MAT 514 - Lecture 6

Limits and Continuity

Definition: Let $f: \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ be a function, x an accumulation point of $\mathcal{U}$. We say that $L \in \mathbb{C}$ is the limit of f at x, written

$$\lim_{z \to x} f(z) = L,$$

if for any $\varepsilon > 0$, there exists $r > 0$ so that if

$$z \in D(x, r) \cap \mathcal{U}$$

then

$$|f(z) - L| < \varepsilon.$$

- Examples

(1) $f(z) = c$, a constant

$$\lim_{z \to x} f(z) = \lim_{z \to x} c = c.$$

(2) $f(z) = a \cdot z + b$, a, b complex, $a \neq 0$

$$\lim_{z \to x} f(z) = ax + b.$$

(3) $f(z) = 1/z$, $z \neq 0$.

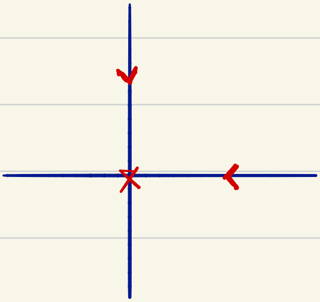
$$\lim_{z \to 0} f(z) = \infty.$$

An example without a limit

(4) $f : \mathbb{C} - \{0\} \longrightarrow \mathbb{C}$

$$f(z) = \frac{\bar{z}}{z}$$

Claim: f does not have a limit as $z \to 0$.



Suppose that $z \to 0$ via the real axis. Then $\bar{z} = z$, so

$$\lim_{\substack{z \to 0, \\ z \in \mathbb{R}}} \frac{\bar{z}}{z} = \lim_{\substack{z \to 0, \\ z \in \mathbb{R}}} 1 = 1.$$

Next suppose $z \to 0$ via the imaginary axis. In this case $\bar{z} = -z$, so

$$\lim_{\substack{z \to 0 \\ iz \in \mathbb{R}}} \bar{z}/z = \lim_{\substack{z \to 0 \\ iz \in \mathbb{R}}} -1 = -1.$$

This function doesn't have a limit, for if it did, its value would be unique, in contrast to what we found above.

Remark: Even if all limits along lines agree, the function may not have a limit in the complex sense (see Homework 2 for an example).

Limits and Algebra

Let f and g be functions defined on a common domain $\mathcal{U}$, and x an accumulation point of $\mathcal{U}$. Suppose that f and g have limits at x.* Then

(a) $\lim_{z \to x} [a f(z) + b g(z)] = a \cdot \left(\lim_{z \to x} f(z)\right) + b\left(\lim_{z \to x} g(z)\right)$

where a, b are complex constants.

(b) $\lim_{z \to x} [f(z) \cdot g(z)] = \left(\lim_{z \to x} f(z)\right) \cdot \left(\lim_{z \to x} g(z)\right)$

(c) if $\lim_{z \to x} g(z) \neq 0$, then

$$\lim_{z \to x} \frac{f(z)}{g(z)} = \frac{\lim_{z \to x} f(z)}{\lim_{z \to x} g(z)}.$$

*: this means the limits are honest complex numbers, not the symbol ∞.

<u>Additional examples</u>

⑤ $\lim_{z \to i} z^2 = \lim_{z \to i} z \cdot z$

$= \left(\lim_{z \to i} z\right)\left(\lim_{z \to i} z\right)$

$$\lim_{z \to i} z^2 = i \cdot i = -1.$$

⑥ $$\lim_{z \to i} z^4 + 3z^3 + 4z^2 + z + 1$$

$$= \left(\lim_{z \to i} z^4\right) + \left(\lim_{z \to i} 3z^3\right) + \left(\lim_{z \to i} 4z^2\right) + \left(\lim_{z \to i} z\right) + \left(\lim_{z \to i} 1\right).$$

$$= \left(\lim_{z \to i} z\right)^4 + 3\cdot\left(\lim_{z \to i} z\right)^3 + 4\left(\lim_{z \to i} z\right)^2 + \left(\lim_{z \to i} z\right) + \left(\lim_{z \to i} 1\right)$$

$$= i^4 + 3 \cdot i^3 + 4 \cdot i^2 + i + 1$$

$$= 1 - 3i - 4 + i + 1$$

$$= -2 - 2i.$$

⑦ $$\lim_{z \to x} p(z) = p(x), \text{ for any}$$

polynomial function.

<u>Definition</u>: Consider a function
$$f : G \subset \mathbb{C} \to \mathbb{C}$$
and a point $x \in G$. We say that f is continuous at x if either:

i) x is an isolated point, or;

ii) x is an accumulation point and
$$\lim_{z \to x} f(z) = f(x).$$

We say that f is continuous on $E \subset G$ if it is continuous at every point of E.

<u>Examples</u>

(8) Polynomials are continuous (refer to example 7).

(9) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$, is a

continuous function.

Let $x \in \mathbb{C}$. Then given $\varepsilon > 0$, for all z with

$$|z - x| < \varepsilon$$

we have

$$|f(z) - f(x)| = |\bar{z} - \bar{x}|$$
$$= |\overline{z-x}|$$
$$= |z-x|$$
$$< \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, this shows that

$$\lim_{z \to x} f(z) = f(x)$$

or

$$\lim_{z \to x} \bar{z} = \bar{x}.$$

(10) A discontinuous function. Let $g : \mathbb{C} \to \mathbb{C}$ be given

$$g(z) = \begin{cases} \dfrac{\bar{z}}{z}, & \text{if } z \neq 0 \\ 1, & \text{if } z = 0. \end{cases}$$

According to example 9, g is not continuous at 0, as

$$\lim_{z \to 0} g(z)$$

doesn't exist.

Continuity and Algebra

Let f, g be continuous at $x \in \mathbb{C}$. Then

(a) $af + bg$ is continuous at x, where a, b are constants.

(b) fg is continuous at x.

(c) f/g is continuous provided $g(x) \neq 0$.

Remark: the quotient may still be continuous even if $g(x) = 0$. An example is

$$f(z) = \begin{cases} \frac{\sin(z)}{z}, & z \neq 0 \\ 1, & z = 0. \end{cases}$$

Continuity and compositions

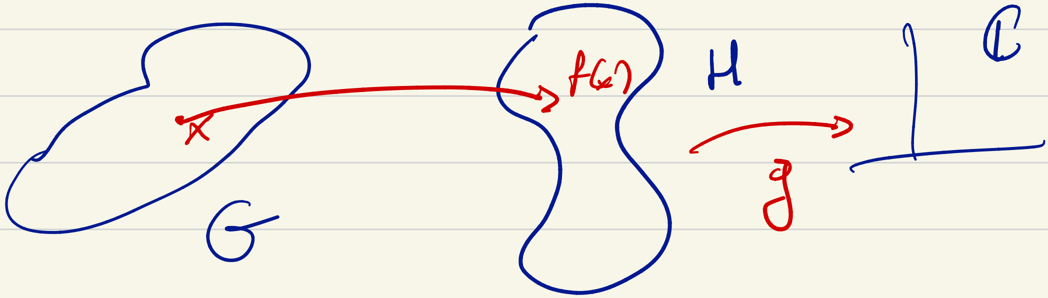
Consider functions

$$f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$$

and

$$g : H \subset \mathbb{C} \to \mathbb{C},$$

and also $x \in G$ such that $f(x) \in H$.



Suppose also that f is continuous at x, and g is continuous at $f(x)$. Then

$$g \circ f$$

is continuous at x.

<u>Example 11:</u> Consider the rational function

$$f(z) = \frac{2z+3}{z-4}.$$

Both numerator and denominator are continuous. It follows from property (e) that so long as $z \neq 4$, f is continuous at z.

As $z \to 4$,

$$\frac{2z+3}{z-4}$$

is so that the numerator is bounded, say $|2z+3| \sim 11$, while the denominator converges to 0, hence the fraction is unbounded, that is

$$\lim_{z \to 4} \frac{2z+3}{z-4} = \infty,$$

There is no value we can assign to f at 4 which would make it continuous.