MAT 303 RECITATIONS: WEEK 10

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Section 3.2: General solutions of linear equations

In the first few problems of Homework 6 you're tasked with finding coefficients of a complementary solution to an inhomogeneous equation of type

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots p_{n-1}(x)y^{'} + p_n(x) = f(x).$$

Solutions to such equations are obtained by combining a *particular solution*, y_p, to solutions y_c of the associated homogeneous equation,

$$y(x) = y_c(x) + y_p(x).$$

The example below is extracted from problem 3.2.23 in our textbook.

Example 1.

$$\begin{cases} y'' - 2y' - 3y & = 6, \\ y(0) & = 3, \\ y'(0) & = 11. \end{cases}$$

You are given the forms of complementary and particular solutions,

$$y_c = c_1 e^{-x} + c_2 e^{3x}; \quad y_p = -2$$

Combining these with the initial conditions, a solution to the problem is obtained by solving the algebraic system

$$\begin{cases} c_1 + c_2 & = 5, \\ -c_1 + 3c_2 & = 11. \end{cases}$$

The solutions are $c_1 = 1, c_2 = 4$*, thus* $y(x) = e^{-x} + 4e^{3x} - 2$.

Problem 3.2.33 concerns the Wronskian criterion, which we recall below:

Theorem 1. *Let* $y_1, y_2, \cdots, y_n$ *be functions defined on an interval* I*, <u>assumed to solve an n-th order homogeneous, linear differential equation, with continuous coefficients</u>*

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots p_{n-1}(x)y^{'} + p_n(x) = 0.$$

Then either

(a) Their Wronskian is identically zero on I, in which case the functions are L.D, or
(b) Their Wronskian is nowhere zero on I, and the funcions are L.I.

The following example, extracted from problem 3.2.30, illustrates well the subtleties of this criterion.

Example 2. *Consider the equation*

$$x^2y^{''} - 2xy^{'} + 2y = 0.$$

It is easy to verify that the functions $y_1(x) = x, y_2(x) = x^2$ *are solutions on* $\mathbb{R}$. *These functions are independent, as if the combination* $y(x) = ax + bx^2$ *is null both coefficients must vanish. Meanwhile,*

$$W(y_1, y_2) = \det \begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix} = 2x^2 - x^2 = x^2,$$

a function which vanishes when $x = 0$. *Why is this not a contradiction?*

Warning: *the Wronskian criterion applies only to* <u>*solutions*</u> *of equations with* <u>*continuous coefficients*</u>, *when written in* <u>*standard form*</u>. *The standard form of the previous equation is*

$$y^{''} - \frac{2}{x}y^{'} + \frac{2}{x^2}y = 0,$$

discontinuous at $x = 0$, *precisely where the criterion fails.*

As a consequence there is no linear, homogeneous, second-order differential equation defined on $\mathbb{R}$ *admitting* x *and* x^2 *as solutions! Think about problem 3.2.33 in this context. Why is the Wronskian criterion applicable to the three functions involved? Can you think of a differential equation whose solutions are given by such functions?*

Section 3.3: Homogeneous Equations with Constant Coefficients

The method of characteristic equations assigns to an equation of type

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots a_1 y^{'} + a_0 = 0$$

its characteristic polynomial:

$$p(r) = a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0,$$

where each a_i is a constant, $a_n \neq 0$,

To each (possibly complex) root r of the polynomial, there corresponds a solution of the differential equation of the form e^{rx}. If the root has a multiplicity, then new solutions may be generated by monomial multiplication: $xe^{rx}, x^2e^{rx}, \cdots, x^{m-1}e^{rx}$, where m is the multiplicity.

The following example is extracted from problem 3.3.12 in our textbook.

Example 3.

$$y^{(4)} - 3y^{(3)} + 3y^{''} - y^{'} = 0.$$

Its characteristic polynomial is

$$p(r) = r^4 - 3r^3 + 3r^2 - r = r(r-1)^3.$$

The roots are $r_1 = 0$, *with multiplicity one, and* $r_2 = 1$, *with multiplicity three. The general solution of the equation takes the form*

$$y(x) = A + Be^x + Cxe^x + Dx^2e^x.$$

The next example is a slight modification of problem 3.3.23.

Example 4. *Consider the equation*

$$y'' - 8y' + 25y = 0,$$

with initial conditions $y(0) = 1, y'(0) = 2$.

The characteristic equation of the problem is

$$r^2 - 8r + 25 = 0,$$

and its solutions are the complex numbers $r_1 = 4 + 3i$, $r_2 = 4 - 3i$. *The general real solution of the D.E. takes the form*

$$y(x) = e^{4x}(A\cos(3x) + B\sin(3x)).$$

To find the specific values of A *and* B*, we impose initial conditions. Since*

$$y'(x) = 4y(x) - 3e^{4x}(A\sin(3x) - B\cos(3x)),$$

we have at $x = 0$*:*

$$\begin{aligned} A &= 1 \\ 4A + 3B &= 2, \end{aligned}$$

thus the desired solution is

$$y(x) = e^{4x}\left(\cos(3x) - \frac{2}{3}\sin(3x)\right).$$

In problems 3.3.44 and 3.3.46, you are confronted with differential equations with *complex coefficients*. In this case the roots of the characteristic polynomial need not be conjugates to each other, as the following example (problem 3.3.45) shows.

Example 5. *The differential equation*

$$y'' - 2iy' + 3y = 0$$

has characteristic equation $r^2 - 2ir + 3 = 0$, *whose solutions are* $-i$ *and* $3i$. *The complex-valued solutions to the D.E. are:*

$$y_1(x) = e^{-ix} = \cos(x) - i\sin(x),$$

and

$$y_2(x) = e^{3ix} = \cos(3x) + i\sin(3x).$$

In problem 3.3.52 you are tasked with solving an Euler equation by means of a logarithmic substitution. Let's see this procedure in practice by solving problem 3.3.53.

Example 6. *Consider the equation*

$$x^2y'' + 7xy' + 25y = 0.$$

Using the substitution $v = \log(x), x > 0$, *we get*

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dx^2}$$

Rephrasing the original equation in terms of v *yields*

$$\frac{d^2y}{dv^2} + 6\frac{dy}{dv} + 25y = 0,$$

whose solutions (obtained via characteristic equation) take the form

$$y(v) = e^{-3v}(A\cos(4v) + B\sin(4v)),$$

or, in terms of x,

$$y(x) = x^{-3}(A\cos(4\log(x)) + B\sin(4\log(x))).$$

References

[1] C. Henry Edwards, David E. Penney and David T. Calvis, *Differential Equations and Boundary Value Problems: Computing and Modelling*, 5th edition, Pearson, 2014.

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