MAT 127
Summer II 2015
Midterm
07/23/15
Time Limit: 2 hours.

Name (Print): ____________________

ID number ____________________

Instructions

- This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.
- You may *not* use your books, notes, or any device that is capable of accessing the internet on this exam(e.g., smartphones, tablets). You may *not* use a calculator.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** In the practice part of the exam, a correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. Determine whether or not the sequences below converge, and if so, calculate their limit.

(a) (3 points)

$$a_n = 1 + \left(\frac{-2}{e}\right)^n$$

The sequence is convergent. Its limit is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \left(-\frac{2}{e}\right)^n\right) = 1.$$

This follows from $\lim_{n\to\infty}\left(-\frac{2}{e}\right)^n = 0$, since $\left|-\frac{2}{e}\right| < 1$.

(b) (3 points)

$$a_n = \frac{e^m + e^{-m}}{e^{2n} - 1}$$

Correction: $a_n = \dfrac{e^{n} + e^{-n}}{e^{2n} - 1}$.

The sequence is convergent, for

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n + e^{-n}}{e^{2n}-1} = \lim_{n\to\infty} e^{-n}\cdot\left(\frac{e^{2n}+1}{e^{2n}-1}\right)$$

$$= \left(\lim_{n\to\infty} e^{-n}\right)\cdot \lim_{n\to\infty}\left(\frac{e^{2n}+1}{e^{2n}-1}\right) = 0\cdot 1 = 0.$$

(c) (4 points)

$$a_n = \frac{(2n-1)!}{(2n+1)!}$$

The sequence converges. Notice that

$$\frac{(2n-1)!}{(2n+1)!} = \frac{1}{(2n+1)\cdot 2n},$$

so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(2n+1)\cdot 2n} = 0.$$

2. Consider the sequence defined recursively by:

$$a_1 = 2, \quad a_{n+1} = \sqrt{12 + a_n}$$

(a) (3 points) Show that this sequence is bounded above by 4.

We shall prove this by induction.
$a_1 = 2 \leq 4$.
Assume that $a_n \leq 4$. Then

$$a_{n+1} = \sqrt{12 + a_n} \leq \sqrt{12+4} \leq 4,$$

as desired.

(b) (3 points) Show that this sequence is increasing. (Hint: consider the quadratic polynomial associated to the recursive formula).

Notice that

$$a_{n+1}^2 = 12 + a_n.$$

Assume, by contradiction, that $a_{n+1} < a_n$. Then we have
$12 + a_n < a_n^2 \Rightarrow a_n^2 - a_n - 12 > 0$. (*)
The quadratic polynomial $p(x) = x^2 - x - 12$ can be written as $p(x) = (x+3)(x-4)$. Since $0 \leq a_n \leq 4$, by part a, we have $p(a_n) = a_n^2 - a_n - 12 \leq 0$, a contradiction with (*).
Therefore the sequence $\{a_n\}$ must be increasing.

(c) (4 points) Explain why the sequence is convergent, and find its limit.

The sequence is bounded above and increasing, so by the Monotone convergence Theorem it is Convergent.
Let $L = \lim_{n\to\infty} a_n$. From the equation

$$a_{n+1} = \sqrt{12 + a_n},$$

we get, in the limit as $n \to \infty$,

$$L = \sqrt{12 + L}.$$

The only positive solution to this equation is $L = 4$.

3. Calculate the values of the following series:

(a) (5 points)

$$\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$$

This is a geometric series, with first term $\left(\frac{9}{10}\right)$ and ratio $\left(\frac{9}{10}\right)$, therefore

$$\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n = \frac{\left(\frac{9}{10}\right)}{\left(1-\frac{9}{10}\right)} = \frac{\frac{9}{10}}{\frac{1}{10}} = 9.$$

(b) (5 points)

$$\sum_{n=1}^{\infty} \log\left(\frac{n^2+n}{n^2+3n+2}\right)$$

First notice that the terms $a_n = \log\left(\frac{n^2+n}{n^2+3n+2}\right)$ can be simplified by factoring the polynomials:

$$\frac{n^2+n}{n^2+3n+2} = \frac{n(n+1)}{(n+1)(n+2)} \Rightarrow \frac{n^2+n}{n^2+3n+2} = \frac{n}{(n+2)}$$

Now by the properties of logarithms:

$$a_n = \log\left(\frac{n^2+n}{n^2+3n+2}\right) = \log\left(\frac{n}{n+2}\right) = (\log n) - (\log(n+2)).$$

The series then becomes a telescoping sum:

$$\sum_{n=1}^{\infty} \log\left(\frac{n^2+n}{n^2+3n+2}\right) = (\log 1 - \cancel{\log 3}) + (\log 2 - \cancel{\log 4}) + (\cancel{\log 3} - \log 5) + \dots$$

$$= \cancel{\log 1}^{\,0} + \log 2$$

$$= \log 2$$

4. Decide whether the series below are convergent or divergent. Explain your answers. In each case, clearly indicate what convergence test you used.

   (a) (3 points)

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{n^2}$$

Notice that $0 < \sin\left(\frac{\pi}{n}\right) \le 1$, $\forall\, n \in \mathbb{N}$. We can use the Comparison Test: $\frac{\sin\left(\frac{\pi}{n}\right)}{n^2} < \frac{1}{n^2}$, $\forall\, n \in \mathbb{N}$.
Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{n^2}$.

   (b) (3 points)

$$\sum_{n=1}^{\infty} \frac{\log n}{n}$$

Consider the continuous function $f(x) = \frac{\log x}{x}$, $x \in [1, \infty)$.
Its derivative is $f'(x) = \frac{1 - \log x}{x^2}$, so that $f'(x) < 0$ if $x \geq 3$.

Moreover, $\int_1^{+\infty} \frac{\log x}{x}\, dx = \lim_{s \to +\infty} \int_1^{s} \frac{\log x}{x}\, dx = \lim_{s \to +\infty} \int_0^{\log s} u\, du = \lim_{s \to +\infty} (\log s) = +\infty$.
By the integral convergence test, $\sum_{n=1}^{\infty} \frac{\log n}{n}$ diverges.

   (c) (4 points)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$$

This alternating series can be written as
$\sum_{n=1}^{\infty} (-1)^{n+1}\, b_n$, with $b_n = \frac{n^2}{n^3+1}$.
Notice that $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{n^2}{n^3+1} = 0$. Moreover, the sequence $\{b_n\}$ is decreasing. Indeed, a simple calculation shows that $b_1 < b_2$, and the function $f(x) = \frac{x^2}{x^3+1}$, is decreasing for $x > 2$ (its derivative is $f'(x) = \frac{-x\,(x^3-2)}{(x^3+1)^2}$).
It follows from the Alternating series convergence Test that the series converges.

5. This question is related to the following power series:

$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

(a) (5 points) Calculate the radius of convergence.

In this power series, $a_n = \frac{n}{3^{n+1}}$. It follows that

$$R = \left(\lim_{n\to\infty} \left(\frac{n+1}{3^{n+2}}\right) \cdot \frac{3^{n+1}}{n} \right)^{-1} \Rightarrow R = \left(\lim_{n\to\infty} \frac{1}{3} \frac{n+1}{n} \right)^{-1} \Rightarrow R = 3.$$

(b) (5 points) Describe what happens in the border line case, i.e., when $|x+2| = R$, the radius of convergence you found on part (a).

If $(x+2) = 3$, then

$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{3^n \cdot n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{n}{3},$$

so the sum diverges, by the divergence test.

If $(x+2) = -3$, then

$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{(-3)^n \cdot n}{3^{n+1}} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{3},$$

which again diverges, by the divergence test.

6. Express the following functions as power series centered at the points indicated.

(a) (3 points)

$$f(x) = \frac{2}{3-x}, \quad x_0 = 1.$$

Notice that $\frac{2}{3-x} = \frac{2}{2}\left(\frac{1}{1-\left(\frac{x-1}{2}\right)}\right)$, so we can express f as a power series centered at 1 by

$$f(x) = \frac{2}{3-x} = \sum_{n=0}^{+\infty} (-1)^n \left(\frac{x-1}{2}\right)^n = \sum_{n=0}^{+\infty} \left(-\frac{1}{2}\right)^n (x-1)^n.$$

(b) (3 points)

$$g(x) = \log(1+x), \quad x_0 = 0$$

Recall that $\frac{dg}{dx} = \frac{1}{1+x}$, $x > -1$. In turn, the power series expansion of $\frac{1}{1+x}$ centered at 0 is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{+\infty} (-1)^n x^n,$$

with radius of convergence 1. By integration, the power series for g is

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}, \text{ or } g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}; \ |x| < 1.$$

(c) (4 points)

$$h(x) = \arctan(x), \quad x_0 = 0$$

Recall that $\frac{dh}{dx} = \frac{d}{dx}(\arctan)(x) = \frac{1}{1+x^2}$. In turn, the power series expansion of $\frac{1}{x^2+1}$ centered at 0 is

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{+\infty} (-1)^n \cdot x^{2n},$$

with radius of convergence 1 (obtainable by applying the root test). By integration

$$h(x) = \int_0^x \frac{1}{1+s^2}\,ds = \int_0^x \sum_{n=0}^{+\infty} (-1)^n s^{2n}\,ds = \sum_{n=0}^{+\infty} (-1)^n \int_0^x s^{2n}\,ds \Rightarrow$$

$$h(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \quad |x| < 1.$$