MAT 132 - Calculus II

Summer II 2019
Mathematics Department
Stony Brook University

Instructor: Marlon de Oliveira Gomes

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Course syllabus

Pre-requisite

C or higher in AMS 151, MAT 131, MAT 141, or level 7 on Placement Exam.

Topics

Symbolic Integration;
Numerical Integration;
Applications to Geometry;
Introduction to Differential Equations and Modelling;
Sequences and Series.

Textbook

Single Variable Calculus, by James Stewart (4th Stony Brook edition).
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Times and dates

Lectures: Mondays, Wednesdays, and Thursdays, 6:00 pm to 9:05pm
Exams:
Q1: 7/15, between 7:00 pm and 7:50 pm.
Q2: 7/24, between 7:00 pm and 7:50 pm.
Q3: 7/31, between 7:00 pm and 7:50 pm.
Q4: 8/8, between 8:15 and 9:05pm.
Final, part 1: 8/12, between 7:55 pm and 9:05 pm.
Final, part 1 (retake): 8/14, between 7:55 pm and 9:05 pm.
Final, part 2: 8/15, between 6:00 pm and 9:05 pm.
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Homework

Four problem sets, available on course webpage. Homework will not be graded. Problems will be solved in class. All problems come from the textbook.

Exams

Four quizzes. Three best scores will count towards final grade (20% each).

Final exam, divided into two parts:

Part one: minimum competency test. Students who obtain 15/20 automatically pass the course (minimum grade C). Does not count towards final grade.

Part two: Counts towards 40% of final grade.

Grades

Pass criteria: 15/20 on part one of final exam, or 60/100 on final grade (if failure on part one).

Letter grade thresholds:

TextGrid[{{“Letter grade”, A, “A-”, “B+”, B, “B-”, “C+”, C, “C-”, “D+”, D, F},

{“Grade threshold”, 94.3, 88.6, 82.9, 77.2, 71.5, 65.8, 60, 54.3, 48.6, 40, 0}},

Frame → All]

Letter grade	*A*	A-	B+	*B*	B-	C+	*C*	C-	D+	*D*	*F*
Grade threshold	94.3	88.6	82.9	77.2	71.5	65.8	60	54.3	48.6	40	0

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Integration

Motivation from geometry: the Area Problem.

Area Problem: to find the area of a planar region bounded by the graph of a positive function.

Simple examples:

A constant function

Plot[1, {x, 0, 5}, Filling → Bottom, PlotLegends → Placed[{“f(x)=1”}, Above]]

A first-degree polynomial

```
Plot[2x + 3, {x, 0, 5}, Filling → Bottom, PlotLegends → Placed[{"f(x)=2x+3"}, Above]]
```

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Integration

Motivation from geometry: the Area Problem.

How to estimate area under a the graph of a general positive function?
Idea: use rectangles as an elementary approximation.
Archimedes' example: the area under a parabola.

```
Manipulate[

Show[

DiscretePlot[t^2,

{t, 0, 1, 1/2^n},

ExtentSize → Right,

Frame → True,

PlotRange → {{0, 1}, {0, 1}},

PlotRangeClipping → True],

Plot[t^2,

{t, 0, 1},

Frame → True,

PlotRange → {{0, 1}, {0, 1}},

PlotRangeClipping → True,

PlotLegends → Placed[{"f(x)=x^2"}, Above]

]

],

{n, 1, 8, 1},

FrameLabel → Style["Left-endpoint approximations", 13]]
```

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Integration

Motivation from Geometry: the Area Problem.

Archimedes' example (continued): the area under a parabola.

```
Manipulate[
Show[
DiscretePlot[t^2,
{t, 0, 1, 1/2^n},
ExtentSize → Left,
Frame → True,
PlotRange → {{0, 1}, {0, 1}},
PlotRangeClipping → True],
Plot[t^2,
{t, 0, 1},
Frame → True,
PlotRange → {{0, 1}, {0, 1}},
PlotRangeClipping → True,
PlotLegends → Placed[{"f(x)=x^2"}, Above]
]
],
{n, 1, 8, 1},
FrameLabel → Style["Right-endpoint approximations", 13]]
```

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Integration

Motivation from Geometry: the Area Problem

The idea of approximating the area under graphs was formalized by Riemann. To approximate the area underneath the graph of a positive function “f” within the interval $[a, b]$, we:

subdivide the interval into “n” equally sized subintervals.

select a sample point in each subinterval, whose image is the height of the approximating rectangle.

add the areas of the approximating rectangles.

These sums take the following form:

```
TraditionalForm [HoldForm [Sum ["f(x_i^*)Δx", {i, 1, n}]]]
```

$\sum_{i=1}^{n} \mathrm{f}(x_i^*)\Delta \mathrm{x}$

Notation:

“f” is the function whose graph bounds the region in question.

“x_i^*” are the *sample points*, one for each rectangle.

“Δx” is the width of each rectangle.

Typical sampling points: left-endpoints, midpoints, right-endpoints.

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Integration

Motivation from Geometry: the Area Problem

A comparison between samplings

```
Manipulate[

Show[

DiscretePlot[t^2,

{t, 0, 1, 1/n},

ExtentSize → Right,

Frame → True,

PlotRange → {{0, 1}, {0, 1}},

PlotRangeClipping → True],

DiscretePlot[t^2,

{t, 0, 1, 1/n},
```

ExtentSize → Left,

Frame → True,

PlotRange → {{0, 1}, {0, 1}},

PlotRangeClipping → True],

Plot[*t*^2,

{*t*, 0, 1},

Frame → True,

PlotRange → {{0, 1}, {0, 1}},

PlotRangeClipping → True,

PlotLegends → Placed[{“f(x)=x^2”}, Above]

]

],

{*n*, 1, 20, 1}

]

This example illustrates the fact that the approximations by left-endpoints (more opaque) and right-endpoints (less opaque) seem to converge to each other, as the number of rectangles grows.

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Integration

Motivation from Geometry: the Area Problem

One might wonder whether our intuitive idea that all of these approximations will *eventually* (as the number of rectangles grows) lead to a number. After all, the number of summands grows infinitely large, and there is no telling where their sum will go. To test it, we compute the limits of the approximations, as the number of rectangles goes to infinity:

TraditionalForm [HoldForm [Limit [Sum ["f(x_i^*)Δx", {*i*, 1, *n*}] , *n* → ∞]]]

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \mathrm{f}(x_i^*)\Delta \mathrm{x}\right)$$

Such limits are called *Riemann Sums.* They can also be defined for functions that change sign, although in this case they no longer represent *areas,* but rather *signed areas,* in which case the area of regions below the axis is counted with a negative sign.

Definition

A function is called integrable on an interval when *all of its Riemann sums* (that is, for all possibilities of sampling points) exist and coincide.

An immediate question is how to test integrability - given that testing each sampling is impractical. We will not answer this question in this course. Instead, we shall contempt ourselves with the fact that *all continuous functions are integrable.*

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Integration

The Definite Integral

Plot[Sin[x + Sin[2x]],

{x, 0, Pi},

Filling → Bottom,

PlotLegends → Placed["y=f(x)", Above]]

If a function "f" is integrable in an interval $[a, b]$, we call the limit of its Riemann sums therein its *integral* on the interval $[a, b]$, represented by

TraditionalForm[Integrate[f[x], {x, a, b}]]

$\int_a^b f(x)\, dx$

The above integral is read "integral of 'f', from 'a' to 'b', relative to 'x'".

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Integration

The Definite Integral

Back to Archimedes: the area under a parabola, by Riemann sums.

In Archimedes example, $f(x) = x^2$, $a = 0$, $b = 1$. Subdividing this interval into n rectangles yields a width size

$\Delta \mathrm{x} = \frac{1}{n}$.

Since f is continuous, we can approximate it by any sampling (in the limit, the results are all the same). We use right-endpoints

$x_i = \frac{i}{n}$.

The Riemann sum is

TraditionalForm[

HoldForm[Integrate[x^2, {x, 0, 1}] == Limit[Sum[i^2/n^3, {i, 1, n}], $n \to \infty$] ==

Limit[Sum[(n * (n + 1) * (2n + 1))/(6n^3), {i, 1, n}], $n \to \infty$] == 1/3]]

$$\int_0^1 x^2\, dx = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{i^2}{n^3}\right) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{n(n+1)(2n+1)}{6n^3}\right) = \frac{1}{3}$$

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Integration

The Definite Integral

Computing definite integrals from Riemann sums is hard. We'll study other means of computing integrals in what follows.

Properties of integration

Linearity: "f" and "g" integrable functions, "t" a constant

TraditionalForm[Integrate[(f[x] + tg[x]), {x, a, b}] ==

(Integrate[f[x], {x, a, b}]) + t(Integrate[g[x], {x, a, b}])]

$$\int_a^b (f(x) + \mathrm{tg}(x))\, dx = \int_a^b f(x)\, dx + t \int_a^b g(x)\, dx$$

Domain additivity: if $a < c < b$, and the function "f" is integrable on $[a, b]$, then

TraditionalForm[Integrate[f[x], {x, a, b}] ==

(Integrate[f[x], {x, a, c}]) + (Integrate[f[x], {x, c, b}])]

$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

Comparison: if $a < b$, and $f(x) \leq g(x)$, for all $a \leq x \leq b$, then

TraditionalForm[Integrate[f[x], {x, a, b}] ≤ (Integrate[g[x], {x, a, b}])]

$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$

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Integration

The Definite Integral

The Mean Value Theorem

Suppose the integrand $f(x)$ is a continuous function on the interval $[a, b]$. Then there exists a number c, between a and b, for which

TraditionalForm[Integrate[f[x], {x, a, b}] == f[c]($b - a$)]

$\int_a^b f(x)\, dx = (b - a) f(c)$

The idea behind this theorem is the *Intermediate Value Theorem* for continuous functions.

Manipulate[

Show[

Plot[Sin[x + Sin[2x]],

{x, 0, Pi},

PlotLegends → Placed["y=f(x) vs y=πm", Above],

PlotRange → {{0, Pi}, {0, Pi/2}},

PlotRangeClipping → True],

Plot[Pi ∗ m,

{x, 0, Pi},

Filling → Bottom,

PlotRange → {{0, Pi}, {0, Pi/2}},

PlotRangeClipping → True

]],

```
{m, 0, 1/Pi}]
```

Integration

The Indefinite Integral

Having seen a few examples and properties of integrals, we turn to study *Integration*, on its own right.

Given a function $f(x)$, integrable on the interval $[a, b]$, we may consider intervals in intermediate regions.

```
DynamicModule[{pts = {{0, 0}, {Pi, 0}}}, LocatorPane[Dynamic[pts, (pts[[1]] = {#[[1, 1]], 0};
pts[[2]] = {#[[2, 1]], 0})&],
Dynamic[
Framed@
Show@
{Plot[Sin@x, {x, 0, 2Pi},
PlotLabel → ToString@TraditionalForm[Integrate[sin[x], {x, pts[[1, 1]], pts[[2, 1]]}]] <>

" = " <> ToString[Integrate[Sin@x, {x, pts[[1, 1]], pts[[2, 1]]}]]],
Plot[Sin@x, {x, pts[[1, 1]], pts[[2, 1]]}, PlotRange → {−1, 1}, Filling → Axis]}]]]
```

That is, we may define a function by integrating another:

```
TraditionalForm[g[y] == Integrate[f[x], {x, a, y}]]
```

$g(y) = \int_a^y f(x)\, dx$

The function g is called an *indefinite integral* of f.

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Integration

The indefinite integral