MAT 132
Summer II 2019
Quiz 4
08/08/19
Time Limit: 50 minutes

Name (Print): ____________________

ID number ____________________

Instructions

- This exam contains 10 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.
- You may *not* use your books, notes, or any device that is capable of accessing the internet on this exam (e.g., smartphones, smartwatches, tablets). You may use a calculator.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.

Problem	Points	Score
1	2	
2	4	
3	2	
4	2	
5	2	
6	2	
7	3	
8	3	
Total:	20	

1. (2 points) Determine whether the sequence

$$a_n = \frac{n\sin(n)}{n^2+1}$$

converges or diverges. If it converges, finds its limit.

Solution: The sequence is bounded as follows

$$\frac{-n}{n^2+1} \leq a_n \leq \frac{n}{n^2+1}.$$

The bounds have the same limit, as $n \to \infty$,

$$\lim_{n\to\infty} \frac{-n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2+1} = 0,$$

hence the limit of the sequence is

$$\lim_{n\to\infty} a_n = 0.$$

2. Determine the value of the following series:

   (a) (2 points)

$$\sum_{n=1}^{\infty} \frac{3^n}{4^{n+2}}$$

Solution: This is a geometric series. It can be written in standard form as follows,

$$\sum_{n=1}^{\infty} \frac{3^n}{4^{n+2}} = \sum_{n=1}^{\infty} \frac{1}{16}\left(\frac{3}{4}\right)^n.$$

We recognize the ratio as $\frac{3}{4}$. Since the ratio is, in absolute value, less than 1, the series converges, and its value is given by

$$\sum_{n=1}^{\infty} \frac{1}{16}\left(\frac{3}{4}\right)^n = \frac{\frac{1}{16}\left(\frac{3}{4}\right)}{1-\frac{3}{4}} = \frac{3}{16}.$$

(b) (2 points)

$$\sum_{n=1}^{\infty} \frac{1}{n^2+7n+10}$$

Solution: The denominator of the fractions can be factored as

$$n^2+7n+10 = (n+2)(n+5).$$

The compound fraction can be written in terms of partial fractions as

$$\begin{aligned}\frac{1}{n^2+7n+10} &= \frac{A}{n+2} + \frac{B}{n+5} \\ &= \frac{A(n+5)+B)n+2)}{(n+2)(n+5)} \\ &= \frac{(A+B)n+(5A+2B)}{(n+2)(n+5)},\end{aligned}$$

for constants A and B to be determined. These constants satisfy the following system of equations

$$\begin{cases} A+B &= 0 \\ 5A+2B &= 1 \end{cases}$$

The solutions of the system are $A = \frac{1}{3}$ and $B = -\frac{1}{3}$.

Now we can rewritte the series as

$$\sum_{n=1}^{\infty} \frac{1}{n^2+7n+10} = \sum_{n=1}^{\infty} \frac{1}{3}\left[\frac{1}{n+2} - \frac{1}{n+5}\right].$$

We recognize the series as a telescopic series, with cancellations every three terms. The value of the series is therefore the sum of its three elements,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+7n+10} = \frac{1}{9} + \frac{1}{12} + \frac{1}{15} = \frac{47}{180}.$$

3. (2 points) Use the Integral Test to determine whether the following series converges

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Solution: The function we use for comparison is

$$f(x) = \frac{1}{x^2+1}.$$

This function is decreasing, for $x > 0$, and the improper integral

$$\int_1^{\infty} \frac{1}{x^2+1}\ dx.$$

converges, as the computation below shows,

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1}\ dx &= \lim_{t\infty} \int_1^{t} \frac{1}{x^2+1}\ dx \\
&= \lim_{t\to\infty} \left(\arctan(x)\Big|_1^t \right) \\
&= \lim_{t\to\infty} (\arctan(t) - \arctan(1)) \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}.
\end{aligned}$$

Therefore, by the Integral Comparison Test, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges.

4. (2 points) Use the Comparison Test to determine whether the following series converges

$$\sum_{n=1}^{\infty} \frac{1+\sin(n)}{3^n}$$

Solution: The terms of the series are bounded as follows,

$$0 \leq \frac{1+\sin(n)}{3^n} \leq \frac{2}{3^n}.$$

The series formed by the upper bounds,

$$\sum_{n=1}^{\infty} \frac{2}{3^n},$$

is a geometric series whose ratio is $\frac{1}{3}$, thus it converges. Therefore, by comparison, the series

$$\sum_{n=1}^{\infty} \frac{1+\sin(n)}{3^n}$$

converges.

5. (2 points) Use the Limit Comparison Test to determine whether the following series converges

$$\sum_{n=1}^{\infty} \frac{n^2 - 1}{3n^4 + 1}$$

Solution: This series is comparable to

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Indeed, the limit

$$\lim_{n\to\infty} \frac{\frac{n^2-1}{3n^4+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4 - n^2}{3n^4 + 1} = \frac{1}{3}$$

exists and is a positive number. By the Limit Comparison Test, the two series have the same behavior relative to convergence. Since $\sum \frac{1}{n^2}$ is known to converge, the series

$$\sum_{n=1}^{\infty} \frac{n^2 - 1}{3n^4 + 1}$$

also converges.

6. (2 points) Use the Alternating Series Test to determine whether the following series converges

$$\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{3n-1}$$

Solution: The sequence of absolute values

$$a_n = \frac{2n+1}{3n-1}$$

is decreasing. Indeed, the auxilliary function

$$f(x) = \frac{2x+1}{3x-1},$$

whose values coincide with the sequence a_n for $x = n$, has derivative

$$f'(x) = \frac{-5}{(3x-1)^2},$$

which is negative for all values of x for which it is defined ($x \neq \frac{1}{3}$). In particular, this function decreases for $x \geq 1$.

However, the limit of the sequence is not zero:

$$\lim_{n\to\infty} a_n = \lim_{n\to 0} \frac{2n+1}{3n-1} = \frac{2}{3},$$

hence the alternating series diverges.

7. (3 points) Find the radius and interval of convergence of the following series

$$\sum_{n=1}^{\infty} \frac{n(x+1)^n}{4^n}$$

Solution: We shall use the Ratio Test to determine the radius of convergence. The limitting ratio is

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\frac{(n+1)(x+1)^{n+1}}{4^{n+1}}}{\frac{n(x+1)^n}{4^n}} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)(x+1)}{4n} \right| \\ &= \frac{|x+1|}{4}. \end{aligned}$$

This limit is less than 1 if

$$\begin{aligned} \frac{|x+1|}{4} &< 1 \\ |x+1| &< 4, \end{aligned}$$

so the radius of convergence of the series is 4.

To determine the interval of convergence, we test for convergence at the endpoints.

When $x = -5$, the series becomes

$$\sum_{n=1}^{\infty} n \left(\frac{-4}{4} \right)^n = \sum_{n=1}^{\infty} (-1)^n n$$

This is a divergent alternating series, as the sequence of absolute values,

$$a_n = n$$

diverges to infinity.

When $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} n \left(\frac{4}{4} \right)^n = \sum_{n=1}^{\infty} n$$

This series diverges, as the general term, $b_n = n$, diverges to infinity.

The interval of convergence of the power series is $(-5, 3)$.

8. (3 points) Find a power series representation of the following function

$$f(x) = \frac{x}{9+x^2}.$$

Solution: We can rewrite this function as

$$\frac{x}{9+x^2} = \frac{x}{9}\left[\frac{1}{1-\left(\frac{-x^2}{9}\right)}\right].$$

In this way, we recognize the expression on the rightmost fraction as a a geometric sum (for $|x| < 3$), hence

$$\begin{aligned} f(x) &= \frac{x}{9}\sum_{n=0}^{\infty}\left(\frac{-x^2}{9}\right)^n \\ &= \frac{x}{9}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{9^n} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{9^{n+1}}, \end{aligned}$$

for $|x| < 3$.