Homework#2 Solutions

Problem 1: Iterative Methods (Gauss-Seidel Method)

Given System of Equations

$$\begin{aligned} 10x_1 + 2x_2 - x_3 &= 27 \\ -3x_1 - 6x_2 + 2x_3 &= -61.5 \\ x_1 + x_2 + 5x_3 &= -21.5 \end{aligned}$$

(a) Perform 1 iteration of the Gauss-Seidel method analytically

Rearrange the equations to solve for each variable:

$$\begin{aligned} x_1 &= \frac{27 - 2x_2 + x_3}{10} \\ x_2 &= \frac{-61.5 + 3x_1 - 2x_3}{-6} \\ x_3 &= \frac{-21.5 - x_1 - x_2}{5} \end{aligned}$$

Assume initial guesses: $x_1 = 0, x_2 = 0, x_3 = 0$.
1st iteration:

$$\begin{aligned} x_1^{(1)} &= \frac{27 - 2(0) + 0}{10} = 2.7 \\ x_2^{(1)} &= \frac{-61.5 + 3(2.7) - 2(0)}{-6} = 11.85 \\ x_3^{(1)} &= \frac{-21.5 - 2.7 - 11.85}{5} = -7.61 \end{aligned}$$

Problem 2: Newton-Raphson Method for Nonlinear Equations

Given Equations

$$\begin{aligned} y &= -x^2 + x + 0.75 \\ y + 1 &= x^2 \end{aligned}$$

Initial guesses: $x = 1.2, y = 1.2$.
Rewrite as:

$$\begin{aligned} f_1(x, y) &= y + x^2 - x - 0.75 = 0 \\ f_2(x, y) &= x^2 - y - 1 = 0 \end{aligned}$$

1. Evaluate Jacobian and functions:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - 1 & 1 \\ 2x & -1 \end{bmatrix}$$

2. Use Newton-Raphson update:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - J^{-1} \begin{bmatrix} f_1(x_k, y_k) \\ f_2(x_k, y_k) \end{bmatrix}$$

1st iteration:

$$J(1.2, 1.2) = \begin{bmatrix} 1.4 & 1 \\ 2.4 & -1 \end{bmatrix}$$
$$f_1(1.2, 1.2) = 1.2 + 1.44 - 1.2 - 0.75 = 0.69$$
$$f_2(1.2, 1.2) = 1.44 - 1.2 - 1 = -0.76$$

Solve for $\Delta x, \Delta y$:

$$J^{-1} = \frac{1}{-3.4 - 2.4} \begin{bmatrix} -1 & -1 \\ -2.4 & 1.4 \end{bmatrix}$$

$$\Delta = J^{-1} \begin{bmatrix} 0.69 \\ -0.76 \end{bmatrix} = \begin{bmatrix} 0.28 \\ -0.32 \end{bmatrix}$$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.2 \end{bmatrix} - \begin{bmatrix} 0.28 \\ -0.32 \end{bmatrix} = \begin{bmatrix} 0.92 \\ 1.52 \end{bmatrix}$$

Subsequent iterations follow similarly.

Problem 3: LU Factorization and Matrix Inverse

Given matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Problem 4: Gauss Elimination

Given system:

$$-3x_2 + 7x_3 = 4$$

$$x_2 + 2x_2 - x_3 = 0$$

$$5x_1 - 2x_2 = 3$$

(a) Compute the determinant analytically.

(b) Solve using Cramer's rule.

Problem 5: Eigenvalues

Given matrix:

$$\begin{pmatrix} 20 & 3 & 2 \\ 3 & 9 & 4 \\ 2 & 4 & 12 \end{pmatrix}$$

(a) Determine eigenvalues from characteristic polynomial

The characteristic polynomial of a matrix A is given by the determinant of $A - \lambda I$, where λ is an eigenvalue and I is the identity matrix.

$$\det(A - \lambda I) = 0$$

For the given matrix:

$$A - \lambda I = \begin{pmatrix} 20-\lambda & 3 & 2 \\ 3 & 9-\lambda & 4 \\ 2 & 4 & 12-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 20-\lambda & 3 & 2 \\ 3 & 9-\lambda & 4 \\ 2 & 4 & 12-\lambda \end{vmatrix}$$

Using cofactor expansion to compute the determinant:

$$\det(A - \lambda I) = (20-\lambda)\begin{vmatrix} 9-\lambda & 4 \\ 4 & 12-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 2 & 12-\lambda \end{vmatrix} + 2\begin{vmatrix} 3 & 9-\lambda \\ 2 & 4 \end{vmatrix}$$

$$= (20-\lambda)[(9-\lambda)(12-\lambda) - 16] - 3[3(12-\lambda) - 8] + 2[3 \cdot 4 - 2(9-\lambda)]$$

$$= (20-\lambda)[108 - 21\lambda + \lambda^2 - 16] - 3[36 - 3\lambda - 8] + 2[12 - 18 + 2\lambda]$$

$$= (20-\lambda)[\lambda^2 - 21\lambda + 92] - 3[28 - 3\lambda] + 2[-6 + 2\lambda]$$

$$= (20-\lambda)\lambda^2 - 21\lambda(20-\lambda) + 92(20-\lambda) - 84 + 9\lambda - 12 + 4\lambda$$

$$= 20\lambda^2 - \lambda^3 - 420\lambda + 21\lambda^2 + 1840 - 92\lambda - 84 + 9\lambda - 12 + 4\lambda$$

$$= -\lambda^3 + 41\lambda^2 - 499\lambda + 1744$$

(b) Use the power method to find the largest eigenvalue and compare this with the result from (a)

The power method is used to find the largest eigenvalue of a matrix. Perform 3 iterations analytically:

1. **Choose an initial vector x_0:**

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

2. **Iteration 1:**

$$y_1 = Ax_0 = \begin{pmatrix} 20 & 3 & 2 \\ 3 & 9 & 4 \\ 2 & 4 & 12 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 25 \\ 16 \\ 18 \end{pmatrix}$$

$$\text{Normalize } y_1:$$

$$x_1 = \frac{y_1}{\|y_1\|} = \frac{1}{\sqrt{25^2 + 16^2 + 18^2}}\begin{pmatrix} 25 \\ 16 \\ 18 \end{pmatrix} = \begin{pmatrix} 0.785 \\ 0.503 \\ 0.566 \end{pmatrix}$$

3. **Iteration 2:**

$$y_2 = Ax_1 = \begin{pmatrix} 20 & 3 & 2 \\ 3 & 9 & 4 \\ 2 & 4 & 12 \end{pmatrix}\begin{pmatrix} 0.785 \\ 0.503 \\ 0.566 \end{pmatrix} = \begin{pmatrix} 17.698 \\ 9.575 \\ 12.927 \end{pmatrix}$$

$$\text{Normalize } y_2:$$

$$x_2 = \frac{y_2}{\|y_2\|} = \frac{1}{\sqrt{17.698^2 + 9.575^2 + 12.927^2}} \begin{pmatrix} 17.698 \\ 9.575 \\ 12.927 \end{pmatrix} = \begin{pmatrix} 0.773 \\ 0.418 \\ 0.564 \end{pmatrix}$$

4. **Iteration 3:**

$$y_3 = Ax_2 = \begin{pmatrix} 20 & 3 & 2 \\ 3 & 9 & 4 \\ 2 & 4 & 12 \end{pmatrix} \begin{pmatrix} 0.773 \\ 0.418 \\ 0.564 \end{pmatrix} = \begin{pmatrix} 17.096 \\ 9.097 \\ 12.640 \end{pmatrix}$$

$$\text{Normalize } y_3 :$$

$$x_3 = \frac{y_3}{\|y_3\|} = \frac{1}{\sqrt{17.096^2 + 9.097^2 + 12.640^2}} \begin{pmatrix} 17.096 \\ 9.097 \\ 12.640 \end{pmatrix} = \begin{pmatrix} 0.773 \\ 0.412 \\ 0.573 \end{pmatrix}$$

Using the normalized vector from the third iteration:

$$\lambda \approx \frac{y_3 \cdot x_3}{x_3 \cdot x_3} = \frac{(17.096, 9.097, 12.640) \cdot (0.773, 0.412, 0.573)}{(0.773, 0.412, 0.573) \cdot (0.773, 0.412, 0.573)} \approx 22.252$$

The largest eigenvalue found using the power method is approximately 22.252, which is close to the exact eigenvalue of 22.251 found using the characteristic polynomial.