AMS-511 Foundations of Quantitative Finance

Fall 2020 — Lecture 04 — 2020-09-23 Wednesday

CAPM, Factor Models, Risk Measures

The Market Portfolio

The Market Portfolio, which here we mean of assets other than the risk free asset, is the proportional holdings of all such assets. That is, it is the capital-weighted sum of the proportions in all investors' portfolios.

It is essentially a theoretical construct. Assets are assumed to be liquid, available to every investor, and infinitely divisible. All means all and includes stocks, bonds, precious metals, real estate, and so forth (See http://en.wikipedia.org/wiki/Market_portfolio).

While the market portfolio theoretically includes everything, in practice surrogates consisting of portfolios of widely available and liquid assets are typically used. Examples of stock indices might be the

- S&P 500 Index (http://en.wikipedia.org/wiki/S%26 P_ 500) for the United States or
- MSCI World Index (http://en.wikipedia.org/wiki/MSCI_World) for a more global perspective.

The marker portfolio is in practice not known, but an appropriate surrogate is often used as representative of the market portfolio and serves as a benchmark against which investors can assess their performance.

Optimality of the Market Portfolio

We will not attempt a formal proof that the market portfolio is optimal. Instead we present a reasonable intuitive argument. However, that argument (as would a more formal proof) depends heavily on our assumptions:

- All investors are:
  - informed (knowing of all available investments along with their means and covariances),
  - rational (being risk adverse), and
  - mean-variance optimizers,
- The market portfolio consists of assets that are
  - tradable at no cost,
  - completely liquid, and
  - infinitely divisible.

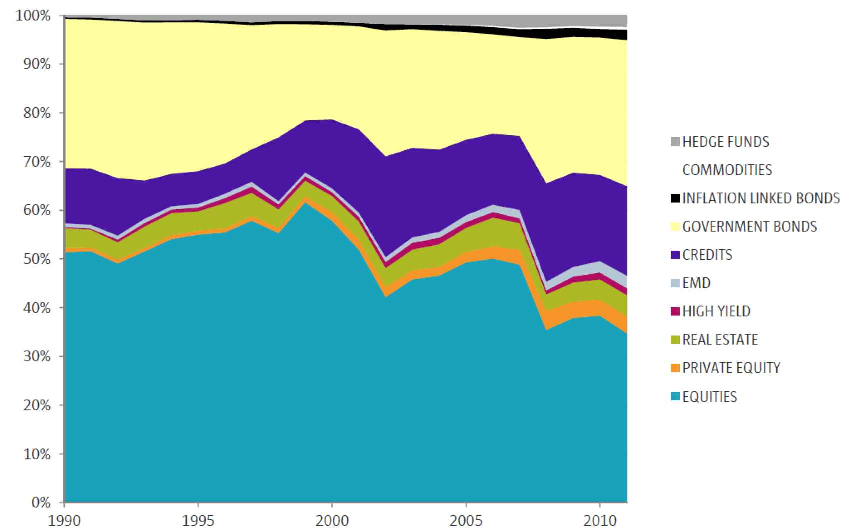
As covered earlier in class, investors, given a collection of risky assets need only consider a single allocation, the tangent portfolio. The risk-reward trade-offs appropriate for the investors are reflected by the adjusting their cash positions—deleveraging to decrease risk by holding cash and investing less in the tangent portfolio or leveraging to increase risk by borrowing cash and funding a larger holding in the tangent portfolio.

Why would investors only consider the full market portfolio and not a subset? Adjusting an investment cannot increase the Sharpe ratio of a tangent portfolio; that would contradict its optimality. There is likewise no benefit in excluding an asset from consideration.

All investors select, therefore, the identical tangent portfolio. That tangent portfolio is itself, therefore, the market portfolio. Thus, the market portfolio is not only efficient; it is the portfolio with the highest possible Sharpe ratio—it is *the* tangent portfolio.

Example - An Estimate of the Global Market Portfolio

Here is a graph which plots estimates of the global market portfolio of liquid investable assets. The authors' methodology is described in the paper cited below.



Strategic Asset Allocation : The Global Multi – Asset Portfolio 1959 – 2011,

by Ronald Q. Doeswijk, Trevin W. Lam, and Laurens A. P. Swinkels, 02 Nov 2012.

Retrieved 02 Mar 2014 from SSRN : http : // papers.ssrn.com / sol3 / papers.cfm ? abstract_id = 2 170 275.

Relevance of the Market Portfolio

Classroom discussion…

The Capital Asset Pricing Model

The Capital Asset Pricing Model (CAPM) is used to determine a theoretical rate of return for an asset based on that portion of its risk that cannot be diversified away (See http://en.wikipedia.org/wiki/Capital_asset_pricing _model).

The CAPM decomposes the risk of an asset into two components: systematic risk (which captures its covariance with the market portfolio) and idiosyncratic risk (which is unique to

the asset).

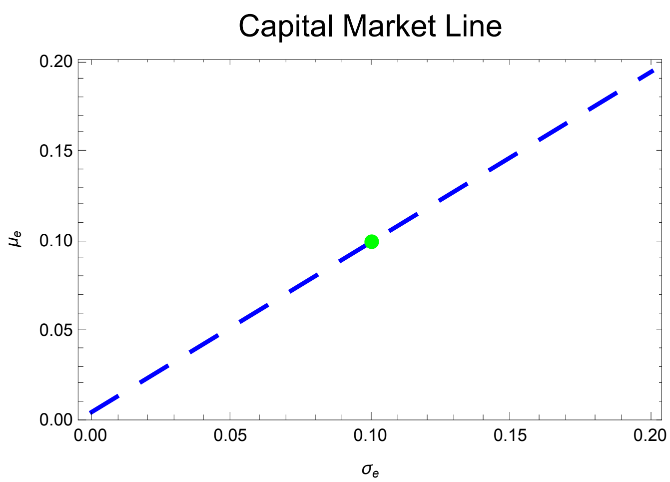
The Capital Market Line (Risk-Reward of Efficient Assets)

A consequence of the CAPM is that an efficient portfolio must lie on the (σ, μ)-line from the risk free rate through the market (tangent) portfolio. This is the *Capital Market Line*. For any efficient asset e, its expected return and risk are governed by the following relationship:

$$\mu_e = r_f + \frac{\mu_M - r_f}{\sigma_M} \sigma_e$$

$$\frac{\mu_e - r_f}{\sigma_e} = \frac{\mu_M - r_f}{\sigma_M}$$

In (σ, μ)-space, an example of the Capital Market Line is



The Security Market Line (Market Covariance and Expected Return)

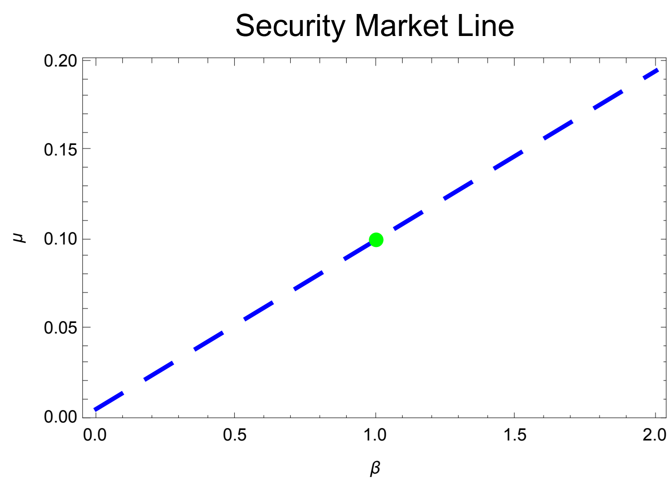
One implication of the CAPM and the security market line is that $\mu_i \propto \beta_i \propto \sigma_{i,M}$. Thus, in an efficient market only risk that is covariant with the market is compensated for in expected return.

$$r_i(t) - r_f = \beta_i \left(r_M(t) - r_f \right) + \epsilon_i(t)$$

$$\beta_i = \frac{\sigma_{i,M}}{\sigma_M{}^2}$$

The economic notion that is behind this is that idiosyncratic risks are uncorrelated with one another and can be largely diversified away. Systematic risks, however, cannot.

The line embodying this relationship is called the *Security Market Line*:



Also note that the value of $\beta_i = \sigma_{i,M} / \sigma_M{}^2$ is the solution to the linear regression problem with independent variable $\left(r_M(t) - r_f\right)$ and dependent variable $\left(r_i(t) - r_f\right)$.

CAPM Theorem

If the market portfolio M is efficient then for a given asset i its return satisfies

$$E\left[r_i(t) - r_f\right] = \beta_i\left(r_M(t) - r_f\right) + \epsilon_i(t)$$

$$\mu_i - r_f = \beta_i\left(\mu_M - r_f\right)$$

where

$$\beta_i = \frac{\sigma_{i,M}}{\sigma_M{}^2}$$

Proof : Consider a portfolio I composed of $0 \leq \alpha \leq 1$ invested in an asset i and (1-α) in the market portfolio; the mean and standard deviation of its return is

$$\mu_I = \alpha\,\mu_i + (1 - \alpha)\,\mu_M$$

$$\sigma_I = \sqrt{\alpha^2\,\sigma_i{}^2 + 2\,\alpha\,(1 - \alpha)\,\sigma_{i,M} + (1 - \alpha)^2\,\sigma_M{}^2}$$

Note that $\alpha = 0$ corresponds to the market portfolio itself. *Note that for any value of $\alpha \neq 0$, the mean and standard deviation of the portfolio I must be below the capital market line; otherwise, the efficiency of the market portfolio would be contradicted*. Consider the following derivatives at $\alpha = 0$.

$$\frac{d\mu_I}{d\alpha} = \mu_i - \mu_M$$

$$\frac{d\sigma_I}{d\alpha} = \frac{\alpha\,\sigma_i{}^2 + (1 - \alpha)\,\sigma_{i,M} + (\alpha - 1)\,\sigma_M{}^2}{\sigma_\alpha}\bigg|_{\alpha=0} = \frac{\sigma_{i,M} - \sigma_M{}^2}{\sigma_M}$$

$$\frac{d\mu_I}{d\sigma_I}\bigg|_{\alpha=0} = \frac{(\mu_i - \mu_M)\,\sigma_M}{\sigma_{i,M} - \sigma_M{}^2}$$

As noted, the $\{\sigma_I, \mu_I\}$ curve is tangent to the security market line at the point of the market portfolio, $\alpha = 0$. Thus,

$$\frac{(\mu_i - \mu_M)\,\sigma_M}{\sigma_{i,M} - \sigma_M{}^2} = \frac{\mu_M - r_f}{\sigma_M}$$

Solving for $\mu_i - r_f$

$$\mu_i - r_f = \frac{\sigma_{i,M}}{{\sigma_M}^2}\left(\mu_M - r_f\right) = \beta_i\left(\mu_M - r_f\right)$$

Applications of the CAPM

The CAPM has many important applications in financial analysis, particularly in pricing and performance analysis. Although the real-world market certainly does not meet all of its assumptions, it has proved to be a useful approximation to guide investors. As with any model, slavishly following its prescriptions can lead to disaster.

Pricing Forms

Consider pricing an asset at time t based on its price one period hence.

$$\frac{\mathrm{E}[P(t+1)] - P(t)}{P(t)} = \mu_i = r_f + \beta\left(\mu_M - r_f\right) \implies P(t) = \frac{\mathrm{E}[P(t+1)]}{1 + r_f + \beta\left(\mu_M - r_f\right)}$$

$$\mu_i = r_f + \beta\left(\mu_M - r_f\right) \implies \mathrm{E}[P(t+1)] = P(t)\left(1 + r_f + \beta\left(\mu_M - r_f\right)\right)$$

Thus, $r_f + \beta\left(\mu_M - r_f\right)$ represents a risk-adjusted interest rate that tells us how to price ($i.e.$, determine the present value of) the expected pay-off at $t+1$.

Jensen's Alpha

The basic form of the CAPM has no intercept term.

$$r_i(t) - r_f = \beta_i\left(r_M(t) - r_f\right) + \epsilon_i(t)$$

However, given actual data we can perform a linear regression an estimate the following regression line which includes an intercept term called Jensen's alpha.

$$r_i(t) - r_f = \alpha_i + \beta_i\left(r_M(t) - r_f\right) + \epsilon_i(t)$$

The value of α_i indicates how far above or below the security market line μ_i falls. Thus, it is a measure of whether the security i beats the market in risk-adjusted terms.

$$\mu_i - r_f = \alpha_i + \beta_i\left(\mu_M - r_f\right)$$

$$\text{BAD: intercept without risk adjustment} = \alpha_i + (1 - \beta_i)\, r_f$$

Sharpe Ratio

A positive α does not imply efficiency. A related measure is called the Sharpe ratio does measure relative efficiency by measuring the risk-to-reward relative to the capital market line.

$$\psi_i = \frac{\mu_i - r_f}{\sigma_i}$$

Mean-Variance Optimization under the CAPM

The CAPM provides us with a framework for better statistical estimation of the parameters required for mean-variance optimization. This not only leads to better estimates but

the structure of the CAPM can be exploited to produce more numerically stable and rapid portfolio optimizations. This is due to the fact the stability of both the likelihood function and the mean-variance quadratic program depend heavily on the structre of the covariance matrix.

Mathematical Development

$$r_i(t) - r_f = \beta_i \left(r_M(t) - r_f\right) + \epsilon_i(t), \quad \text{for } i = \{1, \ldots, n\}$$

The $\epsilon_i(t)$s are mean zero error terms

$$\mathrm{E}[\epsilon_i(t)] = 0, \quad \forall\, i$$

that are uncorrelated to the market and to each other.

$$\mathrm{Cov}[r_M(t), \epsilon_i(t)] = 0, \quad \forall\, i$$

$$\mathrm{Cov}\left[\epsilon_i(t), \epsilon_j(t)\right] = 0, \quad \forall\, i \neq j$$

Under the assumptions of the CAPM, the mean return of each asset is, therefore,

$$\mu_i - r_f = \beta_i \left(\mu_M - r_f\right)$$

and, noting how the cross-product terms are zero and drop out, the variance of each asset's return is

$$\sigma_i^{\,2} = (\sigma_M\, \beta_i)^2 + \sigma_{\epsilon_i}^{\,2}$$

The *systematic risk* is that portion that arises out of its covariance with the market, $(\sigma_M\, \beta_i)^2$, and the *idiosyncratic risk* is that portion that is unique to the asset, $\sigma_{\epsilon_i}^{\,2}$.

Their covariances are based solely on their mutual connection to the market:

$$\sigma_{i,j} = \sigma_M^{\,2}\, \beta_i\, \beta_j$$

Thus, a complete description of the market returns for the purposes of Markowitz mean-variance portfolio optimization involves the risk free rate r_f, the market expected return μ_M and standard deviation σ_M and the vector of betas $\boldsymbol{\beta}$ and a diagonal covariance matrix of error variances $\mathbf{D}$:

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_n \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} \sigma_{\epsilon_1}^{\,2} & & & & \\ & \ddots & & & \\ & & \sigma_{\epsilon_i}^{\,2} & & \\ & & & \ddots & \\ & & & & \sigma_{\epsilon_n}^{\,2} \end{pmatrix}$$

From these we can easily construct the mean vector and covariance matrix for portfolio optimization:

$$\boldsymbol{\mu} - r_f\, \mathbb{1} = \left(\mu_M - r_f\right) \boldsymbol{\beta}$$

$$\boldsymbol{\Sigma} = \sigma_M^{\,2}\, \boldsymbol{\beta}\boldsymbol{\beta}^T + \mathbf{D}$$

Example - Constructing a Covariance Matrix

Consider a simple three asset portfolio modeled under the CAPM. Their covariance matrix can be constructed as follows:

In[∘]:=
```
Print[MatrixForm[σM^2 KroneckerProduct[{β1, β2, β3}, {β1, β2, β3}]],
  " + ", MatrixForm[DiagonalMatrix[{σϵ1, σϵ2, σϵ3}^2]]]
```

$$\begin{pmatrix} \beta_1^2\,\sigma_M^2 & \beta_1\,\beta_2\,\sigma_M^2 & \beta_1\,\beta_3\,\sigma_M^2 \\ \beta_1\,\beta_2\,\sigma_M^2 & \beta_2^2\,\sigma_M^2 & \beta_2\,\beta_3\,\sigma_M^2 \\ \beta_1\,\beta_3\,\sigma_M^2 & \beta_2\,\beta_3\,\sigma_M^2 & \beta_3^2\,\sigma_M^2 \end{pmatrix} + \begin{pmatrix} \sigma_{\epsilon_1}^2 & 0 & 0 \\ 0 & \sigma_{\epsilon_2}^2 & 0 \\ 0 & 0 & \sigma_{\epsilon_3}^2 \end{pmatrix}$$

In[∘]:=
```
MatrixForm[
  σM^2 KroneckerProduct[{β1, β2, β3}, {β1, β2, β3}] + DiagonalMatrix[{σϵ1, σϵ2, σϵ3}^2]
]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \beta_1^2\,\sigma_M^2 + \sigma_{\epsilon_1}^2 & \beta_1\,\beta_2\,\sigma_M^2 & \beta_1\,\beta_3\,\sigma_M^2 \\ \beta_1\,\beta_2\,\sigma_M^2 & \beta_2^2\,\sigma_M^2 + \sigma_{\epsilon_2}^2 & \beta_2\,\beta_3\,\sigma_M^2 \\ \beta_1\,\beta_3\,\sigma_M^2 & \beta_2\,\beta_3\,\sigma_M^2 & \beta_3^2\,\sigma_M^2 + \sigma_{\epsilon_3}^2 \end{pmatrix}$$

Parsimony

The parameters for n assets required for a mean-variance estimation are n estimates for the mean, $n(n+1)/2$ unique estimates for the covariance, and the risk free rate, a total of $n(n+1)/2+n+1$. The CAPM requires the market mean and standard deviation, the risk free rate, n betas and n idiosyncratic (error) standard deviations, a total of $2\,n+3$.

A quick analysis shows how a direct estimate from the raw data compares with the CAPM:

Parameter Count – *n* Assets

	Raw Data	CAPM
100	5151	203
250	31 626	503
500	125 751	1003
1000	501 501	2003

From many perspectives, parsimony is a significant advantage. The CAPM estimates are more statistically efficient. As long as it is a reasonable representation of the behavior of asset returns (which, alas, is not the best job we can do), it makes sense to use it as the basis for modeling returns, Recall that mean-variance optimization requires that the covariance matrix be inverted and the covariance matrix based on the CAPM tends to be much more stable numerically.

This is especially the case for large n where the resulting covariance is often singular or near-singular. This is not surprising. For large n unless we have very long time series a direct estimate may require us to estimate more parameters than we have data points. The results aren’t likely to make much sense.

One possible deficiency of the CAPM is that it models systematic returns as being based on a single factor β. As we will see shortly, we can build similar models based on multiple factors that have the same benefits of parsimony but are more representative of the true character of asset returns.

Solution to a Simple Markowitz Model for the CAPM

Consider the following simple quadratic program representing the portfolios on the Capital Market Line and its solution to proportionality.

$$\min\left\{\frac{1}{2}\mathbf{x}^T\,\boldsymbol{\Sigma}\,\mathbf{x}-\lambda\left(\boldsymbol{\mu}-r_f\,\mathbf{1}\right)^T\mathbf{x}\right\}$$

$$\boldsymbol{\Sigma}\,\mathbf{x}-\lambda\left(\boldsymbol{\mu}-\mathbf{1}\,r_f\right)=\mathbf{0}\implies\lambda\left(\boldsymbol{\mu}-\mathbf{1}\,r_f\right)=\boldsymbol{\Sigma}\,\mathbf{x}$$

If we assume the CAPM, then we can rewrite the covariance matrix as follows.

$$\left(\boldsymbol{\mu}-r_f\,\mathbf{1}\right)\propto\sigma_M{}^2\,\boldsymbol{\beta}\,\boldsymbol{\beta}^T\,\mathbf{x}+\mathbf{D}\,\mathbf{x}$$

Next we multiply both sides with the inverse of the error covariance matrix.

$$\mathbf{D}^{-1}\left(\boldsymbol{\mu}-r_f\,\mathbf{1}\right)\propto\sigma_M{}^2\,\mathbf{D}^{-1}\,\boldsymbol{\beta}\,\boldsymbol{\beta}^T\,\mathbf{x}+\mathbf{x}$$

If we had a solution for $\boldsymbol{\beta}^T\,\mathbf{x}$, then we could solve for $\mathbf{x}$. Thus, we form an equation of the above, multiply both sides by $\boldsymbol{\beta}^T$ and solve for $\boldsymbol{\beta}^T\,\mathbf{x}$.

$$\boldsymbol{\beta}^T\,\mathbf{D}^{-1}\left(\boldsymbol{\mu}-r_f\,\mathbf{1}\right)=\sigma_M{}^2\,\boldsymbol{\beta}^T\,\mathbf{D}^{-1}\,\boldsymbol{\beta}\,\boldsymbol{\beta}^T\,\mathbf{x}+\boldsymbol{\beta}^T\,\mathbf{x}$$

$$\boldsymbol{\beta}^T\,\mathbf{x}=\frac{\boldsymbol{\beta}^T\,\mathbf{D}^{-1}\left(\boldsymbol{\mu}-r_f\,\mathbf{1}\right)}{1+\sigma_M{}^2\,\boldsymbol{\beta}^T\,\mathbf{D}^{-1}\,\boldsymbol{\beta}}$$

We can now substitute the solution for $\boldsymbol{\beta}^T\,\mathbf{x}$ into the original expression and solve for $\mathbf{x}$ then drop the constant term that does not affect proportionality.

$$\mathbf{D}^{-1}\left(\boldsymbol{\mu}-r_f\,\mathbf{1}\right)\propto\sigma_M{}^2\,\mathbf{D}^{-1}\,\boldsymbol{\beta}\left(\frac{\boldsymbol{\beta}^T\,\mathbf{D}^{-1}\left(\boldsymbol{\mu}-r_f\,\mathbf{1}\right)}{1+\sigma_M{}^2\,\boldsymbol{\beta}^T\,\mathbf{D}^{-1}\,\boldsymbol{\beta}}\right)+\mathbf{x}$$

$$\mathbf{x}\propto\left(1-\frac{\sigma_M{}^2\,\boldsymbol{\beta}^T\,\mathbf{D}^{-1}\,\boldsymbol{\beta}}{1+\sigma_M{}^2\,\boldsymbol{\beta}^T\,\mathbf{D}^{-1}\,\boldsymbol{\beta}}\right)\mathbf{D}^{-1}\left(\boldsymbol{\mu}-r_f\,\mathbf{1}\right)\implies\mathbf{x}\propto\mathbf{D}^{-1}\left(\boldsymbol{\mu}-r_f\,\mathbf{1}\right)$$

We can substitute $\left(\mu_M-r_f\right)\boldsymbol{\beta}$ for $\left(\boldsymbol{\mu}-r_f\,\mathbf{1}\right)$, rearrange terms and again drop the constant term.

$$\mathbf{x}\propto\left(\mu_M-r_f\right)\mathbf{D}^{-1}\,\boldsymbol{\beta}\implies\mathbf{x}\propto\mathbf{D}^{-1}\,\boldsymbol{\beta}\implies x_i\propto\frac{\beta_i}{\sigma_{\epsilon_i}{}^2}$$

If we note that $\beta_i=\sigma_{i,M}/\sigma_M{}^2$, then to proportionality we can drop the constant $1/\sigma_M{}^2$ and an alternate form is

$$x_i\propto\frac{\beta_i}{\sigma_{\epsilon_i}{}^2}\implies x_i\propto\frac{\sigma_{i,M}}{\sigma_{\epsilon_i}{}^2}$$

Obviously, if we normalize the solution to add to unity, then we have the market portfolio.

Example - CAPM Allocation

Consider the following CAPM parameters.

In[◦]:=
```
nRiskFree = 0.035;
nMarketReturn = 0.115;
nMarketVar = 0.009;
vnBeta = {0.85, 1.20, 0.95};
vnErrorVar = {0.021, 0.012, 0.015};
```

The expected return vector and covariance matrix can then be computed.

In[∘]:=
```
vnExpectedReturn = (nMarketReturn - nRiskFree) vnBeta + nRiskFree;
MatrixForm[vnExpectedReturn]

mnCovariance = nMarketVar Outer[Times, vnBeta, vnBeta] + DiagonalMatrix[vnErrorVar];
MatrixForm[mnCovariance]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0.103 \\ 0.131 \\ 0.111 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0.0275025 & 0.00918 & 0.0072675 \\ 0.00918 & 0.02496 & 0.01026 \\ 0.0072675 & 0.01026 & 0.0231225 \end{pmatrix}$$

We can use the complete form, $\mathbf{x} \propto \mathbf{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\,\mathbf{1}\right)$, to solve for the optimal portfolio, normalizing the values so that they add to unity.

In[∘]:=
```
Timing[#/Total[#] &[Inverse[mnCovariance].(vnExpectedReturn - nRiskFree)]]
```

Out[∘]= {0.00074, {0.198598, 0.490654, 0.310748}}

The alternate computation, $x_i \propto \beta_i / \sigma_{\epsilon_i}{}^2$, yields the same portfolio allocations.

$$x_i \propto \frac{\beta_i}{\sigma_{\epsilon_i}{}^2}$$

In[∘]:=
```
Timing[#/Total[#] &[vnBeta / vnErrorVar]]
```

Out[∘]= {0.000031, {0.198598, 0.490654, 0.310748}}

In[∘]:=
```
%%[[1]] / %[[1]]
```

Out[∘]= 23.871

Inverting the covariance matrix is a costly operation that is roughly cubic in the number of the securities in the portfolio, but the simplified solution scales linearly. Even with only 3 assets, exploiting the structure of the covariance matrix results in an order of magnitude speed-up. For a larger number of assets the speed-up can be many orders of magnitude. We will show later how a special covariance structure exists when we have more than just a single market factor and how it can be similarly exploited.

Multi-Factor Models

The CAPM has a single factor that explains the systematic exposure experienced by an asset or portfolio of assets. A better representation of returns may be realized by introducing multiple factors, e.g., a factor for each industry group or a factor to distinguish firms by an economic or market statistics such as total capitalization and book-to-market.

Let $\mathcal{I}$ denote the set of assets or securities, $\mathcal{K}$ denote the set of underlying factors with $f_k(t)$ the k^{th} factor's return,

$b_{i,k}$ the *factor loading* of asset i to factor k, a_i an asset dependent intercept and $\xi_i(t)$ a zero mean error term, then the general form for a multi-factor model for an asset i is

$$r_i(t) = a_i + \sum_{k \in \mathcal{K}} b_{i,k} f_k(t) + \xi_i(t), \quad i \in \mathcal{I}$$

In matrix terms, with **B** the factor loading matrix,

$$\mathbf{r} = \mathbf{a} + \mathbf{B}\, \boldsymbol{f} + \boldsymbol{\xi}$$

The cardinality of $\mathcal{K}$ is less than than of $\mathcal{I}$, typically much less, i.e. $|\mathcal{K}| << |\mathcal{I}|$. Multi-factor models are used extensively in finance, particularly in the management of liquid market securities such as stocks and bonds, although their usefulness is by no means limited to those applications.

The CAPM is an example of a single factor model. Some factor models have only a small number of factors, such as the 3-factor Fama-French model described below. Others may have many more factors, such as the 72-factor BARRA Factor Model (See http://www.msci.com/resources/factsheets/Barra_US_Equity _Model _USE4.pdf) which has 60 industry factors and 12 style factors.

The Fama-French Model

The Fama-French Model (FFM) is a widely used three-factor model (see http://en.wikipedia.org/wiki/Fama–French_three-factor_model). Two groups of stocks seem to perform significantly better than expected from the CAPM.

- Stocks with smaller market capitalization seen to outperform ones with higher capitalization; hence, the return r_S which refers to the return of small-cap minus large cap stocks.
- Higher book-to-market ratios (value stocks) seem to outperform lower book-to-market ratios (growth stocks); hence, the return r_V which refers to the return of value stocks minus growth stocks (high minus low book-to-market).

The Fama-French model adds these two returns to the CAPM to produce a three-factor model:

$$r_i(t) - r_f = \alpha_i + \beta_i\left(r_M(t) - r_f\right) + \gamma_i\, r_S(t) + \delta_i\, r_V(t) + \epsilon_i(t), \quad i \in \mathcal{I}$$

Given the values of the returns r_M, r_S, and $r_V$as independent variables and an stock return r_i as the dependent variable, the parameters of the FFM are fit by linear regression. Studies have repeatedly shown that the FFM explains significantly more of a portfolio’s return than does the CAPM.

There have been extensions to the FFM. For example, the Cahart Four-Factor Model (see http://en.wikipedia.org/wiki/Carhart_four-factor_model) adds a momentum term, identifying last month’s winners and losers.

The APT

The Arbitrage Pricing Theory (APT) starts with the assumption that returns follow a multi-factor model

$$r_i(t) = a_i + \sum_{k \in \mathcal{K}} b_{i,k} f_k(t) + \xi_i(t), \quad i \in \mathcal{I}$$

The APT argues that market forces at equilibrium will stabilize exposure to a factor proportional to its *risk-adjusted return*, i.e., its return net of the risk-free rate, an exposure that cannot be removed by diversification. The idiosyncratic error, however, can be diversified away and an investor is not compensated for that exposure. These factor exposures will yield the risk-adjusted return of the asset. The final form of the APT is, therefore,

$$r_i(t) - r_f = \sum_{k \in \mathcal{K}} b_{i,k}\left(f_k(t) - r_f\right) + \xi_i(t), \quad i \in \mathcal{I}$$

Merging the CAPM and the APT

The CAPM is similar to the APT (See https://en.wikipedia.org/wiki/Arbitrage_pricing_theory), but it assumes each investor trades off risk and reward in mean-variance terms. This leads naturally to the notion of the tangent portfolio and, consequently, the market portfolio. It is driven by the demands of the investors.

The APT on the other hand estimates beta coefficients that reflect the sensitivity of assets to various economic factors with arbitrage forces in the market driving over- and under-priced assets to their equilibrium values, but its assumptions are less restrictive and does not necessarily assume all investors preferences can be expressed by a single portfolio.

We can, however, relate the CAPM and APT as follows. First, model the factors by the CAPM, denoting the respective noise terms with η_k.

$$f_k(t) - r_f = \beta_k\left(r_M(t) - r_f\right) + \eta_k(t), \quad k \in \mathcal{K}$$

Apply the APT to the securities.

$$r_i - r_f = \sum_{j \in \mathcal{K}} b_{i,k}\left(f_k - r_f\right) + \xi_i, \quad i \in \mathcal{I}$$

Substitute the factor-level CAPM into the security-level APT.

$$\beta_k\left(r_M(t) - r_f\right) + \eta_k(t) \to f_k(t) - r_f, \quad k \in \mathcal{K}$$

$$r_i - r_f = \sum_{k \in \mathcal{K}} b_{i,k}\, \beta_k\left(r_M - r_f\right) + \sum_{k \in \mathcal{K}} b_{i,k}\, \eta_k + \xi_i, \quad i \in \mathcal{I}$$

The noise terms, being mean-zero and uncorrelated, can be coalesced into a single error term, ϵ_i.

$$r_i - r_f = \left(\sum_{k \in \mathcal{K}} b_{i,k}\, \beta_k\right)\left(r_M - r_f\right) + \epsilon_i, \quad i \in \mathcal{I}$$

The underlying relationships are linear, and we can, therefore, estimate the β of the security by

$$\beta_i = \sum_{k \in \mathcal{K}} b_{i,k}\, \beta_k, \quad i \in \mathcal{I}$$

and end up with a relationship consistent with the CAPM.

$$\mu_i - r_f = \beta_i\left(\mu_M - r_f\right)$$

Applications of Factor Models

Performance Analysis

We can extend the concept of Jensen's alpha introduced above for the CAPM to the APT

$$r_i(t) - r_f = \alpha_i + \sum_{k \in \mathcal{K}} b_{i,k}\left(f_k(t) - r_f\right) + \xi_i(t), \quad i \in \mathcal{I}$$

As before, the value of α_i indicates how far above or below the plane of the APT r_i falls. Thus, it is a measure of whether the asset i beats the market in risk-adjusted and factor-adjusted terms. By the term security we mean either individual securities or portfolios of securities.

Style Analysis

Given an allocation $\mathbf{x}$ for a portfolio $\mathcal{P}$, we can perform a style analysis which tells us the sources of return for $\mathcal{P}$. Let α be the overall portfolio alpha and y_k be the total exposure of $\mathcal{P}$ to factor k. This can be computed by weighting each asset's loading by each asset's allocation. This results in the expression below and its generalization in matrix form.

$$y_k = \sum_{k \in \mathcal{K}} b_{i,k}\, x_i \quad \Longrightarrow \quad \mathbf{y} = \mathbf{B}^T \mathbf{x}$$

Mean-Variance Optimization

$$\mu_i = \sum_{k \in \mathcal{K}} b_{i,k}\, \phi_k, \quad i \in \mathcal{I}$$

$$\boldsymbol{\mu} = \mathbf{B}\, \boldsymbol{\phi}$$

Let, as usual, $\mathbf{x}$ denote the allocation of assets in a portfolio $\mathcal{P}$. Let y_k be the total exposure of $\mathcal{P}$ to factor k. This can be computed by weighting each asset's loading to that factor by each asset's allocation. This results in the expression below and its generalization in matrix form.

$$y_k = \sum_{k \in \mathcal{K}} b_{i,k}\, x_i \quad \Longrightarrow \quad \mathbf{y} = \mathbf{B}^T \mathbf{x}$$

If the covariance matrix of factors is $\mathbf{C}$, then the systematic variance associated with the factors is $\mathbf{y}^T\, \mathbf{C}\, \mathbf{y}$. Thus, we can substitute one expression into the other and get

$$\mathbf{y} = \mathbf{B}^T \mathbf{x} \quad \text{and} \quad \mathbf{y}^T\, \mathbf{C}\, \mathbf{y} \quad \Longrightarrow \quad \left(\mathbf{B}^T \mathbf{x}\right)^T \mathbf{C} \left(\mathbf{B}^T \mathbf{x}\right) \quad \Longrightarrow \quad \mathbf{x}^T \left(\mathbf{B}\, \mathbf{C}\, \mathbf{B}^T\right) \mathbf{x}$$

The idiosyncratic variance is captured by the diagonal matrix of error variances $\mathbf{D}$. The covariance matrix can therefore be expressed as

$$\boldsymbol{\Sigma} = \mathbf{B}\, \mathbf{C}\, \mathbf{B}^T + \mathbf{D}$$

The variance of a portfolio can be expressed by any of the three expression below.

$$\mathbf{x}^T\, \boldsymbol{\Sigma}\, \mathbf{x} \; = \; \mathbf{x}^T\, \mathbf{B}\, \mathbf{C}\, \mathbf{B}^T\, \mathbf{x} + \mathbf{x}^T\, \mathbf{D}\, \mathbf{x} \; = \; \mathbf{y}^T\, \mathbf{C}\, \mathbf{y} + \mathbf{x}^T\, \mathbf{D}\, \mathbf{x}$$

Consider a mean-variance optimization where the allocations must satisfy some constraint set $\mathcal{S}$

$$\mathcal{M} = \min_{\mathbf{x}} \left\{ \frac{1}{2}\, \mathbf{x}^T\, \boldsymbol{\Sigma}\, \mathbf{x} - \lambda\, \boldsymbol{\mu}^T\, \mathbf{x} \,\middle|\, \mathbf{x} \in \mathcal{S} \right\}$$

We can rewrite this in terms of both $\mathbf{x}$ and $\mathbf{y}$ by adding "accounting constraints" to ensure that $\mathbf{y}$ is constrained to be the factor exposures

$$\mathcal{M} = \min_{\mathbf{x},\mathbf{y}} \left\{ \frac{1}{2} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^T \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} - \lambda \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{0} \end{pmatrix}^T \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \,\middle|\, \begin{pmatrix} \mathbf{B}^T & -\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{0} \wedge \mathbf{x} \in \mathcal{S} \right\}$$

While it may seem that we have complicated things, the solution to a quadratic program essentially involves operations equivalent to inverting the covariance matrix. Simplifying the covariance matrix allows us to exploit its special structure and speed up solutions. We can simply things further by employing what we will call an *orthonormal multi-factor model* ($\perp$ MFM), i.e., where the factors are of unit variance and uncorrelated (hence, their covariance matrix is the identity matrix). The covariance matrix has the form

$$\Sigma = \mathbf{B}\,\mathbf{B}^T + \mathbf{D}$$

and the portfolio optimization can be restated in a form in which the quadratic term is diagonalized

$$\mathcal{M} = \min_{\mathbf{x},\mathbf{y}} \left\{ \frac{1}{2} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^T \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} - \lambda \begin{pmatrix} \mu \\ \mathbf{0} \end{pmatrix}^T \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \middle| \begin{pmatrix} \mathbf{B}^T & -\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{0} \land \mathbf{x} \in \mathcal{S} \right\}$$

We will demonstrate a closed-form solution for the above which exploits the special structure of the covariance matrix. Also, optimization code that is sophisticated enough to take advantage of sparse matrix structure will run considerably faster with the above formulation, often several orders of magnitude faster.

Parsimony

The parameters for n assets required for a mean-variance estimation are n estimates for the mean, $n(n-1)/2$ unique estimates for the covariance, and the risk-free rate, a total of $n(n-1)/2 + n + 1$. A multi-factor model requires the risk-free rate, k factor means and $k(k-1)/2$ covariances, the $n \times k$ factor loading matrix, n idiosyncratic (error) standard deviations, a total of $2\,n + 3$.

A quick analysis shows how a direct estimate from the raw data compares with a four-factor model:

Parameter Count
4 Factors – *n* Assets

	Raw Data	Factor Model
100	5151	515
250	31 626	1265
500	125 751	2515
1000	501 501	5015

As with the CAPM, parsimony is a significant advantage. The benefits of statistical efficiency and numerical stability are comparable to those delineated above for the CAPM.

Solution to a Simple Markowitz Model for the ⊥APT

Consider its solution to proportionality of the following simple quadratic program representing the tangent portfolio under an orthonormal APT-like multi-factor model (⊥APT). The linear algebra here closely follows the similar problem for the CAPM with only slight complications

$$\min \left\{ \frac{1}{2} \mathbf{x}^T \,\Sigma\, \mathbf{x} - \lambda \left(\mu - r_f\, \mathbf{1}\right)^T \mathbf{x} \right\}$$

$$\Sigma\,\mathbf{x} - \lambda \left(\mu - \mathbf{1}\, r_f\right) = \mathbf{0} \implies \lambda \left(\mu - \mathbf{1}\, r_f\right) = \Sigma\,\mathbf{x}$$

If we assume the ⊥ APT, then we can rewrite the covariance matrix as follows.

$$\left(\mu - r_f\, \mathbf{1}\right) \propto \mathbf{B}\,\mathbf{B}^T\,\mathbf{x} + \mathbf{D}\,\mathbf{x}$$

Next we multiply both sides with the inverse of the error covariance matrix.

$$\mathbf{D}^{-1}\left(\mu - r_f\, \mathbf{1}\right) \propto \mathbf{D}^{-1}\,\mathbf{B}\,\mathbf{B}^T\,\mathbf{x} + \mathbf{x}$$

If we had a solution for $\mathbf{B}^T\,\mathbf{x}$, then we could solve for $\mathbf{x}$. Thus, we form an equation of the above, multiply both sides by $\mathbf{B}^T$ and solve for $\mathbf{B}^T\,\mathbf{x}$.

$$\mathbf{B}^T\,\mathbf{D}^{-1}\left(\mu - r_f\, \mathbf{1}\right) = \mathbf{B}^T\,\mathbf{D}^{-1}\,\mathbf{B}\,\mathbf{B}^T\,\mathbf{x} + \mathbf{B}^T\,\mathbf{x}$$

$$\mathbf{B}^T\,\mathbf{x} = \left(\mathbf{B}^T\,\mathbf{D}^{-1}\,\mathbf{B} + \mathbf{I}\right)^{-1}\,\mathbf{B}^T\,\mathbf{D}^{-1}\left(\mu - r_f\,\mathbf{1}\right)$$

We can now substitute the solution for $\mathbf{B}^T\,\mathbf{x}$ into the original expression and solve for $\mathbf{x}$

$$\mathbf{D}^{-1}\left(\mu - r_f\,\mathbf{1}\right) \propto \mathbf{D}^{-1}\,\mathbf{B}\left(\mathbf{B}^T\,\mathbf{D}^{-1}\,\mathbf{B} + \mathbf{I}_{\mathcal{K}}\right)^{-1}\,\mathbf{B}^T\,\mathbf{D}^{-1}\left(\mu - r_f\,\mathbf{1}\right) + \mathbf{x}$$

$$\mathbf{x} \propto \mathbf{D}^{-1}\left(\mathbf{I}_{\mathcal{I}} - \mathbf{B}\left(\mathbf{B}^T\,\mathbf{D}^{-1}\,\mathbf{B} + \mathbf{I}_{\mathcal{K}}\right)^{-1}\,\mathbf{B}^T\,\mathbf{D}^{-1}\right)\left(\mu - r_f\,\mathbf{1}\right)$$

We finally normalize the solution to sum to unity, and we have our solution.

It may seem that we have swapped a simple and straightforward equation for a tortured complicated one. The fact is that the solution above can be computed far faster than the original. Consider

- The matrix we must invert, $\left(\mathbf{B}^{\mathbf{T}}\,\mathbf{D}^{-1}\,\mathbf{B} + \mathbf{I}_{\mathcal{K}}\right)$, has only the number of rows and columns as the number of factors; e.g., 3×3 in the FFM versus perhaps 1,000×1,000 in directly representing the assets.
- Code will run much faster if it exploits the fact that multiple diagonal matrices ($\mathbf{D}$, $\mathbf{I}_{\mathcal{I}}$, and $\mathbf{I}_{\mathcal{K}}$) appear throughout the computations; e.g., inverting $\mathbf{D}$ is trivial, and it is not necessary to explicitly represent diagonal matrices in their full form with extraneous zeros.
- Ordering the linear algebra can dramatically reduce both the number of operations and memory requirements. This is also the case when dealing with other computations involving the factor model; e.g., computing $\left(\mathbf{B}^T\,\mathbf{x}\right)^T\left(\mathbf{B}^T\,\mathbf{x}\right)$ requires materially less computation time and working memory than does the mathematically identical $\mathbf{x}^T\left(\mathbf{B}\,\mathbf{B}^T\right)\mathbf{x}$.

Demonstration - Using the Quadratic Programming Package

Installing and Loading the Package

Download and install the QuadraticProgramming.m file. The approach used here is to assume that it is located in the same directory as the notebook. An OS-independent way of referencing its path is

```
In[∘]:= FileNameJoin[{NotebookDirectory[], "QuadraticProgramming.m"}]
```

```
Out[∘]= /Users/robertjfrey/Documents/Work/Stony Brook
          University/AMS/QF/CourseWork/AMS 511/2019 Fall/Class05/QuadraticProgramming.m
```

The package can be loaded into the notebook using Get[ ]

```
In[∘]:= Get[FileNameJoin[{NotebookDirectory[], "QuadraticProgramming.m"}]]
```

Overview of Functions

There is minimal documentation for the main routine, xQuadraticProgramming[]. It assumes that the objective function is to be minimized. Its input form, however is virtually identical to LinearProgramming[] with the exception that the first argument is a list containing a vector of linear coefficients and matrix of quadratic coefficients. Constraints, right-hand sides, and bounds are per LinearProgramming[], although unlike LinearProgramming[] there are no defaults or simplified input forms and the entire problem must be specified. The function uses FindMinimum[] internally and accepts the same optional arguments as does that function. The output is a list whose first element is the value of the optimum and the second is the solution vector.

In[∘]:= `? xQuadraticProgramming`

Out[∘]=

Symbol

xQuadraticProgramming[{lin, quad}, cons, rhs, bound]

The objective function minimized is "0.5 x.quad.x + lin.x".

As a learning aid there is a display function, xDisplayQP[], which produces a lay-out of a quadratic programming problem. It is only usable for small problems, but it is useful for a new user to check how input is interpreted to create an optimization problem. It has two additional initial arguments, a title string to label the overall problems, and a list of lists of row and column names to label the constraints and variables, respectively.

In[∘]:= `? xDisplayQP`

Out[∘]=

Symbol

xDisplayQP[name, {rows, cols}, {lin, quad}, cons, rhs, bounds]

<>

xQuadraticProgramming[] Examples

We'll next run through a series of examples to illustrate how various problems can be expressed in xQuadraticProgramming[]. Note that if the linear or quadratic term of the objective function is missing it must be represented by and empty list: {}.

Optimization I - Min Risk, No Shorts, Upper Bounds: The problem is formulated as one in which we wish to minimize variance. Further, short positions are not permitted and the maximum allocation in any one asset is 50%.

$$\mathcal{M} = \min_{\mathrm{x}} \left\{ \frac{1}{2}\,\mathbf{x}^T\,\Sigma\,\mathbf{x} \;\middle|\; \mathbb{1}^T\,\mathbf{x} \;=\; 1 \land 0 \le \mathbf{x} \le 0.5 \right\}$$

In[∘]:=
```
xDisplayQP["Min Risk at Target Reward", {{"Capital"}, {"x1", "x2", "x3"}},
  {{}, 0.5 mnCovariance}, {{1, 1, 1}}, {{1, 0}}, {{0, 0.5}, {0, 0.5}, {0, 0.5}}]
```

Out[∘]//MatrixForm=

Min Risk at Target Reward

Objective

Lin: {}

Quad:

	x1	x2	x3
x1	0.0137513	0.00459	0.00363375
x2	0.00459	0.01248	0.00513
x3	0.00363375	0.00513	0.0115612

Constraints

	x1	x2	x3		RHS
Capital	1	1	1	==	1

Bounds

	LO	UP
x1	0	0.5
x2	0	0.5
x3	0	0.5

In[∘]:=
```
vxSolMinVar = xQuadraticProgramming[{{}, 0.5 mnCovariance},
  {{1, 1, 1}}, {{1, 0}}, {{0, 0.5}, {0, 0.5}, {0, 0.5}}]
```

Out[∘]=
```
{0.00356752, {0.318071, 0.296164, 0.385765}}
```

In[∘]:=
```
vnExpectedReturn.Last[vxSolMinVar]
```

Out[∘]=
```
0.114379
```

Optimization II - Target Return, No Shorts, Upper Bounds: The problem is formulated as one in which we wish to minimize variance subject to a target return. Further, short positions are not permitted and the maximum allocation in any one asset is 50%.

$$\mathcal{M} = \min_{\mathrm{x}} \left\{ \frac{1}{2} \mathbf{x}^T \mathbf{\Sigma} \mathbf{x} \;\middle|\; \boldsymbol{\mu}^T \mathbf{x} \geq \tau \wedge \mathbb{1}^T \mathbf{x} = 1 \wedge 0 \leq \mathbf{x} \leq 0.5 \right\}$$

In[∘]:=
```
xDisplayQP["Min Risk at Target Reward",
 {{"Capital", "Target"}, {"x1", "x2", "x3"}}, {{}, 0.5 mnCovariance},
 {{1, 1, 1}, vnExpectedReturn}, {{1, 0}, {0.12, 1}}, {{0, 0.5}, {0, 0.5}, {0, 0.5}}]
```

Out[∘]//MatrixForm=

Min Risk at Target Reward

Objective — Lin: {}

Objective — Quad:

	x1	x2	x3
x1	0.0137513	0.00459	0.00363375
x2	0.00459	0.01248	0.00513
x3	0.00363375	0.00513	0.0115612

Constraints:

	x1	x2	x3		RHS
Capital	1	1	1	==	1
Target	0.103	0.131	0.111	>=	0.12

Bounds:

	LO	UP
x1	0	0.5
x2	0	0.5
x3	0	0.5

In[∘]:=
```
vxSolTarget = xQuadraticProgramming[{{}, 0.5 mnCovariance},
  {{1, 1, 1}, vnExpectedReturn}, {{1, 0}, {0.12, 1}}, {{0, 0.5}, {0, 0.5}, {0, 0.5}}]
```

Out[∘]=
```
{0.00389942, {0.124998, 0.499999, 0.375003}}
```

In[∘]:=
```
vnExpectedReturn.Last[vxSolTarget]
```

Out[∘]=
```
0.12
```

Optimization III - λ Trade-off Parameter, No Shorts, Upper Bounds: This is the same problem as above but in the form of risk-reward trade-off parameter λ. After some experimentation we found a value for λ, 0.25, which produces results close to those above. Note that the linear portion also has a negative sign to make it consistent with the problem formulation.

$$\mathcal{M} = \min_{\mathrm{x}} \left\{ \frac{1}{2} \mathbf{x}^T \mathbf{\Sigma} \mathbf{x} - \lambda\, \boldsymbol{\mu}^T \mathbf{x} \;\middle|\; \mathbb{1}^T \mathbf{x} = 1 \wedge 0 \leq \mathbf{x} \leq 0.5 \right\}$$

In[∘]:=
```
xDisplayQP["Min[Risk - λ Reward]", {{"Capital"}, {"x1", "x2", "x3"}},
  {-0.25 vnExpectedReturn, 0.5 mnCovariance},
  {{1, 1, 1}}, {{1, 0}}, {{0, 0.5}, {0, 0.5}, {0, 0.5}}]
```

Out[∘]//MatrixForm=

Min[Risk - λ Reward]

Objective

Lin

x1	-0.02575
x2	-0.03275
x3	-0.02775

Quad

	x1	x2	x3
x1	0.0137513	0.00459	0.00363375
x2	0.00459	0.01248	0.00513
x3	0.00363375	0.00513	0.0115612

Constraints

	x1	x2	x3		RHS
Capital	1	1	1	==	1

Bounds

	LO	UP
x1	0	0.5
x2	0	0.5
x3	0	0.5

In[∘]:=
```
vxSolLambda = xQuadraticProgramming[{-0.25 vnExpectedReturn, 0.5 mnCovariance},
  {{1, 1, 1}}, {{1, 0}}, {{0, 0.5}, {0, 0.5}, {0, 0.5}}]
```

Out[∘]= {-0.0261006, {0.12379, 0.499999, 0.376211}}

In[∘]:=
```
vnExpectedReturn.Last[vxSolLambda]
```

Out[∘]= 0.12001

Optimization IV - Max Return, No Shorts, Upper Bounds: This is a linear programming problem and we'll show the solution returned by both functions

$$\mathcal{M} = \min_{x} \left\{ -\boldsymbol{\mu}^T \mathbf{x} \;\middle|\; \mathbb{1}^T \mathbf{x} = 1 \wedge 0 \le \mathbf{x} \le 0.5 \right\}$$

In[∘]:=
```
xDisplayQP["Max Reward", {{"Capital"}, {"x1", "x2", "x3"}},
  {-vnExpectedReturn, {}}, {{1, 1, 1}}, {{1, 0}}, {{0, 0.5}, {0, 0.5}, {0, 0.5}}]
```

Out[∘]//MatrixForm=

Max Reward

Objective:

Lin:

x1	-0.103
x2	-0.131
x3	-0.111

Quad: {}

Constraints:

	x1	x2	x3		RHS
Capital	1	1	1	==	1

Bounds:

	LO	UP
x1	0	0.5
x2	0	0.5
x3	0	0.5

In[∘]:=
```
vxSolMaxRet = xQuadraticProgramming[{-vnExpectedReturn, {}},
   {{1, 1, 1}}, {{1, 0}}, {{0, 0.5}, {0, 0.5}, {0, 0.5}}]
```

Out[∘]= {-0.121, {0., 0.5, 0.5}}

In[∘]:=
```
vnExpectedReturn.Last[vxSolMaxRet]
```

Out[∘]= 0.121

The solution returned by LinearProgramming[] is, of course, the same.

In[∘]:=
```
LinearProgramming[-vnExpectedReturn,
  {{1, 1, 1}}, {{1, 0}}, {{0, 0.5}, {0, 0.5}, {0, 0.5}}]
```

Out[∘]= {0., 0.5, 0.5}

Nature of Risk

Investment involves the search for return; however, higher returns usually come with increased levels of risk. A prudent investor prefers less risk to more. Thus, investors must develop some means measuring risk so that they can trade-off risk and reward when making investment decisions.

In studying mean-variance optimization we used the uncertainty associated with return, as measured by its variance (or, equivalently, standard deviation), as our measure of risk. Risk, however, is a far more complex subject.

The events associated uncertainty are not always bad; some of the outcomes may be desirable. One way to think about risk to evaluate the possible outcomes in terms of their consequences. We can think of the “risk equation” as

RISK = UNCERTAINTY + CONSEQUENCES

There are many potential measures of risk. One example used in prior lectures was the Sharpe ratio ψ, the expected return less the risk-free rate divided by the standard deviation of return

$$\psi = \frac{\mu - r_f}{\sigma}$$

This measure was motivated by the CAPM, where the efficient portfolio with the highest Sharpe ratio represented the tangent portfolio. The tangent portfolio, through holding or borrowing cash at the risk-free rate, could produce a range of portfolios, all with the same Sharpe ratio, that were preferable to any other combination of risky assets. It made no sense, therefore, to hold any other portfolio of risky assets and this tangent portfolio must, therefore, be the market portfolio.

Example — Determining Leverage

We have funds to invest for a customer and are given the following parameters: The annual market return $\mu_M = 0.08$ and standard deviation $\sigma_M = 0.10$. The risk-free rate $r_f = 0.01$ and we are able to lend and borrow at that rate. This gives a Sharpe ratio of $\psi = 0.7$. We make the (not quite realistic) assumption that the market's returns are Normally distributed and that the CAPM holds. Thus, we know that the only decision to make is how much cash $0 \leq \lambda$ we will lend and borrow with result being invested into the market.

Based on feedback from the customer we decide that we wish to invest so that the probability that we lose more than 10% of capital in any given year is less than 1%. We know that if λ is the cash leverage (i.e., $(1-\lambda)$ is the cash position), then the resulting portfolio $\mathcal{P}$ will have a mean and standard deviation:

$$\mu_{\mathcal{P}} = (1-\lambda)\, r_f + \lambda\, \mu_M$$

$$\sigma_{\mathcal{P}} = \lambda\, \sigma_M$$

If $F[r \mid \mu,\ \sigma]$ is the CDF of a Normal distribution, then we wish to solve for λ such that

$$F[-0.10 \mid \mu_{\mathcal{P}}, \sigma_{\mathcal{P}}] = 0.01$$

Setting this up in *Mathematica*:

```
nMktMean = 0.08;
nMktSdev = 0.10;
nRiskFree = 0.01;
FindRoot[
 CDF[NormalDistribution[((1 - λ) nRiskFree + λ nMktMean), λ nMktSdev], -0.1] == 0.01,
 {λ, 1.}]
```

```
{λ → 0.676362}
```

Thus, this requirement appears somewhat conservative and means that we would invest $\lambda = 67.6\,\%$ of our assets into the market and hold 32.4% in cash. This is a fairly conservative result and to achieve it the mean return of our portfolio is $\lambda\, \mu_M = 5.4\,\%$.

In practice, the distribution of market returns are leptokurtotic, i.e., the tails of their distribution are heavier than those of a Normal distribution. A frequently used alternative is a Student *t* distribution. We could repeat the analysis above with such a distribution with 4 degrees of freedom. We set the location parameter to the mean but need to adjust the scale parameter to match the desired standard deviation:

```
StandardDeviation[StudentTDistribution[μ, σ, ν]]
```

$$\begin{cases} \sqrt{\frac{\nu}{-2+\nu}}\,\sigma & \nu > 2 \\ \text{Indeterminate} & \text{True} \end{cases}$$

```
Simplify[StandardDeviation[StudentTDistribution[μ, Sqrt[(ν - 2)/ν] σ, ν]],
 Assumptions → ν > 2]
```

σ

```
FullSimplify[Mean[StudentTDistribution[m, Sqrt[(ν - 2)/ν] s, ν]]]

FullSimplify[StandardDeviation[StudentTDistribution[m, Sqrt[(ν - 2)/ν] s, ν]]]
```

$$\begin{cases} m & \nu > 1 \\ \text{Indeterminate} & \text{True} \end{cases}$$

$$\begin{cases} s & \nu > 2 \\ \text{Indeterminate} & \text{True} \end{cases}$$

Making the required adjustment and setting up the problem to use a Student *t* distribution in *Mathematica*:

```
FindRoot[
 CDF[StudentTDistribution[((1 - λ) nRiskFree + λ nMktMean), Sqrt[(4 - 2)/4] λ nMktSdev, 4],
   -0.1] == 0.01, {λ, 1.}]
```

{λ → 0.56425}

As expected, using this more representative distribution with fatter tails gives the more conservative result of investing $\lambda = 56.4\,\%$ of our assets into the market.

The reasoning that went into the above example is related to that of Value-at-Risk, usually abbreviated as VaR.

Value-at-Risk (VaR)

Value-at-Risk (VaR) is a widely used technique in the finance and banking industry for quantifying the risk of a collection of financial positions.

Although there are questions and concerns with its effectiveness, it does represent an effective means of quantifying risk in terms of the probability that loss will exceed a certain size at a certain specified level of confidence.

The Value-at-Risk or VaR (See http://en.wikipedia.org/wiki/Value_at_risk.) is the magnitude of the loss expected over a given time period at a given confidence level. Note there are two parameters to the VaR: the time period δ and the confidence level χ. If F_δ is the CDF of returns over a period of length δ, then the $\text{VaR}_{\delta,\chi}$ is

$$\text{VaR}_{\delta,\chi} = \underset{r}{\text{argmin}}[1 - F_\delta(r) \leq 1 - \chi]$$

If we solve the above for the VaR, then

$$\text{VaR}_{\delta,\chi} = F_\delta^{-1}[1 - \chi]$$

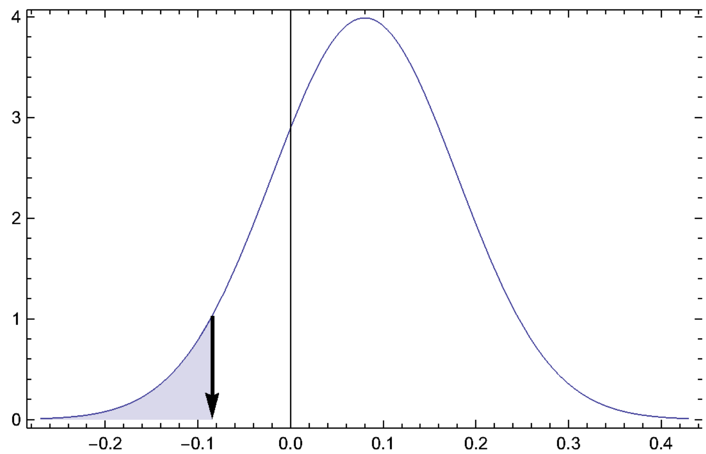
Example — Market's VaR

If we return to the immediately previous example, then the $\text{VaR}_{1,0.95}$ for the market (not the portfolio with cash position) is 8.4 % as computed in *Mathematica*:

```
nVaR = InverseCDF[NormalDistribution[nMktMean, nMktSdev], 1 - 0.95]
```

```
-0.0844854
```

VaR is illustrated in the plot below where the shaded region shows the lower 5% of the probability distribution of the market's return. The VaR is the upper bound of that region and is indicated by the black arrow.



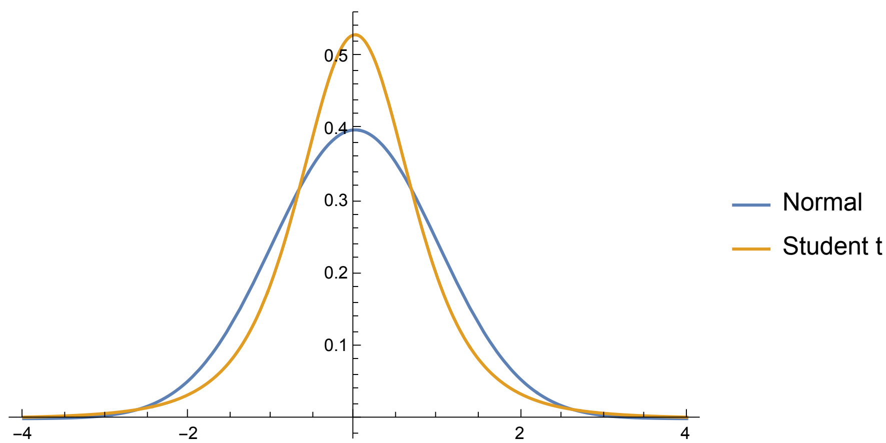
Again, if we are concerned that the Normal distribution is not the best choice because its tails underestimate the probability of large losses we can choose a Student *t* distribution with 4 degrees of freedom and get a VaR = 7.1 %.

$$\texttt{nVaRAlt = InverseCDF}\left[\texttt{StudentTDistribution}\left[\texttt{nMktMean}, \sqrt{\frac{4-2}{4}}\ \texttt{nMktSdev}, 4\right], 1 - 0.95\right]$$

```
-0.0707443
```

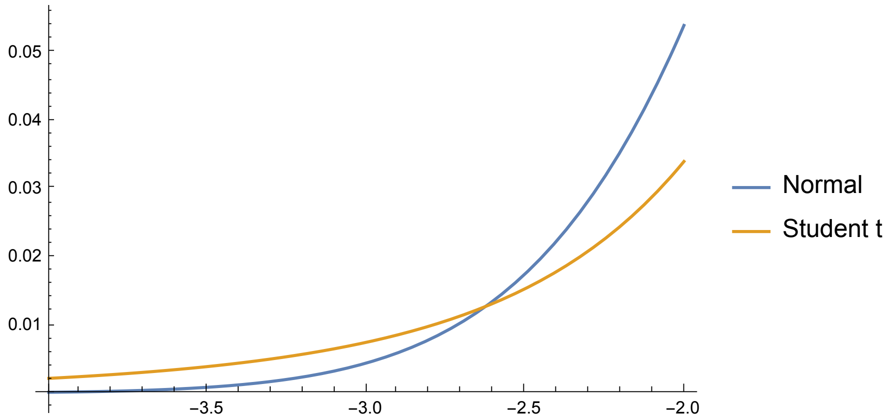
In[∘]:=
```
Plot[
 Evaluate[{
   PDF[NormalDistribution[], x],
   PDF[StudentTDistribution[0, Sqrt[(4 - 2)/4], 4], x]
  }],
 {x, -4, 4},
 PlotLegends → {"Normal", "Student t"}
]
```

Out[∘]=



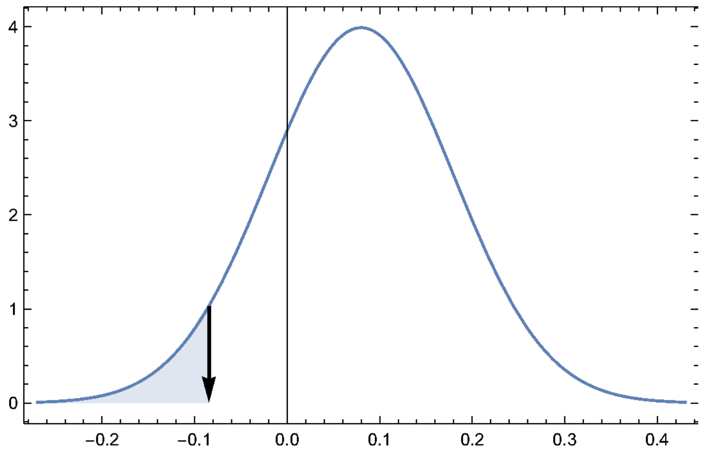
In[∘]:=
```
Plot[
 Evaluate[{
   PDF[NormalDistribution[], x],
   PDF[StudentTDistribution[0, Sqrt[(4 - 2)/4], 4], x]
  }],
 {x, -4, -2},
 PlotLegends → {"Normal", "Student t"}
]
```

Out[∘]=



Here's a sample of the *Mathematica* code used to plot the VaR illustration.

```
Show[
 Plot[Evaluate[PDF[NormalDistribution[nMktMean, nMktSdev], r]],
  {r, -3.5 nMktSdev + nMktMean, 3.5 nMktSdev + nMktMean}, Frame → True],
 Plot[Evaluate[PDF[NormalDistribution[nMktMean, nMktSdev], r]],
  {r, -3.5 nMktSdev + nMktMean, Evaluate[
    InverseCDF[NormalDistribution[nMktMean, nMktSdev], 0.05]]}, Filling → 0],
 Graphics[{Thick, Black, Arrow[
    {{nVaR, PDF[NormalDistribution[nMktMean, nMktSdev], nVaR]}, {nVaR, 0}}]}]
]
```



Conditional VaR or Expected Shortfall (CVaR)

Conditional-Value-at-Risk (CVaR) is another widely used technique in the finance and banking industry for quantifying the risk of a collection of financial positions that is closely related to the VaR and is many respects considered superior to it.

As with VaR, CVaR also represents an effective means of quantifying risk in terms of the expected loss, given a specified VaR shortfall has been met or exceeded.

The expected shortfall, sometimes called the conditional VaR or CVaR, is the expected magnitude of a loss that exceeds the VaR:

$$\mathrm{CVaR}_{\delta,\chi} = \mathrm{E}\left[\, r \;\middle|\; r \le \mathrm{VaR}_{\delta,\chi} \right] = \frac{1}{1-\chi} \int_{-\infty}^{\mathrm{VaR}_{\delta,\chi}} r \, dF_{\delta}[r]$$

The VaR gives the upper bound of the loss region at the given confidence level. The CVaR is the expected loss, given that the loss is greater than or equal to the VaR. Many risk analysts feel that the CVaR is a better risk measure than VaR. As we saw above, using the fatter-tailed Student *t* distribution actually gives a smaller VaR. As we see below, the CVaR computations shows that the fatter-tailed distribution produces a larger CVaR.

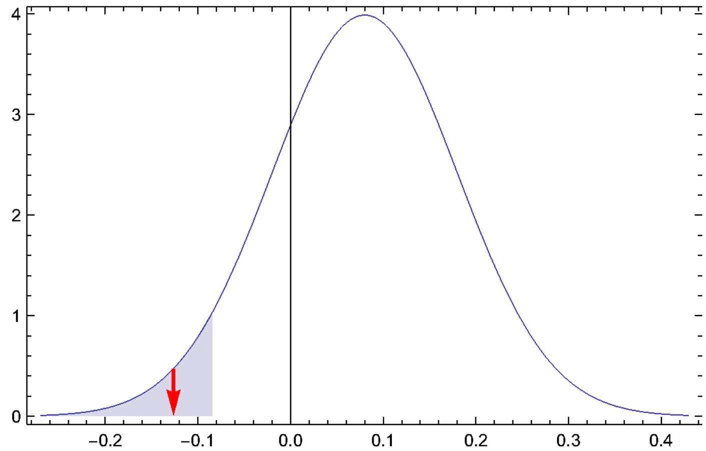
Example — Market’s CVaR

If we return to the previous example, then the $\mathrm{CVaR}_{1,0.95}$ for the market is 12.6 % as computed in *Mathematica*:

```
nCVaR =
  1/(1 - 0.95) Integrate[r PDF[NormalDistribution[nMktMean, nMktSdev], r], {r, -∞, nVaR}]
```

-0.126271

This is illustrated in the plot below where the shaded region shows the lower 5% of the probability distribution of the market' s return. The CVaR is the conditional expectation of the region and is indicated by the red arrow.



Using the more conservative Student t distribution gives the larger CVaR of 14.6 %.

```
nCVaRAlt = 1/(1 - 0.95) Integrate[
   r PDF[StudentTDistribution[nMktMean, Sqrt[(4 - 2)/4] nMktSdev, 4], r], {r, -∞, nVaRAlt}]
```

-0.146477

Alternatives to VaR and CVaR

Coming under the general term of "stress testing" there are many alternative risk measure approaches. In some cases these are devised by the management of a firm for their own purposes and in others they represent regulatory requirements imposed on firms such as banks.

Limits of Analytical Models

The model-based approach is often criticized as not representing an accurate computation of VaR and CVaR. Various distributional assumptions can result in very different tail behavior, and it is often difficult to decide which distribution is the best fit. While it is prudent to employ models with conservative (i.e., larger) loss assumptions, over-estimating risk will cause an investor to realize lower returns.

One advantage models do have is that they can model tail behavior that would otherwise be very difficult to observe. For example, if one only had monthly returns available, then even if there were 20 years of data available, 120 data points, then estimating a 99% confidence level for the monthly VaR directly from the data would be based on a single observation. A model would also allow a risk analyst more scope, *e.g.*, to estimate a daily VaR from monthly returns by multiplying the monthly standard deviation by $\sim \sqrt{20}$ and using an appropriate distribution.

Event Risks and Simulation

There is also the issue of what is called *event risk*. That is the occurrence of a major market or political event that causes large losses. An example would be the default of a country's sovereign debt. For example, an individual may invest in the bonds of a small foreign country which has never defaulted on its debt. However, there may be countries of similar size and with comparable economic conditions that have defaulted. The possibility, therefore, cannot be ignored.

A sound approach would be to examine the histories of similar countries and the current economic health of the subject country to estimate a default probability. The effects on bond prices when defaults did happen in similar countries would be analyzed to produce a distribution of loss during a default. These insights would then be combined with the known history of the bonds of the country in question. A simulation could then be constructed in which the "normal" behavior based on the observed history would be punctuated by random "default events". Multiple runs of such a simulation would be analyzed to produce VaR and CVaR estimates.

Empirical VaR and CVar

If we have enough data, then the best approach may be to use the data directly rather than basing our estimates of VaR and CVaR on fitting a distribution. The empirical approach is best illustrated and contrasted with the distribution based approach with an example.

Example — VaR and CVaR of Daily S&P 500 Returns

A firm has traders who take short-term market positions with extremely high leverage. Even though daily market moves are generally small, occasional large moves occur that could bankrupt the firm if they went against the leveraged positions taken by the trading desks. Therefore, the managers of the firm want to know the daily VaR and CVaR of the market at a 99.9% confidence level.

We start by downloading the value of the S&P 500 index using *Mathematica*'s built-in FinancialData[] function.

```
In[◦]:= mnSP500 = FinancialData["^GSPC", {1976, 11, 4}];
```

This is over four decades of data:

```
In[◦]:= {First@First[#], First@Last[#]} &[mnSP500]

Out[◦]= {{1976, 11, 4}, {2018, 9, 25}}
```

```
In[◦]:= mnSP500[[1 ;; 5]]

Out[◦]= {{{1976, 11, 4}, 102.41}, {{1976, 11, 5}, 100.82},
  {{1976, 11, 8}, 99.6}, {{1976, 11, 9}, 99.32}, {{1976, 11, 10}, 98.81}}
```

```
In[◦]:= Length[mnSP500]

Out[◦]= 10 566
```

We can check for Missing[ ] values used FreeQ[ ]:

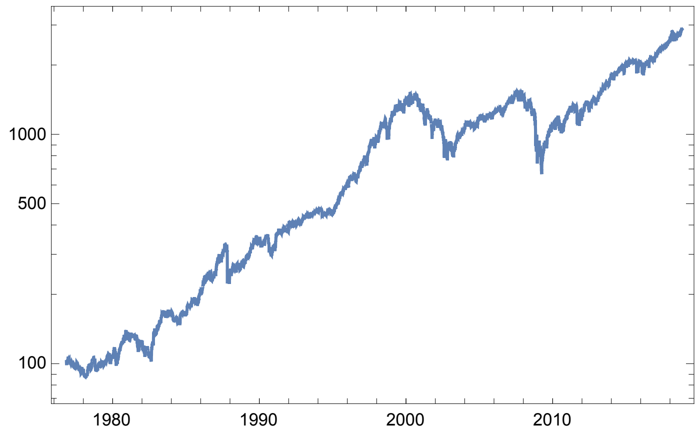
```
In[◦]:= FreeQ[mnSP500[[All, 2]], _Missing]

Out[◦]= True
```

A plot of the log index value is :

```
In[•]:= DateListLogPlot[mnSP500, Joined → True]
```

Out[•]=

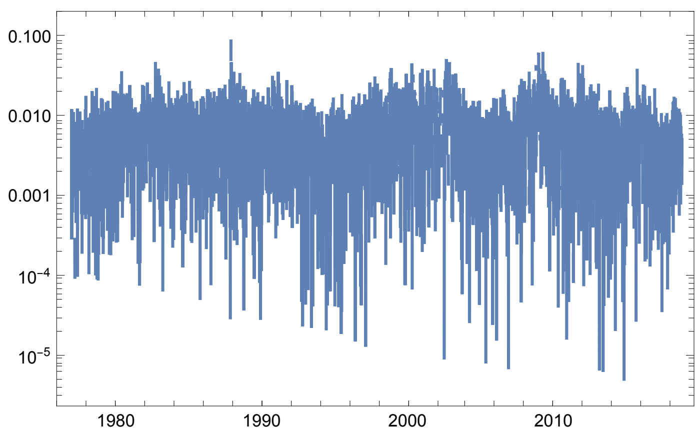


We can compute the daily returns as follows:

```
In[•]:= mnSP500Daily = {Rest[mnSP500[[All, 1]]], (Rest[#] - Most[#])/Most[#] &[mnSP500[[All, 2]]]}ᵀ;
```

```
In[•]:= DateListLogPlot[mnSP500Daily, Joined → True, PlotRange → All]
```

Out[•]=



The mean, standard deviation, skewness, and kurtosis of the sample are:

```
In[•]:= {nSP500Mean, nSP500Sdev, nSP500Skew, nSP500Kurt} =
         Through[{Mean, StandardDeviation, Skewness, Kurtosis}[mnSP500Daily[[All, 2]]]]
Out[•]= {0.000374735, 0.0107154, -0.738733, 23.951}
```

We can compute the daily VaR and CVaR at a 99.9% confidence level assuming a Normal distribution.

```
In[•]:= nSP500VaR = InverseCDF[NormalDistribution[nSP500Mean, nSP500Sdev], 1 - 0.999]
Out[•]= -0.0327383
```

```
In[•]:= nSP500CVaR = 1/(1 - 0.999) Integrate[
            r PDF[NormalDistribution[nSP500Mean, nSP500Sdev], r], {r, -∞, nSP500VaR}]
Out[•]= -0.035705
```

In[•]:=
```
distSP500Daily =
  EstimatedDistribution[mnSP500Daily[[All, 2]], StudentTDistribution[m, s, d]]
```

Out[•]=
```
StudentTDistribution[0.000535447, 0.00667545, 3.08701]
```

We can do the same but use the Student *t* distribution

In[•]:=
```
nSP500VaRAlt = InverseCDF[distSP500Daily, 1 - 0.999]
```

Out[•]=
```
-0.0648916
```

In[•]:=
```
nSP500CVaRAlt =
  1/(1 - 0.999) NIntegrate[Evaluate[r PDF[distSP500Daily, r]], {r, -∞, nSP500VaRAlt}]
```

Out[•]=
```
-0.0968483
```

However, we have a large number of actual observations—more than 10K—so the issue of the absence of rare events in the observed data is less of a concern—although it still ought be carefully considered. We can compute the VaR and CVaR empirically by sorting the returns in ascending order and then selecting the first 0.1%, 10 observations, to produce a sorted lower tail sample.

Note, even with an fairly large number of observations, our conclusions will only be based on 10. This is a fundamental problem with VaR and CVaR. Even if one uses a theoretical distribution, deciding how well that distribution fits the extreme values is inherently difficult.

Also, when dealing with fat-tailed distributions, it is always dangerous to rely solely on the empirical data even if it at first seems that you have enough. The problem is that the tail events you are interested in may never settle down into something representative. With, for example, a Normal distribution, you have a good read on the standard deviation after 30 or so observations and will rarely observe events outside $\pm 3\,\sigma$.

In[•]:=
```
nSP500LowerTail =
  Take[Sort[mnSP500Daily[[All, 2]]], Floor[0.001 Length[mnSP500Daily]]]
```

Out[•]=
```
{-0.204669, -0.0903498, -0.0892952, -0.0880678, -0.0827895,
 -0.0761671, -0.0686568, -0.0680141, -0.067683, -0.0671229}
```

The empirical VaR is the upper bound of the sorted lower tail, and the empirical CVaR is the mean of the sorted lower tail.

In[•]:=
```
nSP500VaREmp = Last[nSP500LowerTail]
```

Out[•]=
```
-0.0671229
```

In[•]:=
```
nSP500CVaREmp = Mean[nSP500LowerTail]
```

Out[•]=
```
-0.0902816
```

Comparing these approaches in the table below it is abundantly clear, however, that a Normal distribution is not a viable choice. Unfortunately, many analysts use a Normal distribution without carefully checking whether or not this choice is a reasonable one.

In[◦]:=
```
Grid[{
  {Style[Column[{"Comparison of VaR & CVaR Methods", "S&P 500 1976-2018"}, Center],
    FontSize → 14]},
  {TableForm[{{nSP500VaR, nSP500CVaR}, {nSP500VaRAlt, nSP500CVaRAlt},
     {nSP500VaREmp, nSP500CVaREmp}},
    TableHeadings → {{"Normal", "Student t", "Empirical"}, {"VaR", "CVaR"}}]}
 },
 Frame →
  All]
```

Out[◦]=

Comparison of VaR & CVaR Methods
S&P 500 1976–2018

	VaR	CVaR
Normal	−0.0327383	−0.035705
Student t	−0.0648916	−0.0968483
Empirical	−0.0671229	−0.0902816

The empirical approach, which is after all based on only 10 observations, may not be a good estimate of the risks faced. In fact, if we looked at the US equity market going further back in history, we would find periods, such as the Great Depression, which suggest that even these empirical results, despite being based on > 40 years of data, still probably underestimate losses.

Criticisms of VaR and CVaR

Although VaR and CVaR have been widely adopted as a risk management tool, they have some undesirable properties. It is important to understand what these problems are and how they may occur if VaR is to be used effectively.

Although VaR and CVaR have been widely adopted as a risk management tool, it has some undesirable properties. It is important to understand what these problems are and how they may occur if VaR is to be used effectively.

Diversification Failure

The *subadditive property* for a risk measure ρ applied to two correlated investments x_1 and $x_2$is

$$\rho(x_1 + x_2) \leq \rho(x_1) + \rho(x_2)$$

VaR for Normally distributed returns does satisfy the subadditive property.

Poor Assessment of Risk

VaR gives us the upper bound of the loss region we are interested, *i.e.*, losses that occur in the confidence region. It does not, however, tell us much about the losses within that region. Two investments with the same VaR may have very different loss behaviors, e.g., their CVaRs or expected losses below the VaR threshold can be very different. Thus, it is possible for an investment to have a lower VaR than another but a higher CVaR.

Example — Misleading Results and Inconsistencies

In two of the example above we computed the VaR and CVaR of a market using both a Normal distribution and Student *t* distribution. The VaRs were:

```
nVaR = InverseCDF[NormalDistribution[nMktMean, nMktSdev], 1 - 0.95]
```

```
-0.0844854
```

$$\texttt{nVaRAlt = InverseCDF}\left[\texttt{StudentTDistribution}\left[\texttt{nMktMean, }\sqrt{\frac{4-2}{4}}\texttt{ nMktSdev, 4}\right]\texttt{, 1 - 0.95}\right]$$

```
-0.0707443
```

Using the more conservative Student *t* distribution resulted in a smaller VaR, even though it has a much heavier loss tale than the Normal distribution. Computing the CVaRs for the two cases:

$$\texttt{nCVaR = }\frac{1}{1-0.95}\texttt{ Integrate}\left[\texttt{r PDF}\left[\texttt{NormalDistribution[nMktMean, nMktSdev], r}\right]\texttt{, }\{\texttt{r, }-\infty\texttt{, nVaR}\}\right]$$

```
-0.126271
```

$$\texttt{nCVaRAlt = }\frac{1}{1-0.95}\texttt{ Integrate}\left[\texttt{r PDF}\left[\texttt{StudentTDistribution}\left[\texttt{nMktMean, }\sqrt{\frac{4-2}{4}}\texttt{ nMktSdev, 4}\right]\texttt{, r}\right]\texttt{, }\{\texttt{r, }-\infty\texttt{, nVaRAlt}\}\right]$$

```
-0.146477
```

We see that the Student *t* distribution has a much higher CVaR even though its VaR is smaller. If we were comparing two investments with the same mean and standard deviation but differing distributions, the VaR might actually lead us to choose that would be reasonably called the riskier investment.

Sensitivity to Confidence Level

The value of the VaR can be extremely sensitive to the choice of confidence level. This is especially true in the case of discrete distributions where the effects of rounding can include and exclude large deviations in a difficult to assess manner.

Coherent Risk Measures

In order to develop and evaluate risk measures it is useful to create a list of desirable properties, or axioms, we believe such measures should satisfy. In what follows a risk measure ρ is a function that takes a position x (which may represent anything from a single security to a portfolio of different types of securities) and maps it onto $\mathbb{R}$. A risk-free asset is also assumed to exist. The risk measure partitions the set of all possible positions into to two classes. If $\rho[x] \leq 0$, then the position x is acceptable; otherwise, it is unacceptable.

A risk measure ρ is a *coherent risk measure* if $\rho[x]$ is finite and satisfies the following four axioms:

1. **Translation Invariance:** For all positions x and risk-free capital pay-offs c there holds $\rho[x+c] = \rho[x] - c$.
2. **Subadditivity:** For all positions x_1 and x_2 there holds $\rho[x_1 + x_2] \leq \rho[x_1] + \rho[x_2]$.

3. **Positive Homogeneity:** For all $\lambda \geq 0$ and all x there holds $\rho[\lambda\, x] = \lambda\, \rho[x]$.
4. **Monotonicity:** For all $x_1 \leq x_2$ there holds $\rho[x_2] \leq \rho[x_1]$.

Implications of the Coherency Axioms

- **Translation Invariance** implies that if a position is unacceptable, $\rho[x] > 0$, then it can be made acceptable by adding a sufficient amount of the risk-free asset.
- **Subadditivity** ensures that if two positions are acceptable, then their sum is also acceptable.
- **Positive Homogeneity** states that risk scales proportionally with position size.
- **Monotonicity** states that if the first position has a larger pay-off than the second in every possible state of nature, then the first position is less risky than the first.
- **Diversification Effect:** One implication of Axioms 2 (subadditivity) and 3 (positive homogeneity) is that

$$\rho[\lambda\, x_1 + (1 - \lambda)\, x_2] \leq \lambda\, \rho[x_1] + (1 - \lambda)\, \rho[x_2]$$

This means that ρ must be convex and that diversification is never disadvantageous.

Example — A Coherent Risk Measure

The function $\mathrm{E}[-x]$ is a coherent risk measure. While the axioms of coherency above are reasonable properties to expect of a risk measure, the simple fact that a risk measure is coherent does not mean it is adequate to an investor’s needs.

Example — Basel III Regulatory Framework

An international framework for risk management is the Third Basel Accord, usually known as Basel III. An informative Wikipedia article can be found at https://en.wikipedia.org/wiki/Basel_III. It attempts to create standards for capital adequacy, stress testing, and market liquidity risk that must be followed by financial institutions. If the are unfamiliar terms in this article, they are usually given as links to further reading.

An understanding of Basel III is important because it affects the behavior and robustness of financial institutions handling tens of trillions of dollars of capital. The article cited here covers some of the criticisms that many have leveled against Basel III and this portion of the article should be carefully read.