MAT 203
Summer I 2020
Midterm
06/11/20
Time Limit: 3 hours and 25 minutes

Name (Print): ____________________

ID number ____________________

Instructions

- This exam contains 9 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.
- You may *not* use your books, notes, or any device that is capable of accessing the internet on this exam (e.g., smartphones, smartwatches, tablets). You may not use a calculator.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.

Problem	Points	Score
1	30	
2	20	
3	10	
4	20	
5	20	
Total:	100	

1. In each of the following problems, determine if the statements are true or false. Explain your reasoning (correct answers without an explanation will be worth only 2 points per statement).

   (a) (5 points) Two vectors in space always determine a unique plane.

   Solution: This is false. Collinear vectors do not determine a unique plane.

   (b) (5 points) Two planes in space always intersect.

   Solution: This is false. Parallel planes do not intersect.

   (c) (5 points) The dot product can be used to detect whether two non-zero vectors in space are aligned.

   Solution: This is true. Two vectors are aligned if $(u \cdot v)^2 = (u \cdot u)(v \cdot v)$.

   (d) (5 points) Let $r(t)$ and $s(t)$ be curves in the plane, such that neither has a limit as t converges to 0. Then their cross-product $r(t) \times s(t)$ does not have a limit at 0 either.

   Solution: This is false. Neither of the curves $r(t) = \frac{1}{t}i$ and $s(t) = \frac{2}{t}i$, defined for $t \neq 0$, has a limit as t goes to 0, but their cross-product $(r \times s)(t)$ is equal to zero for all $t \neq 0$, since the curves are collinear, hence has limit 0 as t goes to 0.

   (e) (5 points) If all directed limits a scalar-valued, multivariable function at a point exist and coincide, then the function has a limit at the point, in the multivariable sense.

   Solution: This is false. One counter-example, as seen in the supplemental notes on continuity, is the function $f(x, y) = \frac{x^2y}{x^4+y^2}$, which has directed limits equal to 0 at the origin, along any direction, but whose limit at the origin does not exist in the multivariable sense, since limits along other curves, such as the parabolae $y = kx^2$ yield inconsistent results, for different values of k.

   (f) (5 points) If a scalar-valued, multivariable function is separately continuous with respect to each variable, then it is continuous in the multivariable sense.

   Solution: This is false. One counter-example, as seen in the notes, is the function defined by

$$f(x, y) = \begin{cases} \frac{y}{x} - y & \text{if } 0 \leq y < x \leq 1 \\ \frac{x}{y} - x & \text{if } 0 \leq x < y \leq 1 \\ 1 - x & \text{if } 0 < x = y \leq 1 \\ 0 & \text{elsewhere,} \end{cases}$$

   which is separately continuous with respect to x and y, but not continuous at the origin, for its directed limit along the ray $y = x, x > 0$, as x tends to 0, does not coincide with the value of the function at the origin.

2. Consider the lines whose parametric equations are given by

$$L_1\colon x = 2t, y = 4t, z = 6t.$$
$$L_2\colon x = 1 - s, y = 4 + s, z = -1 + s.$$

(a) (5 points) Write symmetric equations for each line

Solution: We can rewrite the equations of line 1 by eliminating the variable t as

$$x = \frac{y}{2} = \frac{z}{3}$$

Meanwhile, equations for line 2 can be written as

$$1 - x = y - 4 = z + 1$$

(b) (5 points) Explain why these lines do not intersect.

Solution: If the lines intersect then the following system of equations has a solution,

$$\begin{aligned} 2t &= 1 - s \\ 4t &= 4 + s \\ 6t &= -1 + s. \end{aligned}$$

However, these equations are incompatible with each other: solving equations 1 and 2 for t yields $t = \frac{5}{6}$; solving equations 1 and 3 for t yields $t = 0$.

(c) (5 points) Explain why these lines are not parallel.

Solution: Line L_1 is parallel to the vector $(2,4,6)$. Line L_2 is parallel to the vector $(-1,1,1)$. These vectors are not multiples of each other, so the lines are not parallel.

(d) (5 points) Find the general equations of two parallel planes, Π_1 and Π_2, containing lines L_1 and L_2, respectively.

Solution: The vector $(2,4,6)\times(-1,1,1)=(-2,-8,6)$ is normal to both lines, so it defines a plane parallel to both, given by equation

$$\Pi_1\colon\quad -2x-8y+6z=0.$$

Notice that this plane contains the line L_1. To find a parallel plane Π_2 containing the line L_2, we have to translate this plane. The amount of translation can be computed by inputing the coordinates of points in the line into the left-hand side of the equation of Π_1: since

$$-2(1)-8(4)+6(-1)=-40,$$

the equation of the plane Π_2 is

$$\Pi_2\colon\quad -2x-8y+6z=-40$$

3. (10 points) Compute the trajectory of a curve whose velocity vector is given by

$$r'(t) = e^{-t}i + t^2 j + \frac{1}{1+t^2}k,$$

and such that $r(0) = (1,1,1)$.

Solution: By the Fundamental Theorem of Calculus for Curves,

$$\begin{aligned} r(t) &= r(0) + \int_0^t r'(s)\ ds \\ &= i + j + k + \left(\int_0^t e^{-s} ds\right) i + \left(\int_0^t s^2 ds\right) j + \left(\int_0^t \frac{1}{1+s^2} ds\right) k \\ &= (2 - e^{-t})i + \left(1 + \frac{t^3}{3}\right) j + (1 + \arctan(t))k \end{aligned}$$

4. Consider the function

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) (5 points) What are the directed limits of the function along lines $y = kx$, as $x \to 0$?
Solution: If $y = kx$, then

$$f(x,kx) = \frac{kx^2(x^2 - k^2x^2)}{x^2 + k^2x^2} = x^2 \left(\frac{k(1-k^2)}{1+k^2} \right),$$

and the limit of this expression as $x \to 0$ is 0, regardless of the value of k. *[2 points for setting up the limit, 3 points for correct evaluation.]*

(b) (5 points) Is this function continuous at the origin?
Solution: Yes, it is. The term

$$\frac{x^2 - y^2}{x^2 + y^2}$$

is bounded above and below,

$$-1 \leq \frac{x^2 - y^2}{x^2 + y^2} \leq 1.$$

Thus the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

exists, and equals the limit

$$\lim_{(x,y)\to(0,0)} xy = 0,$$

the same as the value of the function at the origin.

(c) (5 points) What is the partial derivative of this function relative to x, at points other than the origin?

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{(3x^2 - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}.\end{aligned}$$

(d) (5 points) What is the partial derivative of this function relative to y, at points other than the origin?

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{(3x^3 - 3y^2x)(x^2+y^2) - (x^3y - xy^3)(2y)}{(x^2+y^2)^2} \\ &= \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}.\end{aligned}$$

5. Recall that the polar coordinate system in the plane is related to Cartesian coordinates by means of the equations

$$x = r\cos(\theta), \tag{1}$$
$$y = r\sin(\theta). \tag{2}$$

(a) (6 points) Differentiate equation (1) relative to x to find a relation between $\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$.
Solution: By the product rule,

$$1 = \frac{\partial r}{\partial x}\cos(\theta) - r\sin(\theta)\frac{\partial \theta}{\partial x}$$

(b) (6 points) Differentiate equation (2) relative to x to find a second relation between $\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$.
Solution:

$$0 = \frac{\partial r}{\partial x}\sin(\theta) + r\cos(\theta)\frac{\partial \theta}{\partial x}$$

(c) (8 points) By solving the system of equations obtained in the previous two steps, compute the derivatives $\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$.

Solution: This problem can be solved by elimination. Multiplying equation 1 by $\sin(\theta)$ and equation 2 by $-\cos(\theta)$ and adding the resulting equations, we obtain

$$\frac{\partial \theta}{\partial x} = -\frac{\sin(\theta)}{r}$$

Substituting this value within one of the equations of the system yields

$$\frac{\partial r}{\partial x} = \cos(\theta)$$