MAT 514 - Lecture 10

- ## Complex logarithms

The exponential was defined as the unique solution to

$$\begin{cases} f'(z) = f(z) \\ f(0) = 1. \end{cases}$$

We denote this function by e^z or $\exp(z)$.

In the real context, the exponential has as its range $(0, +\infty)$. Its inverse is the logarithm

$$\log : (0, +\infty) \longrightarrow \mathbb{R}.$$

In the complex setting: if

$$z = a + ib, \quad a, b \in \mathbb{R}$$

then

$$e^z = e^a \cdot e^{ib}$$
$$= \underbrace{e^a}_{\text{norm}} \cdot \underbrace{[\cos(b) + i\sin(b)]}_{\text{angular component.}}$$

In contrast to the real exponential function, The complex exponential is not injective. For example,

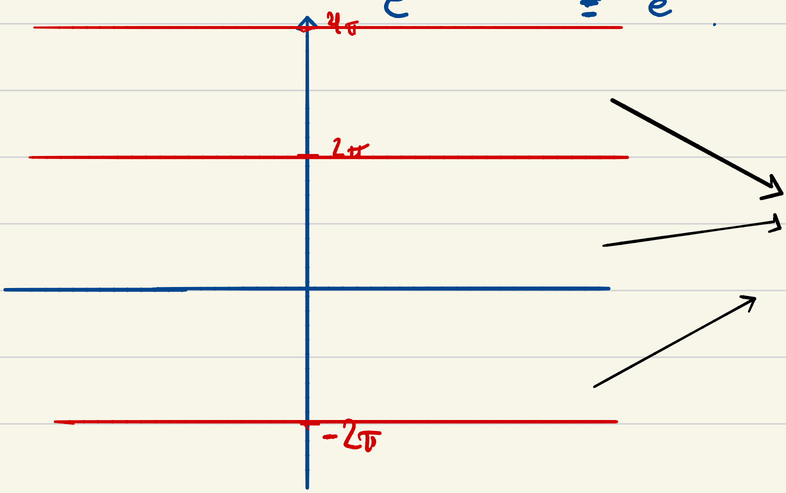
$$e^{2\pi i} = \cos(2\pi) + i\sin(2\pi)$$
$$= 1 + i\cdot 0$$
$$= 1,$$

is the same as

$$e^{0\cdot i} = \cos(0) + i\sin(0)$$
$$= 1 + i\cdot 0$$
$$= 1.$$

This function is periodic in the imaginary part, with period 2π:

$$e^{z+2\pi i} = e^{z}.$$



exp is injective in each of these bands

In fact the exponential is injective in any band of height less than 2π.

The "inverse", the complex logarithm, is a multi-valued function, taking one value in each such band.

Often we choose to work with a <u>branch</u> of the logarithm, inverting it on a specific band.

Recall that

$$e^{a+ib} = e^{a} \cdot [\cos(b) + i\sin(b)],$$

inverting this function amounts to:

i) inverting the norm to find a;

ii) inverting the trigonometric component, to find b.

<u>Example</u>: Finding a logarithm of

$$2+2i$$

can be done as follows.

1) Write $2+2i$ in polar form. Its norm

is

$$|2+2i| = \sqrt{4+4}$$
$$= 2\sqrt{2}.$$

thus

$$2+2i = 2\sqrt{2}.\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$
$$= 2\sqrt{2}.\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$
$$= 2\sqrt{2}.\left[\cos\left(\frac{9\pi}{4}\right) + i\sin\left(\frac{9\pi}{4}\right)\right].$$
$$= 2\sqrt{2}.\left[\cos\left(\frac{-7\pi}{4}\right) + i\sin\left(\frac{-7\pi}{4}\right)\right]$$
$$\vdots$$

We have a multitude of polar representations, differing from each other by their angular components, whose difference is a multiple of 2π.

All the numbers

$$\log(2\sqrt{2}) + i.\left(\frac{\pi}{4} + 2k\pi\right)$$

for $k \in \mathbb{Z}$ are logarithms of $(2+2i)$. Amongst these, we call the principal logarithm the one whose angle is between $-\pi$ and π (ends excluded). Certain definitions include π, but not $-\pi$ (the only caveat is that such functions are discontinuous).

We will write

$$\mathcal{L}og(z)$$

to denote a branch of the logarithm, while the principal branch will be denoted by

$$\mathrm{Log}(z).$$

The principal branch is defined on $\mathbb{C}$ minus the non-positive real numbers (in the textbook's convention, principal logarithms of negative numbers are defined, with angle π).

In certain contexts the polar angle is

called the <u>argument</u>.

<u>Properties</u>:

1) The real logarithm role
$$\log(xy) = \log(x) + \log(y)$$
may fail in $\mathbb{C}$. An example is given by

$$x = i$$
$$y = i - 1.$$

Their logarithms are:

(a) $\operatorname{Log}(i) = \log(|i|) + i \operatorname{Arg}(i)$
$$= 0 + i \cdot \frac{\pi}{2}$$
$$= \frac{i\pi}{2}.$$

(b) $\operatorname{Log}(i-1) = \log(|i-1|) + i \operatorname{Arg}(i-1)$
$$= \log(\sqrt{2}) + i \cdot \left(\frac{3\pi}{4}\right)$$
$$= \log(\sqrt{2}) + i \cdot \frac{3\pi}{4}.$$

Meanwhile,

$$
\begin{aligned}
x \cdot y &= i(i-1) \\
&= i^2 - i \\
&= -1 - i
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Log}(xy) &= \operatorname{Log}(-1-i) \\
&= \log(|-1-i|) + i \operatorname{Arg}(-1-i) \\
&= \log(\sqrt{2}) - i \cdot \frac{3\pi}{4}
\end{aligned}
$$

Overall,

$$
\begin{aligned}
\operatorname{Log}(i(i-1)) &= \operatorname{Log}(i) + \operatorname{Log}(i-1) \\
&= \frac{2\pi i}{4} + \log(\sqrt{2}) + \frac{3\pi i}{4} - 2\pi i.
\end{aligned}
$$

The formula

$$\operatorname{Log}(xy) = \operatorname{Log}(x) + \operatorname{Log}(y)$$

is only correct up to multiples of $2\pi i$.

$$\boxed{\operatorname{Log}(xy) = \operatorname{Log}(x) + \operatorname{Log}(y) + 2k\pi i},$$

for some integer k.

2) Given $z \in \mathbb{C} - \mathbb{R}_{\leq 0}$. We wish to compute

$$\operatorname{Log}(e^z).$$

Let $z = x + iy$. Then

$$
\begin{aligned}
e^z &= e^{x+iy} \\
&= e^x \cdot e^{iy} \\
&= e^x \cdot [\cos(y) + i\sin(y)].
\end{aligned}
$$

Hence the principal logarithm is

$$
\begin{aligned}
\operatorname{Log}(e^z) &= \log(|e^x|) + i\operatorname{Arg}(e^z) \\
&= x + i \cdot \left(y + 2k\pi \right),
\end{aligned}
$$

for the unique value of k such that

$$-\pi < y + 2k\pi < \pi.$$

For instance

$$
\begin{aligned}
\operatorname{Log}(e^{\frac{5\pi}{2} i}) &= \frac{\pi i}{2} \\
&= \left(\frac{5\pi}{2} - 2 \cdot 1 \cdot \pi\right) i \quad (\text{i.e., } k = -1).
\end{aligned}
$$

3) For any branch of the logarithm,

$$\frac{d}{dz}(\mathcal{L}og(z)) = \frac{1}{z}, \quad \underline{z \neq 0}.$$

Property 2 can be restated for any branch of the logarithm: if $z = x + iy$,

$$\mathcal{L}og(e^z) = x + i(y + 2\pi k),$$

where k is the unique integer so that $y + 2\pi k$ belong to the range of arguments of the chosen branch of the logarithm.

Write, for any z in the domain of the chosen branch

$$\begin{aligned} e^{\mathcal{L}og(z)} &= e^{\log(x) + i(y + 2k\pi)} \\ &= e^x \cdot e^{iy} \cdot \underbrace{e^{2k\pi i}}_{1} \\ &= e^{x+iy} = z. \end{aligned}$$

Hence

$$[e^{\mathrm{Log}(z)}]' = 1,$$

or by the Chain rule

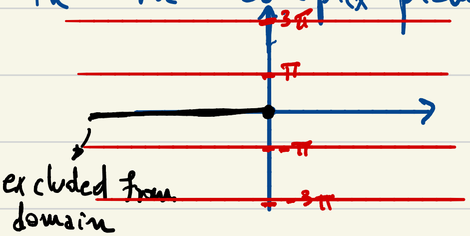
$$e^{\mathrm{Log}(z)} \cdot \mathrm{Log}'(z) = 1$$

$$z \cdot \mathrm{Log}'(z) = 1$$

$$\boxed{\mathrm{Log}'(z) = \frac{1}{z}}.$$

What we've concluded is that if Log has a derivative, then its value is $\frac{1}{z}$. We should argue that this derivative exists. At present we don't have the tools to prove this, so we will postpone this until the <u>Inverse Function Theorem</u>.

We wish to understand how does Log behave in the complex plane. Due to the restriction on the argument, Log(z) has discontinuities at each of the horizontal lines



$$y = \operatorname{Im}(z) = \pi + 2k\pi,$$

where $k \in \mathbb{Z}$.

Auxilliary complex functions

1) Exponentials with different bases:

$$f(z) = a^z$$

where a is a complex constant.

In the real setting, we can define

$$a^x = e^{\log(a) \cdot x},$$

so long as $a > 0$.

In the complex setting the logarithm is not uniquely defined, thus the meaning of an expression such as

$$e^{\operatorname{Log}(a) \cdot x}$$

is questionable.

Say we work with the principal logarithm, and $a = m + ni$. Then

$$\operatorname{Log}(a) = \log|a| + i \operatorname{Arg}(a)$$

$$\mathrm{Log}(a) = \log\left(\sqrt{m^2+n^2}\right) + i\,\mathrm{Arg}(a).$$

As an example,

$$a = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i,$$

so $\mathrm{Log}(a) = i\pi/4$. Then

$$e^{\mathrm{Log}(a)x} = e^{\frac{i\pi}{4}x}$$

$$= \cos\left(\frac{\pi x}{4}\right) + i\sin\left(\frac{\pi x}{4}\right).$$

If we chose a branch $\mathcal{L}og$ where the argument lies between π and 3π, then

$$\mathcal{L}og(a) = i\left(\frac{9\pi}{4}\right).$$

Therefore

$$e^{\mathcal{L}og(a)\cdot x} = e^{i\frac{9\pi}{4}x}$$

$$= \cos\left(\frac{9\pi x}{4}\right) + i\sin\left(\frac{9\pi x}{4}\right).$$

This isn't the same as before: if $x = \frac{1}{2}$, then

$$\cos\left(\frac{\pi}{4}\cdot\frac{1}{2}\right) = \cos\left(\frac{\pi}{8}\right)$$

$$\cos\left(\frac{9\pi}{4}\cdot\frac{1}{2}\right) = \cos\left(\frac{9\pi}{8}\right) = -\cos\left(\frac{\pi}{8}\right).$$

To make it unambiguous, we will choose to always work with the principal Logarithm

<u>Definition</u>: For $a \in \mathbb{C} - \mathbb{R}_{\leq 0}$, we can define

$$\boxed{a^z := e^{\mathrm{Log}(a)\cdot z}.}$$

This function is differentiable, with derivative

$$\begin{aligned}(a^z)' &= \left(e^{\mathrm{Log}(a)\cdot z}\right)' \\ &= e^{\mathrm{Log}(a)\cdot z}\cdot[\mathrm{Log}(a)]. \\ &= \mathrm{Log}(a)\cdot a^z.\end{aligned}$$

Previously we studied derivatives of polynomials. Now we're ready to study functions such as

$$z^c,$$

where c is a complex constant ($c \neq 0$). We can write this function as

$$e^{\mathrm{Log}(z)\cdot c},$$

so its derivative can be computed by the Chain Rule:

$$\begin{aligned}
(z^c)' &= [e^{\mathrm{Log}(z)\cdot c}]' \\
&= [e^{\mathrm{Log}(z)\cdot c}] \cdot [\mathrm{Log}(z)\cdot c]' \\
&= z^c \cdot \frac{c}{z} \\
&= c \cdot z^{c-1},
\end{aligned}$$

i.e., the "power rule" works for complex exponents.