MAT 203 - Lecture 4

Lesson plan: Calculus of vector-valued functions

Definition: A single-variable, vector-valued function is a function whose domain is $\mathbb{R}$ (or some interval) and range is a fixed vector space, say $\mathbb{R}^2$, $\mathbb{R}^3$.

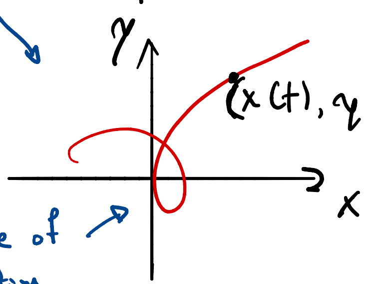
Examples:

1) Movement of a point-particle in the plane.

Input variable: t (time)

Output variables: position, velocity ...

not a graph



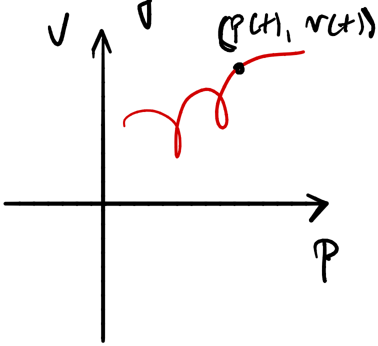
Trace of function.

2) Characteristics of an ideal gas:

- P : pressure
- T : temperature
- V : volume

$$P \cdot V = \underset{\text{some constant.}}{C} \cdot T$$

Tracking evolution of gas system over time



- Algebra of vector-valued functions:

1) We can add vector-valued functions whose outputs have the same dimension:

$$f(t) = (x(t), y(t))$$
$$g(t) = (w(t), z(t))$$

$$(f+g)(t) = (x(t) + w(t), y(t) + z(t))$$

2) We can multiply by scalars: $(\alpha f)(t) = \alpha \cdot f(t)$

We can also form:

3) dot product of two vector-valued functions

$r_1(t) = (x_1(t), y_1(t), z_1(t))$

$r_2(t) = (x_2(t), y_2(t), z_2(t))$

$$f(t) = r_1(t) \cdot r_2(t)$$
$$= (x_1(t), y_1(t), z_1(t)) \cdot (x_2(t), y_2(t), z_2(t))$$
$$= x_1(t) \cdot x_2(t) + y_1(t) \cdot y_2(t) + z_1(t) \cdot z_2(t)$$

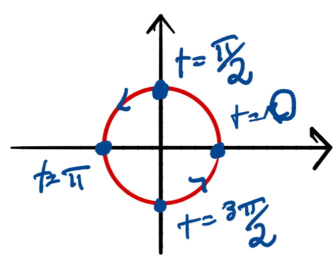
Remark: dot product is scalar-valued.

4) cross-product of two 3D vector-valued function is again a <u>3D vector-valued function.</u>

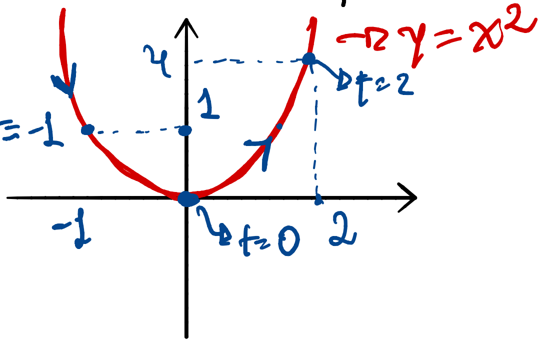
<u>Examples of vector-valued functions</u>

1) $r(t) = (\cos(t), \sin(t))$.

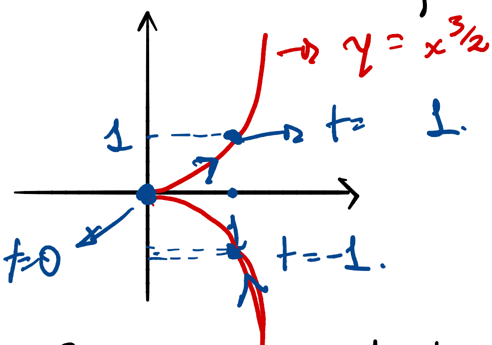
$$d(r(t), (0,0)) = \sqrt{\cos^2(t) + \sin^2(t)}$$
$$= \sqrt{1} = 1.$$



2) $r(t) = (t, t^2) \to \underline{y = x^2}$.



3) $r(t) = (t^2, t^3) \to y = x^{3/2}$.



$r(-1) = ((-1)^2, (-1)^3)$
$= (1, -1)$.

$x^3 = y^2 \to |y| = x^{3/2}$.

- <u>Continuity and limits</u>

If a vector-valued function
$$r(t) = (x(t), y(t))$$
has a limit (to be defined) then

each coordinate function should have a limit. We would like

$$\lim_{t \to a} r(t) = \left(\lim_{t \to a} x(t), \lim_{t \to a} y(t) \right).$$

<u>Remark</u>: In this course we say a scalar-value function "has a limit" if the limit is a real number.

If: $\lim_{t \to a} f(t) = +\infty, -\infty$ or DNE

we say the function does not converge (diverges).

<u>Example 4</u>: $r(t) = (2t)\,i + (4t^2)\,j + (6t^3)\,k.$

$$\lim_{t \to 1} r(t) = 2i + 4j + 6k.$$

<u>Example 5</u>: $r(t) = (\log(t), t^2), \; t > 0.$

$\lim_{t \to 0^+} r(t)$ does not exist, since

$$\lim_{t \to 0^+} (\log(t)) = -\infty.$$

Example 6: Two curves are given,

$$r_1(t) = \left(-\frac{1}{t},\ t\right), \quad t > 0$$

$$r_2(t) = \left(\frac{1}{t},\ e^t\right), \quad t > 0.$$

Neither r_1, nor r_2 have limits as $t \to 0+$, for

$$\lim_{t \to 0+} \frac{1}{t} = +\infty.$$

However, $(r_1 + r_2)(t) = (0,\ t + e^t)$ has a limit, $\lim_{t \to 0+} (r_1 + r_2)(t) = (0, 1)$.

That is: if a sum has a limit you cannot conclude each summand has a limit!

Exercise 1: Given vector-valued functions $r_1(t) = (t, 0, 1)$, $r_2(t) = (1/t,\ 1/t^2,\ e^t)$.
Does the dot product $r_1 \cdot r_2$ have a limit as $t \to 0$?

$r_1(t) = (t, 0, 1) \to \lim_{t \to 0} r_1(t) = (0, 0, 1)$.

$r_2(t) = (1/t, 1/t^2, e^t)$. $\to$ has no limit as $t \to 0$.

$$r_1 \cdot r_2 (t) = t \cdot 1/t + 0 \cdot 1/t^2 + 1 \cdot e^t$$

$$= 1 + 0 + e^t$$

$$= 1 + e^t$$

$$\lim_{t \to 0} (r_1 \cdot r_2)(t) = \lim_{t \to 0} 1 + e^t = 2.$$

Exercise 2: Consider vector-valued functions

$r_1(t) = (\log(t), 0, 0)$, $t > 0$

$r_2(t) = (0, \sqrt{t}, 0)$, $t \geq 0$.

Does the cross product $r_1 \times r_2$ have a limit as $t \to 0+$.

Note that r_1 does not have a limit as $t \to 0+$.

Cross-product

$$r_1 \times r_2 = (\log(t), 0, 0) \times (0, \sqrt{t}, 0).$$
$$= (\log(t)\, i) \times (\sqrt{t}\, j).$$
$$= \log(t) \cdot \sqrt{t}\;\; i \times j$$
$$= \log(t)\sqrt{t}\; K.$$

$$\lim_{t \to 0+} \log(t)\sqrt{t} = \lim_{t \to 0+} \frac{[\log(t)]}{\left[\left(\frac{1}{\sqrt{t}}\right)\right]}. \qquad \to t^{-\frac{1}{2}}$$

By L' Hôspital's rule:

$$\lim_{t \to 0+} \log(t)\sqrt{t} = \lim_{t \to 0+} \frac{[1/t]}{[-\frac{1}{2} t^{-3/2}]} \qquad \to t^{-1}$$
$$= \lim_{t \to 0+} -2\, t^{-1-(-3/2)}$$
$$= \lim_{t \to 0+} -2\, t^{1/2}$$
$$= 0.$$

Definition: A vector-valued function $r(t)$ is continuous at a if

$$\lim_{t \to a} r(t) = r(a).$$

In particular, the coordinates are continuous.

Read examplele 12.1.6.

· Derivatives of vector-valued functions.

We say a vector-valued function $r(t)$ has a derivative (vector) at a if

$$\lim_{t \to 0} \frac{r(a+t) - r(a)}{t}$$

exists.

In particular, its coordinates have derivatives and

$$r'(t) = (x'(t), y'(t))$$

or

$$r'(t) = (x'(t), y'(t), z'(t)).$$

Application: measure velocity vectors

Likewise: $r''(t)$ is called accelation.

Exercise 3: Find the velocity of the curve
$r(t) = (t\sin(t), t\cos(t), t)$.

Solution:
$r'(t) = ((t\sin(t))', (t\cos(t))', (t)')$
$= (\sin(t) + t\cos(t), \cos(t) - t\sin(t), 1)$.

How do derivatives interact with products?

Dot product:
$(r_1 \cdot r_2)'(t) = r_1'(t) \cdot r_2(t) + r_1(t) \cdot r_2'(t)$.

Cross-product:
$(r_1 \times r_2)'(t) = r_1'(t) \times r_2(t) + r_1(t) \times r_2'(t)$.

Explaining cross-product rule:

$$r_1(t) = (t i + t^2 j + t^3 k)$$
$$r_2(t) = (1\cdot i + t^4 j + (2t) k).$$

1) $$(r_1 \times r_2)(t) = t \cdot \cancel{i \times i} + t \cdot t^4 \cdot i \times j + t \cdot (2t) \cdot i \times k$$
$$+ t^2 j \times i + t^2 \cdot t^4 \cancel{j \times j} + t^2 \cdot (2t) j \times k$$
$$+ t^3 k \times i + t^3 \cdot t^4 k \times j + t^3 \cdot (2t) \cancel{k \times k}.$$
$$= (t^5 - t^2) k + (-2t^2 + t^3) j$$
$$(2t^3 - t^7) i$$
$$= (2t^3 - t^7) i + (-2t^2 + t^3) j + (t^5 - t^2) k.$$

Derivatives:

$$\boxed{(r_1 \times r_2)'(t) = (6t^2 - 7t^6) i + (-4t + 3t^2) j + (5t^4 - 2t) k.}$$

2) $$r_1'(t) = 1 \cdot i + (2t) j + (3t^2) k$$
$$r_2'(t) = 0 i + (4t^3) \cdot j + 2k.$$

Product rule: $r_1'(t) \times r_2(t) + r_1(t) \times r_2'(t).$

$$r_1' \times r_2 = (i + (2t) j + (3t^2) k) \times (i + t^4 j + (2t) k).$$
$$= (t^4 - 2t) k - (2t - 3t^2) j + (4t^2 - 3t^6$$

$$r_1' \times r_2 = (4t^2 - 3t^6)\,i + (3t^2 - 2t)\,j + (t^4 - 2t)\,k.$$

$$r_1 \times r_2' = (t\,i + t^2 j + t^3 k) \times (4t^3 j + 2k)$$
$$= (4t^4)k - (2t)j + (2t^2 - 4t^6)i$$
$$= (2t^2 - 4t^6)\,i - (2t)\,j + (4t^4)\,k.$$

$$r_1' \times r_2 + r_1 \times r_2' = (6t^2 - 7t^6)\,i + (3t^2 - 4t)\,j + (5t^4 - 2t)\,k.$$

- <u>Integration of vector-valued functions</u>

Given a curve $r(t)$, it's indefinite integral is another curve:

$$\int r(t)\,dt = \left(\int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt\right),$$

if each component integral exists.

Definite integrals yield vectors:

$$\int_a^b r(t)\,dt = \left(\int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt\right).$$

$$\int r_1 \times r_2(t) \neq \left(\int r_1\,dt\right) \times \left(\int r_2\,dt\right)$$

Warning: integrals and products do not interact nicely.

Applications: Velocity, speed, acceleration, displacement, distance.

If $r(t)$ models the motion a point particle, then

$$\begin{cases} r'(t): & \text{velocity.} \\ \|r'(t)\|: & \text{speed.} \\ r''(t): & \text{acceleration (vector)} \\ \int_a^b r'(t)\,dt: & \text{displacement (vector)} \\ \int_a^b \|r'(t)\|\,dt: & \text{distance (arc length).} \end{cases}$$

Exercise 4: Find the speed of the curve $r(t) = (\cos(t), \sin(t))$.

Solution: speed $= \|r'(t)\|$.

$$r'(t) = (-\sin(t), \cos(t)).$$

$$\|r'(t)\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2}$$
$$= \sqrt{\sin^2(t) + \cos^2(t)}$$
$$= 1.$$

Exercise 5: Find the acceleration vector of the curve given above.

Solution: Acceleration is $r''(t)$

$$r''(t) = (-\cos(t), -\sin(t)).$$
$$= -r(t).$$

Exercise 6: Find the displacement of $r(t)$ between $t=0$, $t=\pi$.

<u>Solution:</u> $r(t) = (\cos(t), \sin(t))$

Displacement between $t=0$, $t=\pi$.

$r(\pi) - r(0) = (\cos(\pi), \sin(\pi)) - (\cos(0), \sin(0))$

$= (-1, 0) - (1, 0)$

$= (-2, 0)$.

- $\Delta r = \displaystyle\int_0^\pi r'(t)\, dt$

$$= \int_0^\pi (-\sin(t), \cos(t))\, dt$$

$$= \left(\int_0^\pi -\sin(t)\, dt, \int_0^\pi \cos(t)\, dt \right)$$

$$= \left(\cos(\pi) - \cos(0), \sin(\pi) - \sin(0) \right).$$

$= (-2, 0)$.

<u>Exercise 7:</u> Find the distance covered by $r(t)$ from $t=0$ to $t=\pi$.

<u>Solution</u> distance $= \displaystyle\int_0^\pi 1\, dt = \pi$.

Homework 1 Review

Problem 27 (as a hint for 28).

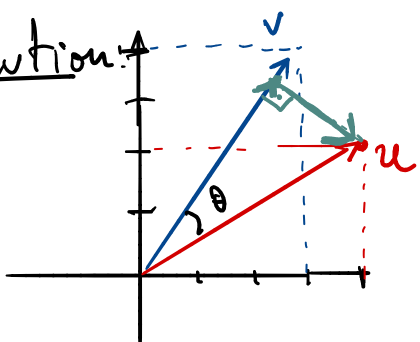
$u = 4i + 2j$

$v = 3i + 4j \longrightarrow \|v\| = 5.$

(a) Find proj(u, v)

(b) Find orthogonal complement.

Solution:



$$\|\text{proj}(u,v)\| = \|u\| \cdot \cos\theta$$

$$= \cancel{\|u\|} \cdot \frac{(u \cdot v)}{\cancel{\|u\|} \cdot \|v\|}.$$

$$= \frac{u \cdot v}{\|v\|}.$$

$$\|\text{proj}(u,v)\| = \frac{(4 \cdot 3 + 2 \cdot 4)}{\sqrt{3^2 + 4^2}}$$

$$= 20/5$$

$$= 4.$$

$$\text{proj}(u,v) = 4/5 \cdot v = 12/5\, i + 16/5\, j$$

(b) $\text{proj}(u, v) + w = u$

$$\left(\frac{12}{5} i + \frac{16}{5} j\right) + w = 4i + 2j$$

$$\boxed{w = \left(\frac{8}{5}\right) i + \left(-\frac{6}{5}\right) j .}$$

Check orthogonality:

$$v \cdot w = (3i + 4j) \cdot \left(\frac{8}{5} i - \frac{6}{5} j\right)$$

$$= \frac{24}{5} - \frac{24}{5}$$

$$= 0.$$

Hint for problem 40:

Find line through $(0,0,1)$ parallel to the line given by

$$\boxed{\frac{x-2}{3} = \frac{y-3}{4} = \frac{z}{1}.}$$

Finding a base point.

Say $z=1$. Then

$$\frac{y-3}{4} = 1 \Rightarrow y-3=4 \Rightarrow \boxed{y=7.}$$

Also $$\frac{x-2}{3} = 1 \Rightarrow x-2=3 \Rightarrow \boxed{x=5}$$

The point $(5,7,1)$ belongs to the line.

Say $z=0$. Then

$$\frac{y-3}{4} = 0 \Rightarrow y=3.$$

$$\frac{x-2}{3} = 0 \Rightarrow x=2.$$

The point $(2,3,0)$ belongs to the line

Direction:

$v = (5,7,1) - (2,3,0) = \underline{\underline{(3,4,1)}}.$

<u>Parametric equation</u>:

Basepoint + (parameter). direction

$(x,y,z) = (0,0,1) + \lambda \cdot (3,4,1)$

<u>Symmetric equation</u>:

$$\begin{cases} x = 3\lambda & \rightarrow & x/3 = \lambda. \\ y = 4\lambda & \rightarrow & y/4 = \lambda \\ z = 1 + \lambda & \rightarrow & z-1 = \lambda \end{cases}$$

$$\frac{x}{3} = \frac{y}{4} = \frac{z-1}{1}.$$

Same denominators, different subtrahends.