Spring 2020 MAT303 Recitations

Week of 4/20/20: Sections 4.1 and 4.2

Section 4.1: First-order systems and applications

A system of differential equation consists of a finite collection of differential equations in indeterminates $x_1(t), x_2(t), \cdots, x_n(t)$, depending on a parameter t. In this chapter, we will deal with linear systems.

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One's first encounter wiht systems of differential equations arises from higher-order scalar equations. One way of solving them is by reducing such equations to systems of first-order problems, as we shall see next. .

Section 4.1: First-order systems and applications

This example is extracted from problem 4.1.3 in our textbook. Consider the third-order equation

$$tx^{(3)} - 2t^2x^{''} + 3tx^{'} + 5x = \ln(t)$$

By using the substitutions $x_1 = x$, $x_2 = x^{'}$, $x_3 = x^{''}$, we can rewrite this equation as a system in three variables

$$\begin{aligned} tx_3^{'} - 2t^2x_3 + 3tx_2 + 5x_1 &= \ln(t) \\ x_2^{'} &= x_3 \\ x_1^{'} &= x_2 \end{aligned}$$

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To solve for y, we use the first equation in our system,

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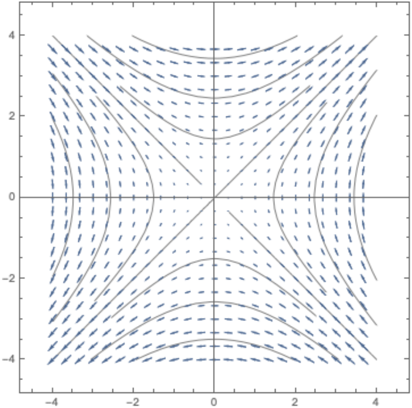
Below is a plot of the slope field for this equation, with several solution curves outlined.



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The characteristic polynomail of this equation is $r^2 + r - 6r = (r-2)(r+3)$, with roots $r = 2, r = -3$. The general solution takes the form

$$x(t) = Ae^{2t} + Be^{-3t}$$

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$$A + B = 1$$
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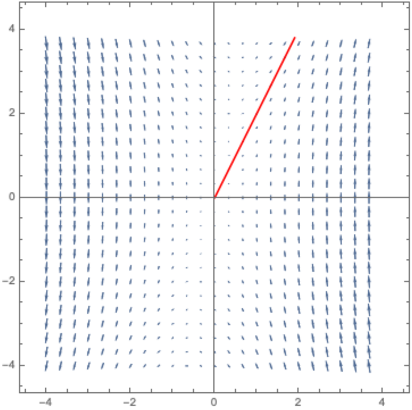
whose solutions are $A = 1, B = 0$. As a result, $x(t) = e^{2t}$, and

$$y(t) = x'(t) = 2e^{2t}$$

Section 4.1: First-order systems and applications

Below is a plot of the direction field, as well as the solution curve corresponding to the given initial condition (in red).



Section 4.2: The method of elimination

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We will rewrite the system by using the short-hand notation D to mean derivatives relative to time, $D = \frac{d}{dt}$,

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By adding the two equations we find

$$(D-1)(D+3)x + 8x = 0,$$

which in usual notation for derivatives is

$$\begin{aligned} (D-1)(D+3)x + 8x &= 0 \\ (D-1)(x' + 3x) + 8x &= 0 \\ (x' + 3x)' - (x' + 3x) + 8x &= 0 \\ x'' + 3x' - x' - 3x + 8x &= 0 \\ x'' + 2x' + 5x &= 0 \end{aligned}$$

This equation can be solved bby the method of characteristic equations, and I will leave the details to you. Its solutions take the form

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To find the solution $y(t)$ we may use the first equation of the system,

$$\begin{aligned}
y(t) &= -\frac{x' + 3x}{4} \\
&= -\frac{e^{-t}[(2B - A)\cos(2t) - (2A + B)\sin(2t)] + 3e^{-t}(A\cos(2t) + B}{4} \\
&= -\frac{e^{-t}[(A + B)\cos(2t) + (B - A)\sin(2t)]}{2}
\end{aligned}$$