MAT 511 - Summer 1
June 20, 2019

This homework is due on Friday, June 28, by 7:00 pm.

Homework 4

Exercise 1 In what follows, determine whether the relations are functions or not.
(a) $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} | 2x^2 - y = 1\}$.

(b) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | x^2 + y = 2\}$.

(c) $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} | xy \text{ is even}\}$.

(d) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x^2 = y\}$.

(e) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x = y^2\}$.

Exercise 2 Describe the images of the following functions
(a) $f : \mathbb{N} \longrightarrow \mathbb{N}$, given by
$$f(n) = 2n$$

(b) $g : \mathbb{Z} \longrightarrow \mathbb{Z}$, given by
$$g(z) = z^2$$

(c) $h : \mathbb{R} \longrightarrow \mathbb{R}$, given by
$$h(x) = x^3$$

(d) $\sin : [0, \pi] \longrightarrow \mathbb{R}$.

(e) $\exp : \mathbb{R} \longrightarrow \mathbb{R}$, given by exponentiation, $\exp(x) = e^x$.

Exercise 3 In each of the problems below, you are given a pair of functions, whose composition $g \circ f$ cannot be defined on the entire domain of f. Describe the largest subset of the domain of f for which the composition is a function.
(a) The functions $f : \mathbb{N} \longrightarrow \mathbb{N}$, given by
$$f(n) = 2n,$$
and $g : \{m \in \mathbb{N} | \ m \text{ is divisible by } 3\} \longrightarrow \mathbb{N}$, given by
$$f(m) = m/3$$

(b) The functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, given by

$$f(x) = x^2,$$

and $g : \{y \in \mathbb{R} | y \leq 9\} \longrightarrow \mathbb{R}$, given by

$$g(y) = \sqrt{9 - y}.$$

(c) The functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, given by

$$f(x) = \frac{x}{x^2 + 1},$$

and the natural logarithm, $\ln : \{y \in \mathbb{R} | 0 < y\} \longrightarrow \mathbb{R}$.

Exercise 4 Find sets A, B and C, and functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ such that

(a) f is surjective, but $g \circ f$ is not surjective.

(b) g is surjective, but $g \circ f$ is not surjective.

(c) $g \circ f$ is surjective, but f is not surjective.

(d) f is injective, but $g \circ f$ is not injective.

(e) g is injective, but $g \circ f$ is not injective.

(f) $g \circ f$ is injective, but g is not injective.

Exercise 5 For each of the bijections below, find the inverse function. Verify your answer by computing the composite of the function and its inverse.

(a) $f : (0, \infty) \longrightarrow (0, \infty)$, given by

$$f(x) = \frac{1}{x}.$$

(b) $g : (-2, \infty) \longrightarrow (-\infty, 4)$, given by

$$g(x) = \frac{4x}{x + 2}.$$

(c) $h : \mathbb{R} \longrightarrow (0, \infty)$, given by

$$h(x) = e^{x+3}$$

(d) $i : (3, \infty) \longrightarrow (5, \infty)$, given by

$$i(x) = \frac{5(x - 1)}{x - 3}.$$

Exercise 6 For each of following statements about cardinality of sets, determine whether the statement is true or false. If true, prove it using the definitions of finiteness and infiniteness given in class. If false, disprove itby exhibiting a counter-example.

(a) If A is finite and $B \subset A$, then B is finite.

(b) if A is infinite and $B \subset A$, then B is infinite.

(c) If A is infinite and $A \subset B$, then B is infinite.

(d) If A is infinite and B is finite, then $A - B$ is infinite.

(e) if A is infinite and B is infinite, then $A - B$ is finite.

(f) If $A \cup B$ is infinite, then A or B is infinite.

(g) If $A \cap B$ is finite, then A or B is finite.

Exercise 7 Show that the following characterizations of a finite set A are equivalent:

- P: "any injective function $f : A \longrightarrow A$ is surjective"
- Q: "any surjective function $f : A \longrightarrow A$ is injective".

Exercise 8 You have been promoted to junior manager of The Grand Hilbert Hotel, a hotel with a countably infinite number of rooms. Your task is simple: keep the hotel full, but make sure there is always room for more guests. Luckily, as you start your first shift, the hotel is fully booked: there is one guest in every room. Describe a way to accomodate new guests in the following scenarios (remember: upper management does not want empty rooms!).

(a) A bus containing 40 guests arrives.

(b) A countably infinite caravan of buses arrives, each bus containing 40 guests.

(c) A countably infinite horde of caravans arrives, each containing countably infinitely many buses, each of which contains 40 guests.