Solutions to Quiz 8

Problem 1 Find a power series representation for

$$\frac{1}{1-z}$$

centered at $z = 4$.

Solution: To obtain this power series representation we will use a change of variables trick, by substituting $w = z - 4$. In terms of w, this function can be written as

$$\frac{1}{1-(w+4)} = \frac{1}{-3-w} = -\frac{1}{3+w} = \frac{-\frac{1}{3}}{1-\left(\frac{-w}{3}\right)}.$$

The last expression can be represented as a geometric series, so long as $|\frac{w}{3}| < 1$,

$$\begin{aligned}\frac{1}{1-z} &= \frac{-\frac{1}{3}}{1-\left(\frac{-w}{3}\right)} \\ &= \left(-\frac{1}{3}\right)\sum_{k=0}^{\infty}\left(-\frac{w}{3}\right)^k \\ &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}w^k}{3^{k+1}} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}(z-4)^k}{3^{k+1}}\end{aligned}$$

Problem 2 Determine the first three terms of the Taylor series for

$$\frac{e^z}{1-z}$$

centered at $z = 0$. Determine the radius of convergence of this power series.

Solution: The first three terms of this Taylor series may be determined by Cauchy's Integral Formulas or via derivatives. In what follows, we use the second approach.

- $a_0 = \frac{e^0}{1-0} = 1.$
- $a_1 = \left(\frac{e^z}{1-z}\right)'(0) = \frac{-e^z(z-2)}{1-z}(0) = 2$
- $a_2 = \left(\frac{1}{2}\right)\left(\frac{e^z}{1-z}\right)''(0) = \left(\frac{1}{2}\right)\left[\frac{e^z}{1-z} + \frac{2e^z}{(1-z)^2} + \frac{2e^z}{(1-z)^3}\right](0) = \frac{5}{2}.$

The series converges for $|z| < 1$, as this is the largest disk centered at $z = 0$ not including the singularity at $z = 1$.