

MAT324: Real Analysis – Fall 2014
ASSIGNMENT 2 – SOLUTIONS

Due Thursday, **September 25**, in class.

Problem 1: Construct a *Cantor-like* closed set $\mathcal{C} \subset [0, 1]$ so that at the k^{th} stage of the construction one removes 2^{k-1} centrally situated open intervals each of length ℓ_k , with

$$\ell_1 + 2\ell_2 + \dots + 2^{k-1}\ell_k < 1.$$

Suppose ℓ_k are chosen small enough so that $\sum_{k=1}^{\infty} 2^{k-1}\ell_k < 1$.

- a) Show that $m(\mathcal{C}) = 1 - \sum_{k=1}^{\infty} 2^{k-1}\ell_k$ and conclude that $m(\mathcal{C}) > 0$.
- b) Give an example of a sequence $(\ell_k)_{k \geq 1}$ that verifies the hypothesis.

SOLUTION.

- a) The intervals removed are disjoint. If C_n denotes what is left in $[0, 1]$ after the n -th process, then $m(C_n) = 1 - \sum_{k=1}^n 2^{k-1}\ell_k$. Furthermore, by construction we have $C_{n+1} \subset C_n$. Apply theorem 2.13.
- b) Let $\ell_k = 4^{-k} = 2^{-2k}$. Then

$$\sum_{k=1}^{\infty} 2^{k-1}2^{-2k} = \sum_{k=1}^{\infty} 2^{-1-k} = \frac{1}{2}$$

Problem 2: Let $E_1, E_2, \dots, E_{2014} \subset [0, 1]$ be measurable sets such that $\sum_{k=1}^{2014} m(E_k) > 2013$. Show □
that $m\left(\bigcap_{k=1}^{2014} E_k\right) > 0$.

SOLUTION. Let $F_n = [0, 1] \setminus E_n$, for each $1 \leq n \leq 2014$. Notice that

$$m(\cup_{n=1}^{2014} F_n) = 1 - m(\cap_{n=1}^{2014} E_n).$$

Use subadditivity and the inequality provided to show that $m(\cup F_n) < 1$. Combined with the result in the previous paragraph, $m(\cap E_n) > 0$. □

Problem 3: Suppose $A \in \mathcal{M}$ and $m(A \Delta B) = 0$. Show that $B \in \mathcal{M}$ and $m(A) = m(B)$.

SOLUTION. See page 52 in the textbook. □

Problem 4: Suppose $A \subset E \subset B$ where A and B are measurable sets of finite measure. Show that if $m(A) = m(B)$, then E is measurable.

SOLUTION. Notice that

$$m(B) = m(A) + m(B \setminus A)$$

Since $m(A) = m(B) < \infty$, we can subtract on both sides to get $m(B \setminus A) = 0$. Since $E \setminus A \subset B \setminus A$, completeness of the Lebesgue measure shows that $E \setminus A$ is measurable, but then so is $E = A \cup (E \setminus A)$. \square

Problem 5: Suppose $E \in \mathcal{M}$ and $m(E) > 0$. Prove that there exists an open interval I such that

$$m(E \cap I) > 0.99 \cdot m(I).$$

Hint: Argue by contradiction, using Theorem 2.17.

SOLUTION. We'll show that in fact a more general result holds.

Claim 1 *If $E \in \mathcal{M}$ and $m(E) > 0$, then for any $0 < \alpha < 1$, there exists an interval I such that*

$$m(E \cap I) > \alpha \cdot m(I).$$

PROOF OF Claim 1. We'll use a slight modification of Theorem 2.19, which the reader can prove as an exercise. This is the

Lemma 1 *If $E \in \mathcal{M}$, then*

$$m(E) = \sup\{m(K) \mid K \subset E, K \text{ is compact}\}$$

With this the reader can easily prove that if E has finite measure, we can find a finite union of disjoint open intervals $A = \cup_{n=1}^N I_n$ such that $m(E \Delta A) < \epsilon$ (consider a suitable open cover of K by open intervals, and extract a finite subcover).

Let $\epsilon = (1 - \alpha)m(E)$, and let A be the set given by the lemma. Since A is a measurable set,

$$\begin{aligned} m(E) &= m(E \cap A) + m(E \cap A^c) \\ m(E) &\leq m(E \cap A) + m(E \Delta A^c) \\ m(E) &< m(E \cap A) + (1 - \alpha)m(E) \\ \alpha m(E) &< m(E \cap A) \end{aligned}$$

Since E is a measurable set,

$$\begin{aligned} m(A) &= m(A \cap E) + m(A \cap E^c) \\ m(A) &\leq m(A \cap E) + (1 - \alpha)m(E) \\ m(A) &< m(A \cap E) + \frac{1 - \alpha}{\alpha} m(E \cap A) \\ m(A) &< \frac{1}{\alpha} m(E \cap A) \end{aligned}$$

Now we notice that

$$m(A) = \sum_{n=1}^N m(I_n)$$

$$m(A \cap E) = \sum_{n=1}^N m(E \cap I_n)$$

This yields,

$$\sum_{n=1}^N m(I_n) < \frac{1}{\alpha} \left(\sum_{n=1}^N m(E \cap I_n) \right)$$

And this proves the claim if $m(E) < \infty$ (argue by contradiction). Now if $m(E) = +\infty$, take $E' \subset E$ with $m(E') < \infty$ and proceed in the same way to get the result for E' . Apply monotonicity to get the claim in its general form. \square

Problem 6: Suppose that E_1, E_2, \dots are measurable subsets of \mathbb{R} . Show that if $E_k \supset E_{k+1}$ for all $k \geq 1$ and $m(E_N) < \infty$ for some N , then

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k \rightarrow \infty} m(E_k).$$

SOLUTION. Let $A_1 = \bigcap_{k=1}^{N-1} E_k$, and $A_n = E_n$, if $n \geq N$. Then the collection A_n satisfies the hypothesis of theorem 2.13 (ii). \square