MAT 514 - Summer II 2020
July 30, 2020

Solutions to Long Quiz 3

The objective of this quiz is to use methods of complex integration to solve a real integral,

$$\int_0^{2\pi} \frac{1}{2+\sin(\phi)}\, d\phi.$$

The problems below will guide you through the solution.

Problem 1 By expressing the sine function as a combination of complex exponentials, rewrite the integrand as a function of $e^{i\phi}$.

Solution: By means of the expression

$$\sin(\phi) = \frac{e^{i\phi}-e^{-i\phi}}{2i},$$

we may rewrite the integrand in terms of complex exponentials as

$$\begin{aligned} \frac{1}{2+\sin(\phi)} &= \frac{1}{2+\frac{e^{i\phi}-e^{-i\phi}}{2i}} \\ &= \frac{2i}{4i+e^{i\phi}-e^{-i\phi}}. \end{aligned}$$

Problem 2 Use the substitution $z = e^{i\phi}$ to turn the real-valued integral into a line integral of a rational function in the complex variable z. Your answer should take the form

$$\int_{C[0,1]} \frac{A}{p(z)}\, dz,$$

where A is a constant and $p(z)$ is a quadratic polynomial on z.

Solution: Using the suggested substitution, as well as the relation $dz = ie^{i\phi}d\phi$, or equivalently $d\phi = \frac{1}{iz}dz$, we find that the integral may be written as

$$\int_{C[0,1]} \left(\frac{2i}{4i+z-\frac{1}{z}}\right)\left(\frac{1}{iz}\right) dz = \int_{C[0,1]} \frac{2}{z^2+4iz-1}\, dz.$$

Problem 3 Factor the polynomial $p(z)$ found in problem 2. Write the integrand from problem 2 as a sum of partial fractions whose denominators have degree one.

Solution: Using the quadratic formula, we may factor the denominator from problem 2 as

$$z^2 + 4iz - 1 = (z + i(2 + \sqrt{3}))(z - i(\sqrt{3} - 2))$$

A partial fractions decomposition of the integrand in problem 2 takes the form

$$\frac{2}{z^2 + 4iz - 1} = \frac{A}{z + i(2+\sqrt{3})} + \frac{B}{z - i(\sqrt{3}-2)},$$

where A and B are complex constants satisfying the following system,

$$\begin{aligned} A + B &= 0, \\ (2 - \sqrt{3})iA + (2 + \sqrt{3})iB &= 2. \end{aligned}$$

A solution to this system may be easily found by elimination, $A = \frac{i\sqrt{3}}{3}$, $B = -\frac{i\sqrt{3}}{3}$, thus

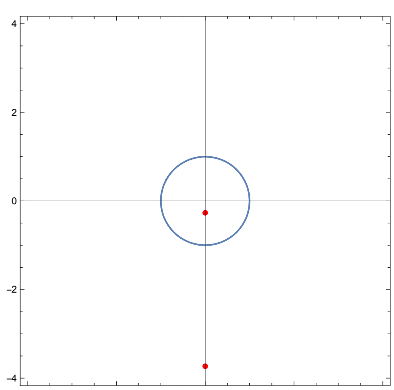
$$\frac{2}{z^2 + 4iz - 1} = \frac{i\sqrt{3}}{3z + 3i(2+\sqrt{3})} - \frac{i\sqrt{3}}{3z - 3i(\sqrt{3}-2)}.$$

Problem 4 Use Cauchy's Theorem and Cauchy's integral formula to solve the integral

$$\int_0^{2\pi} \frac{1}{2 + \sin(\phi)}\, d\phi.$$

via the method developed in problems 1 through 3.

Solution: Below is a plot of the two roots of the polynomial $p(z)$ in the complex plane, as well as the path of integration.



We see that one of the roots is contained in the disk bounded by the domain of integration, namely $(\sqrt{3}-2)i$, while the other root $-(2+\sqrt{3})i$, is located outside the disk. We may, therefore, simplify one of the integrals of the partial fractions decomposition by means of Cauchy's Theorem,

$$\int_{C[0,1]} \frac{i\sqrt{3}}{3z+3i(2+\sqrt{3})}\,dz = 0.$$

The second component in the partial fractions decomposition is not amenable to this trick. We need to use a homotopy argument and Cauchy's integral formula. Using a homotopy of translations, we may change the path of integration,

$$\int_{C[0,1]} \frac{i\sqrt{3}}{3z-3i(\sqrt{3}-2)}\,dz = \int_{C[(\sqrt{3}-2)i,1]} \frac{\frac{i\sqrt{3}}{3}}{z-i(\sqrt{3}-2)}\,dz$$

The value of the latter integral is

$$\int_{C[(\sqrt{3}-2)i,1]} \frac{\frac{i\sqrt{3}}{3}}{z-i(\sqrt{3}-2)}\,dz = 2\pi i\left(\frac{i\sqrt{3}}{3}\right) = -\frac{2\pi\sqrt{3}}{3}$$

according to Cauchy's Integral Formula. In summary,

$$\begin{aligned}\int_0^{2\pi} \frac{1}{2+\sin(\phi)}\,d\phi &= \int_{C[0,1]} \frac{i\sqrt{3}}{3z-3i(\sqrt{3}-2)}\,dz - \int_{C[(\sqrt{3}-2)i,1]} \frac{\frac{i\sqrt{3}}{3}}{z-i(\sqrt{3}-2)}\,dz \\ &= \frac{2\pi\sqrt{3}}{3}.\end{aligned}$$