Solutions to Homework 3

Problem 3.44(d) Determine all complex numbers for which the function $f(z) = \exp(\overline{z})$ is holomorphic.

Solution: Let us write this function in terms of the real and imaginary parts of z,

$$\begin{aligned} f(x+iy) &= \exp(x-iy) \\ &= e^x[\cos(-y) + i\sin(-y)] \\ &= e^x[\cos(y) - i\sin(y)], \end{aligned}$$

thus f has real and imaginary parts

$$u(x,y) = e^x\cos(y),\ v(x,y) = -e^x\sin(y),$$

respecively. The Cauchy-Riemann equations are thus

$$\begin{aligned} e^x\cos(y) &= -e^x\cos(y) \\ -e^x\sin(y) &= e^x\cos(y). \end{aligned}$$

This set of equations has no solutions, thus f is not complex-differentiable anywhere.

Problem 3.45(b) Find all solutions to the equation

$$\mathrm{Log}(z) = \frac{3\pi i}{2}$$

Solution: This equation has no solutions, as the argument of the principal branch of the logarithm is constrained between $-\pi$ and π.

Problem 4.3 Integrate the function $f(z) = \overline{z}$ over the paths given below

(a) $\gamma(t) = t + it$, $0 \le t \le 1$.

(b) $\gamma(t) = t + it^2$, $0 \le t \le 1$.

(c) The path ξ, juxtaposition of the paths $\gamma_1(t) = t$, $0 \leq t \leq 1$, and $\gamma_2(t) = 1 + it$, $0 \leq t \leq 1$.

Solution:

(a) The velocity vector of the path is $\gamma'(t) = 1 + i$, hence the integral is

$$\begin{aligned}
\int_\gamma \overline{z}\, dz &= \int_0^1 (t - it)(1 + i)\, dt \\
&= \int_0^1 (t + it - it - i^2 t)\, dt \\
&= \int_0^1 2t\, dt \\
&= 1
\end{aligned}$$

(b) The velocity vector of the path is $\gamma'(t) = 1 + 2it$, thus the integral is

$$\begin{aligned}
\int_\gamma \overline{z}\, dz &= \int_0^1 (t - it^2)(1 + 2it)\, dt \\
&= \int_0^1 (t + 2it^2 - it^2 - 2i^2 t^3)\, dt \\
&= \frac{t^2}{2} + \frac{it^3}{3} + \frac{t^4}{2}\bigg|_{t=0}^{t=1} \\
&= 1 + \frac{i}{3}.
\end{aligned}$$

(c) This integral is split the sum of the integrals in the two paths, whose respective derivatives are $\gamma_1'(t) = 1$ and $\gamma_2'(t) = i$.

$$\begin{aligned}
\int_\xi \overline{z}\, dz &= \int_{\gamma_1} \overline{z}\, dz + \int_{\gamma_2} \overline{z}\, dz \\
&= \int_0^1 t\, dt + \int_0^1 (1 - it)i\, dt \\
&= \int_0^1 (2t + i)\, dt \\
&= t^2 + it\bigg|_{t=0}^{t=1} \\
&= 1 + i.
\end{aligned}$$

Problem 4.4 Compute $\int_\gamma \frac{dz}{z}$, where γ is the unit circle oriented counterclockwise. More generally, show that for any $w \in \mathbb{C}$ and $r > 0$,

$$\int_{C[w,r]} \frac{dz}{w - z} = 2\pi i.$$

Solution: Let us parametrize the unit circle by $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, with velocity $\gamma'(t) = ie^{it}$. The first integral may be evaluated as

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = \int_0^1 i\,dt = 2\pi i.$$

More generally, consider the circle of center w and radius r, oriented counterclockwise, parametrized by $\xi(t) = w + re^{it}$, for $0 \le t \le 2\pi$, with velocity $\xi'(t) = rie^{it}$. The second integral may be evaluated as

$$\int_{C[w,r]} \frac{dz}{w - z} = \int_0^{2\pi} \frac{rie^{it}}{w + re^{it} - w} = dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

Problem 4.10 Prove the following integration by parts statement: let f and g be holomorphic in G, and suppose $\gamma \subset G$ is a piecewise smooth path from $\gamma(a)$ to $\gamma(b)$. Then

$$\int_\gamma fg' = f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a)) - \int_\gamma f'g$$

Solution: If f and g are holomorphic, so is their product fg, whose derivative is

$$(fg)' = f'g + fg'.$$

Said it another way, (fg) is an antiderivative of $(f'g + fg')$. Applying theorem 4.11 to $(f'g + fg')$, we obtain the desired conclusion,

$$\int_\gamma f'g + fg' = (fg)(\gamma(b)) - (fg)(\gamma(a))$$
$$\int_\gamma fg' = f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a)) - \int_\gamma f'g.$$