MAT324: Real Analysis – Fall 2014
Assignment 5 – Solutions

Problem 1: Compute the following limits if they exist and justify the calculations:

a) $\lim\limits_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx$

b) $\lim\limits_{n\to\infty} \int_0^\infty \frac{n^2 x e^{-n^2x^2}}{1+x} dx.$

c) $\lim\limits_{n\to\infty} \int_1^\infty \frac{n^2 x e^{-n^2x^2}}{1+\sqrt[n]{x}} dx.$

Solution.

a) The integrand is dominated by
$$g(x) = \frac{1}{(1+\frac{x}{2})^2},$$
which is integrable (check). In addition, the integrand goes to zero for every x. The dominated convergence theorem gives that the integral is zero.

b) The limit exists and is zero. Check that the integrands are dominated by the L^1 function $g(x) = n^2 x e^{-n^2x^2}$. Apply dominated convergence.

c) The limit exists and is zero. Check that the integrands are dominated by the same L^1 function as before.

Problem 2: Suppose $E \in \mathcal{M}$. Let (g_n) be a sequence of integrable functions which converges a.e. to an integrable function g. Let (f_n) be a sequence of measurable functions which converge a.e. to a measurable function f. Suppose further that $|f_n| \le g_n$ a.e. on E for all $n \ge 1$. Show that if $\int_E g\, dm = \lim\limits_{n\to\infty} \int_E g_n\, dm$, then $\int_E f\, dm = \lim\limits_{n\to\infty} \int_E f_n\, dm$.

Hint: Rework the proof of the Dominated Convergence Theorem.

Solution. Apply Fatouś lemma to the sequences $g_n - f_n$ and $g_n + f_n$. □.

Problem 3: Let $E \in \mathcal{M}$. Let (f_n) be a sequence of integrable functions which converges a.e. to an integrable function f. Show that $\int_E |f_n - f|\, dm \to 0$ as $n \to \infty$ if and only if $\int_E |f_n|\, dm \to \int_E |f|\, dm$ as $n \to \infty$.

Solution. For one side, use the reverse triangle inequality,
$$||f_n(x)| - |f(x)|| \le |f_n(x) - f(x)|$$

and integrate. For the converse, apply the previous problem to the sequence $|f_n - f|$ which is dominated by the sequence of functions $g_n = |f_n| + |f|$. $\square$.

Problem 4: Consider two functions $f, g : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if} \quad x = \frac{p}{q} \in \mathbb{Q}, \text{ where } p \text{ and } q \text{ are relatively prime} \\ 0 & \text{if} \quad x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Show that f is Riemann integrable on $[0,1]$, but g is not Riemann integrable on $[0,1]$.

Solution. For f, check example 4.6 on the textbook. For g, notice that the set of discontinuities is the whole interval $[0,1]$. Then use theorem 4.23, part (i). $\square$

Problem 5: Consider the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if} \quad x > 0 \\ 1 & \text{if} \quad x = 0. \end{cases}$$

Show that f has an improper Riemann integral over the interval $[0,\infty)$, but f is not Lebesgue integrable.

Solution. Notice that if $a, A > 0$,

$$\int_a^A \frac{\sin x}{x} dx = \frac{\cos a}{a} - \frac{\cos A}{A} - \int_a^A \frac{\cos x}{x^2}$$

The integral on the right-hand side is convergent, whereas the diference also is, hence the Riemann integral exists. However, f is not Lebesgue integrable. Indeed, $|f|$ is not integrable since

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \geq \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x| dx = \frac{2}{(k+1)\pi}$$

and therefore

$$\lim_{n\ to\infty} \int_1^n \frac{|\sin x|}{x} dx \geq \frac{2}{\pi} \sum_{k=2}^{\infty} \frac{1}{k} = \infty$$

diverges. $\square$