MAT 203 - Lecture 6

Remark: Existence of limits along lines does not imply the existence of a multivariable limit.

Example: $$f(x, y) = \frac{x^2 y}{x^4 + y^2}$$

limits along lines:

i) y-axis $x = 0$:

$$f(0, y) = 0 \qquad (y \neq 0).$$

ii) $y = kx$,

$$f(x, kx) = \frac{x^2 \cdot kx}{x^4 + (kx)^2}$$

$$f(x, kx) = \frac{kx^3}{x^4 + k^2x^2}$$

$\boxed{k \neq 0}$

$$\cong \frac{kx^3}{k^2x^2}$$

$$\cong x/k.$$

As $x \to 0$, $k \neq 0$, $f(x, kx) \to 0$.

If $k = 0$: $\frac{kx^3}{x^4 + k^2x^2} = \frac{0 \cdot x^3}{x^4 + 0} = 0.$

Hint: study limits along parabolae,
$$y = kx^2,$$
for different values of k.

Back to derivatives

Yesterday we discussed the first form of the Chain rule: if $f(x, y, z)$ and x, y, z depend on a common (independent) parameter, say t. Then

Chain Rule I:
$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}.$$

Example 1: Suppose the cost of manufacturing and sale value of a product depend on time:
$$c(t) = 10 + 2t$$
$$s(t) = 50 - 0.01t^2.$$
We want to find how the profit changes

over time.

profit: $p(c,s) = s - c$ $\quad \dfrac{\partial p}{\partial c} = -1$ $\quad \dfrac{\partial p}{\partial s} = 1.$

As a function of time:

$$p(t) = s(t) - c(t)$$

$$= 50 - 0.01t^2 - 10 - 2t.$$

$$= 40 - 2t - 0.01t^2.$$

$$\frac{dp}{dt} = 0 - 2 - 0.01 \cdot 2t$$

$$\boxed{\frac{dp}{dt} = -2 - 0.02t}$$

Second approach: using chain rule,

$$\frac{dp}{dt} = \frac{\partial p}{\partial c} \cdot \frac{dc}{dt} + \frac{\partial p}{\partial s} \cdot \frac{ds}{dt}$$

$$\frac{dp}{dt} = (-1)\frac{d}{dt}(10+2t) + 1 \cdot \frac{d}{dt}(50 - 0.01t^2)$$

$$dp/dt = (-1)\cdot 2 - 0.02t$$

$$\boxed{dp/dt = -2 - 0.02t}.$$

Exercise 1: Given functions

$$x(t) = e^t$$

$$y(t) = \sin(t)$$

and $f(x, y) = x \cdot y + x^2$.

Find $\frac{df}{dt}$ by using the chain rule.

Solution:

$$\frac{dx}{dt} = e^t \quad ; \quad \frac{dy}{dt} = \cos(t).$$

$$\frac{\partial f}{\partial x} = y + 2x \quad ; \quad \frac{\partial f}{\partial y} = x$$

$$\frac{df}{dt} = (y + 2x) \cdot e^t + x \cdot \cos(t).$$

$$= (\sin(t) + 2e^t)\, e^t + e^t \cdot \cos(t).$$

$$\boxed{\frac{df}{dt} = 2e^{2t} + e^t(\sin(t) + \cos(t))}$$

Example 2: Suppose dependent variables x, y, z are related by

$$x^2+y^2+z^2=1.$$

In addition, assume

$$x(t) = \cos(2t) \to \frac{dx}{dt} = -2\sin(2t)$$
$$y(t) = \sin(t). \to \frac{dy}{dt} = \cos(t)$$

Question: What is

$$\frac{dz}{dt}$$

when $z = 1/2$?

Solution: $x^2+y^2+z^2=1$

Implicitly differentiating (relative to t) we find

$$2x(t)\cdot\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0.$$

$$2\cos(2t)(-2\sin(2t)) + 2\sin(t)\cos(t) + 2\cdot\frac{1}{2}\cdot\frac{dz}{dt} = 0$$

$$-4\sin(2t)\cos(2t) + 2\sin(t)\cos(t) + \frac{dz}{dt} = 0.$$

$$\boxed{\frac{dz}{dt} = 4\sin(2t)\cos(2t) - 2\sin(t)\cos(t).}$$

• Back to equation (z as a function of (x,y))

$$x^2 + y^2 + z^2 = 1.$$

Differentiate relative to x:

$$\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(z^2) = \frac{\partial}{\partial x}(1)$$

$$2x + 0 + 2z \cdot \frac{\partial z}{\partial x} = 0.$$

$$\boxed{2x + 2z\frac{\partial z}{\partial x} = 0.}$$

Similarly,

$$\boxed{2y + 2z\frac{\partial z}{\partial y} = 0.}$$

At $z = \frac{1}{2}$:

$$\frac{\partial z}{\partial x} = -2x \quad ; \quad \frac{\partial z}{\partial y} = -2y.$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$, same equation.

<u>Chain rule II</u>: the case of multiple independent variables.

Setting: $f(x,y)$ is given.

Each of x, y depend on parameters p, q.

Question: how to express $\partial f/\partial p$ or $\partial f/\partial q$ in terms of known derivatives?

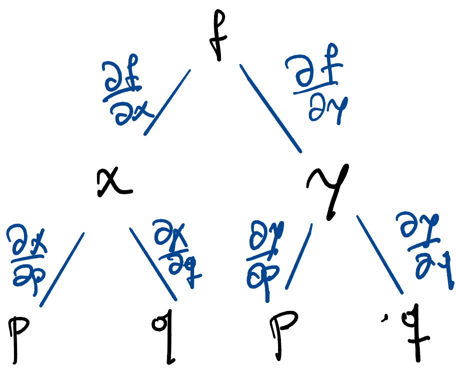
<u>Chain rule II</u>

$$\frac{\partial f}{\partial p} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial p} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial p}$$

and

$$\frac{\partial f}{\partial q} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial q}.$$

<u>Diagram</u>:

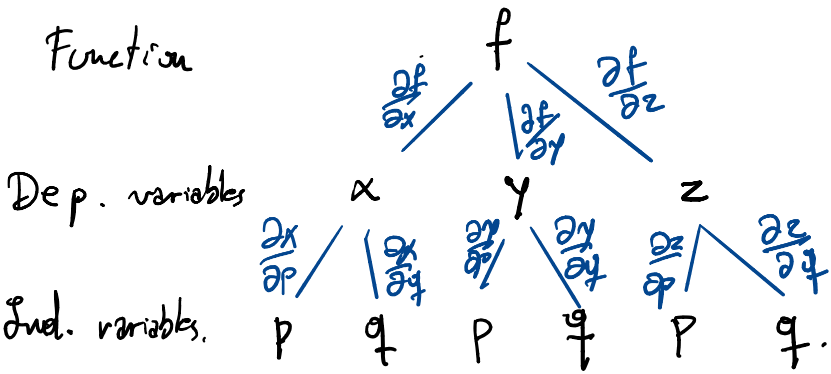


To find $\partial f/\partial p$ add all paths starting at f, ending at p.

Another example:

$f(x, y, z)$ is given

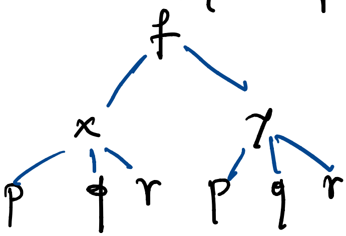
x, y, z depend on p, q.



$$\boxed{\frac{\partial f}{\partial q} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial q} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial q} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial q}}$$

Last one! $f(x, y)$ is given

x, y depending on p, q, r.



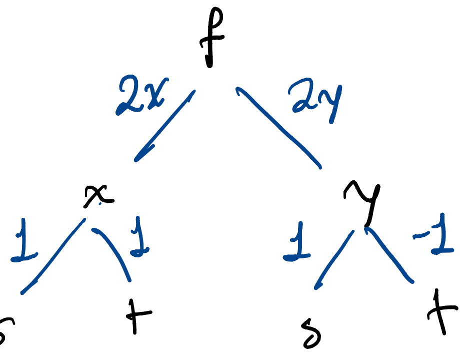
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}.$$

Exercise 2:

$$f(x,y) = x^2 + y^2$$
$$x = s + t$$
$$y = s - t.$$

$$\frac{\partial f}{\partial s} = ?$$

Solution:



$$\frac{\partial f}{\partial s} = 2x \cdot 1 + 2y \cdot 1,$$
$$= 2(s+t) + 2(s-t)$$
$$= 2s + 2t + 2s - 2t$$
$$= 4s.$$

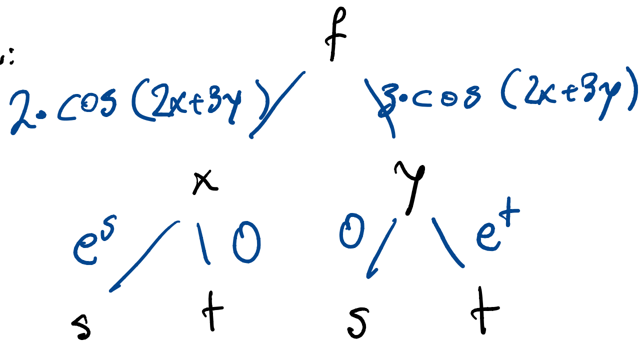
Exercise 3: $f(x,y) = \sin(2x+3y)$

$x(s,t) = e^s$

$y(s,t) = e^t$.

$$\frac{\partial f}{\partial t} = ?$$

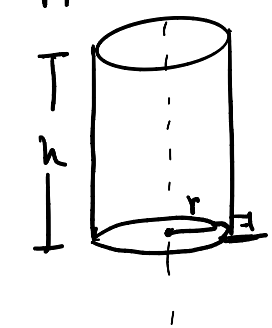
Solution:



$$\frac{\partial f}{\partial t} = \cancel{2\cos(2x+3y)\cdot 0} + 3\cos(2x+3y)\cdot e^t.$$

$$\frac{\partial f}{\partial t} = 3 \cdot \cos(2e^s + 3e^t) \cdot e^t.$$

Application



A right circular cilinder has time-dependent radius and height:

- $\dfrac{dr}{dt} = 6$ in / min
- $\dfrac{dh}{dt} = -4$ in / min,

When $r = 12$, $h = 36$

(a) What is the rate of change of volume over time?

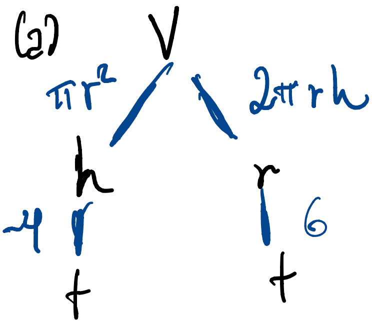
(b) what is the rate of change of the surface area over time?

Solution: $V = \pi r^2 h$

$A = 2\pi r h + 2\pi r^2.$

- $\dfrac{\partial V}{\partial r} = 2\pi r \cdot h$ $\qquad \dfrac{\partial V}{\partial h} = \pi r^2$
- $\dfrac{\partial A}{\partial r} = 2\pi h + 4\pi r$ $\qquad \dfrac{\partial A}{\partial h} = 2\pi r.$

(a)

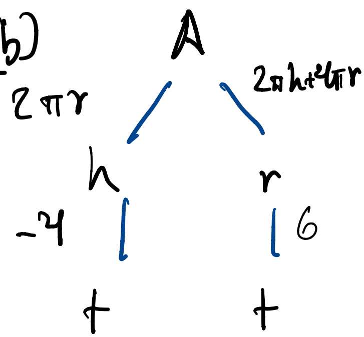


$$\frac{dV}{dt} = \pi r^2 \cdot (-4) + 2\pi r h \cdot 6.$$

$$= \pi \cdot (12)^2 (-4) + 2\pi (12)(36) \cdot 6.$$

= homework.

(b)



$$\frac{dA}{dt} = (2\pi r) \cdot (-4) + (2\pi h + 4\pi r)$$

$$= (2\pi \cdot 12)(-4)(2\pi \cdot 36 + 4 \cdot \pi \cdot 12).$$

= homework

Exercise 4:

$$f(x,y,z) = x^2 + y^2 + z^2.$$

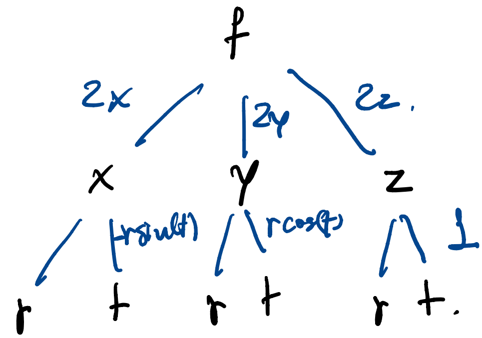
$$x(r,t) = r\cos(t) \rightarrow \frac{\partial x}{\partial t} = -r\sin(t)$$

$$y(r,t) = r\sin(t) \rightarrow \frac{\partial y}{\partial t} = r\cos(t)$$

$$z(r,t) = t. \rightarrow \frac{\partial z}{\partial t} = 1.$$

Find $\frac{\partial f}{\partial t}$.

Solution:



$$\frac{\partial f}{\partial t} = 2x \cdot (-r\sin(t)) + 2y \cdot r\cos(t) + 2z \cdot 1.$$

$$= 2(r\cos(t)) \cdot (-r\sin(t)) + 2 \cdot (r\sin(t)) \cdot r\cos(t) + 2t$$

$$= \cancel{-2r^2\sin(t)\cos(t)} + \cancel{2r^2\sin(t)\cos(t)} + 2t.$$

$$\boxed{\frac{\partial f}{\partial t} = 2t.}$$

Second derivatives and the Chain rule

Case 1: One independent variable.

Given $f(x,y)$, $x(t)$, $y(t)$. How to compute $\frac{d^2f}{dt}$?

- Example: $f(x,y) = 2x^2 + y^2$
  $x(t) = \cos(t)$
  $y(t) = \sin(t)$.

- $$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

$$= 4x \cdot (-\sin(t)) + 2y \cdot \cos(t).$$

$$= 4\cos(t) \cdot (-\sin(t)) + 2\sin(t) \cdot \cos(t).$$

$$= -4\sin(t)\cos(t) + 2\sin(t)\cos(t)$$

$$= -2\sin(t) \cdot \cos(t).$$

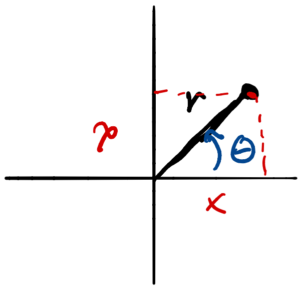
$$\frac{d^2f}{dt^2} = -2\left(\frac{d}{dt}\sin(t)\right) \cdot \cos(t) - 2\sin(t)\left(\frac{d}{dt}\cos(t)\right).$$

$$\frac{d^2f}{dt^2} = -2 \cdot \cos(t) \cdot \cos(t) + 2\sin(t) \cdot \sin(t).$$

$$\frac{d^2f}{dt^2} = -2\cos^2(t) + 2\sin^2(t).$$

Application: Changes to polar coordinates.

In the plane we can use another coordinate system:



r : length of line segment connecting point to origin

θ : angle between line segment and positive x-axis.

Relation to Cartesian coordinates:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Exercise: compute the partials

$$\frac{\partial x}{\partial r}, \quad \frac{\partial x}{\partial \theta}, \quad \frac{\partial y}{\partial r}, \quad \frac{\partial y}{\partial \theta}.$$

Solution:

$$\frac{\partial x}{\partial r} = \cos(\theta), \quad \frac{\partial x}{\partial \theta} = -r\sin(\theta)$$

$$\frac{\partial y}{\partial r} = \sin(\theta), \quad \frac{\partial y}{\partial \theta} = r\cos(\theta)$$

Exercise: $f(x,y) = x \cdot y$

Find polar derivatives

$$\frac{\partial f}{\partial r}, \quad \frac{\partial f}{\partial \theta}.$$

Solution:

$$\begin{tikzcd}
 & & f \arrow[dl, "y"'] \arrow[dr, "x"] & & \\
 & x \arrow[dl, "\cos(\theta)"'] \arrow[d, "-r\sin(\theta)"] & & y \arrow[d, "\sin(\theta)"'] \arrow[dr, "r\cos(\theta)"] & \\
r & \theta & & r & \theta
\end{tikzcd}$$

$$\frac{\partial f}{\partial r} = y \cdot \cos(\theta) + x \cdot \sin(\theta).$$

$$= r\sin(\theta)\cos\theta + r\cos(\theta) \cdot \sin(\theta)$$

$$= 2r\sin(\theta)\cos\theta$$

$$= r\sin(2\theta).$$

$$\frac{\partial f}{\partial \theta} = y \cdot (-r\sin(\theta)) + x \cdot r\cos(\theta)$$

$$= r\sin(\theta) \cdot (-r\sin(\theta)) + r\cos(\theta) \cdot r\cos(\theta)$$

$$= -r^2\sin^2\theta + r^2\cos^2\theta = r^2\cos(2\theta).$$