MAT 511 - Summer 1
June 4, 2019

Homework 2: solutions to selected problems.

Exercise 1 Prove the uniqueness of the empty set, that is, if A and B are empty sets, show that $A = B$.

Solution: Solved in class.

Exercise 2 The following claim is true, but the argument presented as its proof contains a mistake. Find the mistake, and fix it.

Claim: If A is a set, then $A \in \mathcal{P}(A)$.

Proof. Suppose $x \in A$. Then $\{x\} \subset A$. Thus $\{x\} \in \mathcal{P}(A)$. Therefore, $A \in \mathcal{P}(A)$. □

Solution: Solved in class.

Exercise 3 Let A and B be sets. Prove that $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$.

Exercise 4 Let A, B, C and D be sets. Prove that

(a) $A \subset B$ if and only if $A - B = \emptyset$.

(b) If $A \subset B \cup C$ and $A \cap B = \emptyset$, then $A \subset C$.

(c) If $A \subset C$ and $B \subset C$, then $A \cup B \subset C$.

(d) If $C \subset A$ and $D \subset B$, then $C \cap D \subset A \cap B$.

(e) If $A \cup B \subset C \cup D$, $A \cap B = \emptyset$, and $C \subset A$, then $B \subset D$.

Solution: In each part, we may take as the Universe of Discourse the union of the sets A, B, C and D, according to their appearance in the corresponding statement.

(a) Universe: $A \cup B$.

Consider the predicates

- P:"it belongs to A";
- Q: "it belongs to B".

In terms of these predicates, what we wish to prove is that

$$[\forall x\ (P(x) \Rightarrow Q(x))] \Leftrightarrow [\neg\exists x\ (P(x) \wedge \neg Q(x))]$$

By Material Implication,

$$P(x) \Rightarrow Q(x) \Leftrightarrow \neg P(x) \vee Q(x).$$

The negation of $[\forall x\ (P(x) \Rightarrow Q(x))]$ can thus be written as

$$\begin{aligned}[\exists x\ \neg(P(x) \Rightarrow Q(x))] &\Leftrightarrow [\exists x\ \neg(\neg P(x) \vee Q(x))] \\ &\Leftrightarrow [\exists x\ (P(x) \wedge \neg Q(x))],\end{aligned}$$

thus the desired equivalence follows from the double negation law,

$$\begin{aligned}[\forall x\ (P(x) \Rightarrow Q(x))] &\Leftrightarrow \neg[\neg[\forall x\ (P(x) \Rightarrow Q(x))]] \\ &\Leftrightarrow \neg[\exists x\ \neg(P(x) \Rightarrow Q(x))] \\ &\Leftrightarrow \neg[\exists x\ (P(x) \wedge \neg Q(x))].\end{aligned}$$

(b) Universe: $A \cup B \cup C$.

Consider the predicates

- P:"it belongs to A";
- Q: "it belongs to B";
- R: "it belongs to C.

In terms of these predicates, the statement we seek to prove is that if

$$[\forall x\ (P(x) \Rightarrow Q(x) \vee R(x))] \wedge [\forall x\ (P(x) \Rightarrow \neg Q(x)) \wedge (Q(x) \Rightarrow \neg P(x))],$$

then

$$\forall x\ (P(x) \Rightarrow R(x)).$$

In doing so, the first step is to simplify the premise, extracting the quantifier

$$\forall x\ [(P(x) \Rightarrow Q(x) \vee R(x)) \wedge (P(x) \Rightarrow \neg Q(x)) \wedge (Q(x) \Rightarrow \neg P(x))].$$

From

$$(P(x) \Rightarrow Q(x) \vee R(x)) \wedge (P(x) \Rightarrow \neg Q(x)),$$

we may infer

$$(P(x) \Rightarrow R(x)),$$

thus

$$\forall x\ [(P(x) \Rightarrow R(x)) \wedge (Q(x) \Rightarrow \neg P(x))],$$

from which we infer the statement on the left by simplification,

$$\forall x(P(x) \Rightarrow R(x)),$$

as desired.

(c) Universe: $A \cup B \cup C$.

Consider the predicates

- P:“it belongs to A”;
- Q: “it belongs to B”;
- R: “it belongs to C.

In terms of these predicates, the statement we wish to prove is: if

$$\forall x\ [(P(x) \Rightarrow R(x)) \wedge (Q(x) \Rightarrow R(x))],$$

then

$$\forall x\ [(P(x) \vee Q(x)) \Rightarrow R(x)].$$

This is easily achived by using material implication (applied three times), distributivity, and DeMorgan's laws:

$$\begin{aligned}\forall x\ [(P(x) \Rightarrow R(x)) \wedge (Q(x) \Rightarrow R(x)) &\Leftrightarrow \forall x\ [(\neg(x) \vee R(x)) \wedge (\neg Q(x) \vee R(x))],\\ &\Leftrightarrow \forall x\ [(\neg P(x)) \wedge (\neg Q(x))] \vee R(x)\\ &\Leftrightarrow \forall x\ [\neg(P(x) \vee Q(x))] \vee R(x)\\ &\Leftrightarrow \forall x\ (P(x) \vee Q(x)) \Rightarrow R(x).\end{aligned}$$

(d) Suppose $x \in C \cap D$. Then, in particular, $x \in C$, hence $x \in A$, by inclusion. Likewise, $x \in D$, thus $x \in B$, by inclusion. It follows that $x \in A \cap B$. This shows that $C \cap D \subset A \cap B$.

(e) Solved in class.

Exercise 5 Let A and B be sets. Prove that

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B).$$

Show, by means of an example, that the equality

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$$

need not be true.

Solution: Solved in class.

Exercise 6 Let A, B, C and D be sets.

(a) Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(b) Find an example that show thats the equality

$$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$$

is, in general, false.

Solution: Solved in office hours.

Exercise 7 Use Mathematical Induction to verify the following statements:

(a) For every $n \in \mathbb{N}$, the number

$$\frac{n(n+1)}{2}$$

is an integer.

(b) For every $n \in \mathbb{N}$, the number

$$n(n+1)(2n+1)$$

is divisible by 6.

(c) For all $n \in \mathbb{N}$, the sum of the interior angles of a convex polygon with $(n+2)$-sides is $180 \cdot n$ degrees.

(d) For all $n \in \mathbb{N}$, given n points in a plane, no three of which are collinear, there are exactly

$$\frac{n^2 - n}{2}$$

line segments joining pairs all pairs of points.

Solution:

(a) Solved in class.

(b) Solved in class.

(c) As remarked in class, there was a typo in this problem: the polygon is meant to have $(n+3)$ sides, and the sum of angles is correspondently $180 \cdot (n+1)$ degrees. This is for compatibility with our assumption that $0 \in P$.

Consider the following subset of $\mathbb{N}$,

$$P = \{n \in \mathbb{N} | \text{ the sum of angles of any convex polygon with } (n+3) \text{ sides is } 180 \cdot (n+1)\}.$$

When $n = 0$, this is the statement that the sum of angles of any triangle is 180 degrees, which we know to be true, from Euclidean Geometry. This is a fundamental fact, which we will use during the inductive step.

Next, we assume that $n \in P$, and consider the question of whether $(n+1) \in P$. Let $A_0A_1A_2A_3 \cdots A_{n+2}A_{n+3}$ be a convex $(n+4)$sided polygon, with A_i being the vertices. Consider the line segment $\overline{A_1A_{n+3}}$. It splits the polygon into two others, a triangle: $A_{n+3}A_0A_1$, and a polygon with $(n+3)$-sides, $A_1A_2 \cdots A_{n+2}A_{n+3}$. By the induction hypothesis, the sum of angles in the latter is $180(n+1)$, while the sum of angles in the former is 180. Overall, the sum of angles in the original polygon is

$$180(n+1) + 180 = 180(n+2).$$

Since this procedure can be carried out for any $(n+4)$-sided polygon, we conclude that $(n+1) \in P$, as we wanted to show.

(d) Solved in class.

Exercise 8 The Fibonacci numbers are recursively defined by the relations

$$\begin{aligned} f_1 &= 1, \\ f_2 &= 1, \\ f_{n+2} &= f_{n+1} + f_n. \end{aligned}$$

In this problem, you are required to use Induction (or its variants) to show the following:

(a) Two consecutive terms of this sequence have no common divisors, other than ± 1.

(b) f_{3n} is always even.

(c) f_{4n} is divisible by 3, for all $n \in \mathbb{N}$.

Solution: First, a clarification: the rules above imply that $f_0 = 0$.

(a) Consider the following subset of $\mathbb{N}$:

$$P = \{n \in \mathbb{N} | \ f_n \text{ and } f_{n+1} \text{ have no common divisors, other than} \pm 1\}.$$

We observe that $0 \in P$, as $f_0 = 0$ and $f_1 = 1$ have no common divisors, other than ± 1.

Assume that $n \in P$, that is, f_n and f_{n+1} have no common divisors. Then, we consider the question of whether $(n+1) \in P$. We will prove this is the case by means of contradiction. Suppose that the induction hypothesis holds, but $(n+1) \notin P$, that is, suppose that f_{n+1} and f_{n+2} have a common divisor, say $k \in \mathbb{Z}$, $k \neq \pm 1$. Then,

$$\begin{aligned} f_{n+1} &= ka, \text{ and,} \\ f_{n+2} &= kb, \end{aligned}$$

for certain integers a, b. It follows from the recursion relating f_{n+2}, f_{n+1} and f_n, that

$$f_n = f_{n+2} - f_{n+1} = kb - ka = k(b - a),$$

that is, k also divides f_n. This contradicts our induction hypothesis, so it must be the case that $(n + 1) \in P$.

(b) Consider the following subset of $\mathbb{N}$:

$$Q = \{n \in \mathbb{N} | f_{3n} \text{ is even}\}.$$

As usual, we begin by verifying that 0 belongs to Q, for $f_0 = 0$ is even.

Next we assume that f_{3n} is even. Then, we use the recursion defining the sequence to relate $f_{3(n+1)} = f_{3n+3}$ and f_{3n}:

$$\begin{aligned} f_{3n+3} &= f_{3n+2} + f_{3n+1} \\ &= (f_{3n+1} + f_{3n}) + f_{3n+1} \\ &= 2f_{3n+1} + f_{3n}. \end{aligned}$$

Since f_{3n} is even, and $2f_{3n+1}$ is even, their sum, f_{3n+3}, is also even, as we wanted to prove.

(c) Consider the following subset of $\mathbb{N}$:

$$R = \{n \in \mathbb{N} | \ f_{4n} \text{ is divisible by } 3\}.$$

It is easy to verify that $0 \in R$, as $f_0 = 0$ is divisible by 3.

Assume that $n \in R$, that is, f_{4n} is divisible by 3. Again, we use the recursion to relate $f_{4(n+1)} = f_{4n+4}$ and f_n,

$$\begin{aligned} f_{4n+4} &= f_{4n+3} + f_{4n+2} \\ &= (f_{4n+2} + f_{4n+1}) + (f_{4n+1} + f_{4n}) \\ &= f_{4n+2} + 2f_{4n+1} + f_{4n} \\ &= (f_{4n+1} + f_{4n}) + 2f_{4n+1} + f_{4n} \\ &= 3f_{4n+1} + 2f_{4n}. \end{aligned}$$

Since f_{4n} is divisible by 3, so is $2f_{4n}$. Clearly, $3f_{4n+1}$ is divisible by 3, so the sum

$$f_{4n+4} = 3f_{4n+1} + 2f_{4n}$$

is divisible by 3, as we wanted to show.

Exercise 9 In a certain kind of tournament, every player plays every other player exactly once, and either wins or losess. There are no ties. Define a top player to be a player who, for every other player x, either beats x or beats a player y who beats x.

(a) Show, by means of an example, that there can be more than one top player.

(b) Use Induction to show that every such tournament with n players has a top player.

(c) Use the Well-Ordering Principle to show that every such tournament with n players has a top player.

Solution: Solved in class.