Spring 2020 MAT303 Recitations

Week of 4/6/20: Sections 3.4 and 3.5

Section 3.4: Mechanical Vibrations

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Section 3.4: Mechanical Vibrations

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$$mx''(t) + cx'(t) + kx(t) = F(t),$$

where m, c, k are constants ($m \neq 0$). In today's recitation, we will discuss a few examples of such equations.

Section 3.4: Mechanical vibrations

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where

- $C = \sqrt{A^2 + B^2}$ the amplitude;
- $\omega_1 = \frac{\sqrt{4cm - k^2}}{2m}$ is the relative frequency;
- $\alpha \in [0, 2\pi)$ is the phase, so that $\cos(\alpha) = \frac{A}{C}$ and $\sin(\alpha) = \frac{B}{C}$.

Section 3.4: Mechanical Vibrations

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- $m = 250g = 0.25kg$
- $k = 9N/0.25m = 36N/m$.
- at $t = 0s$, $x(0) = 1m, x'(0) = -5m/s$.

The movement of the system is described by the equation

$$0.25x^{''} + 36x = 0 \Leftrightarrow x^{''} + 144x = 0,$$

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The movement of the system is described by the equation

$$0.25x'' + 36x = 0 \Leftrightarrow x'' + 144x = 0,$$

thus the relative frequency is $\omega = 12$. A solution can be written in form

$$x(t) = C\cos(12t - \alpha),$$

for constants $C > 0$, $\alpha \in [0, 2\pi)$ to be determined.

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$$\alpha \approx 5.89.$$

Thus

$$x(t) \approx \frac{13}{12}\cos(12t - 5.89).$$

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Amplitude: $A = \frac{13}{12}$ meters. Period: $T = \frac{2\pi}{12} = \frac{\pi}{6}$ seconds.

Section 3.4: Mechanical Vibrations

In problem 3.4.14, we have a mass-spring-dashpot system with constants

$$m = 25, c = 10, k = 226,$$

and initial data $x(0) = 20, x'(0) = 41$.

Section 3.4: Mechanical Vibrations

In problem 3.4.14, we have a mass-spring-dashpot system with constants

$$m = 25, c = 10, k = 226,$$

and initial data $x(0) = 20, x'(0) = 41$. Its characteristic polynomial is

$$25r^2 + 10r + 226 = (5r + 1)^2 + 15^2,$$

whose roots are

$$r = -\frac{1}{5} \pm 3i.$$

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whose roots are

$$r = -\frac{1}{5} \pm 3i.$$

Solutions to the differential equation take the form

$$x(t) = Ce^{-\frac{t}{5}} \cos(3t - \alpha).$$

Section 3.4: Mechanical Vibrations

To use the initial data, we compute the first derivative,

$$x'(t) = -Ce^{-\frac{t}{5}}\left[\frac{\cos(3t-\alpha)}{5} + 3\sin(3t-\alpha)\right],$$

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$$\begin{aligned} C\cos(\alpha) &= 20 \\ C\left[-\frac{\cos(\alpha)}{5} + 3\sin(\alpha)\right] &= 41, \end{aligned}$$

from which we infer $C = 25$, $\alpha \approx 0.64$, so

$$x(t) \approx 25e^{-\frac{t}{5}}\cos(3t - 0.64).$$

Section 3.4: Mechanical Vibrations

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Section 3.4: Mechanical Vibrations

In problem 3.4.17, we are meant to contrast the damped motion of a spring-mass-dashpot system with its undamped counterpart. The differential equation modelling this problem is

$$x'' + 8x' + 16x = 0,$$

with initial data $x(0) = 5, x'(0) = -10$.

The characteristic polynomial of the problem is

$$r^2 + 8r + 16 = (r+4)^2.$$

Its root is -4, with multiplicity 2, thus this motion is critically damped.

A solution takes the form

$$x(t) = e^{-4t}(At + B),$$

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Subject to initial conditions $x(0) = 5, x'(0) = -10$, we find

$$x(t) = e^{-4t}(2t + 1).$$

Section 3.4: Mechanical Vibrations

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Its solutions take the form

$$u(t) = C\cos(4t - \alpha).$$

Assuming the same initial conditions $u(0) = 5, u'(0) = -10$, we have

$$u(t) \approx \frac{5\sqrt{5}}{2}\cos(4t - 5.82)$$

In problem 3.4.31, we have a mass-spring-dashpot system with constants $m = 1, c = 10$ and $k = 125$, subject to initial conditions $x(0) = 6$, $x'(0) = 50$. The characteristic polynomial of the system is

$$r^2 + 10r + 125 = (r+5)^2 + 10^2,$$

whose roots are $r = -5 \pm 10i$, characterizing underdamped motion. The corresponding solution to the diferential equation is

$$x(t) = Ce^{-5t}\cos(10t - \alpha),$$

for constants $C > 0$, $\alpha \in [0, 2\pi)$ to be determined in what follows.

Section 3.4: Mechanical Vibrations

The first derivative of the solution takes the form

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The initial conditions amount to

$$\begin{aligned} C\cos(\alpha) &= 6, \\ C[-10\cos(\alpha) - 5\sin(\alpha)] &= 50, \end{aligned}$$

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The initial conditions amount to

$$\begin{aligned} C\cos(\alpha) &= 6, \\ C[-10\cos(\alpha) - 5\sin(\alpha)] &= 50, \end{aligned}$$

from which we infer $C = 10$, $\alpha \approx 0.93$. The solution is thus approximated by

$$x(t) \approx 10e^{-5t}\cos(10t - 0.93).$$

Section 3.4: Mechanical Vibrations

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The characteristic polynomial is $p(r) = r^2 + 125$, with roots $r = \pm 5\sqrt{5}i$. Solutions to this equation can be expressed as

$$u(t) = C_* \cos(5\sqrt{5}t - \alpha_*),$$

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$$u'' + 125u = 0.$$

The characteristic polynomial is $p(r) = r^2 + 125$, with roots $r = \pm 5\sqrt{5}i$. Solutions to this equation can be expressed as

$$u(t) = C_* \cos(5\sqrt{5}t - \alpha_*),$$

where the constants $C_* > 0$ and $\alpha_* \in [0, 2\pi)$ are determined by the initial conditions $u(0) = 6, u'(0) = 50$.

Section 3.4: Mechanical Vibrations

The derivative of a solution takes the form

$$u'(t) = -5\sqrt{5}C_* \sin(5\sqrt{5}t - \alpha_*),$$

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so using the initial date we obtain

$$\begin{aligned} C\cos(\alpha_*) &= 6 \\ 5\sqrt{5}C_* \sin(\alpha_*) &= 50, \end{aligned}$$

and infer $C_* = 2\sqrt{14}$, $\alpha_* \approx 0.64$. Finally, we motion of the undamped system is

$$u(t) \approx 2\sqrt{14}\cos(5\sqrt{5}t - 0.64).$$

Section 3.5: Homogeneous Equations with Constant Coefficients

In problem 3.5.3, we consider the equation

$$y'' - y' - 6y = 2\sin(3x). \tag{1}$$

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$$r^2 - r - 6 = (r+2)(r-3),$$

with roots $r_1 = -2, r_2 = 3$. The corresponding solutions to the complementary equation take the form

$$y_c = A_1 e^{-2x} + A_2 e^{3x}.$$

Section 3.5: Homogeneous Equations with Constant Coefficients

As the solutions to the homogeneous problem and the inhomogeneous term of equation (1) are linearly independent (verify!), we can use the method of undetermined coefficients to guess a particular solution of the form

$$y_p = C_1 \sin(3x) + C_2 \cos(3x).$$

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$$y_p = C_1 \sin(3x) + C_2 \cos(3x).$$

To find the coefficients C_1, C_2, we need to compute the derivatives of the particular solution,

$$\begin{aligned} y_p' &= 3C_1 \cos(3x) - 3C_2 \sin(3x), \\ y_p'' &= -9C_1 \sin(3x) - 9C_2 \cos(3x). \end{aligned}$$

Section 3.5: Homogeneous Equations with Constant Coefficients

Combining all of this data into the equation we get

$$(-15C_1 + 3C_2)\sin(3x) + (-3C_1 - 15C_2)\cos(3x) = 2\sin(3x),$$

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from which we infer $C_1 = -\frac{5}{39}$, $C_2 = \frac{1}{39}$. A general solution to the differential equation takes the form

$$y(x) = A_1 e^{-2x} + A_2 e^{3x} - \frac{5}{39}\sin(3x) + \frac{1}{39}\cos(3x),$$

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$$y(x) = A_1 e^{-2x} + A_2 e^{3x} - \frac{5}{39}\sin(3x) + \frac{1}{39}\cos(3x),$$

for constants A_1 and A_2 to be determined based upon initial conditions.