CS 5720 Design and Analysis of Algorithms
Homework #1

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Problem 1

Determine the order of growth $\Theta(\cdot)$ for each of the following functions:

Generalized justification: The goal is to determine which term in the function/eqution has the deciding power in how the function grows, it is the largest term in the function/equwation that has the deciding power.

(a) $(n^2+1)^{10}$

Justification: The dominant term in the expression is $(n^2)^{10} = n^{20}$. According to polynomial theorem, the order of growth is affected most by n^{20}.
$\therefore$ **Order of growth:** $\Theta(n^{20})$

(b) $\sqrt{10n^2+7n+3}$

Justification: The term $10n^2$ inside the square root dominates as n grows, thus $\sqrt{10n^2} = n\sqrt{10} \approx n$. Lower order terms are insignificant.
$\therefore$ **Order of growth:** $\Theta(n)$

(c) $2n\log((n+2)^2) + (n+2)^2\log(n/2)$

Justification: After simplifying, $2n\log((n+2)^2) = 4n\log(n+2) \approx 4n\log n$ and $(n+2)^2\log(n/2) \approx n^2\log n$. The $n^2\log n$ term dominates.
Order of growth: $\Theta(n^2\log n)$

(d) $2^{n+1} + 3^{n-1}$

Justification: Between $2^{n+1} \approx 2^n$ and $3^{n-1} \approx 3^n$, 3^n grows faster, hence it dominates.
Order of growth: $\Theta(3^n)$

(e) $\lfloor \log_2 n \rfloor$

Justification: The floor function does not change the order of growth, this only depends on how the logarithm is scaling.
Order of growth: $\Theta(\log n)$

Problem 2

Prove the following assertions by using the definitions of the notations involved, or disprove them by giving a specific counterexample.

(a) If $t(n) \in O(g(n))$, then $g(n) \in \Omega(t(n))$.

Proof: By definition of Big-O, $t(n) \in O(g(n))$ implies $\exists$ a constant $c > 0$ and n_0 such that

$$t(n) \leq c \cdot g(n) \quad \text{for all} \quad n \geq n_0.$$

By definition of Big-Omega, $g(n) \in \Omega(t(n))$ implies $\exists$ a constant $c' > 0$ and n_0' such that

$$g(n) \geq c' \cdot t(n) \quad \text{for all} \quad n \geq n_0'.$$

Since $t(n) \leq c \cdot g(n)$, we can rewrite this as $g(n) \geq \frac{1}{c} \cdot t(n)$. Therefore, the statement $t(n) \in O(g(n)) \implies g(n) \in \Omega(t(n))$ is true.

(b) $\Theta(\alpha g(n)) = \Theta(g(n))$, where $\alpha > 0$.

Proof: Since α is a positive constant, multiplying by α does not change the order of growth. By definition of Big-Theta, $f(n) = \Theta(g(n))$ if and only if $f(n)$ is both $O(g(n))$ and $\Omega(g(n))$. Thus,

$$\alpha g(n) \in O(g(n)) \quad \text{and} \quad \alpha g(n) \in \Omega(g(n)).$$

This implies:

$$\Theta(\alpha g(n)) = \Theta(g(n)).$$

Therefore, the statement is true.

(c) $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$.

Proof: By definition, $\Theta(g(n))$ means the function is bounded both above and below by $g(n)$ up to constant factors. Specifically,

$$\Theta(g(n)) = \{f(n) : \exists c_1, c_2 > 0, \text{ and } n_0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0\}.$$

This is equivalent to the intersection of $O(g(n))$ (functions asymptotically upper-bounded by $g(n)$) and $\Omega(g(n))$ (functions asymptotically lower-bounded by $g(n)$). Hence,

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n)).$$

Therefore, the statement is true.

(d) For any two nonnegative functions $t(n)$ and $g(n)$ defined on the set of nonnegative integers, either $t(n) \in O(g(n))$ or $t(n) \in \Omega(g(n))$, or both.

Proof: For any two functions, one will eventually grow faster or they will be proportional. Specifically, given nonnegative functions $t(n)$ and $g(n)$, either:

$$\exists c > 0 \text{ and } n_0 \text{ such that } t(n) \leq c \cdot g(n) \quad \text{for all} \quad n \geq n_0,$$

implying $t(n) \in O(g(n))$, or:

$$\exists c' > 0 \text{ and } n'_0 \text{ such that } g(n) \leq c' \cdot t(n) \quad \text{for all} \quad n \geq n'_0,$$

implying $t(n) \in \Omega(g(n))$. If both conditions hold, then $t(n)$ and $g(n)$ are asymptotically proportional, and both $t(n) \in O(g(n))$ and $t(n) \in \Omega(g(n))$. Therefore, the statement is true.

Problem 3

Determine the order of growth ($\Theta(\cdot)$) for each of the following functions. Show your work.

Generalized justification: It is the same as problem one, but we have to understannd how the $\sum$ unrolls.

(a) $T(n) = \sum_{i=1}^{2n} i$

Justification: Sum of the first $2n$ natural numbers is $\frac{2n(2n+1)}{2} = 2n^2 + n$. The dominant term is $2n^2$.
Order of growth: $\Theta(n^2)$

(b) $T(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} n$

Justification: The inner sum is $(n - i + 1)n$. Summing from $i = 1$ to n, we get $n \cdot \frac{n(n+1)}{2} = \frac{n^3+n^2}{2}$. The dominant term is n^3.
Order of growth: $\Theta(n^3)$

(c) $T(n) = \sum_{i=1}^{n} n^2$

Justification: Summing n^2 for n times gives $n \cdot n^2 = n^3$.
Order of growth: $\Theta(n^3)$

(d) $T(n) = \sum_{i=1}^{n^2} i$

Justificaton: Sum of the first n^2 natural numbers is $\frac{n^2(n^2+1)}{2} \approx n^4$.
Order of growth: $\Theta(n^4)$

(e) $T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \ell$

Justification: The innermost sum is $\sum_{\ell=1}^{n} \ell = \frac{n(n+1)}{2} \approx n^2$. Repeatedly summing n^2 over the remaining loops gives $n \cdot n \cdot n \cdot n^2 = n^5$.
Order of growth: $\Theta(n^5)$