MAT 514 - Lecture 2.

- Supplemental notes on Calculus

  Goals:

  - parametrized curves
  - line integrals
  - double integrals and Fubini's Theorem
  - Green's theorem and Applications.

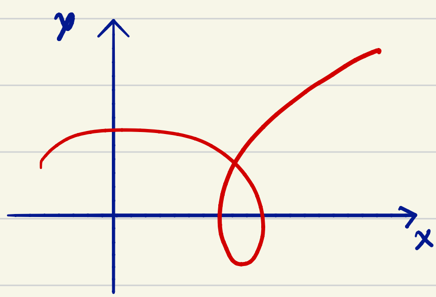
- <u>Parametrized curves</u>

  A parametrized plane curve is a function

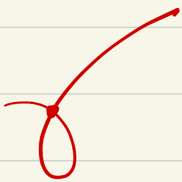
$$r(t) = (x(t), y(t))$$

for $t \in [a, b]$.

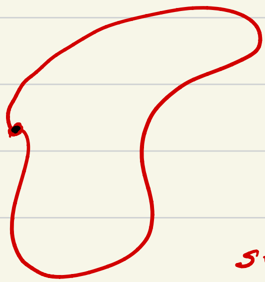
The trace of the curve plots all positions in the plane.



A curve is called simple if it has no self-intersections. In terms of the function defining the curve, it means that $r(t)$ is injective.

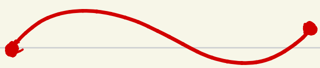


non-simple

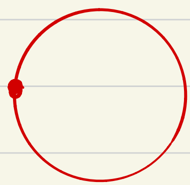


simple.

A closed curve has the same starting point and end point.



non-closed

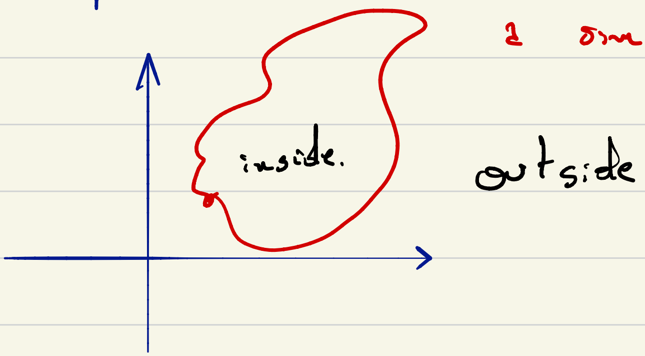


closed.

For most of the course we will deal with simple, closed curves.

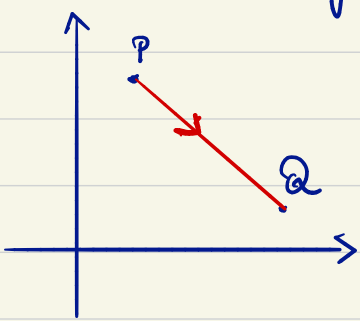
Theorem (Jordan)

A simple, closed curve bounds a region of the plane.



• Examples

1) line segment from P to Q.



$$r(t) = (1-t)\cdot P + t\cdot Q$$
$$= P + t\cdot(Q-P),$$

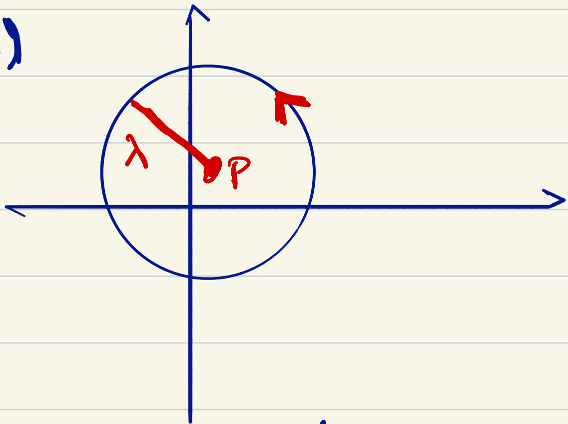
where $t \in [0,1]$.

Concretely, if $P=(1,4)$ and $Q=(3,1)$

then

$$r(t) = (1, 4) + t\cdot(2, -3)$$
$$= (1+2t, 4-3t).$$

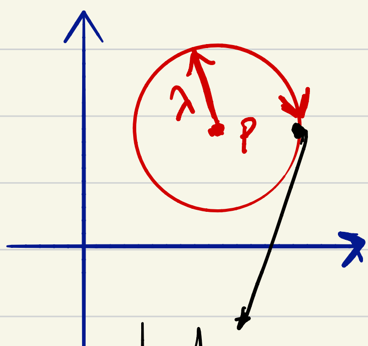
2)



Circle of center P, radius λ, traversed in counterclockwise orientation.

$$r : [0, 2\pi] \longrightarrow \mathbb{R}^2$$
$$r(t) = P + \lambda\cdot(\cos(t), \sin(t)).$$

3)

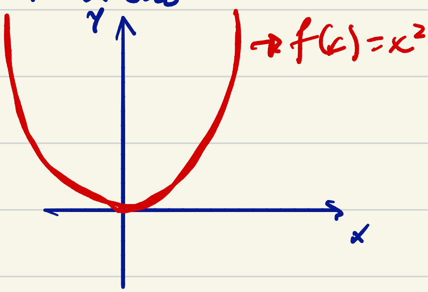


starting point.

Circle of center P, radius λ, traversed in clockwise orientation:

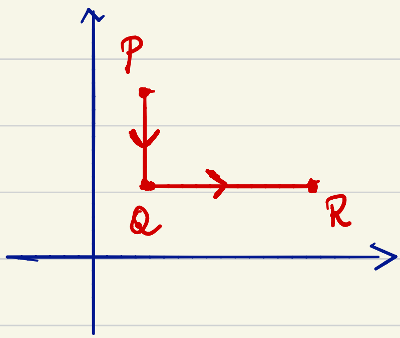
$$r(t) = P + \lambda(\cos(t), -\sin(t))$$

4) Graphs of single-variable, real-valued functions



Given a function
$f: [a,b] \to \mathbb{R}$
its graph can be parametrized as
$r(t) = (t, f(t))$.

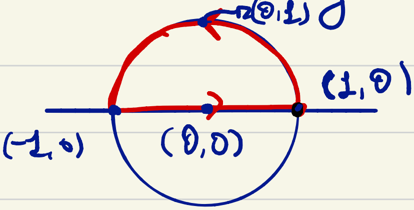
5) Piecewise smooth curves



$r: [0,2] \to \mathbb{R}$

$$r(t) = \begin{cases} P + t(Q-P), & 0 \le t \le 1. \\ Q + (t-1)\cdot(R-Q), & 1 \le t \le 2. \end{cases}$$

6) Parametrizing a semicircle centered at origin with radius 1.

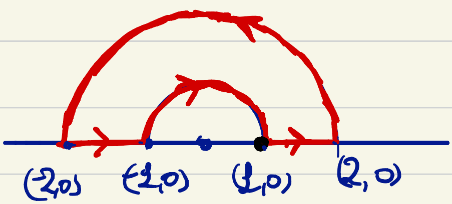


Arc component: $r(t) = (\cos(t), \sin(t))$, $0 \le t \le \pi$.
Segment component: $r(t) = (-1, 0) + (t-\pi) \cdot (2, 0)$
$= (-1 + 2(t-\pi), 0)$,
$\pi \le t \le \pi + 1$.

Overall, $r : [0, \pi+1] \longrightarrow \mathbb{R}^2$

$$r(t) = \begin{cases} (\cos(t), \sin(t)), & \text{if } 0 \le t \le \pi \\ (-1 + 2(t-\pi), 0), & \text{if } \pi \le t \le \pi + 1. \end{cases}$$

7)



The half-annulus.
Inner radius : $\lambda_i = 1$
Outer radius : $\lambda_o = 2$
Center : $(0, 0)$.

Starting point: $(1, 0)$
$r : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$$r(t) = \begin{cases} (1, 0) + t \cdot (1, 0), & 0 \le t \le 1. \\ (2\cos(t-1), 2\sin(t-1)), & 1 \le t \le 1+\pi \\ (-2, 0) + (t-1-\pi) \cdot (1, 0), & 1+\pi \le t \le 2+\pi. \\ (\cos(t-2-\pi), -\sin(t-2-\pi)), & 2+\pi \le t \le 2+2\pi \end{cases}$$

- Velocity and speed.

Given a curve $r(t) = (x(t), y(t))$, which is sufficiently smooth, its velocity vector is

$$r'(t) = (x'(t), y'(t)),$$

wherever defined.

The speed of the curve at time t is the norm of $r'(t)$

$$s(t) = \| r'(t) \|$$

$$s(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}.$$

It is often convenient for computations to parametrize curves so that their speed is 1. In this case, we say that the curve is <u>parametrized by arclength</u>.

<u>Length of a curve</u>

Given a curve $r(t) : [a, b] \to \mathbb{R}^2$, which is sufficiently smooth, we define its length by

$$\ell(r) = \int_a^b \|r'(t)\| \, dt.$$

<u>Useful fact:</u> length is unchanged by reparametrizations.

<u>Examples</u>

8) Length of a line segment from P to Q,

$$r(t) = P + t\,(Q - P), \quad 0 \le t \le 1.$$

The velocity is

$$r'(t) = Q - P$$

The speed is $\|r'(t)\| = \|Q - P\|$, so the length is

$$\ell(r) = \int_0^1 \|Q - P\| \, dt$$

$$= \|Q - P\|.$$

9) Consider the curve

$$r(t) = (3 + 2\cos(t), 3 - 2\sin(t))$$
$$= (3,3) + 2\cdot(\cos(t), -\sin(t))$$

Its velocity is

$$r'(t) = 2\cdot(-\sin(t), -\cos(t))$$
$$r'(t) = -2(\sin(t), \cos(t))$$

The speed is

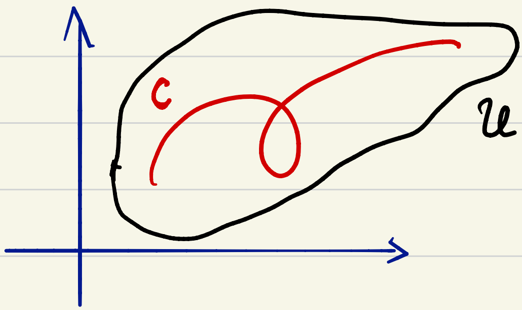
$$\|r'(t)\| = \|(-2\sin(t), -2\cos(t))\|$$
$$= \sqrt{(-2\sin(t))^2 + (-2\cos(t))^2}$$
$$= \sqrt{4\sin^2(t) + 4\cos^2(t)}$$
$$= \sqrt{4}$$
$$= 2.$$

The length is

$$\ell(r) = \int_0^{2\pi} \|r'(t)\| dt$$
$$= \int_0^{2\pi} 2\, dt$$
$$= 4\pi.$$

• Integrals along parametrized curves

Consider a parametrized curve $r : [a,b] \to \mathbb{R}^2$ and a function $f : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$, where $\mathcal{U}$ contains the trace of the curve.



Then we can define the integral of f along the curve C,

$$\int_C f$$

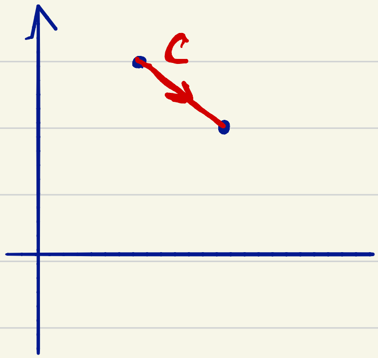
as

$$\int_C f = \int_a^b f(r(t)) \cdot \| r'(t) \| \, dt.$$

Examples:

10) $f(x,y) = x + y$

$C: r(t) = (2+t,\ 3-t)$, $\quad 0 \leq t \leq 1.$



The velocity is:

$r'(t) = (1, -1)$

so the speed is

$\|r'(t)\| = \sqrt{2}.$

$$\int_C f = \int_0^1 [x(t) + y(t)] \cdot \|r'(t)\| \, dt.$$

$$= \int_0^1 [(2+\cancel{t}) + (3-\cancel{t})] \cdot \sqrt{2} \, dt$$

$$= \int_0^1 5\sqrt{2} \, dt$$

$$= 5\sqrt{2}.$$

11) $f(x,y) = x^2 + y^2$

$C: r(t) = (\cos(t), \sin(t))$, $\quad 0 \leq t \leq 2\pi.$

$$r'(t) = (-\sin(t), \cos(t)),$$

$$\|r'(t)\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2}$$

$$= \sqrt{\sin^2(t) + \cos^2(t)}$$

$$= \sqrt{1}.$$

$$\int_C f = \int_0^{2\pi} [x^2(t) + y^2(t)] \cdot \|r'(t)\| \, dt$$

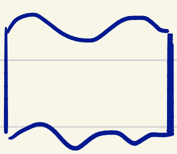
$$= \int_0^{2\pi} [\cos^2(t) + \sin^2(t)] \cdot 1 \, dt$$

$$= \int_0^{2\pi} 1 \, dt$$

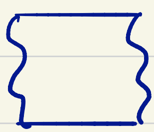
$$= 2\pi.$$

- <u>Double integrals and Fubini's Theorem</u>

Fubini's theorem describes how to integrate two-variable functions on the following regions:



or



Statements:

(a) if the region of integration is bounded by

$$a \le x \le b$$
$$g_1(x) \le y \le g_2(x)$$

then

$$\iint_R f \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx$$

(b) if the region is bounded by

$$c \le y \le d$$
$$h_1(y) \le x \le h_2(y)$$

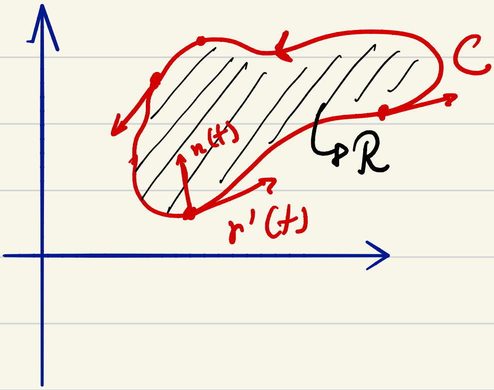
then

$$\iint_R f \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy.$$

In particular, if the region is a rectangle,

$$\int_a^b \int_c^d f(x,y) \, dx \, dy = \int_c^d \int_a^b f(x,y) \, dy \, dx.$$

What if we wish to integrate on a region bounded by other curves?
If the curve is parametrized, closed, simple, then we can turn to Green's theorem.



C has parametrization $r(t)$, so that its starting and end points are the same, and $\|r'(t)\| \neq 0$.

We can define the normal vector by means of the second derivative of the curve.
Convention: if curve is oriented in counter-clockwise orientation, normal points into region bounded by curve; if curve is oriented in clockwise orientation, normal points out of the region bounded by the curve.

One-forms (or implicit derivatives).

Combinations of type

$$p\,dx + q\,dy$$

where p, q are functions of (x, y).

To integrate a one-form along a curve, we compute

$$\int_C p\,dx + q\,dy = \int_a^b \left[p(x(t), y(t)) \cdot x'(t) + q(x(t), y(t)) \cdot y'(t)\right] dt.$$

Examples

1a) $C: r(t) = (2+t, 3-t), \quad 0 \le t \le 1.$

$$\int_C 2x\,dx + y^2\,dy = \int_0^1 (2+t)\cdot(2+t)' + (3-t)^2\cdot(3-t)'\,dt$$

$$= \int_0^1 (2+t)\cdot 1 + (3-t)^2\cdot(-1)\,dt$$

$$= \int_0^1 (2+t) - 9 + 6t - t^2\,dt.$$

$$= \int_0^1 (-t^2 + 7t - 7)\,dt$$

$$\int_C 2x\,dx + y^2 dy = -\frac{t^3}{3} + \frac{7t^2}{2} - 7t \Big|_{t=0}^{t=1}$$

$$= -\frac{1}{3} + \frac{7}{2} - 7$$

13) C: $r(t) = (\cos(t),\ \sin(t))$, $0 \le t \le 2\pi$

$$\int_C y\,dx + x\,dy = \int_0^{2\pi} \sin(t)\cdot(\cos(t))' + \cos(t)(\sin(t))'\,dt$$

$$= \int_0^{2\pi} -\sin^2(t) + \cos^2(t)\,dt.$$

$$= \int_0^{2\pi} \cos(2t)\,dt$$

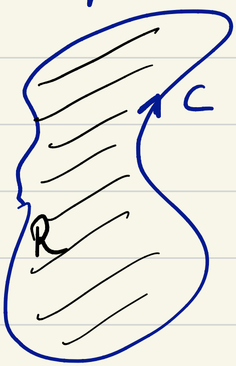
$$= \frac{\sin(2t)}{2}\Big|_{t=0}^{t=2\pi}$$

$$= \frac{\sin(4\pi)}{2} - \frac{\sin(0)}{2}.$$

$$= 0$$

Green's Theorem:

Suppose a region is bounded by a simple, closed curve. Suppose the curve is parametrized by $r(t)$, so that its starting and endpoints agree, $\|r'(t)\| \neq 0$, and the orientation is counterclockwise. Then



$$\boxed{\int_C p\,dx + q\,dy = \iint_R \left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) dA.}$$

In particular, if

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} \quad (*).$$

then $\int_C p\,dx + q\,dy = 0$. One forms satisfying $(*)$ are called closed.