MAT 514 - Lecture 23

Laurent Series

If a function is holomorphic on an annulus,

$$A(c, r_1, r_2) = \{ z \in \mathbb{C} \mid r_1 < |z-c| < r_2 \},$$

then it has a Laurent series centered at c

$$\sum_{k \in \mathbb{Z}} a_k (z-c)^k$$

converges on a domain including $A(c, r_1, r_2)$.

Examples:

1. $f(z) = e^z / z^3$ is holomorphic on

$$\mathbb{C}^* = \{ z \in \mathbb{C} \mid z \neq 0 \}.$$

Its Laurent series centered at 0 is

$$\frac{e^z}{z^3} = \frac{1}{z^3} \cdot \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^{k-3}}{k!}$$

$$f(z) = \sum_{k=-3}^{\infty} z^k / (k+3)!$$

It follows that $z=0$ is a pole of order 3 for e^z/z^3.

Different Laurent series for the same function with the same center may be possible.

Example: Consider the function

$$f(z) = \frac{1}{z(z-1)}$$

with singularities at $z=0$ and $z=1$.

To express it as a Laurent series centered at 0, we can use the following trick:

$$\frac{1}{z(z-1)} = -\frac{1}{z}\cdot\frac{1}{(1-z)}$$

$$= -\frac{1}{z}\cdot\sum_{k=0}^{\infty} z^k,$$

for $0 < z < 1$. That is, for $0 < |z| < 1$,

$$\frac{1}{z(z-1)} = \sum_{k=0}^{\infty} (-1)\, z^{k-1}$$

$$\boxed{\frac{1}{z(z-1)} = \sum_{k=-1}^{\infty} (-1)\, z^{k}, \quad 0<|z|<1}$$

To obtain a representation converging outside of $D[0,1]$, we use

$$\frac{1}{z(z-1)} = \frac{1}{z^2\left(1-\frac{1}{z}\right)}$$

for $\left|\frac{1}{z}\right| < 1$, $\frac{1}{1-\frac{1}{z}}$ can be expressed as a geometric series,

$$\frac{1}{1-\frac{1}{z}} = \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^{k}$$

Therefore, $\boxed{\frac{1}{z(z-1)} = \sum_{k=0}^{\infty} z^{-k-2}, \quad |z|>1}$

Remark: One does not use the second representation to determine the type of the singularity at 0, as it reflect local behavior (its convergence domain does not have 0 as a limit point).

Residues

Recall that

$$\int_{C[w,r]} (z-w)^k \, dz = 0,$$

for $k \neq -1$, and

$$\int_{C[w,r]} (z-w)^{-1} \, dz = 2\pi i.$$

Suppose f has an isolated singularity at w, and a Laurent series

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-w)^k$$

converging on $D^*[w,R]$. Then, if $r<R$,

$$\int_{C[w,r]} f(z)\,dz = \int_{C[w,r]} \sum_{k=-\infty}^{+\infty} a_k (z-w)^k\,dz$$

$$= \sum_{k=-\infty}^{+\infty} a_k \left(\int_{C[w,r]} (z-w)^k\,dz\right)$$

$$= a_{-1}\cdot(2\pi i)$$

$$= 2\pi i \cdot a_{-1}.$$

Amongst all coefficients of a Laurent series, a_{-1} plays a special role. It is called the <u>residue</u> of f at w.

<u>Example</u>:

1. The function e^z/z can be expressed as

$$\frac{1}{z}\cdot\sum_{k=0}^{\infty}\frac{z^k}{k!} = \frac{1}{z}\cdot\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\dots\right)$$

$$= 1\cdot z^{-1} + 1 + \frac{z}{2} + \frac{z^2}{6} + \dots$$

Thus the residue of e^z/z at 0 is 1. In particular,

$$\int_{C[0,r]} \frac{e^z}{z}\,dz = 2\pi i \cdot 1,$$

consistent with Cauchy's Integral Formula applied to e^z.

(2) Consider the function

$$f(z) = e^{1/z}.$$

It has an essential singularity, for its Laurent series is

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{z}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{z^{-k}}{k!}$$

$$= 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

Its residue at $z=0$ is 1, so

$$\int_{C[0,r]} e^{1/z}\,dz = 2\pi i.$$

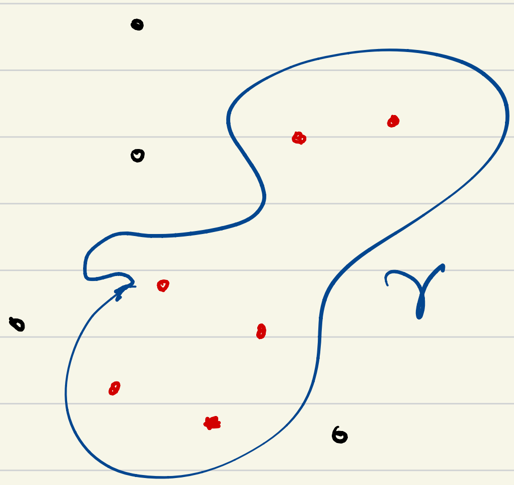
This is not an integral we could solve with Cauchy's Formulas.

The Residue Theorem

Let f be a complex function with isolated singularities. Let γ be a simple, closed, path positively oriented, not passing through any of the function's singularities.

Then

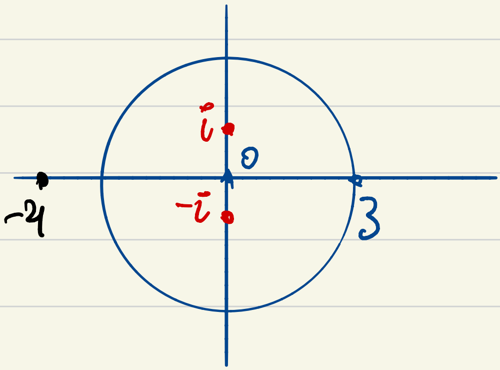
$$\frac{1}{2\pi i}\oint_\gamma f\,dz = \sum_{\substack{p \text{ singularity} \\ \text{in domain bounded by } \gamma}} \operatorname{Res}(f, p)$$



In diagram, red and black points denote singularities. The integral above only counts residues for points marked in red.

Example: Let $\gamma = C[0,3]$. We wish to compute

$$\int_\gamma \frac{1}{(z+4)(z^2+1)}\,dz.$$



By the residue theorem,

$$\int_\gamma \frac{1}{(z+4)(z^2+1)}\,dz = 2\pi i \left\{ \begin{array}{l} \mathrm{Res}(f, i) \\ + \mathrm{Res}(f, -i) \end{array} \right\}$$

Let's find the residues at i and $-i$.

$$\frac{1}{(z+4)(z^2+1)} = \frac{1}{(z+4)(z+i)(z-i)}$$
$$= \frac{A}{z+4} + \frac{B}{z+i} + \frac{C}{z-i},$$

for some constants A, B, C satisfying:

$$1 = A(z+i)(z-i) + B(z+4)(z-i) + C(z+4)(z+i)$$

$$1 = \{(A+B+C)z^2 + ((4-i)B + (4+i)C)z + (A - 4iB + 4iC)\}.$$

Therefore,

$$\begin{cases} A+B+C = 0 \\ (4-i)B + (4+i)C = 0 \\ A - 4iB + 4iC = 1. \end{cases}$$

Solutions ~~are~~:

$$A = \frac{1}{17}, \quad B = -\frac{1}{34} + \frac{2i}{17}, \quad C = -\frac{1}{34} - \frac{2i}{17}.$$

Hence

$$\frac{1}{(z+4)(z^2+1)} = \frac{1}{17(z+4)} + \left(\frac{-1}{34} + \frac{2i}{17}\right)\cdot\frac{1}{z+i} + \left(\frac{-1}{34} - \frac{2i}{17}\right)\cdot\frac{1}{z-i}.$$

Focus on $z=-1$: the first and third

components are holomorphic functions at $z=-1$ and do not contribute towards the residue, so

$$\operatorname{Res}(f, -i) = -\frac{1}{34} + \frac{2i}{17}.$$

Similarly,

$$\operatorname{Res}(f, i) = -\frac{1}{34} - \frac{2i}{17}.$$

It follows that

$$\int_{C[0,3]} \frac{1}{(z+4)(z^2+1)}\,dz = 2\pi i\left\{\left(-\frac{1}{34} + \cancel{\frac{2i}{17}}\right) + \left(-\frac{1}{34} - \cancel{\frac{2i}{17}}\right)\right\}$$

$$= -\frac{\pi i}{17}$$