MAT 514 - Lecture 22

Classification of Isolated Singularities

Recall that we say a function has an isolated singularity at $z_0 \in \mathbb{C}$ if

$$\lim_{z \to z_0} f(z) = \infty,$$

but f is holomorphic on a punctured disk

$$D^*[z_0, r],$$

for some $r > 0$.

Examples:

① $f(z) = \frac{1}{z}$ has an isolated singularity at $z = 0$.

② $f(z) = \tan(z)$ has isolated singularities at $z = (2k+1)\frac{\pi}{2}$, for $k \in \mathbb{Z}$.

Definition: Let z_0 be an isolated singularity of a function $f(z)$. We say that z_0 is a pole if there exists a positive integer m so that

$$f(z)\cdot(z-z_0)^m$$

is holomorphic on a disk $D[z_0, r]$, for some radius $r>0$. If such an integer exist, the least m with this property is called the order of pole of f at z_0.

Examples:

① $\frac{1}{z}$ has a pole of order 1 at $z=0$, for

$$\frac{1}{z}\cdot(z-0)^1 = 1,$$

a holomorphic function on $\mathbb{C}$.

② The function $f(z)=(z+1)/(z^2+1)$ has

isolated singularities at $z = \pm i$, both of which are poles of order 1. Indeed,

$$f(z) = \frac{(z+1)}{(z+i)(z-i)}.$$

Therefore:

a) $f(z)(z+i)^1 = \dfrac{z+1}{z-i}$, a holomorphic function on $D[-i, 2]$.

b) $f(z)(z-i)^1 = \dfrac{z+1}{z+i}$, a holomorphic function on $D[i, 2]$.

Definition: We a singularity is essential if it is neither removable nor a pole.

Example: $f(z) = e^{1/z}$, for $z \neq 0$.
This function has a singularity at $z=0$,

and no limit of type

$$\lim_{n \to \infty} z^k \cdot e^{1/z}$$

exists, therefore no function of type

$$z^k e^{1/z}$$

is holomorphic on a neighborhood of zero, for all $k \in \mathbb{N}$.

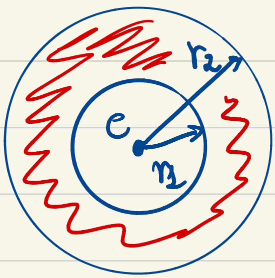
Theorem (Laurent)

Suppose that a function f is holomorphic on an annulus

$$A(c, r_1, r_2) = \{ z \in \mathbb{C} \mid r_1 < |z-c| < r_2 \}.$$

Then f has an expansion as a double power series

$$\sum_{k \in \mathbb{Z}} a_k (z-c)^k,$$



converging on annulus including $A(c, r_1, r_2)$.

Corolary: Suppose that f has a presumptive singularity at z_0, and is holomorphic on an annulus centered at z_0. Then f a Laurent series, and:

i) z_0 is a removable singularity if $a_k = 0$, for all $k < 0$.

ii) z_0 is a pole if $a_k = 0$, for all $k < k_0$, k_0 negative. (i.e, finitely many coefficients of negative index).

iii) z_0 is an essential singularity if the Laurent series has infinitely many coefficients of negative index.

Examples

① $f(z) = \frac{1}{z}$ has a Laurent series expansion around 0,

$$f(z) = 1 \cdot (z-0)^{-1}.$$

Since there only finitely many coefficients of negative order, $z=0$ is a pole.

This Laurent series converges on any annulus

$$A(0, r_1, r_2),$$

with $r_1 < r_2$.

② $f(z) = \frac{\sin(z)}{z}$ has an apparent singularity at $z=0$. It's Laurent series is

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}$$

③ $f(z) = e^z/z$ has a singularity at $z=0$.

Its Laurent series may be obtained by division of Taylor series,

$$f(z) = \frac{1}{z}\left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)$$

$$= \sum_{k=0}^{\infty} \frac{z^{k-1}}{k!}$$

$$= \sum_{k=-1}^{\infty} \frac{z^k}{(k+1)!}$$

$$= \underbrace{\frac{1}{z}}_{\text{singular part.}} + \underbrace{1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots}_{\text{regular part}}$$

The singular part of the Laurent series is finite, hence $z=0$ is a pole. Indeed, it has order 1, for

$$(z-0)\cdot f(z) = e^z$$

is holomorphic on $\mathbb{C}$.

(4) $f(z) = e^{\frac{1}{z}}$ has a Laurent series of type

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^k \Big/ k!$$

$$= \sum_{k=0}^{\infty} \frac{z^{-k}}{k!}$$

$$= \sum_{k=-\infty}^{0} \frac{z^k}{(-k)!}$$

There are infinitely many terms with negative exponents, hence $z=0$ is an essential singularity.

(5) Consider the function

$$f(z) = \frac{\sin(z)}{z^3}$$

with a singularity at $z=0$. The Laurent series of f around 0 is

$$f(z) = \frac{1}{z^3} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2(k-1)}}{(2k+1)!}$$

$$f(z) = \underbrace{\frac{1}{z^2}}_{\text{singular part}} \underbrace{- \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \dots}_{\text{regular part}}$$

This function has a pole of order 2 at 0.

⑥ $f(z) = \dfrac{z}{z^2+1}$ has poles of order 1 at $\pm i$.

We may write

$$\frac{z}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i},$$

for some coefficients A, B, satisfying:

$$\begin{cases} A + B = 1 \\ -Ai + Bi = 0. \end{cases} \Rightarrow A = \frac{1}{2}, \; B = \frac{1}{2}.$$

That is,

$$\frac{z}{z^2+1} = \frac{1}{2} \cdot \left(\frac{1}{z+i}\right) + \frac{1}{2} \cdot \left(\frac{1}{z-i}\right).$$

Laurent series centered at $-i$.

$\frac{1}{2} \cdot \left(\frac{1}{z+i}\right)$ is the singular part

$\frac{1}{2} \cdot \left(\frac{1}{z-i}\right)$ is holomorphic on a neighborhood of $(-i)$, but not written as a power series on $(z+i)$. To adjust it, we will use

$$w = z+i \Leftrightarrow z = w-i.$$

$$\frac{1}{2} \cdot \left(\frac{1}{z-i}\right) = \frac{1}{2} \cdot \left(\frac{1}{w-2i}\right)$$

$$= -\frac{1}{2} \cdot \left(\frac{1}{2i-w}\right)$$

$$= -\frac{1}{2} \cdot \left(\frac{\frac{1}{2i}}{1-\left(\frac{w}{2i}\right)}\right)$$

$$= -\frac{1}{2} \cdot \left[\frac{1}{2i} \cdot \sum_{k \geq 0} \left(\frac{w}{2i}\right)^k\right]$$

$$= -\frac{1}{4i} \sum_{k \geq 0} \frac{(z+i)^k}{(2i)^k}.$$

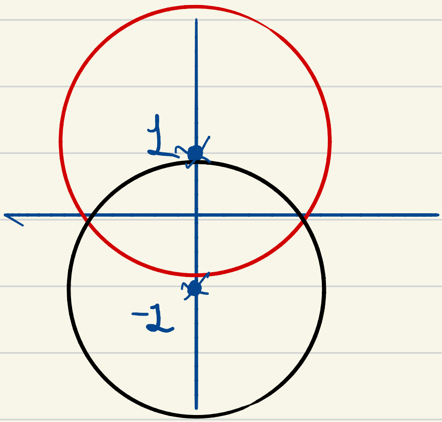
$$\frac{1}{2}\cdot\frac{1}{(z-i)} = \sum_{k\geq 0} \frac{(-1)}{2^{k+2}\cdot i^{k+1}}(z+i)^k.$$

$$f(z) = \underbrace{\frac{1}{2}\cdot(z+i)^{-1}}_{\text{singular part}} + \underbrace{\sum_{k\geq 0} \frac{(-1)}{2^{k+2}\cdot i^{k+1}}(z+i)^k}_{\text{regular part}}$$

Following a similar procedure, we find

$$f(z) = \underbrace{\frac{1}{2}\cdot(z-i)^{-1}}_{\text{singular part.}} + \underbrace{\frac{1}{4i}\sum_{k\geq 0}^{\infty}\left(\frac{-(z-i)}{2i}\right)^k}_{\text{regular part}}.$$

Both $\pm i$ are poles of f order 1.



The Laurent series at $(-i)$ converges on $D^*[-i, 2]$, while the Laurent series at i converges on $D^*[i, 2]$.

<u>Example 7:</u>

The function $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ has singularities at $z=0$, $z=\pm i$. We will find its Laurent series at $z=0$.

Recall that

$$\frac{1}{1+z^2}$$

is holomorphic near 0, and has Taylor series

$$\sum_{k\geq 0} (-1)^k z^{2k} = 1 - z^2 + z^4 - z^6 + z^8 - \dots$$

Hence

$$f(z) = \frac{1}{z^3} \cdot (1+z) \cdot (1 - z^2 + z^4 - z^6 + z^8 - \dots)$$

$$= \frac{1}{z^3} \cdot \left[\begin{array}{l} 1 - z^2 + z^4 - z^6 + z^8 - \dots \\ + z - z^3 + z^5 - z^7 + z^9 - \dots \end{array} \right]$$

$$= \frac{1}{z^3} \cdot \left\{ 1 + z - z^2 - z^3 + z^4 + z^5 - z^6 - z^7 + z^8 + z^9 - \dots \right\}$$

$$= \underbrace{\frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z}}_{\text{singular part}} \; \underbrace{- 1 + z + z^2 - z^3 + \dots}_{\text{regular part}}$$

So $z = 0$ is a pole of order 3.

<u>Proposition:</u> Suppose that f is a quotient of holomorphic functions,

$$f = \frac{g}{h}$$

and f has a presumptive singularity at z_0 (that is, $h(z_0) = 0$).

Then the order of f at z_0 is
$$\mathrm{ord}_{z_0}(f) = \mathrm{ord}_{z_0}(g) - \mathrm{ord}_{z_0}(h).$$
Here we understand that a negative order means order of pole, positive order is order of 0.

Examples:

① $f = \dfrac{z}{z^2+1}$, has presumptive singularities at $z = \pm i$.

$$\mathrm{ord}_i(f) = \mathrm{ord}_i(z) - \mathrm{ord}_i(z^2+1)$$
$$= 0 - 1$$
$$= -1$$

f has a pole of of order 1 at i.

② $f(z) = \dfrac{(z^2+2z+1)}{(z^2-1)}$ has presumptive singularities at $1, -1$.

$z=-1$ is an apparent singularity, for

$$\operatorname{ord}_{-1}(f) = \operatorname{ord}_{-1}(z^2+2z+1) - \operatorname{order}_{-1}(z^2-1)$$
$$= 2 - 1$$
$$= 1.$$

f has a zero of order 1 at $z=-1$.

$$f(z) = \frac{(z+1)^2}{(z+1)(z-1)} = \frac{z+1}{z-1}.$$

f has a pole of order 1 at $z=1$:

$$\operatorname{ord}_1(f) = \operatorname{ord}_1(z^2+2z+1) - \operatorname{ord}_1(z^2-1)$$
$$= 0 - 1$$
$$= -1,$$