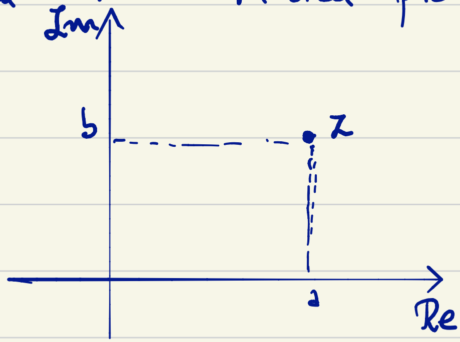
MAT 514 - Lecture 4

The Geometry of Complex Numbers

We introduced complex numbers as ordered pairs of real numbers.

$$z = (a, b) = a + b i.$$

In the Cartesian plane, we represent z by the point with coordinates (a, b).



In the context of complex numbers, the x and y-axis are called real and imaginary axis.

(Often referred to as the Argand-Gauss Plane).

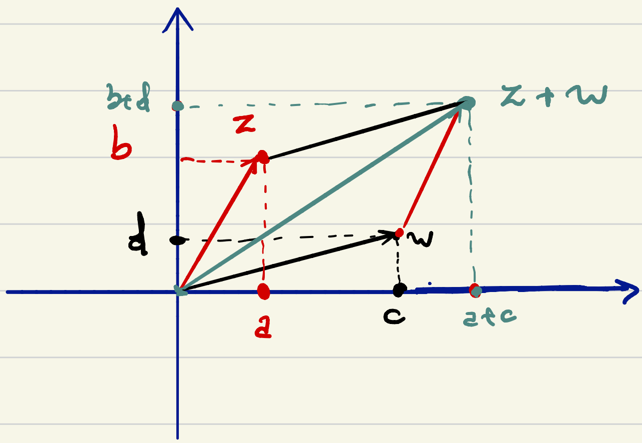
- Relation between geometric representation and algebraic operations.

→ Recall that addition of complex numbers works coordinate-wise: if

$$z = a + bi,$$
$$w = c + di$$

then

$$z + w = (a+c) + (b+d)i.$$



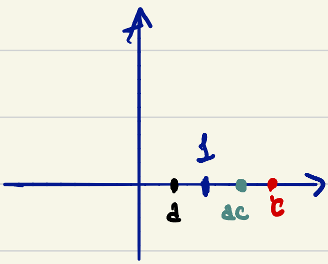
The geometric interpretation of addition of complex numbers is the parallelogram law.

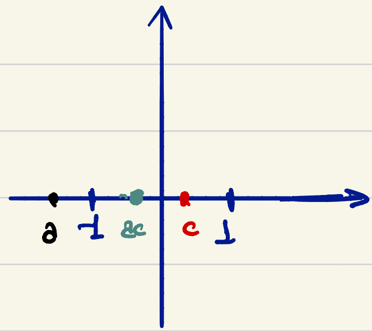
→ Recall that multiplication is given by
$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i.$$

We will interpret this on a case-by-case manner:

i) Multiplication of two real numbers.

$$(a+0i)(c+0i) = a\cdot c$$





Three interpretations:

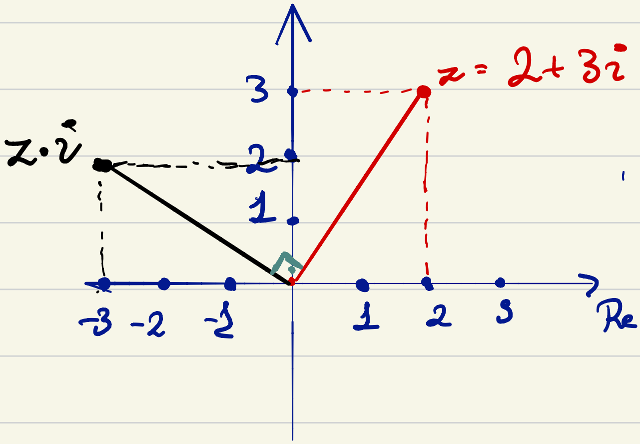
- contractions: multiplications by factors whose absolute value is less than 1.
- dilations: multiplications by factors whose absolute value is greater than 1.
- reflections: multiplications by negative real numbers.

ii) Multiplication by a real number.

Same interpretations as in case i)

Example: Multiplication by i

$$(a+bi)\cdot i = ai + bi^2$$
$$= -b + ai.$$



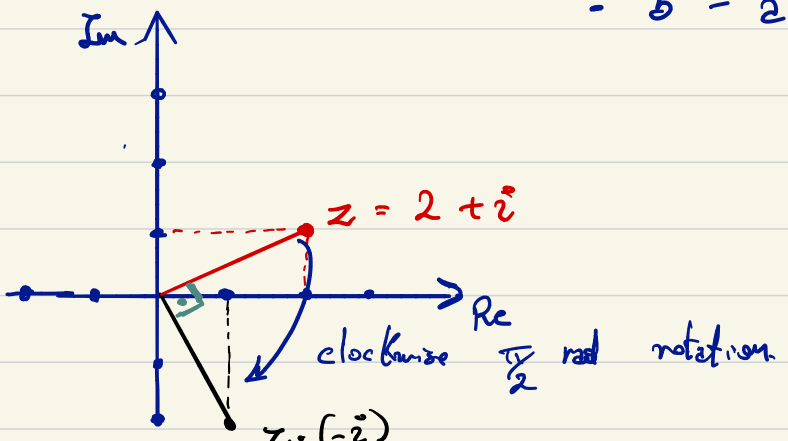
Multiplication by i can be interpreted as a counterclockwise rotation by $\pi/2$ radians (90 degrees).

Remark: Multiplication by -1 is a rotation by π radians. Two successive multiplications by i amount to a multiplication by

1. This is the geometric interpretation of $i^2 = -1$.

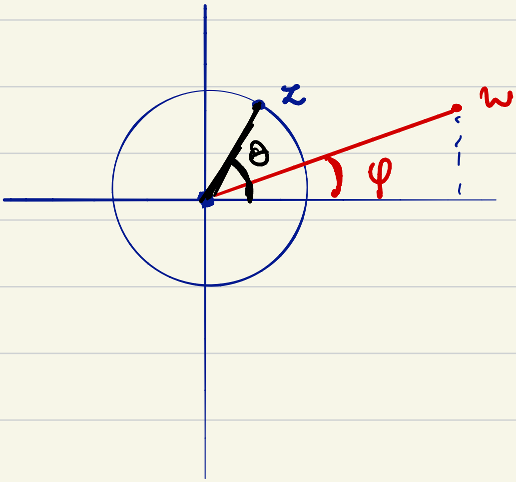
Exercise: convince yourself that multiplication by $(-i)$ is a clockwise rotation by $\pi/2$ radians.

$$(a+bi)(-i) = -ai - bi^2$$
$$= b - ai$$



iii) Multiplication by a unit-norm complex number.

→ Recall that the norm is $|a+bi| = \sqrt{a^2+b^2}$, where a, b are real and imaginary components of z, respectively.



Recall that

$$|z \cdot w| = |z| \cdot |w|$$
$$= 1 \cdot |w|$$
$$= |w|$$

In terms of the norms and polar angles,

$$z = |z| \cdot \cos\theta + |z| \cdot \sin(\theta) i$$
$$= \cos(\theta) + i \cdot \sin(\theta),$$

since we assumed $|z| = 1$. Meanwhile

$$w = |w| \cdot \cos(\varphi) + i|w| \cdot \sin(\varphi)$$
$$= |w| \cdot (\cos(\varphi) + i\sin(\varphi)).$$

These are what we call the polar forms of z and w.

$$z \cdot w = [\cos(\theta) + i\sin(\theta)] \cdot |w| \cdot [\cos(\varphi) + i\sin(\varphi)].$$

$$= |w| \cdot [\cos(\theta)\cos(\varphi) + i\cos\theta\sin(\varphi)$$
$$+ i\sin(\theta)\cos(\varphi) + i^2\sin(\theta)\sin(\varphi)].$$

$$= |w| \cdot [(\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi))$$
$$+ i(\cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi))]$$

$$= |w| \cdot [\cos(\theta + \varphi) + i\sin(\theta + \varphi)]$$

Multiplication by a unit-norm complex number is a rotation.

(iv) Multiplication by a non-unit complex number. This has two components

(a) Dilation / Contraction: multiplication by the norm

(b) Rotation: multiplication by the trigonometric component.

Exercise: Find the polar angle of

$$z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i.$$

Solutions The norm of z is:

$$|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$= \sqrt{1} = 1.$$

We should write

$$z = 1 \cdot (\cos(\varphi) + i \sin(\varphi))$$

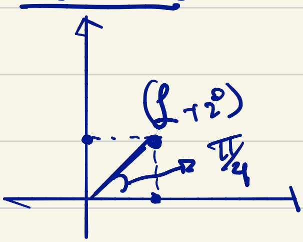
for $\varphi = \pi/4$,

$$\cos\varphi = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}},$$

$$\sin\varphi = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

Exercise: Find the polar angle of
$z = (1+i)^2$

Solution



$(1+i)$ has polar angle $\pi/4$,

so $z = (1+i)^2$ has polar angle
$\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$.

Alternatively,

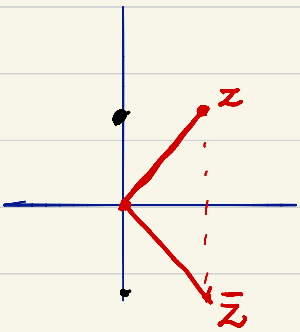
$$z = (1+i)^2 = 1 + 2i + i^2$$
$$= 1 + 2i - 1$$
$$= 2i.$$

$|z| = \sqrt{0^2 + 2^2} = \sqrt{4} = 2.$

Polar angle:

$$\cos(\varphi) = 0, \quad \sin(\varphi) = 1 \Rightarrow \varphi = \frac{\pi}{2}.$$

· Geometric interpretation of conjugation.



Conjugation acts by

$$z = a+bi \longrightarrow \bar{z} = a-bi.$$

As a geometric operation this is a reflection about the real axis.

In particular:

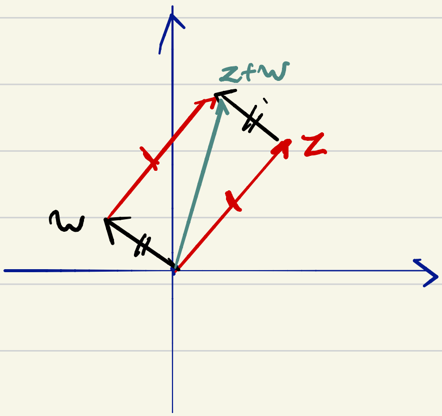
i) a number is <u>real</u> if and only if

$$\bar{z} = z$$

ii) a number is purely imaginary if and only if

$$\bar{z} = -z.$$

Interpretation of the triangle inequality.



$|z+w| < |z| + |w|$.

We can interpret $|z|$, $|w|$, $|z+w|$ as sides of a triangle in the parallelogram law.