NOTES ON LINEAR, FIRST-ORDER DIFFERENTIAL EQUATIONS.

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1. Introduction

A linear, first-order differential equation is an equation of type

$$y'(x) + P(x)y(x) = Q(x), \tag{1.1}$$

where the variable is the function $y(x)$. Throughout this course, we have seen how to solve the simplest types of such equations, as the examples below show.

Example 1.1. *Consider the case when $P(x) = 0$. Such equations were studied in the first week of the course. Here are a few examples,*

$$\begin{aligned}
y'(x) &= 0,\\
y'(x) &= 2,\\
y'(x) &= x^3,\\
y'(x) &= \cos(x),\\
y'(x) &= e^x,\\
y'(x) &= \frac{1}{1+x^2}.
\end{aligned}$$

These equations can all be solved by direct integration, according to the fundamental theorem of Calculus.

Example 1.2. *Consider the case when $Q(x) = 0$. Such equations fall into the category of separable equations, whose solutions we learned this week. Here are a few examples,*

$$\begin{aligned}
y'(x) + xy(x) &= 0,\\
y'(x) + e^{-x}y(x) &= 0,\\
y'(x) - \frac{\ln(x)}{x}y(x) &= 0,\\
y'(x) - \frac{y(x)}{1+x^2} &= 0.
\end{aligned}$$

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2. Integrating Factors

Our goal is to solve a general linear first-order equation, that is, without the assumption that either $P(x)$ or $Q(x)$ is zero. The method we use to solve such equations is called the *Method of Integrating Factors.*

Let's examine the structure of a general equation,

$$y'(x) + P(x)y(x) = Q(x),$$

The left-hand side contains two terms, one involving y, the other involving y'. This resembles the shape of the product rule, but it is not quite the same. The integrating factor of the equation is a function $\mu(x)$ that, when multiplied by the left-hand side, yields the derivative of the product $\mu(x)y(x)$, that is

$$(\mu(x)y(x))' = \mu(x)y'(x) + \mu(x)P(x)y(x).$$

See the examples below.

Example 2.1. *Consider the differential equation*

$$y'(x) + y(x) = 3. \tag{2.1}$$

We wish to find a function $\mu(x)$ satifying the equation

$$(\mu(x)y(x))' = \mu(x)y'(x) + \mu(x)y(x).$$

Applying the product rule to the left-hand side, this turns into

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)y'(x) + \mu(x)y(x),$$

which we can simplify to

$$\mu'(x)y(x) = \mu(x)y(x).$$

One way to solve this last equation is to find a function $\mu(x)$ such that

$$\mu'(x) = \mu(x).$$

This is the Integrating Factor Equation for this problem. A simple solution is well-known, $\mu(x) = e^x$. This is an integrating factor for this differential equation.

Example 2.2. *Consider the equation*

$$y'(x) + 3y(x) = x. \tag{2.2}$$

We wish to find a function $\mu(x)$ satifying the equation

$$(\mu(x)y(x))' = \mu(x)y'(x) + 3\mu(x)y(x).$$

Applying the product rule to the left-hand side, this turns into

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)y'(x) + 3\mu(x)y(x),$$

which we can simplify to

$$\mu'(x)y(x) = 3\mu(x)y(x).$$

One way to solve this last equation is to find a function $\mu(x)$ such that

$$\mu'(x) = 3\mu(x).$$

This is the Integrating Factor Equation for this problem. One solution is $\mu(x) = e^{3x}$*. This is an integrating factor for this problem.*

Example 2.3. *Consider the equation*

$$y'(x) + 2y(x) = \cos(x). \tag{2.3}$$

We wish to find a function $\mu(x)$ *satifying the equation*

$$(\mu(x)y(x))' = \mu(x)y'(x) + 2\mu(x)y(x).$$

Applying the product rule to the left-hand side, this turns into

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)y'(x) + 2\mu(x)y(x),$$

which we can simplify to

$$\mu'(x)y(x) = 2\mu(x)y(x).$$

One way to solve this last equation is to find a function $\mu(x)$ *such that*

$$\mu'(x) = 2\mu(x).$$

An integrating factor is given by $\mu(x) = e^{2x}$*.*

In all the examples we've seen so far, the function $P(x)$ was a constant. The next two examples are a bit more challeging.

Example 2.4. *Consider the equation*

$$y'(x) + 2xy(x) = e^{-x^2}. \tag{2.4}$$

We wish to find a function $\mu(x)$ *satifying the equation*

$$(\mu(x)y(x))' = \mu(x)y'(x) + 2x\mu(x)y(x).$$

Applying the product rule to the left-hand side, this turns into

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)y'(x) + 2x\mu(x)y(x),$$

which we can simplify to

$$\mu'(x)y(x) = 2x\mu(x)y(x).$$

One way to solve this last equation is to find a function $\mu(x)$ *such that*

$$\mu'(x) = 2x\mu(x).$$

This is a separable equation, whose solution, the integrating factor for our problem, is $y(x) = e^{x^2}$*.*

Example 2.5. *Consider the equation*

$$y'(x) + \frac{1}{x}y(x) = \ln(x), \tag{2.5}$$

where $x > 0$*.*

We wish to find a function $\mu(x)$ *satifying the equation*

$$(\mu(x)y(x))' = \mu(x)y'(x) + \frac{1}{x}\mu(x)y(x).$$

Applying the product rule to the left-hand side, this turns into

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)y'(x) + \frac{1}{x}\mu(x)y(x),$$

which we can simplify to

$$\mu'(x)y(x) = \frac{1}{x}\mu(x)y(x).$$

One way to solve this last equation is to find a function $\mu(x)$ such that

$$\mu'(x) = \frac{1}{x}\mu(x).$$

This is a separable equation, whose solution, the integrating factor for our problem, is $y(x) = e^{\ln(x)} = x$.

By examining the examples above more closely, we find that for the standard equation

$$y'(x) + P(x)y(x) = Q(x),$$

the integrating factor is

$$\mu(x) = e^{\int P(x)dx}. \tag{2.6}$$

3. Solving linear, first-order equations

In this section we return to the examples from section 2, to find solutions of the equations.

Example 3.1. *In example 2.1,*

$$y'(x) + y(x) = 3,$$

we found the integrating factor $\mu(x) = e^x$. Multiplying the equation by this factor, we obtain

$$(e^x y(x))' = 3e^x.$$

Integrating both sides, we have

$$e^x y(x) = \int 3e^x \, dx = 3e^x + C. \tag{3.1}$$

Solving this equation for $y(x)$, we have

$$y(x) = 3 + Ce^{-x}.$$

Example 3.2. *In example 2.2,*

$$y'(x) + 3y(x) = x,$$

we found the integrating factor $\mu(x) = e^{3x}$. Multiplying the equation by this factor, we obtain

$$(e^{3x}y(x))' = xe^{3x}.$$

Integrating both sides, we have

$$e^{3x}y(x) = \int xe^{3x} \, dx. \tag{3.2}$$

The integral in the right-hand side can be computed by integration by parts,

$$\begin{aligned}\int xe^{3x}\ dx &= \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}\ dx \\ &= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C.\end{aligned}$$

It follows from equation (3.2) *that*

$$\begin{aligned}e^{3x}y(x) &= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C \\ y(x) &= \frac{x}{3} - \frac{1}{9} + Ce^{-3x}.\end{aligned}$$

Example 3.3. *In example 2.3,*

$$y'(x) + 2y(x) = \cos(x),$$

we found the integrating factor $\mu(x) = e^{2x}$. *Once we multiply the equation by this factor, it becomes*

$$(e^{2x}y(x))' = e^{2x}\cos(x).$$

Integrating both sides, we obtain

$$e^{2x}y(x) = \int e^{2x}\cos(x)\ dx. \tag{3.3}$$

The integral on the right-hand side can be computed by Integration by parts,

$$\begin{aligned}\int e^{2x}\cos(x)\ dx &= e^{2x}\sin(x) - \int 2e^{2x}\sin(x)\ dx \\ &= e^{2x}\sin(x) - 2\left(-e^{2x}\cos(x) + 2\int e^{2x}\cos(x)\ dx\right).\end{aligned}$$

Solving this equation for the desired integral, we have

$$\int e^{2x}\cos(x)\ dx = \frac{e^{2x}(\sin(x) + 2\cos(x))}{5} + C.$$

Going back to equation (3.3), *we find*

$$\begin{aligned}e^{2x}y(x) &= \frac{e^{2x}(\sin(x) + 2\cos(x))}{5} + C \\ y(x) &= \frac{\sin(x) + 2\cos(x)}{5} + Ce^{-2x}.\end{aligned}$$

Example 3.4. *In example 2.4,*

$$y'(x) + 2xy(x) = e^{-x^2},$$

we found the integrating factor $\mu(x) = e^{x^2}$. *Multiplying the equation by this factor, we obtain*

$$(e^{x^2}y(x))' = 1.$$

Integrating both sides,

$$\begin{aligned} e^{x^2} y(x) &= x + C \\ y(x) &= x e^{-x^2} + C e^{-x^2}. \end{aligned}$$

Example 3.5. *In example 2.5,*

$$y'(x) + \frac{1}{x} y(x) = \ln(x),$$

$(x > 0)$, we obtained the integrating factor $\mu(x) = x$. Multiplying the equation by this factor yields,

$$(xy(x))' = x \ln(x).$$

Integrating both sides, we have

$$xy(x) = \int x \ln(x)\ dx. \tag{3.4}$$

The integral on the right-hand side can be computed by integration by parts,

$$\begin{aligned} \int x \ln(x)\ dx &= \frac{x^2 \ln(x)}{2} - \int \frac{x^2}{2} \frac{1}{x}\ dx \\ &= \frac{x^2 \ln(x)}{2} - \int \frac{x}{2}\ dx \\ &= \frac{x^2 \ln(x)}{2} - \frac{x^2}{4} + C. \end{aligned}$$

It follows that

$$\begin{aligned} xy(x) &= \frac{x^2 \ln(x)}{2} - \frac{x^2}{4} + C \\ y(x) &= \frac{x \ln(x)}{2} - \frac{x}{4} + \frac{C}{x}, \end{aligned}$$

for $x > 0$.

In general, the solution to the standard equation

$$y'(x) + P(x) y(x) = Q(x)$$

can be described in terms of the integrating factor,

$$y(x) = \frac{\int \mu(x) Q(x)\ dx}{\mu(x)}. \tag{3.5}$$