This homework is due on Tuesday, June 11, by 7:00 pm.

Homework 2

Exercise 1 Prove the uniqueness of the empty set, that is, if A and B are empty sets, show that $A = B$.

Exercise 2 The following claim is true, but the argument presented as its proof contains a mistake. Find the mistake, and fix it.

Claim: If A is a set, then $A \in \mathcal{P}(A)$.

Proof. Suppose $x \in A$. Then $\{x\} \subset A$. Thus $\{x\} \in \mathcal{P}(A)$. Therefore, $A \in \mathcal{P}(A)$. □

Exercise 3 Let A and B be sets. Prove that $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$.

Exercise 4 Let A, B, C and D be sets. Prove that

(a) $A \subset B$ if and only if $A - B = \emptyset$.

(b) If $A \subset B \cup C$ and $A \cap B = \emptyset$, then $A \subset C$.

(c) If $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.

(d) If $C \subset A$ and $D \subset B$, then $C \cap D \subset A \cap B$.

(e) If $A \cup B \subset C \cup D$, $A \cap B = \emptyset$, and $C \subset A$, then $B \subset D$.

Exercise 5 Let A and B be sets. Prove that

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B).$$

Show, by means of an example, that the equality

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$$

need not be true.

Exercise 6 Let A, B, C and D be sets.

(a) Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(b) Find an example that show thats the equality

$$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$$

is, in general, false.

Exercise 7 Use Mathematical Induction to verify the following statements:

(a) For every $n \in \mathbb{N}$, the number
$$\frac{n(n+1)}{2}$$
is an integer.

(b) For every $n \in \mathbb{N}$, the number
$$n(n+1)(2n+1)$$
is divisible by 6.

(c) For all $n \in \mathbb{N}$, the sum of the interior angles of a convex polygon with $(n+2)$-sides is $180 \cdot n$ degrees.

(d) For all $n \in \mathbb{N}$, given n points in a plane, no three of which are collinear, there are exactly
$$\frac{n^2 - n}{2}$$
line segments joining pairs all pairs of points.

Exercise 8 The Fibonacci numbers are recursively defined by the relations
$$\begin{aligned} f_1 &= 1, \\ f_2 &= 1, \\ f_{n+2} &= f_{n+1} + f_n. \end{aligned}$$

In this problem, you are required to use Induction (or its variants) to show the following:

(a) Two consecutive terms of this sequence have no common divisors, other than ± 1.

(b) f_{3n} is always even.

(c) f_{4n} is divisible by 3, for all $n \in \mathbb{N}$.

Exercise 9 In a certain kind of tournament, every player plays every other player exactly once, and either wins or losess. There are no ties. Define a top player to be a player who, for every other player x, either beats x or beats a player y who beats x.

(a) Show, by means of an example, that there can be more than one top player.

(b) Use Induction to show that every such tournament with n players has a top player.

(c) Use the Well-Ordering Principle to show that every such tournament with n players has a top player.