MAT 203 - Summer I 2020
June 23, 2020

Homework 3 solutions

Exercise 1 Find the directional derivative of the function

$$f(x,y) = \frac{y^2}{4} - x^2,$$

at the point $P = (1,4)$, in the direction of

$$v = 2\mathbf{i} + \mathbf{j}.$$

Solution: We will compute this directional derivative straight from the definition, as a limit:

$$\begin{aligned}
\frac{\partial f}{\partial v}(1,4) &= \lim_{t\to 0} \frac{f((1,4)+tv) - f(1,4)}{t} \\
&= \lim_{t\to 0} \frac{f(1+2t, 4+t) - f(1,4)}{t} \\
&= \lim_{t\to 0} \frac{\frac{(4+t)^2}{4} - (1+2t)^2 - \frac{4^2}{4} + 1^2}{t} \\
&= \lim_{t\to 0} \frac{16 + 8t + t^2 - 4 - 16t - 16t^2 - 4 + 1}{4t} \\
&= \lim_{t\to 0} \frac{9 - 8t - 15t^2}{4t} \\
&= -2.
\end{aligned}$$

Exercise 2 Use the gradient to function the directional derivative of the function

$$w = 5x^2 + 2xy - 3y^2 z$$

at $P = (1,0,1)$, in the direction of

$$v = \mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Solution: The partial derivatives of w are

$$\frac{\partial w}{\partial x} = 10x + 2y,$$
$$\frac{\partial w}{\partial y} = 2x - 6yz,$$
$$\frac{\partial w}{\partial z} = -3y^2.$$

Evaluated at the point $(1, 0, 1)$, the gradient is

$$\nabla f = (10 \cdot 1 + 2 \cdot 0, 2 \cdot 1 - 6 \cdot 0 \cdot 1, -3 \cdot 0^2) = (10, 2, 0)$$

The directional derivative can be computed by means of a dot product of between the direction vector and the gradient,

$$\frac{\partial w}{\partial v} = (\nabla f) \cdot v = (10, 2, 0) \cdot (1, 1, -1) = 12.$$

Exercise 3 Find the gradient of the function

$$z = (e^{-x}) \cos(y)$$

and the maximum value of the directional derivative at the point $\left(0, \frac{\pi}{4}\right)$.

Solution: The partial derivatives of this function are

$$\frac{\partial z}{\partial x} = -e^{-x} \cos(y),$$
$$\frac{\partial z}{\partial y} = -e^{-x} \sin(y).$$

The gradient is thus

$$\nabla z = (-e^{-x} \cos(y), -e^{-x} \sin(y)).$$

Directional derivatives are given by dot products of unit vectors with the gradient. As such, for any unit vector v,

$$\frac{\partial z}{\partial v} = (\nabla z) \cdot v = \|\nabla z\| \cdot \|v\| \cdot \cos(\theta),$$

where θ is the angle between the vectors ∇z and v. Using the fact that $\|v\| = 1$, we see that the value of the directional derivative is maximized when $\cos(\theta) = 1$, i.e., when v is parallel and points in the same direction as ∇z. Such a vector can be found by normalization,

$$\begin{aligned}
v &= \frac{\nabla z}{\|\nabla z\|} \\
&= \frac{(-e^{-x}\cos(y), -e^{-x}\sin(y))}{\sqrt{(-e^{-x}\cos(x))^2 + (-e^{-x}\sin(x))^2}} \\
&= \frac{(-e^{-x}\cos(y), -e^{-x}\sin(y))}{\sqrt{e^{-2x}}} \\
&= \frac{(-e^{-x}\cos(y), -e^{-x}\sin(y))}{e^{-x}} \\
&= (-\cos(y), -\sin(y))
\end{aligned}$$

We now plug in the coordinates of the point at which we seek to maximize the derivative,

$$v_{max} = \left(-\cos\left(\frac{\pi}{4}\right), -\sin\left(\frac{\pi}{4}\right)\right) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

The value of the derivative itself is

$$\frac{\partial z}{\partial v_{max}}\left(0, \frac{\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \cdot \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 1.$$

Exercise 4 Find all relative extrema and saddle points of the function

$$f(x, y) = x^2 - y^2 - 16x - 16y.$$

Use the Second Partials Test where applicable.

Solution: The first and second derivatives of this function are as follows:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2x - 16 \\
\frac{\partial f}{\partial y} &= -2y - 16 \\
\frac{\partial^2 f}{\partial x^2} &= 2 \\
\frac{\partial^2 f}{\partial x \partial y} &= 0 \\
\frac{\partial^2 f}{\partial y^2} &= -2
\end{aligned}$$

The gradient of the function, $\nabla f = (2x - 16, -2y - 16)$ vanishes exactly once, at the point

$(x, y) = (8, -8)$. The Hessian matrix at this point is

$$\text{Hess}(f) = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix}.$$

Its determinant is -4, therefore, by the Second Partials Test, the critical point $(8, -8)$ corresponds to a saddle point.

Exercise 5 A corporation manufactures digital cameras at two locations. The cost of producing x_1 units at location 1 is

$$C_1 = 0.05(x_1)^2 + 15x_1 + 5400,$$

and the cost of producing x_2 units at location 2 is

$$C_2 = 0.03(x_2)^2 + 15x_2 + 6100.$$

The digital cameras sell for \$180 per unit. Find the quantity that should be produced at each location to maximize the profit

$$P(x_1, x_2) = 180(x_1 + x_2) - C_1 - C_2.$$

Solution: The profit function can be measured as

$$P(x_1, x_2) = -0.05x_1^2 - 0.03x_2^2 + 165(x_1 + x_2) - 11500.$$

Its partial derivatives are

$$\frac{\partial P}{\partial x_1} = -0.1x_1 + 165$$
$$\frac{\partial P}{\partial x_2} = 0.06x_2 + 165$$

These vanish simultaneously at the point $(x_1, x_2) = (1650, 2750)$. We shall confirm below that this point is a local maximum by means of the Second Partials Test. The Hessian matrix at $(1650, 2750)$ is

$$\text{Hess}(f)(1650, 2750) = \begin{vmatrix} -0.1 & 0 \\ 0 & -0.06 \end{vmatrix}.$$

Its determinant is $d = 0.006$, a positive number, thus we need the trace of the Hessian matrix (the Laplacian) to determine the type of critical point. As the sum of diagonal elements is negative, $\Delta f(1650, 2750) = -0.16$, the critical point is a local maximum, as expected. .

Exercise 6 Evaluate the definite integral

$$\int_0^2 \int_{x^2}^{2x} (x^2 + 2y)\, dy\, dx.$$

Solution:

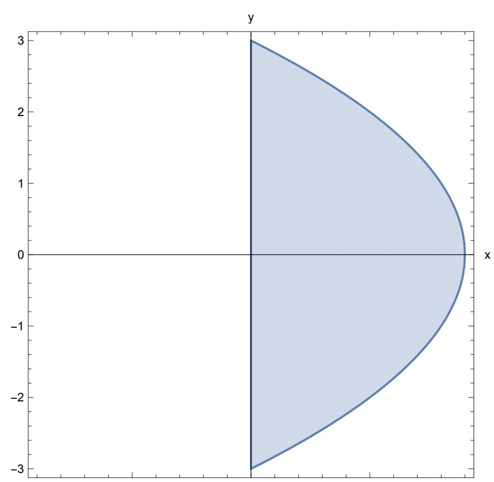
$$\begin{aligned}
\int_0^2 \int_{x^2}^{2x} (x^2 + 2y)\ dy\ dx &= \int_0^2 \left[x^2 y + y^2 \Big|_{y=x^2}^{y=2x} \right]\ dx \\
&= \int_0^2 \left[x^2(2x) + (2x)^2 - x^2 \cdot x^2 - (x^2)^2 \right]\ dx \\
&= \int_0^2 \left[2x^3 + 4x^2 - 2x^4 \right]\ dx \\
&= -\frac{2x^5}{5} + \frac{2x^4}{4} + \frac{4x^3}{3} \Big|_{x=0}^{x=2} \\
&= \frac{88}{15}
\end{aligned}$$

Exercise 7 Sketch the region whose area is given by the iterated integral

$$\int_{-3}^{3} \int_0^{9-y^2} dxdy.$$

Change the order of integration and show that both orders yield the same area.

Solution: Below is a plot of the region whose area we seek to compute,



It is bounded to the left by the y-axis, and to the right by the parabola $x = 9 - y^2$. The value of the area is

$$\begin{aligned}
\int_{-3}^{3}\int_{0}^{9-y^2} dx\,dx &= \int_{-3}^{3}\left[x\Big|_{x=0}^{x=9-y^2}\right] dy \\
&= \int_{-3}^{3}(9-y^2)\,dx \\
&= 9y - \frac{y^3}{3}\Big|_{y=-3}^{y=3} \\
&= 36
\end{aligned}$$

Exercise 8 Use a double integral to find the volume of the tetrahedron bounded by the xy, xz, yz planes and the plane given by the equation

$$x + y + z = 2.$$

Solution: A similar problem was solved in class. The key point here is to understand and parametrize the region of integration, in this case a triangle in the xy-plane, with vertices $(0,0,0)$, $(2,0,0)$, $(0,2,0)$. As such, we can compute the volume by means of the integral

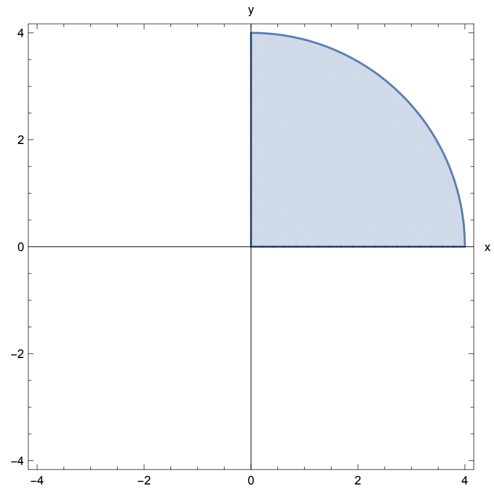
$$\begin{aligned}
\int_{0}^{2}\int_{0}^{2-x}(2-x-y)\,dy\,dx &= \int_{0}^{2}\left[2y - xy - \frac{y^2}{2}\Big|_{y=0}^{y=2-x}\right] dx \\
&= \int_{0}^{2}\left[2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\right] dx \\
&= \int_{0}^{2}\left[2 - 2x + \frac{x^2}{2}\right] dx \\
&= \left[2x - x^2 + \frac{x^3}{6}\Big|_{x=0}^{x=2}\right] \\
&= \frac{4}{3}
\end{aligned}$$

Exercise 9 Evaluate the iterated integral

$$\int_{0}^{4}\int_{0}^{\sqrt{16-y^2}}(x^2+y^2)\,dx\,dy$$

by converting it to polar coordinates.

Solution: Below is a plot of the region of integration,



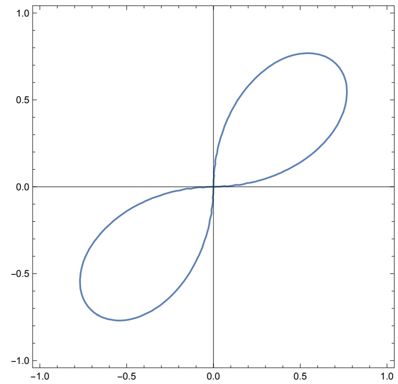
This region can be parametrized in polar coordinates by: $0 \leq r \leq 4$, $0 \leq \theta \leq \frac{\pi}{2}$. Thus the integral can be written as

$$\begin{aligned}
\int_0^4 \int_0^{\sqrt{16-y^2}} (x^2+y^2)\,dx\,dy &= \int_0^{\frac{\pi}{2}} \int_0^4 (r^2)r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \Big|_{r=0}^{r=4} \right] d\theta \\
&= \int_0^{\frac{\pi}{2}} 64\,d\theta \\
&= 32\pi.
\end{aligned}$$

Exercise 10 Use a double integral to find the area of the region bounded by the equation

$$r = 2\sin(2\theta).$$

Solution: A plot of the curve (following the convention announced explained in the supplemental course notes on polar coordinates) is outlined below.



This curve is defined for $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$. By means of symmetry, we may describe the area of the region enclosed by this curve as twice the area of the petal in the first quadrant, that is,

$$\begin{aligned}
\text{Area} &= 2\int_0^{\frac{\pi}{2}} \int_0^{2\sin(2\theta)} r\,dr\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2} \Bigg|_{r=0}^{r=2\sin(2\theta)} \right] d\theta \\
&= \int_0^{\frac{\pi}{2}} 4\sin^2(2\theta)\,d\theta \\
&= \int_0^{\frac{\pi}{2}} 2(1-\cos(4\theta))d\theta \\
&= \left[2\theta - 2\sin(4\theta) \Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}} \right] \\
&= \pi.
\end{aligned}$$