AMS-511 Foundations of Quantitative Finance

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Question 1.

The hyperbolic absolute risk aversion (HARA) class of utility functions has the form

$$U[w] = \frac{1-\gamma}{\gamma} \left(\frac{a\,w}{1-\gamma} + b \right)^{\gamma}, \quad b > 0$$

Show that the parameters of the HARA utility function can be chosen to obtain the following special cases:

- Risk neutral: $U[w] = w$
- Exponential: $U[w] = -e^{-a\,w}$ [*Hint*: Consider the limit $\gamma \to -\infty$]
- Power: $U[w] = a\,w^{\gamma}$
- Log: $U[w] = \ln[w]$ [*Hint*: Consider $U[w] = (1-\gamma)^{1-\gamma}\,((x^{\gamma}-1)/\gamma)$ and the limit $\gamma \to 0$]

Solution

In[93]:= `hara =` $\frac{1-\gamma}{\gamma}\left(\frac{a\,w}{1-\gamma} + b\right)^{\gamma}$ `;`

Linear

In[94]:= `Limit[hara /. a → 1, γ → 1]`

Out[94]= `w`

Exponential

In[95]:= `Limit[hara /. b → 1, γ → -∞]`

Out[95]= $-e^{-a\,w}$

Power

In[96]:= `Limit[hara /. a → 1, b → 0]`

Out[96]= $\dfrac{w\left(\frac{w}{1-\gamma}\right)^{-1+\gamma}}{\gamma}$

… which can be simplified to

$$\frac{(1-\gamma)^{1-\gamma}}{\gamma}\, w^{\gamma} = c\, w^{\gamma}$$

Log

In[97]:= $\mathtt{Limit}\Big[\dfrac{(1-\gamma)^{1-\gamma}}{\gamma}\,(w^{\gamma}-1),\ \gamma \to 0\Big]$

Out[97]= `Log[w]`

Question 2.

You are given a random variate X which has a Student t distribution with PDF with $0 < v$ degrees of freedom. Let $\mathbb{E}[X^n]$ represent the nth moment of X. Show mathematically which moments of X exist and which are indeterminate.

Solution

The PDF of a Student t distributed random variable X with v degrees of freedom is

$$f[x \mid v] = \left(\frac{v}{x^2 + v}\right)^{\frac{1+v}{2}} \Big/ \sqrt{v}\ B\Big[\frac{v}{2}, \frac{1}{2}\Big]$$

where $B[a,\ b]$ is the Euler beta function.

For $|x|$ sufficiently large and an appropriately chosen constant c

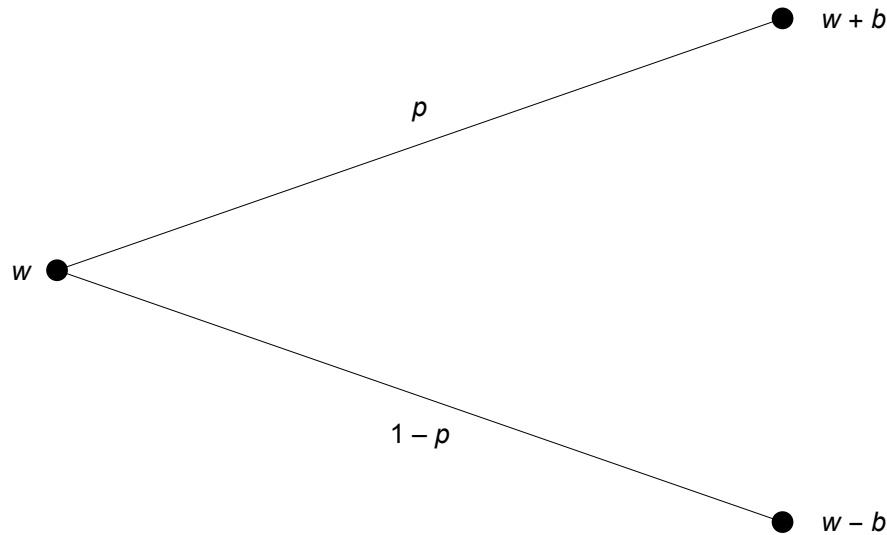$$f[x \mid v] = c\left(\frac{1}{x^2 + v}\right)^{\frac{1+v}{2}} \tilde{\propto}\ x^{-(1+v)}$$

The nth moment is

$$\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n\, f[x \mid v]\, dx \ \tilde{\propto}\ \int_{-\infty}^{\infty} x^{n-(1+v)}\, dx$$

For even moments, this converges for if $n < v$. For odd moments, by symmetry, are zero if $v > 1$, but are indeterminate for $0 < v \le 1$.

Question 3.

Consider the betting game below that we used to illustrate Kelly betting where we sought to maximize the growth rate.

This led to solving for the bet b^*_{Kelly} which maximized the expected log wealth; *i.e.*,

$$b^*_{\text{Kelly}} = \underset{b}{\text{argmax}}[p \log[w + b] + (1 + p) \log[w - b]]$$

Now consider this game in the more general setting of utility theory where our goal will be to maximize the expected utility $\mathbb{E}[U[X]]$; *i.e.*,

$$b^* = \underset{b}{\text{argmax}}[p \, U[w + b] + (1 + p) \, U[w - b]]$$

Let $U[x] = \sqrt{x}$.

- Solve for the optimal bet for a player with square root utility for the game above.

- Assume that the probability of winning $p = 0.6$. Compare the Kelly bet with the square root utility bet.

- For this game is the Kelly bet more aggressive, less aggressive, or sometimes more and sometimes less aggressive than the square root bet? Show how you came up with your answer.

Solution

The solution for b^*_{sqrt} can be calculated by

In[98]:= **Solve$\left[\partial_b \left(\text{p } \sqrt{\text{w + b}} + (\text{1 - p}) \, \sqrt{\text{w - b}}\right) == 0, \text{b}\right]$**

Out[98]= $\left\{\left\{\text{b} \rightarrow \dfrac{(-1 + 2 \, \text{p}) \, \text{w}}{1 - 2 \, \text{p} + 2 \, \text{p}^2}\right\}\right\}$

Thus,

$$b^*_{\text{sqrt}} = \frac{2p - 1}{2p^2 - 2p + 1}$$

The optimal bet for $p = 0.6$ is to bet 38.5% of wealth at each round. The equivalent Kelly bet is 20%.

In[99]:= $\dfrac{(-1 + 2 \, \text{p}) \, \text{w}}{1 - 2 \, \text{p} + 2 \, \text{p}^2}$ **/. p → 0.6**

Out[99]= $0.384615 \, \text{w}$

Note that

$$\frac{b^*_{sqrt}}{b^*_{Kelly}} = \frac{1}{2\,p^2 - 2\,p + 1}$$

If we examine this ratio for $0.5 < p < 1$ we see that the square root utility is more aggressive.

In[100]:= `Plot[(2 p^2 - 2 p + 1)^-1, {p, 0.5, 1}, AxesOrigin -> {0.5, 0}]`

Out[100]=