MAT 514 - Lecture 12

Technical aspects of complex integration.

In Multivariable Calculus, you learned about line integrals,

$$\int_\gamma F\,dr \quad , \quad \int_\gamma p\,dx + q\,dy .$$

Examples: $\gamma(t) = (\cos(t), \sin(t))$, $0 \leq t \leq 2\pi$
(in complex notation we'd write γ as
$\gamma(t) = e^{it}$ or $\gamma(t) = \cos(t) + i\sin(t)$.)
Given the vector field
$$F(x,y) = (x+y, x-y)$$
The velocity of γ is
$$\gamma'(t) = (-\sin(t), \cos(t)).$$

$$\int_\gamma F\,dr = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\,dt$$

$$\int_\gamma F\,dr = \int_0^{2\pi} (\cos(t)+\sin(t), \cos(t)-\sin(t))\cdot(-\sin(t), \cos(t))\,dt$$

$$= \int_0^{2\pi} \left\{ \begin{array}{l} -\cos(t)\sin(t) - \sin^2(t) \\ +\cos^2(t) - \cos(t)\sin(t) \end{array} \right\} dt.$$

$$= \int_0^{2\pi} \left\{ \begin{array}{l} (\cos^2(t) - \sin^2(t)) \\ \quad - 2\sin(t)\cos(t) \end{array} \right\} dt.$$

$$= \int_0^{2\pi} [\cos(2t) - \sin(2t)]\,dt.$$

$$= \frac{\sin(2t)}{2} + \frac{\cos(2t)}{2} \Bigg|_{t=0}^{t=2\pi}$$

$$= 0.$$

<u>Example 2</u>: $p(x,y) = y$, $q(x,y) = x$

$\gamma(t) = (1, 1+t)$, $0 \le t \le 1$.

The velocity vector is $\gamma'(t) = (0, 1)$, thus

$$\int_\gamma y\,dx + x\,dy = \int_0^1 (1+t)\cdot 0 + 1\cdot 1\,dt.$$

$$\int_\gamma y\,dx + x\,dy = t\Big|_0^1 = 1.$$

Closed, conservative vector fields, and closed exact forms.

<u>Definition:</u> A plane vector field $F(x,y)$ is called closed

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

The vector field is called conservative in a region $\mathcal{U} \subset \mathbb{R}^2$ if there exists a function $f: \mathcal{U} \to \mathbb{R}$ so that

$$\nabla f = F.$$

Suppose f is continuously differentiable. Then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Hence $F = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ satisfies

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \rightarrow \text{closedness condition.}$$

<u>Definition:</u> Given a differential form

$$p(x,y)\,dx + q(x,y)\,dy$$

we say that is form is closed

$$\boxed{\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}} \rightarrow \text{closedness.}$$

We say that this form is exact if there exists a function $f(x,y)$ satisfying

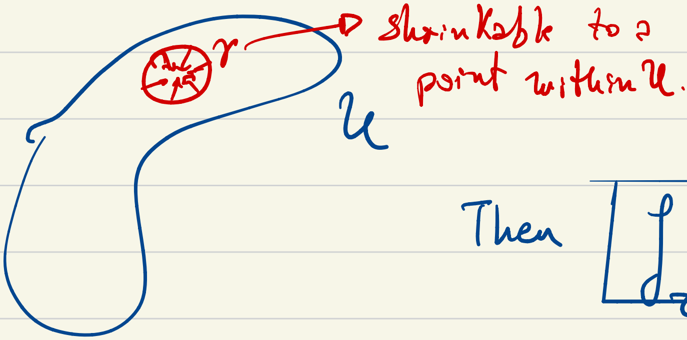
$$\frac{\partial f}{\partial x} = p \,; \quad \frac{\partial f}{\partial y} = q.$$

These conditions are often written as

$$\boxed{df = p\,dx + q\,dy} \rightarrow \text{exactness}$$

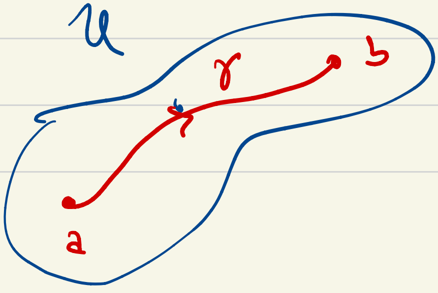
- Fundamental Theorems of Calculus for line integrals

i) Let F be a closed vector field defined on an open set U, and γ a simple curve contained in U which is U-contractible.



Then $\boxed{\int_\gamma F\,dr = 0.}$

ii) Let F be a conservative vector field on an open set U, and γ a path contained within U, joining a to b.



Let f be a potential for F, that is $\nabla f = F$.

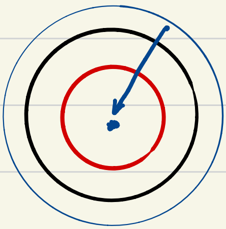
Then
$$\boxed{\int_\gamma F\,dr = f(b) - f(a).}$$

iii) Let
$$p\,dx + q\,dy$$
be a closed form on an open set U.
Also consider a U-contractible curve γ.
Then
$$\boxed{\int_\gamma p\,dx + q\,dy = 0.}$$

iv) Let $p\,dx + q\,dy$ be an exact form, with
$$df = p\,dx + q\,dy.$$
Let γ be a path joining a to b. Then
$$\boxed{\int_\gamma p\,dx + q\,dy = f(b) - f(a).}$$

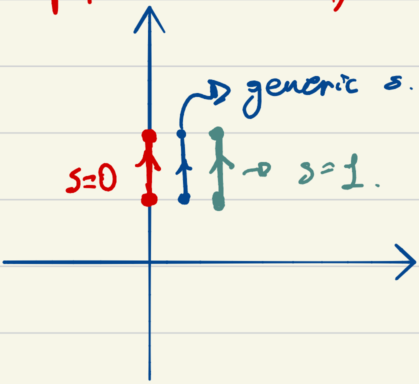
Contractibility

A homotopy between two curves (broadly interpreted to include points) is a family of curves interpolating them.



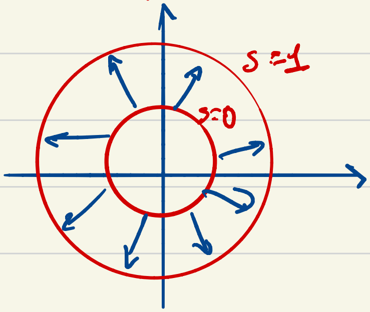
A family of concentric circles is an example.

Example: $H(s,t) = (s, 1+t)$ is a homotopy in s, $s \in [0,1]$, $t \in [0,1]$.



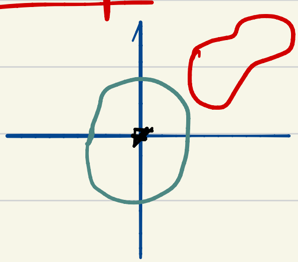
This is a family of vertical line segments moving from left to right as s varies from 0 to 1

Example: $H(s,t) = (s+1)e^{it} \in \mathbb{C}$, where $s \in [0,1]$, $t \in [0, 2\pi]$.



A *contraction of a curve* is a homotopy between it and a point. A *U-contraction*, where U is an open set in the plane, is a contraction constrained to U.

Example: $U = \mathbb{C} - \{0\}$.



Red is U-contractible

Green is not U-contractible.

Green is contractible, but any contraction must pass through

The origin.

In the complex setting there are analogues of closed vector fields and forms.

Cauchy's Theorem

If a function $f: \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ is holomorphic in $\mathcal{U}$ then for any $\mathcal{U}$-contractible closed path γ,

$$\boxed{\int_\gamma f\,dz = 0}$$

Examples: Let $f(z) = z$ (holomorphic on $\mathbb{C}$), and

$$\gamma(t) = e^{it},$$

where $t \in [0, 2\pi]$. The integral is

$$\int_\gamma f\,dz = \int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t)\,dt.$$

↳ complex multiplication.

$$\int_\gamma f\,dz = \int_0^{2\pi} e^{it}\cdot(ie^{it})\,dt.$$

$$= i\int_0^{2\pi} e^{2it}\,dt.$$

$$= i\cdot\frac{1}{2i}e^{2it}\Big|_{t=0}^{t=2\pi}.$$

$$= 0,$$

as expected, by Cauchy's Theorem.

<u>Example</u>: $f(z) = x$, $\gamma(t) = 1 + it$, $0 \le t \le 1$.

Then

$$\int_\gamma f\,dz = \int_0^1 \gamma(t)\cdot\gamma'(t)\,dt.$$

$$= \int_0^1 (1+it)\,i\,dt.$$

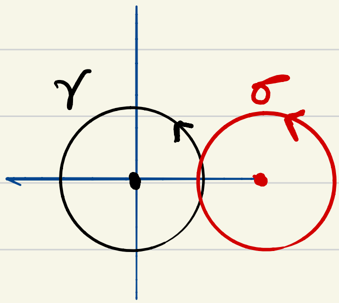
$$= i\int_0^1 (1+it)\,dt$$

$$= i\left[t + \frac{it^2}{2}\Big|_{t=0}^{t=1}\right] = i\left[1+\frac{i}{2}\right].$$

This is not zero, and it does not contradict Cauchy's Theorem, which as stated holds for closed paths.

Example: $f(z) = \frac{1}{z}$, $z \neq 0$, holomorphic away from $z=0$.

Choose $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then



$$\int_\gamma f\,dz = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}} \cdot (i\cancel{e^{it}})\,dt$$

$$= \int_0^{2\pi} i\,dt$$

$$= 2\pi i.$$

Now consider the curve

$$\sigma(t) = 2 + e^{it}, \quad t \in [0, 2\pi]$$

$$\int_\sigma f\,dz = \int_0^{2\pi} \frac{1}{2+e^{it}} \cdot (ie^{it})\,dt.$$

$$= \int_0^{2\pi} \frac{i\,e^{it}}{2+e^{it}}\,dt.$$

conjugate of denominator

$$\int_\sigma f\,dz = \int_0^{2\pi} \frac{i e^{it}}{2+e^{it}} \cdot \frac{(2+e^{-it})}{(2+e^{-it})}\,dt.$$

$$= \int_0^{2\pi} \frac{2ie^{it} + i}{4 + 2e^{-it} + 2e^{it} + 1}\,dt.$$

$$= \int_0^{2\pi} \frac{2ie^{it} + i}{5 + 2\cos(-t) \cancel{+ i\sin(-t)} + 2\cos(t) \cancel{+ i\sin(t)}}\,dt$$

$$= \int_0^{2\pi} \frac{2ie^{it} + i}{5 + 4\cos(t)}\,dt.$$

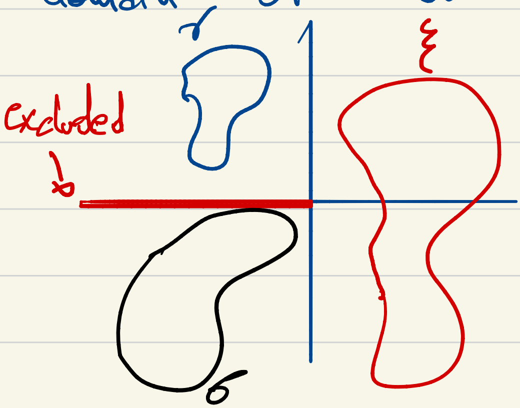
$$= \int_0^{2\pi} \frac{2i(\cos(t) + i\sin(t)) + i}{5 + 4\cos(t)}\,dt$$

$$= \int_0^{2\pi} \frac{2i\cos(t) - 2\sin(t) + i}{5 + 4\cos(t)}\,dt.$$

$$= 0,$$

since σ is contractible within the domain where f is holomorphic.

The principal branch of the logarithm has argument between $-\pi$ and π. It's domain of holomorphicity is $\mathbb{C} - \mathbb{R}_{\leq 0}$.



By Cauchy's Theorem,

$$\int_\gamma \mathrm{Log}(z)\,dz = 0$$

$$\int_\xi \mathrm{Log}(z)\,dz = 0$$

$$\int_\sigma \mathrm{Log}(z)\,dz = 0.$$

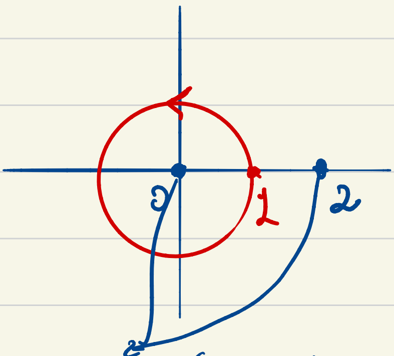
In fact, any closed path in $U = \mathbb{C} - \mathbb{R}_{\leq 0}$ is U-contractible, so for any such path γ

$$\int_\gamma \mathrm{Log}(z)\,dz = 0.$$

Example: $f(z) = \dfrac{1}{z^2 - 2z}$,

defined f $z \neq 0$ and $z \neq 2$, and holomorphic away from these points.

Choose $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.



Singularities (or poles) of f.

$$\int_\gamma f\,dz = \int_\gamma \frac{dz}{z^2 - 2z}$$

The path γ is not $\mathcal{U}$-contractible, where $\mathcal{U} = \mathbb{C} - \{0, 2\}$.

In essence, the real problem is the pole at 0. To solve it, we will use a partial fractions decomposition for f,

$$\frac{1}{z^2 - 2z} = \frac{A}{z} + \frac{B}{z-2} = \frac{A(z-2) + Bz}{z(z-2)}$$

for some complex constants A and B.

$$\frac{1}{z^2-2z} = \frac{(A+B)z - 2A}{z^2-2z}.$$

The constants satisfy

$$\begin{cases} A+B = 0 \\ -2A = 1. \end{cases} \Rightarrow \begin{array}{l} B = \frac{1}{2}. \\ A = -1/2. \end{array}$$

So

$$\int_\gamma \frac{dz}{z^2-2z} = -\frac{1}{2}\int_\gamma \frac{dz}{z} + \frac{1}{2}\int_\gamma \frac{dz}{z-2}.$$

Relative to the open set

$$G = \mathbb{C} - \{2\},$$

where $1/(z-2)$ is holomorphic, the path γ is contractible, so by Cauchy's Theorem

$$\int_\gamma \frac{dz}{z-2} = 0.$$

Meanwhile, from the previous example

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i.$$

It follows from the partial fractions decomposition that

$$\int_\gamma \frac{dz}{z^2-2z} = -\frac{1}{2}\int_\gamma \frac{dz}{z} + 0$$

$$= -\frac{1}{2}\cdot(2\pi i)$$

$$= -\pi i.$$

<u>Remark</u>: Factoring rational functions to explore the relative position between their poles and the path of integration is a useful technique.