MAT 514 - Lecture 11

Integration

Recall that a parametrized plane curve is function

$$\gamma : [a,b] \longrightarrow \mathbb{C} \cong \mathbb{R}^2.$$

Its velocity at t is

$$\gamma'(t).$$

Its speed at t is

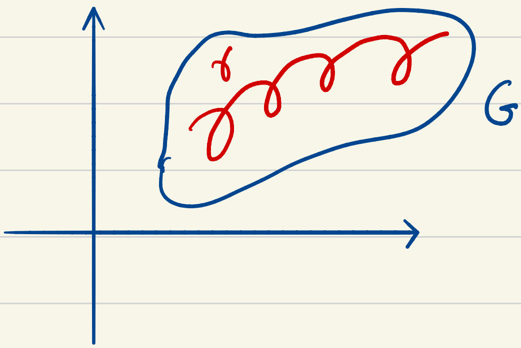
$$|\gamma'(t)|.$$

The length of a curve is

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt.$$

Our goal in this lecture is to learn how to integrate complex functions along plane curves.

Definition: Let $f: G \subset \mathbb{C} \to \mathbb{C}$ be a continuous complex function, and let $\gamma: [a,b] \to G$ be a smooth curve contained within G. Define



$$\int_\gamma f\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$

complex multiplication

Example 1: $f: \mathbb{C} \to \mathbb{C}$

$x+iy \mapsto x$ (real-valued function)

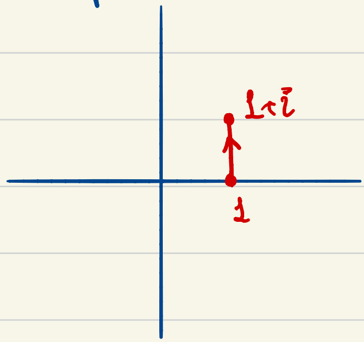
$\gamma: [0,1] \to \mathbb{C}$ $\qquad \gamma'(t) = 1 = (1,0)$

$t \mapsto t + 0i = (t,0)$.

$$\int_\gamma f\,dz = \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\,dt$$

$$= \int_0^1 t \cdot 1\,dt = \left.\frac{t^2}{2}\right|_{t=0}^{t=1} = \frac{1}{2}.$$

Example 2: $f(z) = z$, $\gamma(t) = 1 + ti$, $0 \le t \le 1$.



The velocity is

$$\gamma'(t) = i.$$

Thus

$$\int_\gamma f\,dz = \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\,dt.$$

$$\Rightarrow \int_\gamma f\,dz = \int_0^1 \gamma(t) \cdot \gamma'(t)\,dt.$$

$$= \int_0^1 (1 + ti) \cdot i\,dt.$$

$$= \int_0^1 (i + ti^2)\,dt$$

$$= \int_0^1 (i - t)\,dt.$$

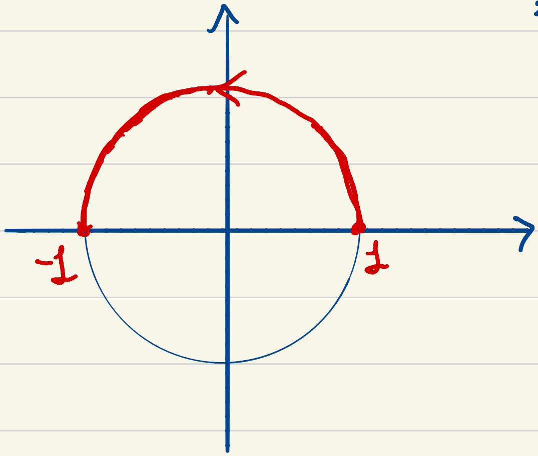
$$= \left(it - \frac{t^2}{2} \Big|_{t=0}^{t=1} \right)$$

$$\boxed{\int_\gamma f\,dz = \left(i - \frac{1}{2} \right)}$$

Example 3: $f(z) = iz$,

$\gamma(t) = e^{it}, \; 0 \leq t \leq \pi.$

$= \cos(t) + i\sin(t), \; 0 \leq t \leq \pi.$



The velocity is

$\gamma'(t) = (e^{it})'$

$= (e^{it}) \cdot (it)'$

$= i e^{it}.$

$$\int_\gamma f \, dz = \int_0^\pi i \, \gamma(t) \cdot \gamma'(t) \, dt.$$

$$= \int_0^\pi i e^{it} \cdot i e^{it} \, dt.$$

$$= -\int_0^\pi e^{2it} \, dt.$$

$$= -\left[\frac{1}{2i} \cdot e^{(2it)} \Big|_{t=0}^{t=\pi} \right]$$

$$= -\left[\frac{e^{2\pi i}}{2i} - \frac{e^0}{2i} \right] = 0.$$

<u>Example 4</u>: $f(z) = z^2$, $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$.

The velocity is

$$\gamma'(t) = i e^{it}.$$

The integral is

$$\int_\gamma f\,dz = \int_0^{2\pi} (e^{it})^2 \cdot (i e^{it})\,dt.$$

$$= \int_0^{2\pi} e^{2it} \cdot i e^{it}\,dt.$$

$$= i \int_0^{2\pi} e^{3it}\,dt.$$

$$= i \cdot \left[\frac{1}{3i} e^{3it} \Big|_{t=0}^{t=2\pi} \right].$$

$$= i \cdot \left[\frac{e^{6\pi i}}{3i} - \frac{e^0}{3i} \right]$$

$$= 0.$$

How does an integral change under a change in orientation? What about reparametrizations?

Example 5: Consider the curve from example 2,

$$\gamma(t) = 1 + it,$$

$0 \le t \le 1$. In opposite orientation, this curve is

$$\sigma(s) = (1+i) - is,$$

$0 \le s \le 1$.

Let's integrate $f(z) = z$ along σ.

$$\int_\sigma f\,dz = \int_0^1 f(\sigma(s)) \cdot \sigma'(s)\,ds$$

$$= \int_0^1 \sigma(s) \cdot \sigma'(s)\,ds$$

$$= \int_0^1 [(1+i) - is] \cdot (-i)\,ds.$$

$$= \int_0^1 [-i - i^2 + i^2 s]\,ds.$$

$$= \int_0^1 [-i + 1 - s]\,ds.$$

$$= -is + s - \frac{s^2}{2}\Big|_{s=0}^{s=1} = -i + 1 - \frac{1}{2} = \boxed{\frac{1}{2} - i}.$$

Property 1: Under a change in orientation the value of the integral changes by a sign.

Method to change orientations

If you are given

$$\gamma : [a,b] \to \mathbb{C}$$

then you can construct

$$\sigma : [0,1] \to \mathbb{C}$$

with reverse orientation by

$$\boxed{\sigma(s) = \gamma(b + (a-b)\cdot s)}$$

So: $\sigma(0) = \gamma(b + (a-b)\cdot 0) = \gamma(b)$.

$\sigma(1) = \gamma(b + (a-b)\cdot 1) = \gamma(\cancel{b} + a - \cancel{b})$

$= \gamma(a)$

Definition: Let $\gamma : [a,b] \to \mathbb{C}$ be a curve and $h : [c,d] \to [a,b]$ be a bijective function. Then

$$\xi : [c,d] \longrightarrow \mathbb{C}$$

given by

$$\xi(s) = \gamma(h(s))$$

is a reparametrization of γ

Example 6: Given any curve

$$\gamma : [a,b] \to \mathbb{C}$$

we can reparametrize it so its domain is $[0,1]$. This is done via

$$h : [0,1] \to [a,b]$$

$$h(s) = a + (b-a)s.$$

Remark: Some sources require preservation of orientation for reparametrizations.

Example 7: Curves whose speed is one are said to be parametrized by arc length. Consider the curve $\gamma(t) = e^{2\pi i t}$, $0 \le t \le 1$. This is not parametrized by arc length, for

$$
\begin{aligned}
|\gamma'(t)| &= |(e^{2\pi i t})'| \\
&= |2\pi i \cdot e^{2\pi i t}| \\
&= |2\pi i| \cdot |e^{2\pi i t}| \\
&= 2\pi
\end{aligned}
$$

The speed is constant, equal to 2π. To reparametrize it so that it has unit speed use

$$h(s) = \frac{s}{2\pi}, \quad 0 \leq s \leq 2\pi.$$

That is, $\xi : [0, 2\pi] \longrightarrow \mathbb{C}$

$$
\begin{aligned}
\xi(s) &= \gamma(h(s)) \\
&= \gamma(s/2\pi) \\
&= \exp\left(\cancel{2\pi} i \cdot \frac{s}{\cancel{2\pi}}\right) \\
&= e^{is}.
\end{aligned}
$$

Check: $|\xi'(s)| = |i e^{is}| = 1.$

Observation:

$$e^{ix} = \cos(x) + i\sin(x).$$

hence

$$|e^{ix}| = \sqrt{\cos^2(x) + \sin^2(x)}$$
$$= \sqrt{1}$$
$$= 1.$$

Example 8: Consider $f(z) = iz$, $\xi(s) = e^{i\pi s}$, $0 \le s \le 1$. (reparametrization of example 3, with $h(s) = \pi s$).

$$\int_{\xi} f\,dz = \int_0^1 (i \cdot \xi(s)) \cdot \xi'(s)\,ds$$
$$= \int_0^1 i \cdot e^{i\pi s} \cdot (i\pi e^{i\pi s}) \cdot ds$$
$$= -\pi \int_0^1 e^{2\pi i s}\,ds.$$
$$= -\pi \cdot \left[\frac{1}{2\pi i} e^{2\pi i s} \Big|_{s=0}^{s=1} \right] = -\pi \left[\frac{e^{2\pi i}}{2\pi i} - \frac{e^{0}}{2\pi i} \right] = 0$$

<u>Property 2</u>: Under an <u>orientation-preserving</u> reparametrization, the value of an integral doesn't change.