MAT 203 - Lecture 3

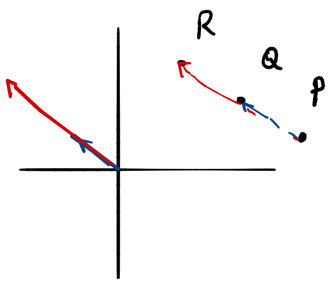
Vectors and analytic geometry

- Equations of lines

Recall that points P, Q, R are aligned if $\overrightarrow{PQ}$, $\overrightarrow{PR}$ are parallel. In other words,

$$\overrightarrow{PQ} = \lambda \cdot \overrightarrow{PR},$$

for some constant λ.



Given points $P = (1, 2)$, $Q = (2, 3)$. Find a condion on R so that P, Q, R are aligned.

Set $R = (x, y)$, then

$$\overrightarrow{PQ} = Q - P = (2, 3) - (1, 2) = (1, 1).$$

$$\overrightarrow{PR} = R - P = (x, y) - (1, 2) = (x-1, y-2)$$

We want (x, y) so that

$$(x-1, y-2) = \lambda (1, 1) \rightarrow \text{Parametric form}$$

$$\boxed{(x-1, y-2) = (\lambda, \lambda)} \rightarrow \text{Symmetric form.}$$

$(x-1,\ y-2) = (\lambda,\ \lambda).$

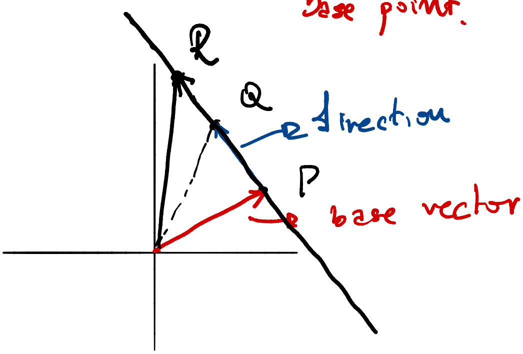
- Parametric form:

$$(x,y) - (1,2) = (\lambda, \lambda)$$

$$(x,y) - (1,2) = \lambda\cdot(1,1)$$

$$\boxed{(x,y) = \underbrace{(1,2)}_{\text{base point}} + \lambda\cdot\underbrace{(1,1)}_{\text{direction}}.}$$

"base point + scalar · direction"



- Symmetric form:

$$\boxed{x-1 = y-2.}$$

$$\begin{cases} x-1 = \lambda \\ y-2 = \lambda. \end{cases}$$

- General form (in 2D)

$$y = 1\cdot x + 1 \iff y - x - 1 = 0.$$

Exercise 1: Find parametric and symmetric equations for the line passing through

$P = (1, 1, 1)$

$Q = (0, 1, -1)$.

Solution: P as base point, $\overrightarrow{PQ}$ as direction

$$(x, y, z) = (1, 1, 1) + \lambda \cdot (-1, 0, -2).$$

$$(x-1, y-1, z-1) = (-\lambda, 0, -2\lambda)$$

$$\begin{cases} x - 1 = -\lambda \xrightarrow{\div(-1)} 1 - x = \lambda \\ y - 1 = 0 \\ z - 1 = -2\lambda \xrightarrow[\div(-2)]{} \dfrac{1-z}{2} = \lambda. \end{cases}$$

$$1 - x = \frac{1-z}{2} \quad ; \quad y = 1.$$

<u>Exercise 2</u>: Find equations (parametric and symmetric) for the line through

$P = (1, 2, 3)$

$Q = (5, 7, 9)$.

<u>Solution</u>:

- Parametric equations: P base, $\overrightarrow{PQ}$ direction.

$$(x, y, z) = (1, 2, 3) + \lambda (4, 5, 6)$$

- Symmetric equations

$$(x-1, y-2, z-3) = (4\lambda, 5\lambda, 6\lambda)$$

$$\begin{cases} x - 1 = 4\lambda \xrightarrow{\div 4} (x-1)/4 = \lambda \\ y - 2 = 5\lambda \xrightarrow{\div 5} (y-2)/5 = \lambda \\ z - 3 = 6\lambda \xrightarrow{\div 6} (z-3)/6 = \lambda. \end{cases}$$

$$\boxed{\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}}$$

Exercise 3: Check whether $R=(1,0,1)$ belongs to the line through $P=(1,2,3)$ and $Q=(5,7,9)$

- Solution:

Parametric form: $(x,y,z)=(1,2,3)+\lambda(4,5,6)$

Symmetric form: $\dfrac{x-1}{4}=\dfrac{y-2}{5}=\dfrac{z-3}{6}$.

- Using parametric form

$(1,0,1)=(1,2,3)+\lambda(4,5,6)$

$(0,-1,-2)=\lambda(4,5,6) \to$ no solutions.

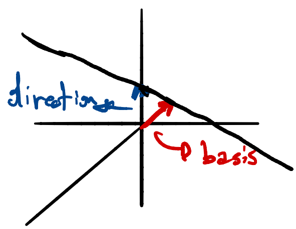
- Using symmetric form,

$\dfrac{x-1}{4}=\dfrac{1-1}{4}=0$; $\dfrac{y-2}{5}=\dfrac{0-2}{5}=-\dfrac{2}{5}$; $\dfrac{z-3}{6}=\dfrac{1-3}{6}=-\dfrac{1}{3}$.

Terms don't match $\Rightarrow$ R does not belong to the line.

Exercise 4: Find the point closest to the origin on the line

$$(x,y,z) = (1,1,1) + \lambda(0,0,1).$$



Hint: Find distance to origin as a function of λ, $s(\lambda)$, find a critical point

$$\frac{ds}{d\lambda} = 0.$$

Solution:

Distance function:

$$(x,y,z) = (1,1,1+\lambda)$$

$$\text{dist}((x,y,z),(0,0,0)) = \sqrt{1^2+1^2+(1+\lambda)^2}$$

$$s(\lambda) = \sqrt{3+2\lambda+\lambda^2}$$

Finding derivative:

$$\frac{ds}{d\lambda} = \frac{1}{2\sqrt{3+2\lambda+\lambda^2}} \cdot (2+2\lambda)$$

To get $\frac{ds}{d\lambda} = 0$ we need $\boxed{\lambda = -1}$

Finding coordinates of closest point:

$$(x, y, z) = (1,1,1) + (-1)(0,0,1)$$
$$= (1,1,1) - (0,0,1)$$
$$\boxed{(x, y, z) = (1,1,0)}$$

Exercise 5: Find the closest point to the origin among points on the line
$$(x, y, z) = (0,1,0) + \lambda(-1, 0, 1).$$

Solution: $(x, y, z) = (-\lambda, 1, \lambda)$.

Distance to origin:
$$s(\lambda) = \sqrt{(-\lambda)^2 + 1^2 + (\lambda)^2}.$$
$$s(\lambda) = \sqrt{2\lambda^2 + 1}.$$

Differentiate: $\frac{ds}{d\lambda} = \frac{(4\lambda)}{2\sqrt{2\lambda^2+1}}$ $\to$ min at $\lambda = 0$,

$(x, y, z) = (0, 1, 0)$.

Remark: Suppose a line does not pass through the origin. Then the closest point to origin, P, is so that $\vec{OP}$ is perpendicular to the direction of the line.

Exercise 6: Find the closest point to $(1,1)$ along the line

$$(x,y) = (1,0) + \lambda(-1,2).$$

Solution: $(x,y) = (1-\lambda, 2\lambda)$.

$$s(\lambda) = \sqrt{(\cancel{1}-\lambda-\cancel{1})^2 + (2\lambda-1)^2}$$
$$= \sqrt{\lambda^2 + 4\lambda^2 - 4\lambda + 1}$$
$$= \sqrt{5\lambda^2 - 4\lambda + 1}.$$

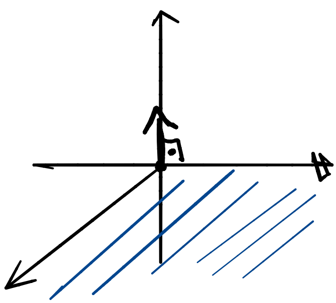
$$\frac{ds}{d\lambda} = \frac{(10\lambda - 4)}{2\sqrt{5\lambda^2 - 4\lambda + 1}} \to \text{critical point at } 10\lambda - 4 = 0 \Rightarrow \lambda = \frac{2}{5}.$$

Going back to equation of line:

$$(x,y) = (1,0) + \frac{2}{5}(-1,2) = \left(\frac{3}{5}, \frac{4}{5}\right).$$

Remark: Finding the distance between a point and a line <u>means</u> finding the distance between the <u>closest</u> point on the line and the given point.

- <u>Equations of planes</u>



ways of describing a plane:

i) given 3 points on the plane, not aligned.

ii) given a point in the plane and a normal vector.

Example: Find equations for a plane containing the points

$P = (0,1,1)$

$Q = (1,0,0)$

$R = (0,1,0)$.

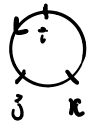
Checking for alignment:
$P = (0,1,1)$, $Q = (1,0,0)$, $R = (0,1,0)$.

$\overrightarrow{PQ} = (1,-1,-1)$
$\overrightarrow{PR} = (0,0,-1)$. $\rightarrow$ not scalar multiples, points not aligned.

<u>Parametric form:</u>
Base point: P
Two directions: $\overrightarrow{PQ}$, $\overrightarrow{PR}$
Two parameters: λ, γ.

$$(x,y,z) = P + \lambda \cdot \overrightarrow{PQ} + \gamma \cdot \overrightarrow{PR}$$
$$(x,y,z) = (0,1,1) + \lambda \cdot (1,-1,-1) + \gamma \cdot (0,0,-1)$$
$$(x,y,z) = (\lambda, 1-\lambda, 1-\lambda-\gamma).$$

<u>Question</u>: Can you find a vector which is perpendicular to the plane?
Hint: $\perp$ plane $\Rightarrow$ $\perp$ to direction vectors.



Base point: $P=(0,1,1)$

Directions: $\overrightarrow{PQ}=(1,-1,-1)$, $\overrightarrow{PR}=(0,0,-1)$

$\overrightarrow{PQ}= i-j-k$; $\overrightarrow{PR}=-k$

Cross product:

$$\begin{aligned}\overrightarrow{PQ}\times\overrightarrow{PR} &= (i-j-k)\times(-k).\\ &= -(i\times k)+j\times k+k\times k.\\ &= j+i+0\\ &= i+j\\ \vec{n} &= (1,1,0).\end{aligned}$$

<u>Finding general equation:</u> Given any other point in the plane, (x,y,z) the difference $(x,y,z)-(0,1,1)=(x,y-1,z-1)$ is also perpendicular to $\vec{n}$.

Using dot product to verify orthogonality

$$(x,y-1,z-1)\cdot(1,1,0)=0.$$

$$x\cdot 1+(y-1)\cdot 1+(z-1)\cdot 0=0$$

$$\boxed{x+y-1=0}$$

$$x\cdot 1+y\cdot 1+z\cdot 0=1$$

Plug in $(0,1,1)$: $0\cdot 1+1\cdot 1+1\cdot 0=1$

Exercise 7: Find the equation of the plane containing $\underbrace{(1,0,0)}_{P}$, $\underbrace{(0,1,0)}_{Q}$, $\underbrace{(0,0,1)}_{R}$.

Solution:

- Parametric form

Base point: $P=(1,0,0)$

Directions: $\overrightarrow{PQ}=(-1,1,0)$; $\overrightarrow{PR}=(-1,0,1)$.

$(x,y,z)=(1,0,0)+\lambda\cdot(-1,1,0)+\gamma\cdot(-1,0,1)$.

$$\boxed{(x,y,z)=(1-\lambda-\gamma,\ \lambda,\ \gamma)}$$

- Non-parametric form

normal vector:

$n=(-i+j)\times(-i+k)$.

$=\cancel{i\times i}-i\times k-j\times i+j\times k$.

$=j+k+i$

$=(1,1,1)$.

Tentative form: $1\cdot x+1\cdot y+1\cdot z=D$

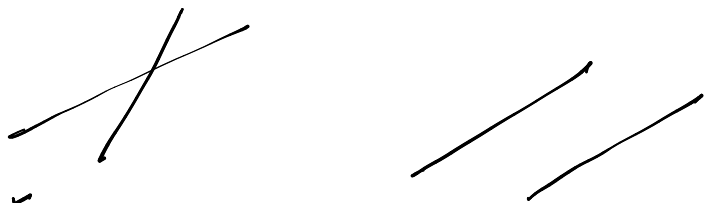
Finding D: replace coordinates of base point

$1\cdot 1+1\cdot 0+1\cdot 0=D\Rightarrow D=1$.

$$\boxed{x+y+z=1.}$$

Relative positions of lines and planes

* In 2D:

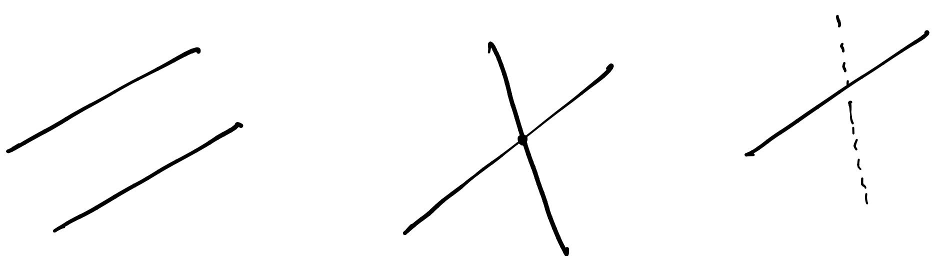


Transverse parallel.

Parallel if directions are scalar multiples of one another, transverse otherwise.

* In 3D:

  - Line vs Line:

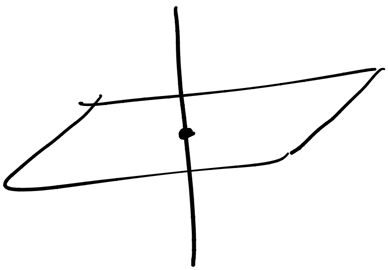


Parallel Intersecting Skew

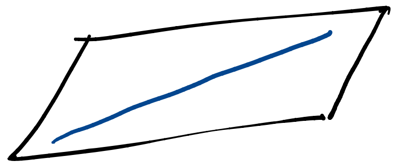
→ Parallel lines have direction vectors which are multiples of each other.

→ Otherwise lines are intersecting or skew depending on whether they have a point in common

- <u>Line vs Plane.</u>



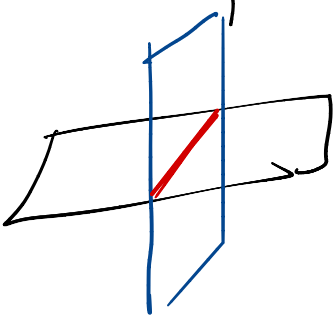
Transverse



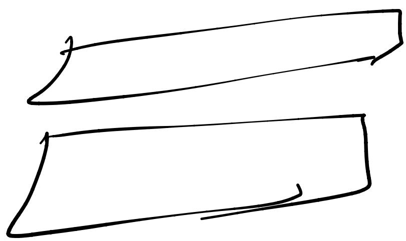
Contained.

Parallel

- Plane vs plane



Intersecting



Parallel.

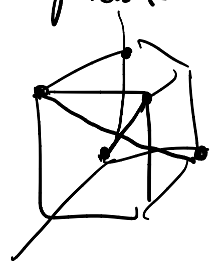
Exercise 8: Describe the relative position of lines below:

1) Line through $(0,1,0)$ and $(1,0,1)$

2) Line through $(0,0,0)$ and $(1,1,1)$.

Solution.

Line 1: $(1,-1,1)$ direction
Line 2: $(1,1,1)$ direction
$\longrightarrow$ not multiples, lines not parallel.



· Equations for line 1:

Parametric: $(x,y,z) = (0,1,0) + \lambda(1,-1,1)$

$$\boxed{(x,y,z) = (\lambda, 1-\lambda, \lambda)}.$$

Non-parametric:

$$\begin{cases} x = \lambda \\ y = 1-\lambda \Rightarrow 1-y=\lambda. \\ z = \lambda \end{cases}$$

$$\boxed{x = 1-y = z.}$$

· Equations for line 2:

Parametric: $(x,y,z) = (0,0,0) + \gamma \cdot (1,1,1)$

$$(x,y,z) = (\gamma, \gamma, \gamma)$$

Non-parametric: $x=y=z$.

Solutions*: $\gamma=\lambda=\frac{1}{2}$. $\Rightarrow$ Lines intersect.

Edited after lecture to fix a mistake.

Exercise 9: Find relative position of the lines
1) passing through $(1,0,0)$ and $(0,1,0)$
2) passing through $(-1,0,-1)$ and $(0,1,1)$.

Solution:

Line 1 direction: $(-1,1,0)$
Line 2 direction: $(1,1,2)$.
$\to$ not multiples, lines not parallel.

Equation for line 1:
$$(x,y,z)=(1,0,0)+\lambda\cdot(-1,1,0)$$
$$=(1-\lambda,\lambda,0).$$

Equation for line 2:
$$(x,y,z)=(-1,0,-1)+\gamma\cdot(1,1,2)$$
$$=(\gamma-1,\gamma,2\gamma-1).$$

For equality: $0=2\gamma-1\Rightarrow 2\gamma=1\Rightarrow\gamma=\frac{1}{2}$; incompatible with first and second entries.

Exercise 10: Find relative position of:

1) Line through $(0,0,1)$ and $(1,2,3)$

2) Plane through $(1,0,0)$, $(1,0,1)$, $(0,1,0)$.

Solution:

Direction for line: $(1,2,2)$.

Equations for line:

$$(x,y,z) = (0,0,1) + \lambda \cdot (1,2,2).$$

$$(x,y,z) = (\lambda, 2\lambda, 1+2\lambda)$$

$$\begin{cases} x = \lambda \\ y = 2\lambda \xrightarrow{\div 2} \frac{y}{2} = \lambda \\ z = 1+2\lambda \xrightarrow[-1]{} z-1 = 2\lambda \xrightarrow[\div 2]{} \frac{z-1}{2} = \lambda \end{cases}$$

$$\boxed{x = \frac{y}{2} = \frac{z-1}{2}}$$

2) Plane through $(1,0,0)$, $(1,0,1)$, $(0,1,0)$

Direction vectors:

$$u = (1,0,1) - (1,0,0) = (0,0,1).$$

$$v = (0,1,0) - (1,0,0) = (-1,1,0).$$

Cross-product: $n = u \times v$

$$= k \times (-i + j)$$

$$= -j - i$$

$$= (-1,-1,0).$$

Tentative equation:

$$(-1)\cdot x + (-1)\cdot y + 0\cdot z = D$$

Plug in $(1,0,0)$:

$$(-1)\cdot 1 + (-1)\cdot 0 + 0\cdot 0 = -1.$$

Equation: $-x - y = -1 \Rightarrow \boxed{x + y = 1.}$

$$\begin{cases} x = y/2 = z - 1/2 \\ x + y = 1. \end{cases}$$

Transverse!

- $\frac{y}{2} + y = 1 \Rightarrow \frac{3y}{2} = 1 \Rightarrow y = \frac{2}{3}.$

$\Rightarrow x = \frac{1}{3} \Rightarrow \frac{z-1}{2} = \frac{1}{3} \Rightarrow z - 1 = 2/3 \Rightarrow z = \frac{5}{3}$

__Exercise 11:__ Find the relative position of

(1) Plane through $(0,0,0)$, $(1,0,0)$, $(0,0,1)$.

(2) Plane through $(1,0,1)$, $(2,0,1)$, $(1,0,2)$.

__Solution:__

Normal to plane 1:

Directions: $u = (1,0,0)$, $v = (0,0,1)$.

$$\begin{aligned} n_1 &= u \times v \\ &= i \times k \\ &= -j \\ &= (0,-1,0). \end{aligned}$$

Normal to plane 2:

Directions: $w = (2,0,1) - (1,0,1) = (1,0,0)$

$y = (1,0,2) - (1,0,1) = (0,0,1)$

$$\begin{aligned} n_2 &= w \times y \\ &= -j = (0,-1,0). \end{aligned}$$

Normals are parallel, hence so are the planes.