Spring 2020 MAT303 Recitations

Week of 4/13/20: Sections 3.5 and 3.6

Section 3.5: Nonhomogeneous Equations

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where P, Q are constants. In today's lecture we will learn how to solve this via Variation of Parameters, by perturbing its complementary solutions.

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$$y(x) = u_1(x) y_1(x) + u_2(x) y_2(x),$$

where u_1, u_2 are *functions* of x to be determined.

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$$y(x) = u_1(x) \cos(x) + u_2(x) \sin(x),$$

for unknown functions u_1, u_2.

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This is the first equation of our system.

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It follows that

$$\begin{aligned} y^{''}(x) + y(x) &= -u_1^{'}(x)\sin(x) + u_2^{'}(x)\cos(x) \\ e^x &= -u_1^{'}(x)\sin(x) + u_2^{'}(x)\cos(x). \end{aligned}$$

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The desited system of linear, first-order equations on u_1, u_2 is

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$$\begin{aligned} u_1(x) &= \frac{e^x(\cos(x) - \sin(x))}{2} + A_1, \\ u_2(x) &= \frac{e^x(\sin(x) + \cos(x))}{2} + A_2. \end{aligned}$$

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The final step is to combine the auxilliary functions u_1, u_2 and the complementary solutions, to obtain

$$\begin{aligned} y(x) &= u_1(x)\cos(x) + u_2(x)\sin(x) \\ &= \frac{e^x}{2} + A\cos(x) + B\sin(x) \end{aligned}$$

Try to confirm this answer by Undetermined Coefficients!

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By elimination we find

$$\begin{aligned} u_1'(x) &= \log(\cot(x) + \csc(x)) + A_1, \\ u_2'(x) &= -\csc(x) + A_2. \end{aligned}$$

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So the final solution is

$$y(x) = A_1\cos(x) + A_2\sin(x) + \log(\cot(x) + \csc(x))\cos(x) - 1.$$

Section 3.6: Forced Oscillations

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which gives

$$x(t) = \frac{3}{2}\sin(2t) - \sin(3t),$$

a periodic function with period 2π.

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Below is a plot of its graph, the marked points corresponding to periods.