MAT 514 - Lecture 8

· Relation between real and complex differentiability.

We can view complex functions

$$f: G \subset \mathbb{C} \to \mathbb{C}$$

as multivariable function

$$f: G \subset \mathbb{R}^2 \to \mathbb{R}^2.$$

The notion of continuity is the same in both viewpoints, but differentiability differs.

Recall that a multivariable function

$$f: G \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is differentiable at an interior point p of G if there exists a matrix $f'(p)$ such that:

error term

$$f(p+h) = f(p) + f'(p)\cdot h + O(|h|^2),$$

where $O(|h|^2)$ is an error whose order is at least quadratic.

Example: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f(x,y) = (x^2, xy)$$

We wish to study differentiability at $p=(1,1)$.

At p,

$$f(1,1) = (1^2, 1\cdot 1)$$
$$= (1,1).$$

Next let h be a vector in $\mathbb{R}^2$, say

$$h=(x,y).$$

Then

$$f(p+h) = f(1+x, 1+y)$$
$$= \left((1+x)^2, (1+x)(1+y)\right)$$
$$= (1+2x+x^2, 1+x+y+xy).$$

We wish to compare $f(p+h)$ and $f(p)$.

$$f(p+h) - f(p) = \left((1+2x+x^2), 1+x+y+xy\right) - (1,1).$$
$$= (2x+x^2, x+y+xy)$$
$$= \underbrace{(2x, x+y)}_{\text{linear}} + \underbrace{(x^2, xy)}_{\text{higher order.}}$$

In terms of a matrix: $f'(p)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ x+y \end{pmatrix}.$

Therefore

$$f'(1,1) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

Another way to obtain f':

$$f'(1,1) = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix},$$

where

$$f_1(x,y) = x^2,$$
$$f_2(x,y) = xy.$$

Meanwhile, for a complex function to be differentiable at a point p, we need that

$$\lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$

① Are complex-differentiable functions differentiable in the real, multivariable sense?

② Are real, multivariable differentiable functions complex-differentiable?

The answer to the first question is yes; all complex-differentiable functions are differentiable in the multivariable sense. However, not all real-differentiable functions are complex-differentiable, as we'll see below. A set of necessary conditions on a real, multi-variable function

$$f: G \subset \mathbb{R}^2 \to \mathbb{R}^2,$$

called the Cauchy-Riemann equations is related to complex-differentiability.

<u>Example</u>: The function $f: \mathbb{C} \to \mathbb{C}$

$$f(z) = z$$

is complex-differentiable, with derivative $f'(z) = 1$. In real coordinates,

$$f(x, y) = (x, y).$$

It Jacobian is

$$Jf(x,y) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example: The function $f: \mathbb{C} \to \mathbb{C}$,

$$f(z) = z^2$$

is complex differentiable.

In terms of real coordinates,

$$(x+iy)^2 = x^2 + 2ixy + (iy)^2$$
$$= (x^2 - y^2) + i(2xy),$$

or $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) = (x^2 - y^2, 2xy).$$

Its Jacobian is

$$Jf(x,y) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}$$
$$= \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

Example: Consider the function $f: \mathbb{C} \to \mathbb{C}$

$$f(z) = \bar{z}.$$

Recall from the previous lecture that f is not complex-differentiable.

In real coordinates, $f(x,y) = (x, -y)$. Its

Jacobian

$$Jf = \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

<u>Example</u>: Consider the function $f: \mathbb{C} \to \mathbb{C}$

$$f(z) = z^3, \text{ or}$$

$$f(x+iy) = (x+iy)^3$$

$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= x^3 + 3ix^2y - 3xy^2 - iy^3$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3).$$

So as a function $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) = (x^3 - 3xy^2,\ 3x^2y - y^3)$$

Its Jacobian is

$$Jf = \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix}$$

$$= \begin{bmatrix} (3x^2 - 3y^2) & (-6xy) \\ (6xy) & (3x^2 - 3y^2) \end{bmatrix}.$$

The Cauchy-Riemann equations

If $f(z) = u(z) + i\,v(z)$ is a complex-differentiable, with u, v its real and imaginary parts, $z = x+iy$, then

$$\boxed{\begin{aligned} \partial_x u &= \partial_y v \\ \partial_y u &= -\partial_x v \end{aligned}}$$

Theorem

Assume a multivariable function

$$f: G \subset \mathbb{R}^2 \to \mathbb{R}^2$$

has continuous partial derivatives at $p \in G$, satisfying the Cauchy-Riemann equations. Then this function is complex-differentiable.

Example: Consider $f: \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = (\bar{z})^2 \Rightarrow$$

$$\begin{aligned} f(x+iy) &= (x-iy)^2 \\ &= (x^2-y^2) - 2ixy, \end{aligned}$$

so $u(x,y) = x^2-y^2$ and $v(x,y) = -2xy$.

Let's compute the derivatives in the Cauchy-Riemann equations:

$$\partial_x u = 2x$$
$$\partial_y u = -2y \qquad Jf = \begin{bmatrix} 2x & -2y \\ -2y & -2x \end{bmatrix}.$$
$$\partial_x v = -2y$$
$$\partial_y v = -2x$$

The Cauchy-Riemann equations are

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \Rightarrow \begin{cases} 2x = -2x \\ -2y = 2y \end{cases},$$

a system whose unique solution is $x=0, y=0$. This is compatible with what we learned in the previous lecture, as f is complex differentiable at 0. Since the Cauchy-Riemann equations are not satisfied elsewhere, this function is not complex-differentiable, except at the origin.

<u>Example</u>: Consider the function $f: \mathbb{C} \to \mathbb{C}$

$f(x+iy) = 2x + i\, x y^2, \quad x, y \in \mathbb{R}.$

The components are $u(x,y) = 2x$, $v(x,y) = xy^2$.

The partial derivatives are

$$\partial_x u = 2, \quad \partial_y u = 0, \quad \partial_x v = y^2, \quad \partial_y v = 2xy.$$

The Cauchy-Riemann equations

$$\begin{cases} 2 = 2xy \\ 0 = -y^2 \Rightarrow y = 0. \end{cases}$$

have no solutions

<u>Example</u>: $f(x+iy) = x^2 + iy^2$.

The components are

$$u(x,y) = x^2$$
$$v(x,y) = y^2,$$

with partial derivatives

$$\partial_x u = 2x, \quad \partial_y u = 0, \quad \partial_x v = 0, \quad \partial_y v = 2y.$$

The Cauchy-Riemann equations are

$$\begin{cases} 2x = 2y \Rightarrow \underline{x = y}. \\ 0 = -0 \end{cases}$$

This function is complex-differentiable along the line $\boxed{y = x}$.

Example: Consider the function

$$f(z) = |z|^2 \Rightarrow$$

$$f(x+iy) = (x^2+y^2) + i\cdot 0,$$

so $u(x,y) = x^2+y^2$, $v(x,y) = 0$. The partial derivatives of u, v are

$$\partial_x u = 2x, \quad \partial_y u = 2y, \quad \partial_x v = 0, \quad \partial_y v = 0.$$

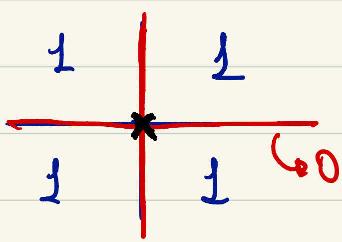
The Cauchy-Riemann equation are:

$$\begin{cases} 2x = 0 \\ 2y = 0 \end{cases} \Rightarrow x = y = 0.$$

The norm-squared function is only complex differentiable at $z = 0$.

Example: Consider the function $f: \mathbb{C} \to \mathbb{R}$

$$f(z) = \begin{cases} 0, & \text{if } x\cdot y = 0 \\ 1, & \text{otherwise.} \end{cases}$$



This function has discontinuities along the axes.

The component of the multivariable form of f are

$$u(x,y) = \begin{cases} 0, & \text{if } x \cdot y = 0 \\ 1, & \text{otherwise.} \end{cases}$$

$$v(x,y) = 0.$$

The imaginary component v is everywhere differentiable, with partials

$$\partial_x v = 0, \quad \partial_y v = 0.$$

The real component is not differentiable along the axes, but it has partial derivatives

$$\partial_x u(0,0) = 0$$

$$\partial_y u(0,0) = 0.$$

The Cauchy-Riemann equations are thus satisfied:

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \Rightarrow \quad \begin{aligned} 0 &= 0 \\ 0 &= 0, \end{aligned}$$

at the origin.

However, the function is not even continuous, let alone differentiable.

Proposition: Let $f: G \subset \mathbb{C} \to \mathbb{C}$ be a complex-differentiable function, whose values are all real ($v(x,y)=0$). Then f is a constant.

Proof: Writing

$$f(x+iy) = u(x,y) + i v(x,y),$$

we get the Cauchy - Riemann equations,

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$

Since v is assumed 0, then

$$\partial_x v = \partial_y v = 0.$$

From the Cauchy - Riemann equations,

$$\partial_x u = \partial_y u = 0.$$

It follows that u is a constant, hence so is f.

$\blacksquare$

Proposition: Let $f(z) = u(x) + i v(y)$ be a complex-differentiable function. Then f is an affine function
$$f(z) = az + b,$$
where a, b are constants, $a \in \mathbb{R}$.

Proof: The Cauchy-Riemann equations imply
$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \Rightarrow \begin{cases} \partial_x u = \partial_y v \\ 0 = -0. \end{cases}$$
Since u only depends on x,
$$u(x) - u(0) = \int_0^x \partial_x u(s)\, ds$$
the same holds for v:
$$v(y) - v(0) = \int_0^y \partial_y v(t)\, dt.$$
It follows that u, v differ by a constant, $u = v + a$, where a is real.
From CR equations
$$\partial_x u(x) = \partial_y v(y),$$

so both $\partial_x u$ and $\partial_y v$ are constants,
say $\partial_x u = \partial_y v = c \in \mathbb{R}$, hence

$$u(x) = c \cdot x + d_1$$
$$v(y) = c \cdot y + d_2$$

$$\begin{aligned} f(z) &= u(x) + i v(y) + (d_1 + i d_2) \\ &= c x + i c y. + (d_1 + i d_2) \\ &= c \cdot (x + i y) + (d_1 + i d_2) \\ &= c \cdot z + d. \end{aligned}$$

Application

Suppose $f : G \subset \mathbb{C} \to \mathbb{C}$ is complex-differentiable with
$$f(z) = u(x,y) + i v(x,y),$$
where
$$u(x,y) \cdot v(x,y) = 1.$$
Claim: f is a constant.

Proof: From the Cauchy-Riemann equations,

$$\begin{cases} \partial_x u = \partial_y v & (1) \\ \partial_y u = -\partial_x v & (2) \end{cases}$$

Meanwhile, differentiating the constraint equation,

- $(\partial_x u)\cdot v + u\cdot(\partial_x v) = 0.$ (3)
- $(\partial_y u)\cdot v + u\cdot(\partial_y v) = 0.$ (4)

Using equation (2) on (3):

$$(\partial_x u)\cdot v + u\cdot(-\partial_y u) = 0.$$

$$(\partial_x u)\cdot v - u\cdot(\partial_y u) = 0. \quad (5).$$

Multiplying (5) by v:

$$v^2\cdot(\partial_x u) - uv(\partial_y u) = 0.$$

$$v^2\cdot(\partial_x u) - u\cdot(-u\,\partial_y v) = 0.$$

$$v^2\cdot(\partial_x u) + u^2(\partial_y v) = 0.$$

From equation 1, we find

$$v^2 \cdot (\partial_x u) + u^2 \cdot (\partial_x u) = 0$$
$$(u^2+v^2)(\partial_x u) = 0.$$

Thus, either

$$u^2 + v^2 = 0$$

or

$$\partial_x u = 0.$$

The first alternative is impossible, as u, v are real, both would be 0, contradicting $u \cdot v = 1$.

The second alternative implies:

- $\partial_y v = 0,$

from the Cauchy-Riemann equations;

- $\partial_x v = 0,$

from equation 3,

- $\partial_y u = 0,$

from equation 5.

Remark: we used continuity to reach the last

Two conclusions.

Since all partials are zero, both u and v are constants, hence so is f.

- Harmonic conjugate

Given a function $u(x,y)$ under which conditions there exists $v(x,y)$ so that the pair satisfies the Cauchy-Riemann equations?

<u>Example</u>: Consider $u(x,y) = x^2 - y^2$. Its partial derivatives are

$$\partial_x u = 2x$$
$$\partial_y u = -2y$$

If such a function v exists, then

$$\partial_x v = -\partial_y u = 2y$$
$$\partial_y v = \partial_x u = 2x.$$

One example of such function is

$$v(x,y) = 2xy.$$

Other examples include

$$v(x,y) = 2xy + c,$$

where c is a real constant.

Example: $u(x,y) = \dfrac{x}{x^2+y^2}$;

- $$\partial_x u = \frac{1\cdot(x^2+y^2) - x\cdot(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}.$$

- $$\partial_y u = \frac{0\cdot(x^2+y^2) - x\cdot(2y)}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}.$$

From the Cauchy-Riemann equations

- $\partial_x v = -\partial_y u$
$$= \frac{2xy}{(x^2+y^2)^2}.$$

- $\partial_y v = \partial_x u$
$$= \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Take $v = \dfrac{-y}{x^2+y^2}$. Then the differentials are satisfied.