MAT 514 - Lecture 19

- Analytic functions

Recall from Calculus that certain functions had local power series representations (Taylor series). A few examples:

- $e^x = \sum_{k\geq 0} \frac{x^k}{k!}$, for all $x \in \mathbb{R}$.
- $\sin(x) = \sum_{k\geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, for all $x \in \mathbb{R}$
- $\cos(x) = \sum_{k\geq 0} \frac{(-1)^k x^{2k}}{(2k)!}$, for all $x \in \mathbb{R}$

Question: Are there similar representations for the complex analogs of these functions? If so, what are their convergence properties?

Example: Consider the power series

$$\sum_{k\geq 0} z^k/k!.$$

We will show that this is a convergent power series, for all values of z.

Recall that the Ratio Test can be used to compute radius of convergence. Schematically

$$\text{“}\ \frac{1}{R} = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \ \text{”}$$

provided the limit is a positive real number. If the limit is zero, then $R=\infty$, and if the limit is ∞, then $R=0$.

For the exponential series,

$$a_k = \frac{1}{k!},$$

so

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{1/(k+1)!}{1/k!}$$

$$= \lim_{k\to\infty} \frac{k!}{(k+1)!}$$

$$= \lim_{k\to\infty} \frac{1}{(k+1)} = 0.$$

It follows from the Ratio Test that the exponential series

$$\sum_{k\geq 0}' \frac{z^k}{k!}$$

is convergent, for all values of $z \in \mathbb{C}$.

Next we want to identify the function represented by this series (hopefully, it is the complex exponential).

Derivatives of power series

Let $\sum_{k\geq 0}' a_k (z-c)^k$ be a power series which converges absolutely on a disk $D[c, r]$. Then we can define a function

$$f : D[c, r] \longrightarrow \mathbb{C}$$

$$f(z) = \sum_{k\geq 0}' a_k (z-c)^k.$$

Then f is holomorphic on $D[c, r]$, and its derivative is given by

$$f'(z) = \sum_{k\geq 1} k a_k (z-c)^{k-1}$$

$$= a_1 (z-c)^0 + 2a_2 (z-c)^1 + 3a_3 (z-c)^2 + \dots$$

and the radius of convergence of this new power series is at least r.

Remark: if r is the radius of convergence of

$$\sum_{k\geq 0} a_k (z-c)^k$$

then it is also the radius of convergence of

$$\sum_{k\geq 1} k a_k (z-c)^{k-1}.$$

Claim 1: f is holomorphic.

The sequence of partial sums

$$f_n(z) = \sum_{k\geq 0}^{n} a_k (z-c)^k$$

converges uniformly to f on $D[c, r]$. Each of the f_n is holomorphic (after all, they are just polynomials). Their derivatives are

$$f_n'(z) = \sum_{k \geq 1}^{n} k a_k (z-c)^{k-1}$$

a power series whose radius of convergence is

$$\left[\lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right|\right]^{-1} = \left[\lim_{k\to\infty} \left|\frac{(k+1) a_{k+1}}{k a_k}\right|\right]^{-1}$$

$$= \left[\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|\right]^{-1}$$

the same as that of f_n. Let's call the limit $g(z) = \sum_{k\geq 1} k a_k (z-c)^{k-1}$. The claim is then a consequence of $f' = g$. This follows from the fact that under uniform convergence, we can interchange limits.

Example: Derivative of the exponential series.

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

$$\left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)' = 0 + 1 + \frac{\cancel{2} z}{\cancel{2} \cdot 1!} + \frac{\cancel{3} z^2}{\cancel{3} \cdot 2!} + \frac{\cancel{4} z^3}{\cancel{4} \cdot 3!} + \dots$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

It turns out that

$$\left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)' = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Moreover,

$$\sum_{k=0}^{\infty} \frac{0^k}{k!} = 1 + 0 + 0 + 0 + \dots$$

$$= 1.$$

Hence the function defined by $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ satisfies

$$\begin{cases} f'(z) = f(z) \\ f(0) = 1. \end{cases}$$

This is, by definition, $f(z) = e^z$. In short,

$$\boxed{e^z = \sum_{k=0}^{\infty}{}' \frac{z^k}{k!}}$$ Exponential series.

for all values $z \in \mathbb{C}$.

<u>Example</u>: Recall the geometric series

$$\sum_{k=0}^{\infty}{}' z^k$$

Its radius of convergence is 1, so it defines a function on $D[0,1]$. In fact, on $D[0,1]$,

$$\sum_{k=0}^{\infty}{}' z^k = \frac{1}{1-z}.$$

Note that $\frac{1}{(1-z)}$ is defined and holomorphic on $\mathbb{C} - \{1\}$, however, away from $D[0,1]$ it is not represented by the geometric series.

Recall the trigonometric series,

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \quad \text{(Cosine Series)}$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \quad \text{(Sine series)}$$

<u>Exercise:</u> Show that

$$\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right)'' = -\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right).$$

Solution:

$$\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right)' = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right)'$$

$$= \left(0 - \frac{\cancel{2}z}{\cancel{2}\cdot 1!} + \frac{\cancel{4}z^3}{\cancel{4}\cdot 3!} - \frac{\cancel{6}z^5}{\cancel{6}\cdot 5!} + \dots\right)$$

$$= -\frac{z}{1!} + \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \dots$$

$$\left[\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right]' = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{(2k+1)}}{(2k+1)!} \qquad *$$

The second derivative is

$$\left[\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right]'' = \left(\frac{-z}{1!} + \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots \right)'$$

$$= -1 + \frac{\cancel{3} z^2}{\cancel{3}\cdot 2!} - \frac{\cancel{5} z^4}{\cancel{5}\cdot 4!} + \frac{\cancel{7}\cdot z^6}{\cancel{7}\cdot 6!} - \cdots$$

$$= -1 + \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots$$

$$= -\left[\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right] . *$$

<u>Remark</u>: Typically, a derived power series starts at a different point,

$$\left(\sum_{k=0}^{\infty} a_k (z-c)^k\right)' = \left(\sum_{k=1}^{\infty} k\cdot a_k (z-c)^{k-1}\right)$$

The points marked as (*) are occurrences of degree-shifting. For instance, given a series

$$\sum_{k=1}^{\infty} b_k (z-c)^{k-1}$$

we can represent it as

$$\sum_{k=0}^{\infty} b_{k+1} \cdot (z-c)^{k}.$$

Downshifting the beginning of summation upshifts the indices in the sum by the same amount, e.g.

$$\left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)' = \sum_{k=1}^{\infty} \frac{k}{k!} z^{k-1}.$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} z^{k-1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1-1)!} z^{k+1-1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot z^{k}$$

Follow-up exercise: show that

$$\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right)' = -\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

Solution:

$$\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right)' = \sum_{k=1}^{\infty} \frac{(-1)^k (2k) \cdot z^{2k-1}}{(2k)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k-1}}{(2k-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2(k+1)-1}}{[2(k+1)-1]!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \cdot z^{2k+1}}{(2k+1)!}$$

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

Or $\cos'(z) = -\sin(z)$.

Taylor Series for a Holomorphic Function.

Suppose that $f : G \subset \mathbb{C} \to \mathbb{C}$ is holomorphic on G, and $D[c, r] \subset G$. Then f can be represented as a power series on $D[c, r]$,

$$f(z) = \sum_{k \geq 0}^{!} \frac{f^{(k)}(c)}{k!} (z - c)^k ,$$

where $f^{(k)}$ denotes the k-th order derivative of f. Moreover, from Cauchy's formulas,

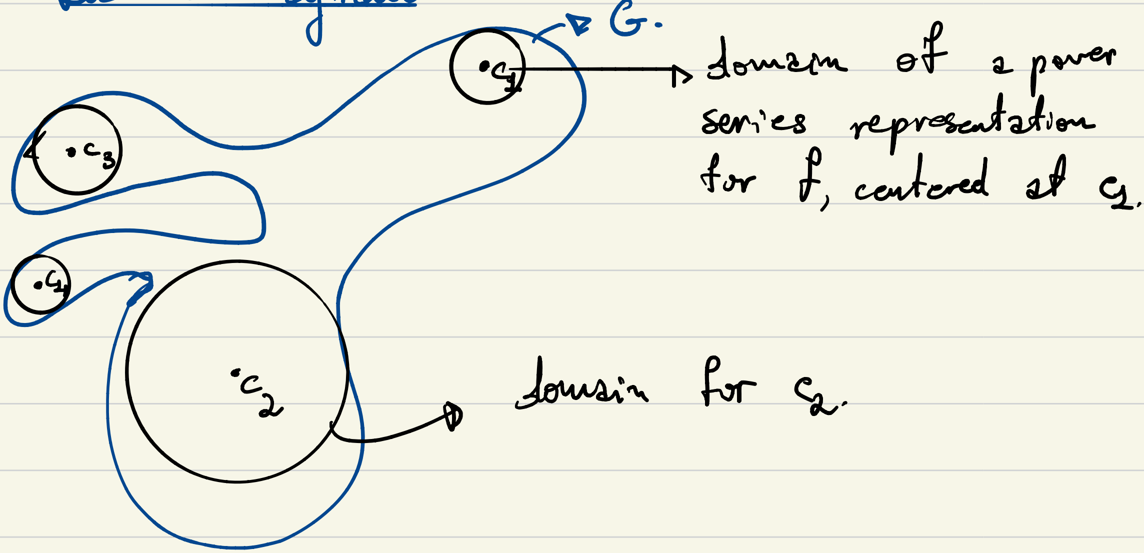
$$f^{(k)}(c) = \frac{k!}{2\pi i} \int_{C[c, r]} \frac{f(c)}{(w-c)^{k+1}} \, dw .$$

Combining these expressions,

$$\boxed{f(z) = \sum_{k \geq 0}^{!} \left(\frac{1}{2\pi i} \int_{C[c, r]} \frac{f(c)}{(w - c)^{k+1}} \, dw \right) (z-c)^k .}$$

for $z \in D[c, r]$.

In a diagram:



This is why

$$\frac{1}{1-z}$$

cannot be represented by the geometric series for $|z|>1$, as any disk of radius $r>1$ centered at the origin would include the singularity $z=1$.