MAT 293 - Lecture 10

Integration of multivariable functions.

Integrals of single-variable functions can be used to compute total variation, or signed area under their graphs.

Iterated integrals

Suppose you are given a function of two variables. We can compute marginal integrals

$$\int f(x,y)\,dx\,, \quad \int f(x,y)\,dy.$$

In these integrals we consider all variables other than the ones emphasized as constants.

Examples:

i) $\displaystyle\int 2xy\,dx = 2y\int x\,dx = 2y\cdot\left(\frac{x^2}{2}+C\right)$

$$\boxed{\int 2xy\,dx = x^2y + \tilde{C}\cdot y,}$$

where $\tilde{C} = 2C.$

2) $$\int (x^2 + y)\,dy = \int x^2\,dy + \int y\,dy$$

$$= x^2\cdot y + \frac{y^2}{2} + C$$

3) $$\int y\cdot\sin(xy)\,dx = y\cdot\left(\int \sin(xy)\,dx\right)$$

$$= -y\cdot\left(\frac{\cos(xy)}{y} + C\right)$$

$$= -\cos(xy) - C\cdot y.$$

- Relation between marginal integrals and partial derivatives.

In example 1, the output is

$$F(x, y) = x^2 y + \tilde{C} \cdot y.$$

Its partial derivative relative to x is:

$$\frac{\partial F}{\partial x} = 2xy + 0 = 2xy.$$

Meanwhile:

$$\frac{\partial F}{\partial y} = x^2 + \tilde{C}$$

In example 2 we had integrand $g(x, y) = x^2 + y$ and output

$$G(x, y) = x^2 y + \frac{y^2}{2} + C.$$

Derivation relative to y confirms the result:

$$\frac{\partial G}{\partial y} = x^2 + y + 0 = x^2 + y.$$

Similarly for example 3,

$$h(x,y) = y \sin(xy)$$

and

$$H(x,y) = -\cos(xy) - C\cdot y$$

Derivation relative to x yields

$$\frac{\partial H}{\partial x} = y\cdot\sin(xy) + 0 = h(x,y).$$

<u>Remark:</u> $\tilde{H}(x,y) = -\cos(xy) - Cy + D$ has the same derivative:

$$\frac{\partial \tilde{H}}{\partial x} = y\sin(xy) + 0 + 0$$

$$= y\sin(xy).$$

$$H^*(x,y) = -\cos(xy) - Cy + y^2.$$

$$\frac{\partial H}{\partial x} = y\sin(xy).$$

Remark: "Constants" of integration are merely function of the remaining variables.

Example 4: Given the function
$$K(x,y) = x^2 + 2xy + y^2.$$
Its partial relative to x is
$$\partial_x K = 2x + 2y.$$
The x-marginal of ∂K is
$$\int (2x + 2y)\,dx = x^2 + 2xy + C$$
numerical constant.

Fundamental Theorem of Calculus (version I)

If a function $F(x,y)$ has continuous partial derivatives then
$$\int \frac{\partial F}{\partial x}\,dx = F(x,y) + g(y)$$
$$\int \frac{\partial F}{\partial y}\,dy = F(x,y) + h(x).$$

Exercise 1: Find the x-marginal of

$$f(x,y) = 2xy + 3y^2.$$

Solution:

$$\int f(x,y)\,dx = \int (2xy + 3y^2)\,dx$$

$$= x^2y + 3xy^2 + C(y).$$

Examples of "$C(y)$": $0,\ 4,\ y^2$.

Definite marginal integrals

$$\int_0^1 (x^2 + y)\,dy = \left[x^2y + \frac{y^2}{2} + C(x) \Big|_{y=0}^{y=1} \right]$$

$$= x^2 \cdot 1 + \frac{1^2}{2} + \cancel{C(x)}$$

$$- \left[\cancelto{0}{x^2 \cdot 0} + \cancelto{0}{\frac{0^2}{2}} + \cancel{C(x)} \right]$$

$$= x^2 + \frac{1}{2}$$

Exercise 2: Compute

$$\int_1^3 xy^2\,dy$$

Solution: An y-antiderivative for xy^2 looks like

$$\frac{xy^3}{3} + f(x).$$

By the Net Change Theorem:

$$\int_1^3 xy^2\,dy = \left[\frac{xy^3}{3} + f(x)\Big|_{y=1}^{y=3}\right]$$

$$= x\cdot\frac{3^3}{3} + \cancel{f(x)} - \left(\frac{x}{3} + \cancel{f(x)}\right)$$

$$= \frac{26x}{3}$$

Iterated integrals

Marginal integrals relative to different variables performed in sequence.

Example 5: function $f(x, y) = x + y$.

$$\int \left[\int f(x, y)\, dx \right] dy.$$

$$= \int \left[\int (x + y)\, dx \right] dy$$

$$= \int \left[\frac{x^2}{2} + y \cdot x + C \right] dy$$

$$= \frac{x^2 y}{2} + \frac{x \cdot y^2}{2} + Cy + D$$

assumed numerical

Remark: Different choices of constant of integration would result in different indefinite integrals.

Example 6: $\int_0^1 \int_0^1 (x+y)\,dy\,dx$

$$= \int_0^1 \left[xy + \frac{y^2}{2} + C(x) \Big|_{y=0}^{y=1} \right] dx$$

$$= \int_0^1 \left[\left(x \cdot 1 + \frac{1^2}{2} + \cancel{C(x)} \right) - \left(\cancel{x \cdot 0} + \cancel{\frac{0^2}{2}} + \cancel{C(x)} \right) \right] dx$$

$$= \int_0^1 \left(x + \frac{1}{2} \right) dx$$

$$= \frac{x^2}{2} + \frac{x}{2} + D \Big|_{x=0}^{x=1}$$

$$= \left(\frac{1}{2} + \frac{1}{2} \right) - \left(\frac{0^2}{2} + \frac{0}{2} \right)$$

$$= 1.$$

Exercise: $$\int_0^{\pi/4}\int_0^1 y\cos(x)\,dy\,dx.$$

Solution: $$\int_0^{\pi/4}\left[\int_0^1 y\cos(x)\,dy\right]dx$$

$$= \int_0^{\pi/4}\cos(x)\cdot\left[\int_0^1 y\,dy\right]dx$$

$$= \int_0^{\pi/4}\cos(x)\cdot\left[\frac{y^2}{2}\Big|_{y=0}^{y=1}\right]dx$$

$$= \int_0^{\pi/4}\cos(x)\cdot\left(\frac{1^2}{2}-\frac{0^2}{2}\right)dx$$

$$= \int_0^{\pi/4}\frac{\cos(x)}{2}\,dx$$

$$= \frac{\sin(x)}{2}\Big|_{x=0}^{x=\pi/4}$$

$$= \frac{1}{2}\sin\left(\frac{\pi}{4}\right) - \frac{1}{2}\sin(0).$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot 0 = \frac{\sqrt{2}}{4}$$

Example 7: Iterated integrals with variable bounds.

$$\int_0^1 \left[\int_0^x \sqrt{1-x^2}\, dy\right] dx = \int_0^1 \left[\sqrt{1-x^2}\left(\int_0^x 1\, dy\right)\right] dx$$

$$= \int_0^1 \left[\left(\sqrt{1-x^2}\right)\cdot\left(y\Big|_{y=0}^{y=x}\right)\right]\cdot dx$$

$$= \int_0^1 \left[\sqrt{1-x^2}\cdot x\right] dx$$

$$= \int_0^1 x\sqrt{1-x^2}\, dx$$

Substitute $u = 1-x^2$, $du = -2x\, dx$

$$\int_0^1 x\sqrt{1-x^2}\, dx = -\int_1^0 \frac{\sqrt{u}}{2}\, du$$

$$= \int_0^1 \frac{\sqrt{u}}{2}\, du \qquad \sqrt{u} = u^{\frac{1}{2}}$$

$$= \frac{1}{\cancel{2}}\cdot u^{3/2}\cdot\frac{\cancel{2}}{3}\Big|_{u=0}^{u=1}$$

$$= \frac{1^{3/2}}{3} - \frac{0^{3/2}}{3} = \frac{1}{3}.$$

In definite integrals bounds cannot depend on:

i) current variable of integration

ii) variables previously integrated.

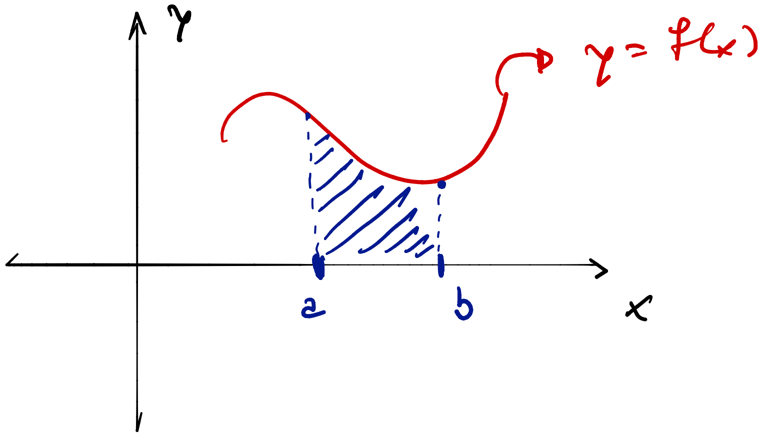
Examples of badly-formed definite integrals:

8) $$\int_0^3 \int_0^{z^2} \int_0^{y} (x+y+z)\, dz\, dx\, dy$$

9) $$\int_0^{y} \int_0^{x} x^2 y \, dx\, dy.$$

Geometric meaning of multiple integrals

In single-variable Calculus, integrals reflect net change over an interval:



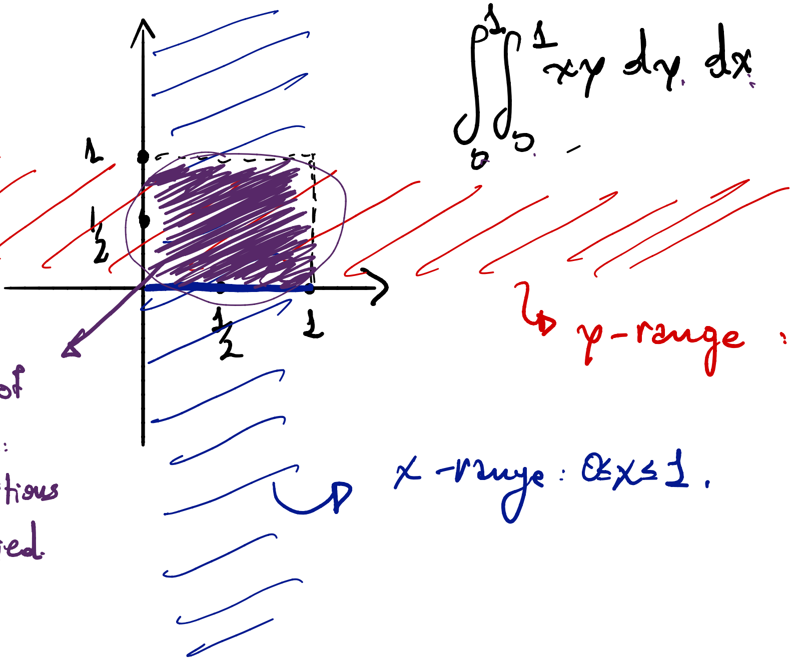
(Signed) Area $= \int_a^b f(x)\,dx$

For a function of two variables, the integral refers to signed <u>volume</u> between its graph and the xy-plane.

- Interpretation of bound in iterated integrals.

$\int_0^1 \int_0^1 xy \, dx \, dy$ means integrating along a square.

$$\int_0^1\int_0^1 xy\,dy\,dx$$



↳ y-range : $0 \le y \le 1$.

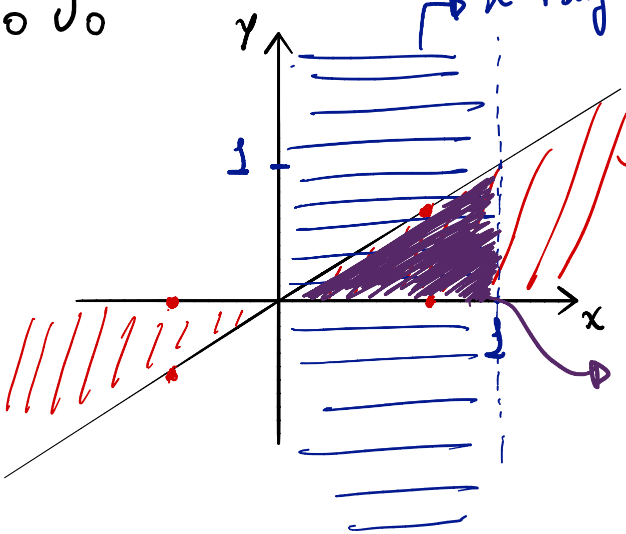
range of integration: both conditions are satisfied.

↳ x-range : $0 \le x \le 1$.

- Variable bounds

$$\int_0^1\int_0^x f(x,y)\,dy\,dx$$

↳ x-range: $0 \le x \le 1$.



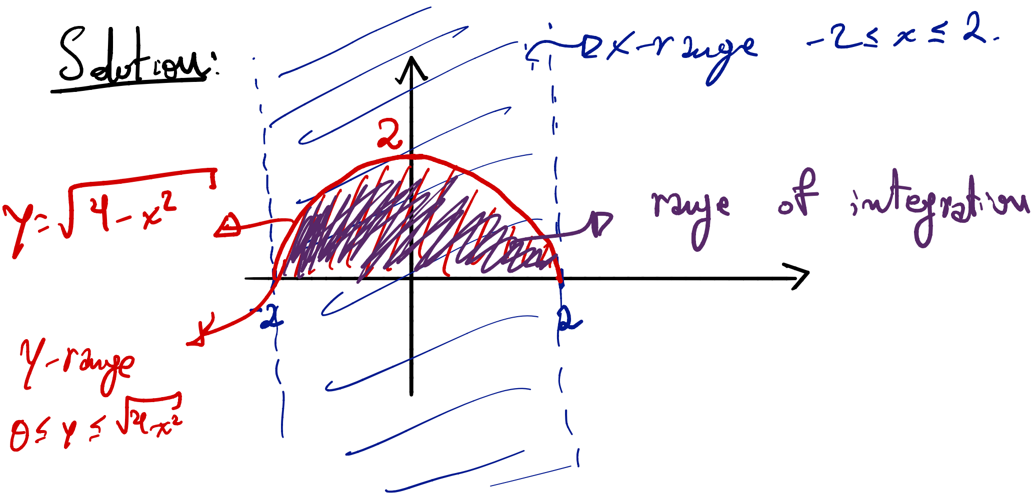
↳ y-range: between 0 and x.

↳ region of integration

Exercise: Describe geometrically the region of integration

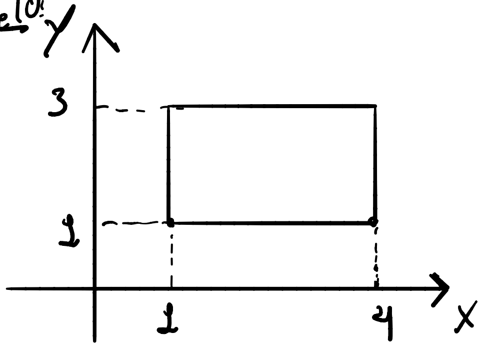
$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

Solution:



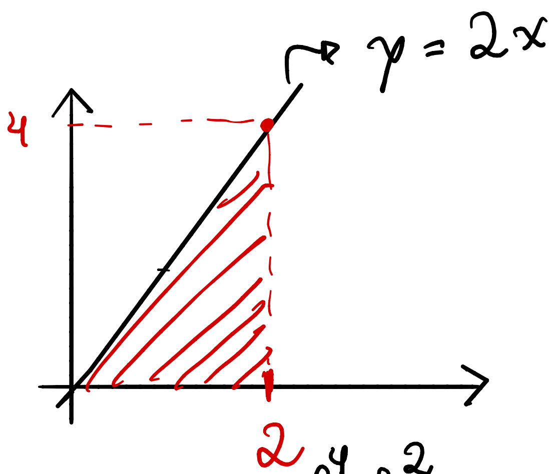
- Changing order of integration

Example 10:



$$\int_1^4\int_1^3 f(x,y)\,dy\,dx = \int_1^3\int_1^4 f(x,y)\,dx\,dy.$$

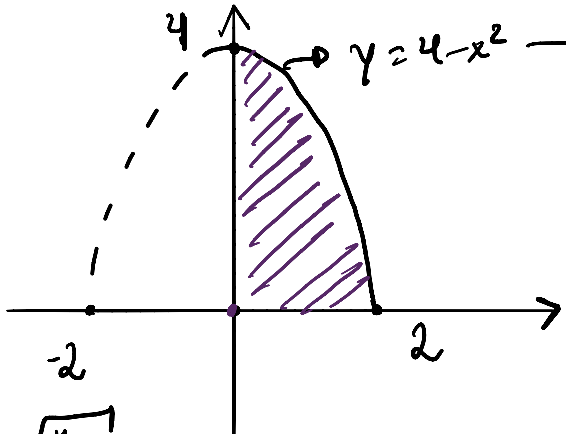
Example 11:



$$\int_0^2\int_0^{2x} f(x,y)\,dy\,dx = \int_0^4\int_{y/2}^2 f(x,y)\,dx\,dy$$

Exercise: Exchange order of integration

$$\int_0^2 \int_0^{4-x^2} f(x,y)\,dy\,dx.$$



$y - 4 = -x^2$

$4 - y = x^2$

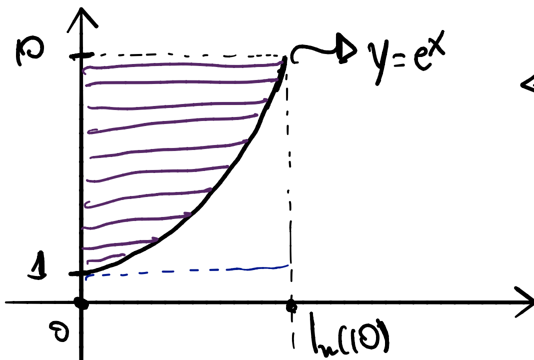
$\Rightarrow x = \sqrt{4-y}$

$$\int_0^4 \int_0^{\sqrt{4-y}} f(x,y)\,dx\,dy$$

Exercise: Exchange the order of integration

$$\int_1^{10} \int_0^{\ln(y)} f(x,y)\,dx\,dy$$

Solution:



$$x = \ln(y)$$
$$\Leftrightarrow \quad y = e^x$$

$$\int_0^{\ln(10)} \int_{e^x}^{10} f(x,y)\,dx\,dy.$$