MAT 303 RECITATIONS: WEEK 15

MARLON DE OLIVEIRA GOMES

Section 5.2: The Eigenvalue Method for Homogeneous Systems

In this section we consider again systems of differential equations, and how to solve them via the eigenvalue method. We will deal with larger matrices and complex eigenvalues.

Example 1. *This example is extracted from problem 5.1.20 in our textbook. Consider the first-order system*

$$\begin{aligned} x_1' &= 5x_1 + x_2 + 3x_3 \\ x_2' &= x_1 + 7x_2 + x_3 \\ x_3' &= 3x_1 + x_2 + 5x_3 \end{aligned}$$

The coefficient matrix is

$$P = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix}$$

Its characteristic polynomial is

$$p(r) = \det(A - rI) = 108 - 84r + 17r^2 - r^3,$$

with roots $r_1 = 2, r_2 = 6, r_3 = 9$. The associated eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

respectively. The general solution to the problem is

$$\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + C_3 e^{9t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example 2. *Consider the system of differential equations extracted from problem 5.2.27,*

$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{5} & 0 \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \mathbf{x},$$

with initial condition

$$x(0) = \begin{bmatrix} 15 \\ 0 \end{bmatrix}.$$

The characteristic equation of the problem is

$$r^2 + \frac{3r}{5} + \frac{4}{25} = 0,$$

with roots $r_1 = -\frac{2}{5}, r_2 = -\frac{1}{5}$. *The associated eigenvectors are*

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

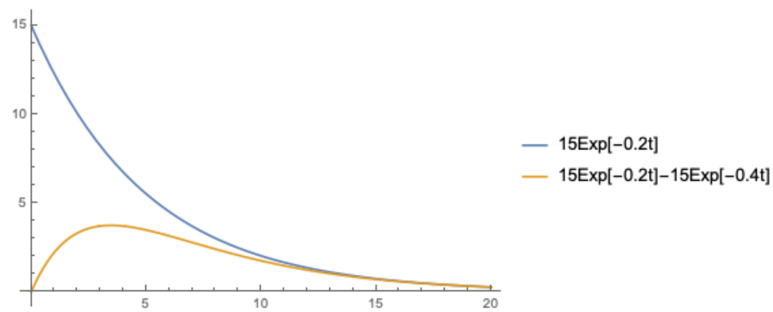
A general solution to the problem takes the form

$$\mathbf{x}(t) = C_1 e^{-\frac{2t}{5}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 e^{-\frac{t}{5}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To match the initial data, we need $C_1 = 15, C_2 = -15$, *thus the solutions* x_1, x_2 *are*

$$x_1(t) = 15e^{-\frac{t}{5}}$$
$$x_2(t) = 15e^{-\frac{t}{5}} - 15e^{\frac{-2t}{5}}$$

The function x_2 *attains a maximum value of* $\frac{15}{4}$ *at* $t = 5\log(2)$. *Below is a comparative plot of the graphs of the solutions*



Example 3. *In this example, extracted from problem 5.1.36, we are given the system*

$$\mathbf{x}' = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix} \mathbf{x},$$

with initial condition

$$\mathbf{x}(0) = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}.$$

The characteristic polynomial is

$$p(r) = -\frac{9r}{20} - \frac{6r^2}{5} - r^3,$$

with roots $r_1 = 0, r_2 = -\frac{3}{5} + \frac{3i}{10}, r_3 = -\frac{3}{5} - \frac{3i}{10}$*, and associated eigenvectors*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \frac{5}{2} \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{2} - \frac{3i}{2} \\ -\frac{1}{2} + \frac{3i}{2} \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -\frac{1}{2} + \frac{3i}{2} \\ -\frac{1}{2} - \frac{3i}{2} \\ 1 \end{bmatrix}.$$

The general solution of the problem can be written in terms of complex exponentials as

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ \frac{5}{2} \\ 1 \end{bmatrix} + C_2 e^{-\frac{3t}{5} + \frac{3it}{10}} \begin{bmatrix} -\frac{1}{2} - \frac{3i}{2} \\ -\frac{1}{2} + \frac{3i}{2} \\ 1 \end{bmatrix} + C_3 e^{-\frac{3t}{5} - \frac{3it}{10}} \begin{bmatrix} -\frac{1}{2} + \frac{3i}{2} \\ -\frac{1}{2} - \frac{3i}{2} \\ 1 \end{bmatrix}$$

At time $t = 0$*, we observe*

$$\begin{bmatrix} C_1 + C_2\left(-\frac{1}{2} - \frac{3i}{2}\right) + C_3\left(-\frac{1}{2} + \frac{3i}{2}\right) \\ \frac{5C_1}{2} + C_2\left(-\frac{1}{2} + \frac{3i}{2}\right) + C_3\left(-\frac{1}{2} - \frac{3i}{2}\right) \\ C_1 + C_2 + C_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$$

By inverting the coefficient matrix (or alternatively, solving it by elimination) we find

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 + 4i \\ -2 - 4i \end{bmatrix},$$

thus we find that

$$\mathbf{x}(t) = 4 \begin{bmatrix} 1 \\ \frac{5}{2} \\ 1 \end{bmatrix} + (-2+4i) e^{-\frac{3t}{5} + \frac{3it}{10}} \begin{bmatrix} -\frac{1}{2} - \frac{3i}{2} \\ -\frac{1}{2} + \frac{3i}{2} \\ 1 \end{bmatrix} + (-2-4i) e^{-\frac{3t}{5} - \frac{3it}{10}} \begin{bmatrix} -\frac{1}{2} + \frac{3i}{2} \\ -\frac{1}{2} - \frac{3i}{2} \\ 1 \end{bmatrix}.$$

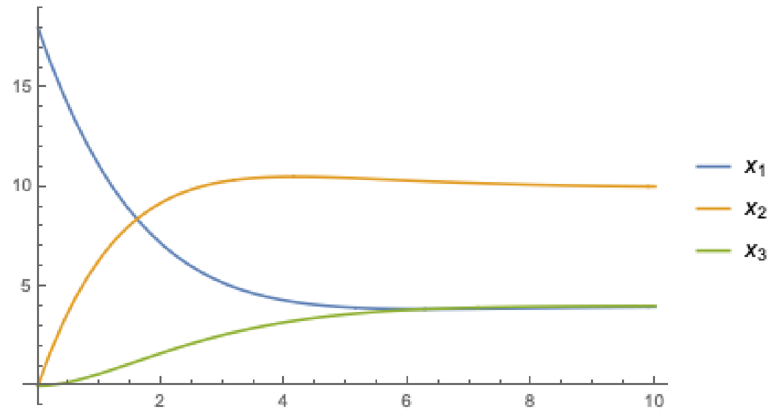
By converting complex exponentials into trigonometric functions via Euler's formula, we find the real form of the solution,

$$\mathbf{x}(t) = \begin{bmatrix} 4 + 2e^{-3t} 5\left(7\cos\left(\frac{3t}{10}\right) - 2\sin\left(\frac{3t}{10}\right)\right) \\ 10 - 10e^{\frac{-3t}{5}}\left(\cos(\left(\frac{3t}{10}\right) - \sin\left(\frac{3t}{10}\right)\right) \\ 4 - 4e^{\frac{3t}{5}}\left(\cos(\left(\frac{3t}{10}\right) - 2\sin\left(\frac{3t}{10}\right)\right) \end{bmatrix}$$

The limiting amounts of salt in each tank are

$$\lim_{t\to\infty} x_1(t) = 4, \ \lim_{t\to\infty} x_1(t) = 10, \ \lim_{t\to\infty} x_3(t) = 4.$$

Below is a comparative plot of the graphs of the solutions



Section 5.4: Second-Order Systems and Mechanical Applications.

In section 5.4 we discuss mechanical systems described by second-order systems of type

$$M\mathbf{x}^{'} = K\mathbf{x},$$

where M is a diagonal matrix describing the masses of the system, while K is called the stiffness matrix, and encodes the resulting spring constants. For simplicity, we assume that the matrix $A = M^{-1}K$ has only negative eigenvalues.

Example 4. *Consider the following system, extracted from problem 5.4.4, we are given masses $m_1 = m_2 = 1$, and stiffness matrix*

$$K = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$$

The characteristic equation for this problem is

$$r^2 + 6r + 5 = 0,$$

with roots $r_1 = -5, r_2 = -1$, and associated eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solutions can thus be written as

$$x(t) = \begin{bmatrix} -C_1 \cos(\sqrt{5}t) - C_2 \sin(\sqrt{5}t) \\ C_1 \cos(\sqrt{5}t) + C_2 \sin(\sqrt{5}t) \end{bmatrix} + \begin{bmatrix} C_3 \cos(t) + C_4 \sin(t) \\ C_3 \cos(t) + C_4 \sin(t) \end{bmatrix}$$

References

[1] C. Henry Edwards, David E. Penney and David T. Calvis, *Differential Equations and Boundary Value Problems: Computing and Modelling*, 5th edition, Pearson, 2014.

Mathematics Department 3-101, Stony Brook University, 100 Nicolls Road, Math Tower, Stony Brook, NY, 11794, USA

E-mail address: `mgomes@math.stonybrook.edu`