MAT 514 - Lecture 20

Power series representations

If $f : G \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic and $D[c,r] \subset G$. Then f can be represented as a convergent power series in $D[c,r]$:

$$f(z) = \sum_{k \geq 0} a_k (z-c)^k,$$

where the a_k can be computed as

$$a_k = \frac{f^{(k)}(c)}{k!} = \frac{1}{2\pi i} \int_{C[c,r]} \frac{f(w)}{(w-c)^{k+1}} \, dw.$$

Example: Let $c \in \mathbb{C} \setminus \{1\}$. We wish to represent the holomorphic function

$$f(z) = \frac{1}{1-z}, \quad z \neq 1$$

as a power series centered at c. We will do this by a change of variables, $w = z - c$.

$$\frac{1}{1-z} = \frac{1}{1-c-(z-c)} = \frac{1}{(1-c)-w}$$

$$= \frac{1/(1-c)}{1-\left(\frac{w}{1-c}\right)} \quad (*)$$

So long as

$$\left|\frac{w}{1-c}\right| < 1,$$

we can express $(*)$ as a geometric series

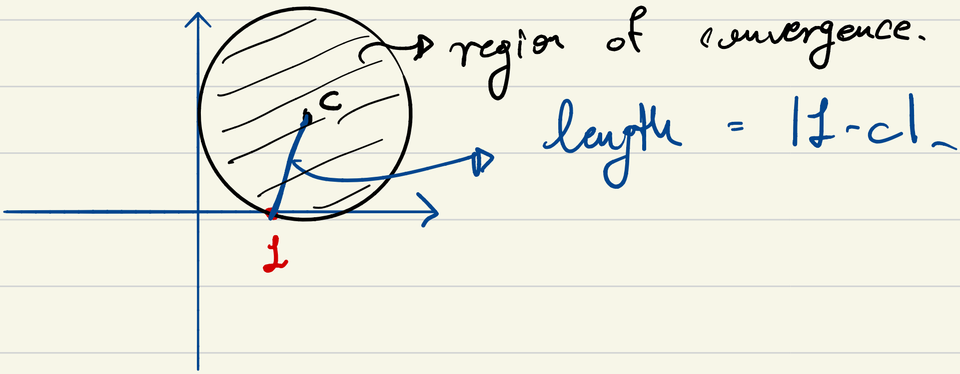
$$\frac{1/(1-c)}{1-w/(1-c)} = \sum_{k\geq 0} \frac{1}{(1-c)}\left(\frac{w}{1-c}\right)^k$$

$$= \sum_{k\geq 0} \frac{1}{(1-c)^{k+1}} \cdot (z-c)^k.$$

<u>Convergence condition</u>

$$\left|\frac{z-c}{1-c}\right| < 1$$

$$\Rightarrow \quad |z-c| < |1-c|.$$



Direct methods, involving either computation of derivatives

$$a_k = \frac{f^{(k)}(c)}{k!}$$

or integrals

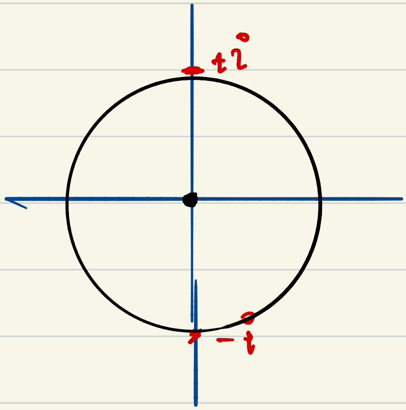
$$a_k = \frac{1}{2\pi i} \int_{C[c,r]} \frac{f(w)}{(w-c)^{k+1}}\, dw.$$

<u>Example</u>: Consider the holomorphic function

$$f(z) = \frac{1}{1+z^2},$$

defined for $z \neq \pm i$.

We will express f as a power series with center $c = 0$.



Use a substitution $w = -z^2$. Then

$$\frac{1}{1+z^2} = \frac{1}{1-w}$$

$$= \sum_{k \geq 0}' w^k \text{, so long as } |w| < 1$$

$$= \sum_{k \geq 0}' (-z^2)^k$$

$$= \sum_{k \geq 0}' (-1)^k \cdot z^{2k}$$

If we apply the root test, the radius

of convergence is 1.

Example: We wish to express sin(z) as a power series with center $c=\pi$. The key point is periodicity of derivatives:

$$\sin^{(1)}(z) = \cos(z)$$
$$\sin^{(2)}(z) = -\sin(z)$$
$$\sin^{(3)}(z) = -\cos(z)$$
$$\sin^{(4)}(z) = \sin(z).$$

and repeats every four iterations.

At $c=\pi$, the values are

$$\sin^{(4k)}(\pi) = \sin(\pi) = 0$$
$$\sin^{(4k+1)}(\pi) = \cos(\pi) = -1$$
$$\sin^{(4k+2)}(\pi) = -\sin(\pi) = 0$$
$$\sin^{(4k+3)}(\pi) = -\cos(\pi) = 1.$$

Hence the power series representation is

$$\sin(z) = 0 - \frac{1}{1!}(z-c) + 0 + \frac{1}{3!}(z-c)^3 - 0 - \frac{1}{5!}(z-c)^5 + 0 + \frac{1}{7!}(z-c)^7$$

$$\sin(z) = \sum_{k\geq 0}' \frac{(-1)^{k-1} (z-c)^{2k+1}}{(2k+1)!},$$

for all z.

Algebraic properties of power series

1) Suppose we are given two power series reps,
$f(z) = \sum_{k\geq 0}' a_k (z-c)^k$, $g(z) = \sum_{k\geq 0}' b_k (z-c)^k$,
with radii of convergence R_1 and R_2. Then $(f+g)(z)$ can be represented as a power series on $D[c, R]$, where

$$R = \min\{R_1, R_2\},$$

as

$$(f+g)(z) = \sum_{k\geq 0}' (a_k + b_k)(z-c)^k$$

2) If we are given power series as

above, then f_g can be represented as a power series on $D[c, R]$, by

$$f_g(z) = \sum_{k \geq 0} m_k (z-c)^k,$$

where

$$m_k = \sum_{p=0}^{k} a_p \cdot b_{k-p}$$

For instance,

$$m_0 = a_0 b_0$$

$$m_1 = a_0 b_1 + a_1 b_0$$

$$m_2 = a_0 b_2 + a_1 b_1 + a_2 b_0.$$

$$\vdots$$

Analytic operations

3) If a function $f(z)$ has representation

$$f(z) = \sum_{k \geq 0} a_k (z-c)^k$$

on $D[c, R]$, then its derivative has representation

$$f'(z) = \sum_{k\geq 1}' k a_k (z-c)^{k-1}$$

Downshift-upshift trick.

$$= \sum_{k\geq 0}' (k+1) a_{k+1} (z-c)^k,$$

converging on $D[c, R]$.

4) Suppose f has a power series representation

$$f(z) = \sum_{k\geq 0}' a_k (z-c)^k$$

on $D[c, R]$. Then its antiderivatives take the form

$$F(z) = C + \sum_{k\geq 0}' \frac{a_k}{k+1} (z-c)^{k+1}.$$

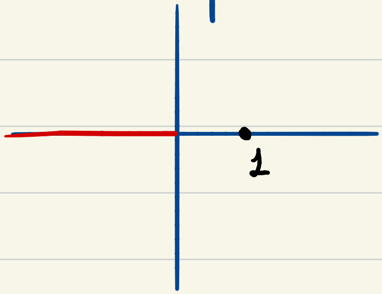
Downshift-upshift

$$= C + \sum_{k\geq 1}' \frac{a_{k-1}}{k} (z-c)^k,$$

on $D[c, R]$.

__Example:__ Consider the function
$$f(z) = \frac{z}{1+z}, \quad z \neq -1.$$
To obtain its power series representation with center 0, we may take the product representation of the components:
$$z = z, \quad \text{on } \mathbb{C} = \text{"}D[0,\infty]\text{"}$$
$$\frac{1}{1+z} = \sum_{k\geq 0} (-1)^k z^k, \quad \text{on } D[0,1].$$
Hence
$$\frac{z}{1+z} = \sum_{k\geq 0} (-1)^k z^{k+1}, \quad \text{on } D[0,1].$$

__Example:__ We wish to obtain a power series representation for $\mathrm{Log}(z)$, a function defined on $\mathbb{C}-\mathbb{R}_{\leq 0}$



We will choose as a center $c=1$, which means we expect our representation

to have radius of convergence 1.
We will do this indirectly, by computing a representation for

$$g(z) = \frac{1}{z} \quad , \quad z \neq 0$$

on $D[1,1]$. Use a variable $w = (z-1)$, so

$$\frac{1}{z} = \frac{1}{z-1+1} = \frac{1}{1+w}.$$

$$= \sum_{k\geq 0}' (-w)^k$$

$$= \sum_{k\geq 0}' (-1)^k w^k$$

$$= \sum_{k\geq 0}' (-1)^k (z-1)^k, \text{ for } |z-1| < 1.$$

Term-by-term integration leads to an antiderivative

$$G(z) = \sum_{k\geq 0}' (-1)^k \int (z-1)^k dz$$

$$G(z) = \sum_{k \geq 0}' \frac{(-1)^k (z-1)^{k+1}}{k+1}$$

We need to compare $\mathrm{Log}(z)$ and $G(z)$. They differ at most by a constant, so we can test their values at $z=1$.

$\mathrm{Log}(1) = 0$, and $\displaystyle\sum_{k \geq 0}' \frac{(-1)^k (1-1)^{k+1}}{k+1} = 0$,

Therefore,

$$\mathrm{Log}(z) = \sum_{k \geq 0}' \frac{(-1)^k (z-1)^{k+1}}{(k+1)}$$

$$= \sum_{k \geq 1}' \frac{(-1)^{k-1} (z-1)^k}{k}$$

Downshift_ Upshift

for $z \in D[1,1]$.