MAT324: Real Analysis – Fall 2014

Assignment 1 – Solutions

Problem 1: Let $\mathcal{C}$ be the Cantor middle-thirds set constructed in the textbook. Show that $\mathcal{C}$ is compact, uncountable, and a null set.

Solution. The textbook proves that $\mathcal{C}$ is a null set (page 19). To check that $\mathcal{C}$ is compact, notice that it is bounded, $\mathcal{C} \subset [0,1]$, and each $\mathcal{C}_n$ constructed in the definition of $\mathcal{C}$ is closed, so that

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}_n$$

is a closed set. Hence, by the Heine-Borel Theorem, $\mathcal{C}$ is closed. To prove that $\mathcal{C}$ is uncountable, consider for each $x \in \mathcal{C}$ its infinite ternary expansion[1]

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}.$$

As shown in the textbook, since $x \in \mathcal{C}$, $a_k = 0$ or 2, for each $k \in \mathbb{N}$. Suppose there is an enumeration of the Cantor set, $\mathcal{C} = \{x_1, x_2, \cdots, \}$, where

$$x_1 = \sum_{k=1}^{\infty} \frac{a_{1k}}{3^k}$$

$$x_2 = \sum_{k=1}^{\infty} \frac{a_{2k}}{3^k}$$

$$\vdots$$

Then $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$ where, $a_k = |2 - a_{kk}|$ is not on the list, and belongs to the Cantor middle-third set, hence $\mathcal{C}$ is uncountable (check that the a_k's are not eventually zero!). $\square$

Problem 2: Let A be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $m(A)$.

Solution. Let $x = \sum_{k=1}^{\infty} \frac{x_k}{10^k}$ be the infinite decimal representation of x. By a similar argument given in the construction of the Cantor middle-third set, one can construct A by the following procedure:

1. Let $A_0 = [0,1]$.
2. Define A_1 by removing from A_0 the set $(\frac{4}{10}, \frac{5}{10})$, i.e., all the numbers x whose infinite decimal representation is such that $x_1 = 4$.

[1] Recall that in such an expansion, if $a_k = 0$, for all $k > N$, for some $N \in \mathbb{N}$ and $a_N \neq 0$, we replace the a_k by $\overline{a}_k = 2$, if $K > N$, and a_N by $\overline{a}_N = a_N - 1$.

3. Define A_2 by removing from A_1 the sets $(\frac{4+10k}{100}, \frac{5+10k}{100})$, where $0 \le k \le 9$ thus removing all the numbers left in A_1 such that $x_2 = 4$.

4. Assume A_n has been defined. Define A_{n+1} removing from A_n all the intervals of the form

$$\left(\frac{4+10^n k}{10^{n+1}}, \frac{5+10^n k}{10^{n+1}}\right),$$

for $0 \le k \le 10^k - 1$, thus removing from A_n all the numbers left in A_1 such that $x_{n+1} = 4$.

Now this description is not optimal, since some of the intervals have already been removed in the previous steps. In fact, in the n-th step we remove 9^{n-1} disjoint intervals, each of them with lenght 10^{-n}. In addition, $A = \cap_{n\in\mathbb{N}} A_n$ and

$$m(A_n) = 1 - \sum_{k=1}^{n} \frac{9^{n-1}}{10^n}$$

Since the A_n form a descreasing sequence (i.e., $A_n \supset A_{n-1}$),

$$m(A) = m(\cap A_n) = \lim_{n\to\infty} m(A_n) = 1 - \sum_{k=1}^{\infty} \frac{9^{n-1}}{10^n} = 0$$

$\square$

Problem 3: Let A be a null set. Show that $m^*(A \cup B) = m^*(B)$ for any set B.

Solution. By monotonicity,

$$m^*(B) \le m^*(A \cup B).$$

By subadditivity,

$$m^*(A \cup B) \le m^*(A) + m^*(B) = m^*(B).$$

$\square$

Problem 4: Let $E_1, E_2, \ldots, E_n$ be disjoint measurable sets. Show that for all $A \subseteq \mathbb{R}$, we have

$$m^*\left(A \cap \left(\bigcup_{j=1}^{n} E_j\right)\right) = \sum_{j=1}^{n} m^*\left(A \cap E_j\right).$$

Solution. Notice that since the E_j's are measurable,

$$m^*\left(A \cap \left(\bigcup_{j=1}^{n} E_j\right)\right) = m^*\left(\left[A \cap \left(\bigcup_{j=1}^{n} E_j\right)\right] \cap E_n\right) + m^*\left(\left[A \cap \left(\bigcup_{j=1}^{n} E_j\right)\right] \cap (E_n)^c\right)$$

Now since the E_j are disjoint,

$$m^*\left(A \cap \left(\bigcup_{j=1}^{n} E_j\right)\right) = m^*\left(A \cap E_n\right) + m^*\left(A \cap \left(\bigcup_{j=1}^{n-1} E_j\right)\right)$$

Therefore the result can be proved by induction on n. $\square$

Problem 5: Suppose $E_1, E_2 \subseteq \mathbb{R}$ are measurable sets. Show that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Solution. Notice that $E_1 \cup E_2$ can be expressed as a union of disjoint measurable sets

$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_1 \cap E_2) \cup (E_2 \setminus E_1).$$

Additivity implies that

$$\begin{aligned} m(E_1 \cup E_2) &= m(E_1 \setminus E_2) + m(E_1 \cap E_2) + m(E_2 \setminus E_1) \\ &= m(E_1 \cap E_2) + m(E_1 \cap (E_2)^c) + m(E_2 \cap (E_1)^c) \end{aligned}$$

Hence

$$\begin{aligned} m(E_1 \cup E_2) + m(E_1 \cap E_2) &= [m(E_1 \cap E_2) + m(E_1 \cap (E_2)^c)] + [m(E_2 \cap (E_1)^c) + m(E_1 \cap E_2)] \\ &= m(E_1) + m(E_2) \end{aligned}$$

$\square$