MAT 203 - Lecture 16

• Vector Calculus

Vector fields: functions whose inputs and outputs are vectors of same dimension.

Examples:

① $V(x,y) = (x+y,\ x-y)$
$= (x+y)\,i + (x-y)\,j.$

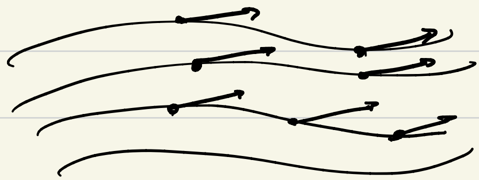
② $V(x,y) = (1,\ x)$
$= i + x\,j.$

③ $V(x,y,z) = (1,1,z)$
$= i + j + z\,k$

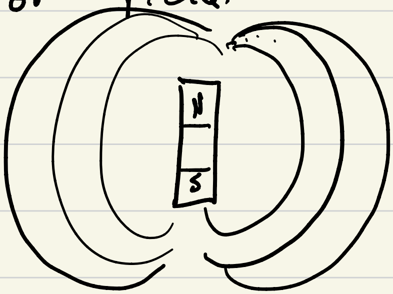
④ $V(x,y,z) = (x+y,\ y+z,\ z+x)$
$= (x+y)\,i + (y+z)\,j + (z+x)\,K.$

Common uses:

- Fluid dynamics: use vector fields to represent velocity vectors of fluid.



- Electromagnetism: represent EM field by a vector field.



- Gravitation: represent gravitational field by a vector field.

- Optimization: given a function $f(x,y)$ we can construct its gradient vector field

$$\nabla f(x,y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j.$$

At every point this vector field represents direction of greatest change in value.

<u>Definition:</u> A vector field V is called <u>conservative</u> within a given region R if there exists a function

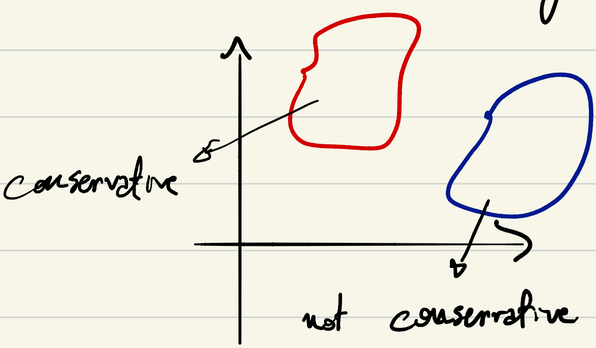
$$f: R \to \mathbb{R}$$

so that

$$V = \nabla f$$

If this is the case, f is called a potential function.

<u>Remark:</u> potential functions are not unique, they can be changed by additive constants



<u>Remark 2:</u> A vector field may be conservative in one region but not in another

Finding potential functions:

Example 1: $V(x,y) = (1,1)$.

We seek a function $f(x,y)$ such that

$$\partial_x f = 1$$
$$\partial_y f = 1$$

One such function $f(x,y) = x+y$. This is a potential on $\mathbb{R}^2$.

Example 2: $V(x,y) = (x,-y)$

We seek a function satisfying

$$\partial_x f = x$$
$$\partial_y f = -y.$$

One example:

$$f(x,y) = \frac{x^2 - y^2}{2}$$

This is a potential function on all of $\mathbb{R}^2$.

Exercise 1: Find a potential function for
$$V(x,y) = 3x^2y^3 i + 3x^3y^2 j$$

Solution: One solution is
$$f(x,y) = x^3y^3.$$
$$\partial_x f = 3x^2y^3$$
$$\partial_y f = 3x^3y^2.$$

Exercise 2: Try to find a potential function for
$$V(x,y) = 3x^3y^2 i + 3x^2y^3 j$$

Solution: This is not possible. Suppose a potential function $f(x,y)$ exists, that
$$\partial_x f = 3x^3y^2$$
$$\partial_y f = 3x^2y^3.$$
Then the second derivatives are:

$$\partial^2_{xx} f = 9x^2y^2 \qquad \partial^2_{xy} f = 6xy^3$$
$$\partial^2_{yx} = 6x^3y \qquad \partial^2_{yy} f = 9x^2y^2.$$

not the same, contradicting continuity of these.

<u>Closedness criterion</u>: A 2D vector field

$$V(x,y) = M(x,y)\,i + N(x,y)\,j$$

is closed

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

A 3D vector field

$$V(x,y,z) = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k$$

is closed if

$$\boxed{\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x}, \\ \frac{\partial P}{\partial z} &= \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial z} &= \frac{\partial R}{\partial y} \end{aligned}}$$

<u>Examples</u>

③ Consider the vector field

$$V(x,y) = \frac{x}{x^2+y^2}\,i + \frac{y}{x^2+y^2}\,j.$$

$$M(x,y) = \frac{x}{x^2+y^2} \quad ; \quad N(x,y) = \frac{y}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = x \cdot \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)$$

$$= \frac{-x \cdot (2y)}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = y \cdot \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

$$= \frac{-y \cdot (2x)}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence V is closed.

Example 4:

$$V(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$P(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$Q(x,y,z) = \frac{y}{(x^2+y^2+z^2)^{3/2}}$$

$$R(x,y,z) = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

- $$\frac{\partial P}{\partial y} = x \cdot \frac{\partial}{\partial y}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{-x \cdot \frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}} \cdot (2y)}{(x^2+y^2+z^2)^3}$$

$$= \frac{-3xy}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

- $$\frac{\partial Q}{\partial x} = y \cdot \frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{-y \cdot \frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}} \cdot (2x)}{(x^2+y^2+z^2)^3}$$

$$\frac{\partial Q}{\partial x} = \frac{-3xy}{(x^2+y^2+z^2)^{5/2}}.$$

Thus $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Similarly, for other entries

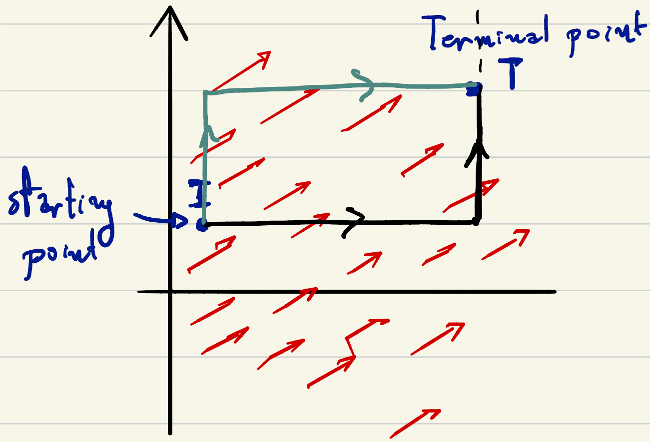
$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},$$

so vector field is closed.

<u>Remark</u>: The vector field of example 4 is what we use to model gravitational and electric fields of idealized point-particles.

Proposition: A continuous conservative vector field is closed.

Suppose a vector field is closed. How to find potentials?



We can integrate the vector field along paths, to find variation of potential function.

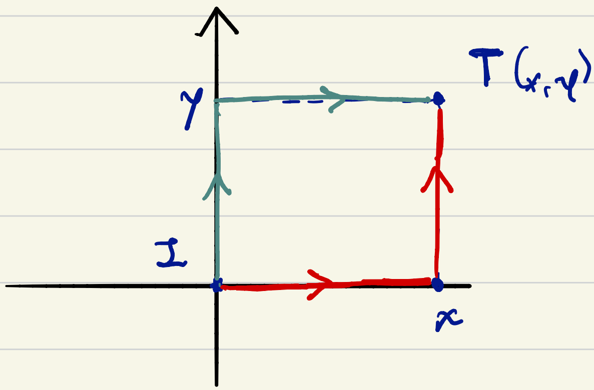
Example:

(5) $v(x, y) = x\,i - y\,j$

Choose starting point: $I = (0, 0)$

Terminal point: $T = (x, y)$.

v is closed: $\dfrac{\partial (x)}{\partial y} = 0 = \dfrac{\partial (-y)}{\partial x}$.



$$V = \nabla f$$

$$x = \frac{\partial f}{\partial x} \;;\; -y = \frac{\partial f}{\partial y}.$$

Step 1: Horizontal integration

Let $f(x,y)$ denote the potential function.

By the Fundamental Theorem of Calculus

$$f(x,0) - f(0,0) = \int_0^x \frac{\partial f}{\partial x}(t,0)\,dt$$

$$= \int_0^x t\,dt.$$

$$= \frac{t^2}{2}\Big|_{t=0}^{t=x}$$

$$= \frac{x^2}{2}.$$

Step 2: Vertical integration.

By the fundamental theorem of Calculus

$$f(x,y) - f(x,0) = \int_0^y \frac{\partial f}{\partial y}(x,s)\, ds.$$

$$= \int_0^y (-s)\, ds$$

$$= -\frac{s^2}{2} \Big|_{s=0}^{s=y}$$

$$= -\frac{y^2}{2}.$$

Step 3: Total variation

$$f(x,y) - f(0,0) = \underbrace{[f(x,y) - f(x,0)]}_{\text{vertical variation}} + \underbrace{[f(x,0) - f(0,0)]}_{\text{horizontal variation}}$$

$$= -\frac{y^2}{2} + \frac{x^2}{2}.$$

$$\boxed{f(x,y) = f(0,0) + \frac{x^2}{2} - \frac{y^2}{2}.}$$

Exercise 3: Find a potential for

$$V(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$$

by integrating vertically, then horizontally from $(0,0)$.

Solution:

Step 1: Vertical integration

$$f(0,y) - f(0,0) = \int_0^y \frac{\partial f}{\partial y}(0,s)\,ds$$

$$= \int_0^y (-s)\,ds$$

$$= -\frac{s^2}{2}\Bigg|_{s=0}^{s=y}$$

$$= -\frac{y^2}{2}.$$

Step 2: Horizontal integration

$$f(x,y) - f(0,y) = \int_0^x \frac{\partial f}{\partial x}(t,y)\,dt.$$

$$= \int_0^x t\,dt$$

$$\int_0^x M(t,0)\,dt = \int_0^x 0\,dt = 0.$$

Step 2: $$\int_0^y N(x,s)\,ds = \int_0^y 3x^2 s^3\,ds.$$

$$\Rightarrow \int_0^y N(x,s)\,dx = 3x^2 \int_0^y s^3\,ds$$

$$= \left.\frac{3x^2 s^4}{4}\right|_{s=0}^{s=y}$$

$$= \frac{3x^2 y^4}{4}.$$

Total variation:

$$\int_0^x M(t,0)\,dt + \int_0^y N(x,s)\,ds = \frac{3x^2y^4}{4}.$$

Method 2: Vertical $\rightarrow$ Horizontal integration

Step 1: $$\int_0^y N(0,s)\,ds = \int_0^y 3\cdot 0^2\cdot s^3\,ds.$$

$$\int_0^y N(0,s)\,ds = 0.$$

Step 2:

$$\int_0^x M(t,y)\,dt = \int_0^x 3t^3 y^2\,dt.$$

$$= 3y^2 \int_0^x t^3\,dt$$

$$= \frac{3y^2 t^4}{4}\Big|_{t=0}^{t=x}$$

$$= \frac{3x^4 y^2}{4}.$$

Total variation:

$$\int_0^y N(0,s)\,ds + \int_0^x M(t,y)\,dy = \frac{3x^4y^2}{4}.$$

We get results whose difference

is not a constant!

<u>Example 7:</u> A singular, closed vector field.

Consider the vector field

$$V(x,y) = \frac{x}{(x^2+y^2)^{3/2}}\,i + \frac{y}{(x^2+y^2)^{3/2}}\,j.$$

Here:

$$M(x,y) = \frac{x}{(x^2+y^2)^{3/2}}$$

$$N(x,y) = \frac{y}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial M}{\partial y} = x \cdot \frac{\partial}{\partial y}\left(\frac{1}{(x^2+y^2)^{3/2}}\right) = \frac{-2xy}{(x^2+y^2)^{5/2}}.$$

$$\frac{\partial N}{\partial x} = y \cdot \frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2)^{3/2}}\right) = \frac{-2xy}{(x^2+y^2)^{5/2}}.$$

The vector field is closed: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

A potential function is

$$f(x,y) = \frac{-1}{\sqrt{x^2+y^2}} = \frac{-1}{(x^2+y^2)^{1/2}}$$

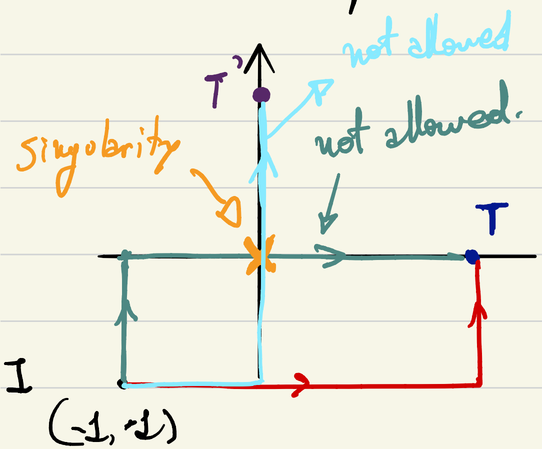
defined except at the origin.

$$\frac{\partial f}{\partial x} = -\left[\left(\frac{-1}{2}\right)\frac{(2x)}{(x^2+y^2)^{3/2}}\right] = \frac{x}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial f}{\partial y} = -\left[\left(\frac{-1}{2}\right)\frac{2y}{(x^2+y^2)^{3/2}}\right] = \frac{y}{(x^2+y^2)^{3/2}}.$$

Choose starting point: $I = (-1,-1)$

Terminal point: $T = (x,y)$



Certain orders of integration are obstructed:

i) if T is along x-axis, we cannot integrate vertically → horizontally

ii) if T is along y-axis, we cannot integrate horizontally + vertically.

Other that the two scenarios, we can choose any order, results will be the same.

<u>Example 8</u>: A closed, non-conservative vector field.

$$V(x,y) = \frac{-y}{(x^2+y^2)}\, i + \frac{x}{(x^2+y^2)}\, j.$$

$$M(x,y) = \frac{-y}{(x^2+y^2)} = -y \cdot (x^2+y^2)^{-1}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-y) \cdot (x^2+y^2)^{-1} + (-y) \cdot \frac{\partial}{\partial y}\left[(x^2+y^2)^{-1}\right]$$

$$= (-1) \cdot (x^2+y^2)^{-1} + (-y) \cdot \left[(-1)(x^2+y^2)^{-2} \cdot (2y)\right]$$

$$= \frac{-1}{(x^2+y^2)} + \frac{2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial M}{\partial y} = \frac{-(x^2+y^2) \quad + 2y^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2 + y^2}{(x^2+y^2)^2}$$

$$N(x, y) = \frac{x}{x^2+y^2} = x \cdot (x^2+y^2)^{-1}$$

$$\frac{\partial N}{\partial x} = \frac{\partial x}{\partial x} \cdot (x^2+y^2)^{-1} + x \cdot \frac{\partial [(x^2+y^2)^{-1}]}{\partial x}$$

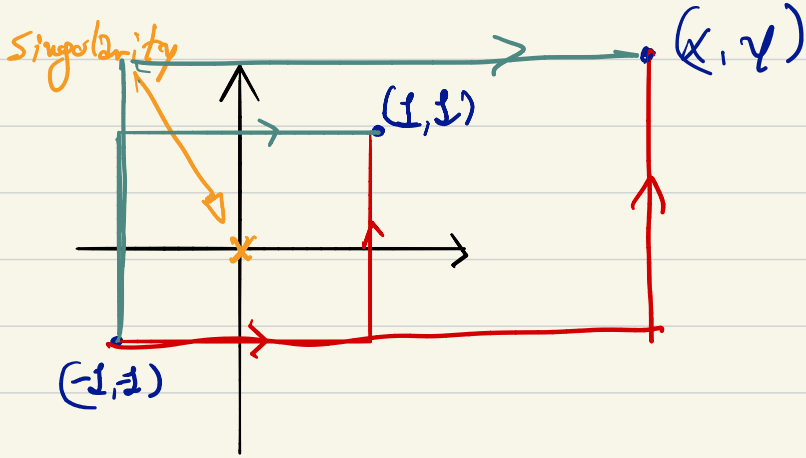
$$= (x^2+y^2)^{-1} + x \cdot (-1) \cdot (x^2+y^2)^{-2} \cdot (2x)$$

$$= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2 \quad - 2x^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2}.$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, V is closed.



Computing variation between $(-1,-1)$ and $(1,1)$:

Method 1: Horizontal $\to$ vertical.

Step 1 Horizontal

$$\int_{-1}^{1} M(t,-1)\,dt = \int_{-1}^{1} \frac{-(-1)}{t^2+(-1)^2}\,dt$$

$$= \int_{-1}^{1} \frac{1}{1+t^2}\,dt.$$

$$= \arctan(t)\Big|_{t=-1}^{t=1}$$

$$= \arctan(1) - \arctan(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

Step 2: Vertical

$$\int_{-1}^{1} N(1, s)\,ds = \int_{-1}^{1} \frac{1}{1^2 + s^2}\,ds$$

$$= \arctan(s)\Big|_{s=-1}^{s=1}$$

$$= \frac{\pi}{2}.$$

Total variation:

$$\int_{-1}^{1} M(t, -1)\,dt + \int_{-1}^{1} N(1, s)\,ds = \pi.$$

Method 2: Vertically $\to$ Horizontally

Step 1: Vertical variation

$$\int_{-1}^{1} N(-1, s)\,ds = \int_{-1}^{1} \frac{(-1)}{(-1)^2 + s^2}\,ds$$

$$\int_{-1}^{1} N(-1,s)\,ds = \int_{-1}^{1} \frac{-1}{1+s^2}\,ds$$

$$= -\arctan(s)\Big|_{s=-1}^{s=1}$$

$$= -\frac{\pi}{2}.$$

Step 2: Horizontal varition

$$\int_{-1}^{1} M(t,1)\,dt = \int_{-1}^{1} \frac{-(1)}{t^2+1^2}\,dt$$

$$= \int_{-1}^{1} \frac{-1}{1+t^2}\,dt$$

$$= -\arctan(t)\Big|_{t=-1}^{t=1}$$

$$= -\frac{\pi}{2}.$$

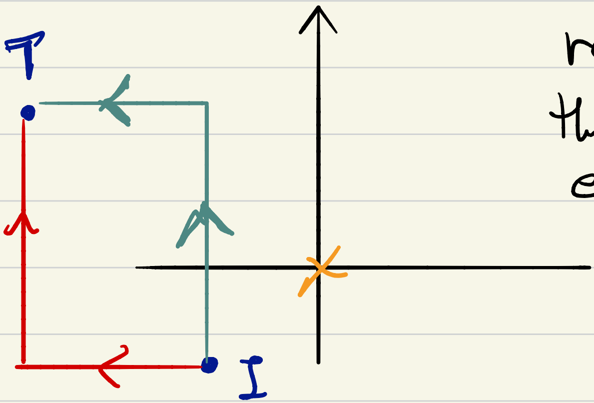
Step 3: total variation

$$\int_{-1}^{1} N(-1,s)\,ds + \int_{-1}^{1} M(t,1)\,dt = -\pi.$$

The difference is (-2π)!

This is a closed, non-conservative vector field on $\mathbb{R}^2 - \{(0,0)\}$.

<u>Remark</u>: Had the point been elsewhere the could be no difference.



This happens if the region bounded by the paths does not enclosed the singularity.

We will pick up from this example on Wednesday.