Evolving the FOSH-SW

system

Goal:

Simulate two pulses approaching & passing through each other.

The scalar wave equation:

$$\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial x^2} = 0$$

We convert this into a first-order system of equations by introducing the auxiliary fields Π and Φ.

$$\left.\begin{aligned}
\frac{\partial \psi}{\partial t} &= -\Pi \\
\frac{\partial \Pi}{\partial t} &= -\frac{\partial \Phi}{\partial x} \\
\frac{\partial \Phi}{\partial t} &= -\frac{\partial \Pi}{\partial x}
\end{aligned}\right\}$$

Equations written in conservation form:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = S$$

$$\mathbf{u} = \begin{pmatrix} \psi \\ \Pi \\ \Phi \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 \\ \Phi \\ \Pi \end{pmatrix}, \quad S = \begin{pmatrix} -\Pi \\ 0 \\ 0 \end{pmatrix}$$

We must specify three initial profiles:

$$\psi_0, \ \Pi_0, \ \Phi_0.$$

For the first-order advection equation, we only needed to specify the initial profile. (One degree of freedom.)

Does the second-order scalar wave eqn really have three degrees of freedom?

No, One eqn. is a constraint.

ψ is set freely.

$\pi = -\partial_t \psi$ is also set freely

$\Phi = \partial_x \psi$ is a constraint eqn.

Φ is determined by ψ and cannot be set freely.

$$\left.\begin{aligned} \frac{\partial \Pi}{\partial t} &= -\frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial t} &= -\frac{\partial \Pi}{\partial x} \end{aligned}\right\}$$ are not advection equations, they cannot be evolved with an upwind scheme.

Now we will obtain a higher-order method. We start with FTCS:

forward-time-centered-space.

Our first attempt at amending the upwind scheme, since now wave fronts can move forwards <u>and</u> backwards, is to average the upwind and downwind derivatives.

upwind (for $v > 0$)

$$\left.\frac{\partial \psi}{\partial x}\right|_x^{up} = \frac{\psi(x) - \psi(x - \Delta x)}{\Delta x} + \mathcal{O}(\Delta x)$$

downwind (for $v < 0$)

$$\left.\frac{\partial \psi}{\partial x}\right|_x^{dwn} = \frac{\psi(x + \Delta x) - \psi(x)}{\Delta x} + \mathcal{O}(\Delta x)$$

$$\left.\frac{\partial \psi}{\partial x}\right|_x^{\text{centered}} = \frac{1}{2}\left(\left.\frac{\partial \psi}{\partial x}\right|_x^{\text{up}} + \left.\frac{\partial \psi}{\partial x}\right|_x^{\text{dwn}} \right)$$

$$= \frac{\psi(x+\Delta x) - \psi(x-\Delta x)}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

Note, the $\mathcal{O}(\Delta x)$ terms cancel out, leaving $\mathcal{O}(\Delta x^2)$ as the error (see book)

Psuedocode for $\partial\psi/\partial x$:

```
def centered_finite_difference(ψ):
    dψ_dx = [0]
    for i in range(1, len(ψ)-1):
        dψ_dx.append((ψ[i+1]-ψ[i-1]) / 2Δx)
    dψ_dx.append(0)
    return dψ_dx
```

Recall our update equations:

$$\frac{\partial \psi}{\partial t} = -\Pi$$

$$\frac{\partial \Pi}{\partial t} = -\frac{\partial \Phi}{\partial x}$$

$$\frac{\partial \Phi}{\partial t} = -\frac{\partial \Pi}{\partial x}$$

(at a point x)

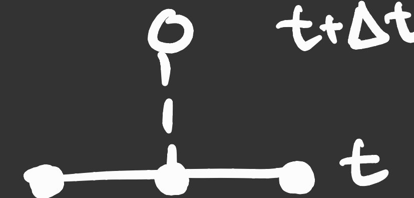
$$\psi(t+\Delta t) \approx \psi(t) - \Pi \Delta t$$

$$\Pi(t+\Delta t) \approx \Pi(t) - \partial_x \Phi \Delta t$$

$$\Phi(t+\Delta t) \approx \Phi(t) - \partial_x \Pi \Delta t$$

these can be computed using a centered finite difference.

$$\Psi(x,t+\Delta t) \approx \Psi(x,t) - \Pi\,\Delta t$$



$$\Pi(x,t+\Delta t) \approx \Pi(x,t) - \frac{\Phi(x+\Delta x,t) - \Phi(x-\Delta x,t)}{2\Delta x}\,\Delta t$$

$$\Phi(x,t+\Delta t) \approx \Phi(x,t) - \frac{\Pi(x+\Delta x,t) - \Pi(x-\Delta x,t)}{2\Delta x}\,\Delta t$$

This is the FTCS scheme for solving the scalar wave system.

It is not stable.

How can we anticipate this instability? von Neumann stability analysis.

First, the adv. eqn. w/ the upwind scheme.

Suppose the field u has linear evolution equations (the principle of superposition applies)

Then,

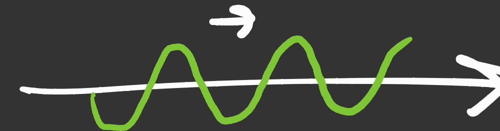
$$u(x,t) \approx \sum_{i=0}^{N} \xi_i(t)\, e^{ik_i x}$$

The profile u can be represented as a sum of modes evolving independently, superposed.

We can analyze one mode at a time.

$$u_k(x,t) = \xi_k(t)\, e^{ikx}$$

adv. eqn.



u_k

at the next time step,

$$u_k(x,t+\Delta t) = u_k(x,t) - v\frac{\Delta t}{\Delta x}\left(u_k(x,t) - u_k(x-\Delta x,t)\right)$$

$$\xi_k(t+\Delta t)\, e^{ikx} = \xi(t)\, e^{ikx} - v\frac{\Delta t}{\Delta x}\left(\xi_k(t)\, e^{ikx} - \xi_k(t)\, e^{ik(x-\Delta x)}\right)$$

$$\xi_k(t+\Delta t) = \xi_k(t) - v\frac{\Delta t}{\Delta x}\left(\xi_k(t) - \xi_k(t)\, e^{-ik\Delta x}\right)$$

$$\xi_k(t+\Delta t) = \xi_k(t) - v\frac{\Delta t}{\Delta x}\left(\xi_k(t) - \xi_k(t)e^{-ik\Delta x}\right)$$

$$\frac{\xi_k(t+\Delta t)}{\xi_k(t)} = 1 - v\frac{\Delta t}{\Delta x}\left(1 - e^{-ik\Delta x}\right)$$

$$\frac{\xi_k(t+\Delta t)}{\xi_k(t)} = 1 - \alpha + \alpha e^{-ik\Delta x}$$

The amplification factor is:

$$\left|1 - \alpha + \alpha e^{-ik\Delta x}\right| \leq 1$$ required for stability

in the worst case, $e^{-ik\Delta x} = -1$,

$$|1 - \alpha - \alpha| \leq 1 \Rightarrow |1 - 2\alpha| \leq 1$$

$$\Rightarrow -1 \leq 1 - 2\alpha \leq 1 \Rightarrow -2 \leq -2\alpha \leq 0$$

$$\Rightarrow -1 \leq -\alpha \leq 0$$

$$\Rightarrow \boxed{0 \leq \alpha \leq 1}$$ CFL condition

If the CFL condition is met, the upwind scheme is stable for all modes k!

We can do VNSA for the adv. eqn. with the FTCS scheme.

We find $$\frac{\xi(t+\Delta t)}{\xi(t)} = 1 - i\frac{v\Delta t}{\Delta x}\sin(k\Delta x)$$

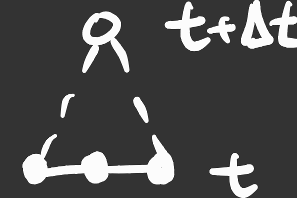
which has modulus > 1 for <u>all</u> modes k.

We then proceed to amend the FTCS scheme with a replacement:

$$\psi(x,t) \longrightarrow \psi(x,t) \quad \text{(why?)}$$

$$\Pi(x,t) \longrightarrow \frac{1}{2}\left(\Pi(x+\Delta x) + \Pi(x-\Delta x)\right)$$

$$\Phi(x,t) \longrightarrow \frac{1}{2}\left(\Phi(x+\Delta x) + \Phi(x-\Delta x)\right)$$



With this replacement, the FTCS scheme becomes the Lax-Friedrichs scheme.

VNSA returns

$$\left|\frac{\xi(t+\Delta t)}{\xi(t)}\right|^2 = 1 - \sin^2(k\Delta x)\left(1 - \frac{v^2\Delta t^2}{\Delta x^2}\right)$$

$\Rightarrow$ stable as long as $(1-\alpha^2) \geq 0$

$\Rightarrow$ $-1 \leq \alpha \leq 1$ (CFL condition)

Why is Lax-Friedrichs stable over FTCS?

lets look at the first update eqn.

$$\Pi(x, t+\Delta t) \approx \Pi(x,t) - \frac{\Phi(x+\Delta x, t) - \Phi(x-\Delta x, t)}{2\Delta x}\Delta t$$

← FTCS

This update equation becomes...

Lax-Friedrichs

$$\Pi(x, t+\Delta t) \approx \frac{1}{2}\left(\Pi(x+\Delta x) + \Pi(x-\Delta x)\right) - \frac{\Phi(x+\Delta x, t) - \Phi(x-\Delta x, t)}{2\Delta x}\Delta t$$

$$\frac{\Pi(x, t+\Delta t)}{\Delta t} \approx \frac{\left(\Pi(x+\Delta x) + \Pi(x-\Delta x)\right)}{2\Delta t} - \frac{\Phi(x+\Delta x, t) - \Phi(x-\Delta x, t)}{2\Delta x}$$

$$\frac{\Pi(x,t+\Delta t)}{\Delta t} \approx \frac{\Pi(x+\Delta x)+\Pi(x-\Delta x)+2\Pi(x,t)-2\Pi(x,t)}{2\Delta t} - \frac{\Phi(x+\Delta x,t)-\Phi(x-\Delta x,t)}{2\Delta x}$$

$$\frac{\Pi(x,t+\Delta t)-\Pi(x,t)}{\Delta t} \approx \frac{\Pi(x+\Delta x)-2\Pi(x,t)+\Pi(x-\Delta x)}{2\Delta t} - \frac{\Phi(x+\Delta x,t)-\Phi(x-\Delta x,t)}{2\Delta x}$$

$$\frac{\Pi(x,t+\Delta t)-\Pi(x,t)}{\Delta t} \approx \frac{\Delta x^2\left(\Pi(x+\Delta x)-2\Pi(x,t)+\Pi(x-\Delta x)\right)}{2\Delta t\,\Delta x^2} - \frac{\Phi(x+\Delta x,t)-\Phi(x-\Delta x,t)}{2\Delta x}$$

finite difference for 2nd derivative!

$$\rightarrow \frac{\partial \Pi}{\partial t} = \underbrace{\frac{\Delta x^2}{2\Delta t}\frac{\partial^2 \Pi}{\partial x^2}}_{\text{numerical diffusion}} - \frac{\partial \Phi}{\partial x}$$

is what we are evolving — a diffusion eqn!

$\Rightarrow$ The Lax-Friedrichs eqns evolve:

$$\frac{\partial \psi}{\partial t} = -\Pi$$

$$\left.\begin{aligned}
\frac{\partial \Pi}{\partial t} &= -\frac{\partial \Phi}{\partial x} + \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 \Pi}{\partial x^2} \\
\frac{\partial \Phi}{\partial t} &= -\frac{\partial \Pi}{\partial x} + \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 \Phi}{\partial x^2}
\end{aligned}\right\}$$

The numerical dissipation goes to zero faster than truncation error $\rightarrow$ solution converges.

Again, the update equations are:

$$\psi(x, t+\Delta t) \approx \psi(x,t) - \Pi(x,t)\,\Delta t$$

$$\Pi(x,t+\Delta t) \approx \frac{1}{2}\left(\Pi(x+\Delta x) + \Pi(x-\Delta x)\right) - \frac{\Phi(x+\Delta x,t) - \Phi(x-\Delta x,t)}{\Delta x}\,\Delta t$$

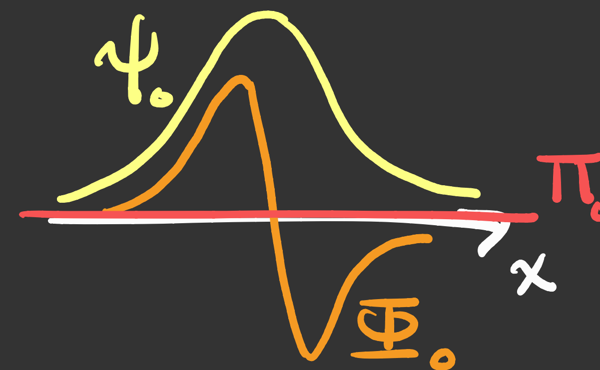
$$\Phi(x,t+\Delta t) \approx \frac{1}{2}\left(\Phi(x+\Delta x) + \Phi(x-\Delta x)\right) - \frac{\Pi(x+\Delta x,t) - \Pi(x-\Delta x,t)}{\Delta x}\,\Delta t$$

Let's code these up!

Initial profiles, ex:

$$\psi_0 = \frac{A}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$\pi_0 = 0$$



$$\Phi_0 = \partial_x \psi_0 = \frac{-A(x-\mu)}{\sigma^3\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} = \frac{-(x-\mu)}{\sigma^2}\psi_0$$

Test setting π_0 to different values.
Set Φ_0 such that the constraint is violated.
What happens?

A moving Gaussian has profile:

$$\psi(x,t) = \frac{A}{\sigma\sqrt{2\pi}} e^{\frac{-(x-v t-\mu)^2}{2\sigma^2}} \qquad v = \pm 1$$

To initialize a moving Gaussian, Π_0 must be specified correctly!

$$\Pi_0 = -\partial_t \psi = \frac{v(x-vt-\mu)}{\sigma^2}\psi = v\partial_x\psi$$

$$\Phi_0 = \partial_x \psi = \frac{(x-vt-\mu)}{\sigma^2}\psi$$

We've now evolved $\partial_\mu \partial^\mu \psi = 0$

$$\partial_t^2 \psi - \partial_x^2 \psi = 0$$

Homework!

evolve $$\partial_t^2 \psi - \partial_x^2 \psi = \rho$$

with $\rho(x,t)$ a sinusoidally moving Gaussian

Hint: first find first-order formulation, then use previous slide for hint on moving Gaussian.