MAT 132 **Name (Print):** ____________________
Summer II 2017
Final Exam
08/17/17
Time Limit: 3 hours and 5 minutes ID number ____________________

Instructions

- This exam contains 12 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.
- You may *not* use your books, notes, or any device that is capable of accessing the internet on this exam (e.g., smartphones, smartwatches, tablets). You may use a scientific calculator.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. In each of the following problems, determine if the sequences are convergent or divergent. If the sequences converge, find their limits.

   (a) (5 points) $a_n = (-1)^n n$

   (b) (5 points) $a_n = \frac{2n-3}{3n^2+4}$.

(c) (5 points) $a_1 = 1$, $a_{n+1} = 4 - a_n$, for $n \geq 1$.

(d) (5 points) $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$, for $n \geq 1$.

2. A population P grows with respect to time t according to the following differential equation.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)\left(1 - \frac{200}{P}\right),$$

where $P(t)$ is assumed to be positive, for all values of t.

This model corresponds to logistic growth with: a maximum environmental capacity M a minimum population m, below which reproduction is ineffective and the population tends to become extinct.

(a) (2 points) Find the equilibrium solutions of this equation.

(b) (3 points) Sketch the direction field for this equation.

(c) (5 points) Describe the behavior of the solution to this differential equation with initial condition $P(0) = 100$ as $t \to \infty$. Explain your reasoning.

(d) (5 points) Describe the behavior of the solution to this differential equation with initial condition $P(0) = 500$ as $t \to \infty$. Explain your reasoning.

(e) (5 points) Describe the behavior of the solution to this differential equation with initial condition $P(0) = 1500$ as $t \to \infty$. Explain your reasoning.

3. A family of curves is given, in terms of a real parameter c, by the following equations:

$$x^2 + y^2 = 2cx.$$

(a) (3 points) Describe geometrically the elements in this family, for each value of c. (Hint: consider the case $c = 0$ separately).

(b) (3 points) Compute the slope of an element in this family at a point (x, y) in the plane such that: $y \neq 0$, $x \neq 0$, and $x \neq \pm y$. The slope should be written in terms of the variables x and y, but not the parameter c (use the equation of the family to write c in terms of x and y).

(c) (4 points) Set up the differential equation for the orthogonal trajectories of this family, at a point (x, y) such that $x \neq 0$, $y \neq 0$, $x \neq \pm y$. Once this step is done, use the substitution $z = \frac{y}{x}$ to simplify this differential equation, by writing a new equation involving z and x only.

(d) (10 points) Find the orthogonal trajectory to the family through the point $(3, 4)$ by solving the equation found in part (c) for the variable z. Remember to go back to the original variables x, y once the solution to the differential equation is found. Hint: compute the constant of integration only after writing the solution in the original variables x and y.

4. In this problem we will use Taylor polynomials to estimate the value of the function $f(x) = x^2 sin(x)$ at the point $x = \frac{\pi}{6}$, with error less than 0.0001.

   (a) (5 points) Write down the Maclaurin series for this function. What is its radius of convergence?

   (b) (5 points) Find upper bounds M_n for the $n-th$ derivatives in the interval $0 \leq x \leq \frac{\pi}{6}$. Your upper bounds do not need to be the maximum of the corresponding derivatives.

(c) (10 points) Find the least order n of the Taylor polynomial of f at 0 which approximates $f(\frac{\pi}{6})$, with error less than 0.0001.

5. A solid is obtained by rotating the region bounded by the curves $y = x \arctan(x^2)$, $x = 1$, $y = 0$ around the y-axis.

   (a) (5 points) Write an integral that computes the volume of the solid using cylindrical shells. **Do not try to evaluate the integral.**

   (b) (5 points) Compute the Maclaurin series of the above integrand.

(c) (5 points) Compute the radius of convergence of the Maclaurin series from part (b).

(d) (5 points) Express the integral from part (a) as a numerical series, by integrating the series on part (b) between the appropriate endpoints. **Do not try to evaluate the numerical series.**