AMS-511 Foundations of Quantitative Finance

Fall 2020 — Solutions 06

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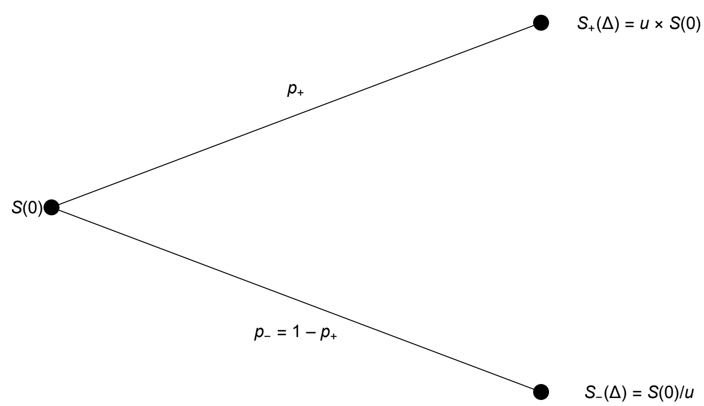
Question 1.

We are given a single-step geometric binomial pricing lattice as the model for the price dynamics of a stock with current price $S(0)$:



Consider a case in which $S(0) = 95$, $\Delta = 0.25$, $u = 1.1$, and the risk free rate of return $r = 1.5\%$, then...

- Show that the market for $S(t)$ is arbitrage free.
- What is the risk neutral measure?
- What is the price at $t = 0$ for an at-the-money put?

Solution

The state prices must be strictly positive:

$$\begin{pmatrix} 1 \\ S(t) \end{pmatrix} = \begin{pmatrix} e^{r\Delta} & e^{r\Delta} \\ u\,S(t) & S(t)/u \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 95 \end{pmatrix} = \begin{pmatrix} e^{0.015\times 0.25} & e^{0.015\times 0.25} \\ 1.1\times 95 & 95/1.1 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \implies \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0.494014 \\ 0.502243 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus, this market is arbitrage-free. Also note that

In[∘]:= `1 / 1.1`

Out[∘]= `0.909091`

$$u^{-1} \le e^{r\Delta} \le u$$

$$0.9091 \le 1.0038 \le 1.1000$$

The risk neutral measure Q can be derived by normalizing the state prices

$$\begin{pmatrix} q_+ \\ q_- \end{pmatrix} = \frac{1}{\psi_+ + \psi_-} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0.4959 \\ 0.5041 \end{pmatrix}$$

The price of an at-the-money put option expiring at Δ is

$$P(t) = e^{-r(T-t)}\, \mathrm{E}_Q[P(T)] = e^{r\times\Delta} \begin{pmatrix} \max[K - u\,S(t), 0] \\ \max[K - S(t)/u, 0] \end{pmatrix}^T \begin{pmatrix} q_+ \\ q_- \end{pmatrix}$$

$$P(0) = e^{-0.015\times 0.25} \begin{pmatrix} \max[95 - 95\times 1.1, 0] \\ \max[95 - 95/1.1, 0] \end{pmatrix}^T \begin{pmatrix} 0.49587 \\ 0.50413 \end{pmatrix} = 8.60407$$

Question 2.

For an underlying whose price dynamic follows the following SDE

$$dS(t) = \mu\, S(t)\, dt + \sigma\, S(t)\, dW(t)$$

Let

$$X(t) = S(t)^{-1}$$

Use Itô's lemma to determine the SDE which describes the price dynamics of $X(t)$, assuming $S(0) > 0$.

Solution

Given the SDE

$$dS(t) = a(S(t), t)\, dt + b(S(t), t)\, dW(t)$$

and suitably differentiable function

$$X(t) = g(S(t), t)$$

Itô's lemma states

$$dX(t) = \left(\frac{\partial g}{\partial S} a + \frac{\partial g}{\partial t} + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} b^2 \right) dt + \frac{\partial g}{\partial S} b\, dW(t)$$

For the problem presented we have $a = \mu\, S(t)$, $b = \sigma\, S(t)$, and $g = 1/S(t)$; therefore,

$$\frac{\partial g}{\partial S} a = -\frac{\mu}{S}$$

$$\frac{\partial g}{\partial t} = 0$$

$$\frac{1}{2} \frac{\partial^2 g}{\partial S^2} b^2 = \frac{\sigma^2}{S}$$

$$\frac{\partial g}{\partial S} b = -\frac{\sigma}{S}$$

yields

$$dX(t) = \left(\sigma^2 - \mu\right) \frac{1}{S(t)} dt - \sigma \frac{1}{S(t)} dW(t)$$

Finally, making the substitution $S(t)^{-1} \to X(t)$ to put the expression in terms of $X(t)$

$$dX(t) = \left(\sigma^2 - \mu\right) X(t)\, dt - \sigma X(t)\, dW(t)$$

Question 3.

Consider a European *capped-call option* with strike price K, cap C, and expiration date T . With $S(t)$ the price of the underlying and $F(t)$ the price of the option, the capped-call gives the holder the right to exercise the option at expiry with value of a put with strike K whose value, however, is capped at a maximum pay-out C where $K < C$. Thus, the value at expiry can be expressed by

$$F(T) = \min[\, \max[\, S(T) - K,\ 0\,],\ C\,]$$

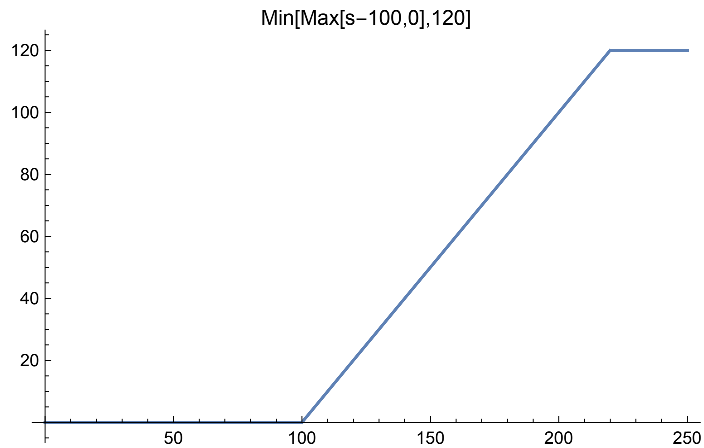
Assume the risk free rate is r with continuous compounding. Express the value of the option at time $t < T$ in terms of vanilla European options with appropriately chosen parameters and, if necessary, any cash position.

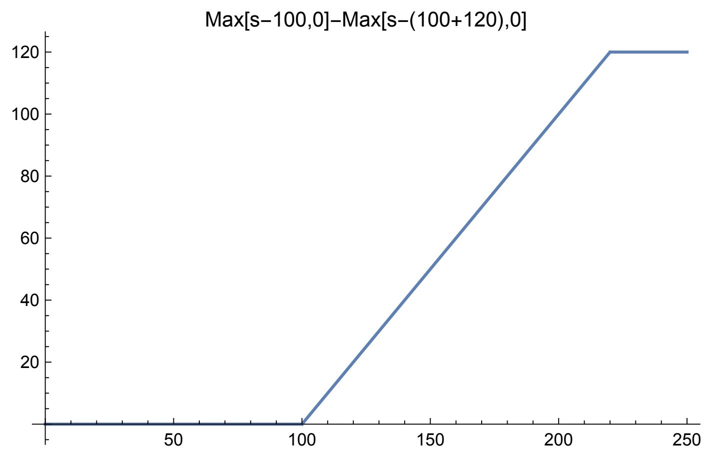
Solution

An equivalent statement of the price at expiry is:

$$F[T] = \max[S(T) - K,\ 0] - \max[S(T) - (K + C), 0]$$

This is easily seen if the plot of $F(T)$ as a function of $S(T)$ is sketched. Consider an example with $K = 100$ and $C = 120$





This a long position in a call with strike K and a short position in a call with strike $K+C$. Thus,

$$F(t) = C(t \mid K, T) - C(t \mid K+C, T)$$

Question 4.

Consider a security priced $S(t)$ with $\sigma = 0.22$ where the risk free rate $r = 0.01$.

- Use FinancialDerivative[ ] to plot the values of a European call option for a strike $K = 102$ and expiry $T = 0.5$ at $t = 0.25$ for $60 \leq S(t) \leq 140$.
- With the same parameters above, use FinancialDerivative[ ] to plot the values of an at-the-money European call option at $t = 0.25$ for values of $0.1 \leq \sigma \leq 0.3$.

Solution

Note that the two calls to FinancialDerivative[ ] are equivalent. In the first case, where a “ReferenceTime” is given, the time to expiry is the “Expiration” less the “ReferenceTime”. In the second case, where no “ReferenceTime” is given, the “Expiration” represents the time to expiry.

In[•]:=
```
FinancialDerivative[{"European", "Call"},
  {"StrikePrice" → 102.00, "Expiration" → 0.5}, {"InterestRate" → 0.01,
   "Volatility" → 0.22, "ReferenceTime" → 0.25, "CurrentPrice" → 102}]
```

Out[•]= 4.59678

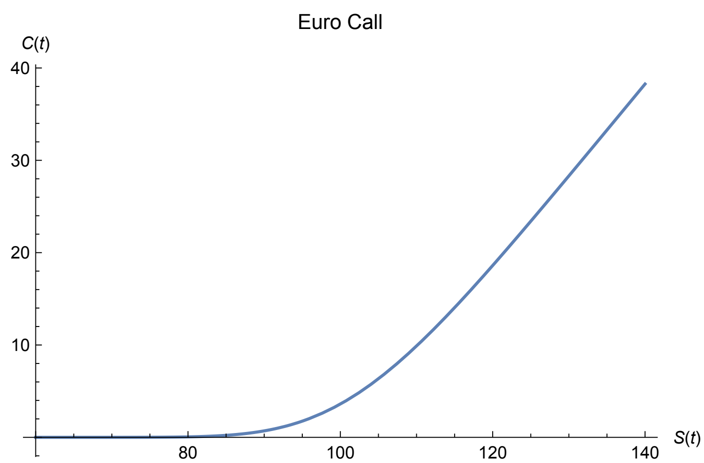
In[•]:=
```
FinancialDerivative[{"European", "Call"},
  {"StrikePrice" → 102.00, "Expiration" → 0.25},
  {"InterestRate" → 0.01, "Volatility" → 0.22, "CurrentPrice" → 102}]
```

Out[•]= 4.59678

The first plot shows the value of the call at for various prices of the underlying.

In[•]:=
```
Plot[FinancialDerivative[{"European", "Call"},
   {"StrikePrice" → 102.00, "Expiration" → 0.5}, {"InterestRate" → 0.01,
    "Volatility" → 0.22, "ReferenceTime" → 0.25, "CurrentPrice" → s}],
  {s, 60, 140}, PlotLabel → "Euro Call", AxesLabel → {"S(t)", "C(t)"}]
```

Out[•]=



The next plot shows the effects of varying the volatility on the option price,

In[•]:=
```
Plot[FinancialDerivative[{"European", "Call"},
  {"StrikePrice" → 102.00, "Expiration" → 0.5}, {"InterestRate" → 0.01,
   "Volatility" → σ, "ReferenceTime" → 0.25, "CurrentPrice" → 102.00}],
 {σ, 0.1, 0.3}, PlotLabel → "Euro Call (at the money)", AxesLabel → {"σ", "C(t)"}]
```

Out[•]=