MAT 203 - Lecture 24

- Applications of triple integrals and other coordinate systems.

Mass of solids with variable density

Mass of a solid S with density $\rho(x,y,z)$

$$m = \iiint_S \rho(x,y,z)\, dV$$

Example 1: Solid: unit cube with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$.

Density: $\rho(x,y,z) = x^2+y^2+z^2$.

$$m = \iiint_S (x^2+y^2+z^2)\, dV = \int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\, dx\, dy\, dz$$

$$m = \int_0^1\int_0^1 \left[\frac{x^3}{3} + y^2x + z^2x \Big|_{x=0}^{x=1} \right] dy\, dz.$$

$$= \int_0^1\int_0^1 \left[\frac{1}{3} + y^2 + z^2 \right] dy\, dz$$

$$= \int_0^1 \left[\frac{y}{3} + \frac{y^3}{3} + z^2 y \Big|_{y=0}^{y=1} \right] dz$$

$$= \int_0^1 \left[\frac{1}{3} + \frac{1}{3} + z^2 \right] dz$$

$$= \int_0^1 \left(\frac{2}{3} + z^2 \right) dz$$

$$= \left[\frac{2z}{3} + \frac{z^3}{3} \Big|_{z=0}^{z=1} \right]$$

$$= \frac{2}{3} + \frac{1}{3}$$

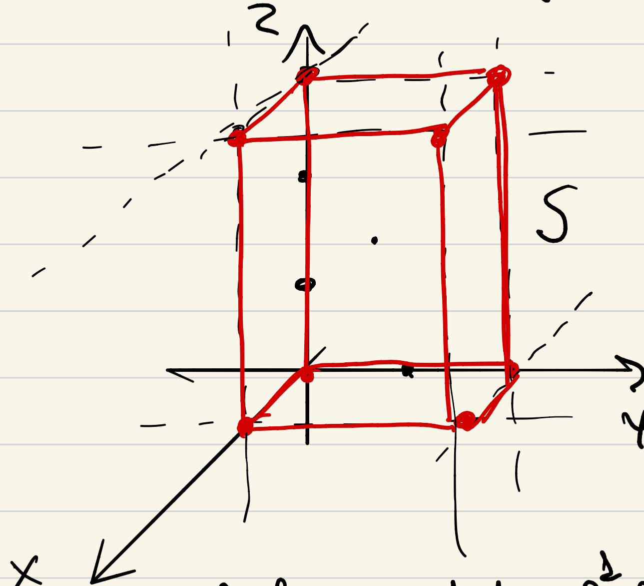
$$= 1.$$

Compare to volume computed in the previous lecture.

Example 2: Solid bounded by

$$\begin{cases} x = 0 \\ x = 1 \\ y = 0 \\ y = 2 \\ z = 0 \\ z = 3 \end{cases}$$

with density $\rho(x,y,z) = x \cdot y$.



$$m = \iiint_S xy \, dV = \int_0^1 \int_0^2 \int_0^3 xy \, dz \, dy \, dx$$

$$m = \int_0^1 \int_0^2 \left[xyz \Big|_{z=0}^{z=3} \right] dy\, dx$$

$$= \int_0^1 \int_0^2 3xy \, dy\, dx.$$

$$= \int_0^1 \left[\frac{3xy^2}{2} \Big|_{y=0}^{y=2} \right] dx$$

$$= \int_0^1 6x \, dx$$

$$= 3x^2 \Big|_{x=0}^{x=1}$$

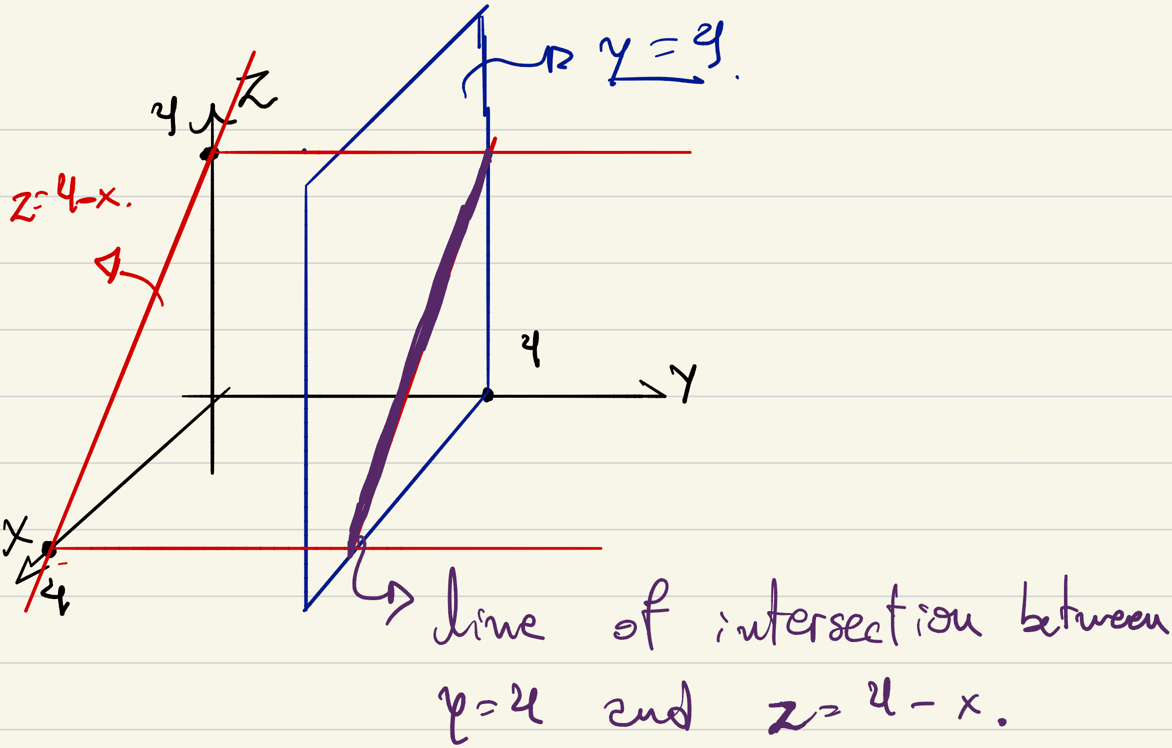
$$= 3.$$

<u>Example 3</u>: The solid S bounded by

$$z = 4 - x, \quad z = 0$$

$$y = 0, \; y = 4, \; x = 0.$$

Density: $\rho(x, y, z) = x.$



Parametrization:

Choose y as a free variable.
$0 \le y \le 4$

x-Range: $0 \le x \le 4.$

z-Range: $0 \le z \le 4 - x.$

$$m = \iiint_S x \, dV = \int_0^4 \int_0^4 \int_0^{4-x} x \, dz \, dx \, dy.$$

$$m = \int_0^4 \int_0^4 \left[x \cdot z \Big|_{z=0}^{z=4-x} \right] dx\, dy$$

$$= \int_0^4 \int_0^4 \left[x \cdot (4-x) \right] dx\, dy$$

$$= \int_0^4 \int_0^4 (4x - x^2)\, dx\, dy$$

$$= \int_0^4 \left[2x^2 - \frac{x^3}{3} \Big|_{x=0}^{4} \right] dy$$

$$= \int_0^4 \left[2 \cdot 4^2 - \frac{4^3}{3} \right] dy$$

$$= \int_0^4 \left[32 - \frac{64}{3} \right] dy$$

$$= \int_0^4 \left(\frac{96 - 64}{3} \right) dy$$

$$= \int_0^4 \frac{32}{3}\, dy$$

$$= \frac{32y}{3} \Big|_{y=0}^{y=4}$$

$$m = \frac{32 \cdot 4}{3} \Rightarrow$$

$$\boxed{m = \frac{128}{3}.}$$

Exercise 1: Compute the mass of the solid bounded by

$$x=0, \quad x=4$$
$$y=0, \quad y=2$$
$$z=0, \quad z=4-y^2$$

with density

$$\rho(x,y,z)=4-z.$$

Solution:

$$m=\int_0^4\int_0^2\int_0^{4-y^2}(4-z)\,dz\,dy\,dx$$

$$=\int_0^4\int_0^2\left[4z-\frac{z^2}{2}\Big|_{z=0}^{z=4-y^2}\right]dy\,dx.$$

$$=\int_0^4\int_0^2\left[4\cdot(4-y^2)-\frac{(4-y^2)^2}{2}\right]dy\,dx$$

$$=\int_0^4\int_0^2\left[16-4y^2-\frac{(16-8y^2+y^4)}{2}\right]dy\,dx$$

$$=\int_0^4\int_0^2\left[\frac{32-\cancel{8y^2}-16+\cancel{8y^2}-y^4}{2}\right]dy\,dx$$

$$m = \int_0^4 \int_0^2 \frac{16 - y^4}{2} \, dy \, dx.$$

$$= \int_0^4 \int_0^2 8 - \frac{y^4}{2} \, dy \, dx.$$

$$= \int_0^4 \left[8y - \frac{y^5}{10} \Bigg|_{y=0}^{y=2} \right] dx.$$

$$= \int_0^4 \left[8 \cdot 2 - \frac{2^5}{10} \right] dx.$$

$$= \int_0^4 \left[16 - \frac{32}{10} \right] dx$$

$$= \int_0^4 \left[\frac{160 - 32}{10} \right] dx$$

$$= \int_0^4 \frac{128}{10} \, dx.$$

$$= \int_0^4 \frac{64}{5} \, dx$$

$$= \frac{256}{5}.$$

Center of mass

Setting: we are given a solid with mass distribution given by a density function

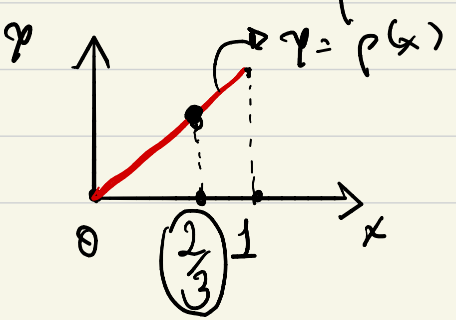
$$\rho(x, y, z).$$

Rough idea: We wish to find the center of the mass distribution.

One-dimensional analog

Supposed we're given a line segment $I = [0, 1]$ with density

$$\rho(x) = x$$



Center of mass

$$\bar{x} = \frac{\int x \cdot \rho(x)\, dx}{m}$$

- $m = \int_0^1 \rho(x)\, dx$

$$= \int_0^1 x\, dx$$

$$= \frac{x^2}{2}\Big|_{x=0}^{x=1}$$

$$= \frac{1}{2}$$

- $\bar{x} = \dfrac{\int_0^1 x \cdot \rho\, dx}{m}$

$$\bar{x} = \frac{\int_0^1 x \cdot x\, dx}{\frac{1}{2}}$$

$$\bar{x} = 2 \cdot \int_0^1 x^2\, dx = 2 \cdot \left(\frac{x^3}{3} \Big|_{x=0}^{x=1} \right)$$

$$\bar{x} = \frac{2}{3}$$

Mass to the left of $\bar{x} = 2/3$:

$$m_l = \int_0^{2/3} \rho(x)\, dx$$

$$= \int_0^{2/3} x\, dx$$

$$= \left. \frac{x^2}{2} \right|_{x=0}^{x=2/3}$$

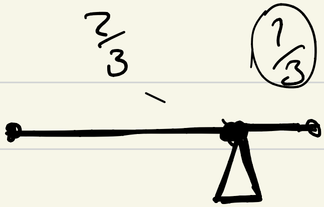
$$= \frac{1}{2} \cdot \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{2} \cdot \frac{4}{9}$$

$$m_l = \frac{2}{9}$$

Mass to right of $\bar{x}$:

$$m_r = m - m_l = \frac{1}{2} - \frac{2}{9} = \frac{9-4}{18} = \frac{5}{18}$$



Higher dimensions

First moments

$$M_{yz} = \iiint_S x\, \rho(x,y,z)\, dV$$

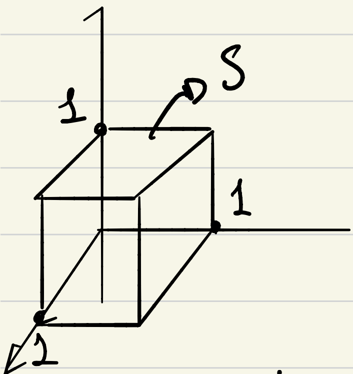
$$M_{xz} = \iiint_S y\, \rho(x,y,z)\, dV$$

$$M_{xy} = \iiint_S z\, \rho(x,y,z)\, dV.$$

Center of mass has coordinates

$$\bar{x} = \frac{M_{yz}}{m}\,;\quad \bar{y} = \frac{M_{xz}}{m}\,;\quad \bar{z} = \frac{M_{xy}}{m}.$$

Example 4: Back to example 1, we a unit cube, with density



$$\rho(x,y,z) = x^2+y^2+z^2.$$

- $$M_{yz} = \iiint_S x \cdot \rho(x,y,z)\, dV$$

$$= \int_0^1\int_0^1\int_0^1 x\cdot(x^2+y^2+z^2)\cdot dx\,dy\,dz.$$

$$= \int_0^1\int_0^1\int_0^1 (x^3+xy^2+xz^2)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left[\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^2z^2}{2} \Big|_{x=0}^{x=1} \right] dy\,dz$$

$$= \int_0^1\int_0^1 \left[\frac{1}{4} + \frac{y^2}{2} + \frac{z^2}{2} \right] dy\,dz.$$

$$M_{yz} = \int_0^1 \left[\frac{y}{4} + \frac{y^3}{6} + \frac{yz^2}{2} \Big|_{y=0}^{y=1} \right] dz.$$

$$= \int_0^1 \left[\frac{1}{4} + \frac{1}{6} + \frac{z^2}{2} \right] dz$$

$$= \int_0^1 \left[\frac{3+2}{12} + \frac{z^2}{2} \right] dz$$

$$= \int_0^1 \left(\frac{5}{12} + \frac{z^2}{2} \right) dz$$

$$= \frac{5z}{12} + \frac{z^3}{6} \Big|_{z=0}^{z=1}$$

$$= \frac{5}{12} + \frac{1}{6}$$

$$= \frac{5+2}{12}$$

$$= \frac{7}{12}$$

x- coordinate of center of mass:

$$\bar{x} = \frac{M_{yz}}{m} \Rightarrow \boxed{\bar{x} = \frac{7}{12}.}$$

Similarly,

$$\bar{y} = \frac{7}{12}, \quad \bar{z} = \frac{7}{12}.$$

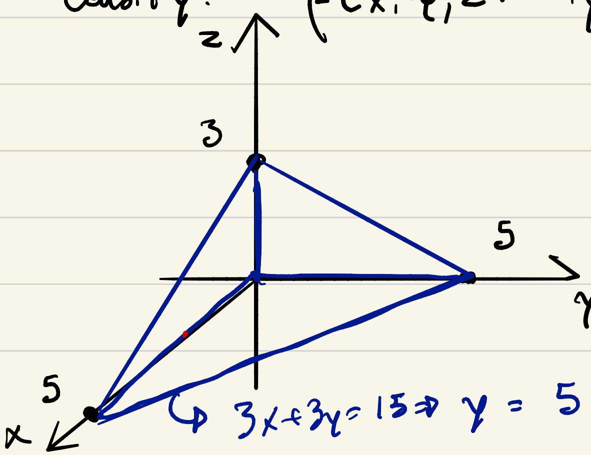
Center of mass:

$$\left(\frac{7}{12}, \frac{7}{12}, \frac{7}{12} \right)$$

<u>Example 5</u>: Solid bounded by

$$\begin{cases} 3x + 3y + 5z = 15 \rightsquigarrow z = 3 - \frac{3x}{5} - \frac{3y}{5}. \\ x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

Density: $\rho(x, y, z) = y$.



$\hookrightarrow 3x + 3y = 15 \Rightarrow y = 5 - x.$

<u>Solution:</u> Solid is a tetrahedron.

Parametrization:

- free variable x: $0 \leq x \leq 5$.

- y-range: $0 \le y \le 5 - x$.
- z-range: $0 \le z \le 3 - \frac{3x}{5} - \frac{3y}{5}$.

<u>Finding mass:</u>

$$m = \int_0^5 \int_0^{5-x} \int_0^{3 - \frac{3x}{5} - \frac{3y}{5}} y \; dz\, dy\, dx.$$

$$= \int_0^5 \int_0^{5-x} \left[yz \Big|_{z=0}^{z = 3 - \frac{3x}{5} - \frac{3y}{5}} \right] dy\, dx.$$

$$= \int_0^5 \int_0^{5-x} \left[y \cdot \left(3 - \frac{3x}{5} - \frac{3y}{5} \right) \right] dy\, dx.$$

$$= \int_0^5 \int_0^{5-x} \left[3y - \frac{3xy}{5} - \frac{3y^2}{5} \right] dy\, dx$$

$$= \int_0^5 \left[\frac{3y^2}{2} - \frac{3xy^2}{10} - \frac{y^3}{5} \Big|_{y=0}^{y=5-x} \right] dx$$

$$= \int_0^5 \left[\frac{3}{2} \cdot (5-x)^2 - \frac{3x(5-x)^2}{10} - \frac{(5-x)^3}{5} \right] dx$$

$$m = \int_0^5 \left[\left(\frac{3}{2} - \frac{3x}{10}\right)(25 - 10x + x^2) - \frac{(125 - 75x + 15x^2 - x^3)}{5} \right] dx$$

$$= \int_0^5 \left[\frac{75}{2} - \cancel{15x} + \frac{3x^2}{2} - \frac{15x}{2} + \cancel{3x^2} - \frac{3x^3}{10} - 25 + \cancel{15x} - \cancel{3x^2} + \frac{x^3}{5} \right] dx$$

$$= \int_0^5 \left[\frac{25}{2} - \frac{15x}{2} + \frac{3x^2}{2} - \frac{x^3}{10} \right] dx$$

$$= \left[\frac{25x}{2} - \frac{15x^2}{4} + \frac{x^3}{2} - \frac{x^4}{40} \Bigg|_{x=0}^{x=5} \right]$$

$$= \left[\frac{125}{2} - \frac{15 \cdot 25}{4} + \frac{125}{2} - \frac{5^4}{40} \right]$$

$$= \left[125 - \frac{375}{4} - \frac{625}{40} \right] = \frac{5000 - 3750 - 625}{40} = \frac{625}{40}$$

$$\boxed{m = \frac{125}{8}}$$

Moments:

1) $$M_{yz} = \int_0^5 \int_0^{5-x} \int_0^{3-\frac{3x}{5}-\frac{3y}{5}} x \cdot y \, dz\, dy\, dx.$$

$$= \int_0^5 \int_0^{5-x} xy\left(3 - \frac{3x}{5} - \frac{3y}{5}\right) dy\, dx$$

$$= \int_0^5 \int_0^{5-x} \left[3xy - \frac{3x^2y}{5} - \frac{3xy^2}{5}\right] dy\, dx$$

$$= \int_0^5 \left[\frac{3xy^2}{2} - \frac{3x^2y^2}{10} - \frac{xy^3}{5}\Bigg|_{y=0}^{y=5-x}\right] dx$$

$$= \int_0^5 \left[\frac{3x(5-x)^2}{2} - \frac{3x^2(5-x)^2}{10} - \frac{x(5-x)^3}{5}\right] dx.$$

$$= \int_0^5 \Bigg[\frac{3x}{2} \cdot (25 - 10x + x^2)$$

$$- \frac{3x^2}{10} \cdot (25 - 10x + x^2)$$

$$- \frac{x}{5} \cdot (125 - 75x + 15x^2 - x^3)\Bigg] dx$$

$$= \int_0^5 \left[\left(\frac{75}{2} - 25\right)x + \left(-15 - \frac{15}{2} + 25\right)x^2 + \left(\frac{3}{2} + 3 - 3\right)x^3 + \left(\frac{-3}{10} + \frac{1}{5}\right)x^4\right] dx.$$

$$= \int_0^5 \left(\frac{25x}{2} + \frac{5x^2}{2} + \frac{3x^3}{2} - \frac{x^4}{10} \right) dx.$$

$$= \frac{25x^2}{4} + \frac{5x^3}{6} + \frac{3x^4}{8} - \frac{x^5}{50} \Big|_{x=0}^{x=5}$$

$$= \frac{25 \cdot 25}{4} + \frac{5 \cdot 125}{6} + \frac{3 \cdot 625}{8} - \frac{625}{10}$$

$$= 625 \cdot \left(\frac{1}{4} + \frac{1}{6} + \frac{3}{8} - \frac{1}{10} \right)$$

$$= 625 \cdot \left(\frac{30 + 20 + 45 - 12}{120} \right)$$

$$= \frac{625 \cdot 83}{120}$$

$$= \frac{125 \cdot 83}{24}$$

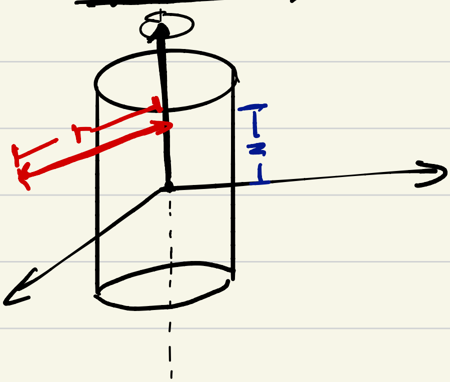
$$\bar{x} = \frac{M_{yz}}{} = \left(\frac{125 \cdot 83}{24} \right) \Big/ \left(\frac{125}{8} \right)$$

$$= \frac{\cancel{125} \cdot 83}{\cancel{24}_{3}} \cdot \frac{\cancel{8}}{\cancel{125}} = \frac{83}{3}$$

Details of $\bar{y}, \bar{z}$ in notes to follow.

- <u>Other coordinate systems</u>

<u>Cylindrical</u>



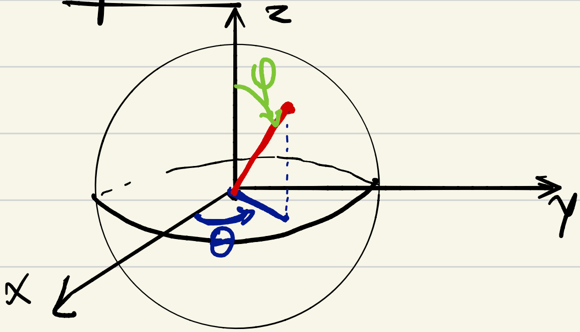
3 coordinates:

- r : distance to axis of symmetry } polar on
- θ : angle with positive x-axis. } xy-plane.
- z: height relative to xy-plane.

<u>Relations:</u>

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z; \quad dV = r\,dr\,d\theta\,dz.$$

Spherical



3 coordinates:

- r: distance to origin
- θ: angle of vertical projection with xy-axis (longitude).
- φ: angle with positive z-axis (latitude)

Relations:

- $x = r\sin\varphi \cdot \cos\theta$
- $y = r\sin\varphi \cdot \sin\theta$
- $z = r\cos\varphi$

Ranges: $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$
$0 \leq \varphi \leq \pi$.

$$dV = r^2 \sin\varphi \, dr \, d\varphi \, d\theta.$$

- **Simplifying integrals by changing coordinates**

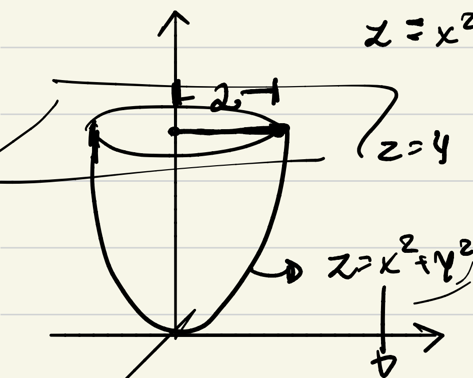
Example 6: Calculate

$$\iiint_S (x^2+y^2)\sqrt{x^2+y^2}\, dV$$

on the solid bounded by

$$z = 4$$

$$z = x^2+y^2$$



In cylindrical coordinates:

r - Range: $0 \le r \le 2$

θ - Range: $0 \le \theta \le 2\pi$

z - Range: $r^2 \le z \le 4$

$$z = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(\cos^2\theta + \sin^2\theta) = r^2.$$

Remark: For the same integral in a different order, consult example 3, section 14.7 of textbook.

$$\iiint_S (x^2+y^2)\sqrt{x^2+y^2}^{\,3}\, dV = \int_0^2\int_0^{2\pi}\int_{r^2}^4 (r^3)(r\,dz\,d\theta\,dr)$$

$$= \int_0^2\int_0^{2\pi}\int_{r^2}^4 r^4\, dz\, d\theta\, dr$$

$$= \int_0^2\int_0^{2\pi}\left(r^4 z \Big|_{z=r^2}^{4} \right) d\theta\, dr$$

$$= \int_0^2\int_0^{2\pi}\left(4r^4 - r^6 \right) d\theta\, dr$$

$$= \int_0^2 \left(2\pi \cdot (4r^4 - r^6) \right) dr$$

$$= 2\pi \int_0^2 (4r^4 - r^6)\, dr$$

$$= 2\pi \cdot \left[\frac{4r^5}{5} - \frac{r^7}{7} \Big|_{r=0}^{2} \right]$$

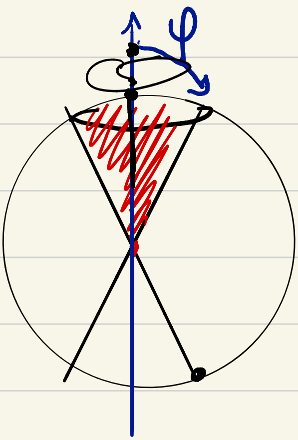
$$= 2\pi \cdot \left[\frac{4 \cdot 2^5}{5} - \frac{2^7}{7} \right].$$

Example 7: Compute the volume of solid bounded above by

$$x^2+y^2+z^2=9 \to \text{radius } 3$$

and below by

$$z^2=x^2+y^2.$$



Choose r as free-variable.

$$0 \le r \le 3$$

Intersection between sphere and cone happens when:

(z - coor. of sphere)^2 = (z - coor. of cone)^2

$$9-x^2-y^2 = x^2+y^2$$

$$\Rightarrow 2x^2+2y^2=9$$

$$x^2+y^2=\frac{9}{2}$$

The corresponding value of z is:

$$z=\sqrt{x^2+y^2}=\sqrt{\frac{9}{2}}=\frac{3\sqrt{2}}{2}.$$

From z, r we can find φ:

$$z = r\cos\varphi$$

$$\frac{3\sqrt{2}}{2} = 3 \cdot \cos(\varphi)$$

$$\cos\varphi = \frac{\sqrt{2}}{2} \Rightarrow \boxed{\varphi = \pi/4}.$$

φ-range: $\quad 0 \leq \varphi \leq \pi/4$.

θ-range: $\quad 0 \leq \theta \leq 2\pi$.

$$V = \iiint_S 1 \, dV$$

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 r^2 \sin\varphi \, dr \, d\varphi \, d\theta.$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{r^3}{3} \sin\varphi \Big|_{r=0}^{3} \right] d\varphi \, d\theta$$

$$V = \int_0^{2\pi} \int_0^{\pi/4} 9 \sin\varphi \, d\varphi \, d\theta .$$

$$= \int_0^{2\pi} \left[-9 \cos\varphi \Big|_{\varphi=0}^{\varphi=\pi/4} \right] d\theta$$

$$= \int_0^{2\pi} \left[-\frac{9\sqrt{2}}{2} + 9 \right] d\theta$$

$$= \left(9 - \frac{9\sqrt{2}}{2} \right) \cdot 2\pi .$$