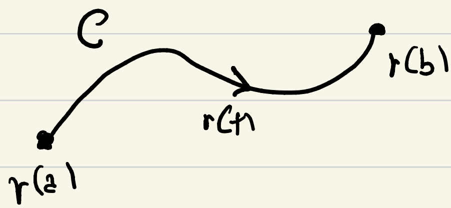
MAT 203 - Lecture 17

Classical Integration Theorems of Vector Calculus.

- Line integrals

Integrate along parametrized curves



Suppose we're given a curve: $r: [a,b] \longrightarrow \mathbb{R}^2$ or $\mathbb{R}^3$. We're also given a vector field V defined along the curve.

i) integration of functions along r

Speed of curve.

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \cdot \| r'(t) \| \, dt.$$

Similarly, in three dimensions,

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) \cdot \| r'(t) \| dt.$$

ii) integration of a vector field.

In 2D:

velocity of curve

$$\int_C V \cdot dr = \int_a^b V(x(t), y(t)) \cdot r'(t)\, dt.$$

In 3D:

$$\int_C V \cdot dr = \int_a^b V(x(t), y(t), z(t)) \cdot r'(t)\, dt.$$

Examples:

1) C: $r(t) = (\cos(t), \sin(t), t)$, $0 \le t \le 2\pi$. (Helix curve)

$f(x,y,z) = z$.

$$\int_C f\, ds = \int_0^{2\pi} t \cdot \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2}\, dt$$

$$= \int_0^{2\pi} t \cdot \sqrt{2}\, dt$$

$$= \frac{\sqrt{2}}{2} t^2 \Big|_{t=0}^{t=2\pi} = 4\pi^2 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}\pi^2.$$

2) $f(x,y) = x \cdot y \quad ; \quad r(t) = (4t)i + (3t)j, \; 0 \leq t \leq 1$

$r'(t) = 4i + 3j$

$\|r'(t)\| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$

$$\int_C f \, ds = \int_0^1 (4t) \cdot (3t) \cdot 5 \, dt$$

$$= \int_0^1 60t^2 \, dt.$$

$$= 20t^3 \Big|_{t=0}^{t=1}$$

$$= 20.$$

3) $V(x,y) = xi + (2y)j$

$C: \quad r(t) = ti + t^3 j \quad , \quad 0 \leq t \leq 2$

$r'(t) = i + (3t^2)j$

$$\int_C V \cdot dr = \int_0^2 (ti + (2t^3)j) \cdot (i + (3t^2)j) \, dt.$$

$$= \int_0^2 [t + 6t^5] \, dt.$$

$$\int_C v \cdot dr = \frac{t^2}{2} + t^6 \Big|_{t=0}^{t=2}$$

$$= \frac{2^2}{2} + 2^6$$

$$= 2 + 64$$

$$= 66.$$

4) $V(x, y, z) = x\,i + y\,j - 5z\,k$

$C: \; r(t) = 2\cos(t)\,i + 2\sin(t)\,j + t\,k, \quad 0 \le t \le 2\pi.$

$r'(t) = -2\sin(t)\,i + 2\cos(t)\,j + k$

$$\int_C V\,dr = \int_0^{2\pi} (2\cos(t)\,i + 2\sin(t)\,j - 5t\,k)(-2\sin(t)\,i + 2\cos(t)\,j + k)\,dt$$

$$= \int_0^{2\pi} [\cancel{-4\sin(t)\cos(t)} + \cancel{4\sin(t)\cos(t)} - 5t]\,dt.$$

$$= -\frac{5t^2}{2} \Big|_{t=0}^{2\pi}$$

$$= -\frac{5}{2} \cdot 4\pi^2 = -10\pi^2.$$

<u>Exercise 1:</u> Compute the integral of

$$f(x,y) = 3(x-y)$$

along the curve

$$C: \; r(t) = t\,\mathbf{i} + (2-t)\,\mathbf{j}, \quad 0 \le t \le 2.$$

<u>Solution:</u>

$$r'(t) = \mathbf{i} - \mathbf{j}$$

$$\|r'(t)\| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

$$\int_C f\, ds = \int_0^2 3(t - (2-t)) \cdot \sqrt{2}\, dt.$$

$$= \int_0^2 3 \cdot (2t - 2)\sqrt{2}\, dt.$$

$$= 3\sqrt{2} \int_0^2 (2t - 2)\, dt$$

$$= 3\sqrt{2} \left[t^2 - 2t \Big|_{t=0}^{t=2} \right]$$

$$= 3\sqrt{2} \cdot \left[4 - 4 - 0 - 0 \right].$$

$$= 0.$$

Exercise 2: $V(x,y,z) = (yz)i + (xz)j + (xy)k$.
C: line segment from $(0,0,0)$ to $(5,3,2)$.

Solution:

Parametrizing segment:

Base point: $(0,0,0)$

Direction: $(5,3,2) - (0,0,0) = (5,3,2)$.

$$r(t) = (0,0,0) + t\cdot(5,3,2)$$
$$= (5t, 3t, 2t)$$
$$= (5t)\,i + (3t)\,j + (2t)\,k.$$

- $r'(t) = 5i + 3j + 2k = (5,3,2)$.

$$\int_C V\,dr = \int_0^1 (6t^2 i + 10t^2 j + 15t^2 k)(5t\,i + 3t\,j + 2t\,k)\,dt$$
$$= \int_0^1 [30t^3 + 30t^3 + 30t^3]\,dt$$
$$= \int_0^1 90t^3\,dt = \frac{90t^4}{4}\Bigg|_{t=0}^{t=1} = \frac{45}{2}.$$

Properties of line integrals

- Fundamental Theorem of line integrals

Suppose that a vector field V is **conservative** in a region containing the curve $r(t)$, $a \le t \le b$, that is, there exists a potential function such that

$$V = \nabla f.$$

Then

$$\int_C V \, dr = f(r(b)) - f(r(a)).$$

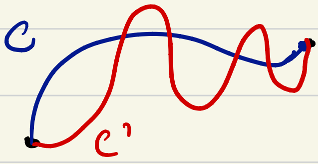
or

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a)).$$

In particular, the integral of a conservative vector field along a closed curve (initial point = final point) is zero.

Independence of path

The integral of a conservative vector field between two points does not depend on the chosen path joining them.



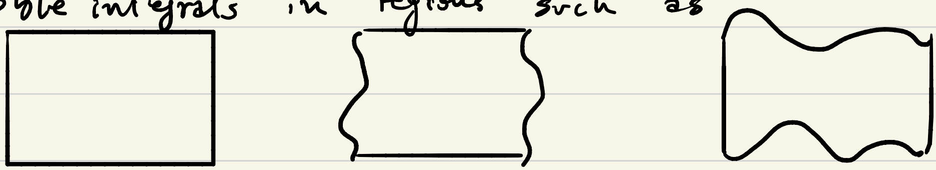
$$\int_C V \cdot dr = \int_{C'} V \cdot dr$$

Remarks. If vector field is merely closed, difference can be a non-zero constant. If the vector field is not even closed, the difference can be arbitrary.

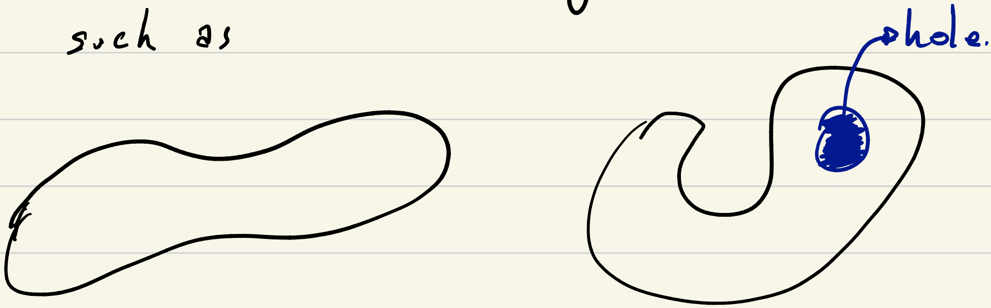
Conversely, if the integral is independent of path, then the vector field is conservative.

Green's Theorem

Fobini's theorem allows for computations of double integrals in regions such as



Green's theorem will allow for the computation of double integrals in other regions, such as



It does so by converting double integrals into line integrals and vice-versa.

Simply-connected regions: "regions without holes".

<u>Green's theorem</u>: Suppose a vector field

$$V(x,y) = M(x,y)\,i + N(x,y)\,j$$

is so that

$$\frac{\partial M}{\partial x},\ \frac{\partial M}{\partial y},\ \frac{\partial N}{\partial x},\ \frac{\partial N}{\partial y}$$

are continuous on a region R. Suppose also that R is simply-connected, and its boundary is parametrized by a curve $C: r(t)$ in <u>counterclockwise</u> sense, <u>running through it once</u>.

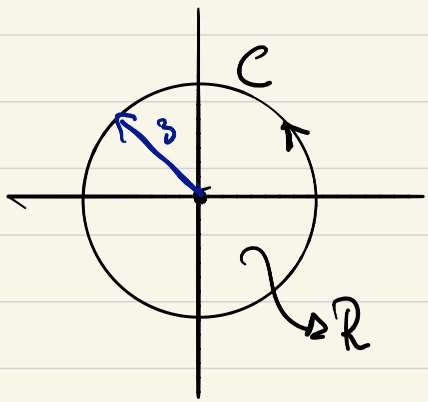
Then

$$\int_C V\,dr = \iint_R \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{}\,dA.$$

the difference we use to test for closedness: for a closed vector field, it is zero.

Example 5: From line integrals to double integrals.

Vector field: $V(x,y) = y^3 \hat{i} + (x^3 + 3xy^2)\hat{j}$.
C: $r(t) = 3\cos(t)\hat{i} + 3\sin(t)\hat{j}$, $0 \le t \le 2\pi$.



- $M(x,y) = y^3$

$$\frac{\partial M}{\partial y} = 3y^2$$

- $N(x,y) = x^3 + 3xy^2$

$$\frac{\partial N}{\partial x} = 3x^2 + 3y^2.$$

This vector field is not closed!
Using Green's Theorem:

$$\int_C V \cdot dr = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

$$= \iint_R (3x^2 + 3y^2 - 3y^2)\, dA$$

$$= \iint_R 3x^2 \, dA.$$

$$\int_C V \cdot dr = \int_0^{2\pi}\int_0^{3} 3r^2\cos^2\theta \cdot r\,dr\,d\theta$$

Here we used polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

$$\int_C V\,dr = \int_0^{2\pi}\left(\frac{3r^4}{4}\cos^2\theta\,\Big|_{r=0}^{r=3}\right)d\theta.$$

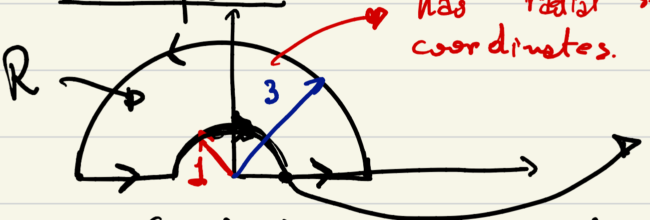
$$= \int_0^{2\pi}\frac{243}{4}\cos^2\theta\,d\theta.$$

$$= \frac{243}{4}\int_0^{2\pi}\frac{1+\cos(2\theta)}{2}\,d\theta.$$

$$= \frac{243}{8}\left[\theta + \frac{\sin(2\theta)}{2}\Big|_{\theta=0}^{\theta=2\pi}\right]$$

$$= \frac{243\pi}{4}.$$

Example 6:



has radial symmetry, easier in polar coordinates.

starting point of parametrization

$$V(x,y) = [\arctan(x) + y^2]\,i + [e^y - x^2]\,j.$$

$$r(t) = \begin{cases} (1+2t)\,i, & 0 \le t \le 1. \\ 3\cos(\pi(t-1))\,i + 3\sin(\pi(t-1))\,j, & 1 \le t \le 2. \\ -3 + 2(t-2), & 2 \le t \le 3. \\ \cos(\pi + \pi(t-3))\,i - \sin(\pi + \pi(t-3))\,j, & 3 \le t \le 4. \end{cases}$$

Much easier to integrate

$$\int_C V\,dr$$

via Green's Theorem. Here

$$M(x,y) = \arctan(x) + y^2$$
$$N(x,y) = e^y - x^2.$$

and

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = -2x.$$

$$\int_C V\,dr = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (-2x - 2y)\, dA.$$

$$= \int_0^{\pi}\int_1^3 -2r.(\cos\theta + \sin\theta)\; r\,dr\,d\theta$$

$$= -2\int_0^{\pi}\left[\frac{r^3}{3}(\cos\theta + \sin\theta)\Big|_{r=1}^{r=3}\right]d\theta$$

$$= -2\int_0^{\pi}\frac{26}{3}(\cos\theta + \sin\theta)\,d\theta.$$

$$= -\frac{52}{3}\int_0^{\pi}(\cos\theta + \sin\theta)\,d\theta$$

$$= -\frac{52}{3}\left[\sin\theta - \cos\theta\Big|_{\theta=0}^{\theta=\pi}\right]$$

$$= -\frac{52}{3}\left[-\cos\pi + \cos 0\right].$$

$$= -\frac{104}{3}.$$

Finding a vector field

$$V(x,y) = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$$

with prescribed value of

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

is simple, but ambiguous.

One special case happens for

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

We can choose

$$M = -\frac{y}{2}, \quad N = \frac{x}{2}.$$

Into Green's theorem:

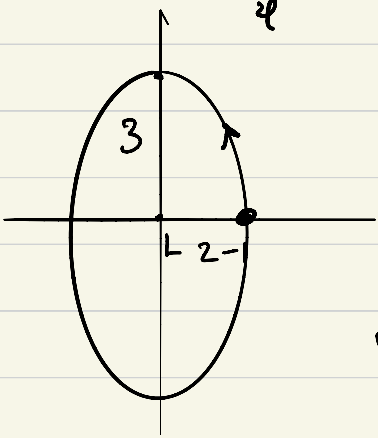
$$\frac{1}{2}\int_C (-y\,\mathbf{i} + x\,\mathbf{j})\,dr = \iint_R 1\,dA$$

or

$$\text{Area} = \frac{1}{2}\int_C (-y\,\mathbf{i} + x\,\mathbf{j})\cdot dr = \frac{1}{2}\int_C -y\,dx + x\,dy.$$

Example 7: Area of the elipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$



Recall vector field used for area: $-\frac{y}{2}\,\mathbf{i} + \frac{x}{2}\,\mathbf{j}$.

Parametrization of boundary.

$$r(t) = 2\cos(t)\,\mathbf{i} + 3\sin(t)\,\mathbf{j}.$$

$$r'(t) = -2\sin(t)\,\mathbf{i} + 3\cos(t)\,\mathbf{j}.$$

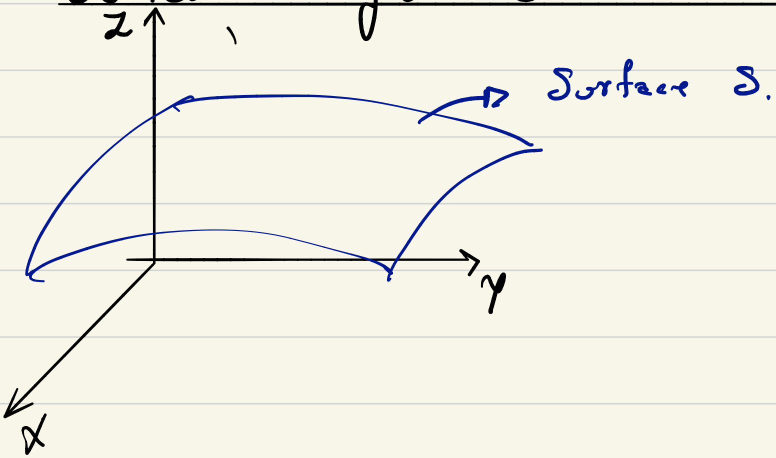
$$\text{Area} = \int_C \left(-\frac{y}{2}\,\mathbf{i} + \frac{x}{2}\,\mathbf{j}\right)\cdot dr$$

$$= \int_0^{2\pi}\left(-\frac{3\sin(t)}{2}\,\mathbf{i} + \frac{2\cos(t)}{2}\,\mathbf{j}\right)\cdot\left(-2\sin(t)\,\mathbf{i} + 3\cos(t)\,\mathbf{j}\right)dt$$

$$= \int_0^{2\pi}\left(3\sin^2(t) + 3\cos^2(t)\right)dt.$$

$$= \int_0^{2\pi} 3\,dt = 6\pi.$$

Surface integrals and Gauss' Theorem



Typically S is either the graph of a two-variable function

$$S: \; z = f(x,y),$$

or a level set of a three-variable function

$$S: \; g(x,y,z) = c.$$

We wish to know how to integrate functions and vector fields on S.

Suppose the surface has a parametrization

$$r(u,v) = x(u,v)\,i + y(u,v)\,j + z(u,v)\,k. \quad (*)$$

<u>Example 7</u>: A plane can be parametrized as

$r(u,v) =$ (Base point) + $u\cdot$(direction 1) + v(direction 2)

so long as directions 1 and 2 are not parallel. Expanding the coordinates of base point and direction leads to an expression like $(*)$.

<u>Example 8</u>: The upper hemisphere of a sphere with radius 1 about the origin can be parametrized via spherical coordinates.

$$r(\theta,\phi) = \cos(\theta)\sin(\phi)\,i + \sin(\theta)\sin(\phi)\,j + \cos(\phi)\,k.$$

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/2.$$

Normal vectors

1) If the surface is a level set, then take the gradient of the corresponding function.

2) Cross-product between

$$\frac{\partial r}{\partial u} \text{ and } \frac{\partial r}{\partial v}.$$

Example 8: Consider a plane with parametric equations

$$r(u,v) = 2i + 3uj + 4vk;$$

$-\infty \le u \le \infty$, $-\infty \le v \le \infty$.

Via method 2:

$$\frac{\partial r}{\partial u} = 3j \quad ; \quad \frac{\partial r}{\partial v} = 4k$$

$$n = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$$

$$n = (3j) \times (4k) = 12\, j \times k = 12i.$$

Via method 2: we need to find a function $f(x, y, z)$ such that the plane is a level set.
Function is linear,

$$f(x, y, z) = x + ay + bz - c.$$

and satisfies:

i) $(2, 0, 0)$ belongs to plane
ii) $(2, 3, 0)$ belongs to plane.
iii) $(2, 0, 4)$ belongs to plane.

i) $2 + a \cdot 0 + b \cdot 0 - c = 0$

$\boxed{c = 2}$

ii) $2 + 3 \cdot a + 0 b - c = 0$

$2 + 3 \cdot a + 0 - 2 = 0 \Rightarrow \boxed{a = 0}$

iii) $2 + 0 \cdot a + 4 \cdot b - c = 0 \Rightarrow \boxed{b = 0}$

Equation

$$f(x,y,z) = x - 2$$

Gradient:

$$\nabla f = 1i + 0j + 0k.$$

$$n = \nabla f$$
$$= (1, 0, 0)$$

is normal to the plane.

parallel to vector found by method 2

Surface integrals

- For functions

$$\iint_S f(x,y,z)\, dS \rightarrow \text{representation}$$

Over a graph

$$z = g(x,y)$$

we compute

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,g(x,y)) \cdot \|n\|\, dS,$$

where : R is projection onto xy-plane,

$$n = \left(\frac{\partial g}{\partial x}\right) i + \left(\frac{\partial g}{\partial y}\right) j + K.$$

- For a vector field $V(x,y,z)$

$$\iint_S V \cdot n\, dS = \iint_R V \cdot n\, dA.$$

dot product with normal vector.

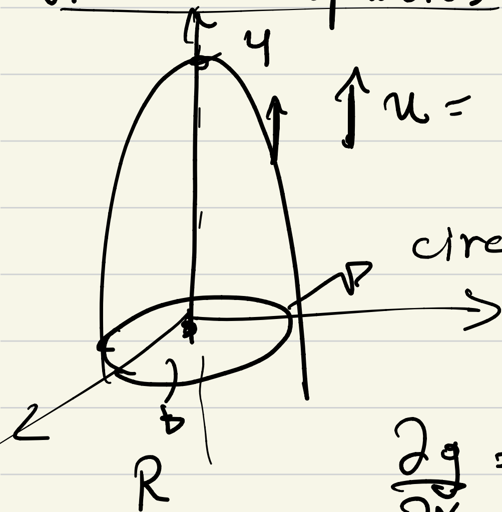
__Example 10:__

$$V(x,y,z) = x\,i + y\,j + z\,k.$$

$$S: \quad z = g(x,y) = 4 - x^2 - y^2;$$

oriented upwards.

$$H(x,y,z) = z - g(x,y).$$



$$n = (-\partial_x g, -\partial_y g, 1)$$

circle of radius 2: $x^2+y^2=4.$
centered at $(0,0,0)$.

$$\frac{\partial g}{\partial x} = -2x$$

$$\frac{\partial g}{\partial y} = -2y.$$

$$\iint_S V\cdot n\, dS = \iint_R (x,y,z)\cdot(2x\ \ 2y, 1)\,dA$$

$$= \iint_R 2x^2\ \ 2y^2 + z\ \ dA.$$

$$= \iint_R (2x^2 + 2y^2 + 4 - x^2 - y^2)\, dA$$

$$= \iint_R (4 + x^2 + y^2)\, dA.$$

$$= \int_0^{2\pi}\int_0^{2} (4 + r^2)\, r\, dr\, d\theta$$

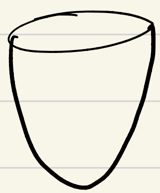
$$= \int_0^{2\pi} \left[2r^2 + r^4/4 \Big|_{r=0}^{r=2} \right] d\theta$$

$$= \int_0^{2\pi} \left[8 + 4 \right] d\theta$$

$$= \int_0^{2\pi} 12\, d\theta$$

$$= 24\pi.$$

Integration on closed surfaces.



not closed. closed.

Over closed surfaces we can turn surface integrals into triple integrals.

Divergence of a vector field:

In 2D: $V(x,y) = M(x,y)\,i + N(x,y)\,j$

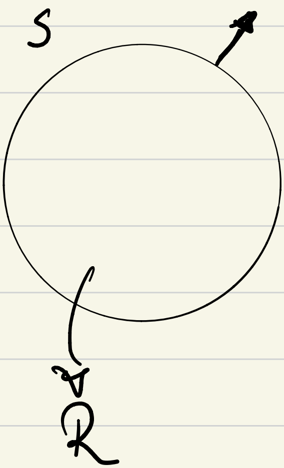
$$\operatorname{div}(V) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

In 3D: $V(x,y,z) = L(x,y)\,i + M(x,y)\,j + N(x,y)\,k.$

$$\operatorname{div}(V) = \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}.$$

Divergence (or Gauss') Theorem

Suppose the normal vector to closed surface is oriented outwards, and are normalized (have unit length). Then



$$\iint_S F \cdot n \, dS = \iiint_R \operatorname{div}(F) \, dV.$$