Solutions to Homework 2

Problem 2.2.(a) Evaluate the limit

$$\lim_{z \to i} \frac{iz^3 - 1}{z + i},$$

or explain why it does not exist.

Solution: This limit exists, and it may be computed by direct substitution, as the function in question is continuous at i

$$\lim_{z \to i} \frac{iz^3 - 1}{z + i} = \frac{i^4 - 1}{2i} = 0.$$

Problem 2.6 Proposition 2.2. is useful for showing that limits **do not** exist, but it is not at all useful for showing that a limit **does** exist. For example, define

$$f(z) = \frac{x^2 y}{x^4 + y^2},$$

where$z = x + iy \neq 0$. Show that the limits of f at 0 along all straight lines through the origin exist and are equal, but

$$\lim_{z \to 0} f(z)$$

does not exist (Hint: consider the limit along the parabola $y = x^2$).

Solution: Let $k \in \mathbb{R}$, and consider the line through the origin with slope k, $y = kx$. Along this line, for $x \neq 0$, the function reduces to

$$\begin{aligned} \frac{x^2 y}{x^4 + y^2} &= \frac{x^2 (kx)}{x^4 + (kx)^2} \\ &= \frac{kx^3}{x^4 + k^2 x^2} \\ &= \frac{kx}{x^2 + k^2}. \end{aligned}$$

The limit along such a line as $x \to 0$ is

$$\lim_{x\to 0} \frac{kx}{x^2+k^2} = 0,$$

irrespective of k.

Meanwhile, the limit along the parabola $y = x^2$ is

$$\lim_{x\to 0} \frac{x^2(x^2)}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}.$$

It follows that the limit of f at 0 does not exist, as it depends on the curve of approach.

Problem 2.6 Consider the function $f\colon \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}$, given by $f(z) = \frac{1}{z}$. Apply the definition of the derivative to give a direct proof that $f'(z) = -\frac{1}{z^2}$.

Solution: Applying the definition via Newton quotients,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0} \frac{\frac{1}{z+h}-\frac{1}{z}}{h} \\
&= \lim_{h\to 0} \frac{\frac{z-(z+h)}{z(z+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{zh(z+h)} \\
&= \lim_{h\to 0} -\frac{1}{z(z+h)} \\
&= -\frac{1}{z^2}.
\end{aligned}$$

Problem 2.15 Find the derivative of the function $T(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$, with $ad - bc \neq 0$. when is $T'(z) = 0$?

Solution: Throughout the solution, we assume $cz + d \neq 0$. Applying the quotient rule, we obtain

$$\begin{aligned}
T'(z) &= \frac{(az+b)'(cz+d)-(az+b)(cz+d)'}{(cz+d)^2} \\
&= \frac{a(cz+d)-(az+b)c}{(cz+d)^2} \\
&= \frac{ad-bc}{(cz+d)^2},
\end{aligned}$$

hence, by assumption, $T'(z) \neq 0$, for all z within its domain of definition.

Problem 2.18 Where are the following funcitons differentiable? Where are they holomorphic? Determine their derivatives at points where thery are differentiable.

(b) $f(z) = 2x + ixy^2$

(f) $f(z) = \Im(z)$

(h) $f(z) = z\Im(z)$

(l) $f(z) = z^2 - (\overline{z})^2$

Solution:

(b) The real and imaginary components of this function are $u(x+iy) = 2x$, $v(x+y) = xy^2$, respectively. The corresponding Cauchy-Riemann equations are

$$\begin{aligned}
2 &= 2xy \\
0 &= -y^2,
\end{aligned}$$

a system without solutions. It follows that the function $f(z) = 2x + ixy^2$ is nowhere complex-differentiable.

(f) The real and imaginary components of this function are $u(x+y) = y$ and $v(x+iy) = 0$. The corresponding Cauchy-Riemann equations are

$$\begin{aligned}
0 &= 1 \\
0 &= 0,
\end{aligned}$$

a system without solutions. It follows that the function $f(z) = \Im(z)$ is nowhere-complex-differentiable.

(h) The real and imaginary parts of this function are $u(x+iy) = xy$, $v(x+iy) = y^2$, respectively. The corresponding Cauchy-Riemann equations are

$$\begin{aligned} y &= 2y \\ x &= -0, \end{aligned}$$

a system whose only solution is $x+iy = 0$. It follows that the function $f(z) = z\Im(z)$ is complex-differentiable at 0, but nowhere holomorphic. Next we apply Newton quotients to compute the derivative at 0,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

We will use the fact that the function is differentiable at 0, i.e. this limit exists, to choose a suitable direction for computation, say along the real axis. With this restriction, the limit becomes

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= 0. \end{aligned}$$

(l) The real and imaginary parts of this function are $u(x+iy) = 0$, $v(x+iy) = 4xy$. The corresponding Cauchy-Riemann equations are

$$\begin{aligned} 0 &= 4x \\ 0 &= -4y, \end{aligned}$$

a system whose only solution is $x+iy = 0$. It follows that the function $f(z) = z^2-(\overline{z})^2$ is complex-differentiable at 0, but nowhere holomorphic.Next we apply Newton quotients to compute the derivative at 0,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

We will use the fact that the function is differentiable at 0, i.e. this limit exists, to choose a suitable direction for computation, say along the real axis. With this restriction, the limit becomes

$$\begin{aligned}
f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\
&= \lim_{h \to 0} \frac{0}{h} \\
&= 0.
\end{aligned}$$

Problem 2.25 For each of the following funcitons u, find a function v such that $u + iv$ is holomorphic in some region. Maximize that region.

(a) $u(x, y) = x^2 - y^2$.

(d) $u(x, y) = \frac{x}{x^2+y^2}$.

Solution:

(a) The function has derivatives

$$\frac{\partial u}{\partial x} = 2x, \; \frac{\partial u}{\partial y} = -2y.$$

If $u + iv$ is to be holomorphic, it must satify the Cauchy-Riemann equations,

$$\begin{aligned}
2x &= \frac{\partial v}{\partial y} \\
-2y &= -\frac{\partial v}{\partial x}.
\end{aligned}$$

A simple solution to such equations, defined on the entire plane, is $v(x, y) = 2xy$. Other solutions can be obtained by adding a (real) constant.

(d) The function u has derivatives

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \; \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2}.$$

A companion funciton v such that $u = iv$ is holomorphic must satisfy the Cauchy-Riemann equations,

$$\begin{aligned}
\frac{y^2 - x^2}{(x^2 + y^2)^2} &= \frac{\partial v}{\partial y} \\
-\frac{2xy}{(x^2 + y^2)^2} &= -\frac{\partial v}{\partial x}.
\end{aligned}$$

The function

$$v(x,y) = -\frac{y}{x^2+y^2}$$

is a solution, defined on $\mathbb{C} \setminus \{0\}$.