MAT 514 - Lecture 18

- In lecture 16, we studied power series and convergence.
- In lecture 17, we saw that every holomorphic function admits all higher-order derivatives.

Question: Is there a local power series representation to a holomorphic function?

Recall that a power series
$$\sum_{k\geq 0}' a_k (z-c)^k$$
is said to converge at z if there exists $L \in \mathbb{C}$ such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ for which
$$\left| L - \sum_{k=1}^{N}' a_k (z-c)^k \right| < \varepsilon.$$

This is called pointwise convergence, for the threshold N depends not only on ε, but also on z.

Example: Recall that the partial sums of

$$\sum_{k\geq 0}' z^k$$

is given by

$$S_N = 1 + z + z^2 + \dots + z^N$$
$$= \frac{1 - z^{N+1}}{1 - z}$$

We also know that for $|z| < 1$,

$$\sum_{k\geq 0}' z^k = \frac{1}{1-z}$$

Choose $z = 1/2$, so that

$$S_N = \frac{1 - (1/2)^{N+1}}{1 - 1/2} = \frac{1 - (1/2)^{N+1}}{1/2} = 2 - \left(\frac{1}{2}\right)^N$$

$$\sum_{k\geq 0}' \left(\frac{1}{2}\right)^k = \frac{1}{1 - 1/2} = 2.$$

The difference

$$\left[\sum_{k\geq 0}\left(\frac{1}{2}\right)^k\right] - s_N$$

is $\left(\frac{1}{2}\right)^N$. Given an error margin $\varepsilon > 0$, if we wish to have

$$\left|\left[\sum_{k\geq 0}\left(\frac{1}{2}\right)^k\right] - s_N\right| < \varepsilon$$

$$\iff \left(\frac{1}{2}\right)^N < \varepsilon$$

$$2^N > \frac{1}{\varepsilon}$$

$$N\cdot\log(2) > \log\left(\frac{1}{\varepsilon}\right)$$

$$N > \frac{\log\left(\frac{1}{\varepsilon}\right)}{\log 2}$$

$$N > -\frac{\log(\varepsilon)}{\log 2} = \frac{\log(\varepsilon)}{\log\left(\frac{1}{2}\right)}$$

Let's now choose $z = 1/4$. Then

$$\cdot\ s_N = \sum_{k=0}^{N}{}' \left(\frac{1}{4}\right)^k = \frac{1 - \left(\frac{1}{4}\right)^{N+1}}{1 - \frac{1}{4}} = \frac{1 - \left(\frac{1}{4}\right)^{N+1}}{3/4}$$

$$= \frac{4}{3} - \frac{1}{3}\cdot\left(\frac{1}{4}\right)^N.$$

$$\cdot\ \sum_{k=0}^{N}{}' \left(\frac{1}{4}\right)^k = \frac{1}{1 - 1/4} = \frac{1}{3/4} = \frac{4}{3}.$$

Choose the same error margin ε. In order to make

$$\left| \left[\sum_{k=0}{}' \left(\frac{1}{4}\right)^k\right] - s_N \right| < \varepsilon$$

we need

$$\left| \frac{4}{3} - \left(\frac{4}{3} - \frac{1}{3}\cdot\left(\frac{1}{4}\right)^N\right) \right| < \varepsilon$$

$$\frac{1}{3}\cdot\left(\frac{1}{4}\right)^N < \varepsilon$$

$$4^N > \frac{1}{3\varepsilon}$$

$$N \cdot \log(4) > -\log(3\varepsilon)$$

$$N > \frac{-\log(3\varepsilon)}{\log 4} = \frac{\log(3\varepsilon)}{\log\left(\frac{1}{4}\right)}$$

Question: Is the threshold found for one value of z enough to guarantee the margin of error for the other?

Test: Take $\varepsilon = 0.1 = 1/10$.

- $\frac{\log(\varepsilon)}{\log(1/2)} \cong \frac{-2.3}{-0.69} \cong 3.33$

for $N \geq 4$, the sum

$$\sum_{k=0}^{N} \left(\frac{1}{2}\right)^N$$

is within 0.1 of $\sum_{k \geq 0} \left(\frac{1}{2}\right) = 2$. In fact, for $N = 4$,

$$2 - \sum_{k=0}^{4} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^4 = 0.0625.$$

- $\dfrac{\log(3\varepsilon)}{\log(1/4)} = \dfrac{\log(0.3)}{\log(0.25)} \simeq \dfrac{-1.2}{-1.39} \simeq 0.86.$

for $N \geq 1$, the sum

$$\sum_{k=0}^{N} \left(\frac{1}{4}\right)^k$$

is within 0.1 of

$$\sum_{k\geq 0} \left(\frac{1}{4}\right)^k = \frac{4}{3}.$$

Indeed,

$$\frac{4}{3} - \sum_{k=0}^{1} \left(\frac{1}{4}\right)^k = \frac{4}{3} - 1 - \frac{1}{4} \cong 0.08$$

The threshold $N=1$ serves the error margin $\varepsilon = 0.1$ for $z = 1/4$, but not for $z = \frac{1}{2}$.

Motivated by this phenomenon, we introduce a new mode of convergence.

<u>Definition</u>: A sequence of functions $f_n : G \subset \mathbb{C} \to \mathbb{C}$

is said to converge <u>uniformly</u> to a function

$$f: G \subset \mathbb{C} \to \mathbb{C}$$

if given an error margin $\varepsilon > 0$, there exists a threshold N such that

$$|f(z) - f_n(z)| < \varepsilon$$

for all $n \geq N$ and <u>all $z \in G$</u>.

<u>Remark</u>: The key difference to pointwise convergence is that here the same threshold can be chosen for all points.

<u>Example</u>: Consider the sequence of functions

$$f_n: D[0,1] \to \mathbb{C}$$

$$f_n(z) = z^n.$$

<u>Claim</u>: the sequence f_n converges pointwise to the zero function.

<u>Proof</u>: Let $z \in D[0,1] \setminus \{0\}$. Then

$$\lim_{n \to \infty} z^n = 0.$$

Indeed,

$$\left|\lim_{n\to\infty} z^n\right| \leq \lim_{n\to\infty} |z^n| \leq \lim_{n\to\infty} |z|^n = 0.$$

$\blacksquare$

<u>Example 2:</u> Let $f_n : D[0, \frac{1}{2}] \longrightarrow \mathbb{C}$ be given by $f_n(z) = z^n$.

<u>Claim:</u> (f_n) converges uniformly to the zero function.

<u>Proof:</u> Consider a margin of error $\varepsilon > 0$. The difference

$$\left| f_n(z) - 0(z) \right| = |z^n|$$

is bounded by ε so long as

$$|z^n| < \varepsilon \Rightarrow |z|^n < \varepsilon$$

If $z = 0$ this is certainly satisfied. Otherwise we can use logarithms

$$n \cdot \log|z| < \log(\varepsilon)$$

$$\Rightarrow n > \frac{\log(\varepsilon)}{\log|z|} = -\frac{\log(\varepsilon)}{\log\left(1/|z|\right)}$$

* the inequality reversal happens because $|z| < 1$, hence $\log|z| < 0$.

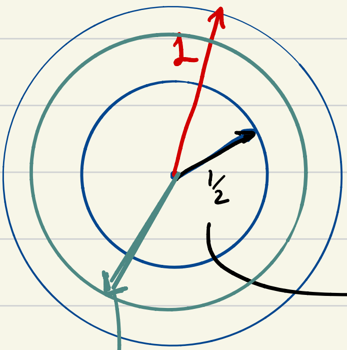
Choose $z \neq 0$ in $D[0, \tfrac{1}{2}]$ to minimize $\log\left(1/|z|\right)$, that is, $|z| = \frac{1}{2}$, so that

$$\log\left(\frac{1}{|z|}\right) = \log(2).$$

So long as $N > -\dfrac{\log(\varepsilon)}{\log(2)}$, we have

$$|f_n(z) - 0| < \varepsilon,$$

for all $z \in D[0, \tfrac{1}{2}]$.



$N > -\frac{\log(\varepsilon)}{\log(2)}$ is enough.

radius $\frac{2}{3} = \frac{1}{\left(\frac{3}{2}\right)}$

<u>Example</u>: Consider the sequence of functions

$$f_n : D[0, \tfrac{2}{3}] \longrightarrow \mathbb{C}$$
$$f_n(z) = z^n$$

<u>Claim</u>: f_n converges uniformly to the zero function

<u>Proof</u>: As in the previous example, we have an estimate

$$n > -\log(\varepsilon) / \log(1/|z|)$$

which is enough to achieve

$$|f_n(z)| < \varepsilon.$$

To achieve the best bound, which serves all z, we choose $|z| = 2/3$, so that

$$n > \frac{-\log(\varepsilon)}{\log(3/2)}$$

This threshold is greater than

$$\frac{-\log(\varepsilon)}{\log(2)}$$

As the radius of the disk grows closer to 1, the lower bound on N increases,

$$N > \frac{-\log(\varepsilon)}{\log(1/r)}$$

on $D[0,r]$ (for $r<1$). As $r \to 1-$

$\log\left(\frac{1}{r}\right) \to 0+$, thus

$$-\frac{\log(\varepsilon)}{\log(1/r)} \longrightarrow +\infty$$

- ε is meant to be small, so that $\log(\varepsilon) < 0$.

We can't choose a uniform bound: the closer we get to the boundary of $D[0,1]$ the closer the bounds on N are to $+\infty$.

In summary: $z^n \to 0$ pointwise on $D[0,1]$, but not uniformly. By contrast for any $0<r<1$, $z^n \to 0$ uniformly on $D[0,r]$.

Consequences of uniform convergence

① If $f_n : G \subset \mathbb{C} \to \mathbb{C}$ is a sequence of functions converging uniformly to $f: G \to \mathbb{C}$ then

$$\lim_{n\to\infty} \lim_{k\to\infty} f_n(z_k) = \lim_{k\to\infty} f(z_k)$$

for all sequences $\{z_k\} \subset G$.

② If $f_n : G \to \mathbb{C}$ is a sequence of continuous functions which converges uniformly to a function $f: G \to \mathbb{C}$, then f is continuous.

Remark about property 2: if the convergence is not uniform, continuity of the limit may fail.

Example: Consider the sequence

$$f_n : \overline{D[0,1]} \longrightarrow \mathbb{C}$$

given by $f_n(z) = |z^n|$. Then in poitwise sense,

$$\lim_{n\to\infty} f_n(z) = \begin{cases} 0, & \text{if } z \in D[0,1] \\ 1, & \text{if } z \in \partial D[0,1]. \end{cases}$$

This limit is discontinuous. Since each f_n is continuous, this shows that the convergence on $\overline{D[0,1]}$ is not uniform.

(3) Suppose $f_n : G \to \mathbb{C}$ is a sequence of continuous functions converging uniformly to $f : G \to \mathbb{C}$. Assume that γ is a piece-wise smooth path. Then

$$\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f$$

(4) <u>Weierstrass M-test</u>

Suppose $f_k : G \to \mathbb{C}$ is a sequence of bounded functions, i.e., for each $k \in \mathbb{N}$ there exist a real constant M_k for which

$$|f_k(z)| < M_k, \quad \text{for all } z \in G.$$

Suppose further that

$$\sum_{k\geq 0}{}' M_k$$

converges. Then

$$\sum_{k\geq 0}{}' |f_k| \;, \quad \sum_{k\geq 0}{}' f_k$$

converge uniformly.

In case the Weierstrass test holds, we say that $\sum' f_k$ converges absolutely and uniformly.