Solutions to Long Quiz 1

Problem 1 Determine the real and imaginary parts[1] of the complex number

$$z = \frac{3i + 2}{12 + 5i}$$

Solution: We begin by simplifying the expression,

$$\begin{aligned} z &= \frac{3i+2}{12+5i} \\ &= \left(\frac{3i+2}{12+5i}\right)\left(\frac{12-5i}{12-5i}\right) \\ &= \frac{36i + 15 + 24 - 10i}{169} \\ &= \frac{39 + 26i}{169} \\ &= \frac{3}{13} + \frac{2i}{13} \end{aligned}$$

Therefore the real and imaginary parts of z are $\Re(z) = \frac{3}{13}$ and $\Im(z) = \frac{2}{13}$, respectively.

Problem 2 Find the conjugate, norm, and polar angle of the complex number

$$z = \frac{4}{\sqrt{3} - i}$$

Solution: We begin by simplifying z,

$$\begin{aligned} z &= \frac{4}{\sqrt{3}-i} \\ &= \left(\frac{4}{\sqrt{3}-i}\right)\left(\frac{\sqrt{3}+i}{\sqrt{3}+i}\right) \\ &= \frac{4\sqrt{3}+4i}{4} \\ &= \sqrt{3} + i. \end{aligned}$$

[1] We will use the symbols $\Re(z), \Im(z)$ for the real and imaginary parts of a complex number z, respectively.

Its conjugate is $\overline{z} = \sqrt{3} - i$. Its norm is

$$\|z\| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2.$$

To find the polar angle, we write the complex number in polar form

$$2(\cos(\theta) + i\sin(\theta)) = \sqrt{3} + i,$$

hence

$$\cos(\theta) = \frac{\sqrt{3}}{2},$$
$$\sin(\theta) = \frac{1}{2},$$

from which we infer $\theta = \frac{\pi}{6}$.

Problem 3 Sketch the set

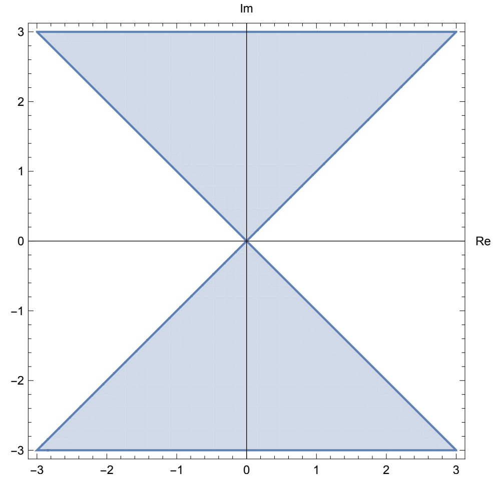
$$\{z \in \mathbb{C} | \Re(z^2) \leq 0\}$$

in the complex plane. Determine its basic topological properties: is it open, closed, or neither; bounded; compact; connected?

Solution: To better understand this region, we will express it in rectangular coordinates. Let $z = x + iy$, so that

$$z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy),$$

so $\Re(z^2) = (x^2 - y^2)$. The region is thus a cone, which we plot below for values of $\|x\|, \|y\| \leq 3$.



It is important to remark that this is not an entire plot, as the region is unbounded. Its topological properties are:

- it is closed;

- it is not bounded;
- it is not compact;
- it is connected.

Problem 4 Determine whether the limit below exists:

$$\lim_{z \to (1-i)} \Re(z) + i(2\Re(z) + \Im(z)).$$

If it exists, compute it. Otherwise, explain why it does not exist.

Solution: The real and imaginary part functions are continuous, therefore, the limits may be computed by substitition:

$$\begin{aligned}\lim_{z \to (1-i)} \Re(z) + i(2\Re(z) + \Im(z)) &= \Re(1-i) + i(2\Re(1-i) + \Im(1-i)) \\ &= 1 + i(2-1) \\ &= 1 + i.\end{aligned}$$