MAT 203 - Lecture 12

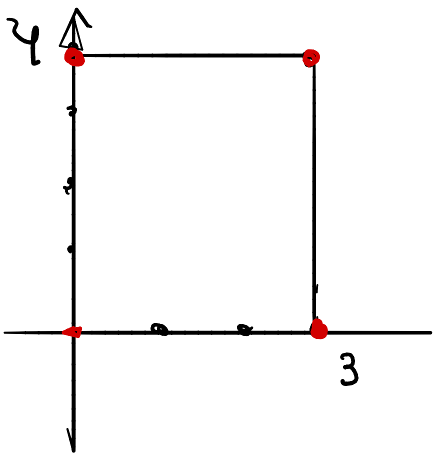
- Areas and volumes

Double integrals by definition compute signed volumes under graphs.
They can also be used to compute areas:

$$\text{Area}(R) = \iint_R 1 \, dA.$$

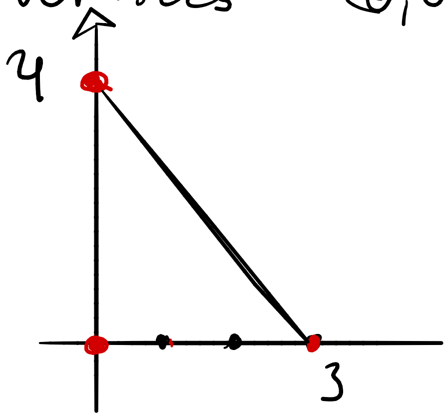
<u>Examples:</u>

1) Let R be the rectangle with vertices $(0,0)$, $(0,4)$, $(3,0)$, $(3,4)$.



$$\iint_R 1 \, dA = \int_0^3 \int_0^4 1 \, dy \, dx$$
$$= \int_0^3 4 \, dx$$
$$= 12.$$

2) R is the triangle with vertices (0,0), (3,0), (0,4).



$$\iint_R 1\, dA = \int_0^3 \int_0^{-\frac{4}{3}x+4} 1 \; dy\, dx$$

Equation of line:

$$\frac{4-0}{0-3} = \frac{y-0}{x-3}$$

$$-\frac{4}{3} = \frac{y}{(x-3)}$$

$$-\frac{4}{3}(x-3) = y$$

$$y = -\frac{4}{3}x + 4.$$

$$\int_0^3 \int_0^{-\frac{4}{3}x+4} 1 \, dy \, dx = \int_0^3 \left[y \Big|_{y=0}^{y=-\frac{4x}{3}+4} \right] dx$$

$$= \int_0^3 \left[-\frac{4}{3}x + 4 \right] dx$$

$$= -\frac{2x^2}{3} + 4x \Big|_{x=0}^{x=3}$$

$$= -\frac{2 \cdot 3^2}{3} + 4 \cdot 3 - \cancel{\left(\frac{-2 \cdot 0^2}{3} + 4 \cdot 0 \right)}^{\,0}$$

$$= -6 + 12$$

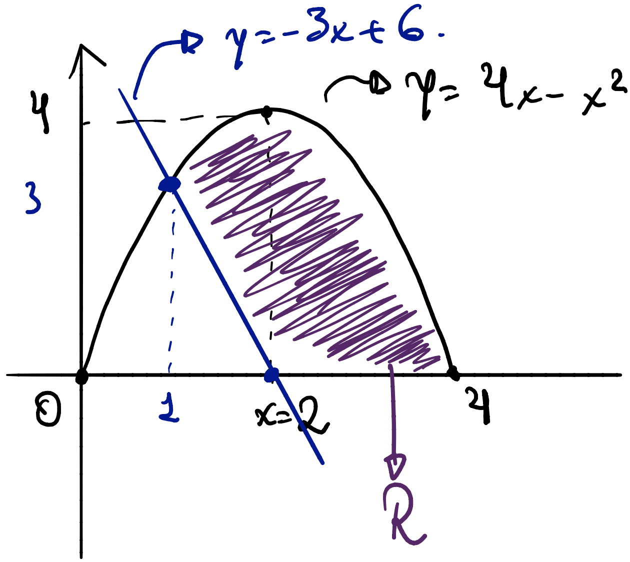
$$= 6.$$

<u>Example 3</u>: R is the region lying below the parabola

$$y = 4x - x^2$$

above the x-axis, and above the line

$$y = -3x + 6.$$



$$\text{Area} = \iint_R 1 \, dA.$$

$$= \int_1^2 \int_{-3x+6}^{4x-x^2} 1 \, dy \, dx + \int_2^4 \int_0^{4x-x^2} 1 \, dy \, dx$$

$$= \int_1^2 \left(y \Big|_{y=-3x+6}^{y=4x-x^2} \right) dx + \int_2^4 \left(y \Big|_{y=0}^{4x-x^2} \right) dx$$

$$\text{Area} = \int_1^2 \left(4x - x^2 + 3x - 6 \right) dx$$
$$+ \int_2^4 \left(4x - x^2 - 0 \right) dx$$
$$= \int_1^2 \left(-x^2 + 7x - 6 \right) dx$$
$$+ \int_2^4 \left(-x^2 + 4x \right) dx$$
$$= \left(-\frac{x^3}{3} + \frac{7x^3}{2} - 6x \Big|_{x=1}^{x=2} \right)$$
$$+ \left(-\frac{x^3}{3} + 2x^2 \Big|_{x=2}^{x=4} \right)$$
$$= \left(-\frac{8}{3} + 14 - 12 + \frac{1}{3} - \frac{7}{2} + 6 \right)$$
$$+ \left(-\frac{64}{3} + 32 + \frac{8}{3} - 8 \right)$$
$$= \frac{15}{2}.$$

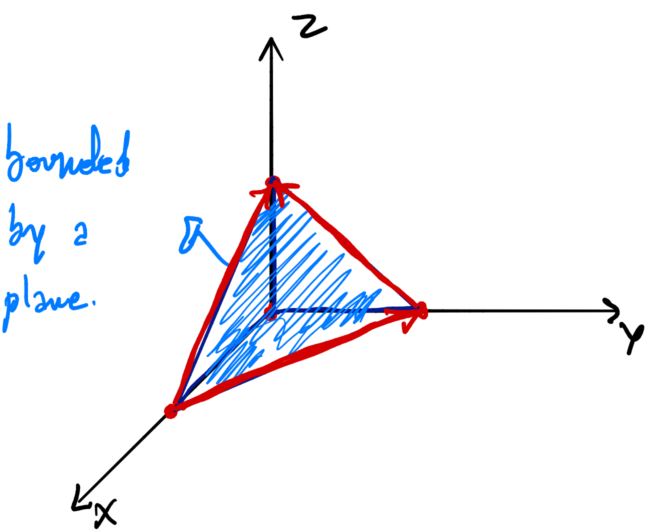
Volumes

To find the volume of a region we use the "formula"

$$\text{Volume} = \iint (\text{height})\, dA.$$

Examples

4) Find the volume of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.



Height: z-coordinate along light-blue plane

Finding equation of plane:

- directed line segments between

$(1,0,0)$ and $(0,1,0)$;
$(1,0,0)$ and $(0,0,1)$;
$(0,1,0)$ and $(0,0,1)$;
are parallel to plane.
Corresponding vectors:

$$u = (1,0,0) - (0,1,0)$$
$$= (1,-1,0).$$
$$v = (1,0,0) - (0,0,1)$$
$$= (1,0,-1)$$
$$w = (0,1,0) - (0,0,1)$$
$$= (0,1,-1)$$

We will choose u, v are direction vectors.
Cross product is normal to plane:

$$n = u \times v$$
$$= (i - j) \times (i - k)$$
$$= i \times i - i \times k - j \times i + j \times k.$$
$$= 0 + j + k + i = (1,1,1).$$

Equation of plane is:

$$1 \cdot x + 1 \cdot y + 1 \cdot z = D$$

To find the constant, substitute coordinates of a point in the plane, say $(1,0,0)$:

$$1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 = D \Rightarrow D = 1.$$

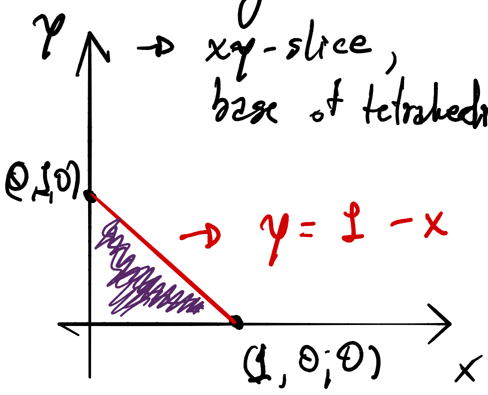
Equation of plane:

$$\boxed{x + y + z = 1.}$$

$$z = 1 - x - y.$$ → function we will integrate.

Next we need to describe the region of integration.

→ xy-slice, base of tetrahedron.



$$\text{Volume} = \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx.$$

$$\text{Volume} = \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

$$= \int_0^1 \left(y - xy - \frac{y^2}{2} \Big|_{y=0}^{y=1-x} \right) dx$$

$$= \int_0^1 \left((1-x) - x(1-x) - \frac{(1-x)^2}{2} - 0 \right) dx$$

$$= \int_0^1 \left(1 - x - x + x^2 - \frac{(1-2x+x^2)}{2} \right) dx$$

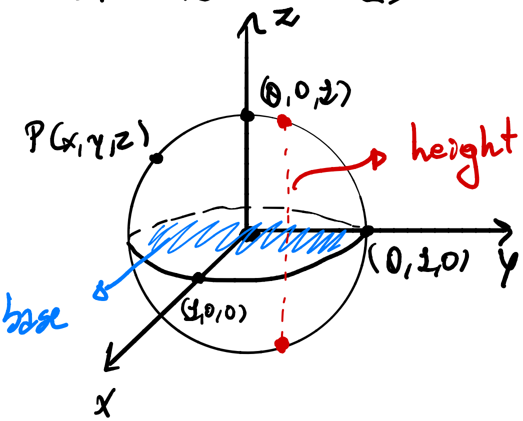
$$= \int_0^1 \left(\frac{x^2}{2} - x + \frac{1}{2} \right) dx$$

$$= \frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2} \Big|_{x=0}^{x=1}.$$

$$= \frac{1^3}{6} - \frac{1^2}{2} + \frac{1}{2}$$

$$= \frac{1}{6}.$$

Example 5: Find the volume of a sphere of radius 1.



Choose base as the disk along equatorial plane (xy-plane).
Height: difference between z-value of northern and southern hemispheres.

Characterization of points along the sphere: all are situated at distance 1 from the origin.

$$d((x,y,z),(0,0,0)) = 1$$

$$\sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = 1.$$

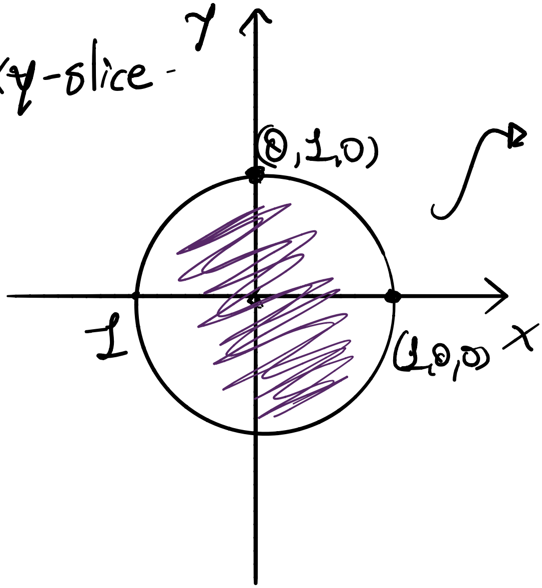
$$\sqrt{x^2+y^2+z^2} = 1.$$

$$\boxed{x^2+y^2+z^2=1.}$$ eq. for sphere.

$$z^2 = 1-x^2-y^2 \quad \begin{cases} z_+ = \sqrt{1-x^2-y^2} & \text{north} \\ z_- = -\sqrt{1-x^2-y^2} & \text{south} \end{cases}$$

$$\text{height} = z_+ - z_-$$
$$= \sqrt{1-x^2-y^2} - \left(-\sqrt{1-x^2-y^2}\right)$$
$$= 2\sqrt{1-x^2-y^2}.$$

xy-slice.



equation of circle:

$$x^2+y^2=1.$$
$$y_+ = \sqrt{1-x^2}$$
$$y_- = -\sqrt{1-x^2}$$

$$\text{volume} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\,dy\,dx.$$

Anti-derivative of

$$2\sqrt{(1-x^2)-y^2}$$

relative to y.

Recall: in the inner integral we think of x as a constant.

Substitute:

$$y = \sqrt{(1-x^2)} \cdot \sin\theta$$
$$dy = \sqrt{1-x^2} \cdot \cos\theta \, d\theta.$$

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy = \int_{-\pi/2}^{\pi/2} \sqrt{(1-x^2)-(1-x^2)\sin^2\theta} \cdot \sqrt{1-x^2}\cos\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{(1-x^2)(1-\sin^2\theta)} \cdot \sqrt{1-x^2}\cos\theta\, d\theta.$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{(1-x^2)\cos^2\theta} \cdot \sqrt{1-x^2}\cos\theta\, d\theta.$$

$$= \int_{-\pi/2}^{\pi/2} (1-x^2) \cdot \cos^2\theta\, d\theta.$$

$$= (1-x^2) \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta.$$

$$= (1-x^2) \cdot \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos(2\theta)}{2}\right) d\theta.$$

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy = (1-x^2)\int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos(2\theta)}{2}\right) d\theta$$

$$= (1-x^2)\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\Bigg|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\right)$$

$$= (1-x^2)\cdot\left(\frac{\pi}{4} + \cancel{\frac{\sin(\pi)}{4}} + \frac{\pi}{4} + \cancel{\frac{\sin(-\pi)}{4}}\right)$$

$$= \frac{(1-x^2)\pi}{2}.$$

Recall :

$$\text{Volume} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\, dy\, dx$$

$$= \int_{-1}^{1} (1-x^2)\pi \quad dx.$$

$$= \pi\left(x - \frac{x^3}{3}\Bigg|_{x=-1}^{x=1}\right)$$

$$= \pi\cdot\left(1-\frac{1}{3} - \left(-1+\frac{1}{3}\right)\right).$$

$$= \pi\cdot\left(1-\tfrac{1}{3}+1-\tfrac{1}{3}\right) = \frac{4\pi}{3}.$$

Exercise 1: Find the volume of the solid bounded by

$$z = \frac{xy}{2}$$

over the rectangle with vertices $(0,0), (4,0), (4,3), (0,3)$.

Solution:

$$\text{Volume} = \int_0^4 \int_0^3 \frac{xy}{2}\, dy\, dx.$$

$$= \int_0^4 \left(\frac{xy^2}{4} \Big|_{y=0}^{y=3} \right) dx$$

$$= \int_0^4 \left(\frac{9x}{4} - \cancel{\frac{0^2 x}{4}} \right) dx$$

$$= \int_0^4 \frac{9x}{4}\, dx$$

$$= \frac{9x^2}{8} \Big|_{x=0}^{x=4} = \frac{9 \cdot 16}{8} - \frac{9 \cdot 0}{8} = 18.$$

Exercise 2: Find the volume of the solid bounded by:

i) the xy-plane $\rightarrow \boxed{z=0}$

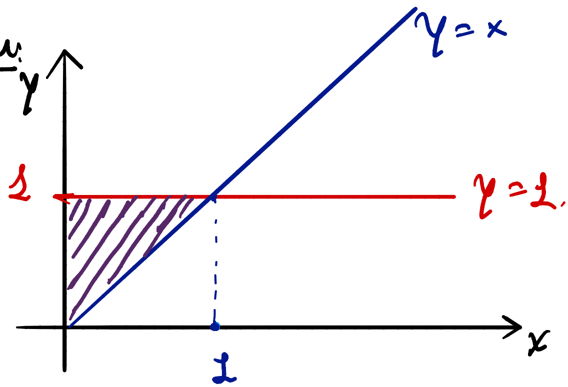
ii) the $y=1$ plane.

iii) the $y=x$ plane, and

iv) the surface $z=1-xy$.

v) $x=0$ plane.

Solution:



$$\text{Volume} = \int_0^1 \int_x^1 (1 - xy)\, dy\, dx$$

$$= \int_0^1 \left(y - \frac{xy^2}{2} \Big|_{y=x}^{y=1} \right) dx$$

$$= \int_0^1 \left(1 - \frac{x}{2} - x + \frac{x^3}{2} \right) dx.$$

Volume $= \int_0^1 \left(1 - \frac{3x}{2} + \frac{x^3}{2}\right) dx$

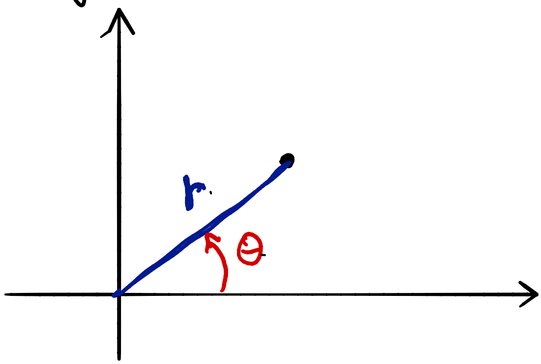
$$= \left(x - \frac{3x^2}{4} + x^4/8 \Big|_{x=0}^{x=1} \right)$$

$$= 1 - 3/4 + 1/8$$

$$= \frac{8 - 6 + 1}{8}$$

$$= 3/8 .$$

Integration in polar coordinates



r: distance to origin
θ: angle with positive x-axis, measured in counter-clockwise sense.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

- Area elements in Cartesian vs Polar coordinates:

$$dx = \frac{\partial}{\partial r}(r\cos\theta)\cdot dr + \frac{\partial}{\partial\theta}(r\cos\theta)\,d\theta.$$

$$\boxed{dx = \cos\theta\cdot dr - r\sin\theta\cdot d\theta}$$

$$dy = \frac{\partial}{\partial r}(r\sin\theta)\,dr + \frac{\partial}{\partial\theta}(r\sin\theta)\,d\theta.$$

$$\boxed{dy = \sin\theta\cdot dr + r\cos\theta\, d\theta}$$

Roles for products of differentials:

i) $dp\,dp = 0$, for a function p.

ii) $dp\,dq = -dq\,dp$, for functions p, q.

$$dx\,dy = (\cos\theta\,dr - r\sin\theta\,d\theta)(\sin\theta\,dr + r\cos\theta\,d\theta).$$

$$= \cancel{\cos\theta\cdot\sin\theta\,dr\,dr}$$

$$+\, r\cos^2\theta\,dr\,d\theta$$

$$-\, r\sin^2\theta\,d\theta\,dr$$

$$-\, \cancel{r^2\sin\theta\cos\theta\,d\theta\,d\theta}.$$

$$= r\cos^2\theta\,dr\,d\theta + r\sin^2\theta\,dr\,d\theta.$$

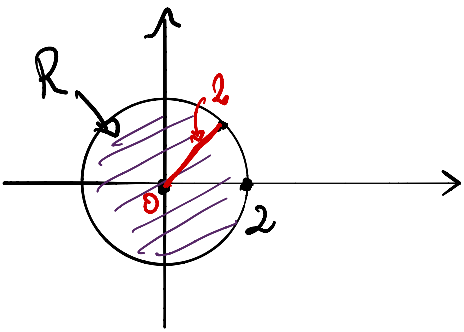
$$= (r\cos^2\theta + r\sin^2\theta)\,dr\,d\theta$$

$$= r\,dr\,d\theta.$$

Area element in polar coordinates:

$$\boxed{dA = r\,dr\,d\theta.}$$

Example 6: Find the area of a disk of radius 2



Parametrizing region of integration in polar coordinates:

r-range: $[0, 2]$

θ-range: $[0, 2\pi]$

$$\text{Area} = \iint_R 1 \, dA \Rightarrow \text{Area} = \int_0^{2\pi}\int_0^2 1 \, (r\,dr\,d\theta)$$

$$= \int_0^{2\pi}\int_0^2 r \, dr \, d\theta.$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} \Big|_{r=0}^{r=2} \right) d\theta.$$

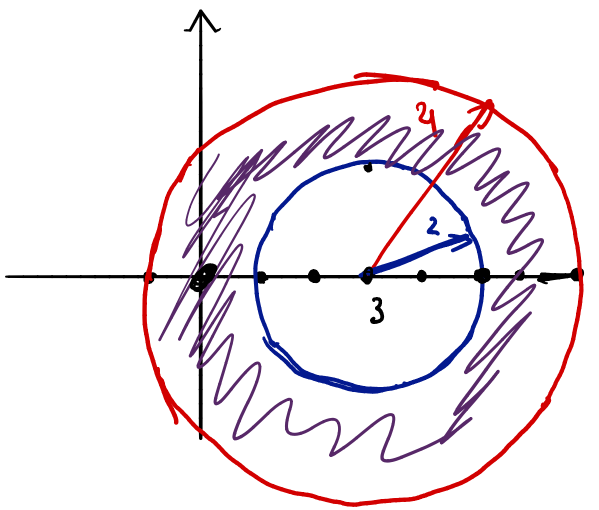
$$= \int_0^{2\pi} \left(2 - 0 \right) d\theta.$$

$$= 2\theta \Big|_{\theta=0}^{\theta=2\pi} = 2\cdot 2\pi - 2\cdot 0 = 4\pi.$$

Example 7: Find the area of the annulus bounded by the circles

$$C1: \quad x^2 - 6x + 9 + y^2 = 4 \to (x-3)^2 + y^3 = 2^2$$

$$C2: \quad x^2 - 6x + 9 + y^2 = 16 \to (x-3)^2 + y^2 = 4^2$$



$$x_1: \quad 3 + 2\cos\theta, \quad y_1 = 3 + 2\sin\theta$$

$$r_1 = \sqrt{x_1^2 + y_1^2}$$

$$= \sqrt{18 + 4(\cos\theta + \sin\theta) + 4} = \sqrt{22 + 4(\cos\theta + \sin\theta)}$$

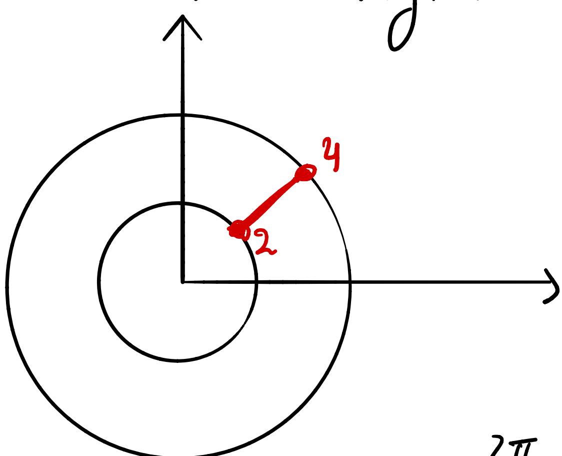
This is a complicated expression of θ!

Instead, use a translation argument to find area: area is the same as the area of the annulus bounded by

$$\tilde{C_1}: \quad x^2+y^2=4$$

$$\tilde{C_2}: \quad x^2+y^2=16,$$

circles obtained by translating centers to origin.



r-range: $[2,4]$.

θ-range: $[0,2\pi]$

$$\text{Area} = \iint_R 1\, dA = \int_0^{2\pi}\int_2^4 r\, dr\, d\theta.$$

$$\text{Area} = \int_0^{2\pi} \int_2^4 r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} \Big|_{r=2}^{r=4} \right) d\theta$$

$$= \int_0^{2\pi} (8 - 2) \, d\theta$$

$$= \int_0^{2\pi} 6 \, d\theta$$

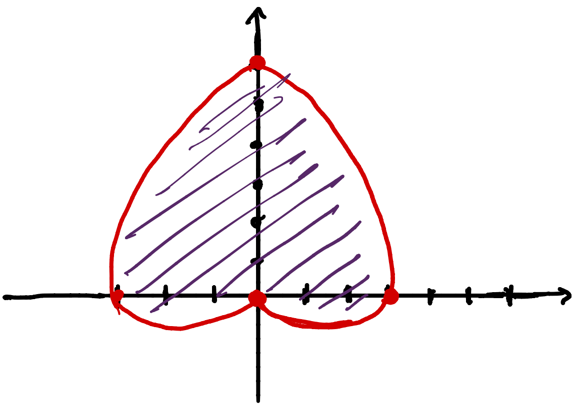
$$= 6 \cdot 2\pi$$

$$= 12\pi.$$

$$= \underbrace{16\pi}_{\text{Area of larger disk}} - \underbrace{4\pi}_{\text{Area of smaller disk}}.$$

Example 8: Find area within the cardioid,

$r = 3 + 3\sin\theta$.



$\theta = 0 \to r = 3$

$\theta = \frac{\pi}{2} \to r = 6$

$\theta = \pi \to r = 3$

$\theta = \frac{3\pi}{2} \to r = 0$.

$$\text{Area} = \iint_R 1 \, dA \Rightarrow \text{Area} = \int_0^{2\pi}\int_0^{3+3\sin(\theta)} r \, dr \, d\theta.$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} \Big|_{r=0}^{r=3+3\sin\theta} \right) d\theta.$$

$$= \int_0^{2\pi} \left(\frac{(3+3\sin\theta)^2}{2} \right) d\theta.$$

$$= \frac{9}{2}\int_0^{2\pi} 1 + 2\sin\theta + \sin^2\theta \, d\theta.$$

1) $\int_0^{2\pi} 1 \, d\theta = 2\pi$.

2) $\displaystyle\int_0^{2\pi} 2\sin(\theta)\,d\theta = 2\int_0^{2\pi} \sin(\theta)\,d\theta$

$$= 2\left(-\cos\theta \Big|_{\theta=0}^{\theta=2\pi}\right)$$

$$= 2(-1+1)$$

$$= 0$$

3) $\displaystyle\int_0^{2\pi} \sin^2(\theta)\,d\theta = \int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\,d\theta$

$$= \int_0^{2\pi} \frac{1}{2} - \frac{\cos(2\theta)}{2}\,d\theta$$

$$= \left(\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\Big|_0^{2\pi}\right)$$

$$= \frac{2\pi}{2} = \pi.$$

$$\text{Area} = \frac{9}{2}\left(2\pi + 0 + \pi\right)$$

$$= \frac{9}{2}(3\pi)$$

$$= \frac{27\pi}{2}.$$