MAT324: Real Analysis – Fall 2014
Assignment 3 – Solutions

Problem 1: Suppose that, for each rational number q, the set $\{x \mid f(x) > q\}$ is measurable. Can we conclude that f is measurable?

Solution. Yes, it is true. Use the density of $\mathbb{Q}$ in $\mathbb{R}$ to show that for any $a \in \mathbb{R}$,

$$f^{-1}((a, +\infty)) = \bigcap_{\{r\in\mathbb{Q}|r>a\}} f^{-1}((r, +\infty))$$

□

Problem 2: Suppose $f, g : E \to \mathbb{R}$ are measurable functions on $E \in \mathcal{M}$. Show that $h : E \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \dfrac{f(x)}{g(x)} & \text{if} \quad g(x) \neq 0 \\ 0 & \text{if} \quad g(x) = 0 \end{cases}$$

is measurable.

Solution. It suffices to show that if g is a measurable function, then so is the function

$$\left(\frac{1}{g}\right)(x) = \begin{cases} \dfrac{1}{g(x)} & \text{if} \quad g(x) \neq 0 \\ 0 & \text{if} \quad g(x) = 0 \end{cases}$$

If this is proven, then the result follows from the fact that the measurable functions are closed under products. Notice that

1. If $a > 0$, then
$$\left(\frac{1}{g}\right)^{-1}((a, +\infty)) = \left\{x|\frac{1}{g(x)} > a\right\} = \left\{x|g(x) < \frac{1}{a}\right\} \cap \{x|g(x) > 0\}$$
2. If $a = 0$, then
$$\left(\frac{1}{g}\right)^{-1}((a, +\infty)) = \{x|g(x) > 0\}$$
3. If $a < 0$, then
$$\left(\frac{1}{g}\right)^{-1}((a, +\infty)) = \left\{x|\frac{1}{g(x)} < a\right\} \cup \{x|g(x) \geq 0\}$$

In each case, the sets are measurable, hence the result follows. □

Problem 3: Let $f : (a, b) \to \mathbb{R}$. If f has a finite derivative at all points then show that f' is measurable.

SOLUTION. For each $n \in \mathbb{N}$, define $f_n : (a,b) \to \mathbb{R}$

$$f_n(x) = \begin{cases} [f(x+\dfrac{1}{n}) - f(x)]n, & \text{if} \quad x + \frac{1}{m} \in (a,b) \\ 0, & \text{otherwise.} \end{cases}$$

For any $x \in (a,b)$, there exists $N = N(x)$ such that $n > N$ implies $x + \frac{1}{n} \in (a,b)$, hence for every $x \in (a,b)$, $f_n(x) \to f`(x)$. Furthermore, each $f_n(x)$ is measurable (check!). Now use corollary 3.8 from the textbook. □

Problem 4: Let $\{f_n\}$ be a sequence of measurable functions defined on $\mathbb{R}$. Show that the sets

$$\begin{aligned} E_1 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) \text{ exists and is finite}\} \\ E_2 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) = \infty\} \\ E_3 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) = -\infty\} \end{aligned}$$

are measurable.

SOLUTION. Theorem 3.5 of the textbook says that if $\{f_n\}$ is a sequence of measurable functions, then the functions $g = \liminf_n f_n$ and $h = \limsup_n f_n$ are measurable.

Notice that $\lim_{n\to\infty} f_n(x) = \infty$, if and only if $\liminf_n f_n(x) = \infty$. Hence,

$$E_2 = \{x | g(x) = \infty\} = \bigcap_{k\in\mathbb{N}} \{x | g(x) > k\}$$

is measurable. Likewise,

$$E_3 = \{x | h(x) = -\infty\} = \bigcap_{k\in\mathbb{N}} \{x | h(x) < -k\}$$

is measurable.

Further, notice that $E_1 = \{x \in | g(x) = h(x)\} \setminus (E_2 \cup E_3)$, hence E_1 is also measurable.

Problem 5: Let $\mathcal{N} \subset [0,1]$ be a non-measurable set. Determine whether the function

$$f(x) = \begin{cases} -x & \text{if} \quad x \in \mathcal{N} \\ x & \text{if} \quad x \notin \mathcal{N} \end{cases}$$

is measurable. Explain.

SOLUTION. The function f is not measurable. Indeed,

$$f^{-1}((-1,0)) = \mathcal{N} \setminus \{0\}$$

is a nonmeasurable set. □