# MAT 514 - Lecture 14

## Harmonic Functions

A harmonic function is a solution to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, \qquad \text{(Laplace's equation)}$$

where $z = x + iy$.

**Examples:**

1. Any constant function is harmonic.
2. The functions
$$Re(x+iy) = x$$
$$Im(x+iy) = y$$
are harmonic
3. The function
$$f : \mathbb{C} \to \mathbb{R}$$
defined by $f(z) = x^2 - y^2$ is harmonic.

This is because
$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = -2$$

hence

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

(4) Another example is

$$f: \mathbb{C} \to \mathbb{R}$$
$$f(x+iy) = x \cdot y.$$

This function satisfies

$$\frac{\partial f}{\partial x} = y \;, \quad \frac{\partial f}{\partial y} = x,$$

so

$$\frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 0;$$

hence

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

(5) The function $f: \mathbb{C} \to \mathbb{R}$ given by

$$f(x+iy) = e^x \cdot \cos(y)$$

Then

$$\frac{\partial f}{\partial x} = e^x \cdot \cos(y), \quad \frac{\partial f}{\partial y} = -e^x \sin(y);$$

so

$$\frac{\partial^2 f}{\partial x^2} = e^x \cdot \cos(y), \quad \frac{\partial^2 f}{\partial y^2} = -e^x \cos(y),$$

and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

**Proposition:** Let $h: G \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function, written as

$$h(z) = u(z) + i\,v(z),$$

where $u$ and $v$ are real-valued (the real and imaginary parts of $h$). Then both $u$ and $v$ are harmonic.

**Proof:** The key to the argument is the $h$ satisfies the Cauchy-Riemann equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}. \end{cases}$$

It follows that

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right). \\ &= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} \\ &= 0, \end{aligned}$$

by Schwarz Theorem. One proves that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

by a similar argument.

∎

Revisiting examples 2 to 5:

(2) The functions studied were $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, which are the real and imaginary components of

$$h : \mathbb{C} \to \mathbb{C}$$

$$h(z) = z,$$

a holomorphic function.

(3) The function from example 3,

$$f(z) = x^2 - y^2$$

is the real part of $h(z) = z^2$, for

$$\begin{aligned} z^2 &= (x+iy)^2 \\ &= (x+iy)(x+iy) \\ &= x^2 + ixy + ixy + i^2y^2 \\ &= (x^2 - y^2) + i(2xy). \end{aligned}$$

(4) In example 4 the function was

$$f(z) = xy$$

which is the imaginary part of

$$h(z) = \frac{1}{2} z^2.$$

⑤ The function $f(z) = e^x \cos(y)$ is the real part of

$$\begin{aligned} h(z) &= e^z \\ &= e^{(x+iy)} \\ &= e^x \cdot e^{iy} \\ &= e^x \cdot (\cos(y) + i\sin(y)). \\ &= e^x \cos(y) + i\,(e^x \sin(y)). \end{aligned}$$

<u>Theorem</u>: Let $u: G \subset \mathbb{C} \to \mathbb{R}$ be a harmonic function defined on a star-shaped domain $G$. Then there exists a harmonic conjugate, that is, a harmonic function

$$v: G \subset \mathbb{C} \to \mathbb{R}$$

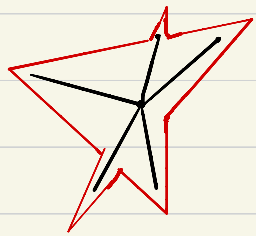
so that

$$h(z) = u(z) + i v(z)$$

is holomorphic.

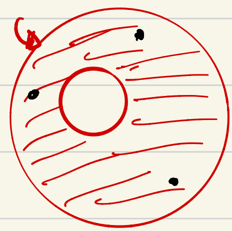
**Definition:** We call a domain $G$ star-shaped if there exists a point $p \in G$ such that any other point $q \in G$ can be joined to $p$ by a line segment in $G$.

star-shaped, in fact convex.



star-shaped, but not convex.

$G$ is the annulus




not star-shaped.

**Examples:**

① The functions

$$\mathrm{Re}(z) = x$$
$$\mathrm{Im}(z) = y$$

are harmonic conjugates, after all

$$h(z) = z = x + iy$$

is holomorphic.

(2) The functions

$$u(z) = x^2 - y^2,$$
$$v(z) = 2xy$$

are harmonic conjugates, for

$$(x^2 - y^2) + i(2xy) = z^2,$$

a holomorphic function.

(3) The functions

$$u(z) = e^x \cos(y),$$
$$v(z) = e^x \sin(y)$$

are also conjugates, for

$$u(z) + iv(z) = e^x \cos(y) + ie^x \sin(y)$$
$$= e^x [\cos(y) + i\sin(y)],$$
$$= e^x \cdot e^{iy}$$
$$= e^{x+iy} = e^z.$$

a holomorphic function.

Let's investigate how to generate harmonic conjugates. Our guiding example is

$$u(z) = x^2 - y^2.$$

We will impose the Cauchy-Riemann equations on

$$u(z) + i v(z):$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} 2x = \dfrac{\partial v}{\partial y} \\ -2y = -\dfrac{\partial v}{\partial x} \end{cases}$$

That is

$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x$$

The mixed partials $\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$ agree, hence we have a chance of solving this system.

Integrating $\frac{\partial v}{\partial x} = 2y$, we find

$$v = 2xy + g(y),$$

for some function $g$ of $y$. By the second equation we find

$$\frac{\partial}{\partial y}\left(2xy + g(y)\right) = 2x$$

$$2x + \frac{\partial g}{\partial y}(y) = 2x$$

$$\frac{\partial g}{\partial y} = 0.$$

Solutions are (real-valued) constants. In particular, we may choose $g = 0$.

So we can turn this naive idea of antiderivatives into proper line integrals. The prescription is to assign an arbitrary value the "star" in a star-shaped set and integrate from this point.

The key point in the proof is Cauchy's theorem:


$$\int_\sigma f\,dz = \int_\xi f\,dz$$

, so long as $f$ is holomorphic in a region containing the region bounded by the curves.

This allows us to extend the proof of Theorem 6.8 in our textbook to star-shaped domains.

**Theorem.** (Mean-value formula for harmonic functions)

If $u$ is harmonic in a region $G$ and $\overline{D}[w, r] \subset G$, then

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{it})\,dt.$$

Proof: On the disk $\overline{D[\omega, r]}$, which is star-shaped, $u$ has a conjugate $v$ so that

$$f(z) = u + iv$$

is holomorphic.

We can apply the Mean-Value Formula for holomorphic functions to $f$:

$$f(\omega) = \frac{1}{2\pi i} \int_{C[\omega, r]} \frac{f(z)}{z-\omega}\, dz.$$

Parametrize the circle $C[\omega, r]$ as

$$\gamma(t) = \omega + re^{it},$$

$0 \leq t \leq 2\pi$. Then

$$\gamma'(t) = rie^{it},$$

while

$$\frac{f(\gamma(t))}{\gamma(t) - \omega} = \frac{f(\omega + re^{it})}{\omega + re^{it} - \omega}$$

$$= \frac{f(\omega + re^{it})}{re^{it}}.$$

Therefore

$$f(w) = \frac{1}{\cancel{2\pi i}} \int_0^{2\pi} \frac{f(w+re^{it})}{re^{it}} \cdot \cancel{i} re^{it} \, dt.$$

$$\boxed{f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{it}) \, dt.}$$

$\blacksquare$

## Consequences

(1) A harmonic function is smooth (infinitely differentiable). As a matter of fact, it is real-analytic.

(2) Maximum principle

Suppose $u : G \subset \mathbb{C} \to \mathbb{R}$ is a harmonic function, $D[w, r] \subset G$. Then $u$ has no strong maximum within $D$, that is given any point $x \in D[w, r]$, there exists another $y \in D[w, r]$ for which

$$u(x) \le u(y).$$

Alternatively: if $u$ is harmonic and has a strong maximum within its domain, then it is constant.

Exercises

(1) Consider the function
$$u(z) = \log(x^2+y^2),$$
for $z \neq 0$.

This function is harmonic. We verify this by direct computation:
$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}.$$
$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$$
So
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) = \frac{2\cdot(x^2+y^2) - 2x\cdot(2x)}{(x^2+y^2)^2}$$
$$= \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}.$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right) = \frac{2\cdot(x^2+y^2)-2y(2y)}{(x^2+y^2)^2}$$

$$= \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

or $u$ is harmonic.

Question: Does $u$ have a harmonic conjugate on $\mathbb{C}\setminus\{0\}$?

This is an interesting question because the domain is not star-shaped.

Answer: No! We are going to prove this by contradiction. Suppose there exists $v: \mathbb{C}\setminus\{0\} \to \mathbb{R}$ harmonic so that $u+iv$ is holomorphic.

Note that

$$\log(x^2+y^2)$$

is the real part of $\mathrm{Log}(z)$. If $f$ exists, then on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, it differs from $\mathrm{Log}(z)$ by a purely imaginary constant. The difference, being holomorphic, cannot be purely imaginary, unless it is $0$. Thus,

$$f = \mathrm{Log}$$

on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. It follows that $f$ cannot be extended holomorphically to $\mathbb{C} \setminus \{0\}$.