MAT 514 - Summer II 2020
July 23, 2020

# Solutions to Long Quiz 2

**Problem 1** Determine at which points the function $f(z) = \frac{1}{\bar{z}}$, defined for $z \neq 0$, is complex-differentiable.

**Solution:** Let us write this function in terms of the real and imaginary parts of $z = x + iy$,

$$\begin{aligned} f(z) &= \frac{1}{\bar{z}} \\ &= \frac{1}{x - iy} \\ &= \left(\frac{1}{x - iy}\right)\left(\frac{x + iy}{x + iy}\right) \\ &= \left(\frac{x + iy}{x^2 + y^2}\right). \end{aligned}$$

Its real and imaginary parts are thus

$$u(x, y) = \frac{x}{x^2 + y^2}, \; v(x, y) = \frac{y}{x^2 + y^2},$$

respectively. The Cauchy-Riemann equations for this function are

$$\begin{aligned} \frac{y^2 - x^2}{(x^2 + y^2)^2} &= \frac{x^2 - y^2}{(x^2 + y^2)^2} \\ -\frac{2xy}{(x^2 + y^2)^2} &= \frac{2xy}{(x^2 + y^2)^2}, \end{aligned}$$

a system without solutions. It follows that $f$ is nowhere complex-differentiable.

**Problem 2** Find a function $v(x, y)$ so that

$$f(x + iy) = (2x^2 + x + 1 - 2y^2) + iv(x, y)$$

safisties the Cauchy-Riemann equations.

**Solution:** The derivatives of the real part of $f$ are

$$\frac{\partial u}{\partial x} = 4x + 1, \; \frac{\partial u}{\partial y} = -4y.$$

The Cauchy-Riemann equations for $v$ thus amount to

$$\begin{aligned}\frac{\partial v}{\partial x} &= 4y\\ \frac{\partial v}{\partial y} &= 4x+1.\end{aligned}$$

The function $v(x,y) = 4xy + y$ is a solution, defined on the entire complex plane.

**Problem 3** Use properties of the exponential function to derive the following relation:

$$\sin(2z) = 2\sin(z)\cos(z).$$

**Solution:** From the definition of the complex sine and cosine functions in terms of complex exponentials,

$$\begin{aligned}\sin(2x) &= \frac{e^{2iz} - e^{-2iz}}{2i}\\ &= \frac{(e^{iz})^2 - (e^{-iz})^2}{2i}\\ &= \frac{(e^{iz} + e^{-iz})(e^{iz} - e^{-iz})}{2i}\\ &= 2\left(\frac{e^{iz} - e^{-iz}}{2i}\right)\left(\frac{e^{iz} + e^{-iz}}{2}\right)\\ &= 2\sin(z)\cos(z).\end{aligned}$$