MAT 203 - Lecture 11.

- Evaluating integrals in different orders.

<u>Example 1</u>:

$$\int_0^2 \int_0^4 (x+y)\,dx\,dy = \int_0^2 \left[\frac{x^2}{2} + y \cdot x \Big|_{x=0}^{x=4} \right] dy$$

$$= \int_0^2 \left[\frac{4^2}{2} + 4y - \cancel{\frac{0^2}{2}} - \cancel{0 \cdot y} \right] dy$$

$$= \int_0^2 (8 + 4y)\,dy$$

$$= 8y + 2y^2 \Big|_{y=0}^{y=2}$$

$$= 8 \cdot 2 + 2 \cdot 2^2 - 8 \cdot 0 - 2 \cdot 0^2$$

$$= 16 + 8 - 0 - 0$$

$$= 24.$$

Change order of integration.

$$\int_0^4 \int_0^2 (x+y)\,dy\,dx = \int_0^4 \left[xy + \frac{y^2}{2} \Big|_{y=0}^{y=2} \right] dx$$

$$\int_0^4\int_0^2 (x+y)\,dy\,dx = \int_0^4 \left[2x+2-0\cdot x-0\right]dx$$

$$= \int_0^4 (2x+2)\,dx$$

$$= x^2+2x\Big|_{x=0}^{x=4}$$

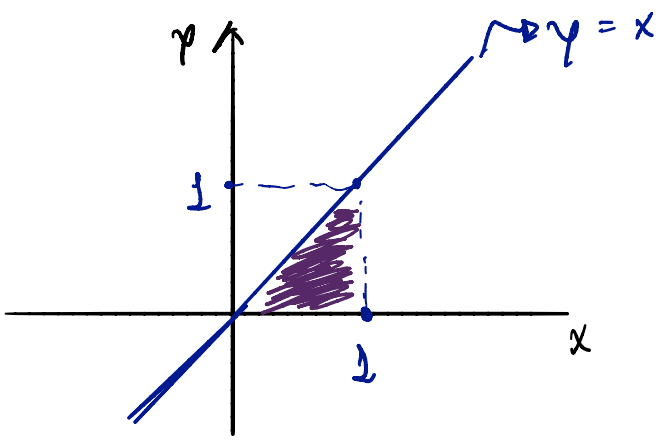
$$= 4^2+2\cdot 4-0^2-2\cdot 0$$

$$= 24.$$

An example with variable bounds.

<u>Example 2:</u>

$$\int_0^1\int_0^x xy\,dy\,dx = \int_0^1\int_y^1 xy\,dx\,dy$$



$$\int_0^1 \int_0^x xy \, dy \, dx = \int_0^1 \left[\frac{xy^2}{2} \Big|_{y=0}^{y=x} \right] dx$$

$$= \int_0^1 \frac{x^3}{2} \, dx$$

$$= \frac{x^4}{8} \Big|_{x=0}^{x=1}$$

$$= 1/8$$

$$\int_0^1 \int_y^1 xy \, dx \, dy = \int_0^1 \left[\frac{x^2 y}{2} \Big|_{x=y}^{x=1} \right] dy$$

$$= \int_0^1 \left[\frac{y}{2} - \frac{y^3}{2} \right] dy.$$

$$= \frac{y^2}{4} - \frac{y^4}{8} \Big|_{y=0}^{y=1}.$$

$$= 1/4 - 1/8$$

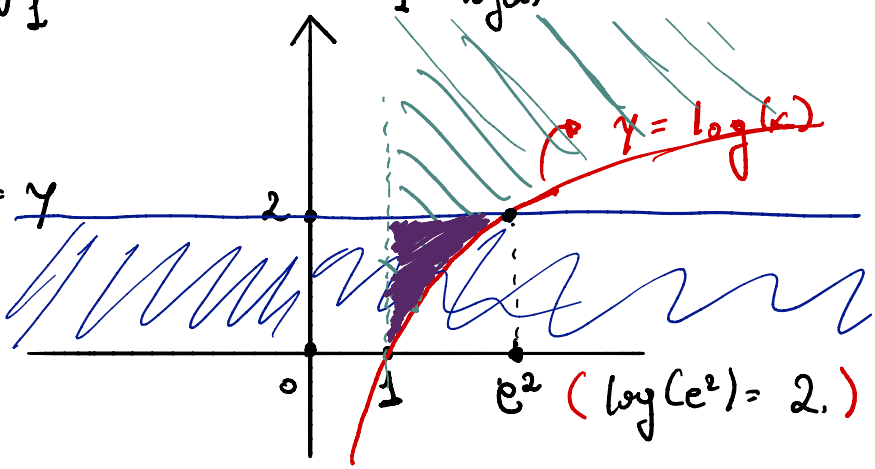
$$= 1/8.$$

Example 3

$$\int_0^2 \int_1^{e^y} x \, dx \, dy = \int_1^{e^2} \int_{\log(x)}^{2} x \, dy \, dx$$

$x = e^y$

$\Leftrightarrow \log(x) = y$



$$\int_0^2 \int_1^{e^y} x \, dx \, dy = \int_0^2 \left[\frac{x^2}{2} \Big|_{x=1}^{x=e^y} \right] dy$$

$$= \int_0^2 \left[\frac{(e^y)^2}{2} - \frac{1^2}{2} \right] dy$$

$$= \int_0^2 \left(\frac{e^{2y}}{2} - \frac{1}{2} \right) dy$$

$$= \frac{e^{2y}}{4} - \frac{y}{2} \Big|_{y=0}^{y=2} = \left(\frac{e^4}{4} - \frac{2}{2} \right) - \left(\frac{e^0}{4} - \frac{0}{2} \right)$$

$$= \frac{e^4}{4} - \frac{5}{4}.$$

$$\int_1^{e^2}\int_{\log(x)}^{2} x \, dy \, dx = \int_1^{e^2}\left[xy \Big|_{y=\log(x)}^{y=2} \right] dx$$

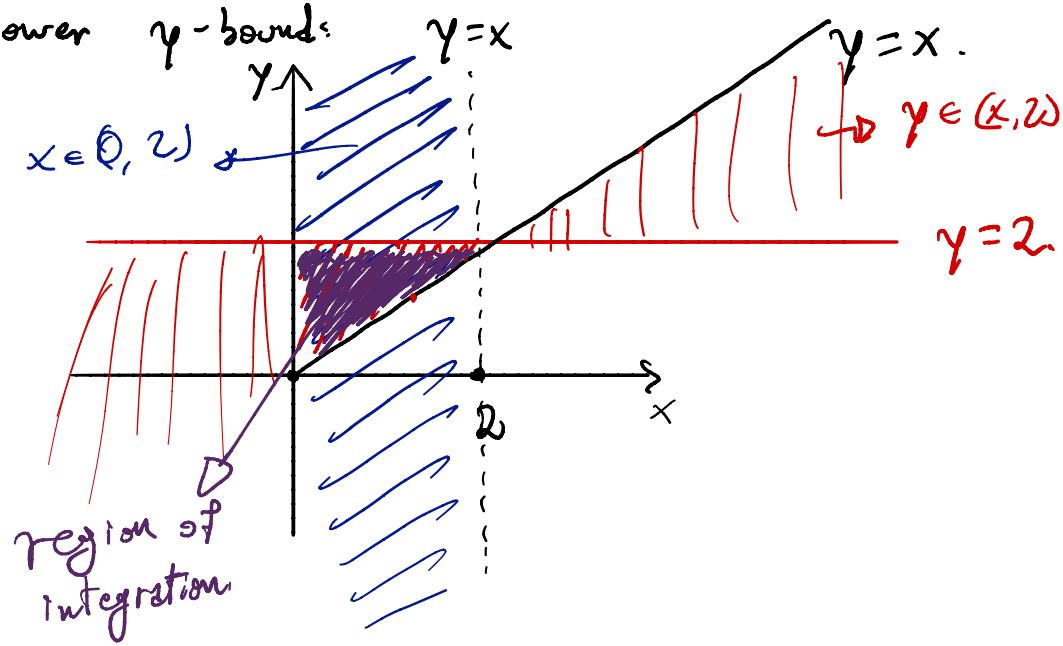
$$= \int_1^{e^2} \left(2x - x\log(x) \right) dx$$

Leads to more computations than necessary.

<u>Example 4</u>

$$\int_0^2 \int_x^2 e^{-y^2} \, dy \, dx = \int_0^2 \int_0^y e^{-y^2} \, dx \, dy$$

lower y-bound:



$$\int_0^2 \int_0^y e^{-y^2}\,dx\,dy = \int_0^2 \left[\left(e^{-y^2}\right) \cdot x \Big|_{x=0}^{x=y} \right] dy$$

$$= \int_0^2 y \cdot e^{-y^2}\,dy$$

Substitute: $u = y^2 \Rightarrow du = 2y\,dy$.

$$\int_0^2 y e^{-y^2}\,dy = \int_0^4 \frac{e^{-u}}{2}\,du$$

$$= -\frac{e^{-u}}{2} \Big|_{u=0}^{u=4}$$

$$= -\frac{e^{-4}}{2} + \frac{e^{-0}}{2}$$

$$= \frac{1}{2} - \frac{e^{-4}}{2}.$$

<u>Exercise 1</u>: Compute the integral below by changing the order of integration

$$\int_{-3}^{3}\int_{0}^{9-y^2} 1 \, dx \, dy.$$

Solution: $$\int_{0}^{9}\int_{-\sqrt{9-x}}^{\sqrt{9-x}} 1 \, dy \, dx$$

$x = 9 - y^2 \Rightarrow x - 9 = -y^2 \Rightarrow y^2 = 9 - x.$

$y_+ = \sqrt{9-x}, \quad 0 \le x \le 9.$

$y_- = -\sqrt{9-x}, \quad 0 \le x \le 9.$

$$\int_{-\sqrt{9-x}}^{\sqrt{9-x}} 1 \, dy = y \Big|_{y=-\sqrt{9-x}}^{y=\sqrt{9-x}}$$

$$= \sqrt{9-x} - (-\sqrt{9-x})$$

$$= 2\sqrt{9-x}$$

$$\int_{0}^{9} 2\sqrt{9-x}\, dx \underset{u=9-x,\ du=-dx}{=} -\int_{9}^{0} 2\sqrt{u}\, du = \int_{0}^{9} 2\sqrt{u}\, du$$

$$\int_0^9 2\sqrt{9-x}\,dx = 2\cdot\frac{u^{3/2}}{3/2}\Bigg|_{u=0}^{u=9}.$$

$$= \frac{4}{3}u^{3/2}\Bigg|_{u=0}^{u=9}$$

$$= \frac{4}{3}\cdot\left(27 - 0^{3/2}\right)$$

$$= 36.$$

Double integrals

Suppose you are given a function of two variables, $f(x,y)$. The signed volume between its graph and the xy-plane over a region R is given by a double integral

$$\iint_R f(x,y)\, dA.$$

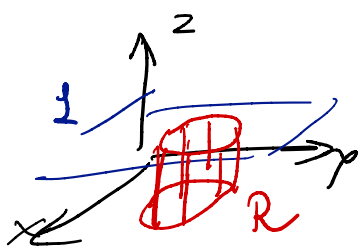
"area element".

Region of integration.

Examples

i) $$\iint_{\mathbb{R}^2} 0 \, dA = 0.$$

ii) $$\iint_R 1 \, dA = \text{area of the region } R.$$

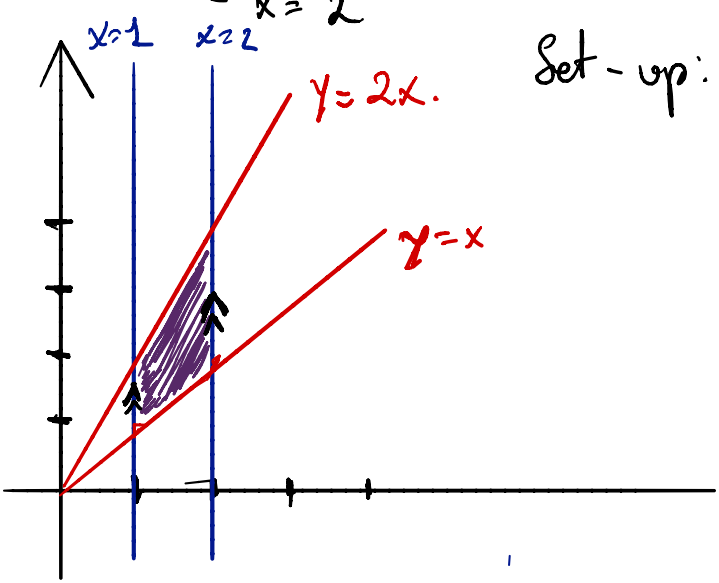


Example 5:

$$\iint_R \frac{y}{x^2+y^2}\, dA$$

where R is the trapezoid bounded by

$$\begin{cases} y = x \\ y = 2x \\ x = 1 \\ x = 2 \end{cases}$$



Set-up: $\displaystyle\int_1^2 \int_x^{2x} \frac{y}{x^2+y^2}\, dy\, dx$

$$u = x^2 + y^2$$
$$du = 2y\,dy$$

$$\int_1^2 \int_x^{2x} \frac{y}{x^2+y^2}\,dy\,dx = \int_1^2 \left(\int_{2x^2}^{5x^2} \frac{1}{2u}\,du \right) dx$$

$$= \int_1^2 \left(\frac{1}{2} \log(u) \Big|_{2x^2}^{5x^2} \right) dx$$

$$= \int_1^2 \left(\frac{1}{2} \cdot \left(\log(5x^2) - \log(2x^2) \right) \right) dx$$

$$= \int_1^2 \left(\frac{1}{2} \cdot \left(\log(5) + \cancel{2\log(x)} - \log(2) - \cancel{2\log(x)} \right) \right) dx.$$

$$= \frac{1}{2} \left(\log\left(\frac{5}{2}\right) \right) \cdot \int_1^2 1\,dx$$

$$= \frac{1}{2} \log\left(\frac{5}{2}\right).$$

Exercise 2:

$$\iint_R -2y \, dA$$

where R is the region bounded by

$$\begin{cases} y = 4 - x^2 \\ y = 4 - x \end{cases}.$$

Solution: Finding bounds.

$$\cancel{4} - x^2 = \cancel{4} - x$$

$$x^2 = x \Rightarrow x = 0 \text{ or } x = 1.$$

Set-up with x as independent variable:

$$\int_0^1 \int_{4-x}^{4-x^2} -2y \, dy \, dx$$

Test to determine upper/lower bounds:

at $x = \frac{1}{2}$: $4 - \left(\frac{1}{2}\right)^2 = 4 - \frac{1}{4} = \frac{15}{4}$,

$$4 - \frac{1}{2} = \frac{14}{4}.$$

Since value for parabola is greater than value for the line, parabola is the upper bound.

$$\int_0^1 \int_{4-x}^{4-x^2} -2y \, dy \, dx$$

$$= \int_0^1 \left(-y^2 \Big|_{y=4-x}^{y=4-x^2} \right) dx$$

$$= \int_0^1 \left(-(4-x^2)^2 + (4-x)^2 \right) dx$$

$$= \int_0^1 \left(\cancel{-16} + 8x^2 - x^4 \cancel{+16} - 8x + x^2 \right) dx.$$

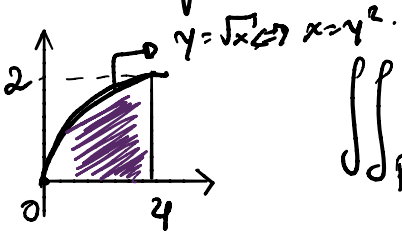
$$= \int_0^1 (-x^4 + 9x^2 - 8x) \, dx$$

$$= -\frac{x^5}{5} + 3x^3 - 4x^2 \Big|_{x=0}^{x=1}.$$

$$= -\frac{1}{5} + 3 - 4 = -\frac{1}{5} - 1 = -\frac{6}{5}.$$

Exercise 3: $\displaystyle\iint_R \frac{y}{1+x^2}\, dA$

R: region bounded by $y=0$, $y=\sqrt{x}$, $x=4$.



$$\iint_R \frac{y}{1+x^2}\, dA = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\, dy\, dx$$

$$= \int_0^2 \int_{y^2}^4 \frac{y}{1+x^2}\, dx\, dy.$$

$$\int_0^4 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\, dy\, dx = \int_0^4 \left[\frac{y^2}{2(1+x^2)} \Big|_{y=0}^{y=\sqrt{x}} \right] dx$$

$$= \int_0^4 \left[\frac{x}{2(1+x^2)} - \cancel{\frac{0^2}{2(1+x^2)}} \right] dx$$

$$= \int_0^4 \frac{x}{2(1+x^2)}\, dx.$$

Substitute: $u = 1+x^2 \Rightarrow du = 2x\,dx$.

$$\int_0^4 \frac{x}{2(1+x^2)}\, dx = \int_1^{17} \frac{1}{4u}\, du = \frac{1}{4} \log(u) \Big|_{u=1}^{u=17}$$

$$= \frac{\log(17)}{4}.$$

$$\int_0^2 \int_{y^2}^4 \frac{y}{1+x^2}\, dx\, dy = \int_0^2 \left(y \cdot \arctan(x) \Big|_{x=y^2}^{x=4} \right) dy$$

$$= \int_0^2 \left(y \cdot \arctan(4) - y \cdot \arctan(y^2) \right) dy.$$

Finish with $u = y^2$; $du = 2y\,dy$.

<u>Example 6</u>: Improper double integrals.

$$\iint_R \frac{1}{(x^2+1)(y^2+1)}\, dA,$$

R: first quadrant. $\rightarrow$ $0 \le x < \infty$, $0 \le y < \infty$.

$$\iint_R \frac{1}{(x^2+1)(y^2+1)}\, dA = \int_0^\infty \int_0^\infty \frac{1}{(x^2+1)(y^2+1)}\, dx\, dy$$

This may not make sense. We will integrate each step at a time:

$$\int_0^\infty \frac{1}{(x^2+1)(y^2+1)}\, dx = \lim_{t\to\infty} \int_0^t \frac{1}{(x^2+1)(y^2+1)}\, dx.$$

$$= \lim_{t\to\infty} \frac{1}{(y^2+1)} \left(\int_0^t \frac{1}{x^2+1}\, dx \right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{y^2+1} \right) \left(\arctan(x) \Big|_{x=0}^{x=t} \right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{y^2+1} \right) \left(\arctan(t) \right)$$

$$\int_0^\infty \frac{1}{(1+x^2)(1+y^2)}\,dx = \lim_{t\to\infty} \frac{\arctan(t)}{1+y^2}$$

$$= \frac{\pi}{2(1+y^2)}$$

Thus

$$\iint_R \frac{1}{(x^2+1)(y^2+1)}\,dA = \int_0^\infty \frac{\pi}{2(1+y^2)}\,dy.$$

$$= \frac{\pi}{2}\left(\lim_{s\to\infty} \int_0^s \frac{1}{(1+y^2)}\,dy\right)$$

$$= \frac{\pi}{2}\cdot\left(\lim_{s\to\infty} \left(\arctan(y)\Big|_{y=0}^{y=s}\right)\right)$$

$$= \frac{\pi}{2}\cdot\left(\lim_{s\to\infty} \arctan(s)\right)$$

$$= \frac{\pi}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi^2}{4}.$$