MAT 514 - Lecture 5

Complex Functions

Definition: A complex function is a function
$$f: G \subset \mathbb{C} \to \mathbb{C}$$
where G is any subset of $\mathbb{C}$.

Typically, G is an open, connected subset of $\mathbb{C}$.

Examples

1) Polynomials:
$$p(z) = a_n z^n + a_{n-1} \cdot z^{n-1} + \dots + a_1 z + a_0,$$
where $a_n, a_{n-1}, \dots, a_1, a_0$ are complex numbers ($a_n \neq 0$).

2) Rational functions: defined as quotients of polynomials, for example
$$r(z) = (3z + z^2 + 1) / (z + 2)$$

3) Möbius transformations: irreducible rational functions whose numerator and denominator have degree one, for example

$$m(z) = \frac{z+2}{3z-1}$$

4) Complex conjugation: $c : \mathbb{C} \longrightarrow \mathbb{C}$

$$c(z) = \bar{z}$$

$$c(a+bi) = a-bi,$$

where a, b are real.

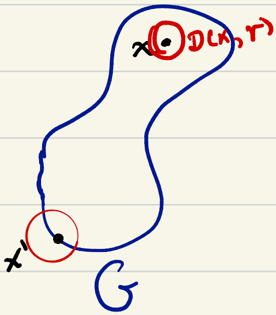
5) Norm: $n : \mathbb{C} \to \mathbb{C}$

$$n(z) = |z|$$

(actually real-valued).

Our goal is to study limits, continuity, differentiability, integration of such functions.

Limits: Consider a function
$$f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$$
and an accumulation point of G, say x.



A complex number L is called the limit of f at x, written
$$L = \lim_{z \to x} f(z),$$
if for every $\varepsilon > 0$, there exists an open disk $D(x, r)$ so that if $z \in D(x, r) \cap G$ then $|f(z) - L| < \varepsilon$.

Examples

(1) Consider a constant function $f : \mathbb{C} \to \mathbb{C}$.
$$f(z) = 1.$$
Choose as our guess $L = 1$. Then for any $x \in \mathbb{C}$, $\varepsilon > 0$ and all $z \in D(x, \varepsilon)$ have have

$$|z - x| < \varepsilon.$$

Meanwhile

$$\begin{aligned} |f(z) - \mathcal{L}| &= |\mathcal{L} - \mathcal{L}| \\ &= 0. \\ &< \varepsilon. \end{aligned}$$

It follows that for any $x \in \mathbb{C}$,

$$\lim_{z \to x} \mathcal{L} = \mathcal{L}.$$

Had $\mathcal{L}$ been replaced by any other complex number, say c, a similar result would hold:

$$\lim_{z \to x} c = c$$

② Consider the polynomial

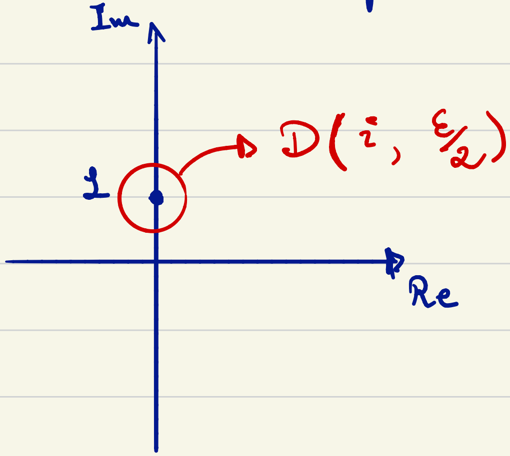
$$f(z) = 2z + i$$

Let us study this function at $x = i$. A natural candidate for a limit is

$$f(i) = 2i + i$$
$$= 3i.$$

This will be our test value, $L = 3i$.

Consider a positive real number $\varepsilon > 0$.



Suppose $z \in D(i, \varepsilon/2)$, that is
$|z - i| < \frac{\varepsilon}{2}$.

Then

$$|f(z) - 3i| = |2z + i - 3i|$$
$$= |2z - 2i|$$
$$= 2|z - i|$$
$$< 2 \cdot \frac{\varepsilon}{2}$$

Therefore,

$$|f(z) - 3i| < \varepsilon.$$

Exercise: You are given the function

$$f(z) = (3+4i)z + 3$$

Its limit at $x=0$ is 3,

$$\lim_{z \to 0} f(z) = 3.$$

Given an error $\varepsilon = 1$, how large can a disk centered at 0 be so that

$$|f(z) - 3| < 1,$$

for all points in this disk?

Solution: The function dilates its input by a factor of

$$|3+4i| = \sqrt{3^2 + 4^2}$$
$$= \sqrt{9+16}$$
$$= \sqrt{25} = 5$$

Choose radius

$$r = \varepsilon/5 = 1/5.$$

Suppose

$$|z - 0| < 1/5$$

$$\iff \quad |z| < 1/5.$$

Let us compare $f(z)$ and $L = 3$.

$$\begin{aligned} |f(z) - 3| &= |(3+4i)\cdot z + 3 - 3| \\ &= |(3+4i)z| \\ &= |3+4i| \cdot |z| \\ &= 5 \cdot |z| \\ &< 1, \end{aligned}$$

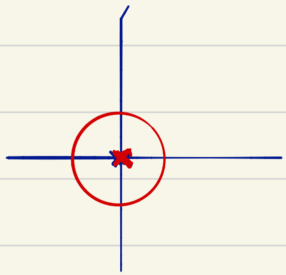
under the assumption $|z| < 1/5$, thus confirming what we knew,

$$\lim_{z\to 0} f(z) = 3.$$

③ $f: \mathbb{C} - \{0\} \longrightarrow \mathbb{C}$

$f(z) = \frac{1}{z}$.

0 is an accumulation point of $\mathbb{C} - \{0\}$, so it is conceivable that f has a limit at 0.



In real-variable Calculus, we'd say that

$$\lim_{x \to 0} \frac{1}{x}$$

does not exist, as the lateral limits

$$\lim_{x \to 0-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0+} \frac{1}{x} = +\infty$$

disagree.

In complex variables, the notions of $+\infty$, $-\infty$ are abandoned (there is no order). There is instead a single concept called infinity, ∞. This is an ideal complex object infinitely distant from any complex number.

Back to the function $f(z) = 1/z$. As $|z| \to 0$, $\left|\frac{1}{z}\right| \to +\infty$*, $\boxed{\lim_{z \to 0} \frac{1}{z} = \infty.}$

*: $\left|1/z\right|$ is a positive real number, therefore it makes sense to talk about convergence to $+\infty$.

Operations with infinity (in the context of limits)

1) Sums and subtractions with complex numbers

$$z + \infty = \infty$$

$$z - \infty = \infty$$

where z is a complex number (not ∞ itself).

2) Multiplications

$$z \cdot \infty = \infty,$$

where z is a non-zero complex number.

3) Division

$$\frac{\infty}{z} = \infty,$$

so long as $z \neq 0$ or $z \neq \infty$.

4) Multiplication of infinities:

$$\infty \cdot \infty = \infty.$$

5) Addition of infinities

$$\infty + \infty = \infty.$$

<u>Remark:</u> Certain operations cannot always be perform or are ambigoous: "$\infty - \infty$", "$0 \cdot \infty$", "$\frac{\infty}{\infty}$".