The FOSH Scalar Wave

System:

Constraints
Characteristics
Boundary Conditions

We now have our scalar wave system: $\partial_t^2 \psi - \partial_x^2 \psi = 0$

$$\Rightarrow \begin{cases} \partial_t \psi = -\Pi \\ \partial_t \Pi = -\partial_x \Phi \\ \partial_t \Phi = -\partial_x \Pi \end{cases}$$

The scalar wave eqn. has two degrees of freedom, but three fields. What gives?

We also have our constraint eqn:

$$\partial_x \psi = \Phi . \quad (\text{c.f. } \partial_t \psi = -\pi)$$

Since ψ and Φ are numerically evolved as different fields, a disagreement can grow b/w $\partial_x \psi$ and Φ. This is a <u>constraint violation</u>. $\partial_x \psi - \Phi = C$.

We have the equations for how the fields evolve. Can we anticipate how the constraints evolve?

What is $\partial_t C = ?$

$$
\begin{aligned}
\partial_t C &= \partial_t \left(\partial_x \psi - \Phi \right) \\
&= \partial_x \partial_t \psi - \partial_t \Phi \\
&= -\partial_x \Pi + \partial_x \Pi \\
&= 0
\end{aligned}
$$

Now, this equation holds true analytically, but not numerically. Numerically, we know that constraints become non-zero and grow.

If we have non-zero constraints, $\partial_t C = 0$ tells us the best we can do is to keep these errors constant — they won't go away!

If we have non-zero constraints, we want the fields (ψ, π, Φ) to evolve such that the C's go down.

Let's aim for the C's to decay exponentially.

Then, $C(t) = C_0 e^{-\gamma t}$.

What differential equation describes this behavior?

$$C(t) = C_0 e^{-\gamma t}$$

$$\Rightarrow \partial_t C = -\gamma C$$

Comparing to our original equation,

$$\begin{aligned}
\partial_t C &= \partial_t(\partial_x \psi - \Phi) \\
&= \partial_x \partial_t \psi - \partial_t \Phi \\
&= -\partial_x \pi - \partial_t \Phi = -\gamma C
\end{aligned}$$

$$\Rightarrow \partial_t \Phi = -\partial_x \pi + \gamma(\partial_x \psi - \Phi)$$

We now have our new evolution eqns, including constraint damping!

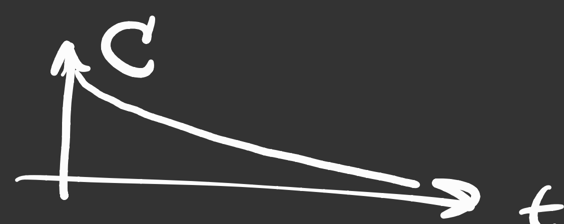
$$\partial_t \psi = -\pi$$

$$\partial_t \pi = -\partial_x \Phi$$

$$\partial_t \Phi = -\partial_x \pi + \gamma \underbrace{(\partial_x \psi - \Phi)}_{C}$$

The constraints now influence the evolution of the fields.

Note that we have modeled the evolution of our constraints analytically. This will still not necessarily be what happens numerically! But now we can see for when the evolution is approximate to analytical, the constraint violations are damped exponentially.



Characteristic fields.

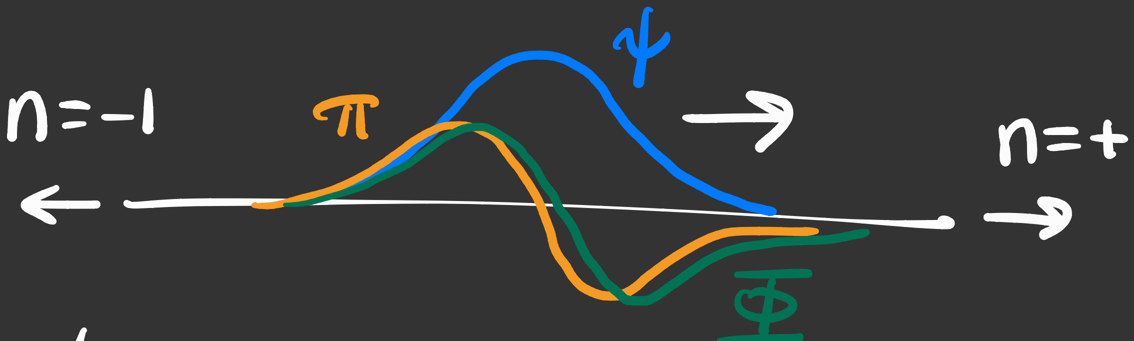
We see the wave profiles moving across the domain. How do we determine the direction of motion of a "wave" in a crowd?

$$\omega^{\psi} = \psi$$

$$\omega^{+} = \pi + n\Phi$$

$$\omega^{-} = \pi - n\Phi$$

$$n = \begin{cases} +1 & \text{right boundary} \\ -1 & \text{left boundary} \end{cases}$$



$\rightarrow$: $\omega^{+}_{\rightarrow} = \pi + \Phi$ $\qquad$ $\omega^{-}_{\rightarrow} = \pi - \Phi$

$\leftarrow$: $\omega^{-}_{\leftarrow} = \pi + \Phi$ $\qquad$ $\omega^{+}_{\leftarrow} = \pi - \Phi$

Where do ω^+, ω^- come from?

Recall:

$$\partial_t \Pi = -\partial_x \Phi \qquad \partial_t \Phi = -\partial_x \Pi \quad \Rightarrow \partial_t \begin{pmatrix} \Pi \\ \Phi \end{pmatrix} = -\partial_x \begin{pmatrix} \Phi \\ \Pi \end{pmatrix}$$

$$\Rightarrow \partial_t \begin{pmatrix} \Pi \\ \Phi \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \Pi \\ \Phi \end{pmatrix} = 0$$

$$\Rightarrow \partial_t u + A \partial_x u = 0$$

$$\partial_t \mathcal{U} + \mathcal{A}\,\partial_x \mathcal{U} = 0$$

$$\Rightarrow \partial_t \mathcal{U} + \partial_x F = 0$$

$$\Rightarrow \partial_t \mathcal{U} + \vec{\nabla} \cdot \vec{F} = 0$$

This is the continuity equation for multiple species! In one basis, the two speceies are π and Φ, with characteristic matrix $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

This matrix is not diagonal!

The fields with which the characteristic matrix is diagonal are the characteristic fields.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix}$$

$$\partial_t u + A R R^{-1} \partial_x u = 0$$

$$\partial_t \underbrace{R^{-1} u}_{u'} + \underbrace{R^{-1} A R}_{A'} \partial_x \underbrace{R^{-1} u}_{u'} = 0$$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has eigenvectors:

$$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 \text{, has roots } \lambda^{\pm} = \pm 1$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda^{+}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow a = b$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda^{-}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ a \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow a = -b$$

eigenvectors: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = R^{-1} A R$$

$$= \tfrac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \tfrac{1}{2} \begin{pmatrix} 1+1 & -1+1 \\ 1-1 & -1-1 \end{pmatrix}$$

$$= \tfrac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} = A'$$

$$u' = R^{-1} u = \tfrac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pi \\ \Phi \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} \pi + \Phi \\ \pi - \Phi \end{pmatrix}$$

$$\Rightarrow \partial_t \tfrac{1}{2} \begin{pmatrix} \pi + \Phi \\ \pi - \Phi \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \tfrac{1}{2} \begin{pmatrix} \pi + \Phi \\ \pi - \Phi \end{pmatrix} = 0$$

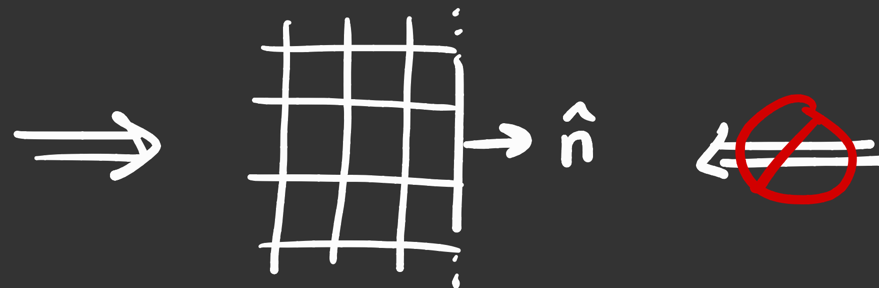
$$\Rightarrow \partial_t \begin{pmatrix} \omega^+ \\ \omega^- \end{pmatrix} + \partial_x \begin{pmatrix} \lambda^+ \omega^+ \\ \lambda^- \omega^- \end{pmatrix} = 0 \quad \checkmark$$

$\Rightarrow$ The system $\begin{cases} \partial_t \Pi = -\partial_x \Phi \\ \partial_t \Phi = -\partial_x \Pi \end{cases}$

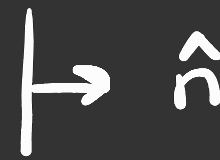
is equivalent to: $\begin{cases} \partial_t \omega^+ = \lambda^+ \partial_x \omega^+ \\ \partial_t \omega^- = \lambda^- \partial_x \omega^- \end{cases}$

After a change of basis, the leftward and rightward moving waves are revealed!

Boundary conditions.



In this basis, we can specify what values of ω^+ and ω^- we want on the boundary. If $\hat{n}$ is $\rightarrow$, and we want no incoming characteristics, what should ω^+ and ω^- be?



no incoming characteristics $\Rightarrow$ $\omega^-_{\rightarrow} = 0$

$$\Rightarrow \quad \omega^-_{\rightarrow} = \pi - \Phi = 0$$

$\omega^+_{\rightarrow} = \pi + \Phi$ is left alone.

at the other end,

$$\omega^-_{\leftarrow} = \pi + \Phi = 0$$

$\omega^+_{\leftarrow} = \pi - \Phi$ is left alone.

Reflecting boundary conditions

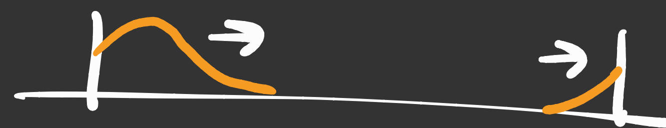
For reflecting boundary conditions at a $\hat{n} = \rightarrow$ boundary, the would-be outgoing characteristic is set to the ingoing one.

$$w^+_{\rightarrow} = w^-_{\rightarrow}$$

$$\Rightarrow \Pi + \Phi = \Pi - \Phi$$

$$\Rightarrow \Phi = 0$$

Periodic boundary conditions



set the incoming char. on the left to be the outgoing char. on the right.

$$w^-_{\leftarrow} = w^+_{\rightarrow}$$

also set

$$w^-_{\rightarrow} = w^+_{\leftarrow}.$$

In terms of the char fields, we have:

$$\psi = \omega^{\psi}$$

$$\pi = \tfrac{1}{2}(\omega^{+} + \omega^{-}) + \gamma\,\omega^{\psi}$$

$$\Phi = \tfrac{1}{2}(\omega^{+} - \omega^{-})\,n$$

$$\omega^{\psi} = \psi$$

$$\omega^{+} = \pi + n\,\underline{\Phi} - \gamma\psi$$

$$\omega^{-} = \pi - n\,\underline{\Phi} - \gamma\psi$$

The scalar wave equation with sources:

$$\partial_t^2 \psi - \partial_x^2 \psi = \rho$$

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if $\Phi = \partial_x \psi$,

$\Pi = -\partial_t \psi$

then $-\partial_t \Pi - \partial_x \Phi = \rho$

$$\Rightarrow \partial_t \pi = -\partial_x \Phi - \rho$$

$$\Rightarrow$$

$$\begin{cases} \partial_t \psi = -\pi \\ \partial_t \pi = -\partial_x \Phi - \rho \\ \partial_t \Phi = -\partial_x \pi + \gamma(\underbrace{\partial_x \psi - \Phi}_{C}) \end{cases}$$

Suppose $\partial_t^2 \psi = 0 \Rightarrow \partial_x^2 \psi = -\rho$

if $\rho = q\,\delta(x - x_0)$

then $\displaystyle\int_{x_0-\epsilon}^{x_0+\epsilon} dx\, \partial_x^2 \psi = \int_{x_0-\epsilon}^{x_0+\epsilon} dx\, -\rho$

$$\Rightarrow \quad \partial_x \psi \Big|_{x_0-\epsilon}^{x_0+\epsilon} = -q$$

Since $\partial_x^2 \psi = 0$ away from source,

$\psi_- = Ax + B$ $\qquad$ $\psi_+ = Cx + D$

$\psi_-' = A$ $\qquad$ $\psi_+' = C$

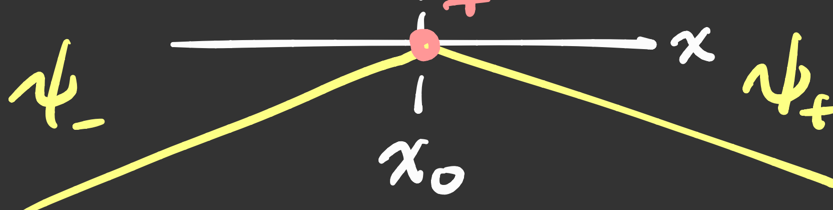
Continuity $\Rightarrow$ $Ax_0 + B = Cx_0 + D$

Derivative condition $\Rightarrow$ $C - A = -g$

$$Ax_0 + B = (A - q)x_0 + D$$

$$B = -qx_0 + D$$

$$\Rightarrow \quad \begin{aligned} \psi_- &= Ax + B & \quad x < x_0 \\ \psi_+ &= (A - q)x + B + qx_0 & \quad x \geq x_0 \end{aligned}$$



Recap:

- Constraint Damping
- Characteristic Speeds
- Boundary Conditions
- Sources

Next time: The vector potential returns!