MAT324: Real Analysis – Fall 2014
ASSIGNMENT 4 – SOLUTIONS

Problem 1: Let $\mathcal{C} \subset [0,1]$ be the Cantor middle-thirds set. Suppose that $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = 0$ for $x \in \mathcal{C}$ and $f(x) = k$ for all x in each interval of length 3^{-k} which has been removed from $[0,1]$ at the k^{th} step of the construction of the Cantor set. Show that f is measurable and calculate $\int_{[0,1]} f dm$.

SOLUTION. Denote by $f_n : [0,1] \to \mathbb{R}$ the function constructed following way: If $\mathcal{C}_k$ denotes the union of the intervals of lenght 3^{-k} removed in the k-th step of the construction of the Cantor middle-third set, let $f_n(x) = k$ for $x \in \mathcal{C}_k$, and zero elsewhere. Then f_n is a simple. function (it only takes $(n+1)$ values). Furthermore, it is easy to see that $f_n \to f$ pointwise, hence f is a measurable function. In addition, the sequence f_n is increasing to f, hence the Monotone Convergence Theorem gives us

$$\begin{aligned} \int_{[0,1]} f dm &= \lim_{n\to\infty} \int_{[0,1]} f_n dm \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} k 2^{k-1} 3^{-k} \right) \\ &= \tfrac{1}{3} \lim_{n\to\infty} \left[\sum_{k=1}^{n} k \left(\frac{2}{3}\right)^{k-1} \right] \end{aligned}$$

The answer up to this point is fine. With a little more effort, one can get the answer 3. This uses the following relation:

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \quad \text{if } \ 0 < |x| < 1$$

There are a number of ways one can use to prove this fact, including Riemman sums and Taylor's formula. $\square$

Problem 2: Suppose $\{f_n\}$ is a sequence of non-negative measurable functions on $E \in \mathcal{M}$. If $\{f_n\}$ decreases to f almost everywhere and $\int_E f_1 dm < \infty$, then show that

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f dm.$$

Hint: Look at the sequence $g_n = f_1 - f_n$.

SOLUTION. Consider the sequence of measurable functions $g_n = f_1 - f_n$. Since $\{f_n\}$ is a decreasing sequence, the sequence $\{g_n\}$ is an increasing sequence of nonnegative measurable functions converging to $g = (f_1 - f)$. By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_E g_n dm = \int_E g dm.$$

On the other hand, since $\int_E f_1 dm < \infty$, and the f_n's decrease, monotonicity gives us $\int_E f_n dm < \infty$. Then, for each $n \in \mathbb{N}$, we have

$$\int_E g_n dm = \int_E (f_1 - f_n) dm = \int_E f_1 dm - \int_E f_n dm.$$

Likewise,

$$\int_E g dm = \int_E (f_1 - f) dm = \int_E f_1 dm - \int_E f dm.$$

The result now follows from cancellation (notice that it is necessary to assume $\int_E f_1 dm < \infty$ for this). $\square$

Problem 3: Suppose $\{f_n\}$ is a sequence of non-negative measurable functions. Show that

$$\int \sum_{n=1}^{\infty} f_n dm = \sum_{n=1}^{\infty} \int f_n dm.$$

Solution. Consider the sequence of measurable functions $g_n = \sum_{k=1}^{n} f_n$. This sequence is increasing, and $g_n(x) \to g(x) = \sum_{k=1}^{\infty}$, for every x (convergence is taken in the extended real line). Apply the monotone convergence theorem. $\square$

Problem 4: Prove that if f is integrable on $\mathbb{R}$ and $\int_E f(x) dm \geq 0$ for every measurable set E, then $f(x) \geq 0$ a.e. x.

Solution. Since f is integrable, it is in particular measurable. Let E be the measurable set $E = \{x | f(x) < 0\}$. By hypothesis, and using monotonicity of the integral

$$0 \leq \int_E f(x) dm \leq \int_E 0 dm = 0 \Rightarrow \int_E f(x) dm = 0$$

Notice that $-f$ is a positive function on E, and

$$\int_E (-f(x)) dm = 0.$$

Now Theorem 4.4 implies that $-f$ is zero almost everywhere. By the definition of E, this happens if and only if E has zero measure. $\square$

Problem 5: Let E be a measurable set. Suppose $f \geq 0$ and let $E_k = \{x \in E \mid 2^k < f(x) \leq 2^{k+1}\}$ for any integer k. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} E_k = \{x \in E \mid f(x) > 0\},$$

and the sets E_k are disjoint.

(a) Prove that f is integrable if and only if $\sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$.

(b) Let $a > 0$ and consider the function

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Use part a) to show that f is integrable on $\mathbb{R}$ if and only if $a < 1$.

Solution.

(a) Suppose f is integrable. Since $f(x) > 2^k$ on E_k, we have

$$\int_{E_k} f dm \geq \int_{E_k} 2^k dm = 2^k m(E_k)$$

Therefore, by the comparison test,

$$\sum_{k=-\infty}^{\infty} 2^k m(E_k) \leq \sum_{k=-\infty}^{\infty} \int_{E_k} f dm = \int_{\mathbb{R}} f dm < \infty$$

Next suppose $\sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$. Then $2\left(\sum_{k=-\infty}^{\infty} 2^k m(E_k)\right) = \sum_{k=-\infty}^{\infty} 2^{k+1} m(E_k) < \infty$. Since $f(x) \leq 2^{k+1}$ on E_k, we have

$$\int_{E_k} f dm \leq \int_{E_k} 2^{k+1} dm = 2^{k+1} m(E_k).$$

Then

$$\int_{\mathbb{R}} f dm = \sum_{k=-\infty}^{\infty} \int_{E_k} f dm \leq \sum_{k=-\infty}^{\infty} 2^{k+1} m(E_k) < \infty,$$

and f is integrable.

(b) Following part a), we need to find the measure of the sets E_k. If $K \geq 0$, then

$$2^k < |x|^{-a} \leq 2^{k+1},$$

and

$$\begin{aligned} 2^{-k} &> |x|^a \geq 2^{-k-1} \\ 2^{\frac{-k}{a}} &> |x| \geq 2^{\frac{-k-1}{a}} \end{aligned}$$

Then $m(E_k) = 2 \cdot 2^{\frac{-k-1}{a}}(2^{\frac{1}{a}} - 1)$. If $k < 0$, then $2^k < |x|^{-a} \leq 2^{k+1}$ implies $|x| \geq 1$, hence $m(E_k) = 0$, if $k < 0$. Thus,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m(E_k) &= \sum_{k=-0}^{\infty} 2^{k+1} \cdot 2^{\frac{-k-1}{a}}(2^{\frac{1}{a}} - 1) \\ \sum_{k=-\infty}^{\infty} 2^k m(E_k) &= (2^{\frac{1}{a}} - 1) \sum_{k=0}^{\infty} 2^{\frac{(k+1)(a-1)}{a}} \\ \sum_{k=-\infty}^{\infty} 2^k m(E_k) &= (2^{\frac{1}{a}} - 1) \sum_{k=0}^{\infty} \left[2^{\frac{(a-1)}{a}}\right]^{(k+1)} \end{aligned}$$

Notice that this geometric series converge if and only if $2^{\frac{(a-1)}{a}} < 1$, and this happens if and only if $a < 1$. $\square$