AMS-511 Foundations of Quantitative Finance

Fall 2020 — Assignment 08

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Question 1.

A *lookback* option is a path dependent option whose value is based on some function of the underlying's price history over the life of the option/ rather than the price at expiry. Read the Wikipedia article https://en.wikipedia.org/wiki/Lookback_option which describes these options in detail.

There are many possible forms; however, the particular case for this assignment is a lookback put with:

$$P(T) = \max\left[\max_t[S(t)] - S(T),\, 0\right] = S_{\max} - S(T)$$

Assume a stock with price $S(0) = \$96.25$ and volatility $\sigma = 28\%$, and a risk free rate $r_f = 0.5\%$. Use a Monte Carlo simulation to price the lookback option above for an expiry of $T = 0.75$. Compare your solution to that returned by FinancialDerivative[ ].

Note: The corresponding *Mathematica* solution is identified as a {"LookbackFloating", "European", "Put"}. Use the Mathematica documentation to determine the required parameters for FinancialDerivative[ ].

Solution

As with the case of the Asian option we need to generate sample paths that we can use to apply an appropriate function to. Let $\boldsymbol{\delta}_\Delta$ denote the random vector below:

$$\delta_\Delta = \left(r - \frac{\sigma^2}{2}\right)\Delta + \sigma\,\text{RandomVariate}\left[\text{NormalDistribution}\left[0.,\, \sqrt{\Delta}\,\right],\, \frac{T}{\Delta}\right]$$

Let $\mathbf{S}(0, T \mid \Delta)$ denote a random price path from $t = 0$ to T in increments of Δ. Then

$$\mathbf{S}(0, T \mid \Delta) = S(0)\,\text{Exp}[\text{FoldList}[\text{Plus}, 0, \boldsymbol{\delta}_\Delta]]$$

We repeat this experiment M times with the j^{th} estimate being

$$P^{(j)}(T) = \max[\max[\mathbf{S}(0, T \mid \Delta)] - S(T), 0]$$

and the risk neutral estimate of the option price being

$$\hat{P}[0] = \frac{\text{Exp}[-r\,T]}{J}\sum_{j=1}^{M} P_{(j)}(T)$$

Code

We only need a single line of code to compute the Monte Carlo estimate, including the simulation's standard error. Whether you express your function as a single line or break it up into multiple lines depends upon both clarity and efficiency. For an experienced programmer figuring out the logic of the code below would not be difficult, but there is no problem breaking it into separate steps if you find that clearer or more maintainable. Here's a rather coarse path using $\Delta = 0.1$ as an illustration. Note that The number of steps is $T/\Delta = 1/0.1$, and we would want to insure that this is an integer.

```
In[69]:= 95. Exp[FoldList[Plus, 0., (0.02 - 0.10^2 / 2.) 0.1 +
             0.10 RandomVariate[NormalDistribution[0., √0.1], 1. / 0.1]]]

Out[69]= {95., 96.4224, 95.3743, 96.1576, 97.8991,
          94.2734, 96.8544, 99.4851, 95.2379, 101.687, 95.8486}
```

In terms of the variables described above we define a function

$$\text{xAsianLookbackPutMonteCarlo}[S(0), \sigma, r, T, \Delta, M]$$

```
In[70]:= xLookbackFloatingPutMonteCarlo[nInitialPrice_,
           nVolatility_, nRiskFree_, nExpiry_, nDeltaT_, iNSamples_] :=
          {Mean[#], StandardDeviation[#] / √iNSamples} &[
           Exp[-nRiskFree nExpiry] ×
            Table[
             (Max[#] - Last[#] &)[nInitialPrice
               Exp[FoldList[Plus, 0., (nRiskFree - nVolatility^2 / 2.) nDeltaT + nVolatility
                   RandomVariate[NormalDistribution[0., √nDeltaT], nExpiry / nDeltaT]]]],
             {iNSamples}
            ]
          ]

In[71]:= xLookbackFloatingPutMonteCarlo[96.25, 0.28, 0.005, 0.75, 0.75 / 100, 1000]

Out[71]= {18.0805, 0.397604}
```

The simulation study below returns a vector of points representing the underlying price at various levels vs. the lower 95% confidence limit, mean, and upper 95% confidence limit for the option price. The vector of such results is then transposed to yield three lines representing the lower confidence level, mean, and upper confidence level.

```
In[72]:= mnSim = Transpose@Block[
            {sol},
            Table[
             sol = xAsianArithPutMonteCarlo[
               price, 0.1, 100., 1., 0.0005, 0.02, 1. / 0.0005, 1000.];
             {{price, sol⟦1⟧ - 1.96 sol⟦2⟧}, {price, sol⟦1⟧}, {price, sol⟦1⟧ + 1.96 sol⟦2⟧}},
             {price, 80., 120., 5.}
            ]
           ]

Out[72]= {{{80., 79.804}, {85., 84.804}, {90., 89.804}, {95., 94.804}, {100., 99.804},
           {105., 104.804}, {110., 109.804}, {115., 114.804}, {120., 119.804}},
          {{80., 80.}, {85., 85.}, {90., 90.}, {95., 95.}, {100., 100.},
           {105., 105.}, {110., 110.}, {115., 115.}, {120., 120.}},
          {{80., 80.196}, {85., 85.196}, {90., 90.196}, {95., 95.196}, {100., 100.196},
           {105., 105.196}, {110., 110.196}, {115., 115.196}, {120., 120.196}}}
```

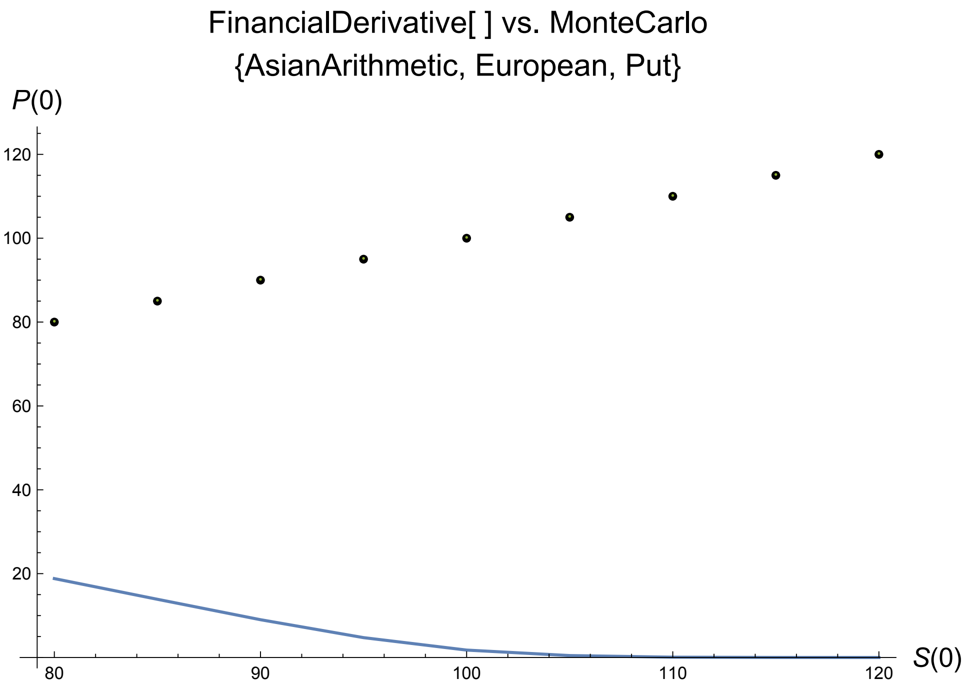
A comparison of our solutions and those of *Mathematica*'s FinancialDerivative[ ] follows.

```
In[73]:= mnSim[[2]]

Out[73]= {{80., 80.}, {85., 85.}, {90., 90.}, {95., 95.},
          {100., 100.}, {105., 105.}, {110., 110.}, {115., 115.}, {120., 120.}}
```

```
In[74]:= Show[
          ListLinePlot[
           Table[{price, FinancialDerivative[{"AsianArithmetic", "European", "Put"},
               {"StrikePrice" → 100., "Expiration" → 1.,
                "Inception" → 0., "AverageSoFar" → 95.},
               {"CurrentPrice" → price, "Volatility" → 0.10, "InterestRate" → 0.02}]},
             {price, 80, 120, 5}], PlotLabel → Style[
              "FinancialDerivative[ ] vs. MonteCarlo\n{AsianArithmetic, European, Put}",
              FontSize → 16], AxesLabel → (Style[#, FontSize → 14] & /@ {"S(0)", "P(0)"})],
          ListPlot[mnSim, PlotStyle → {{PointSize[0]}, {Black, PointSize[0.01]},
              {PointSize[0]}}, Filling → {1 → {3}},
            FillingStyle → Red], PlotRange → All, ImageSize → 500
         ]
```

Out[74]=



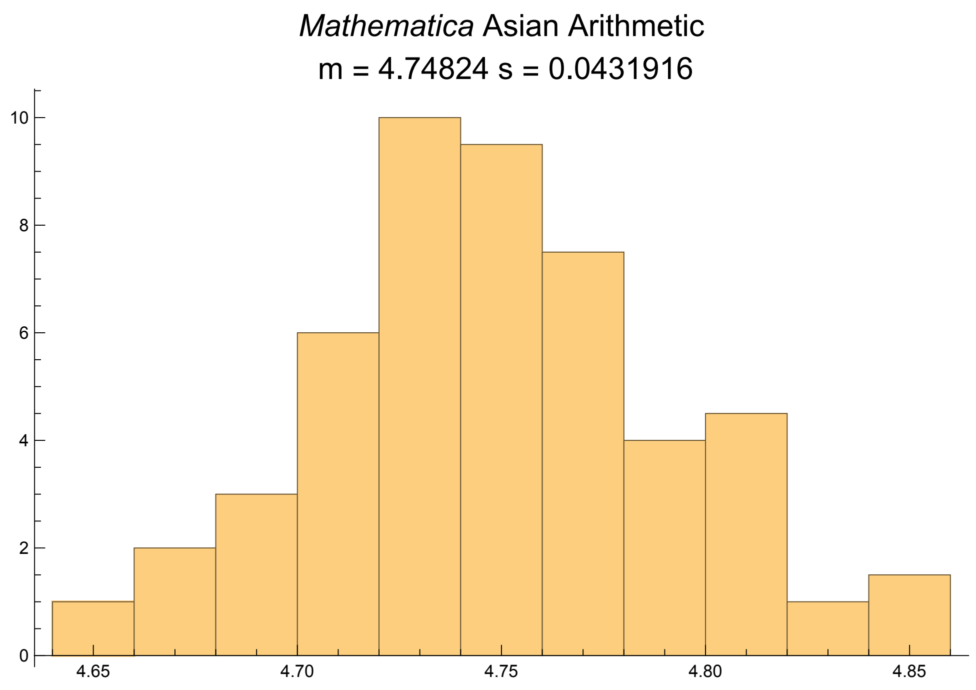
Again, note that the *Mathematica* function itself uses Monte Carlo. It returns a slightly different result each time it is run. We'll sample it to see how much it varies from call to call.

```
In[75]:= vnMmaSim = Table[FinancialDerivative[
             {"AsianArithmetic", "European", "Put"}, {"StrikePrice" → 100.,
              "Expiration" → 1., "Inception" → 0., "AverageSoFar" → 95.},
             {"CurrentPrice" → 95., "Volatility" → 0.10, "InterestRate" → 0.02}], {100}];
```

Note in the histogram below we have plotted the standard deviation of the simulated results, not the standard error of the mean.

```
In[•]:= Histogram[vnMmaSim, Automatic, "PDF", PlotLabel → Style[
          "Mathematica Asian Arithmetic\n m = " <> ToString[Mean[vnMmaSim]] <> " s = " <>
           ToString[StandardDeviation[vnMmaSim]], FontSize → 16], ImageSize → 500]
```

Out[•]=


Mathematica Asian Arithmetic
m = 4.74824 s = 0.0431916