AMS-511 Foundations of Quantitative Finance

Fall 2020 — Solutions 06 — Addendum to Question 3

Robert J. Frey, Research Professor
Stony Brook University, Applied Mathematics and Statistics

Robert.Frey@StonyBrook.edu
http://www.ams.sunysb.edu/~frey

Question 3.

Consider a European *capped-call option* with strike price K, cap C, and expiration date T. With $S(t)$ the price of the underlying and $F(t)$ the price of the option, the capped-call gives the holder the right to exercise the option at expiry with value of a put with strike K whose value, however, is capped at a maximum pay-out C where $K < C$. Thus, the value at expiry can be expressed by

$$F(T) = \min[\,\max[\,S(T) - K,\ 0\,],\ C\,]$$

Express the value of the option at time $t < T$ in terms of vanilla European options with appropriately chosen parameters and, if necessary, any cash position.

Solution

Converting the form above into one consisting of a portfolio of simple instruments is more like, for example, solving an integration problem than a derivative problem. You have a library of tricks, use some pattern recognition, and try different alternatives until you find a promising attack that leads to a solution.

The particular solution that worked was reached after having worked through a few dead ends. We start with

$$\min[\,\max[\,S(T) - K,\ 0\,],\ C\,] \quad \wedge \quad 0 < K \le C$$

We can express the above as a piecewise function

$$\min[\,\max[\,S(T) - K,\ 0\,],\ C\,] = \begin{cases} \max[\,S(T) - K,\ 0\,] - (S[T] - (K + C)) & \mid\ S[T] - K \ge C \\ \max[\,S(T) - K,\ 0\,] & \mid\ S[T] - K < C \end{cases}$$

We can now re-express the conditions on the right-side as follows

$$\min[\,\max[\,S(T) - K,\ 0\,],\ C\,] \;=\; \begin{cases} \max[\,S(T) - K,\ 0\,] - (S[T] - (K + C)) & \mid\ S[T] - (K + C) \ge 0 \\ \max[\,S(T) - K,\ 0\,] & \mid\ S[T] - (K + C) < 0 \end{cases}$$

Thus, when $S[T] - (K + C)$ is greater than zero we subtract it and when it is less than zero we do not (or to put it another way we subtract zero). Clearly, this is equivalent to

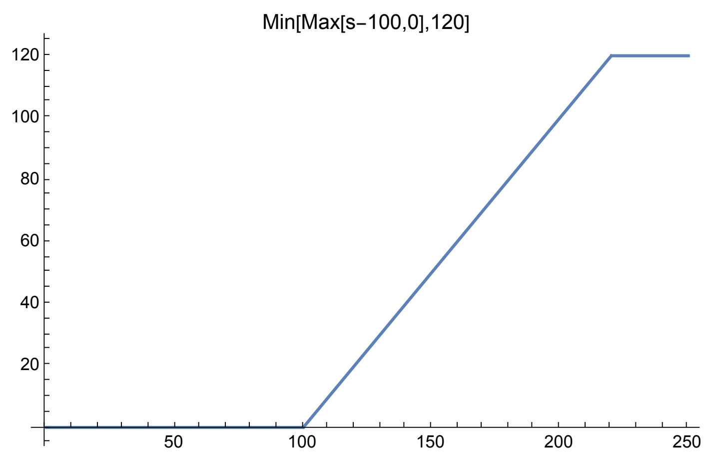
$$\min[\ \max[\ S(T) - K,\ 0\ ],\ C\ ] = \max[S(T) - K,\ 0] - \max[S(T) - (K + C), 0]$$

The solution is, therefore, a long capped call is equivalent to a long vanilla call with strike K and a short vanilla call with strike $K + C$.

$$F(t) = C(t \mid K, T) - C(t \mid K + C, T)$$

It's also a good idea to take a geometric approach and carefully plot the pay-off function. That often allows one to generate a candidate solution which can be checked against the original expression. Even if plotting doesn't immediately lead to a solution, it can often get you started on an algebraic approach. Of course, it's not a bad idea to use one method as a check against the other.

The solution is easily checked if the plot of $F(T)$ as a function of $S(T)$ is sketched. Consider an example with $K = 100$ and $C = 120$



Comparing the above to our solution