MAT 514 - Lecture 9

- Antiderivatives

Definition: We say that $F : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ is an antiderivative of $f : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ if

$$F'(z) = f(z).$$

Examples:

(1) If $f(z) = 0$, then for any complex constant c, $F(z) = c$ is an antiderivative of f.

(2) If $c \in \mathbb{C}$ and $f(z) = c$, then

$$F(z) = cz + d,$$

where d is another constant, is an antiderivative of f.

(3) More generally, if $f(z) = z^n$, then $F(z) = \frac{z^{n+1}}{n+1} + c$, where c is a constant,

$n \neq -1$, is an antiderivative.

connected open set.

Theorem: Let $f : \mathcal{U} \subset \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function. Any two antiderivatives, if they exist, differ by a constant.

Proof: Suppose F_1, F_2 are antiderivatives. Then

$$F_1' = f, \quad F_2' = f$$

so

$$(F_1 - F_2)' = f - f = 0$$

so $F_1 - F_2 = c$, for some constant c. ∎

Define the complex exponential

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}$$

as a function whose derivative is itself, satisfying

$$\exp(0) = 1$$

In practice we write $\exp(z) = e^z$.

Properties:

1. $e^z \cdot e^w = e^{z+w}$
2. $\frac{e^z}{e^w} = e^{z-w}$.
3. $(e^z)^w = e^{z \cdot w}$ *
4. Euler's Formula
$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$
where θ is a real number.

*: We haven't defined exponentiation with complex exponent yet. The function
$$w \longmapsto a^w,$$
can be defined as a function whose derivative is proportional to itself with value 1 at 0. This is only defined away from
$$\{a \in \mathbb{C} \mid a = x + 0i, x \leq 0\}.$$

Some relations

(a) $(e^{2z})' = (e^{2z}) \cdot 2$

Chain rule

(b) $(z \cdot e^z)' = (z') \cdot e^z + z \cdot (e^z)'$

$= 1 \cdot e^z + z \cdot e^z$

$= (1+z) \cdot e^z.$

Exercise 1: Compute the derivative of the exponential function via the Newton quotient definition,

$$(e^z)' = \lim_{h \to 0} \frac{e^{z+h} - e^z}{h}$$

Solution:

$$\lim_{h \to 0} \frac{e^{z+h} - e^z}{h} = \lim_{h \to 0} \frac{e^z \cdot e^h - e^z}{h}$$

$$= \lim_{h \to 0} e^z \cdot \left(\frac{e^h - 1}{h} \right)$$

$$= \lim_{h \to 0} \frac{e^z \cdot e^h}{1} = e^z.$$

Complex Trigonometric Functions

Recall Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \quad (a)$$

where θ is a real number.

Let's find the values of $\cos(\theta)$, $\sin(\theta)$ in terms of complex exponentials.

Observe that

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta) \quad (b)$$

Adding (a), (b) we find:

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \Rightarrow$$

$$\boxed{\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}},$$

where θ is a real number.

Subtracting (a) and (b) we find:

$$e^{i\theta} - e^{-i\theta} = (2i)\sin(\theta).$$

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

where θ is a real numbers.

<u>Definition:</u> If z is a complex number, we set

$$\boxed{\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}}$$

$$\boxed{\cos(z) = \frac{e^{iz} + e^{-iz}}{2}}$$

Exercise 2: Find the values of
$\sin(i)$, $\cos(i)$.

Solution:

$$\sin(i) = \frac{e^{i \cdot i} - e^{-i \cdot i}}{2i}$$

$$= \frac{e^{-1} - e^{1}}{2i}$$

$$= \frac{e^{-1} - e}{2i}$$

$$\cos(i) = \frac{e^{i \cdot i} + e^{-i \cdot i}}{2}$$

$$= \frac{e^{-1} + e}{2}.$$

Exercise 3: Use the relation

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

and properties of exponential to find $\cos(z)'$.

Solution:

$$\cos(z)' = \left(\frac{e^{iz}+e^{-iz}}{2}\right)'$$

$$= \frac{\left(e^{iz}+e^{-iz}\right)'}{2}$$

$$= \frac{\left(e^{iz}\right)'+\left(e^{-iz}\right)'}{2}$$

$$= \frac{e^{iz}\cdot i + \left(e^{-iz}\right)\cdot(-i)}{2}$$

$$= i\cdot\left(\frac{e^{iz}-e^{-iz}}{2}\right)$$

$$= i^2\cdot\left(\frac{e^{iz}-e^{-iz}}{2i}\right)$$

$$= -\sin(z).$$

$\blacksquare$

Complex tangents, secants, cosecants, cotangents can be defined as usual:

$\tan(z) = \frac{\sin(z)}{\cos(z)}$, $\sec(z) = \frac{1}{\cos(z)}$, $\csc(z) = \frac{1}{\sin(z)}$, $\cot(z) = \frac{1}{\tan(z)}$

wherever the denominators are not 0.

Complex Hyperbolic functions

We define

$$\boxed{\sinh(z) = \frac{e^{z} - e^{-z}}{2}}$$

$$\boxed{\cosh(z) = \frac{e^{z} + e^{-z}}{2}.}$$

Note that:

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i \cdot \sin(z).$$

$$\cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos(z).$$

Derivative of $\sinh(z)$:

$$\sinh(z)' = \left(\frac{e^{z} - e^{-z}}{2}\right)'$$

$$= \frac{(e^{z} - e^{-z})'}{2}$$

$$\sinh'(z) = \frac{e^z - (-1)\cdot e^{-z}}{2}$$

$$\sinh'(z) = \frac{e^z + e^{-z}}{2}.$$

$$\boxed{\sinh'(z) = \cosh(z).}$$

Exercise 4: Find the derivative of $\cosh(z)$.

Solution: $$\cosh'(z) = \left(\frac{e^z + e^{-z}}{2}\right)'$$

$$= \frac{(e^z + e^{-z})'}{2}$$

$$= \frac{e^z - e^{-z}}{2}$$

$$\boxed{\cosh'(z) = \sinh(z).}$$