MAT 514 - Summer II 2020
July 23, 2020

Solutions to Quiz 5

Problem 1 Use properties of complex exponentiation to verify the double angle formula

$$\cos(2z) = \cos^2(z) - \sin^2(z).$$

Solution: Recall that we described the complex cosine function in terms of exponentials as

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

It follows that

$$\begin{aligned}
\cos(2z) &= \frac{e^{2iz} + e^{-2iz}}{2} \\
&= \frac{(e^{iz})^2 + (e^{-iz})^2}{2} \\
&= \frac{2(e^{iz})^2 + 2(e^{-iz})^2}{4} \\
&= \left(\frac{(e^{iz})^2 + 2 + (e^{-iz})^2}{4}\right) + \left(\frac{(e^{iz})^2 - 2 + (e^{-iz})^2}{4}\right) \\
&= \left(\frac{(e^{iz})^2 + 2 + (e^{-iz})^2}{4}\right) - \left(\frac{(e^{iz})^2 - 2 + (e^{-iz})^2}{4i^2}\right) \\
&= \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 \\
&= \cos^2(z) - \sin^2(z).
\end{aligned}$$

Problem 2 Find an antiderivative of $\mathrm{Log}(z)$. Confirm your answer by differentiation.

Solution: We draw inspiration from Single-Variable Calculus and set $F(z) = z\mathrm{Log}(z) - z$, defined on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. To confirm that this is an antiderivative, we differentiate it,

$$\begin{aligned}
(z\mathrm{Log}(z) - z))' &= (z)'\mathrm{Log}(z) + z(\mathrm{Log})'(z) - 1 \\
&= \mathrm{Log}(z) + z\left(\frac{1}{z}\right) - 1 \\
&= \mathrm{Log}(z).
\end{aligned}$$