AMS-511 Foundations of Quantitative Finance

Fall 2020 — Solutions 10

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Question 1.

In the lecture notes we used the example of a firm with the characteristics:

$$\begin{aligned} \mu &= 0.07 \\ \sigma &= 0.20 \\ r &= 0.01 \\ K &= 7\,000\,000 \\ V &= 10\,000\,000 \end{aligned}$$

then used the following expression to estimate the default probability.

$$P[V(T) < K] = F_{\text{Normal}}\left[\frac{\log[K/V(0)] - \left(r - \sigma^2/2\right)T}{\sqrt{\sigma\,T}}\right]$$

The example assumed that a default would occur if the value of the firm fell below the face value of the debt at the expiry point; *i.e.*, if $V(T) < K$.

We noted that an alternative approach was to allow defaults to occur at any point $0 \le t \le T$ which introduces path dependence into the default process.

Assume now that a default is said to occur if the value of the firm falls below the face value of the debt at any time up to and including the expiry; *i.e.*; if $V(t) < K$ for $0 \le t \le T$. Construct a Monte Carlo simulation to estimate the risk neutral default probability under that condition.

Solution

The value of T wasn't given here but was in the Lecture Notes.

As with the case of the Asian and lookback options we need to generate sample paths that we can use to apply an appropriate function to. Let δ denote the random function below:

$$\delta = \left(r - \frac{\sigma^2}{2}\right)\Delta + \sigma\,\text{RandomVariate}\left[\text{NormalDistribution}\left[0.,\ \sqrt{\Delta}\right],\ \frac{T}{\Delta}\right]$$

Let $\mathbf{S}(0, T \mid \Delta)$ denote a random price path from $t = 0$ to T in increments of Δ. Then

$$\mathbf{S}(0, T \mid \Delta) = S(0)\,\text{Exp}[\text{FoldList}[\text{Plus}, 0, \delta]]$$

We repeat this experiment M times with the j^{th} estimate being

$$P^{(j)}(T) = \max[\max[\mathbf{S}(0, T \mid \Delta)] - S(T), 0]$$

and the risk neutral estimate of the option price being

$$\hat{P}[0] = \frac{\text{Exp}[-r\,T]}{J} \sum_{j=1}^{M} P_{(j)}(T)$$

In[101]:=
```
nMu = 0.07;
nSD = 0.2;
nRF = 0.01;
nK = 7 × 10^6;
nV = 10^7;
nT = 5.;
nΔ = 0.01;
```

Working from the inside out, we first generate a path, take its minimum, test to see if its < K. We can then use the Boole[ ] function to convert True (default) to 1 and False (no default) to 0. This is wrapped in a Table[ ] to be executed 100,000 times.

In[108]:=
```
mnSim = Table[
    Boole[
     nK > Min[nV Exp[
         FoldList[Plus, 0., (nRF - nSD^2 / 2.) nΔ +
           nSD RandomVariate[NormalDistribution[0., √nΔ], nT / nΔ]]]]
    ],
    {100 000}
   ];
```

The percentage of paths the result in a default is 44.9%.

In[109]:=
```
N[Total[vnSim] / Length[vnSim]]
```

Power: Infinite expression $\frac{1}{0}$ encountered.

Out[109]= ComplexInfinity

This is obviously a quick and dirty solution. If we wanted to compute, for example, the average time to default, then we would have to analyze each sample path in more detail. We could also improve efficiency by stopping further generation of a sample path once a default had occurred.

Question 2.

Consider a 10-year zero coupon bond with a face value $F = 100\,000$. The risk free rate is 0.01. The credit spread for the bond is 150 basis points. Compute the following at $t = 0$:

- The default probability of the bond.
- The value of the bond if there is no recovery on a default.
- The value of the bond if there is a recovery of 15%.
- The value of the bond if there were no possibility of default.

Solution

Based on the intensity model, the default probability is one minus the survival probability:

$$1 - q(T) = 1 - e^{-\lambda T}$$

In[110]:= `1 - e^(-0.015×10)`

Out[110]= `0.139292`

The risk neutral valuation is based on the risk free rate plus the spread.

$$B = e^{-(r+\lambda) T} F$$

In[111]:= `e^(-(0.01+0.015) 10) 100 000`

Out[111]= `77 880.1`

With partial recovery:

$$B = e^{-(r+\lambda) T} F + \left(e^{-r T} - e^{-(r+\lambda) T}\right) f$$

In[112]:= `(e^(-(0.01+0.015) 10) 100 000) + ((e^(-0.01×10) - e^(-(0.01+0.015) 10)) (0.15 × 100 000))`

Out[112]= `79 770.6`

With no credit risk:

$$Z = e^{-r T} F$$

In[113]:= `e^(-0.01×10) 100 000`

Out[113]= `90 483.7`