MAT 203 - Lecture 13

- Guidelines for polar coordinates.

  Two coordinates:

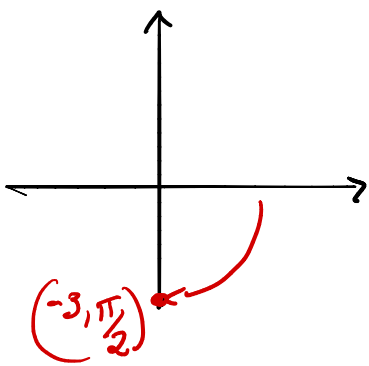
  i) radial component: measuring distance to origin;

  ii) angular component: measuring angle with positive x-axis, in counterclockwise sense

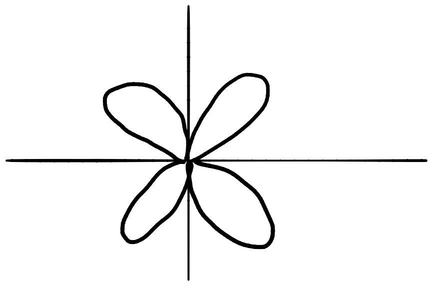
Remark: radial component must be non-negative.

<u>Textbook</u>: negative r-values should be interpreted by reflection:

$(r, \theta) = (-3, \pi/2) \cong \left(3, \frac{3\pi}{2}\right).$

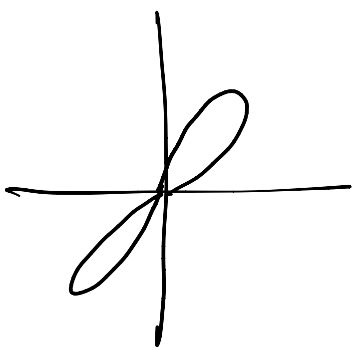


With textbook convention, $r = \sin(2\theta)$



<u>Our convention</u>: If r is negative, do not plot the point.

In this case, $r = \sin(2\theta)$ has two petals.



Let's examine the polar equation

$$r = \sin(2\theta).$$

$$r = 2\sin(\theta)\cdot\cos(\theta)$$

$$\overset{\cdot r^2}{\Rightarrow} \quad r^3 = 2(r\sin(\theta))\cdot(r\cos(\theta))$$

$$(x^2+y^2)^{\frac{3}{2}} = 2xy.$$

From LHS: $x^2+y^2 \geq 0 \to (x^2+y^2)^{3/2} \geq 0$.

This means that RHS ≥ 0. If

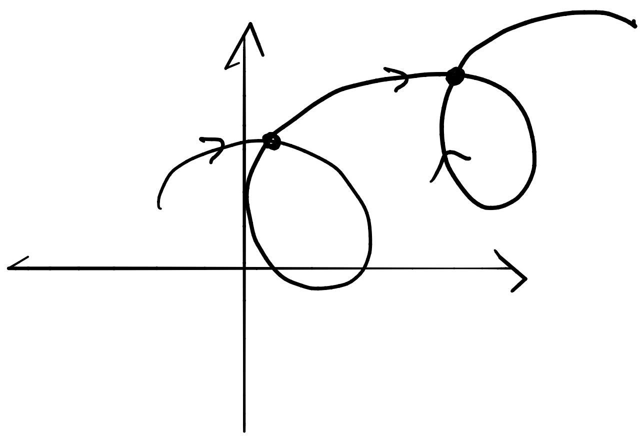
$$xy \geq 0,$$

then x, y have the same sign. That is, the point (x, y) belongs to first or third quadrants.

<u>Recall</u>: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$; $r = \sqrt{x^2+y^2}$.

- Difference between coordinatization and parametrization

- Coordinates must address a point unambiguously.
- Parameters are used to describe multiple instance of the same point.

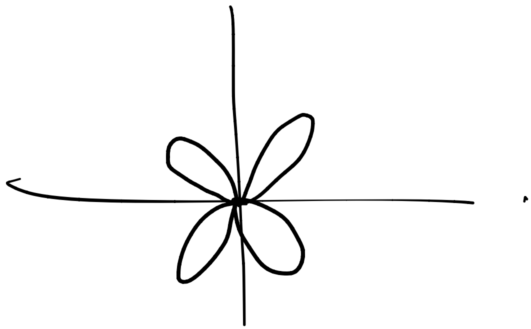


Particle moving along a trajectory, passing through marked points twice.

Using parametric equations,

$x(t) = \sin(2t)\cos(t)$, $y(t) = \sin(2t)\sin(t)$.

leads to a 4-petaled rose,



$$r(t) = \sqrt{\sin^2(2t)\cos^2(t) + \sin^2(2t)\sin^2(t)}$$

$$r(t) = \sqrt{\sin^2(2t)\cdot(\cos^2(t) + \sin^2(t))}$$

$$r(t) = \sqrt{\sin^2(2t)}$$

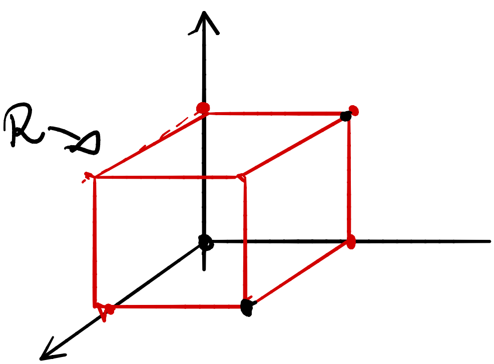
$$\boxed{r(t) = |\sin(2t)|}$$

<u>Triple integrals</u>

A function of three variables, $f(x,y,z)$ can be integrated in regions of space.

In parallelepipeds, integrals can be computed by iteration.

<u>Example</u>: Integration of $f(x,y,z) = x+y+z$ on the cube with vertices: $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ $(0,0,1)$ and $(1,1,1)$.



$$\iiint_R (x+y+z)\,dV = \int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz.$$

$$\iiint_R (x+y+z)\,dV = \int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left[\frac{x^2}{2} + yx + zx \Big|_{x=0}^{x=1} \right] dy\,dz$$

$$= \int_0^1\int_0^1 \left[\frac{1}{2} + y + z \right] dy\,dz.$$

$$= \int_0^1 \left[\frac{y}{2} + \frac{y^2}{2} + zy \Big|_{y=0}^{y=1} \right] dz$$

$$= \int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z \right] dz$$

$$= \int_0^1 (1+z)\,dz.$$

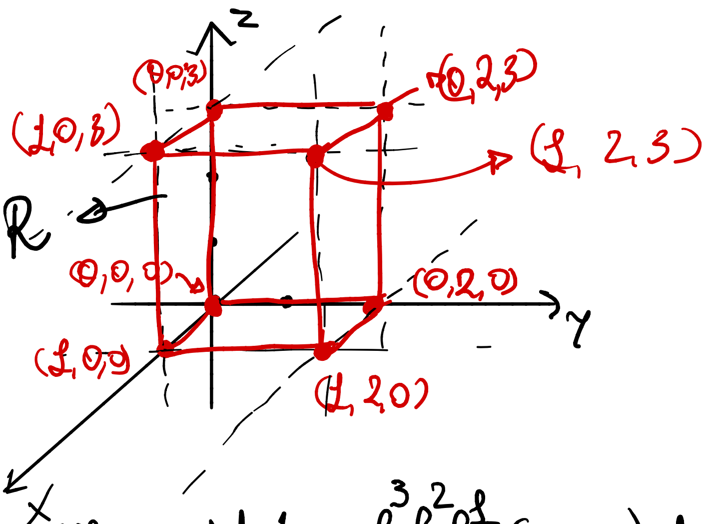
$$= z + \frac{z^2}{2} \Big|_{z=0}^{z=1}$$

$$= 1 + \frac{1}{2}.$$

$$= \frac{3}{2}.$$

Example 2: Integrate $f(x,y,z) = xyz$ on the parallelepiped with vertices

$(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$
$(1,2,0)$, $(1,0,3)$, $(0,2,3)$, $(1,2,3)$.



$$\iiint_R (xyz)\,dV = \int_0^3\int_0^2\int_0^1 (xyz)\,dx\,dy\,dz.$$

$$= \int_0^1\int_0^2\int_0^3 (xyz)\,dz\,dy\,dx.$$

$$= \int_0^2\int_0^3\int_0^1 (xyz)\,dx\,dz\,dy.$$

$$\int_0^3 \int_0^2 \int_0^1 (xyz)\, dx\, dy\, dz$$

$$= \int_0^3 \int_0^2 \left(\frac{x^2yz}{2} \Big|_{x=0}^{x=1} \right) dy\, dz$$

$$= \int_0^3 \int_0^2 \left(\frac{yz}{2} \right) dy\, dz$$

$$= \int_0^3 \left(\frac{y^2z}{4} \Big|_{y=0}^{y=2} \right) dz$$

$$= \int_0^3 z\, dz$$

$$= \frac{z^2}{2} \Big|_0^3 = \frac{9}{2} = 4.5$$

Fubini's Theorem:

Suppose a region of space R can be described by:

$$a < x < b$$
$$g_1(x) < y < g_2(x)$$
$$h_1(x,y) \le z \le h_2(x,y).$$

Then

$$\iiint_R f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Similarly for other constraint orders.

Remark: Triple integrals can be used to compute volumes by integrating the function

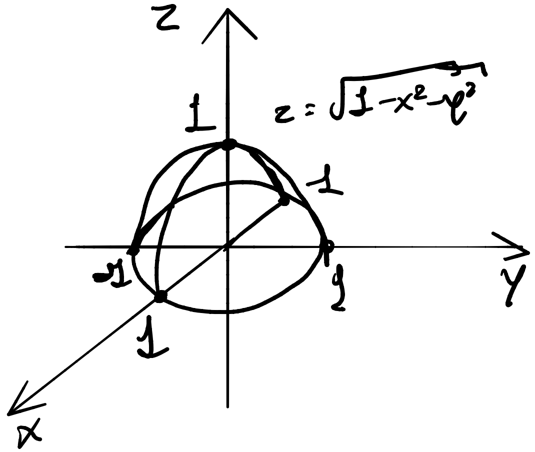
$$f(x,y,z)=1.$$

Example 3: Integrate

$$\iiint_R 1 \, dV$$

where R is the upper hemisphere

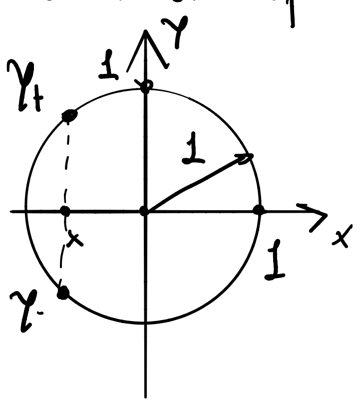
$$z = \sqrt{1 - x^2 - y^2}$$



The radius is 1, hence

$$-1 \le x \le 1.$$

2D - slice of equator:



Equation of circle $(z = 0)$

$$1 - x^2 - y^2 = 0 \Rightarrow$$

$$y^2 = 1 - x^2.$$

Two solutions:

$$y_+ = \sqrt{1 - x^2}, \quad y_- = -\sqrt{1 - x^2}$$

Once x, y are chosen we can find bounds for z:

lower bound: $z_- = 0$

upper bound: $z_+ = \sqrt{1-x^2-y^2}$.

$$\iiint_R 1 \, dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 1 \, dz \, dy \, dx.$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} \, dy \, dx.$$

Evaluated last week. Answer: $\frac{2\pi}{3}$.

Example 4: $$\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)\,dz\,dy\,dx.$$

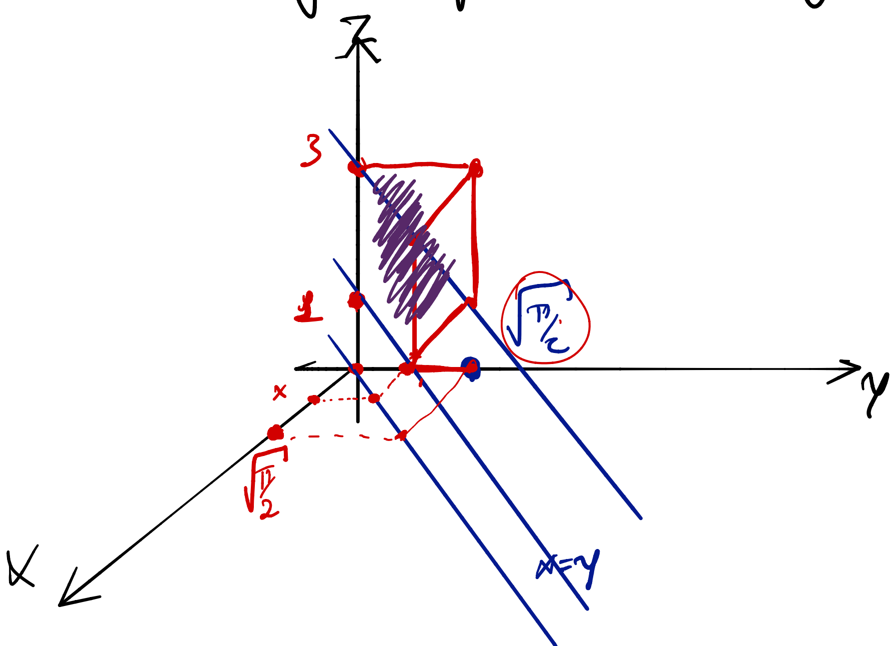
First attempt:

$$\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)\,dz\,dy\,dx = \int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \left(\sin(y^2)\cdot z \Big|_{z=1}^{z=3} \right) dy\,dx$$

$$= \int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} 2\sin(y^2)\,dy\,dx.$$

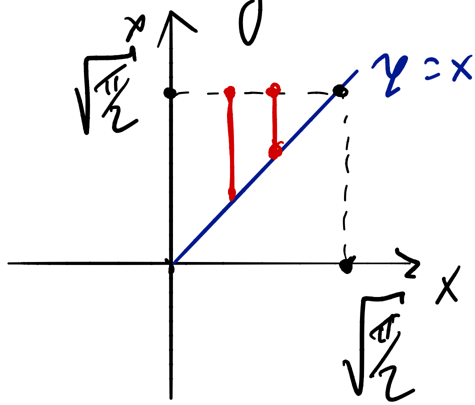
$\int \sin(y^2)\,dy$ is not an elementary function.

Understanding region of integration:



Bounds for z are constants, they can be integrated in any order we choose.

Let's focus on the xy-projection of the region of integration.



$$\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} f(x,y,z)\, dy\, dx = \int_0^{\sqrt{\pi/2}} \int_0^{y} f(x,y,z)\, dx\, dy.$$

Back to problem:

$$\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)\, dz\, dy\, dx = \int_0^{\sqrt{\pi/2}} \int_0^{y} \int_1^3 \sin(y^2)\, dz\, dx\, dy.$$

$$\int_0^{\sqrt{\pi/2}} \int_0^{y} \int_1^{3} \sin(y^2)\, dz\, dx\, dy$$

$$= \int_0^{\sqrt{\pi/2}} \int_0^{y} \left(\sin(y^2) \cdot z \Big|_{z=1}^{z=3} \right) dx\, dy$$

$$= \int_0^{\sqrt{\pi/2}} \int_0^{y} \left(2\sin(y^2) \right) dx\, dy$$

$$= \int_0^{\sqrt{\pi/2}} \cancel{\int} \left[2x \sin(y^2) \Big|_{x=0}^{x=y} \right] dy$$

$$= \int_0^{\sqrt{\pi/2}} \left[2y \sin(y^2) \right] dy.$$

Solve by substitution:

$$u = y^2$$

$$du = 2y\, dy$$

$$\int_0^{\sqrt{\pi/2}} 2y \sin(y^2)\, dy = \int_0^{\pi/2} \sin(u)\, du = -\cos(u) \Big|_{u=0}^{u=\pi/2}$$

$$= -0 + \cos(0)$$

$$= \underline{1}.$$

Exercise 1: Sketch the region of integration and evaluate

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} 1 \, dz \, dy \, dx$$

by changing the order of integration if necessary.

Solution:

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} 1 \, dz \, dy \, dx = \int_0^1 \int_{-1}^0 \left[z \Big|_{z=0}^{z=y^2} \right] dy \, dx$$

$$= \int_0^1 \left(\int_{-1}^0 y^2 \, dy \right) dx$$

$$= \int_0^1 \left(\frac{y^3}{3} \Big|_{y=-1}^{y=0} \right) dx$$

$$= \int_0^1 \frac{1}{3} \, dx$$

$$= \frac{x}{3} \Big|_{x=0}^{x=1}$$

$$= 1/3 .$$

Exercise 2: Evaluate the integral

$$\int_1^4 \int_0^1 \int_0^x 2z \cdot e^{-x^2}\, dy\, dx\, dz,$$

by changing order if necessary.

Solution:

$$\int_1^4 \int_0^1 \int_0^x 2z \cdot e^{-x^2}\, dy\, dx\, dz = \int_1^4 \int_0^1 \left[2zy e^{-x^2} \Big|_{y=0}^{y=x} \right] dx\, dz$$

$$= \int_1^4 \int_0^1 \left[2z x e^{-x^2} \right] dx\, dz$$

Integral relative to x can be done via

$$u = x^2;$$

$$du = 2x\, dx$$

$$\int_1^4 \int_0^1 2zx\, e^{-x^2}\, dx\, dz = \int_1^4 \left(\int_0^1 z \cdot e^{-u}\, du \right) dz$$

$$= \int_1^4 \left(-z e^{-u} \Big|_{u=0}^{u=1} \right) dz$$

$$= \int_1^4 \left(-z e^{-1} + z \cdot e^{0} \right) dz.$$

$$\int_1^4 \int_0^1 2zxe^{-x^2}\,dx\,dz = \int_1^4 z(1-e^{-1})\,dz.$$

$$= (1-e^{-1}) \cdot \int_1^4 z\,dz$$

$$= (1-e^{-1}) \cdot \left[\frac{z^2}{2} \Big|_{z=1}^{z=4} \right]$$

$$= (1-e^{-1}) \cdot \left[\frac{16}{2} - \frac{1}{2} \right]$$

$$= \frac{15(1-e^{-1})}{2}$$

Exercise 3: Find the volume of the solid bounded above by

$$x^2+y^2+z^2=4$$

and below by

$$z^2 = 3x^2+3y^2.$$

Reminder: Volumes may be expressed as

$$\iiint_R 1\, dV.$$

Solution Region of integration is region between cone

$$z^2= 3x^2+3y^2$$

and sphere

$$x^2+y^2+z^2=4.$$

The projection to xy-plane is bounded by the circle obtained when projecting the intersection of the two surfaces. Set z coordinates equal to each other

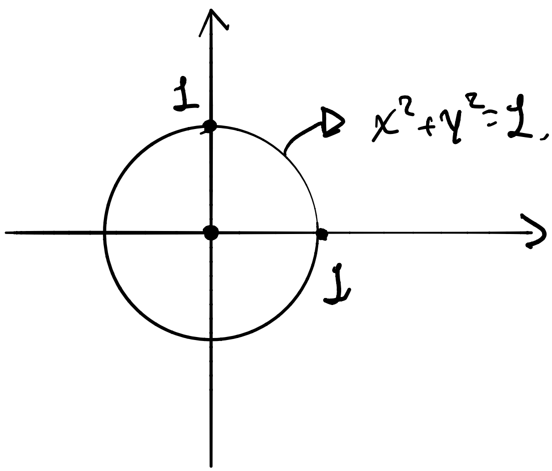
(z - coordinate of sphere)^2 = (z-coordinate of cone)^2

$$4 - x^2 - y^2 = 3x^2 + 3y^2.$$

$$4x^2 + 4y^2 = 4$$

$$\boxed{x^2 + y^2 = 1} \rightarrow y^2 = 1 - x^2$$

Projection onto xy-plane: $y_{\pm} = \pm\sqrt{1-x^2}$



x as free variable:

$$-1 \le x \le 1$$

y is dependent variable

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}.$$

<u>z-bounds:</u> upper hemisphere $\sqrt{4-x^2-y^2} \rightarrow$ upper bound

upper cone $\sqrt{3x^2+3y^2}$. $\rightarrow$ lower bound.

$$\text{Volume} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} 1 \, dz \, dy \, dx.$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(z \Big|_{z=\sqrt{3x^2+3y^2}}^{z=\sqrt{4-x^2-y^2}} \right) dy \, dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\sqrt{4-x^2-y^2} - \sqrt{3x^2+3y^2} \right] dy \, dx.$$

Changing to polar coordinates:

bounds: $0 \leq r \leq 1$

angle: $0 \leq \theta \leq 2\pi$.

$\Rightarrow$ $x = r\cos\theta$, $y = r\sin\theta$.

$dA = dy\,dx = dx\,dy = r\,dr\,d\theta$.

$$\text{Volume} = \int_{0}^{2\pi} \int_{0}^{1} \left(\sqrt{4-r^2} - \sqrt{3r^2} \right) r \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{1} \left[r\sqrt{4-r^2} - r^2\sqrt{3} \right] dr \, d\theta.$$

$$\text{Volume} = \int_0^{2\pi} \left[\left(\int_0^1 r\sqrt{4-r^2}\,dr \right) - \sqrt{3}\left(\int_0^1 r\,dr \right) \right] d\theta$$

First integral: substitute $u = 4 - r^2$

$du = -2r\,dr$

$$\int_0^1 r\sqrt{4-r^2}\,dr = \int_4^3 \frac{\sqrt{u}}{(-2)}\,du$$

$$= \int_3^4 \frac{\sqrt{u}}{2}\,du.$$

$$= \frac{u^{3/2}}{2} \cdot \frac{2}{3} \Bigg|_{u=3}^{u=4}$$

$$= \frac{u^{3/2}}{3} \Bigg|_{u=3}^{u=4}$$

$$= \frac{4^{3/2}}{3} - \frac{3^{3/2}}{3}$$

$$= \frac{8}{3} - \sqrt{3}.$$

Second integral:

$$\int_0^1 r^2 \, dr = \frac{r^3}{3} \Big|_{r=0}^{r=1}$$

$$= \frac{1}{3}.$$

Thus

$$\text{Volume} = \int_0^{2\pi} \left[\frac{8}{3} - \sqrt{3} - \frac{\sqrt{3}}{3} \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{8}{3} - \frac{4\sqrt{3}}{3} \right] d\theta$$

$$= \frac{(8 - 4\sqrt{3}) \cdot 2\pi}{3}$$

Exercise 4: Find the volume of the region bounded above by

$$x^2+y^2+z^2=1$$

and below by

$$z=\frac{1}{2}.$$

Solution: Set z coordinates equal

$$x^2+y^2+\left(\frac{1}{2}\right)^2=1$$

$$x^2+y^2+\frac{1}{4}=1$$

$$x^2+y^2=\frac{3}{4}.$$

Set x as free variable, $-\frac{\sqrt{3}}{2} \le x \le \frac{\sqrt{3}}{2}$.

y as a dependent variable, $-\sqrt{\frac{3}{4}-x^2} \le y \le \sqrt{\frac{3}{4}-x^2}$.

z as function of x, y: $\frac{1}{2} \le z \le \sqrt{1-x^2-y^2}$.

$$\text{Volume} = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}} \int_{\frac{1}{2}}^{\sqrt{1-x^2-y^2}} 1 \, dz \, dy \, dx.$$

In polar coordinates:

$$\text{Volume} = \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{1}{2} \right) r \, dr \, d\theta.$$