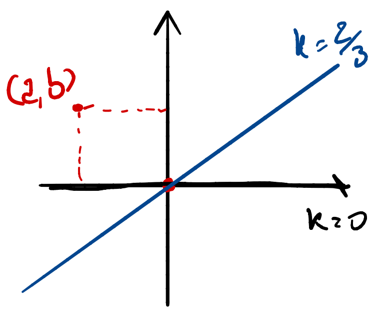
MAT 203 - Lecture 7

- Homework 3 will not be assigned this week.
- Midterm practice on Blackboard this afternoon.
- Solutions posted after office hours.
- Homework 2 graded, grades will be posted this afternoon

- Directed limits.



Directed limits at $(0,0)$:
$x=0$, $y = kx$, for various values of k.

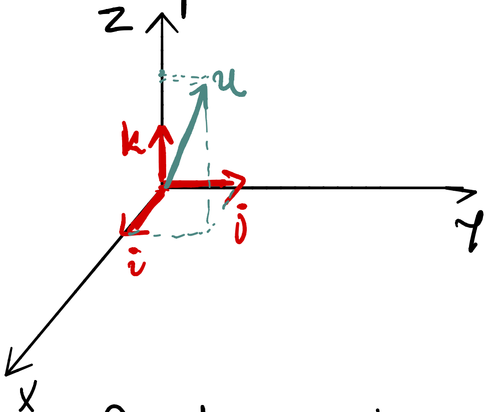
For limits at different points, lines are translated. To compute directed limits at (a, b) we take:

i) line $x=a$.

ii) lines $\boxed{y - b = k(x-a).}$

· Directional derivatives and gradient.

Reminder: partial derivatives we introduced in previous lecture.



∂_x measures rate of change in directions parallel to i. Likewise, $\partial_y \sim j$, $\partial_z \sim k$.

<u>Question</u>: how to measure rate of change along other directions?

<u>Motivation</u>: The ideal gas law states that

$$\text{pressure} \cdot \text{volume} = C \cdot \text{temperature}.$$

$$P \cdot V = C\,T.$$

We can think of temperature as a function of pressure and volume.

Experiment 1: measure how T changes if P varies but V fixed. $\partial T / \partial P$

Experiment 2: measure how T changes if P is fixed but V varies. $\partial T / \partial V$.

Experiment 3: measure how T changes as P and V vary simultaneously with ratio

$$\frac{dV}{dt} = \frac{2}{3} \frac{dP}{dt}.$$

This is epressed by a directional derivative!

Let's go back to partial derivatives.

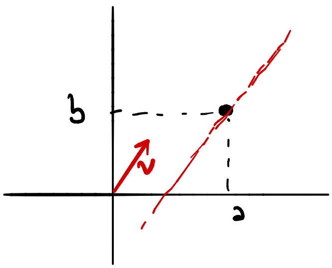
$$\frac{\partial f}{\partial x}(a,b) = \frac{d}{dx} f(a,b)$$

$$= \lim_{t \to 0} \frac{f(a+t,b) - f(a,b)}{t}.$$

$$= \lim_{t \to 0} \frac{f((a,b) + t \cdot \underbrace{(1,0)}_{\vec{v}}) - f(a,b)}{t}$$

If v is a unit vector (direction vector)
then we set

$$\frac{\partial f}{\partial v}(a,b) = \lim_{t \to 0} \frac{f((a,b) + t \cdot v) - f(a,b)}{t}$$



- The unit vector requirement is the textbook convention, which varies across sources.

Example 1: Find the directional derivative of

$$f(x,y) = x^2 + 2y$$

along the direction $v = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, at base-point $(1,0)$.

<u>Solution:</u> $f(x,y) = x^2 + 2y \,;\ (a,b) = (1,0)$,

$v = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$\frac{\partial f}{\partial v}(1,0) = \lim_{t \to 0} \frac{f\left((1,0) + t\cdot\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right) - f(1,0)}{t}.$$

$$= \lim_{t \to 0} \frac{f\left(1 + \frac{t\sqrt{2}}{2}, \frac{t\sqrt{2}}{2}\right) - f(1,0)}{t}$$

$$= \lim_{t \to 0} \frac{\left(1 + \frac{t\sqrt{2}}{2}\right)^2 + t\sqrt{2} - 1}{t}$$

$$= \lim_{t \to 0} \frac{\cancel{1} + t\sqrt{2} + \frac{t^2}{2} + t\sqrt{2} - \cancel{1}}{t}.$$

$$= \lim_{t \to 0} 2\sqrt{2} + t/2.$$

$$= 2\sqrt{2}.$$

$\blacksquare$

__Exercise 1:__ Find the directional derivative of

$$f(x,y) = x \cdot y$$

at $(1,1)$, along the direction $v = \left(\frac{3}{5}, \frac{4}{5}\right)$.

__Solution:__

$$\frac{\partial f}{\partial v} = \lim_{t \to 0} \frac{f\left((1,1) + t \cdot \left(\frac{3}{5}, \frac{4}{5}\right)\right) - f(1,1)}{t}$$

$$= \lim_{t \to 0} \frac{f\left(1 + \frac{3t}{5}, 1 + \frac{4t}{5}\right) - f(1,1)}{t}$$

$$= \lim_{t \to 0} \frac{\left(1 + \frac{3t}{5}\right) \cdot \left(1 + \frac{4t}{5}\right) - 1 \cdot 1}{t}$$

$$= \lim_{t \to 0} \frac{\cancel{1} + \frac{4t}{5} + \frac{3t}{5} + \frac{12t^2}{25} - \cancel{1}}{t}.$$

$$= \lim_{t \to 0} \frac{\frac{7}{5} + \frac{12t}{25}}{1}$$

$$= \frac{7}{5}.$$

Exercise 2: Find directional derivative of

$$f(x,y,z) = x^2+y^2+z^2$$

along $v = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, at the point $(0,0,1)$.

Solution:

$$\frac{\partial f}{\partial v} = \lim_{t\to 0} \frac{f\left((0,0,1)+t\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right) - f(0,0,1)}{t}$$

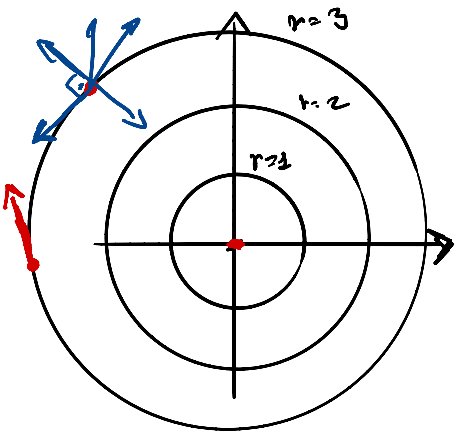
$$= \lim_{t\to 0} \frac{f\left(\frac{t}{\sqrt{3}}, \frac{t}{\sqrt{3}}, 1+\frac{t}{\sqrt{3}}\right) - f(0,0,1)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{t^2}{3} + \frac{t^2}{3} + \left(1+\frac{t}{\sqrt{3}}\right)^2 - 0^2 - 0^2 - 1^2}{t}$$

$$= \lim_{t\to 0} \frac{\frac{2t^2}{3} + \cancel{1} + \frac{2t}{\sqrt{3}} + \frac{t^2}{3} - \cancel{1}}{t}$$

$$= \frac{2}{\sqrt{3}}. \qquad \blacksquare$$

Gradients



$f(x,y) = x^2 + y^2$.
Level curves on the side.

Question: how to characterize directions of maximal change?
We want to maximize

$$\left| \frac{\partial f}{\partial v} \right|$$

relative to the unit vector v.

Looking closer into our examples:
if $v = m\,\hat{i} + n\,\hat{j}$ then

$$\frac{\partial f}{\partial v} = m \cdot \frac{\partial f}{\partial \hat{i}} + n \cdot \frac{\partial f}{\partial \hat{j}} = m\,\partial_x f + n\,\partial_y f.$$

Exercise 3: $f(x,y) = x \cdot y$,

base : $(1,1)$

direction: $v = \frac{3}{5} i + \frac{4}{5} j$.

Compute:

$$\left(\frac{3}{5} \cdot \frac{\partial f}{\partial x} + \frac{4}{5} \frac{\partial f}{\partial y}\right)(1,1)$$

Solution:

$$\frac{\partial f}{\partial x} = y \quad , \quad \frac{\partial f}{\partial y} = x$$

$$\left(\frac{3}{5} \cdot y + \frac{4}{5} \cdot x\right)(1,1) = \frac{3}{5} \cdot 1 + \frac{4}{5} \cdot 1$$

$$= 7/5 .$$

Exercise 4: $f(x,y,z) = x^2+y^2+z^2$

$v = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

$p = (0, 0, 1)$

Compute: $\left(\frac{1}{\sqrt{3}} \frac{\partial f}{\partial x} + \frac{1}{\sqrt{3}} \frac{\partial f}{\partial y} + \frac{1}{\sqrt{3}} \frac{\partial f}{\partial z}\right)(0,0,1).$

Solution:

$\frac{\partial f}{\partial x} = 2x \; ; \quad \frac{\partial f}{\partial y} = 2y \; ; \quad \frac{\partial f}{\partial z} = 2z.$

$\left(\frac{1}{\sqrt{3}} \cdot 2x + \frac{1}{\sqrt{3}} \cdot 2y + \frac{1}{\sqrt{3}} \cdot 2z\right)(0,0,1)$

$= 2/\sqrt{3}.$ ∎

For sufficiently nice functions (polynomials, exponentials, trigonometric functions) the procedure outlined above simplifies computation of directional derivatives.

Consider the vector

$$w = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

Then for any unit vector v,

$$\frac{\partial f}{\partial v} = v \cdot w.$$

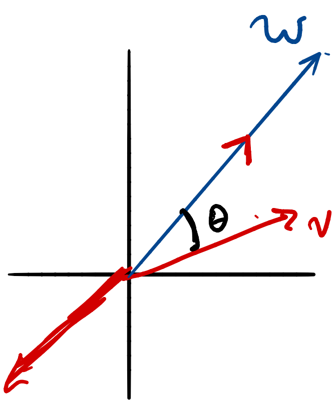
In particular:

$$\left| \frac{\partial f}{\partial v} \right| = | v \cdot w |$$

$$= \|v\| \cdot \|w\| \cdot |\cos \theta|$$

$$= 1 \cdot \|w\| \cdot |\cos \theta|$$

$$= \|w\| \cdot |\cos \theta|$$



To maximize the rate of change (in absolute value) we need $\theta = 0$ or π, that is $v \parallel w$.

We call w the gradient of the function, and henceforth denote it

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$
$$= \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} k$$

Also, if $\theta = \pi/2$, that is the direction v is perpendicular to the gradient ∇f, then the function does not grow along v (at the point we are studying). These are tangent directions.

<u>Gradients are perpendicular to level curves!</u>

Properties of gradients:

i) $\nabla(\lambda \cdot f) = \lambda \cdot \nabla f$, λ is a constant, f is a function.

ii) $\nabla(f+g) = \nabla f + \nabla g$.

iv) $\nabla(f \cdot g) = (\nabla f) \cdot g + f \cdot (\nabla g)$

Exercise 5: Find the gradient of

$$f(x,y,z) = 2xy + 2yz + 4xz^2.$$

Describe it in coordinate form $(\_,\_,\_)$.

Solution:

$$\frac{\partial f}{\partial x} = 2y + 4z^2 \; ; \; \frac{\partial f}{\partial y} = 2x + 2z \; ; \; \frac{\partial f}{\partial z} = 2y + 8xz.$$

$$\nabla f = (2y + 4z^2,\ 2x + 2z,\ 2y + 8xz)$$

$\blacksquare$

<u>Exercise 6:</u> Given

$$f(x,y) = 2xy + y^2$$
$$g(x,y) = 4x^2 + 3y^3$$

Find

$$\nabla(f \cdot g).$$

<u>Solution:</u>

$$\nabla f = (2y,\ 2x+2y)$$
$$\nabla g = (8x,\ 9y^2).$$

$$\nabla(f g) = (2y, 2x+2y)\cdot g + f\cdot(8x, 9y^2).$$
$$= (2y,\ 2x+2y)\cdot(4x^2+3y^3) + (2xy+y^2)(8x, 9y^2)$$
$$= (8x^2y + 6y^4 + 16x^2y + 8xy^2,$$
$$(2x+2y)(4x^2+3y^3) + (2xy+y^2)\cdot 9y^2)$$
$$= (24x^2y + 8xy^2 + 6y^4,$$
$$8x^3 + 6xy^3 + 8yx^2 + 6y^4 + 18xy^3 + 9y^4)$$
$$= (24x^2y + 8xy^2 + 6y^4$$
$$8x^3 + 8x^2y + 24xy^3 + 15y^4).$$

Applications to Optimization

Definition: A critical point of a function f is a point at which its gradient vanishes.

Examples:

1) $f(x,y) = x^2 + y^2$

$\nabla f(x,y) = (2x, 2y)$

Q: where is $\nabla f = 0$?

$(2x, 2y) = 0 \Rightarrow x = y = 0.$

The origin is the only critical point.

2) $f(x,y) = 2x^2 + y^2 + 8x - 6y + 20.$

$\nabla f(x,y) = (4x + 8, 2y - 6)$

Q: where is $\nabla f = 0$?

$(4x+8, 2y-6) = 0 \Rightarrow \boxed{x = -2, \; y = 3}$

Critical point: $(-2, 3)$.

Critical point vs critical value

Critical point is the point at which gradient is zero. The value of the function at such point is the critical value.

- Types of critical points for 2-variable functions

i) local maximum: 

ii) local minimum: 

iii) saddle point: 

Classification (second partials test)

Consider the functions

i) $f(x,y) = -3x^2 - 4y^2 \rightarrow \nabla f = (-6x, -8y)$

ii) $g(x,y) = x^2 + y^2 \rightarrow \nabla g = (2x, 2y)$

iii) $h(x,y) = x^2 - y^2 \rightarrow \nabla h = (2x, -2y)$.

At $(0,0)$, all gradients ∇f, ∇g, ∇h vanish.

- $\nabla f = (-6x, -8y)$

$\partial_{xx} f = -6$; $\partial_{yx} f = 0, \partial_{xy} f = 0, -8)$.

$$\begin{pmatrix} \partial_{xx} f & \partial_{xy} f \\ \partial_{yx} f & \partial_{yy} f \end{pmatrix} = \underbrace{\begin{pmatrix} -6 & 0 \\ 0 & -8 \end{pmatrix}}_{\text{local max}}.$$

- $\nabla g = (2x, 2y)$

$\partial_{xx} g = 2$, $\partial_{yx} g = 0 = \partial_{xy} g$, $\partial_{yy} g = 2$,

$$\begin{pmatrix} \partial_{xx} g & \partial_{xy} g \\ \partial_{yx} g & \partial_{yy} g \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix};$$

$\hookrightarrow$ local min

- $\nabla h = (2x, -2y)$

$\partial_{xx} h = 2$, $\partial_{yx} h = \partial_{xy} h = 0$, $\partial_{yy} h = -2$.

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

$\hookrightarrow$ saddle.

In general, if off diagonal terms are not zero, we need:

$$d = \det \begin{pmatrix} \partial_{xx} f & \partial_{xy} f \\ \partial_{yx} f & \partial_{yy} f \end{pmatrix} = (\partial_{xx} f)(\partial_{yy}) - (\partial_{xy} f)^2$$

If

1) $d > 0$: $\partial_{xx} f > 0$: local minimum
2) $d > 0$: $\partial_{xx} f < 0$: local maximum
3) $d < 0$: is a saddle point.
4) $d = 0$: test is inconclusive.

Exercise 7: Find all critical points of

$$f(x,y) = x^2 - y^2 - x - y$$

and determine their type, if possible.

Solution: $\partial_x f = 2x - 1 \rightarrow$ simultaneously zero
$\partial_y f = -2y - 1.$ if $x = \frac{1}{2}, y = -\frac{1}{2}$

$$\partial_{xx} f = 2 \; ; \; \partial_{xy} f = 0 = \partial_{yx} f; \; \partial_{yy} f = -2.$$

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \rightarrow \text{saddle point } (d = -4)$$

Exercise 8: Find all critical points of

$$f(x,y) = x^2 - 3xy - y^2$$

and classify them.

Solution:

$$\partial_x f = 2x - 3y \rightarrow \text{Simultaneously zero}$$
$$\partial_y f = -3x - 2y.$$

$x = y = 0.$

$$\begin{cases} 2x = 3y \\ -3x = 2y \end{cases} \rightarrow \begin{array}{l} \frac{2}{3}x = y \\ -\frac{3}{2}x = y \end{array}$$

$$\partial_{xx} f = 2; \ \partial_{xy} f = -3 = \partial_{yx} f; \ \partial_{yy} f = -2.$$

$$d = \det\begin{pmatrix} 2 & -3 \\ -3 & -2 \end{pmatrix} \Rightarrow d = 2\cdot(-2) - (-3)\cdot(-3)$$
$$= -4 - 9$$
$$= -13.$$

Saddle point!

Exercise 9: Find and classify all critical points of

$$f(x,y) = \sqrt[3]{(x^2+y^2)} + 2.$$

Solution:
$$\partial_x f = \frac{1}{3}\frac{2x}{\sqrt[3]{(x^2+y^2)^2}}$$
$$\partial_y f = \frac{2y}{3\sqrt[3]{(x^2+y^2)^2}}$$
$\to$ Simultaneously zero if $x=y=0$.

Do $\partial_x f$, $\partial_y f$ make sense at $(0,0)$?

$$\lim_{(x,y)\to(0,0)} \frac{2x}{3\sqrt[3]{(x^2+y^2)^2}} = \text{DNE}.$$

The origin $(0,0)$ is a global minimum, although gradient test fails.