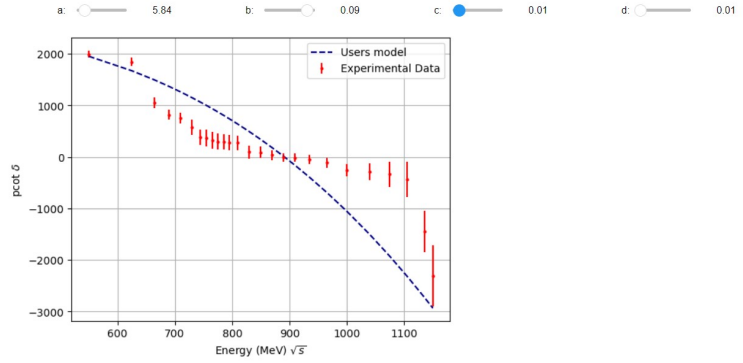


Figure 4: Energy vs pcotδ, Experimental data vs model parameters selected by χ-square

Four adjustable sliders are created for each of the parameters a, b, c, and d. Users can slide the sliders to modify the parameter values.

Interactive Visualization:

The adjustable sliders are displayed in a horizontal container using `widgets.HBox`. The initial plot is displayed using the initial guess values of the parameters.

As users interact with the sliders, the plot updates dynamically based on the adjusted parameter values. The function `update_plot` is called whenever the sliders are adjusted, ensuring that the plot always shows the model corresponding to the selected parameter values.

3 Conclusion

This report aims to study the $\rho(770)$ resonance by analyzing experimental data ($E2$ vs δ) obtained from PDG. The data structure is explored to identify relevant observables like the phase-shift or pcot δ. The selected observable is modelled by primarily parametrizing K-matrix with a particular functional form of s, and then obtaining T matrix form from the K matrix, as well as from the *Breit-Wigner formula*. To fit the model to the data, the Chi-square (χ^2) function is defined; minimizing which yields the best-fit values of the parameters. Predictions include determining the pole position of the $\rho(770)$ resonance using the fitted parameters and estimating uncertainties to account for measurement errors and fitting uncertainties. The work aims at enhancing our understanding of resonant states and scattering processes in particle physics and providing a general theoretical framework in doing so. Much thanks to Prof. Maxim Mai for his invaluable guidance in both the theoretical & computational part.