This homework is due on Tuesday, 7/24, in class, by 1:30 pm.

Homework 2

Exercise 1 Sketch the following plane curves:
(a) $r(t) = (t, t^2)$.

(b) $r(t) = (\cos(t), \sin(t))$.

(c) $r(t) = (t^3 - 4t, t^2 - 4)$.

(d) $r(t) = (t^3, t^2)$.

(e) $r(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$.

Exercise 2 For each of the plane curves below, find the parametric equation of their tangent lines at the points indicated.
(a) $r(t) = (t, t^3)$ at the point $(2, 8)$.

(b) $r(t) = (\cos(t), \sin(t))$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Exercise 3 Consider the curves $r(t) = (t, t, t^2)$ and $s(t) = \left(\frac{1}{t}, \frac{1}{t}, 0\right)$, defined for $t \neq 0$.
(a) Does the curve r have a limit as t goes to 0? If so, what is the limit?

(b) Does the curve s have a limit as t goes to 0? If so, what is the limit?

(c) Compute the dot product of the curves, $r(t) \cdot s(t)$, for $t \neq 0$.

(d) Does the scalar function obtained in part (c) have a limit as t goes to 0? If so, what is this limit?

Exercise 4 An object moves in the plane according to a trajectory described by a smooth curve $r(t)$. Assume that:

1. the curve never passes through the origin, i.e., $r(t) \neq 0$;

2. the velocity vector is never zero, $r'(t) \neq 0$;

3. at time $t = 0$, the curve is at its closest point to the origin.

Explain why the position and velocity vectors $r(0)$ and $r'(0)$ are perpendicular.

Exercise 5 Let $r(t)$ denote a spatial curve, and $r'(t)$, $r''(t)$ its first and second derivatives, respectively. Assume that $r''(t) \neq 0$. If the position $r(t)$ and acceleration $r''(t)$ are colinear, for all times, what can you say about the cross product $r(t) \times r'(t)$?

Exercise 6 As we saw in class, the Fundamental Theorem of Calculus for Curves can be used to compute the displacement vector,

$$\int_a^b r'(t)dt = r(b) - r(a).$$

Use this to describe the trajectory described by a curve with velocity vector

$$r'(t) = \frac{1}{1+t^2}i + tj + e^t k,$$

and which satisfies $r(0) = (1, 0, -1)$.

Exercise 7 Find two vector functions $r(t)$ and $s(t)$ for which

$$\int [r(t) \times s(t)]dt \neq \left(\int r(t)dt\right) \times \left(\int s(t)dt\right)$$

Exercise 8 Consider the function

$$f(x, y) = \frac{x^2 y}{x^4 + y^2},$$

defined for all points (x, y) in the plane except the origin $(0, 0)$. This exercise will study the behavior of this function near the origin.

(a) Compute the directed limits of this function along the lines $y = kx$, in terms of the parameter k.

(b) Compute the limit of this function along the parabola $y = x^2$.

The results of parts a and b should be different. This is to show you that unlike what you studied in single-variable Calculus and what we observed for vector-valued, single-variable functions, the existence and coincidence of directed limits no longer imply the existence of the limit of the function. This marks a sharp contrast between single-variable functions and multivariable functions.

Exercise 9 This exercise is about the following function:

$$f(x,y) = \begin{cases} \frac{y}{x} - y & \text{if } 0 \le y < x \le 1 \\ \frac{x}{y} - x & \text{if } 0 \le x < y \le 1 \\ 1 - x & \text{if } 0 < x = y \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Sketch the graph of this function on the square $[0,1] \times [0,1]$.

(b) What is the value of this function along the boundary of the square?

(c) If the value of x is kept constant, is f a continuous function of y?

(d) If the value of y is kept constant, is f a continuous function of x?

(e) Compute the directed limit of the function as (x,y) approaches the origin along the line $y = x$.

(f) Compare your answer of part (e) with the value of the function at $(0,0)$ which you obtained in part (b). Is this function continuous at $(0,0)$?

Exercise 10 Compute **all** the partial derivatives of the function $f(x,y,z) = ye^x + x\ln(z^2+1)$ up to order two.

Exercise 11 Verify that the functions given satisfy the corresponding equations.

(a) The function $f(t,x) = \sin(x-t)$ satisfies *the wave equation*

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}.$$

(b) The function $f(t,x) = e^{-t}\cos(x)$ satisfies *the heat equation*

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

(c) The function $f(x,y) = e^x \sin(y)$ satisfies *Laplace's equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$