MAT 514 - Summer II 2020
August 11, 2020

Solutions to Homework 4

Problem 5.1(d) Compute the integral

$$\int_{\square} \frac{\exp(z)\cos(z)}{(z-\pi)^3}\, dz$$

where $\square$ is the boundary of the square with vertices at $\pm 4 \pm 4i$, positively oriented.

Solution: The path of integration $\square$ is $\mathbb{C} \setminus \{\pi\}$-homotopic to a circle centered at π, $C(\pi, r)$, for any radius $r > 0$. We may therefore move the integration to such circles and employ Cauchy's Formulas for Derivatives,

$$\begin{aligned}
\int_{\square} \frac{\exp(z)\cos(z)}{(z-\pi)^3}\, dz &= \int_{C[\pi,r]} \frac{\exp(z)\cos(z)}{(z-\pi)^3} \\
&= \pi i [\exp(z)\cos(z)]''(\pi) \\
&= -2\pi i e^{\pi} \sin(\pi) \\
&= 0.
\end{aligned}$$

Problem 5.3(h) Integrate

$$\frac{1}{(z+4)(z^2+1)}$$

over the circle $C[0,3]$.

Solution: This problem may be solved in two ways: by splitting the curve of integration, or via a partial fractions decomposition. We'll take the second approach here. Decomposing the integrand into its partial components,

$$\frac{1}{(z+4)(z^2+1)} = \frac{A}{z+4} + \frac{B}{z+i} + \frac{C}{z-i},$$

we find a system of linear equations on A, B, C:

$$\begin{aligned}
A + B + C &= 0 \\
(4-i)B + (4+i)C &= 0 \\
-4Bi + 4Ci &= 1,
\end{aligned}$$

whose solution is $A = \frac{1}{16}$, $B = -\frac{1}{32} + \frac{i}{8}$, $C = -\frac{1}{32} - \frac{i}{8}$. It follows that

$$\begin{aligned}
\int_{C[0,3]} \frac{1}{(z+4)(z^2+1)}\,dz &= \frac{1}{16}\int_{C[0,3]} \frac{1}{z+4}\,dz + \left(-\frac{1}{32} + \frac{i}{8}\right)\int_{C[0,3]} \frac{1}{z+i}\,dz \\
&\quad + \left(-\frac{1}{32} - \frac{i}{8}\right)\int_{C[0,3]} \frac{1}{z-i}\,dz \\
&= 0 + \left(-\frac{1}{32} + \frac{i}{8}\right)\int_{C[0,3]} 2\pi i + \left(-\frac{1}{32} - \frac{i}{8}\right)\int_{C[0,3]} 2\pi i \\
&= -\frac{\pi i}{8}.
\end{aligned}$$

The values of the integrals on the right-hand side were computed by means of Cauchy's Theorem and Cauchy's Integral Formula: the first integrand, $\frac{1}{z+4}$, is holomorphic on the disk $\overline{D[0,3]}$, hence its integral along the boundary is 0; the latter two integrands contain an isolated singularity in $D[0,3]$, and have numerators equal to 1, hence their integrals are equal to $2\pi i$.

Problem 5.12 Show that a polynomial of odd degree with real conefficients must have a real zero.

Solution: Let $p(z)$ be such a polynomial,

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where all the a_i are real, $a_n \neq 0$, and n is odd. Then, for any complex number z,

$$\begin{aligned}
p(\overline{z}) &= a_0 + a_1\overline{z} + a_2(\overline{z})^2 + \cdots + a_n(\overline{z})^n \\
&= \overline{a_0} + \overline{a_1 z} + \overline{a_2 z^2} + \cdots + \overline{a_n z^n} \\
&= \overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} \\
&\overline{p(z)}.
\end{aligned}$$

It follows that if z is a root of p, then so is $\overline{z}$. This has implications when z is a complex, non-real number, in which case z and $\overline{z}$ are distinct. In summary, complex, non-real roots to a polynomial with real coefficients come in pairs.

An odd degree polynomial has an odd number of complex roots, according to the Fundamental Theorem of Algebra. If such a polynomial has real coefficients, at least one such root must be real, so as to not contradict the conclusion from the last paragraph.

Problem 5.13 Suppose f is entire and $|f(z)| \leq \sqrt{|z|}$ for all $z \in \mathbb{C}$. Prove that f is identically 0.

Solution: We will follows a similar approach to the one employed in the proof of Liouville's Theorem. Let $w \in \mathbb{C}$ be an arbitrary point. By Cauchy's Integral Formulas for Derivatives,

$$f'(w) = \frac{1}{2\pi i} \int_{C[w,r]} \frac{f(z)}{(z-w)^2}\, dz,$$

for all $r > 0$. We will use the estimate provided in the statement to bound the value of $f'(w)$,

$$\begin{aligned}
|f'(w)| &\leq \frac{1}{2\pi} \int_{C[w,r]} \frac{|f(z)|}{|z-w|^2}\, dz \\
&\leq \frac{1}{2\pi} \int_{C[w,r]} \frac{\sqrt{|z|}}{r^2}\, dz \\
&\leq \frac{1}{2\pi} \int_{C[w,r]} \frac{\sqrt{|w|+r}}{r^2}\, dz \\
&\leq \frac{\sqrt{|w|+r}}{r} \\
&\leq \sqrt{\frac{|w|+r}{r^2}}.
\end{aligned}$$

Since this bound is valid for all $r > 0$, and the expression on the right-hand side converges to 0 (regardless of w) as $r \to \infty$, we conclude that $f'(w) = 0$, for all $w \in \mathbb{C}$, i.e., f is constant. In particular, $|f(0)| \leq \sqrt{0} = 0$, the value of f is 0 on the entire plane.

Problem 5.15 Suppose f is entire with bounded real part, i.e., writing

$$f(z) = u(z) + iv(z),$$

there exists $M > 0$ such that

$$|u(z)| \leq M$$

for all $z \in \mathbb{C}$. Prove that f is constant.

Solution: Consider the entire function $g(z) = \exp(f(z)) = e^{u(z)} \cdot e^{iv(z)}$. The bound on the real part of f implies a bound on g,

$$|g(z)| = |e^{u(z)} \leq e^{|u(z)|} \leq e^{M}.$$

It follows that g is a bounded entire function, hence a constant. It follows that its derivative is zero, so

$$g'(z) = f'(z) \exp(f(z)) = 0,$$

thus $f'(z) = 0$. This is valid for all $z \in \mathbb{C}$, therefore f too is a constant function.

Problem 6.8 Is it possible to find a real function $v(x, y)$ so that

$$x^3 + y^3 + iv(x, y)$$

is holomorphic?

Solution: If such a function existed, $u(x, y) = x^3 + y^3$ would be a harmonic function, as the real part of a holomorphic function. As it happens,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6y,$$

therefore u is not harmonic, hence no such a function v exists.

Problem 6.11 Prove that, if u is harmonic and bounded on $\mathbb{C}$, then u is constant.

Solution: As a harmonic function defined on a contractible domain, u has a harmonic conjugate $v \colon \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$f(z) = u(z) + iv(z)$$

is an entire function with bounded real part. By problem 5.13, f must be constant. In particular, its real part u is also constant.