MAT 514 - Lecture 13

Cauchy's Theorem and Cauchy's Integral Formula.

Theorem (Cauchy - Goursat)

Let $f: G \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Let γ be a simple closed path contained in G, which is G-contractible. Then

$$\int_\gamma f \, dz = 0.$$

Other formulation:

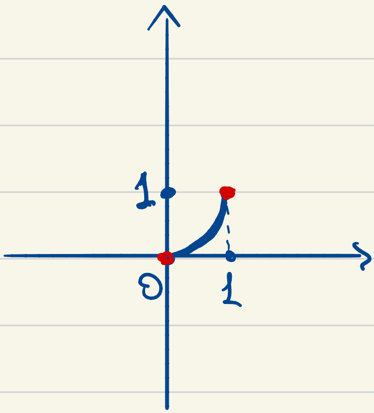
Let f be as above, and γ, ξ be G-homotopic curves (with the same orientation). Then

$$\int_\gamma f \, dz = \int_\xi f \, dz.$$

Remark: in the second formulation the paths don't need to closed, but just have the same starting and endpoint.

<u>Examples:</u> Let's integrate $f(z) = z^2$ along the path

$$\gamma(t) = t + it^3 \quad , \quad 0 \le t \le 1.$$



From the definition

$$\int_\gamma z^2\,dz = \int_0^1 (t + it^3)^2 \cdot (1 + 3t^2 i)\,dt$$

$$= \int_0^1 (t^2 + 2t^4 i + (it^3)^2)(1 + 3t^2 i)\,dt$$

$$= \int_0^1 (t^2 + 2t^4 i - t^6)(1 + 3t^2 i)\,dt$$

$$= \int_0^1 t^2 + 3t^4 i + 2t^4 i + 6t^6 i^2 - t^6 - 3t^8 i\,dt$$

$$= \int_0^1 t^2 + 5t^4 i - 7t^6 - 3t^8 i\,dt.$$

$$= \frac{t^3}{3} + t^5 i - t^7 - \frac{t^9}{8} i \Big|_{t=0}^{t=1}.$$

$$\int_\gamma z^2\,dz = \frac{1}{3} + i - 1 - \frac{1}{3}i$$
$$= \frac{-2}{3} + \frac{2i}{3}.$$

Alternatively, we can integrate along
$$\xi(t) = (1+i)t,$$
$0 \le t \le 1$. This path has velocity
$$\xi'(t) = 1+i;$$
so the integral is

$$\int_\xi z^2\,dz = \int_0^1 [(1+i)t]^2 \cdot (1+i)\,dt.$$
$$= \int_0^1 (1+i)^3 t^2\,dt.$$
$$= (1+i)^3 \frac{t^3}{3}\Big|_{t=0}^{t=1}.$$
$$= \frac{(1+i)^3}{3}$$

$$\int_{\xi} z^2 dz = \frac{(1 + 3i + 3i^2 + i^3)}{3}$$

$$= \frac{1 + 3i - 3 - i}{3}$$

$$= -\frac{2}{3} + \frac{2i}{3}.$$

Cauchy's Integral Formula

Recall that

$$\int_{C[w,r]} \frac{1}{w - z} dz = 2\pi i.$$

Remark: Convention is that $C[w,r]$ is the circle with center w and radius r, traversed once, in counterclockwise sense.

Cauchy's Formula: Suppose that $f: G \to \mathbb{C}$ is holomorphic, and that the disk

$$D[w, r]$$

is entirely contained in the interior of G. Then

$$\boxed{f(w) = \frac{1}{2\pi i} \int_{C[w,r]} \frac{f(z)}{w-z}\, dz.}$$

This is a mean-value formula: it states that the value of f at the center is the mean of its values along the boundary circle. This reflects rigidity of holomorphic functions.

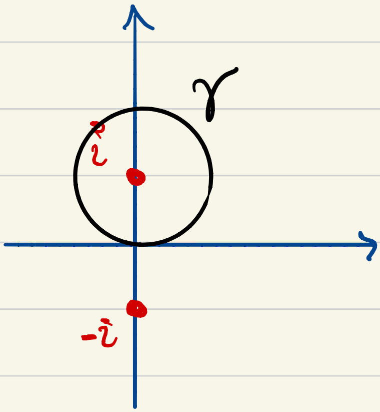
Example: $$\int_{C[i,1]} \frac{1}{z^2+1}\, dz = \int_{C[i,1]} \frac{1}{(z+i)(z-i)}\, dz.$$

$$= \int_{C[i,1]} \frac{\left[1/(z+i)\right]}{z-i}\, dz.$$

The numerator

$$f(z) = \frac{1}{z+i}$$

is holomorphic on $\mathbb{C} \setminus \{-i\}$. In particular, it is holomorphic on $D[i,1]$.



Cauchy's formula then states

$$\frac{1}{2\pi i} \int_{C[i,1]} \frac{f(z)}{z-i}\, dz = f(i)$$

$$= \frac{1}{i+i}$$

$$= \frac{1}{2i}$$

$$= -\frac{i}{2}.$$

So

$$\int_{C[i,1]} \frac{1}{z^2+1}\, dz = 2\pi i \left(-\frac{i}{2}\right) = \pi.$$

Example: $\displaystyle\int_{C[0,3]} \frac{e^z}{z^2-2z}\,dz.$

To apply Cauchy's formula directly, the denominator should be z.

$$\frac{e^z}{z^2-2z} = \frac{e^z}{z(z-2)}$$

$$= \frac{[e^z/(z-2)]}{z}.$$

This trick will not work this time, since the numerator is singular at $z=2$, a point within the disk $D[0,3]$.

Let's try instead to use partial fractions:

$$\frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$$= \frac{A(z-2) + Bz}{z(z-2)}$$

$$= \frac{(A+B)z - 2A}{z(z-2)}.$$

Thus A and B satisfy:

$$\begin{cases} A+B=0 \\ -2A=1 \end{cases} \Rightarrow \quad A=-\frac{1}{2}, \quad B=\frac{1}{2}.$$

In other words,

$$\frac{1}{z(z-2)} = -\frac{1}{2z} + \frac{1}{2(z-2)} \Rightarrow$$

$$\frac{e^z}{z(z-2)} = \frac{-e^z}{2z} + \frac{e^z}{2(z-2)}.$$

Therefore,

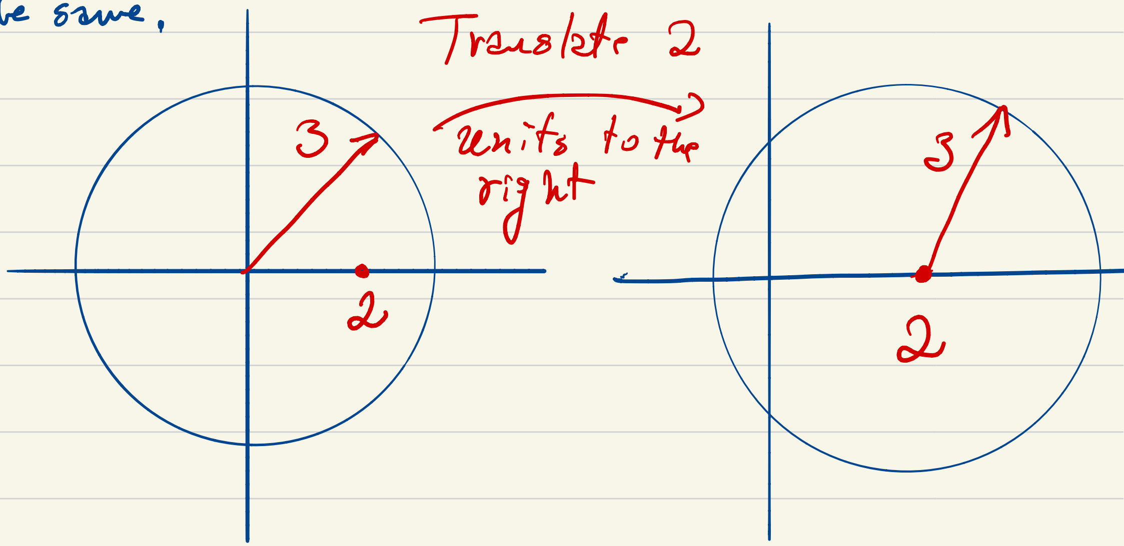
$$\int_{C[0,3]} \frac{e^z}{z(z-2)}\,dz = \frac{-1}{2}\int_{C[0,3]} \frac{e^z}{z}\,dz + \frac{1}{2}\int_{C[0,3]} \frac{e^z}{z-2}\,dz.$$

Cauchy's Formula deals with the first integral on the right:

$$\int_{C[0,3]} \frac{e^z}{z}\,dz = 2\pi i \cdot e^0 = 2\pi i.$$

In the second integral, the singularity

and the center of the circle are not the same.



We can deform by a $(\mathbb{C}\setminus\{2\})$-homotopy the circle

$$C[0,3]$$

into

$$C[2,3].$$

via translations. Hence

$$\int_{C[0,3]} \frac{e^z}{z-2}\,dz = \int_{C[2,3]} \frac{e^z}{z-2}\,dz.$$

For the integral

$$\int_{C[2,3]} \frac{e^z}{z-2}\,dz$$

we may use Cauchy's Formula:

$$\int_{C[2,3]} \frac{e^z}{z-2}\,dz = (2\pi i)\cdot e^2$$

Combining it all:

$$\int_{C[0,3]} \frac{e^z}{z^2-2z}\,dz = -\frac{1}{2}\cdot(2\pi i) + \frac{1}{2}(2\pi i\cdot e^2)$$

$$= -\pi i + e^2\pi i$$

$$= \pi i(e^2-1).$$