MAT 514 - Lecture 16

Power series

Definition: a power series is sequence of polynomials in a complex variable,

$$s_n = a_0 + a_1 (z-c) + \dots + a_n (z-c)^n .$$

Often we use the notation

$$s_n = \sum_{k=0}^{n} a_k (z-c)^k$$

to refer to a specific element, and

$$\sum_{k \geq 0} a_k (z-c)^k \qquad (*)$$

to refer to the entire series.

Remark: In principle the infinite sum $(*)$ is meaningless. We'll study under which conditions it makes sense.

Examples

1 A finite power series is one whose terms after a certain order vanish

$$a_k = 0,$$

for all $k > k_0$. In this case we can identify the series with a polynomial,

$$\sum_{k\geq 0}' a_k (z-c)^k = a_0 + a_1(z-c) + a_2(z-c)^2 + \dots + a_{k_0}(z-c)^{k_0}.$$

2 The exponential series:

$$\sum_{k\geq 0}' \frac{1}{k!} \cdot z^k.$$

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

3 The geometric series:

$$\sum_{k\geq 0}' a(z-c)^k.$$

$$= a + a(z-c) + a(z-c)^2 + \dots$$

This is called a geometric series because its terms are in geometric progression, that is, the ratio between consecutive terms is constant.

Convergence

The power series

$$\sum_{k\geq 0}' a_k (z-c)^k$$

is said to be centered at c. Typically, c = 0, but not always.

At the center, the power series has a clear meaning:

$$a_0 + a_1(c-c) + a_2(c-c)^2 + \dots + a_k(c-c)^k + \dots$$

$$= a_0 + 0 + 0 + \dots + 0 + \dots$$

Property: A power series always converges

at its center, to a value
$$\sum_{k\geq 0}' a_k (c-c)^k = a_0.$$

<u>Definition</u>: Given a power series
$$\sum_{k\geq 0}' a_k(z-c)^k$$
we say that it converges at a point z, to a value L if given $\varepsilon > 0$, we can find $k_0 \in \mathbb{N}$ such that
$$\left| L - \sum_{k=0}^{k_0}' a_k (z-c)^k \right| < \varepsilon.$$

This definition is clear, but impractical. Guessing an appropriate limit is next to impossible.

For the time being we will try to establish criteria for convergence, without worrying about the actual limit.

Example 4:

A finite power series always converges.

Let a finite power series be given

$$\sum_{k\geq 0}' a_k (z-c)^k,$$

with $a_k = 0$ for all $k > K_0$. Then we can form a polynomial

$$L(z) = a_0 + a_1(z-c) + \dots + a_{K_0}(z-c)^{K_0}.$$

Then, for any $z \in \mathbb{C}$ and any $\varepsilon > 0$,

$$\left| L(z) - \sum_{k=0}^{K_0}{}' a_k (z-c) \right|^k = 0 < \varepsilon.$$

So we can choose a uniform threshold K_0 for the error estimate.

Finite series converge to the polynomial they represent.

Example 5: Consider a geometric series

$$\sum_{k\geq 0}' z^k.$$

The question we wish to address is: for which values of z does the sum converge?

Let's take a closer look into the partial sums:

$$S_n = \sum_{k=0}^{n} z^k$$

$$S_n = 1 + z^1 + \dots + z^n$$

We can evaluate S_n as follows:

$$z \cdot S_n = z + z^2 + \dots + z^{n+1}.$$

The difference is

$$S_n - z \cdot S_n = 1 - z^{n+1}$$

$$S_n (1-z) = 1 - z^{n+1}$$

So long as $z \neq 1$,

$$\boxed{S_n(z) = \frac{1 - z^{n+1}}{1-z}}$$

The partial sums formula holds for all $z \neq 1$. Its limit is

$$\lim_{n\to\infty} s_n(z) = \lim_{n\to\infty} \frac{1-z^{n+1}}{1-z}.$$

The numerator $1-z^{n+1}$ has a limit as $n\to\infty$ if and only if $|z|<1$. We have alternatives

$$\lim_{n\to\infty} 1-z^{n+1} = \begin{cases} 1, & \text{if } |z|<1. \\ 0, & \text{if } z=1. \\ \text{Does not exist}, & \text{if } z=-1. \\ \infty, & \text{if } |z|>1. \end{cases}$$

<u>Proposition</u> (convergence of geometric series)

The geometric series

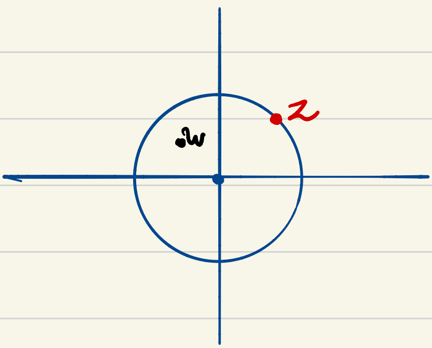
$$\sum_{k\geq 0}' z^k$$

i) converges to $\dfrac{1}{1-z}$, if $|z|<1$.

ii) diverges to ∞ if $z=1$ or $|z|>1$.
iii) does not have a limit at any other point in the unit circle.

<u>Convergence properties</u>

① Comparison principle: if a power series converges for some z, then it converges for every other w <u>closer</u> to its center.



As a consequence, if the power series is centred at c, and

$$|z-c| = r,$$

then the series converges on $D(c,r)$ (open disk, boundary may not be included in the region of convergence).

Definition: The radius of convergence of a power series is either:

a) the greatest non-negative real number r for which the power series is guaranteed to converge on $D[c, r]$.

b) ∞, if no real number satisfies property a.

Example: A polynomial (finite power series) has infinite radius of convergence.

Example: A geometric series

$$\sum_{k\geq 0} z^k$$

has radius of convergence 1.

(2) Boundedness property (Absolute convergence)

If the series of absolute values

$$\sum_{k\geq 0} |a_k (z-c)^k|$$

is a convergent real series, then the complex series

$$\sum_{k \geq 0} a_k (z-c)^k$$

converges.

<u>Example.</u>

The exponential series

$$\sum_{k \geq 0} \frac{z^k}{k!}$$

converges absolutely for $z=1$:

$$\sum_{k \geq 0} \left| \frac{1}{k!} \right| = \sum_{k \geq 0} \frac{1}{k!}$$

$$= e^1$$

$$= e.$$

Therefore the sum converges for any z with $|z|=1$:

$$\sum_{k \geq 0} \left| \frac{z^k}{k!} \right| = \sum_{k \geq 0} \frac{1}{k!}$$

Example: The x-harmonic series is

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

When $x=1$, this is simply the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k},$$

which diverges to $+\infty$. This is because we have estimates:

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$s_3 = 1 + \frac{1}{2} + \frac{1}{3} \geq \frac{3}{2}.$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \geq \frac{4}{2}$$

$$s_5 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \frac{1}{5} \geq \frac{4}{2}$$

$$\vdots$$

$$s_7 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) \geq \frac{4}{2}.$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \geq \frac{5}{2}.$$

$$s_9, s_{10}, \dots, s_{15} \geq \frac{5}{2}.$$

$$s_{16} \geq \frac{6}{2}.$$

$$\vdots$$

$$s_{32} \geq \frac{7}{2}$$

And so on, so that

$$s_{2^k} \geq \frac{2+k}{2}$$

as $k \to +\infty$, the estimates

$$\frac{2+k}{2} \to +\infty.$$

so the sums $s_n \to +\infty$.

Let's compare this to the behavior the alternating harmonic series, obtained by substituting $x = -1$.

$$\sum_{k \geq 1}' \frac{(-1)^k}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \dots$$

Claim: the alternating harmonic series converges.

Let's estimate the partial sums:

$$s_1 = -1.$$
$$s_2 = -1 + \frac{1}{2} = -\frac{1}{2}.$$

$$s_3 = -1 + \frac{1}{2} - \frac{1}{3} = -\frac{1}{2} - \frac{1}{3} = -\frac{5}{6}.$$

$$s_4 = -\frac{5}{6} + \frac{1}{4} = \frac{-20 + 6}{24} = \frac{-14}{24} = \frac{-7}{12}.$$

While the series alternates between

increasing and decreasing, the oscillation goes to 0 as $k \to \infty$.

Eventually, the series achives a limit. (by the Alternating Series Test).

In summary: the z-harmonic series converges for $z = -1$, but not for $z = 1$.

The Alternating Harmonic Series is convergent, but not absolutely convergent.

The Ratio Test

Suppose a series is such that for $k > k_0$ $a_k \neq 0$ and

$$\left| \frac{a_{k+1}}{a_k} \right|$$

has a limit, as $k \to \infty$, say

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = L$$

Then

(a) The series

$$\sum_{k\geq 0}' a_k (z-c)^k$$

converges for $|z-c| < 1/L$, if $L \neq 0$.

(b) The series

$$\sum_{k\geq 0}' a_k (z-c)^k$$

converges for all z, if $L = 0$.

Remark: One often states that the radius of convergence is

$$\frac{1}{L}$$

and interprets

$$\text{“}\frac{1}{0}\text{”}$$

as infinite radius of convergence.

In case (a) the guaranteed to

diverge if

$$|z-c| > 1/L.$$

Along the circle

$$|z-c| = 1/L$$

the test is inconclusive.

<u>Examples</u>: The exponential series

$$\sum_{k\geq 0}' \frac{z^k}{k!}$$

has infinite radius of convergence. This is because

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{\frac{1}{(k+1)!}}{\frac{1}{k!}}\right|$$

$$= \lim_{k\to\infty} \frac{k!}{(k+1)!}$$

$$= \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

Example: On the other hand, the z-harmonic series

$$\sum_{k\geq 1}' \frac{z^k}{k}$$

has radius of convergence $R=1$. In fact,

$$\lim_{k\to\infty} \frac{1/(k+1)}{1/k} = \lim_{k\to\infty} \frac{k}{k+1} = 1.$$