MAT 514 - Lecture 17

- Consequences of Cauchy's Theorem

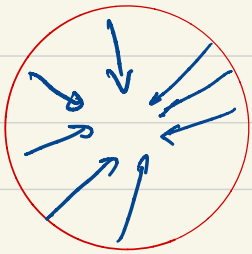
Recall <u>Cauchy-Goursat Theorem</u>:

If $f: G \subset \mathbb{C} \longrightarrow \mathbb{C}$ is a holomorphic function and γ is a G-contractible path, then

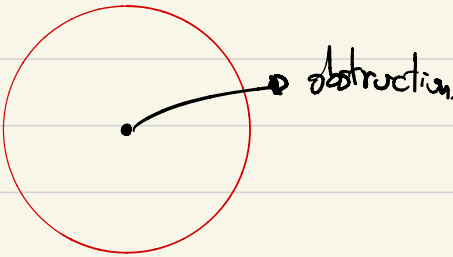
$$\int_\gamma f\,dz = 0.$$

<u>Remark</u>:

The unit circle $\gamma(t) = e^{2\pi t}$ is $\mathbb{C}$-contractible, but not $(\mathbb{C} - \{0\})$-contractible



A $\mathbb{C}$-contraction



No contraction possible

Cauchy Integral Formula:

Let $C[\omega, r]$ be a circle, centered at ω, with radius r, parametrized in counterclockwise sense. Explicitly,

$$C[\omega, r]: \ \gamma(t) = \omega + re^{it}, \quad 0 \le t \le 2\pi.$$

Suppose that f is a holomorphic function in a domain G, and $C[\omega, r] \subset G$. Then

$$f(\omega) = \frac{1}{2\pi i} \int_{C[\omega, r]} \frac{f(z)}{z - \omega} \, dz.$$

Cauchy's formulas for derivatives:

Under the same assumptions as above,

$$f^{(n)}(\omega) = \frac{n!}{2\pi i} \int_{C[\omega, r]} \frac{f(z)}{(z-\omega)^{n+1}} \, dz.$$

Theorem: Suppose $f : G \subset \mathbb{C} \to \mathbb{C}$ is holomorphic. Then all complex derivatives

of f exist and are themselves holomorphic

Let's draw a parallel with the real setting.

<u>Example</u>: Consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0. \\ 0, & \text{if } x = 0. \end{cases}$$

<u>Claim 1</u>: f is continuous.

Continuity on $\mathbb{R} \setminus \{0\}$ is an easy consequence of continuity of $1/x$. It remains to test for continuity at 0: the limit

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

is bounded by

$$\lim_{x \to 0} -x^2 = 0$$

and

$$\lim_{x \to 0} x^2 = 0.$$

Therefore,

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0,$$

hence f is continuous.

<u>Claim 2:</u> f is differentiable

Away from 0, f is a product of differentiable function, hence differentiable with derivative

$$f'(x) = \left(x^2 \cdot \sin\left(\frac{1}{x}\right)\right)'$$

$$= (2x) \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \sin'\left(\frac{1}{x}\right)$$

$$= (2x)\sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{x^2}\right)$$

$$= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

The problem is at $x = 0$. Let compute the Newton quotients at 0:

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(1/h) - 0}{h}$$

$$= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$$

$$= 0,$$

by the same boundedness argument as before.

<u>Claim 3</u>: f' is not continuous at 0.

This amounts to say

$$\lim_{x\to 0} f'(x) \neq f'(0) = 0.$$

Note that

$$\lim_{x\to 0} 2x \sin\left(\frac{1}{x}\right) = 0,$$

by a boundedness argument. However,

$$\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$$

does not exist, therefore $\lim_{x\to 0} f'(x)$ does not

exist.

In summary: f as defined above is a real-differentiable function whose derivative is not only continuous, let alone differentiable.

<u>Remark:</u> The theorem about existence of higher-order complex derivatives requires holomorphicity, not only differentiability at a point.

Taking Cauchy's formulas to the limit

We wish to compute the improper integral

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx.$$

From Calculus 2, we know what to do:

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx = 2\int_{0}^{+\infty} \frac{1}{x^2+1}\,dx.$$

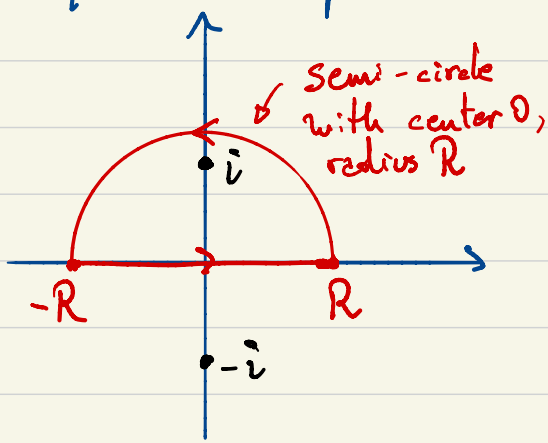
$$= 2\lim_{t\to+\infty}\int_{0}^{t} \frac{1}{x^2+1}\,dx$$

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx = 2 \lim_{t\to+\infty} \left[\arctan(x) \Big|_0^t \right]$$

$$= 2 \cdot \lim_{t\to+\infty} \arctan(t).$$

$$= \pi.$$

We will approach this from the complex-analytic view point.



Let σ_R be the path composed of:

- the real line segment from $-R$ to R
- the upper semi-circle with center 0 and radius R.

We will compute

$$\int_{\sigma_R} \frac{1}{z^2+1}\,dz.$$

The integrand has two singularities, $\pm i$, one of which $(-i)$ is always outside the domain of integration. Relative to R the placement of the other singularity is:

1) Outside the domain of integration, if $R < 1$. In this case, by Cauchy's theorem

$$\int_{\sigma_R} \frac{1}{z^2+1} = 0.$$

2) On the path of integration, if $R=1$. We haven't learned how to deal with such integrals.

3) Inside the domain of integration, if $R>1$. This case can be dealt with by Cauchy's integral formula:

$$\int_{\sigma_R} \frac{1}{z^2+1}\,dz = \int_{\sigma_R} \frac{1}{(z-i)(z+i)}\,dz$$

$$\int_{\sigma_R} \frac{1}{z^2+1}\, dz = \int_{\sigma_R} \frac{\left[1/(z+i)\right]}{z-i}\, dz.$$

The function

$$f(z) = \frac{1}{z+i}$$

is holomorphic on the upper semicircle, so

$$\int_{\sigma_R} \frac{(1/z+i)}{z-i}\, dz = 2\pi i \cdot \left(\frac{1}{i+i}\right)$$

$$\int_{\sigma_R} \frac{1}{z^2+1}\, dz = \pi,$$

for all $R \geq 1$.

The integral is divided between two components: the arc of circle σ_R, and line segment l_R.

$$\int_{\sigma_R} \frac{1}{z^2+1}\,dz = \int_{\gamma_R} \frac{1}{z^2+1}\,dz + \int_{l_R} \frac{1}{z^2+1}\,dz.$$

The integral along l_R is what we care about: along the real axis $z = x + 0i$, and

$$\int_{l_R} \frac{1}{z^2+1}\,dz = \int_{-R}^{R} \frac{1}{x^2+1}\,dx.$$

We wish to take $\int_{\gamma_R} \frac{1}{z^2+1}\,dz$ and $\int_{l_R} \frac{1}{z^2+1}\,dz$ to the limit as $R \to \infty$.

$$\lim_{R\to\infty} \int_{l_R} \frac{1}{z^2+1}\,dz = \int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx.$$

Whereas

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z^2+1}\,dz$$

is unknown.

Note that

$$\int_{\gamma_R} \frac{1}{z^2+1}\,dz \le \int_{\gamma_R} \max_{\gamma_R} \left|\frac{1}{z^2+1}\right| dz.$$

$$\le \left(\max_{\gamma_R} \left|\frac{1}{z^2+1}\right|\right) \cdot \int_{\gamma_R} 1\,dz$$

$$\le \left(\max_{\gamma_R} \left|\frac{1}{z^2+1}\right|\right) \cdot \pi R$$

$$\le \left(\max_{\gamma_R} \frac{1}{|z|^2-1}\right) \cdot \pi R.$$

$$\le \frac{\pi R}{R^2-1},$$

so as $R \to +\infty$

$$\frac{\pi R}{R^2-1} \to 0$$

and

$$\lim_{R\to+\infty} \int_{\gamma_R} \frac{1}{z^2+1}\,dz = 0.$$

It follows that:

$$\lim_{R\to\infty} \int_{\sigma_R} \frac{1}{z^2+1}\,dz = \lim_{R\to\infty} \pi \Rightarrow$$

$$\left(\lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z^2+1}\,dz\right) + \left(\lim_{R\to\infty} \int_{\ell_R} \frac{1}{z^2+1}\,dz\right) = \pi$$

$$0 + \boxed{\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx = \pi.}$$

Remark: This is not the optimal way of computing this integral. This method is best used when antiderivatives are not known.

Other consequences that arise using this method are:

Fundamental Theorem of Algebra

If $p: \mathbb{C} \to \mathbb{C}$ is a polynomial,

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

and p is not constant, then p has a root, that is there exists a number r such that $p(r) = 0$.

Liouville's Theorem

If $f: \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, then f is constant.

Remark: functions which are holomorphic on all of $\mathbb{C}$ are called entire.

Both theorems are proved in our textbook, chapter 5. Their proofs will be left to an asynchronous lecture.

Examples and applications

(1) Consider a holomorphic function

$$f: \mathbb{C} \to \mathbb{C}$$

satisfying the bounds

$$|f(z)| < a|z| + b,$$

for some real constants a and b, $a \geq 0$, $b \geq 0$.

By Cauchy's formula for derivatives:

$$f'(\omega) = \frac{1}{2\pi i}\int_{C[\omega,r]} \frac{f(z)}{(z-\omega)^2}\,dz.$$

$$|f'(\omega)| = \left|\frac{1}{2\pi i}\int_{C[\omega,r]} \frac{f(z)}{(z-\omega)^2}\,dz\right|$$

$$= \frac{1}{2\pi}\cdot\left|\int_{C[\omega,r]} \frac{f(z)}{(z-\omega)^2}\,dz\right|$$

$$\leq \frac{1}{2\pi}\cdot\int_{C[\omega,r]} \left|\frac{f(z)}{(z-\omega)^2}\right|dz$$

$$\leq \frac{1}{2\pi}\cdot\int_{C[\omega,r]} \frac{(a|z|+b)}{|(z-\omega)|^2}\,dz.$$

$$\leq \frac{C}{2\pi},$$

for some positive constant C. It follows that f' is an entire bounded function, thus, according to Liouville's Theorem,

$$f'(z) = A,$$

for some complex constant A. By

integration,

$$f(z) = Az + B,$$

for some complex constant B. In summary, f is a polynomial.