Spherical Kerr-Schild Coordinates

Yitian Chen (yc2377@cornell.edu)

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1 Introduction

This article contains all useful formulae to construct a spinning black-hole (BH) in spherical Kerr-Schild (SphKS) coordinates. Since They are closely related to Kerr-Schild (KS) coordinates, we follow the symbol convention of Kerr-Schild (KS) in the file *KerrSchildCoords.tex* in the same folder. We first describe a rotating BH with spin along z axis in SphKS, and later generalize the spin to any direction. As KS coordinates use symbol $\{t, x, y, z\}$, we denote SphKS coordinates as $\{t, \bar{x}, \bar{y}, \bar{z}\}$.

2 Spin in the z direction

2.1 Transformation

In KS, the Boyer-Lindquist radius r at a point satisfies an equation of spheroid

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1, \tag{1}$$

or equivalently,

$$r^2 = \frac{1}{2}(x^2 + y^2 + z^2 - a^2) + \left(\frac{1}{4}(x^2 + y^2 + z^2 - a^2)^2 + a^2 z^2\right)^{1/2}. \tag{2}$$

$\vec{\bar{x}} = \bar{x}^i = \bar{x}_i = (\bar{x}, \bar{y}, \bar{z})$ is related to $\vec{x} = x^i = x_i = (x, y, z)$ by

$$\rho^2 \equiv r^2 + a^2, \tag{3}$$

$$\left(\frac{\bar{x}}{r}, \frac{\bar{y}}{r}, \frac{\bar{z}}{r}\right) \equiv \left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{r}\right), \tag{4}$$

or more compactly,

$$x^i = P^i{}_j \bar{x}^j, \tag{5}$$

$$\bar{x}^i = Q^i{}_j x^j, \tag{6}$$

$$P^i{}_j \equiv \text{Diagonal}\left(\frac{\rho}{r}, \frac{\rho}{r}, 1\right), \tag{7}$$

$$Q^i{}_j \equiv (P^{-1})^i{}_j = \text{Diagonal}\left(\frac{r}{\rho}, \frac{r}{\rho}, 1\right). \tag{8}$$

Thus, r satisfies the equation of sphere in SphKS

$$r^2 = \vec{\bar{x}} \cdot \vec{\bar{x}} = \bar{x}^2 + \bar{y}^2 + \bar{z}^2. \tag{9}$$

In other words, Euclidean radius coincides with Boyer-Lindquist radius, in SphKS. The Jacobian $T^i{}_j$ is

$$dx^i = T^i{}_j d\bar{x}^j, \tag{10}$$

$$F^i{}_k \equiv -\frac{a^2}{\rho r^3} \cdot \text{Diagonal}(1, 1, 0), \tag{11}$$

$$T^i{}_j = P^i{}_j + F^i{}_k \bar{x}^k \bar{x}_j. \tag{12}$$

Its inverse $S^i{}_j = (T^{-1})^i{}_j$ is

$$
\begin{aligned}
(G_1)^i{}_m &\equiv -\frac{r^2}{\rho} F^i{}_m = \frac{a^2}{\rho^2 r} \cdot \text{Diagonal}(1,1,0), && (13)\\
s &\equiv r^2 + \frac{a^2 z^2}{r^2}, && (14)\\
(G_2)^n{}_j &\equiv \frac{\rho^2}{sr} Q^n{}_j = \frac{\rho}{s} \cdot \text{Diagonal}\left(1,1,\frac{\rho}{r}\right),, && (15)\\
S^i{}_j &= Q^i{}_j + (G_1)^i{}_m \bar{x}^m \bar{x}_n (G_2)^n{}_j. && (16)
\end{aligned}
$$

2.2 Metric

The spatial metric is

$$
\begin{aligned}
g_{ij} &= \bar{\eta}_{ij} + 2H\bar{l}_i\bar{l}_j, && (17)\\
\bar{\eta}_{ij} &= \eta_{mn} T^m{}_i T^n{}_j, && (18)\\
H &= \frac{r^3}{r^4 + a^2 z^2} = \frac{r}{s}, && (19)\\
l_i &= l^i = \frac{r\vec{x} - \vec{a}\times\vec{x} + \frac{(\vec{a}\cdot\vec{x})\vec{a}}{r}}{\rho^2}, && (20)\\
\bar{l}_i &= T^m{}_i l_m, && (21)\\
\bar{l}^i &= S^i{}_m l^m, && (22)
\end{aligned}
$$

and the spacetime metric is

$$
\begin{aligned}
\psi_{\mu\nu} &= \bar{\eta}_{\mu\nu} + 2H\bar{l}_\mu\bar{l}_\nu, && (23)\\
\bar{l}_\mu &= (1, \bar{l}_i), && (24)\\
\bar{l}^\mu &= (-1, \bar{l}^i), && (25)\\
\bar{\eta}_{\mu\nu} &= (-1)\otimes\bar{\eta}_{ij}. && (26)
\end{aligned}
$$

Lapse and shift are

$$
\begin{aligned}
\beta^i &= \frac{2H\bar{l}^i}{1+2H} = 2H\alpha^2\bar{l}^i, && (27)\\
\beta_i &= 2H\bar{l}_i, && (28)\\
\alpha &= (1+2H)^{-1/2}. && (29)
\end{aligned}
$$

2.3 Derivatives

In the following, symbol ∂_i always refers to the derivative relative to $\bar{x}^i$. Derivatives of the Jacobian and its inverse are

$$
\begin{aligned}
D^i{}_j &\equiv \frac{a^2}{\rho^3 r}\cdot \mathrm{Diagonal}(1,1,0), &(30)\\
C^i{}_m &\equiv D^i{}_m - 3F^i{}_m \;=\; \frac{a^2}{\rho r}\left(\frac{1}{\rho^2}+\frac{3}{r^2}\right)\cdot \mathrm{Diagonal}(1,1,0), &(31)\\
\partial_k T^i{}_j &= F^i{}_j\bar{x}_k + F^i{}_k\bar{x}_j + F^i{}_m\bar{x}^m\delta_{jk} + C^i{}_m\frac{\bar{x}_k\bar{x}^m\bar{x}_j}{r^2}, &(32)\\
(E_1)^i{}_m &\equiv -\frac{a^2}{\rho^2}\left(\frac{1}{r^2}+\frac{2}{\rho^2}\right)\cdot \mathrm{Diagonal}(1,1,0), &(33)\\
(E_2)^n{}_j &\equiv \left[-\frac{a^2}{\rho^2 r}-\frac{2}{s}\left(r-\frac{a^2\bar{z}^2}{r^3}\right)\right]\cdot (G_2)^n{}_j + \frac{1}{s}P^n{}_j, &(34)\\
\partial_k S^i{}_j &= D^i{}_j\bar{x}_k + (G_1)^i{}_k\bar{x}_n(G_2)^n{}_j + (G_1)^i{}_m\bar{x}^m(G_2)_{kj}\\
&\quad +(E_1)^i{}_m\frac{\bar{x}_k\bar{x}^m\bar{x}_n}{r}(G_2)^n{}_j + (G_1)^i{}_m\frac{\bar{x}_k\bar{x}^m\bar{x}_n}{r}(E_2)^n{}_j - (G_1)^i{}_m\bar{x}^m\bar{x}_n(G_2)^n{}_j\frac{2a^2\bar{z}}{sr^2}\delta_{k\bar{z}}. &(35)
\end{aligned}
$$

where $(G_2)_{kj} \equiv (G_2)^k{}_j$ and $\delta_{k\bar{z}}$ is 1 if $k=\bar{z}$ but 0 otherwise. Other important derivatives are

$$
\begin{aligned}
\frac{\partial r}{\partial x^i} &= \frac{r^2 x_i + (\vec{a}\cdot\vec{x})a_i}{rs}, &(36)\\
\partial_i H &= HT^m{}_i\left[\frac{3}{r}\frac{\partial r}{\partial x^m} - \frac{4r^3\dfrac{\partial r}{\partial x^m}+2(\vec{a}\cdot\vec{x})a_m}{r^4+(\vec{a}\cdot\vec{x})^2}\right], &(37)\\
\partial_j\bar{l}_i &= T^k_i T^m{}_j\frac{1}{\rho^2}\left[\left(x_k - 2rl_k - \frac{(\vec{a}\cdot\vec{x})a_k}{r^2}\right)\frac{\partial r}{\partial x^m} + r\delta_{km} + \frac{a_k a_m}{r} - \epsilon^{kmn}a_n\right] + l_k\partial_j T^k_i, &(38)\\
\partial_k g_{ij} &= 2\bar{l}_i\bar{l}_j\partial_k H + 4H\bar{l}_{(i}\partial_k\bar{l}_{j)} + T^m{}_j\partial_k T^m{}_i + T^m{}_i\partial_k T^m{}_j, &(39)\\
\partial_k\alpha &= -(1+2H)^{-3/2}\partial_k H \;=\; -\alpha^3\partial_k H, &(40)\\
\partial_k\beta^i &= 2\alpha^2\left[\bar{l}^i\partial_k H + H(S^i{}_jS^m{}_j\partial_k\bar{l}_m + S^i{}_j\bar{l}_m\partial_k S^m{}_j + S^m{}_j\bar{l}_m\partial_k S^i{}_j)\right] - 4H\bar{l}^i\alpha^4\partial_k H, &(41)
\end{aligned}
$$

where ϵ^{kmn} is the antisymetric symbol and $\epsilon^{xyz}=+1$.

3 Spin in an arbitrary direction

3.1 Transformation

Now, we generalize the spin to arbitrary direction. The spin vector is $\vec{a}$ and the unit vector along its direction is $\hat{a}$ (meaningful only if spin is nonzero). In KS, r satisfies

$$
\begin{aligned}
r^2 &= \frac{1}{2}(\vec{x}\cdot\vec{x}-a^2) + \left(\frac{1}{4}(\vec{x}\cdot\vec{x}-a^2)^2 + (\vec{a}\cdot\vec{x})^2\right)^{1/2}, &(42)\\
\rho^2 &\equiv r^2 + a^2, &(43)
\end{aligned}
$$

and we define $\vec{\bar{x}} = \bar{x}^i = \bar{x}_i = (\bar{x},\bar{y},\bar{z})$ in terms of $\vec{x} = x^i = x_i = (x,y,z)$ as

$$
\begin{aligned}
Q^i{}_j &\equiv \frac{r}{\rho}\delta^i{}_j + \frac{1}{(\rho+r)\rho}a^i a_j, &(44)\\
P^i{}_j &\equiv \frac{\rho}{r}\delta^i{}_j - \frac{1}{(\rho+r)r}a^i a_j, &(45)\\
\vec{\bar{x}} &\equiv Q^i{}_j x^j \;=\; Q\vec{x}, &(46)\\
\vec{x} &= P^i{}_j\bar{x}^j \;=\; P\vec{\bar{x}}. &(47)
\end{aligned}
$$

Note that

- as $a \to 0$, both P and Q tend to the identity. If a is nonzero, P and Q can be written in projection matrices:

$$\begin{aligned} Q^i{}_j &= \frac{r}{\rho}(P_\perp)^i{}_j + (P_{//})^i{}_j, & (48)\\ P^i{}_j &= \frac{\rho}{r}(P_\perp)^i{}_j + (P_{//})^i{}_j, & (49)\\ (P_\perp)^i{}_j &\equiv \delta^i{}_j - \hat{a}^i\hat{a}_j, & (50)\\ (P_{//})^i{}_j &\equiv \hat{a}^i\hat{a}_j. & (51) \end{aligned}$$

  One should be careful about the difference between $\hat{a}^i$ and a^i in the above formulae.

- r still satisfies

$$r^2 = \vec{\bar{x}} \cdot \vec{\bar{x}} = \bar{x}^2 + \bar{y}^2 + \bar{z}^2. \qquad (52)$$

- $\vec{a} \cdot \vec{x}$ and $\vec{a} \cdot \vec{\bar{x}}$ give the same result, i.e.

$$\vec{a} \cdot \vec{x} = \vec{a} \cdot \vec{\bar{x}}. \qquad (53)$$

The Jacobian and its inverse are

$$\begin{aligned} dx^i &= T^i{}_j d\bar{x}^j, & (54)\\ F^i{}_k &\equiv -\frac{1}{\rho r^3}(a^2\delta^i{}_j - a^i a_j), & (55)\\ T^i{}_j &= P^i{}_j + F^i{}_k \bar{x}^k \bar{x}_j, & (56)\\ (G_1)^i{}_m &\equiv \frac{1}{\rho^2 r}(a^2\delta^i{}_m - a^i a_m), & (57)\\ s &\equiv r^2 + \frac{(\vec{a}\cdot\vec{x})^2}{r^2}, & (58)\\ (G_2)^n{}_j &\equiv \frac{\rho^2}{sr} Q^n{}_j, & (59)\\ S^i{}_j &= (T^{-1})^i{}_j = Q^i{}_j + (G_1)^i{}_m \bar{x}^m \bar{x}_n (G_2)^n{}_j. & (60) \end{aligned}$$

3.2 Metric

Metrics, lapse and shift formulae are unchanged, but we copy them here for convenience.

$$g_{ij} = \bar{\eta}_{ij} + 2H\bar{l}_i\bar{l}_j, \tag{61}$$

$$\bar{\eta}_{ij} = \eta_{mn}T^m{}_iT^n{}_j, \tag{62}$$

$$H = \frac{r^3}{r^4 + (\vec{a}\cdot\vec{x})^2}, \tag{63}$$

$$l_i = l^i = \frac{r\vec{x} - \vec{a}\times\vec{x} + \dfrac{(\vec{a}\cdot\vec{x})\vec{a}}{r}}{\rho^2}, \tag{64}$$

$$\bar{l}_i = T^m{}_i l_m, \tag{65}$$

$$\bar{l}^i = S^i{}_m l^m, \tag{66}$$

$$\psi_{\mu\nu} = \bar{\eta}_{\mu\nu} + 2H\bar{l}_\mu\bar{l}_\nu, \tag{67}$$

$$\bar{l}_\mu = (1, \bar{l}_i), \tag{68}$$

$$\bar{l}^\mu = (-1, \bar{l}^i), \tag{69}$$

$$\bar{\eta}_{\mu\nu} = (-1)\otimes\bar{\eta}_{ij}, \tag{70}$$

$$\beta^i = \frac{2H\bar{l}^i}{1+2H} = 2H\alpha^2\bar{l}^i, \tag{71}$$

$$\beta_i = 2H\bar{l}_i, \tag{72}$$

$$\alpha = (1+2H)^{-1/2}. \tag{73}$$

3.3 Derivatives

Important derivatives are

$$D^i{}_j \equiv \frac{1}{\rho^3 r}(a^2\delta^i{}_m - a^i a_m), \tag{74}$$

$$C^i{}_m \equiv D^i{}_m - 3F^i{}_m = \frac{1}{\rho r}\left(\frac{1}{\rho^2} + \frac{3}{r^2}\right)(a^2\delta^i{}_m - a^i a_m), \tag{75}$$

$$\partial_k T^i{}_j = F^i{}_j\bar{x}_k + F^i{}_k\bar{x}_j + F^i{}_m\bar{x}^m\delta_{jk} + C^i{}_m\frac{\bar{x}_k\bar{x}^m\bar{x}_j}{r^2}, \tag{76}$$

$$(E_1)^i{}_m \equiv -\frac{1}{\rho^2}\left(\frac{1}{r^2} + \frac{2}{\rho^2}\right)(a^2\delta^i{}_m - a^i a_m), \tag{77}$$

$$(E_2)^n{}_j \equiv \left[-\frac{a^2}{\rho^2 r} - \frac{2}{s}\left(r - \frac{(\vec{a}\cdot\vec{x})^2}{r^3}\right)\right]\cdot(G_2)^n{}_j + \frac{1}{s}P^n{}_j, \tag{78}$$

$$\begin{aligned}\partial_k S^i{}_j = {} & D^i{}_j\bar{x}_k + (G_1)^i{}_k\bar{x}_n(G_2)^n{}_j + (G_1)^i{}_m\bar{x}^m(G_2)_{kj} \\ & + (E_1)^i{}_m\frac{\bar{x}_k\bar{x}^m\bar{x}_n}{r}(G_2)^n{}_j + (G_1)^i{}_m\frac{\bar{x}_k\bar{x}^m\bar{x}_n}{r}(E_2)^n{}_j - (G_1)^i{}_m\bar{x}^m\bar{x}_n(G_2)^n{}_j\frac{2\vec{a}\cdot\vec{x}}{sr^2}a_k,\end{aligned} \tag{79}$$

$$\frac{\partial r}{\partial x^i} = \frac{r^2 x_i + (\vec{a}\cdot\vec{x})a_i}{rs}, \tag{80}$$

$$\partial_i H = HT^m{}_i\left[\frac{3}{r}\frac{\partial r}{\partial x^m} - \frac{4r^3\dfrac{\partial r}{\partial x^m} + 2(\vec{a}\cdot\vec{x})a_m}{r^4 + (\vec{a}\cdot\vec{x})^2}\right], \tag{81}$$

$$\partial_j\bar{l}_i = T^k_iT^m{}_j\frac{1}{\rho^2}\left[\left(x_k - 2rl_k - \frac{(\vec{a}\cdot\vec{x})a_k}{r^2}\right)\frac{\partial r}{\partial x^m} + r\delta_{km} + \frac{a_k a_m}{r} - \epsilon^{kmn}a_n\right] + l_k\partial_j T^k_i, \tag{82}$$

$$\partial_k g_{ij} = 2\bar{l}_i\bar{l}_j\partial_k H + 4H\bar{l}_{(i}\partial_k\bar{l}_{j)} + T^m{}_j\partial_k T^m{}_i + T^m{}_i\partial_k T^m{}_j, \tag{83}$$

$$\partial_k\alpha = -(1+2H)^{-3/2}\partial_k H = -\alpha^3\partial_k H, \tag{84}$$

$$\partial_k\beta^i = 2\alpha^2\left[\bar{l}^i\partial_k H + H(S^i{}_jS^m{}_j\partial_k\bar{l}_m + S^i{}_j\bar{l}_m\partial_k S^m{}_j + S^m{}_j\bar{l}_m\partial_k S^i{}_j)\right] - 4H\bar{l}^i\alpha^4\partial_k H. \tag{85}$$

$$\text{Jacobian Grid Point 1: } \begin{pmatrix} \frac{\rho^2 r^2 - 4a^2}{\rho r^3} & 0 & 0 \\ 0 & \frac{\rho}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Jacobian Grid Point 2: } \begin{pmatrix} \frac{\rho}{r} & 0 & 0 \\ 0 & \frac{\rho^2 r^2 - 4a^2}{\rho r^3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Jacobian Grid Point 3: } \begin{pmatrix} \frac{\rho}{r} & 0 & 0 \\ 0 & \frac{\rho}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$