MAT 514 - Lecture 15

Applications of Cauchy's techiques

1) Using complex analysis to solve real integrals.

$$\int_0^{2\pi} [\cos^2(x) - \sin^2(x)]\,dx$$

Use the relations:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

To obtain

$$\cos^2(x) - \sin^2(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 - \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2$$

$$\cos^2(x) - \sin^2(x) = \frac{e^{2ix} + 2 + e^{-2ix}}{4} - \frac{(e^{2ix} - 2 + e^{-2ix})}{(-4)}$$

$$= \frac{e^{2ix} + \cancel{2} + e^{-2ix}}{4} + \frac{e^{2ix} - \cancel{2} + e^{-2ix}}{4}.$$

$$= \frac{2e^{2ix} + 2e^{-2ix}}{4}$$

$$= \frac{e^{2ix} + e^{-2ix}}{2}$$

$$= \cos(2x).$$

Use the substitution

$$z = e^{ix}$$

$$dz = ie^{ix}\,dx \to \frac{dz}{iz} = dx$$

Describing the domain of integration:

$$\gamma(x) = e^{ix}, \quad 0 \leq x \leq 2\pi,$$

i.e., the unit circle in counterclockwise orientation.

$$\int_0^{2\pi} [\cos^2(x) - \sin^2(x)\, dx] = \int_{C[0,1]} \left(\frac{z^2 + z^{-2}}{2}\right) \frac{dz}{iz}$$

Computing the complex integral

$$\int_{C[0,1]} \frac{z^2 + z^{-2}}{2iz}\, dz = \frac{1}{2i} \int_{C[0,1]} (z + z^{-3})\, dz.$$

- $$\int_{C[0,1]} z\, dz = 0,$$

as z is holomorphic in $D[0,1]$.

⇝ A consequence of Cauchy's formula:

$$\int_{C[0,1]} \frac{z^2}{z}\, dz = 2\pi i \cdot (0^2) = 0.$$

- z^{-3} is not holomorphic in $D[0,1]$ (in fact, it is not even defined at $z=0$).

Using the parametrization

$$\gamma(x) = e^{ix}, \quad 0 \le x \le 2\pi,$$

$$\int_{C[0,1]} z^{-3}\, dz = \int_0^{2\pi} e^{-3ix} \cdot (ie^{ix})\, dx$$

$$= \int_0^{2\pi} i e^{-2ix}\, dx.$$

$$= 0,$$

as $0, -4\pi$ are coterminal.

It follows that

$$\int_0^{2\pi} [\cos^2(x) - \sin^2(x)]\,dx = \int_{C[0,1]} \frac{z^2 - z^{-2}}{2iz}\, dz = 0.$$

2) Derivative formulas

Recall Cauchy's Integral Formula:

$$f(w) = \frac{1}{2\pi i} \int_{C[w,r]} \frac{f(z)}{z-w}\, dz.$$

The derivative formulas are:

$$f'(w) = \frac{1}{2\pi i} \int_{C[w,r]} \frac{f(z)}{(z-w)^2}\, dz.$$

$$f''(w) = \frac{1}{\pi i} \int_{C[w,r]} \frac{f(z)}{(z-w)^3}\, dz$$

More generally,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{C[w,r]} \frac{f(z)}{(z-w)^{n+1}}\, dz,$$

where $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$

Example 1:

$$\int_{C[0,1]} \frac{\sin(z)}{z}\, dz = 2\pi i \cdot \sin(0) = 0.$$

Example 2:

$$\int_{C[0,1]} \frac{\sin(z)}{z^2}\, dz$$

The integrand is not holomorphic in $D[0,1]$. In fact, it is singular at 0:

$$\lim_{z \to 0} \frac{\sin(z)}{z^2} \overset{L'H}{=} \lim_{z \to 0} \frac{\cos(z)}{2z} = \infty .$$

We can alternatively write:

$$\frac{\sin(z)}{z^2} = \frac{\left(\sin(z)/z\right)}{z},$$

in a form where the numerator is holomorphic in $D[0,1]$ and the denominator is singular at the center of $C[0,1]$.

By Cauchy's integral formula,

$$\int_{C[0,1]} \frac{\sin(z)}{z^2}\,dz = 2\pi i \cdot \text{"}\frac{\sin(0)}{0}\text{"}$$

$$= 2\pi i \cdot 1.$$

Used L'Hospital's rule.

We could, instead use Cauchy's formulas for derivatives.

$$\int_{C[0,1]} \frac{\sin(z)}{z^2}\,dz = 2\pi i \cdot [\sin'(0)]$$

$$= 2\pi i \cdot \cos(0)$$

$$= 2\pi i.$$

Example 3: $\displaystyle\int_{C[0,1]} \frac{\sin(z)}{z^3}\,dz$

We can't use Cauchy's Theorem: the integrand is not holomorphic. Writing $\displaystyle\frac{\sin(z)}{z^3} = \frac{\left(\sin(z)/z^2\right)}{z}$, we have an expression with the desired degree in the denominator, but whose numerator is not holomorphic in $D[0,1]$. Cauchy's Integral formula doesn't apply.

We can instead apply Cauchy's formula for the second derivative:

$$\int_{C[0,1]} \frac{\sin(z)}{z^3}\,dz = \pi i \cdot \sin''(0)$$

$$= -\pi i \cdot \sin(0)$$

$$= 0$$

<u>Example 4:</u> $\displaystyle\int_{C[0,1]} \frac{\cos(z)}{z^3}\,dz$

The integrand is not holomorphic, Cauchy's Theorem cannot be applied.

If we rewrite

$$\frac{\cos(z)}{z^3} = \frac{\left(\cos(z)/z^2\right)}{z}$$

we find an expression whose denominator is singular of order 1, but whose numerator is not holomorphic on $D[0,1]$.

We can, however, apply Cauchy's formula for the second derivative:

$$\int_{C[0,1]} \frac{\cos(z)}{z^3}\,dz = \pi i \cdot \cos''(0)$$

$$= \pi i \cdot (-\cos(0))$$

$$= -\pi i.$$