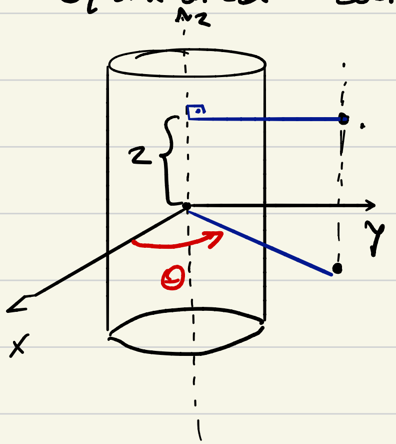
MAT 203 - lecture 15

• Cylindrical coordinates



r: distance to axis of symmetry (z-axis).

θ: angle between projection to xy-plane and positive x-axis (measured counterclockwise).

z: height relative to xy-plane

Cartesian	Cylindrical
x	$r\cos\theta$
y	$r\sin\theta$
z	z
$dV = dx\,dy\,dz$	$dV = r\,dr\,d\theta\,dz$.

A useful relation: $r^2 = x^2 + y^2$.

Ranges: $0 \le r \le +\infty$

$0 \le \theta \le 2\pi$

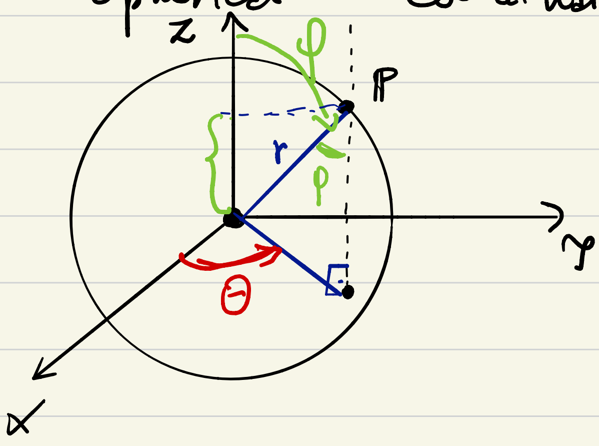
$-\infty \le z \le +\infty$

- **Remark:** as differentials,

$$\begin{cases} dp\, dq = -dq\, dp \\ dp\, dp = 0, \end{cases}$$

where p, q are any functions.

- Spherical Coordinates



- r: distance to origin
- θ (longitude): polar angle of projection to xy-plane.
- φ (latitude): angle between vector and north pole

Ranges:

$$0 \leq r \leq +\infty$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \varphi \leq \pi$$

Cartesian	Spherical
x	$r\sin\phi\cos\theta$
y	$r\sin\phi\sin\theta$.
z	$r\cos\phi$
$dV = dx\,dy\,dz$	$dV = r^2\sin\phi\,dr\,d\phi\,d\theta$

$$r^2 = x^2 + y^2 + z^2.$$

Example 1: Find the volume of the solid bounded above by
$$z = 8 - x^2 - y^2$$
and below by
$$z = x^2 + y^2.$$

Solution: Finding the region of integration
(z-coord. of upper bound) = (z-coord. of lower bnd)

$$8 - x^2 - y^2 = x^2 + y^2.$$

$$2x^2 + 2y^2 = 8 \Rightarrow x^2 + y^2 = 4.$$

A circle with center at (0,0) and radius 2. In cylindrical coordinates:
$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi.$$
z-bounds: $z_{upper} = 8 - x^2 - y^2 = 8 - r^2$
$z_{lower} = r^2.$

$$\text{Volume} = \iiint_S 1 \, dV$$

$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} 1 \cdot r \, dz \, dr \, d\theta.$$

$$= \int_0^{2\pi} \int_0^2 \left[rz \Big|_{z=r^2}^{z=8-r^2} \right] dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left[r \cdot (8-r^2) - r \cdot r^2 \right] dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left[8r - r^3 - r^3 \right] dr \, d\theta$$

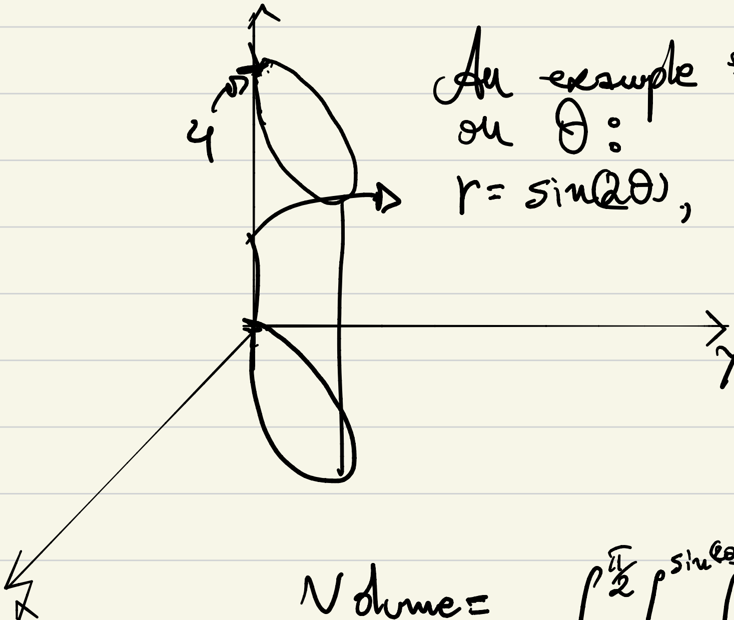
$$= \int_0^{2\pi} \int_0^2 [8r - 2r^3] \, dr \, d\theta.$$

$$= \int_0^{2\pi} \left[4r^2 - \frac{r^4}{2} \Big|_{r=0}^{r=2} \right] d\theta.$$

$$= \int_0^{2\pi} \left[4 \cdot 2^2 - \frac{2^4}{2} \right] d\theta$$

$$= \int_0^{2\pi} \left[16 - 8 \right] d\theta$$

$$= \int_0^{2\pi} 8 \, d\theta = 16\pi.$$



An example in which r depends on θ:

$r = \sin(2\theta)$, $\quad 0 \leq \theta \leq \pi/2$.

$$\text{Volume} = \int_0^{\frac{\pi}{2}} \int_0^{\sin(2\theta)} \int_0^{4} 1 \; r \, dz \, dr \, d\theta$$

Example 2: Find the volume of the solid bounded above

$$x^2+y^2+z^2 = 8 \to z_+ = \sqrt{8-x^2-y^2}$$

and below by

$$z = 2.$$

1st solution: By cylindrical coordinates.

Find region of integration

z-Range: $2 \le z \le \sqrt{8 - x^2 - y^2} = \sqrt{8 - r^2}$.

θ-Range: $0 \le \theta \le 2\pi$.

Finding intersection:

$(z\text{-coor. of plane})^2 = (z\text{-coord. of sphere})^2$

$$4 = 8 - x^2 - y^2$$

$$4 = 8 - r^2$$

$$\Rightarrow r^2 = 4 \Rightarrow \boxed{r = 2},$$

of the intersection.

r-Range: $0 \le r \le 2$.

Volume $= \int_0^{2\pi} \int_0^2 \int_2^{\sqrt{8-r^2}} 1 \cdot r \, dz \, dr \, d\theta$.

$$= \int_0^{2\pi} \int_0^2 \left[rz \Big|_{z=2}^{z=\sqrt{8-r^2}} \right] dr \, d\theta.$$

$$= \int_0^{2\pi} \left[\left(\int_0^2 r \cdot \sqrt{8 - r^2} \, dr \right) - \left(\int_0^2 2r \, dr \right) \right] d\theta.$$

To solve $\int_0^2 r\sqrt{8-r^2}\,dr$

we use $u = 8-r^2$, $du = -2r\,dr$.

$$\int_0^2 r\sqrt{8-r^2}\,dr = \int_8^4 -\frac{\sqrt{u}}{2}\,du$$

$$= \int_4^8 \frac{\sqrt{u}}{2}\,du.$$

$$= \frac{u^{3/2}}{\cancel{2}} \cdot \cancel{2}/3 \Big|_{u=4}^{u=8}$$

$$= \frac{8^{3/2}}{3} - \frac{4^{3/2}}{3}$$

$$= \frac{(2\sqrt{2})^3}{3} - \frac{2^3}{3}$$

$$= \frac{16\sqrt{2}}{3} - 8/3$$

$$= \frac{16\sqrt{2}-8}{3}$$

Meanwhile, $\displaystyle\int_0^2 2r\,dr = r^2\Big|_{r=0}^{r=2} = 4.$

Thus

$$\text{Volume} = \int_0^{2\pi}\left[\frac{16\sqrt{2}-8}{3} - 4\right]d\theta.$$

$$= \int_0^{2\pi} \frac{16\sqrt{2}-20}{3}\,d\theta$$

$$= \left(\frac{16\sqrt{2}-20}{3}\right)\cdot 2\pi.$$

$$= \frac{(32\sqrt{2}-40)\pi}{3}.$$

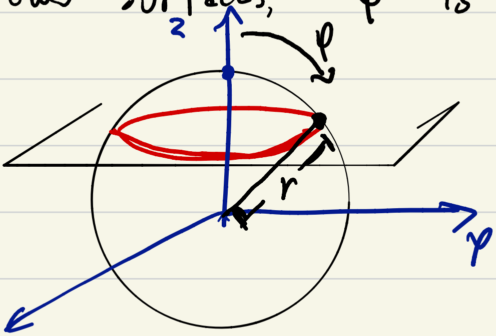
2nd solution: Using spherical coordinates.

θ- range: $0 \le \theta \le 2\pi$. $\to$ free variable

Finding φ - range:

For points along positive z-axis, φ is at its lowest, $\varphi = 0$.

For points on the intersection of the two surfaces, φ is at its greatest value.



Steps:

i) measure radius of sphere

ii) measure height of intersection.

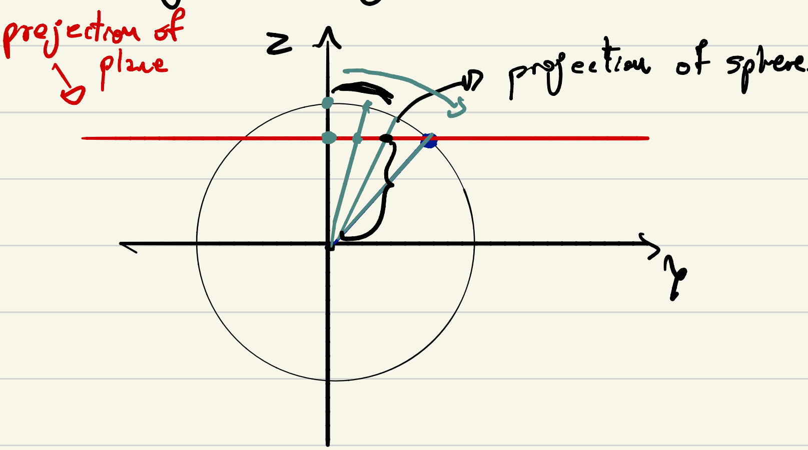
iii) find φ using relations between Cartesian and Spherical coordinates.

i) $x^2 + y^2 + z^2 = 8 \Rightarrow$ radius : $r = \sqrt{8} = \boxed{2\sqrt{2}}$

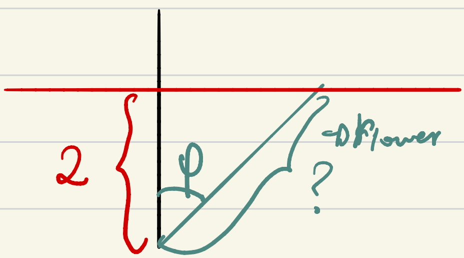
ii) height of intersection : $z = 2$

iii)
$$r\cos(\varphi) = z$$
$$\cancel{2}\sqrt{2}\cos(\varphi) = \cancel{2}$$
$$\sqrt{2}\cos(\varphi) = 1$$
$$\cos(\varphi) = \frac{\sqrt{2}}{2} \Rightarrow \boxed{\varphi = \pi/4}$$

Finding r-range: Consider the yz- projection



Upper bound for r is along sphere; $r_{upper} = 2\sqrt{2}$.
Lower bound for r depends on latitude angle.



$$\cos\varphi = \frac{2}{r_{lower}}$$

$$\Rightarrow r_{lower} = \frac{2}{\cos\varphi}$$

$$\boxed{r_{lower} = 2\sec\varphi.}$$

Setting up integral:

$$\text{Volume} = \iiint_S 1 \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_{2\sec\varphi}^{2\sqrt{2}} r^2 \sin(\varphi) \, dr \, d\varphi \, d\theta.$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{r^3}{3} \sin(\varphi) \Big|_{r=2\sec\varphi}^{r=2\sqrt{2}} \right] d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{(2\sqrt{2})^3}{3} \sin(\varphi) - \frac{8\sin\varphi}{3\cos^3\varphi} \right] d\varphi \, d\theta.$$

Computing integrals within brackets:

$$\int_0^{\pi/4} \frac{(2\sqrt{2})^3}{3} \sin\varphi \, d\varphi = \frac{16\sqrt{2}}{3} \int_0^{\pi/4} \sin\varphi \, d\varphi$$

$$= \frac{16\sqrt{2}}{3} \left(-\cos(\varphi) \Big|_{\varphi=0}^{\varphi=\pi/4} \right)$$

$$= \frac{16\sqrt{2}}{3} \left(-\frac{\sqrt{2}}{2} + 1 \right)$$

$$\int_0^{\pi/4} \frac{(2\sqrt{2})^3}{3} \sin\varphi \, d\varphi = \frac{16}{3}\left(-1+\sqrt{2}\right)$$

The second integral is

$$\int_0^{\pi/4} -\frac{8\sin\varphi}{3\cos^3\varphi} \, d\varphi = \int_1^{\frac{\sqrt{2}}{2}} \frac{8}{3u^3} \, du,$$

where

$$u = \cos\varphi,$$
$$du = -\sin\varphi \, d\varphi.$$

Hence

$$\int_0^{\pi/4} -\frac{8\sin\varphi}{3\cos^3\varphi} \, d\varphi = -\int_{\frac{\sqrt{2}}{2}}^{1} \frac{8}{3} u^{-3} \, du$$

$$= -\frac{8}{3} \cdot \frac{u^{-2}}{(-2)} \Bigg|_{u=\sqrt{2}/2}^{u=1}$$

$$= \frac{4}{3u^2} \Bigg|_{u=\sqrt{2}/2}^{u=1}$$

$$= \frac{4}{3} - \frac{4}{3 \cdot (1/2)}$$

$$\int_0^{\pi/4} -\frac{8\sin\varphi}{3\cos^3\varphi}\,d\varphi = \frac{4}{3} - \frac{8}{3} = -\frac{4}{3}.$$

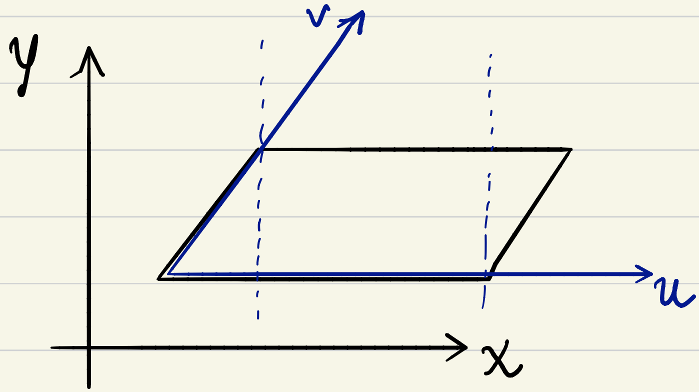
Thus we have

$$\text{Volume} = \int_0^{2\pi}\left[\frac{16}{3}(-1+\sqrt{2}) - \frac{4}{3}\right]d\theta$$
$$= \int_0^{2\pi}\left[-\frac{16}{3} + \frac{16\sqrt{2}}{3} - \frac{4}{3}\right]d\theta$$
$$= \int_0^{2\pi}\left[-\frac{20}{3} + \frac{16\sqrt{2}}{3}\right]d\theta$$
$$= -\frac{40\pi}{3} + \frac{32\pi\sqrt{2}}{3},$$

coinciding with the value we found via cylindrical coordinates.

General changes of coordinates

2D case.



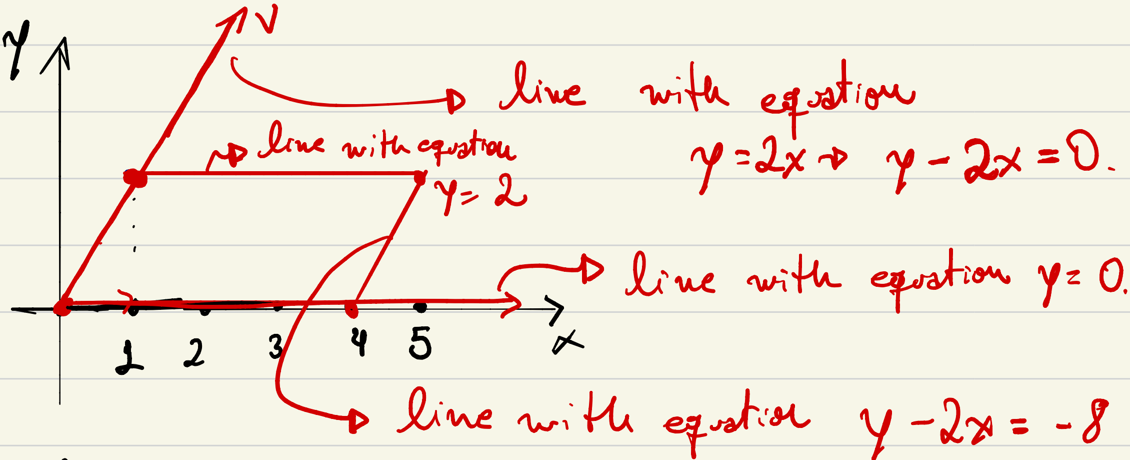
How to integrate over a parallelogram?

Method 1: slicing

Method 2: changing coordinates.

Problem Given descriptions of new "parallelogram coordinates" u, v, how do we integrate in this new coordinate system?

Example: Finding area of parallelogram with vertices $(0,0)$, $(1,2)$, $(4,0)$, $(5,2)$.



New variables:

$$u = y$$
$$v = y - 2x$$

In other words,

$$\boxed{y = u}$$

$$v = u - 2x \Rightarrow$$

$$v - u = -2x \Rightarrow$$

$$\boxed{\frac{u - v}{2} = x.}$$

Relations between differentials:

$$dx = \left(\frac{du - dv}{2}\right)$$

$$dy = du$$

product.

- $$dx \wedge dy = \left(\frac{du - dv}{2}\right) \wedge du$$

$$= \frac{1}{2}\left[du \wedge du - dv \wedge du\right]$$

$$= -\frac{1}{2}\, dv \wedge du.$$

$$\text{Area} = \iint_R 1\, dA$$

$$= \int_0^2 \int_0^{-8} \left(-\frac{1}{2}\right) dv\, du$$

$$= \int_0^2 \left[-\frac{v}{2}\Big|_{v=0}^{v=-8}\right] du$$

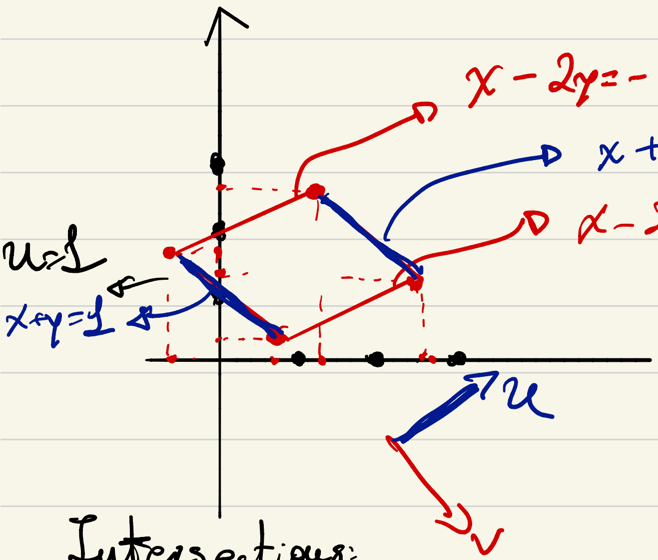
$$= \int_0^2 \left[-\frac{(-8)}{2} - \frac{0}{2}\right] du = \int_0^2 4\, du = 8.$$

Example: Integrate

$$\iint_R 3xy \, dA$$

on the region bounded by

$$\begin{cases} x - 2y = 0 \\ x - 2y = -4 \\ x + y = 4 \\ x + y = 1. \end{cases}$$



v increases as we go down

u increases as we go to the right.

Intersections:

i) $\begin{cases} x - 2y = 0 \Rightarrow x = 2y \\ x + y = 4 \Rightarrow 2y + y = 4 \Rightarrow 3y = 4 \Rightarrow \boxed{y = \frac{4}{3}} \end{cases}$

$$x = 2\cdot\left(\frac{4}{3}\right) \Rightarrow \boxed{x = \frac{8}{3}}$$

ii) $\begin{cases} x - 2y = 0 \\ x + y = 1 \end{cases}$

$$x - 2y = 0 \Rightarrow x = 2y$$

$$x + y = 1 \Rightarrow 2y + y = 1 \Rightarrow 3y = 1 \Rightarrow \boxed{y = \frac{1}{3}}$$

$$\boxed{x = \frac{2}{3}}$$

iii) $\begin{cases} \cancel{x} - 2y = -4 \\ \cancel{x} + y = 4 \end{cases} \ominus$

$$-3y = -8 \Rightarrow \boxed{y = \frac{8}{3}}$$

$$x + \frac{8}{3} = \frac{12}{3} \rightarrow \boxed{x = \frac{4}{3}}$$

iv) $\begin{cases} \cancel{x} - 2y = -4 \\ \cancel{x} + y = 1 \end{cases} \ominus$

$$-3y = -5 \Rightarrow \boxed{y = \frac{5}{3}} \Rightarrow x + \frac{5}{3} = 1 \Rightarrow \boxed{x = -\frac{2}{3}}$$

New variables:

$$u = x + y$$
$$v = x - 2y.$$

Inverting relations:

$$u = \cancel{x} + y$$
$$v = \cancel{x} - 2y \qquad \ominus$$
$$u - v = 3y \Rightarrow$$

$$\boxed{y = \frac{u - v}{3}}$$

Into first equation:

$$u = x + \left(\frac{u - v}{3}\right)$$

$$\Rightarrow u - \left(\frac{u - v}{3}\right) = x.$$

$$\Rightarrow \boxed{\frac{v + 2u}{3} = x.}$$

Relations between differentials:

$$dx = \frac{1}{3}dv + \frac{2}{3}du.$$

$$dy = \frac{1}{3}du - \frac{1}{3}dv.$$

$$dx \wedge dy = \left(\frac{1}{3}dv + \frac{2}{3}du\right) \wedge \left(\frac{1}{3}du - \frac{1}{3}dv\right)$$

$$= \frac{1}{9}dv \wedge du - \cancel{\frac{1}{9}dv \wedge dv}.$$

0, repeated diff.

$$+ \cancel{\frac{2}{9}du \wedge du} \qquad - \frac{2}{9}du \wedge dv.$$

0, repeated diff.

$$= \left(\frac{1}{9} + \frac{2}{9}\right) dv \wedge du.$$

$$= \frac{1}{3} dv \wedge du.$$

Back to integral:

$$\iint_R 3xy\,dA = \int_1^4 \int_{-4}^0 \cancel{3} \cdot \left(\frac{v+2u}{\cancel{3}}\right)\left(\frac{u-v}{3}\right) \frac{1}{3}\,dv\,du$$

x y

$$\iint_R 3xy\,dA = \int_1^4 \int_{-4}^{0} \frac{1}{9}\left(vu - v^2 + 2u^2 - 2uv\right) dv\,du$$

$$= \int_1^4 \int_{-4}^{0} \left(\frac{2u^2 - v^2 - uv}{9}\right) dv\,du$$

$$= \int_1^4 \left[\frac{2u^2 v}{9} - \frac{v^3}{27} - \frac{uv^2}{18} \Big|_{v=-4}^{v=0} \right] du$$

$$= \int_1^4 \left[-\frac{2 \cdot u^2 \cdot (-4)}{9} + \frac{(-4)^3}{27} + \frac{u \cdot (-4)^2}{18} \right] du$$

$$= \int_1^4 \left[\frac{8u^2}{9} - \frac{64}{27} + \frac{8u}{9} \right] du.$$

$$= \left[\frac{8u^3}{27} - \frac{64u}{27} + \frac{4u^2}{9} \Big|_{u=1}^{u=4} \right]$$

$$= \frac{164}{9}$$