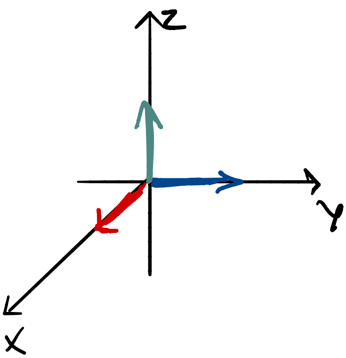
MAT 203 - Lecture 2

Vectors and analytic geometry

Cross product (in $\mathbb{R}^3$): inputs u, v vectors
output vector $\underline{\underline{u \times v}}$

Conceptually: if u, v are given, then their product is perpendicular to both, and it follows the right-hand rule.



Algebraically: Given the coordinates, assemble them into an array, such as

$u = (1, 0, 0)$, $v = (0, 1, 0)$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Three minors: $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$.

Minors: $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$.

determinants: $\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

i) $\det \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \cdot 0 - 0 \cdot 1 = 0$

ii) $\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 1 \cdot 0 - 0 \cdot 0 = 0$.

iii) $\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1$.

$$u \times v = \left(\det \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, -\det \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right)$$

$(1, 0, 0) \times (0, 1, 0) = (0, 0, 1)$.

Exercise 1: Compute the cross product of $u = (1, 2, 3)$, $v = (0, -1, -2)$.

$$\begin{vmatrix} 1 & 0 \\ 2 & -1 \\ 3 & -2 \end{vmatrix}$$

Minors: $\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix}$, $\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix}$, $\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$

Determinants:

i) $2 \cdot (-2) - (-1) \cdot 3 = -4 + 3 = -1$.

ii) $1 \cdot (-2) - 0 \cdot 3 = -2$

iii) $1 \cdot (-1) - 0 \cdot 2 = -1$.

$u \times v = (-1, 2, -1)$.

Exercise 2: Given $v = (1, 2, 3)$ compute $v \times v$.

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{vmatrix}$$

$$\boxed{v \times v = (0, 0, 0)}$$

Minors: $\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$.

$\downarrow$ $2 \cdot 3 - 2 \cdot 3$ $\quad$ $\downarrow$ $1 \cdot 3 - 1 \cdot 3$ $\quad$ $\downarrow$ $1 \cdot 2 - 1 \cdot 2$

Exercise 3: Compute the cross product

$(0, -1, -2) \times (1, 2, 3)$. $= (1, -2, 1)$.

$$\begin{vmatrix} 0 & 1 \\ -1 & 2 \\ -2 & 3 \end{vmatrix}$$

Minors: $\begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix}$, $\begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix}$, $\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}$

determinants:

i) $(-1)\cdot 3 - (-2)\cdot 2 = -3 + 4 = 1$

ii) $0 \cdot 3 - (-2)\cdot 1 = 2$

iii) $0 \cdot 2 - (-1)\cdot 1 = 1$.

$$\boxed{u \times v = -v \times u}$$

<u>Exercise 4</u>: Given

$u = (1,1,1)$, $v = (1,0,1)$, $w = (0,1,0)$.

Compute:

(a) $(u \times v) \times w$

(b) $u \times (v \times w)$

Hint: compute $u \times v$, $v \times w$ first.

<u>Solution</u>:

• $u \times v$: $\begin{vmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{vmatrix}$; Minors: $\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$.

Determinants: $1, 0, -1$.

$$\boxed{u \times v = (1, 0, -1)}.$$

• $v \times w$: $\begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{vmatrix}$; Minors: $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$, $\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$, $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$.

Determinants: $-1, 0, 1$.

$$\boxed{v \times w = (-1, 0, 1)}.$$

- $(u \times v) \times w$:

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{vmatrix}; \quad \text{Minors} \quad \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

Determinants: $1, 0, 1$.

$(u \times v) \times w = (1, 0, 1)$.

- $u \times (v \times w)$:

$$\begin{vmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{vmatrix}; \quad \text{Minors} \quad \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}.$$

Determinants: $1, 2, 1$.

$u \times (v \times w) = (1, -2, 1)$.

In general, $(u \times v) \times w \neq u \times (v \times w)$

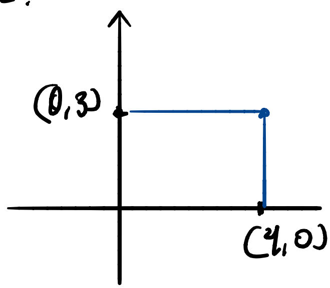
- Properties of cross products:

1) $0 \times u = 0$.
2) $u \times v = -v \times u$.
3) $(\alpha u) \times v = \alpha \cdot (u \times v)$.
4) $(u + v) \times w = u \times w + v \times w$.
5) $u \times u = 0$.

Applications:

1) Finding areas:

Consider vectors $(0,3)$ and $(4,0)$ in the plane:



Area $= 3 \cdot 4 = 12.$

$$\begin{vmatrix} 0 & 4 \\ 3 & 0 \\ 0 & 0 \end{vmatrix}$$

Cross-product:

$(0,3,0) \times (4,0,0) = (0,0,-12).$

$\|(0,0,-12)\| = \sqrt{0^2 + 0^2 + (-12)^2} = 12.$

2) Deciding when vectors are parallel.

$u \times v = 0 \iff u \parallel v.$

__Exercise 4:__ Decide whether $(1,3,2)$ and $(0,1,2)$ are parallel.

$$\begin{vmatrix} 1 & 0 \\ 3 & 1 \\ 2 & 2 \end{vmatrix}; \quad \text{Minors:} \quad \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$$

Determinants: $6-2=4$; $2-0=2$; $1-0=1$.

$(1,3,2)\times(0,1,2) = (4,-2,1)$.

Notation: Standard unit vectors

- in 2D:

$$\vec{i} = (1,0), \quad \vec{j} = (0,1).$$

- in 3D:

$$\vec{i} = (1,0,0), \quad \vec{j} = (0,1,0), \quad \vec{k} = (0,0,1).$$

Every vector in 2D can be written uniquely as as a combination of $\vec{i}, \vec{j}$, and likewise, every vector in 3D can be written uniquely as a combination of $\vec{i}, \vec{j}, \vec{k}$.

Example

- $u = (2,3) = 2\cdot(1,0) + 3\cdot(0,1)$
$= 2i + 3j.$
- $v = (1,-1,5) = 1\cdot(1,0,0) + (-1)\cdot(0,1,0) + 5(0,0,1)$
$= 1i + (-1)j + 5k.$
$= i - j + 5k.$

- Relation to dot products:

If p is a standard unit vector

$$p\cdot p = 1.$$

Example: $\underset{j}{(0,1,0)} \cdot \underset{j}{(0,1,0)} = 0\cdot 0 + 1\cdot 1 + 0\cdot 0 = 1.$

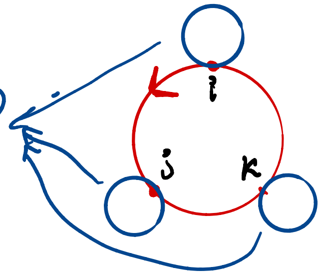
If p, q are distinct standard unit vectors then

$$p\cdot q = 0.$$

Example: $i\cdot j = (1,0,0)\cdot(0,1,0)$
$= 1\cdot 0 + 0\cdot 1 + 0\cdot 0$
$= 0.$

In 3D: relation to cross products.

$i \times i = j \times j = k \times k = 0$



$i \times j = k$, $\quad j \times i = -k$

$j \times k = i$, $\quad k \times j = -i$

$k \times i = j$, $\quad i \times k = -j$.

Exercise: Check that $k \times i = j$.

$$\begin{vmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{vmatrix},$$

Minors: $\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$.

Determinants: $0, -1, 0$.

$k \times i = (0, 1, 0) = j$.

Example: $u = (1, 2, 3)$, $v = (-1, 0, 1)$.

Representing u, v in standard unit vectors:

$$u = 1i + 2j + 3k = i + 2j + 3k.$$

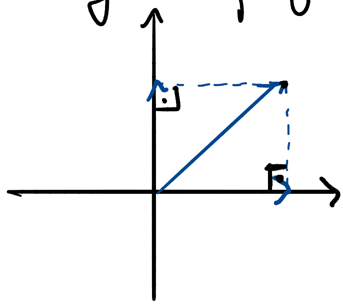
$$v = -1i + 0j + 1k = -i + k.$$

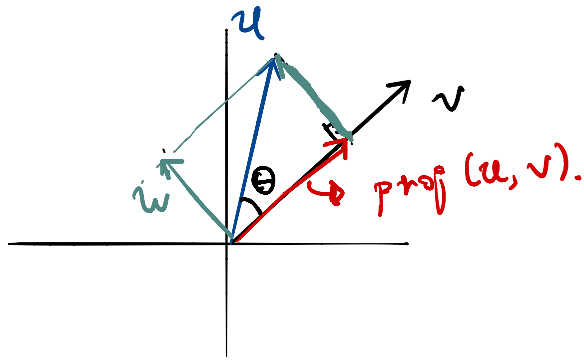
$$u \times v = (i + 2j + 3k) \times (-i + k).$$

$$= -\cancel{i \times i}^{\,0} + i \times k - 2j \times i + 2j \times k - 3k \times i + \cancel{3k \times k}^{\,0}$$

$$= -j + 2k + 2i - 3j = 2i - 4j + 2k.$$

Orthogonal projections





1) $proj(u,v) + w = u$.

2) If θ is the angle between u, v then

$$\cos\theta = \frac{u\cdot v}{\|u\|\cdot\|v\|}.$$

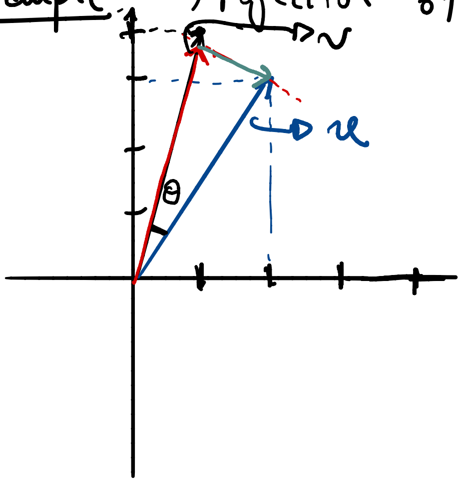
From trigonometry,

$$\|proj(u,v)\| = \|u\|\cdot\cos\theta$$
$$= \cancel{\|u\|}\cdot\frac{(u\cdot v)}{\cancel{\|u\|}\cdot\|v\|}$$

$$\boxed{\|proj(u,v)\| = \frac{u\cdot v}{\|v\|}.}$$

Knowing magnitude, direction and orientation, we can construct the projection.

Example: Projection of $\underbrace{(2,3)}_{u}$ onto $\underbrace{(1,4)}_{v}$.



How long is proj(u,v)?

$$\|\text{proj}(u,v)\| = \frac{u \cdot v}{\|v\|}$$

$u \cdot v = (2,3)\cdot(1,4) = 2\cdot 1 + 3\cdot 4 = 14.$

$\|v\| = \sqrt{1^2+4^2} = \sqrt{17}$

$\|\text{proj}(u,v)\| = 14/\sqrt{17}$

Then $\text{proj}(u,v) = \alpha \cdot v$, with α so that $\|\text{proj}(u,v)\| = 14/\sqrt{17}$.

$$\alpha = \frac{\|\text{proj}(u,v)\|}{\|v\|} = \frac{14/\sqrt{17}}{\sqrt{17}} = \frac{14}{17}$$

Therefore, $\text{proj}(u,v) = \frac{14}{17}\cdot v = \frac{14}{17}\cdot(1,4)$

$$\boxed{\text{proj}(u,v) = \left(\frac{14}{17}, \frac{4\cdot 14}{17}\right).}$$

Orthogonal complement of a projection in the plane.

Recall that the complement satisfies

$$u = \operatorname{proj}(u,v) + \underbrace{w}_{\text{complement}}.$$

$$w = u - \operatorname{proj}(u,v).$$

In our example: $u = (2,3)$, $v = (1,4)$

$$\operatorname{proj}(u,v) = \frac{14}{17}\cdot v = \left(\frac{14}{17}, \frac{56}{17}\right)$$

Thus

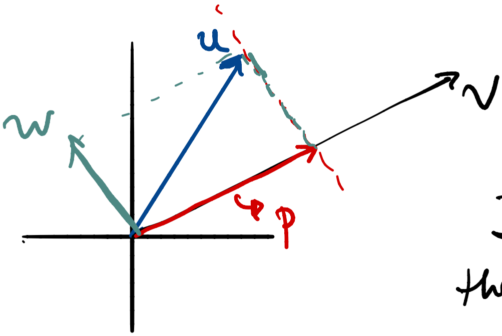
$$w = (2,3) - \left(\frac{14}{17}, \frac{56}{17}\right)$$

$$w = \left(\frac{34}{17}, \frac{51}{17}\right) - \left(\frac{14}{17}, \frac{56}{17}\right)$$

$$\boxed{w = \left(\frac{20}{17}, -\frac{5}{17}\right)}$$

Gram-Schmidt Process

This is a procedure to obtain an orthogonal set of vectors from an initial data set.



Initial pair: (u, v)

Replace it by: (w, v).

In the plane, compute the projection

$p = \text{proj}(u, v)$

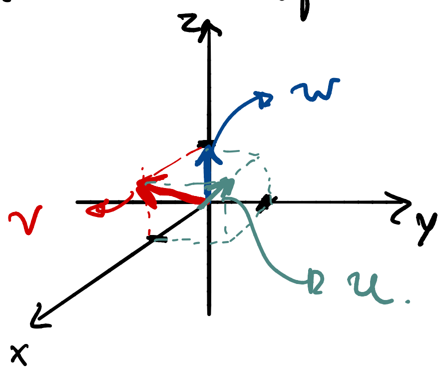
and subtract it from u.

Example: Gram-Schmidt in space.

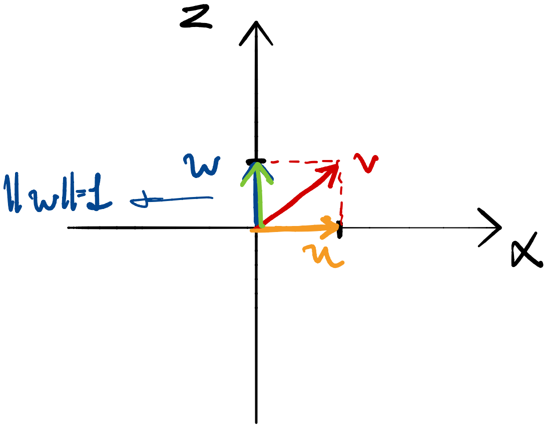
$u = (1, 1, 1)$

$v = (1, 0, 1)$

$w = (0, 0, 1)$.



Choose a reference vector: w.



Projecting v onto w:

$$\|\text{proj}(v, w)\| = \frac{v \cdot w}{\|w\|}$$

$$v \cdot w = (1,0,1)\cdot(0,0,1) = 1.$$

$$\text{proj}(v, w) = 1 \cdot w.$$

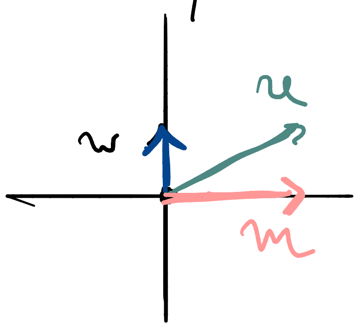
The complement, u, is

$$u = v - \text{proj}(v, w)$$

$$u = (1,0,1) - (0,0,1)$$

$$\boxed{u = (1, 0, 0).}$$

In the plane containing u, w:



Projecting u onto w:

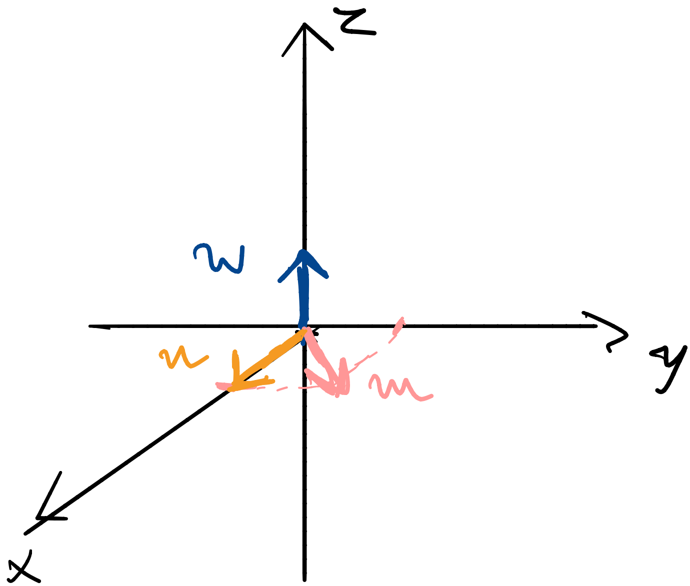
$$\|\text{proj}(u, w)\| = \frac{u \cdot w}{\|w\|}$$

$$u \cdot w = (1,1,1)\cdot(0,0,1) = 1.$$

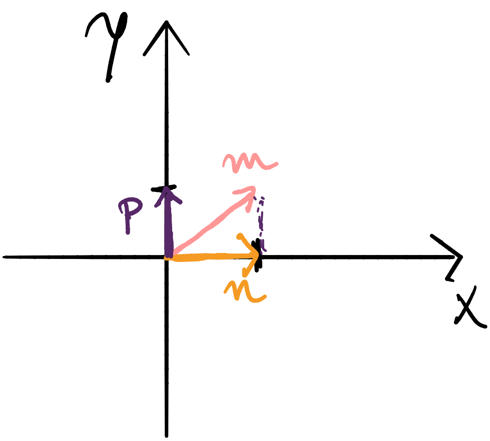
$\|\text{proj}(u, w)\| = 1, \to \text{proj}(u, w) = 1 \cdot w.$

$u = w + m \Rightarrow m = (1,1,1) - (0,0,1)$

$m = (1,1,0).$



We found m, n perpendicular to w, but not to each other.



Choose n as a reference.

$$\|proj(m,n)\| = \frac{m \cdot n}{\|n\|}$$

$$m \cdot n = (1,1,0) \cdot (1,0,0)$$
$$= 1$$

$$p = m - \text{proj}(m, n)$$
$$= (1, 1, 0) - (1, 0, 0)$$
$$= (0, 1, 0).$$

Final set of vectors:

$n = (1, 0, 0)$, $p = (0, 1, 0)$, $w = (0, 0, 1)$.