Basic Probability Rules:

a.

Sample Space Ω for 6-sided die problem: $\{1, 2, 3, 4, 5, 6\}$

b.

Outcome	Probability
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

Table 1: Outcomes and their probabilities for a fair 6-sided die.

c.

Outcome	Probability
1	$\frac{1}{20}$
2	$\frac{18}{80}$
3	$\frac{18}{80}$
4	$\frac{18}{80}$
5	$\frac{18}{80}$
6	$\frac{1}{20}$

Table 2: Outcomes and their probabilities for a biased 6-sided die.

d.

The probability of getting a 3 on a fair die is $\frac{1}{6}$.
The probability of getting a 3 on a biased die is $\frac{18}{80}$.

e.

The probability of rolling an even number on a fair die is $\frac{3}{6}$.
The probability of rolling an even number on a biased die is $\frac{40}{80}$.

f.

The probability of rolling a number divisible by 3 on a fair die is $\frac{2}{6}$.
The probability of rolling a number divisible by 3 on a biased die is $\frac{22}{80}$.

More Interesting Probability Spaces

(a) Sample Space

We are modeling a joint experiment involving:

- A coin flip, with outcomes: $\{\text{Heads}, \text{Tails}\}$,
- A die roll, with outcomes: $\{1, 2, 3, 4, 5, 6\}$.

The sample space Ω consists of all ordered pairs of the form (coin outcome, die outcome):

$$\Omega = \{(\text{Heads}, 1), (\text{Heads}, 2), (\text{Heads}, 3), (\text{Heads}, 4), (\text{Heads}, 5), (\text{Heads}, 6), (\text{Tails}, 1), (\text{Tails}, 2), (\text{Tails}, 3), (\text{Tails},$$

(b) Probabilities for Fair Coin and Fair Die

- $\Pr(\text{Heads}) = \Pr(\text{Tails}) = \frac{1}{2}$
- $\Pr(\text{die outcome } k) = \frac{1}{6}$ for $k \in \{1, 2, 3, 4, 5, 6\}$

Since the coin and die are independent:

$$\Pr(\text{coin outcome, die outcome}) = \Pr(\text{coin}) \cdot \Pr(\text{die}) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

(c) Probabilities for Fair Coin and Biased Die

From Problem 1(c), the biased die probabilities are:

$$\Pr(1) = \frac{1}{20}, \quad \Pr(6) = \frac{1}{20}, \quad \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \frac{18}{80}$$

Also, $\Pr(\text{Heads}) = \Pr(\text{Tails}) = \frac{1}{2}$
Using independence:

$$\Pr(\text{Heads}, 1) = \Pr(\text{Tails}, 1) = \frac{1}{40} \Pr(\text{Heads}, 2) = \Pr(\text{Tails}, 2) = \frac{9}{80} \Pr(\text{Heads}, 3) = \Pr(\text{Tails}, 3) = \frac{9}{80} \Pr(\text{Heads},$$

(d) Are Coin and Die Still Independent (Based on Table 1)?

Given joint probabilities from Table 1:

	1	2	3	4	5	6
Heads	0.08	0.08	0.08	0.14	0.08	0.10
Tails	0.08	0.08	0.08	0.02	0.08	0.10

Compute marginal probabilities:

$$\Pr(\text{Heads}) = 0.08+0.08+0.08+0.14+0.08+0.10 = 0.56 \Pr(\text{Tails}) = 0.08+0.08+0.08+0.02+0.08+0.10 = 0.44$$

$$\Pr(1) = 0.08{+}0.08 = 0.16, \quad \Pr(2) = 0.08{+}0.08 = 0.16, \quad \Pr(3) = 0.08{+}0.08 = 0.16, \quad \Pr(4) = 0.14{+}0.02 = 0$$

Now check if $\Pr(\text{Heads}, 4) = \Pr(\text{Heads}) \cdot \Pr(4)$:

$$\Pr(\text{Heads}) \cdot \Pr(4) = 0.56 \cdot 0.16 = 0.0896 \neq 0.14$$

Hence, the coin and die are **not independent**.

Problem 3 – Expected Value

(a) Flip a fair coin 10 times. $1 per Head. What is the expected payout?

To solve this, I consider each coin flip as a Bernoulli trial with:

$$\Pr(\text{Heads}) = \frac{1}{2}, \quad \Pr(\text{Tails}) = \frac{1}{2}$$

Let X_i represent the payout from the i^{th} flip:

$$X_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ flip is Heads} \\ 0 & \text{if the } i^{\text{th}} \text{ flip is Tails} \end{cases}$$

Then, the expected value for each flip is:

$$\mathbb{E}[X_i] = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$$

Since there are 10 independent flips, the total payout is:

$$X = X_1 + X_2 + \cdots + X_{10}$$

Using linearity of expectation:

$$\mathbb{E}[X] = 10 \cdot \mathbb{E}[X_i] = 10 \cdot \frac{1}{2} = 5$$

So, the expected payout from this game is $5.

(b) Stop flipping at first Tail. Payout is $\$2^{\mathrm{k}}$ if you get k Heads before the first Tail.

In this variant of the game, I only keep flipping the coin until I get the first Tail (or up to 10 flips), and the payout depends on how many consecutive Heads I get before the first Tail.

If the first Tail occurs on the i^{th} flip, then I must have gotten $k = i - 1$ Heads. The probability of this is:

$$\left(\frac{1}{2}\right)^i \quad \text{(since I need } i-1 \text{ Heads followed by a Tail)}$$

The payout for this event is $2^k = 2^{i-1}$. So the expected payout becomes:

$$\mathbb{E}[X] = \sum_{i=1}^{10} \left(\frac{1}{2}\right)^i \cdot 2^{i-1}$$

I simplify the expression:

$$\mathbb{E}[X] = \sum_{i=1}^{10} \left(\frac{1}{2} \cdot \left(\frac{2}{2}\right)^{i-1}\right) = \sum_{i=1}^{10} \frac{1}{2}$$

Which leads to:

$$\mathbb{E}[X] = 10 \cdot \frac{1}{2} = 5$$

So surprisingly, the expected payout in this version is still \$5.

(c) What if the game continues indefinitely? Should I pay \$1000 to play?

Now, the game doesn't have a 10-flip limit — I keep flipping until I get a Tail, no matter how many flips it takes. In this version, the same payout formula applies: $\$2^{\mathrm{k}}$, where k is the number of consecutive Heads before the first Tail.

In this case, the expected payout becomes:

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \cdot 2^{i-1}$$

Again simplifying:

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right) = \infty$$

That means the expected payout is infinite.

So in theory, I would be getting infinite value from this game. Mathematically, it would make sense to pay \$1000 (or any finite amount) to play it.

However this becomes a a confusing here even though the expected value is infinite, most actual outcomes will be small. I could end up getting just \$2 or \$4, and the chances of hitting very high payouts are tiny.

So would I pay \$1000? Mathematically, yes — because the expected value is infinite.

Problem 4 – Linear Search Empirical Verification

In this problem, I empirically verified the theoretical probabilities associated with a linear search algorithm, under the assumption that all elements in an array are independently and uniformly drawn from a fixed-size alphabet.

(a) Choosing M

I selected $M = 5$ so that the alphabet would be small but still yield meaningful variability. A smaller M increases the probability of duplicate values, which is important for testing how likely the key is to appear early in the array.

(b) Choosing the Alphabet

The alphabet chosen was:

$$\text{Alphabet} = \{\text{A}, \text{B}, \text{C}, \text{D}, \text{E}\}$$

This keeps the experimental setup manageable while demonstrating the effects of randomization in element selection.

(c) Choosing the Search Key

I fixed the search key as A throughout the simulation. This decision helps us focus on measuring the behavior of the algorithm and verifying the underlying probability model, rather than being distracted by variation in key selection.

(d) Generating the Dataset

For each array size $n \in \{5, 10, 20, 50\}$, I generated 10,000 arrays of length n, where each element was randomly selected (with replacement) from the defined alphabet. This large number of samples ensures that empirical frequencies closely reflect theoretical expectations, according to the law of large numbers.

(e) Running Linear Search and Collecting Outcomes

For each array, I simulated a linear search:

- If the search key appeared, I recorded the index of its first occurrence.
- If it did not appear, I recorded index -1.

This helped capture the distribution of where the key is found during linear search or whether it's found at all.

(f) Computing Empirical Frequencies and Theoretical Probabilities

The empirical frequencies were obtained by dividing the number of occurrences of each index by the total number of arrays.

The theoretical probabilities were calculated as:

$$p_i = \left(1 - \frac{1}{M}\right)^i \cdot \left(\frac{1}{M}\right) \quad \text{for } i = 0 \text{ to } n - 1$$

$$p_{-1} = 1 - \sum_{i=0}^{n-1} p_i$$

These equations capture the likelihood of the key being found at index i, assuming earlier positions did not contain it, and the probability of not finding the key at all.

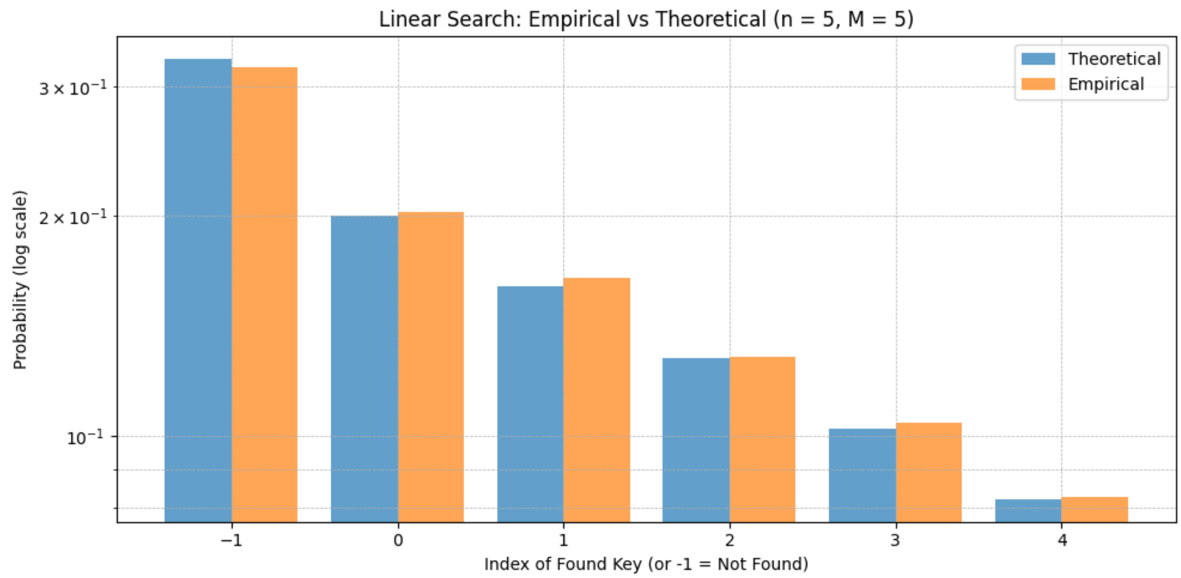

Figure 1: Empirical vs Theoretical Probabilities for $n = 5$

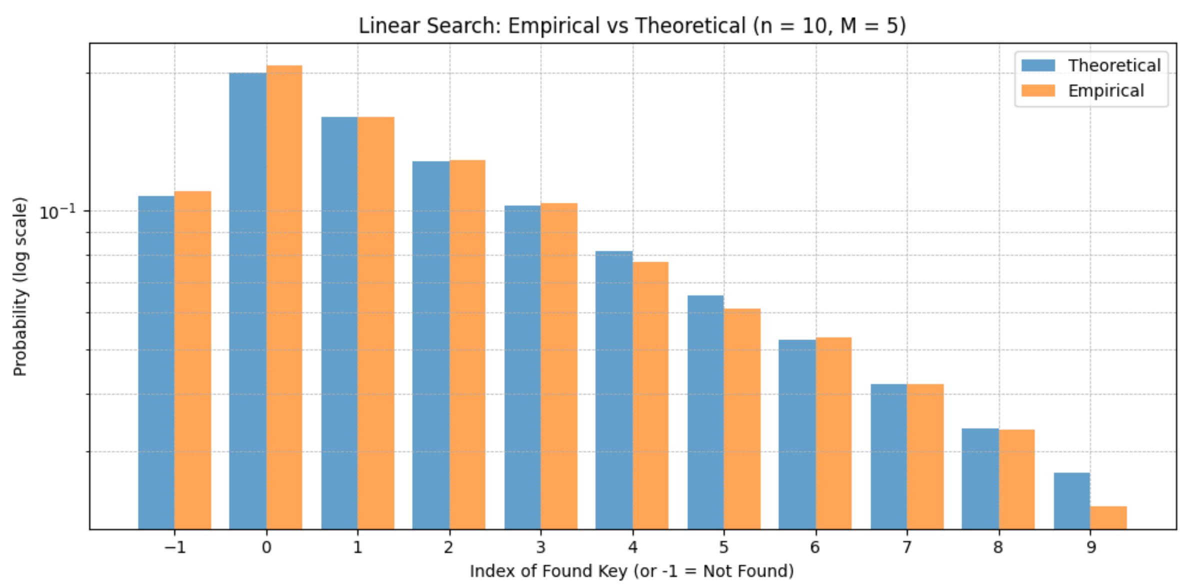

Figure 2: Empirical vs Theoretical Probabilities for $n = 10$

(g) Visualization and Analysis

The following plots compare the theoretical probabilities with the empirical frequencies for each value of n. The y-axis is in logarithmic scale to clearly highlight both high and low probability values.

Discussion

Across all values of n, the empirical frequencies closely follow the theoretical probabilities. The differences are negligible for small n and slightly more noticeable for large n, particularly at higher

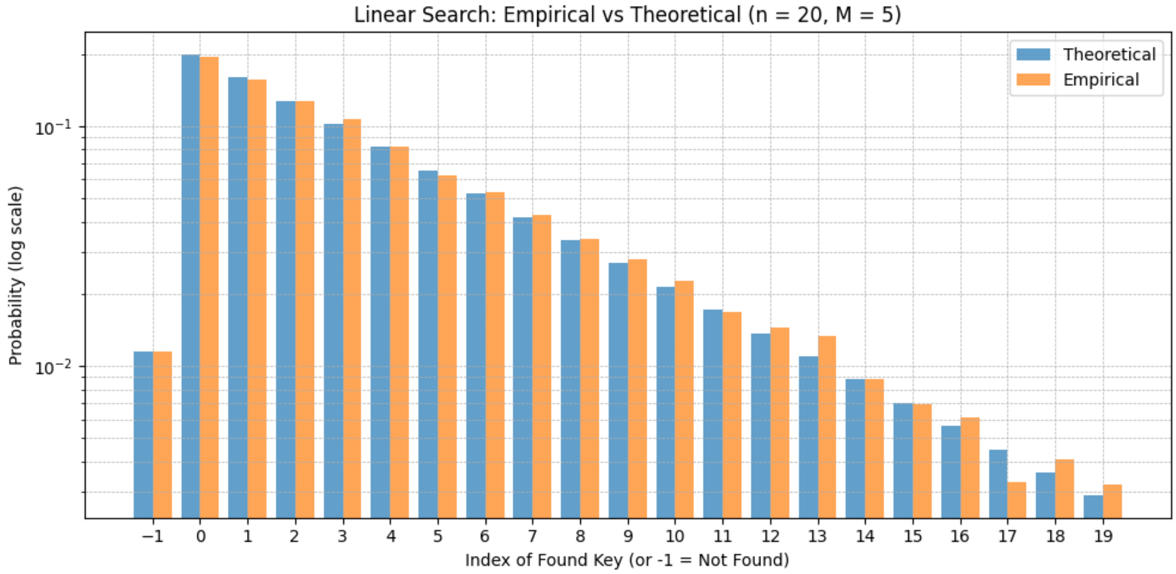

Figure 3: Empirical vs Theoretical Probabilities for $n = 20$

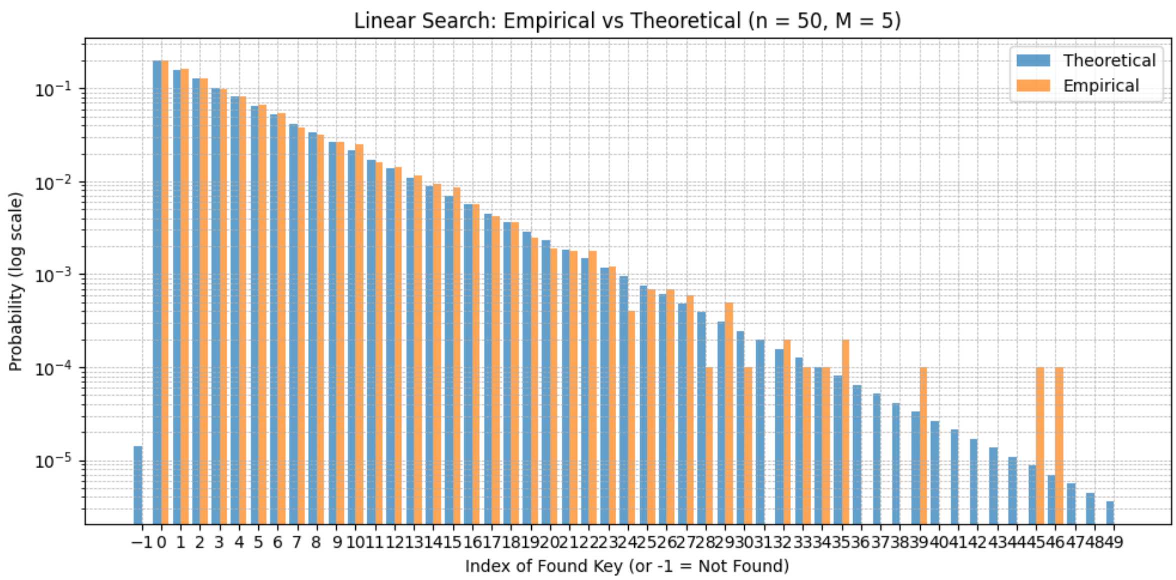

Figure 4: Empirical vs Theoretical Probabilities for $n = 50$

indices where the probabilities become very small. This aligns with expectations, as rare outcomes require larger sample sizes to achieve accurate estimation.

The logarithmic scale on the y-axis was crucial for visualizing the probabilities associated with rare events, especially for $n = 50$.

Here is what I think,The experimental results strongly support the theoretical model. The power of probabilistic reasoning in analyzing algorithms like linear search. Repitition of these trails will eventually make the distribution converge at some point.