MAT 203 - Lecture 8

Applications to Optimization

- Critical points: points at which $\nabla f = 0$
- Second derivative test.

$$\text{Hess}(f) \begin{bmatrix} \partial_{xx} f & \partial_{xy} f \\ \partial_{yx} f & \partial_{yy} f \end{bmatrix} \to \underline{\text{Hessian matrix}}$$

Two characteristics:

1) Hessian determinant

$$\begin{aligned} d &= (\partial_{xx} f)\cdot(\partial_{yy} f) - (\partial_{xy} f)(\partial_{yx} f) \\ &= (\partial_{xx} f)\cdot(\partial_{yy} f) - (\partial_{xy} f)^2 \end{aligned}$$

2) Sum of pure partials:

$$\Delta f = \partial_{xx} f + \partial_{yy} f \quad (\text{Laplacian of } f).$$

Test:

$$\text{If: } \begin{cases} d > 0 \to & \Delta f > 0 : \text{ local min} \\ & \Delta f < 0 : \text{ local max.} \\ d < 0 \to & \text{saddle point.} \\ d = 0 \to & \text{inconclusive.} \end{cases}$$

Exercise 1: The profit of a company is measured by

$$P(x,y) = 8x + 10y - (0.001)(x^2 + xy + y^2) - 10000$$

x: units sold for product 1

y: units sold for product 2

How should the company divide production so as to maximize profit?

Solution: $\frac{\partial P}{\partial x} = 8 - 0.001(2x + y)$

$\frac{\partial P}{\partial y} = 10 - 0.001(x + 2y)$

To find critical point, set $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0$.

$$\begin{aligned} 8 - 0.001(2x+y) &= 0 \\ 10 - 0.001(x+2y) &= 0 \end{aligned} \Rightarrow \begin{cases} 2x + y = 8000 \\ x + 2y = 10000. \quad (-2) \end{cases}$$

$$\begin{aligned} \cancel{2x} + y &= 8000 \\ -\cancel{2x} - 4y &= -20000 \quad \oplus \\ \hline -3y &= -12000 \end{aligned} \Rightarrow \underline{y = 4000} \Rightarrow \underline{x = 2000}$$

Exercise 2: Given a plane with equation

$$x - y + z = 3, \rightarrow z = 3 - x + y$$

and a point $P = (4, 0, 6)$, find the point in the plane which is closest to P.

Hint: minimizing distance is equivalent to minimizing the square of distance.

Solution: The function to be minimized is

$$f(x,y,z) = d((x,y,z),(4,0,6))^2.$$

Trick: reduce dimension

$$\begin{aligned}
g(x,y) &= f(x,y,3-x+y) \\
&= d((x,y,3-x+y),(4,0,6))^2 \\
&= (x-4)^2 + (y-0)^2 + (3-x+y-6)^2 \\
&= (x-4)^2 + y^2 + (-3-x+y)^2. \\
&= x^2 - 8x + 16 + y^2 + \\
&\quad 9 + x^2 + y^2 + 6x - 6y - 2xy. \\
&= 2x^2 + 2y^2 - 2xy - 2x - 6y + 25.
\end{aligned}$$

$$g(x,y) = 2x^2 + 2y^2 - 2xy - 2x - 6y + 25.$$

$$\frac{\partial g}{\partial x} = 4x - 2y - 2$$

$$\frac{\partial g}{\partial y} = 4y - 2x - 6$$

Finding critical point:

$$\begin{cases} 4x - 2y - 2 = 0 \\ 4y - 2x - 6 = 0. \end{cases}$$

$$\begin{cases} 4x - 2y = 2 \\ -2x + 4y = 6 \quad \cdot (2) \end{cases}$$

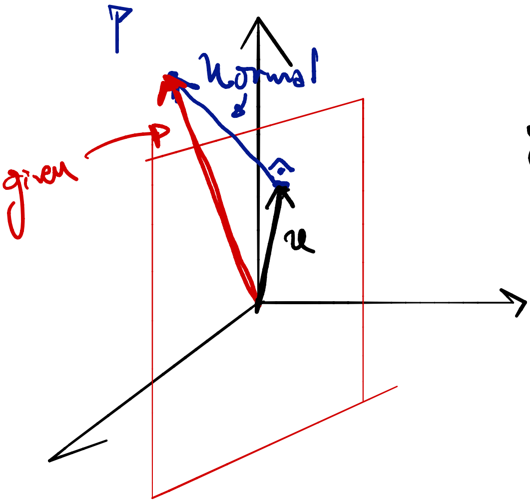
$$\begin{cases} \cancel{4x} - 2y = 2 \\ \cancel{-4x} + 8y = 12 \end{cases} \oplus$$

$$6y = 14 \rightarrow \boxed{y = \frac{14}{6}} = \frac{7}{3}.$$

$$4x - 2\cdot\left(\frac{7}{3}\right) = 2 \Rightarrow 4x - \frac{14}{3} = 2$$

$$12x - 14 = 6$$

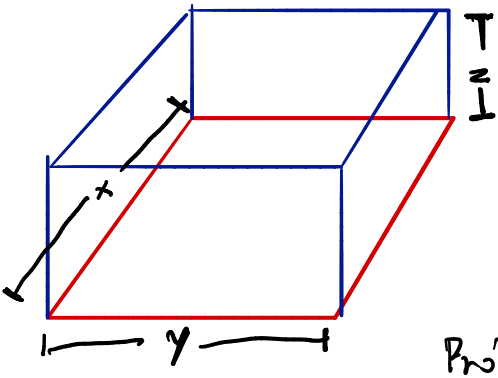
$$12x = 20 \rightarrow \boxed{x = \frac{5}{3}}$$



To minimize distance geometrically: find a normal vector to plane

$$u^t n = p.$$

Exercise 3: Open box whose bottom and sides are made of different materials. Material for base costs 1.5 times as much as material for sides.



Problem: with fixed amount of money to spend, C, find relative dimensions of box of largest volume.

Cost: $\boxed{1.5\, x \cdot y + 2xz + 2yz = C}$

Volume: $V(x,y,z) = x \cdot y \cdot z.$ → wish to maximize

$$\frac{\partial V}{\partial x} = yz, \quad \frac{\partial V}{\partial y} = xz, \quad \frac{\partial V}{\partial z} = xy.$$

Reduce number of variables.

Cost: $1.5 \cdot xy + 2xz + 2yz = C$

Volume: $x \cdot y \cdot z = V$

Get rid of variable z.

$$1.5\,xy + 2(x+y)\cdot z = C$$

$$2(x+y)\cdot z = C - 1.5xy$$

$$z = \frac{C - 1.5xy}{2(x+y)}$$

$$V = x \cdot y \left(\frac{C - 1.5xy}{2(x+y)}\right)$$ (recall: C is constant)

$\partial V/\partial x$, $\partial V/\partial y$, $x > 0$, $y > 0$.

$$\frac{\partial V}{\partial x} = y\left(\frac{C - 1.5xy}{2(x+y)}\right) + xy\,\frac{\partial}{\partial x}\left(\frac{C - 1.5xy}{2(x+y)}\right)$$

$$= y\left(\frac{C - 1.5xy}{2(x+y)}\right) +$$

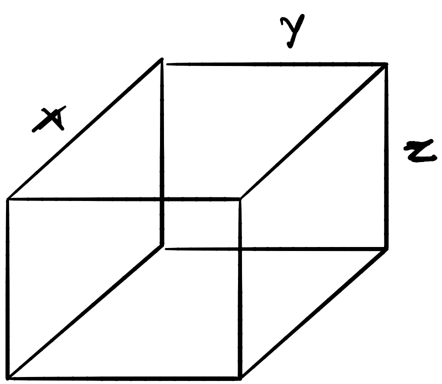
$$xy \cdot \left[\frac{(-1.5y)\cdot(2(x+y)) - (C - 1.5xy)\cdot 2}{4(x+y)^2}\right]$$

<u>Exercise 4</u>: Paint walls and ceiling of rectangular room. Volume of room $668.25\ ft^3$.

Wall paint: \$0.06 per ft^2.

Ceiling paints: \$0.11 per ft^2.

Find dimensions resulting in least cost.



$$V = x \cdot y \cdot z = 668.25$$

$$z = \frac{668.25}{xy}$$

$$C(x,y,z) = 2\cdot(0.06)yz + 2(0.06)xz + 1\cdot(0.11)xy.$$

$$C(x,y,z) = 0.12yz + 0.12xz + 0.11xy$$

$$C(x,y) = 0.12\cancel{y}\cdot\left(\frac{668.25}{x\cancel{y}}\right) + 0.12\cancel{x}\cdot\left(\frac{668.25}{\cancel{x}y}\right) + 0.11xy.$$

$$\boxed{C(x,y) = \frac{80.19}{y} + \frac{80.19}{x} + 0.11xy.}$$

$$80.19\,x^{-1} \qquad 80.19\,y^{-1}$$

$$C(x,y) = \frac{80.19}{x} + \frac{80.19}{y} + 0.11xy.$$

$$\frac{\partial C}{\partial x} = -\frac{80.19}{x^2} + 0.11y$$

$$\frac{\partial C}{\partial y} = -\frac{80.19}{y^2} + 0.11x$$

$$\begin{cases} -\dfrac{80.19}{x^2} + 0.11y = 0 \\ -\dfrac{80.19}{y^2} + 0.11x = 0. \end{cases}$$

$$\begin{cases} 0.11y = \dfrac{80.19}{x^2} \Rightarrow y = \dfrac{729}{x^2} \\ 0.11x = \dfrac{80.19}{y^2}. \end{cases}$$

$$0.11x = \frac{80.19\,x^4}{729^2} \Rightarrow \cancel{0.11}x = \cancel{\left(\frac{80.19}{729}\right)} \cdot \frac{x^4}{729}.$$

$$729x = x^4 \Rightarrow \cancel{(x = 0)} \quad \text{or} \quad x^3 = 729 \Rightarrow \boxed{x = \sqrt[3]{729}} = 9.$$

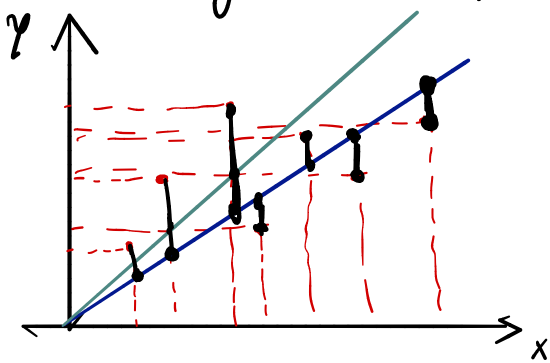
$x = \sqrt[3]{729}$, $y = \frac{729}{x^2}$

Hence $y = \frac{729}{729^{2/3}} \Rightarrow y = \sqrt[3]{729} = 9$

In other words, ceiling is a square.

$z = \frac{668.25}{(\sqrt[3]{729})^2} = \frac{668.25}{81} = 8.25$

Linear regression (by least squares)



Goal: to find a line which best fits data.
Method: by minimizing sum of square-distances.

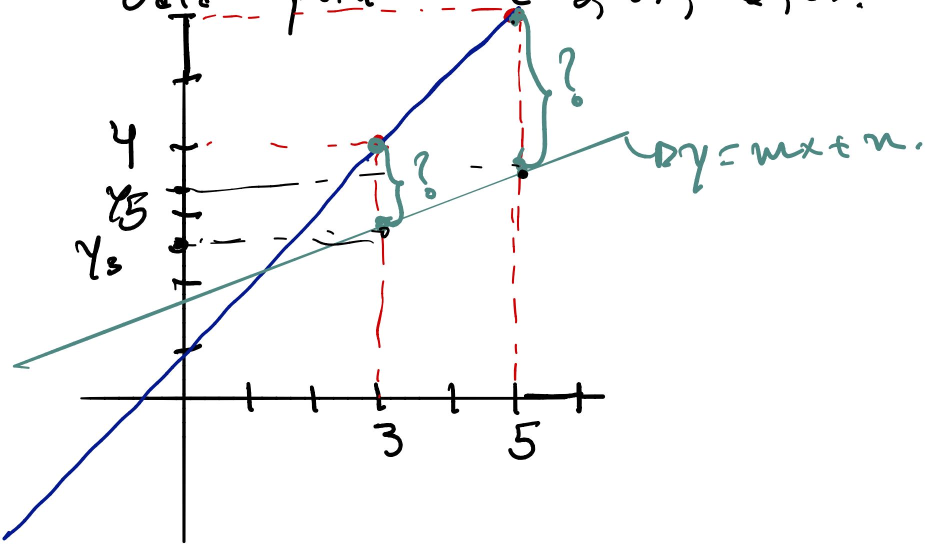
Framework: lines (not vertical) are described by slope "m", y-intercept "n"

$$y = mx + n.$$

We seek best pair (m, n).

Example 1:

Data points: (3, 4), (5, 6).



For a given line $y = mx + n$;

value at $x = 3$: is $y_3 = 3m + n$

value at $x = 5$ is $y_5 = 5m + n$

$$F(m,n) = (4 - 3m - n)^2 + (6 - 5m - n)^2$$

$$\frac{\partial F}{\partial m} = 2(4 - 3m - n)\cdot(-3) + 2(6 - 5m - n)\cdot(-5)$$

$$= -6(4 - 3m - n) - 10(6 - 5m - n)$$

$$\frac{\partial F}{\partial m} = -24 + 18m + 6n \quad -60 + 50m + 10n$$

$$\boxed{\frac{\partial F}{\partial m} = -84 + 68m + 16n}$$

$$F(m,n) = (4 - 3m - n)^2 + (6 - 5m - n)^2$$

$$\frac{\partial F}{\partial n} = 2(4 - 3m - n)\cdot(-1) + 2(6 - 5m - n)\cdot(-1)$$

$$= -2(4 - 3m - n) - 2(6 - 5m - n)$$

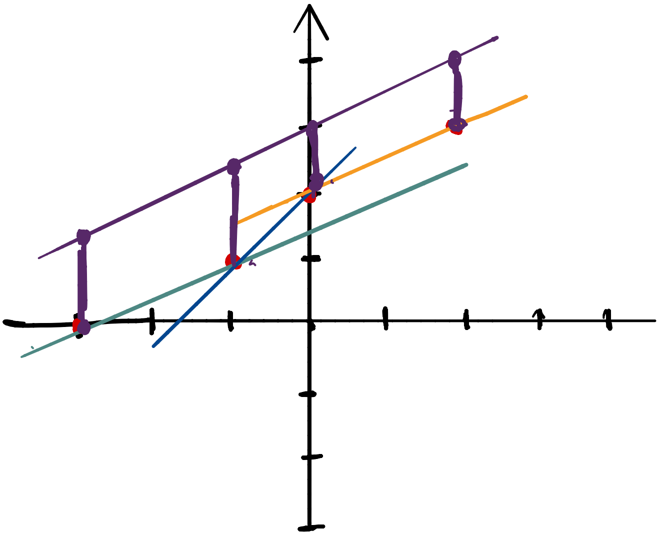
$$= -8 + 6m + 2n - 12 + 10m + 2n$$

$$\boxed{\frac{\partial F}{\partial n} = -20 + 16m + 4n}$$

$$\begin{cases} 68m + 16n = 84 \\ 16m + 4n = 20 \end{cases}$$

Unique solution: $m = 1$, $n = 1$

Example 2: Data points $(-3,0)$, $(-1,1)$, $(0,2)$, $(2,3)$.



First check: points are not all aligned.

Choose a line $y = mx + n$.

Values at $-3, -1, 0$ and 2 are:

$-3m+n$,

$-m+n$,

$0m+n$

$2m+n$.

$$F(m,n) = [0 - (-3m+n)]^2 + [1 - (-m+n)]^2$$
$$[2 - (0m+n)]^2 + [3 - (2m+n)]^2.$$

$$F(m,n) = [3m - n]^2 + [1+m-n]^2 + [2-n]^2 + [3-2m-n]^2.$$

$$F(m,n) = [3m-n]^2 + [1+m-n]^2 + [2-n]^2 + [3-2m-n]^2.$$

$$\frac{\partial F}{\partial m} = \Big\{ 2(3m-n)\cdot 3 + 2(1+m-n)\cdot 1 + \cancel{2\cdot(2-n)\cdot 0} + 2\cdot(3-2m-n)\cdot(-2) \Big\}$$

0

$$\frac{\partial F}{\partial m} = (18m - 6n) + (2 + 2m - 2n) + (-12 + 8m + 4n)$$

$$\boxed{\frac{\partial F}{\partial m} = 28m - 4n - 10.}$$

$$\frac{\partial F}{\partial n} = \Big\{ 2(3m-n)\cdot(-1) + 2(1+m-n)(-1) + 2(2-n)\cdot(-1) + 2(3-2m-n)\cdot(-1) \Big\}$$

$$= -6m + 2n - 2 - 2m + 2n - 4 + 2n - 6 + 4m + 2n.$$

$$\boxed{\frac{\partial F}{\partial n} = -4m + 8n - 12.}$$

$$\begin{cases} 28m - 4n = 10 \quad \cdot 2 \\ -4m + 8n = 12 \end{cases}$$

$$\begin{cases} 56m - \cancel{8n} = 20 \\ -4m + \cancel{8n} = 12 \end{cases} \quad (+)$$

$$52m = 32$$

$$m = \frac{32}{52} = \frac{16}{26} = \frac{8}{13}.$$

$$-4 \cdot \left(\frac{8}{13}\right) + 8 \cdot n = 12$$

$$n = \frac{47}{26}.$$

Best fitting line: $\boxed{y = \frac{8x}{13} + \frac{47}{26}}$.

Least squares method

1) Check all x-values of given data points are different.

2) Find corresponding tentative y-values obtained by replacing given x-values in

$$y = mx + n.$$

3) Measure sum of squared differences:

Sum [(given y-value) - (expected y-value)]^2

↳ over all data points.

4) Find partials, set them equal to 0, solve for m and n.