Homework 2: solutions to selected problems

Exercise 1 Sketch the following plane curves:

(a) $r(t) = (t, t^2)$.

(b) $r(t) = (\cos(t), \sin(t))$.

(c) $r(t) = (t^3 - 4t, t^2 - 4)$.

(d) $r(t) = (t^3, t^2)$.

(e) $r(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$.

Solution

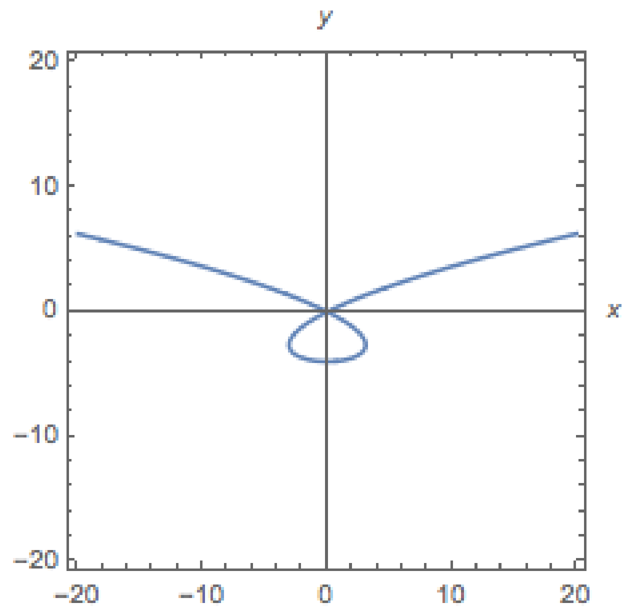
(a) The relationship $y = x^2$ characterizes the trace of this curve as a parabola.

(b) The relationship $x^2 + y^2 = 1$ characterizes the trace of this curve as a circle centered at the origin, with radius 1.

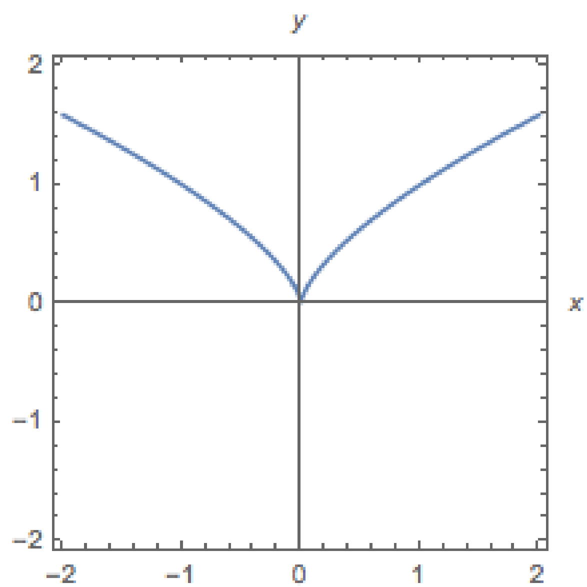
(c) The relationship between the coordinates is that x and y are related by multiplication by t, that is $t = \frac{x}{y}$ (as long as $y \neq 0$) The coordinate y can be expressed by $y = t^2 - 4$. Thus, the parametric equations become

$$y = \left(\frac{x}{y}\right)^2 - 4$$
$$y^3 = x^2 - 4y^2.$$

The last equation makes sense when $y = 0$. You should check that the only valuee of t that makes the y coordinate equal to 0 in the parametric equations are $t = -2, t = 2$, whose corresponding points are $r(-2) = (0, 0)$, $r(2) = (0, 0)$, the same as the output of the non-parametric equations when $y = 0$. A plot of the curve defined by this equation is represented below.



(d) The coordinates satisfy the relation $y^3 = x^2$. One can view this as the graph of the function $y(x) = x^{\frac{2}{3}}$. A plot of this graph is represented below (notice the unusual position of the axes).



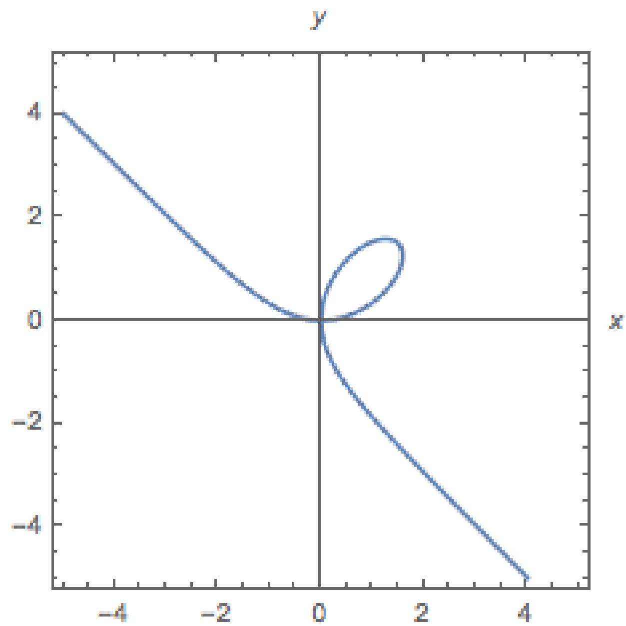
(e) To find a relation between the coordinates that does not involve t will require a bit more work than part (c), but can be done in a similar fashion. Notice that $t = \frac{y}{x}$ (as long as $x \neq 0$) this time, thus

$$x = \frac{3\frac{y}{x}}{1 + \left(\frac{y}{x}\right)^3}$$
$$x = \frac{3y(x^3)}{x(x^3 + y^3)}$$
$$x = \frac{3x^2 y}{x^3 + y^3}$$

Thus the parametric equations are equivalent to

$$x^3 + y^3 - 3xy = 0,$$

an equation which makes sense (and produces the correct output) when $x = 0$. A plot of the curve defined by this equation is represented below.



Exercise 2 For each of the plane curves below, find the parametric equation of their tangent lines at the points indicated.

(a) $r(t) = (t, t^3)$ at the point $(2, 8)$.

(b) $r(t) = (\cos(t), \sin(t))$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Solution

(a) The tangent vector to the curve at t is given by $(r'(t) = (1, 3t^2)$. At the point $(2, 8)$, corresponding to $t = 2$, this tangent vector becomes $r'(2) = (1, 12)$. The tangent line at the point $(2, 8)$ can thus be described by the parametric equation

$$P(s) = (2 + s, 8 + 12s)$$

(b) Similar to part (a).

Exercise 3 Consider the curves $r(t) = (t, t, t^2)$ and $s(t) = \left(\frac{1}{t}, \frac{1}{t}, 0\right)$, defined for $t \neq 0$.

(a) Does the curve r have a limit as t goes to 0? If so, what is the limit?

(b) Does the curve s have a limit as t goes to 0? If so, what is the limit?

(c) Compute the dot product of the curves, $r(t) \cdot s(t)$, for $t \neq 0$.

(d) Does the scalar function obtained in part (c) have a limit as t goes to 0? If so, what is this limit?

Solution

(a) The curve $r(t)$ tends to $(0, 0, 0)$ as $t \to 0$.

(b) The curve $s(t)$ does not have a limit as $t \to 0$, as the limits of its x and y coordinates are undefined.

(c) The dot product of the curves is equal to 2, for all $t \neq 0$.

(d) The limit of the dot product as t approaches 0 is equal to 2.

Exercise 4 An object moves in the plane according to a trajectory described by a smooth curve $r(t)$. Assume that:

1. the curve never passes through the origin, i.e., $r(t) \neq 0$;
2. the velocity vector is never zero, $r'(t) \neq 0$;
3. at time $t = 0$, the curve is at its closest point to the origin.

Explain why the position and velocity vectors $r(0)$ and $r'(0)$ are perpendicular.

Solution The distane between the object and the origin at time t is given by the magnitude of its position vector, $||r(t)||^2$. Minimizing this function and its square are equivalent problems, so we are going to study the latter in order to reduce the computations.

The square of the lentgh of a vector can be computed in terms of the dot product, $||r(t)||^2 = r(t) \cdot r(t)$. Since this scalar function has a minimum at $t = 0$, its derivative at $t = 0$ is equal to 0. By the product rule for the dot product, this implies that

$$\begin{aligned} \frac{d(||r(t)||^2)}{dt}(0) &= \frac{d[r(t) \cdot r(t)]}{dt}(0) \\ 0 &= 2r(0) \cdot r'(0), \end{aligned}$$

hence $r(0) \cdot r'(0) = 0$, that is, position and velocity vectors are perpendicular at $t = 0$.

Exercise 5 Let $r(t)$ denote a spatial curve, and $r'(t)$, $r''(t)$ its first and second derivatives, respectively. Assume that $r''(t) \neq 0$. If the position $r(t)$ and acceleration $r''(t)$ are colinear, for all times, what can you say about the cross product $r(t) \times r'(t)$?

Solution If r and r'' are aligned for all t, then $r(t) \times r''(t) = 0$, for all t. From the product formula for the cross product,

$$\begin{aligned}\frac{d[r(t) \times r'(t)]}{dt} &= r'(t) \times r'(t) + r(t) \times r''(t) \\ &= r(t) \times r''(t) \\ &= 0,\end{aligned}$$

hence the cross product $r(t) \times r'(t)$ is constant.

Exercise 6 As we saw in class, the Fundamental Theorem of Calculus for Curves can be used to compute the displacement vector,

$$\int_a^b r'(t)dt = r(b) - r(a).$$

Use this to describe the trajectory described by a curve with velocity vector

$$r'(t) = \frac{1}{1+t^2} i + tj + e^t k,$$

and which satisfies $r(0) = (1, 0, -1)$.

Solution The trajectory is described by

$$\begin{aligned}r(t) &= r(0) + \int_0^t r'(s)ds \\ &= i - k + \left(\int_0^t \frac{1}{1+s^2} ds\right) i + \left(\int_0^t s ds\right) j + \left(\int_0^t e^s ds\right) k \\ &= (\arctan(t) + 1)i + \frac{t^2}{2} j + e^t k\end{aligned}$$

Exercise 7 Find two vector functions $r(t)$ and $s(t)$ for which

$$\int [r(t) \times s(t)]dt \neq \left(\int r(t)dt\right) \times \left(\int s(t)dt\right)$$

Solution There are infinitely many right answers to this problem. One of them is $r(t) = ti$,

$s(t) = tj$, as you can easily verify by yourself.

Exercise 8 Consider the function

$$f(x,y) = \frac{x^2 y}{x^4 + y^2},$$

defined for all points (x, y) in the plane except the origin $(0, 0)$. This exercise will study the behavior of this function near the origin.

(a) Compute the directed limits of this function along the lines $y = kx$, in terms of the parameter k.

(b) Compute the limit of this function along the parabola $y = x^2$.

The results of parts a and b should be different. This is to show you that unlike what you studied in single-variable Calculus and what we observed for vector-valued, single-variable functions, the existence and coincidence of directed limits no longer imply the existence of the limit of the function. This marks a sharp contrast between single-variable functions and multivariable functions.

Solution

(a) Replacing y by kx and computing the resulting limit with L'Hôpital's rule yields limit 0, regardless of k.

(b) This time replacing y by x^2 and computing the resulting limit yields the result $\frac{1}{2}$.

Exercise 9 This exercise is about the following function:

$$f(x,y) = \begin{cases} \frac{y}{x} - y & \text{if } 0 \le y < x \le 1 \\ \frac{x}{y} - x & \text{if } 0 \le x < y \le 1 \\ 1 - x & \text{if } 0 < x = y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Sketch the graph of this function on the square $[0, 1] \times [0, 1]$.

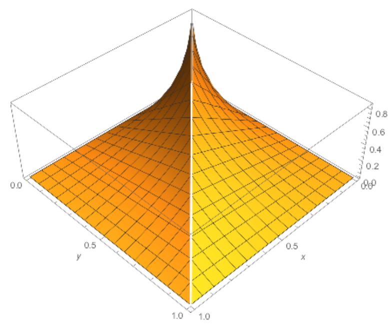
(b) What is the value of this function along the boundary of the square?

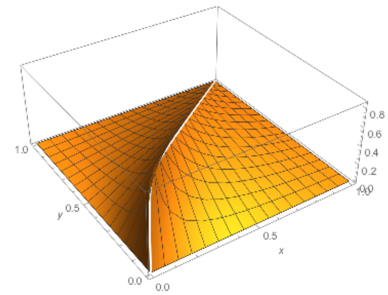
(c) If the value of x is kept constant, is f a continuous function of y?

(d) If the value of y is kept constant, is f a continuous function of x?

(e) Compute the directed limit of the function as (x, y) approaches the origin along the line $y = x$.

(f) Compare your answer of part (e) with the value of the function at $(0, 0)$ which you obtained in part (b). Is this function continuous at $(0, 0)$?





Solution

(a) Below are two plots of the function (notice the slightly unusual position of the axes). The region outside the square $[0,1] \times [0,1]$ was excluded from view, as it is not relevant to the problem at hand.

(b) The value of the function on the boundary of the square is 0, as it can easily be verified by checking the equations defining the function.

(c) The function is continuous relative to the variable y. To verify this, notice that the pieces defining the function are the same when $y = x$, the only point at which a discontinuity could appear within the square. Outside the square, the value of the function is 0, the same as the value on the boundary, thus there is no discontinuity there either.

(d) Analogous to part (c).

(e) The value of the function along the line segment $y = x$, $0 < x < 1$, is given by $1 - x$. The directed limit as x approaches 0 from the positive side is equal to 1. Meanwhile, if x is negative, then the value of the function is zero (since the corresponding point (x, y) is outside of the square), so the directed limit as x approaches 0 from the left is 0. It follows that the function does not have a limit at 0 along the line $y = x$.

(f) The value of the function at $(0,0)$ is 0, in contrast with the fact that the limit of the function as (x, y) approaches zero does not exist. This is an example of a function which is separately continuous with respect to each variable, but not continuous in the multivariable sense.

Exercise 10 Compute **all** the partial derivatives of the function $f(x, y, z) = ye^x + x\ln(z^2+1)$ up to order two.

Solution This can be done by sucessively computing single-variable derivatives. Your answers should be:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= ye^x + \ln(z^2+1) \\
\frac{\partial f}{\partial y} &= e^x \\
\frac{\partial f}{\partial z} &= \frac{2xz}{z^2+1} \\
\frac{\partial^2 f}{\partial x^2} &= ye^x \\
\frac{\partial^2 f}{\partial y^2} &= 0 \\
\frac{\partial^2 f}{\partial z^2} &= \frac{2x-4xz^2}{(z^2+1)^2} \\
\frac{\partial^2 f}{\partial x \partial y} &= e^x \\
\frac{\partial^2 f}{\partial x \partial z} &= \frac{2z}{z^2+1} \\
\frac{\partial^2 f}{\partial y \partial z} &= 0.
\end{aligned}$$

Exercise 11 Verify that the functions given satisfy the corresponding equations.

(a) The function $f(t,x) = \sin(x-t)$ satisfies *the wave equation*

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}.$$

(b) The function $f(t,x) = e^{-t}\cos(x)$ satisfies *the heat equation*

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

(c) The function $f(x,y) = e^x \sin(y)$ satisfies *Laplace's equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Solution Simple computation.