MAT 514 - Lecture 3

Basic topology of the plane.

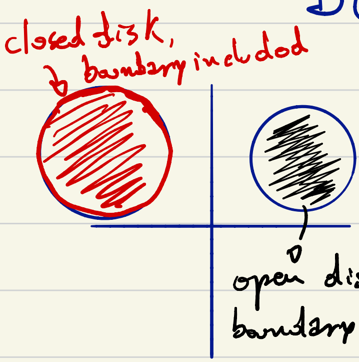
Roughly, topology is a study of "geometry of relative position".

Notation: we denote by

$$D(P, r)$$

the open disk centered at P, with radius r in the plane.

$$D(P,r) := \{ Q \in \mathbb{R}^2 \mid \operatorname{dist}(P,Q) < r \}.$$

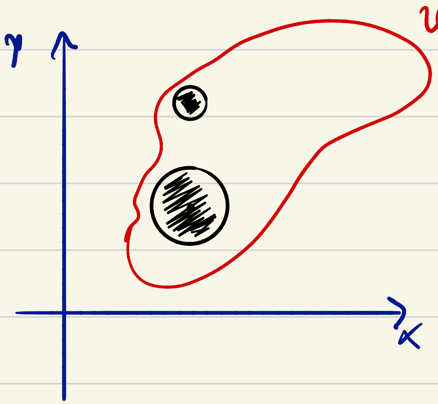


The closed disk with center P and radius r is denoted by

$$\overline{D(P,r)} = \{ Q \in \mathbb{R}^2 \mid \operatorname{dist}(P,Q) \leq r \}.$$

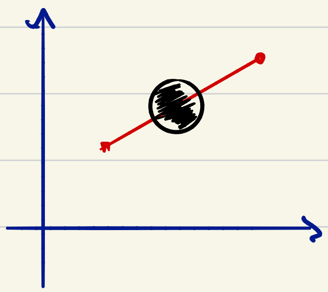
Definition: A set $U \subset \mathbb{R}^2$ is called open if given any of its points, $P \in U$, there exists a positive number r such that

$D(P, r) \subset \mathcal{U}$.



<u>Remark</u>: the choice of radius can, and in most cases will, depend upon the chosen point.

<u>Example 1</u>: a non-open set.



A line segment is not open. Regardless of the chosen point and radius, a disk will not fit into the segment.

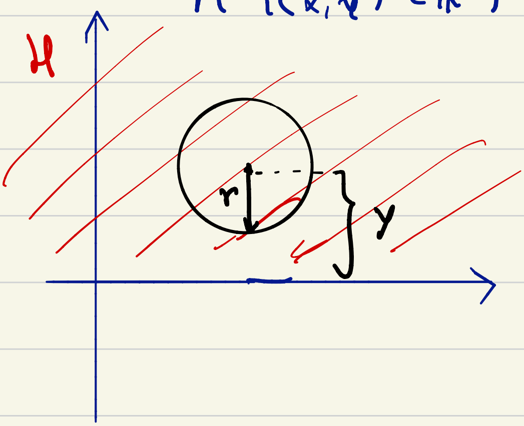
<u>Example 2</u>: the entire plane is open

Given any point in $\mathbb{R}^2$, the disk with radius 1 centered at the point is contained in $\mathbb{R}^2$.

Example 3: the empty set is open, by vacuity, there is no point in ϕ to violate the openness condition.

Example 4: the upper half-plane, H, defined by

$$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}.$$

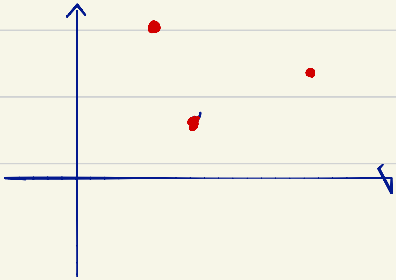


Given any point $P = (x, y)$ on the upper half-plane, the disk $D(P, r)$ fits with H so long as

$$r < y,$$

therefore H is open.

Example 5: Isolated points do not form an open set.



$U = \{P_1, P_2, P_3, \ldots\}$

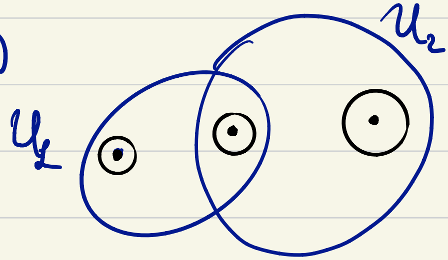
No disk fits within this set.

Properties of open sets:

1) ϕ is open
2) $\mathbb{R}^2$ is open
3) If $\mathcal{U}_1, \mathcal{U}_2$ are open, then $\mathcal{U}_1 \cup \mathcal{U}_2$ is open.
4) If $\mathcal{U}_1, \mathcal{U}_2$ are open, then $\mathcal{U}_1 \cap \mathcal{U}_2$ is open.

Understanding 3), 4):

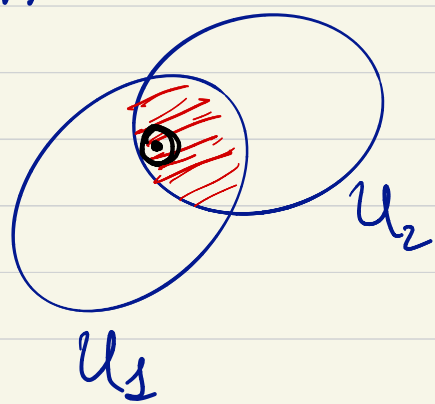
3)



Let $P \in \mathcal{U}_1 \cup \mathcal{U}_2$. Then $P \in \mathcal{U}_1$ or $P \in \mathcal{U}_2$. Since both sets are open, there exists a disk, centered at P which is contained in $\mathcal{U}_1$ or $\mathcal{U}_2$, hence contained in the union. This shows that the union is open.

Remark: This argument extends to an arbitrary number of open sets (including infinitely many).

4)



Choose a point P in $U_1 \cap U_2$. Then P belong to each set. Since each set is open, there exist radii r_1, r_2 such that

$$D(P, r_1) \subset U_1$$
$$D(P, r_2) \subset U_2$$

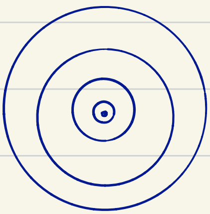
Since the disks are concentric, the disk with smaller radius is contained within the other. Therefore, it is contained within both U_1 and U_2, as well as $U_1 \cap U_2$.

Remark: this argument extends to a finite collection of open sets, but not to infinite collections.

Example: $U_i = D(0, 1/i)$, then

$$\bigcap_{i \in \mathbb{N}} U_i = \{0\}$$

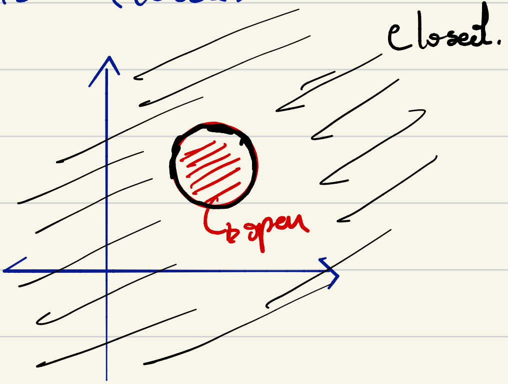
open　　　　not open



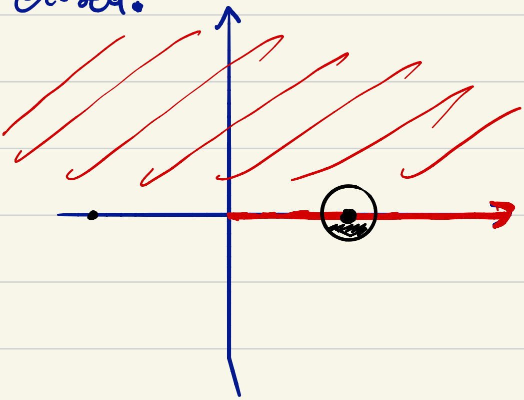
Definition: A set $F \subset \mathbb{R}^2$ is called ***closed*** if its complement is open.

Example 6: $\emptyset$ and $\mathbb{R}^2$ are closed. This is because their complements, $\mathbb{R}^2$ and $\emptyset$, respectively, are open.

Example 7: The complement of an open disk is closed.



Remark: A non-open set is not necessarily closed.



Let U be the union of the upper half-plane and positive x-axis. This is not open: any point chosen in the positive x-axis and and any radius are so that a portion of the disk is below the x-axis.

By symmetry, its complement is also not open, hence, by definition, the set is not closed.

Properties of closed sets

1) $\mathbb{R}^2$ is closed
2) $\emptyset$ is closed
3) If U_1 and U_2 are closed, then so is $U_1 \cap U_2$.
4) If U_1 and U_2 are closed, then so is $U_1 \cup U_2$.

Remark: Statement 3) can be extended to an arbitrary number of closed sets, while statement 4) can only be extended to a finite number.

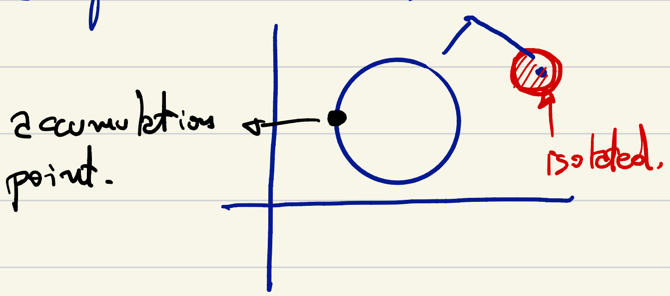
Definitions

(a) An interior point of a set G is one which can be enclosed by a disk entirely contained in G.

(b) A boundary point of a set G is one so that <u>any</u> disk centered at it contains both points within G and outside of G.

(c) An accumulation point of G is one so that any disk centered at contains points of G other than itself.

(d) An isolated point of G is one for which there exists a disk centered at it whose intersection with G is the point itself.



$G = (\text{circle}) \cup (\text{point})$

In terms of these concepts:

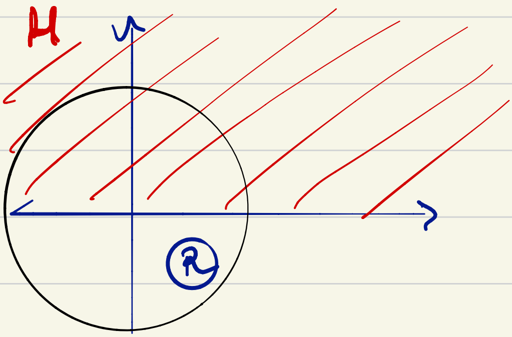
- A set is open if all of its points are interior points.
- A set is closed if it includes all its boundary points.
- A set is closed if it includes all its accumulation points.

<u>Notation:</u>

1) The collection of all interior points of G is called its interior, and denoted $\mathring{G}$.
2) The collection of all boundary points of G is called its boundary, and denoted ∂G.
3) The union of G and its accumulation points (or boundary points) is called the closure of G, denoted $\overline{G} := G \cup \partial G$.

Boundedness and connectedness

Definition: A set is called bounde if it is contain in some disk of finite radius centered at the origin which contains it.



R is bounded, but the upper half-plane is not.

Definition: A set is called compact if it is bounded and closed.

Examples: A closed disk

$$\overline{D(P,r)} = \{ Q \in \mathbb{R}^2 \mid \operatorname{dist}(P,Q) \leq r \}$$

is compact.

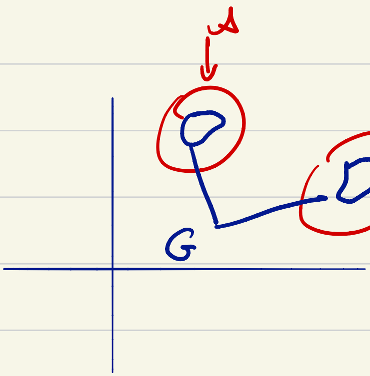
The closure of the upper half-plane, $\overline{H}$

$$\overline{H} = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$$

is closed, but not compact, as it is not bounded.

Definition: A separation of a set G is a collection of disjoint open subsets $A, B \subset \mathbb{R}^2$ such that

$$(A \cup B) \cap G = G.$$



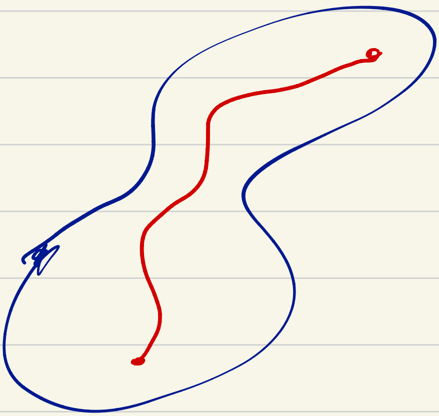
Definition: A set is called connected if it does not admit a separation. If the set is also open, then it is called a region or domain.

Examples: An open disk, the upper half-plane, a line segment, a circle are all connected sets.

Sets containing isolated points are not connected.

Theorem (1.12 on textbook)

If a set G is a region (that is, open and connected) then any two points within it can be joined by a path in G.



Also, if any two points on a set can be joined by a path, the set is connected.

Example:

Consider the set

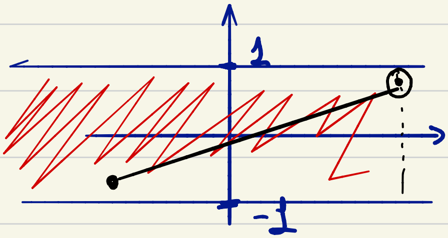
$$D((-3,0), 2).$$

This set is:

- open
- not closed, as it doesn't include the boundary circle.
- bounded (with radius 6 of the origin, for instance).
- not compact, as it isn't closed.
- connected.

Example: Consider the set

$$G = \{ (x, y) \in \mathbb{R}^2 \mid |y| < 1 \}.$$

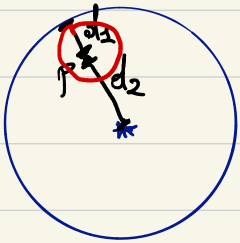


- G is open. Any point within it has a distance to the line $y=1$, d_1, and a distance to the line $y=-1$, d_1. The disk whose radius is the smaller of the two distances is contained within G.

- Not closed, as it doesn't include the boundary lines.
- Not bounded, as there are points with arbitrary x-coordinate in G.
- Not compact.
- Connected, as any two points within it may be joined by a line segment.

<u>Example</u>: A punctured disk

$$D^*(0,1) = \{ P \in \mathbb{R}^2 \mid 0 < dist(0,P) < 1 \}.$$



- D^* is open. Given a point P, it has a distance to boundary circle (along radial line), d_1, and a distance to the origin, d_2. The disk, centered at P with the smaller distance as its radius is contained within D^*.

- D^* is not closed: it doesn't include the boundary circle or the puncture point.
- D^* is bounded: it is contained within a disk of radius 2 centered at the origin.
- D^* not compact: it is bounded but not closed.
- D^* is connected: any two points within D^* can be joined by a collection of line segments.