Spring 2020 MAT303 Recitations

Week of 3/20/20: Sections 3.2 and 3.3

Section 3.2: General solutions of linear equations

In the first few problems of Homework 6 you're tasked with finding coefficients of a complementary solution to an inhomogeneous equation of type

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots p_{n-1}(x)y' + p_n(x) = f(x).$$

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$$y(x) = y_c(x) + y_p(x)$$

Section 3.2: General solutions of linear equations

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$$\begin{cases} y'' - 2y' - 3y &= 6, \\ y(0) &= 3, \\ y'(0) &= 11. \end{cases}$$

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You are given the forms of complementary and particular solutions,

$$y_c = c_1 e^{-x} + c_2 e^{3x}; \quad y_p = -2$$

Combining these with the initial conditions, a solution to the problem is obtained by solving the algebraic system

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The solutions are $c_1 = 1, c_2 = 4$, thus $y(x) = e^{-x} + 4e^{3x} - 2$.

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Let $y_1, y_2, \cdots, y_n$ be functions defined on an interval I, assumed to solve an n-th order homogeneous, linear differential equation, with continuous coefficients

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Then either

(a) Their Wronskian is identically zero on I, in which case the functions are L.D.

(b) Their Wronskian is nowhere zero on I, and the funcions are L.I.

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It is easy to verify that the functions $y_1(x) = x, y_2(x) = x^2$ are solutions on $\mathbb{R}$. These functions are independent, as if the combination $y(x) = ax + bx^2$ is null both coefficients must vanish.

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$$W(y_1, y_2) = \det \begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix} = 2x^2 - x^2 = x^2,$$

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$$W(y_1, y_2) = \det \begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix} = 2x^2 - x^2 = x^2,$$

a function which vanishes when $x = 0$. Why is this not a contradiction?

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As a consequence: there is no linear, second-order differential equation defined on $\mathbb{R}$ whose solutions are x and x^2. Think about problem 33 in this context. Why is the Wronskian criterion applicable to the three functions involved? Can you think of a differential equation whose solutions are given by such functions?

Section 3.3: Homogeneous Equations with Constant Coefficients

The method of characteristic equations assigns to an equation of type

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_1 y' + a_0 = 0$$

its characteristic polynomial:

$$p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0,$$

where each a_i is a constant, $a_n \neq 0$,

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To each (possibly complex) root r of the polynomial, there corresponds a solution of the differential equation of the form e^{rx}. If the root has a multiplicity, then new solutions may be generated by monomial multiplication: $xe^{rx}, x^2 e^{rx}, \cdots, x^{m-1} e^{rx}$, where m is the multiplicity.

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Its characteristic polynomial is

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The roots are $r_1 = 0$, with multiplicity one, and $r_2 = 1$, with multiplicity three.

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The roots are $r_1 = 0$, with multiplicity one, and $r_2 = 1$, with multiplicity three. The general solution of the equation takes the form

$$y(x) = A + Be^x + Cxe^x + Dx^2e^x.$$

Section 3.3: Homogeneous Equations with Constant Coefficients

The next example is a slight modification of problem 3.3.23:

$$y'' - 8y' + 25y = 0,$$

with initial conditions $y(0) = 1, y'(0) = 2$.

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The characteristic equation of the problem is

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and its solutions are the complex numbers $r_1 = 4 + 3i$, $r_2 = 4 - 3i$.

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and its solutions are the complex numbers $r_1 = 4 + 3i$, $r_2 = 4 - 3i$.
The general real solution of the D.E. takes the form

$$y(x) = e^{4x}(A\cos(3x) + B\sin(3x)).$$

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thus the desired solution is

$$y(x) = e^{4x}\left(\cos(3x) - \frac{2}{3}\sin(3x)\right).$$

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In problems 3.3.44 and 3.3.46, you are confronted with differential equations with *complex coefficients*. In this case the roots of the characteristic polynomial need not be conjugates to each other, as the following example (problem 3.3.45) shows.

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has characteristic equation $r^2 - 2ir + 3 = 0$, whose solutions are $-i$ and $3i$. The complex-valued solutions to the D.E. are:

$$y_1(x) = e^{-ix} = \cos(x) - i\sin(x),$$

and

$$y_2(x) = e^{3ix} = \cos(3x) + i\sin(3x).$$

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Using the substitution $v = \log(x), x > 0$, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x}\frac{dy}{dx} \\ \frac{d^2y}{dx^2} &= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dx^2} \end{aligned}$$

Section 3.3: Homogeneous Equations with Constant Coefficients

Rephrasing the original equation in terms of v yields

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whose solutions (obtained via characteristic equation) take the form

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or, in terms of x,

$$y(x) = x^{-3}(A\cos(4\log(x)) + B\sin(4\log(x))).$$