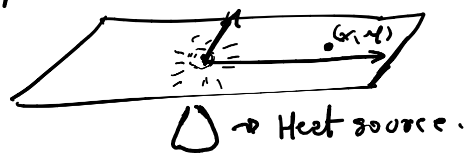
MAT 203 - Lecture 5

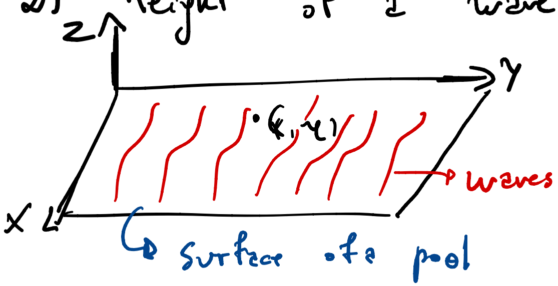
Multivariable functions: scalar-valued functions depending on multiple parameters.

Examples:

1) 

$T(x,y,t)$: temperature at (x,y) at time t.

2) Height of a wave



$h(x,y,t)$ height of wave at time t

3) Profit of a manufactured good:

p(cost of manufacturing, sale value)

Domain and Range

Domain: vector spaces or subsets thereoff.

Recall: domain is the collection of values for the varibles for which the function makes sense.

Range: subsets of real numbers.

Examples:

1) $f(x, y) = x + y^2$

Domain: $\mathbb{R}^2$

Range: $\mathbb{R}$

2) $f(x, y) = y/x$

Domain: $\{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$.

Range: $\mathbb{R}$

3) $f(x, y, z) = \sin(x) \cdot \sin(y) \cdot \sin(z)$

Domain: $\mathbb{R}^3$

Range: $[-1, 1] = \{ w \in \mathbb{R} \mid -1 \leq w \leq 1 \}$.

4) $f(x,y,z) = \sqrt{1-x^2-y^2-z^2}$

Restriction: $(x^2+y^2+z^2) \leq 1$.

Domain $= \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1 \}$.

Range: $[0,1]$.

Factors that impose restrictios

- divisions
- radicals
- logarithms
- trigonometric functions other than sine or cosine.

Representations of multivariable functions

For a single variable function, the graph is a one-dimensional object within a 2-dimensional plane

For a function of two variables, the graph is a two-dimensional object within 3-dimensional space.

For functions of three or more variables graphs become less significant.

Level set:

For a given function $f(\vec{u})$ and a given value c (a scalar) we call the set

$$\{\vec{u} \mid f(\vec{u}) = c\}$$

the c-level set of the function.

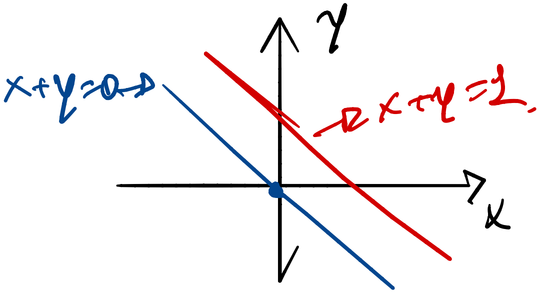
Examples

5) $f(x, y) = x + y.$

$c = 0$

What is the 0-level set? That is, what is the collection of points for which

$$f(x, y) = x + y = 0 \, ?$$



1-level set:

$x + y = 1.$

6) $f(x,y) = x^2+y^2$.

Three types of level sets

i) empty, when $c<0$.

ii) a point, when $c=0$,

$$x^2+y^2 = 0 \Rightarrow (x,y)=(0,0).$$

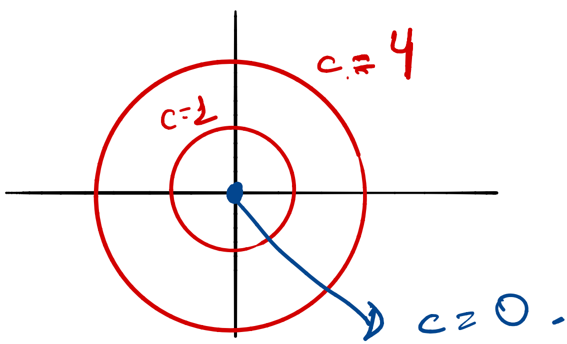
iii) circles, when $c>0$.

$$x^2+y^2 = c$$

$$\sqrt{x^2+y^2} = \sqrt{c}$$

$$\text{dist}((x,y),(0,0)) = \sqrt{c}$$

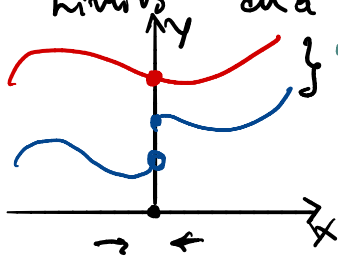
equation for a circle, centered at $(0,0)$, with radius $\sqrt{c}$.



The representation of functions by level sets is called a contour plot.

In 3D, we obtain a concept called a level surface. Consult textbook for pictures.

- Limits and continuity.



one-dimensional graphs of single-variable functions.

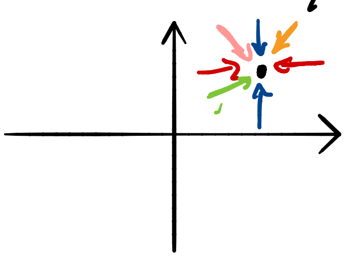
One-variable limits exist when left and right approximations agree.

limit exists

limit does not exist.

In a multivariable context, we can approach points infinitely many directions.

Domain of → a function of two variables



Conceivably, each such approximation could yield a different value.

Examples

7) Consider the function

$$f(x,y) = 2x + 3y.$$

Near $(0,0)$, the values of f are uniformly small:

$$\begin{aligned} |f(x,y)| &= |2x+3y| \\ &\leq |2x| + |3y| \\ &\leq 2|x| + 3|y|. \\ &\leq 5 \cdot \max\{|x|, |y|\}. \end{aligned}$$

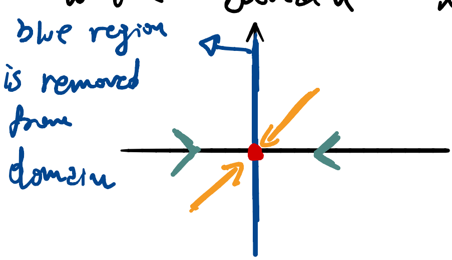
As $x, y \to 0$, $f(x,y) \to 0$. We say

$$\lim_{(x,y)\to 0} 2x + 3y = 0.$$

8) Consider the function

$$f(x,y) = y/x,$$

with domain $x \neq 0$.



We can estimate values of $f(x,y)$ from any direction other than vertical.

For example, along the x-axis: $y = 0$,
hence $f(x, 0) = \frac{0}{x} = 0$ ($x \neq 0$).

$$\lim_{x \to 0} f(x, 0) = 0.$$

Another example: approximating along $y = x$. ($x \neq 0$)

$$f(x, y) = \frac{y}{x} = 1.$$

$$\lim_{x \to 0} f(x, x) = 1.$$

Since approximations from different directions yield different results, we function does not have a <u>limit in multivariable sense</u>.

<u>Definition</u>: A multivariable function has a limit (in multivariable sense) at p if the one-variable approximations from

each direction agree.

Remark: "Each direction" includes oblique directions, not only coordinate axis.

Exercise 1: Determine whether

$$f(x,y) = \frac{xy}{x^2+y^2} \quad ((x,y) \neq (0,0))$$

has a limit as $(x,y) \to (0,0)$.

Hint: Analyze function along lines of different slope.

Solution: Along x-axis ($y=0$), we find

$$f(x,0) = \frac{x \cdot 0}{x^2+0^2} = 0.$$

Along y-axis, $x=0$, so

$$f(0,y) = \frac{0 \cdot y}{0^2+y^2} = 0.$$

Along line with slope c: $y = cx$, we find

$$f(x, cx) = \frac{x \cdot (cx)}{x^2 + (cx)^2}$$

$$= \frac{cx^2}{x^2 + c^2x^2}$$

$$= \frac{c}{(1+c^2)} \cdot \frac{\cancel{x^2}}{x^2}$$

$$= \frac{c}{1+c^2}$$

This function is <u>constant along each line</u> through the origin, but the values of the constants change depending on the slope of the line. The function has no multivariable limit.

<u>Exercise 2</u>: Determine if the function

$$f(x, y) = \frac{x}{x^2 + y^2}$$

has a limit, in multivariable sense, as $(x, y) \to (0, 0)$.

Solutions: limit along y-axis

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0}{0^2+y^2} = 0.$$

Limit along a line of slope c: $y = cx$

$$\lim_{x\to 0} f(x,cx) = \lim_{x\to 0} \frac{x}{x^2+(cx)^2}$$

$$= \lim_{x\to 0} \frac{x}{x^2(1+c^2)}$$

$$= \lim_{x\to 0} \frac{1}{x(1+c^2)} \quad \begin{cases} +\infty, & \text{if } x>0 \\ -\infty, & \text{if } x<0. \end{cases}$$

No limits along lines of slope c, no multivariable limit.

Exercise 3: Compute the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$$

if it exists. If it doesn't exist, describe why.

Solution: Along y-axis,

$$\lim_{y \to 0} \frac{0^2 \cdot y^2}{0^2 + y^2} = 0.$$

Along a line with slope c,

$$\lim_{x \to 0} \frac{x^2 \cdot (cx)^2}{x^2 + (cx)^2} = \lim_{x \to 0} \frac{c^2 x^4}{(1+c^2)x^2}$$

$$= \lim_{x \to 0} \left(\frac{c^2}{1+c^2}\right) \cdot x^2$$

$$= 0.$$

As limits agree in all directions,

$$\boxed{\lim_{(x,y) \to (0,0)} \frac{x^2 y^2}{x^2 + y^2} = 0.}$$

Just like in the single-variable case, we say that a function is continuous if its value agrees with the limits

$$\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b).$$

Properties:

i) the sum of continuous functions is continuous

ii) products of continuous functions are continuous

iii) ratios are continuous wherever the denominator is non-zero.

iv) compositions are continuous in their appropriate domain.

Examples: polynomials, exponentials, sine, cosine.

Warning: Marginal continuity, that continuity in each variable, is not enough to guarantee continuity in the multivariable sense.

Derivatives

Recall that in single-variable calculus, derivatives were defined as limits:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This means that

$$(*) \quad f(x+h) = f(x) + f'(x) \cdot h + \left(\begin{array}{c}\text{error in } h^2 \\ \text{or higher order}\end{array}\right)$$

$h = 0.1 \to \text{error } 0.01$ or lower

$h = 0.01 \to \text{error} \sim 0.0001$ or lower.

In multivariable sense we do something similar to (*). We define a (total) derivative as a linear map satisfying

$$f(x+h) = f(x) + dh(x) \cdot (h) + (\text{higher order})$$

- Linear map: multiplication by a row-matrix.

Example 9: $f(x,y) = 1.$

Then for a point (x,y),

$$df(x,y) = (0 \quad 0)$$

$$f(x+h_x, y+h_y) = f(x,y) + df(x,y)\cdot\begin{pmatrix} h_x \\ h_y \end{pmatrix}$$

$$1 = 1 + \underbrace{(0 \quad 0)\cdot\begin{pmatrix} h_x \\ h_y \end{pmatrix}}_{=0} + \underbrace{\text{error}}_{=0}.$$

Example 10: $f(x,y) = 2x + 3y.$

Claim: $df(x,y) = (2 \quad 3)$

$$f(x+h_x, y+h_y) = 2(x+h_x) + 3(y+h_y)$$

$$f(x+h_x, y+h_y) = (2x+3y) + (2h_x + 3h_y).$$

$$f(x+h_x, y+h_y) = f(x,y) + (2 \quad 3)\begin{pmatrix} h_x \\ h_y \end{pmatrix}$$

$$\boxed{\text{error} = 0}$$

<u>Example 11:</u> $f(x,y) = x^2 + 2y^2$.

$$f(x+h_x, y+h_y) = (x+h_x)^2 + 2(y+h_y)^2$$
$$= x^2 + 2xh_x + h_x^2 + 2y^2 + 4yh_y + 2h_y^2.$$
$$= (x^2 + 2y^2) + (2xh_x + 4yh_y)$$
$$+ (h_x^2 + 2h_y^2).$$
$$= f(x,y) + (2x \;\; 4y)\begin{pmatrix} h_x \\ h_y \end{pmatrix} + \begin{pmatrix} \text{higher} \\ \text{order} \end{pmatrix}$$

Derivative: $df(x,y) = (2x \;\; 4y)$

<u>Partial (marginal) derivatives</u>

Given a function $f(x,y,z)$ we can define its partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$$

(often abbreviated by $\partial_x f$, $\partial_y f$, $\partial_z f$) as derivatives with other variables fixed.

<u>Example 12:</u> Describe partial derivatives of $f(x,y,z) = 2x + 4y + 3xz$.

$$f(x,y,z) = 2x + 4y + 3xz.$$

$$\partial_x f = 2 + 0 + 3z = 2 + 3z.$$
$$\partial_y f = 0 + 4 + 0 = 4$$
$$\partial_z f = 0 + 0 + 3x = 3x.$$

Exercise 4: Find partial derivatives of
$$f(x,y) = 2x + 4xy$$

Solution:
$$\partial_x f = 2 + 4y = 2 + 4y$$
$$\partial_y f = 0 + 4x = 4x$$

Exercise 5: Find partial derivatives of
$$f(x,y) = x^2 + 2y^2$$

Solution:
$$\partial_x f = 2x$$
$$\partial_y f = 4y.$$

The concept of multiple partial derivatives makes sense.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

We write these as

$$\frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial x^2} \rightarrow \text{long notation}$$

or

$$\partial_{xy} f, \quad \partial_{xx} f. \rightarrow \text{short notation.}$$

<u>Example 13</u>: Consider the function

$$f(x,y) = e^x + xy^2.$$

Find all second-order derivatives.

$$\partial_x f = e^x + y^2$$

$$\partial_y f = 2xy.$$

$$\partial_{xx} f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(e^x + y^2) = e^x$$

$$\partial_{xy} f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2xy) = 2y.$$

$$\partial_{yx} f = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(e^x + y^2\right) = 2y.$$

$$\partial_{yy} f = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(2xy\right) = 2x.$$

List: $\partial_{xx} f = e^x$, $\partial_{xy} f = \partial_{yx} f = 2y$, $\partial_{yy} f = 2x$.

Under "mild conditions" mixed partials agree:

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and so on, for other pairs of variables.

Here mild conditions means that $\partial_{xy} f$ and $\partial_{yx} f$ are continuous.

- Properties of partial derivatives (u is a placeholder)

i) $\partial_u (f+g) = \partial_u f + \partial_u g$

ii) $\partial_u (c \cdot f) = c \cdot \partial_u f$, where c is a constant

iii) (Product rule)

$$\partial_u (f g) = (\partial_u f) \cdot g + f \cdot (\partial_u g).$$

iv) (Quotient rule)

$$\partial_u \left(\frac{f}{g} \right) = \frac{(\partial_u f)\, g - f\, (\partial_u g)}{g^2}.$$

- Properties of differentials:

v) $d(f+g) = (df) + (dg)$

vi) $d(c \cdot f) = c \cdot (df)$, where c is constant

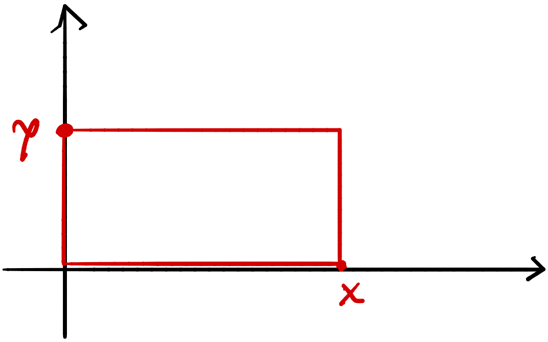
vii) $d(f \cdot g) = (df) \cdot g + f \cdot (dg)$

viii) $d\left(\frac{f}{g} \right) = \frac{(df) \cdot g - f\, (dg)}{g^2}$

<u>Chain rule I</u>

Suppose a function $f = f(x, y)$ is given. Now suppose x, y themselves are functions of an independent variable t: $x = x(t)$, $y = y(t)$.

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$



side lengths x, y, depend on t.

Area of rectangle

$$\begin{cases} A(x,y) = x \cdot y. \\ A(t) = x(t) \cdot y(t). \end{cases}$$

$$\frac{dA}{dt} = \frac{d}{dt}\left(x(t) \cdot y(t)\right)$$

$$= \frac{dx}{dt} \cdot y(t) + x(t) \cdot \frac{dy}{dt}.$$

$$\frac{\partial A}{\partial x} = y, \quad \frac{\partial A}{\partial y} = x.$$

$$\boxed{\frac{dA}{dt} = \frac{\partial A}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial A}{\partial y} \cdot \frac{dy}{dt}.}$$

Comparing orders (in context of derivatives):

if $\lim_{h \to 0} \frac{f(h)}{g(h)} = 0$

then we say that f has higher order than g.

<u>Example</u>: $f(x) = x^2$, $g(x) = x$.

$$\lim_{h \to 0} \frac{f(h)}{g(h)} = \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0.$$

<u>Example</u>: $f(x) = x$, $g(x) = x^2$

$$\lim_{h \to 0} \frac{f(h)}{g(h)} = \lim_{h \to 0} \frac{h}{h^2} = \lim_{h \to 0} \frac{1}{h}, \text{ DNE.}$$

<u>Example</u>: $f(x) = \sin(x)$, $g(x) = x$.

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1.$$

Near 0, $\sin(h) \sim h$.

If limit exists and is non-zero, we call functions comparable.