MAT 514 - Lecture 21

- Singular functions

We've seen that holomorphic functions can be expressed as a convergent power series near a regular point. This representation works until you hit a singularity.

Question: What can we say about a holomorphic function near a singular point.

Singularities come in various types:

1) Isolated: apart from the singular point all other points in a neighborhood are regular

- $f(z) = 1/z$, has an isolated singularity at $z=0$
- $\tan(z)$ has isolated singularities at points of the form

$$z = \frac{\pi}{2}(2k+1), \quad k \in \mathbb{Z}.$$

2) Non-isolated: all non-positive real numbers are singularities for $\log(z)$.

In this course, we will study only isolated singularities.

Riemann's Removable Singularity Theorem

Suppose that f is a holomorphic function on a punctured disk,

$$D^*[c, r] = \{ z \in \mathbb{C} \mid 0 < |z - c| < r \}.$$

If f is bounded on $D^*[c, r]$, then it has a holomorphic extension to $D[c, r]$.

Corollary: True singularities occur when the function is unbounded.

Examples:

1. $f(z) = \dfrac{e^z}{z}$ has a presumptive singularity

at $z=0$.

This is a non-removable singularity.

In a punctured disk centered at 0, $D^*[0,r]$,

the numerator, e^z is bounded by e^r.

Since the denominator goes to 0 as $z \to 0$, the fraction is unbounded.

② $f(z) = \dfrac{\sin(z)}{z}$ this has a presumptive singularity at $z=0$.

This is an apparent singularity.

i) $\lim_{z \to 0} \dfrac{\sin(z)}{z} = 1$, therefore near $z=0$, the function $f(z)$ is bounded, therefore by Riemann's Theorem, $z=0$ is a removable singularity.

ii) Recall the Taylor series for $\sin(z)$:

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$\sin(z) = z \cdot \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots\right)$$

$$\frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}$$

In particular as $z \to 0$, $\sin(z)/z$ is bounded and converges to 1.

③ $\frac{\sin(z)}{z^2}$ has a presumptive singularity at $z = 0$.

This is a true singularity. The limit

$$\lim_{z \to 0} \frac{\sin(z)}{z^2} = \infty,$$

after all

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$\frac{\sin(z)}{z^2} = \underset{\text{diverges to } \infty}{\frac{1}{z}} - \underbrace{\frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + \dots}_{\text{converge to } 0}$$

Exercise: Verify that e^z/z is singular at 0 by means of a power series expansion.

Solution: The Taylor series of e^z is

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

so e^z/z can be written as

$$\frac{1}{z} + \underbrace{1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots}_{\text{regular part, converging to } 1 \text{ as } z \to 0}$$

singular part, diverging to ∞ as $\to 0$.

Definition: Suppose a holomorphic function has a zero at a point c, and suppose it has a power series representation around c,

$$f(z) = 0 + a_1(z-c) + a_2(z-c)^2 + a_3(z-c)^3 + \cdots$$

The order of vanishing at c is the least index k for which a_k is not 0.

Examples:

1. $z^2 - 2z + 1$ has a zero of order 2 at $z = 1$

$$(z^2 - 2z + 1) = (z - 1)^2$$

2. $\dfrac{\sin(z^4)}{z}$ has a zero of order 3 at $z = 0$.

$$\sin(w) = w - \frac{w^3}{3!} + \frac{w^5}{5!} - \frac{w^7}{7!} + \cdots$$

if $w = z^4$,

$$\sin(z^4) = z^4 - \frac{z^{12}}{3!} + \frac{z^{20}}{5!} - \frac{z^{28}}{7!} + \cdots$$

and

$$\frac{\sin(z^4)}{z} = z^{3} - \frac{z^{11}}{3!} + \frac{z^{19}}{5!} - \frac{z^{27}}{7!} + \cdots$$

first exponent with non-zero coefficient.

<u>Exercise</u>: the function
$$f(z) = \sin(z) - \tan(z)$$
has a zero at $z=0$. Compute its order.

<u>Hint</u>: Compute Taylor coefficients via derivatives until you find a non-zero coefficient.

<u>Solution</u>: The first-order Taylor coefficient is
$$a_1 = \frac{f^{(1)}(0)}{1!}$$
$$\begin{aligned} a_1 &= \sin^{(1)}(0) - \tan^{(1)}(0) \\ &= \cos(0) - \sec^2(0) \\ &= 1 - 1 \\ &= 0. \end{aligned}$$
The second-order Taylor coefficient is
$$a_2 = \frac{f^{(2)}(0)}{2!}$$
$$= \frac{\sin^{(2)}(0) - \tan^{(2)}(0)}{2}$$

$$a_2 = \frac{-\sin(0) \quad -(\sec^2)^{(1)}(0)}{2}$$

$$= -\frac{2\sec(0)\cdot\sec(0)\cdot\cancel{\tan(0)}^{\,0}}{2}$$

$$= 0$$

The next coefficient is

$$a_3 = \frac{f^{(3)}(0)}{3!}$$

$$= \frac{\sin^{(3)}(0) \quad - \tan^{(3)}(0)}{6}$$

$$= \frac{-\cos(0) \quad -[2\sec^2\cdot\tan]'(0)}{6}$$

$$= -\frac{1}{6} \quad - \frac{1}{3}\cdot\left\{\cancel{[2\sec^2\tan^2](0)}^{\,0} + 2\overset{1}{\sec^4(0)}\right\}.$$

$$= -\frac{1}{6} - \frac{2}{3}$$

$$\neq 0$$

therefore $\sin(z) - \tan(z)$ vanishes to order 3 at $z = 0$.