Solutions to Quiz 6

Problem 1 Compute the integral

$$\int_\gamma \frac{z+1}{z}\,dz,$$

where γ is the line segment joining 4 to $4i$.

Solution: The function $f(z) = \frac{z+1}{z}$ is singular at the origin, but otherwise holomorphic. An antiderivative is

$$F(z) = z + \mathrm{Log}(z).$$

This anti-derivative is defined and holomorphic in $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$, a domain which includes the path of integration. It follows from Theorem 4.11 in our textbook that

$$\begin{aligned}\int_\gamma \frac{z+1}{z}\,dz &= 4i + \mathrm{Log}(4i) - 4 - \mathrm{Log}(4)\\ &= 4i + \log(4) + \frac{i\pi}{2} - 4 - \log(4)\\ &= -4 + \left(4 + \frac{\pi}{2}\right) i.\end{aligned}$$

Problem 2 Compute the integral

$$\int_{C[-1,2]} \frac{z^2}{4-z^2}\,dz.$$

Solution: We begin by simplifying the integrand,

$$\frac{z^2}{4-z^2} = -1 + \frac{4}{4-z^2}.$$

Next we write the remaining fraction in terms of partial fractions,

$$\frac{4}{4-z^2} = -\frac{1}{z+2} + \frac{1}{z-2}.$$

We can therefore decompose the original integral into

$$\int_{C[-1,2]} \frac{z^2}{4-z^2}\, dz = -\int_{C[-1,2]} 1\, dz - \int_{C[-1,2]} \frac{1}{z+2}\, dz + \int_{C[-1,2]} \frac{1}{z-2}\, dz.$$

Among these integrands, two are holomorphic functions within the disk bounded by the circle $C[-1,2]$, namely (-1) and $\frac{1}{z-2}$. By Cauchy's Theorem, the first and third integrals above vanish. The second integral remains, as its integrand has a singularity at -2, a point situated within the disk $D[-1,2]$. The circle $C[-1,2]$ is $(\mathbb{C} \setminus \{-2\})$-homotopic to the circle $C[-2,2]$, by means of translations, so by Theorem 4.18,

$$\int_{C[-1,2]} \frac{1}{z+2}\, dz = \int_{C[-2,2]} \frac{1}{z+2}\, dz$$

The latter is evaluated by substitution, using the parametrization $\gamma(t) = -2 + 2e^{it}$, $0 \le t \le 2\pi$.

$$\begin{aligned}
\int_{C[-2,2]} \frac{1}{z+2}\, dz &= \int_0^{2\pi} \frac{2ie^{it}}{2 + (-2 + 2e^{it})}\, dt \\
&= \int_0^{2\pi} \frac{2ie^{it}}{2e^{it}}\, dt \\
&= 2\pi i.
\end{aligned}$$