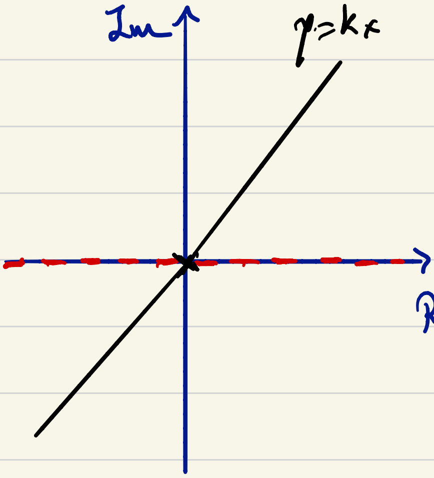
MAT 514 - Lecture 7

Review of Quiz 3

Problem 1: Consider the function
$$f: \{z = x+iy \in \mathbb{C} \mid x, y \in \mathbb{R},\ y \neq 0\} \to \mathbb{C}$$
defined
$$f(x+iy) = \frac{ix + 1}{y}.$$
Determine if this function has a limit at 0.

Solution:



The domain of f consists of the complement of the real axis in $\mathbb{C}$.

Let's study this function along lines $y = kx$.

$$\lim_{x \to 0} \frac{ix+1}{Kx}$$ does not exist, as the lateral limits are different ($-\infty$ and $+\infty$). It follows that f has no complex limit as $z \to 0$.

<u>Problem 2:</u> Determine all poles (infinite discontinuities) of the function

$$f(z) = \frac{z^2+z-2}{2z^2+z-3}$$

<u>Solution:</u> Both numerator and denominator are continuous functions. Continuity is guaranteed at all points for which

$$2z^2+z-3 \neq 0.$$

The only <u>possible</u> issues occur at the roots of

$$2z^2+z-3.$$

Solving $2z^2+z-3=0$, we obtain

$$2z^2+z-3=0 \Rightarrow (z-1)(2z+3)=0,$$

that is, the roots are

$$z=1 \quad \text{and} \quad z=-\frac{3}{2}.$$

<u>Case 1:</u> $$\lim_{z\to\left(-\frac{3}{2}\right)} \frac{z^2+z-2}{2z^2+z-3}.$$

Numerator goes to

$$\left(-\frac{3}{2}\right)^2+\left(-\frac{3}{2}\right)-2$$

$$=\frac{9}{4}-\left(\frac{3}{2}\right)-2$$

$$=\frac{9-6-8}{4}$$

$$=-\frac{5}{4}.$$

Since the numerator is bounded away from zero and the denominator converges to zero, the fraction converges to ∞,

$$\lim_{z \to (-3/2)} \frac{z^2 + z - 2}{2z^2 + z - 3} = \infty.$$

$z = -\frac{3}{2}$ is a pole.

Case 2: As $z \to 1$, the numerator goes to 0:

$$\lim_{z \to 1} z^2 + z - 2 = 0.$$

We have thus an indetermination, which will be dealt with via algebraic simplification

$$\frac{z^2 + z - 2}{2z^2 + z - 3} = \frac{\cancel{(z-1)}(z+2)}{\cancel{(z-1)}(2z+3)} = \frac{z+2}{2z+3}$$

So the function has a limit as $z \to 1$:

$$\lim_{z \to 1} \frac{z^2 + z - 2}{2z^2 + z - 3} = \frac{1 + 2}{2 \cdot 1 + 3} = \frac{3}{5},$$

therefore 1 is not a pole.

• Differentiabity and Holomorphicity

Newton's quotient for a function f, at a point x is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

In real variable calculus, this quotient defines the derivative of f at x.

In the complex setting, we define the derivative of a function

$$f : G \subset \mathbb{C} \to \mathbb{C}$$

at an interior point $x \in G$ by the same quotient (limit being interpreted in the complex sense). In case this limit exists, that is, it is a complex number (not the symbol ∞) we call the function differentiable at x.

Examples

① Let $f: \mathbb{C} \to \mathbb{C}$ be a constant function, $f(z) = c$.

Then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} 0$$

$$= 0.$$

In short, derivatives of constants are zero.

② Consider $f: \mathbb{C} \to \mathbb{C}$, $f(z) = 2z$.

Then

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)-2x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x}+2h-\cancel{2x}}{h}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}}$$

$$= \lim_{h\to 0} 2$$

$$= 2.$$

__Exercise:__ Compute the derivative of

$$f(z) = 3z-5$$

from its definition as a Newton quotient.

__Solution:__

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)-5-(3x-5)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x}+3h-\cancel{5}-\cancel{3x}+\cancel{5}}{h}$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}} = \lim_{h\to 0} 3 = 3.$$

Example 3: $f: \mathbb{C} \to \mathbb{C}$

$f(z) = z^2.$

Its derivative at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\cdot(2x+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x + h = 2x$$

Example 4: Consider the conjugation function

$$f: \mathbb{C} \longrightarrow \mathbb{C}$$
$$f(z) = \bar{z}.$$

Let's study its Newton quotient at $x=0$.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{\bar{h} - \bar{0}}{h}$$
$$= \lim_{h \to 0} \frac{\bar{h}}{h}$$

Recall from previous synchronous lecture that this limit does not exist, so $f(z)$ doesn't not have a complex derivative at 0. In fact, the complex conjugate function is not complex-differentiable at any point. For instance, if we try to compute the derivative at $x=1$, we find

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\overline{1+h} - \overline{1}}{h}$$

$$= \lim_{h\to 0} \frac{(\cancel{1}+\bar{h}-\cancel{1})}{h}$$

$$= \lim_{h\to 0} \frac{\bar{h}}{h}$$

<u>Definition</u>: Let $f: G \subset \mathbb{C} \to \mathbb{C}$ be a continuous function, and $x \in G$ an interior point. If there exists an open disk

$$D(x,r) \subset G$$

on which f is complex-differentiable, then we say that f is holomorphic at x.

<u>Example 5</u>: A differentiable, non-holomorphic function.

Consider $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = (\bar{z})^2$. This function is differentiable at 0, for its Newton quotient is:

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{(\bar{h})^2 - \bar{0}^2}{h}$$

$$= \lim_{h\to 0} \frac{(\bar{h})^2}{h}.$$

This limit is 0. Using polar representations: if $h = |h| \cdot (\cos\theta + i \sin\theta)$, then

$$(\bar{h})^2 = |h|^2 \cdot (\cos(-2\theta) + i\sin(-2\theta)).$$

Comparing polar forms,

$$\frac{(\bar{h})^2}{h} = \frac{|h|^2}{|h|} \cdot \frac{(\cos(-2\theta) + i \sin(-2\theta))}{(\cos(\theta) + i\sin(\theta))}.$$

$\hookrightarrow$ converges to zero. $\hookrightarrow$ bounded

It follows that

$$\lim_{h\to 0} \frac{(\bar{h})^2}{h} = 0.$$

Therefore f is differentiable at 0, with $f'(0) = 0$.

Now let x be any other complex number ($x \neq 0$). The Newton quotient at x is

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(\overline{x+h})^2 - (\bar{x})^2}{h}$$

$$= \lim_{h\to 0} \frac{(\bar{x}+\bar{h})^2 - (\bar{x})^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{(\bar{x})^2} + 2\bar{x}\cdot\bar{h} + (\bar{h})^2 - \cancel{(\bar{x})^2}}{h}$$

$$= \lim_{h\to 0} \frac{2\bar{x}\cdot\bar{h} + (\bar{h})^2}{h}$$

$$= \lim_{h\to 0} \left[\frac{2\bar{x}\cdot\bar{h}}{h} + \frac{(\bar{h})^2}{h} \right]$$

has no limit, ($x \neq 0$).

has 0 as its limit

The Newton quotient does not exist.