MAT 511 - Summer 1
June 14, 2019

This homework is due on Thursday, June 20, by 7:00 pm.

Homework 3

Exercise 1 Let $A = \{1, 2, 3\}$. Construct a relation on $A \times A$ satisfying the following properties:

(a) It is not reflexive, not symmetric, and not transitive.

(b) It is reflexive, not symmetric, and not transitive.

(c) It is not reflexive, symmetric, and not transitive.

(d) It is reflexive, symmetric, and not transitive.

(e) It is not reflexive, not symmetric, and transitive.

(f) It is reflexive, not symmetric, and transitive.

(g) It is not reflexive, symmetric, and transitive.

(h) It is reflexive, symmetric, and transitive.

Exercise 2 Define the following relation on $\mathbb{R} \times \mathbb{R}$: a point (a, b) is related to (x, y) if

$$y - b = x - a$$

Show that this relation is an equivalence, i.e., it is reflexive, symmetric, and transitive. What are the equivalence classes?

Exercise 3 Let $A = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. This is the set of pairs of integers, in which the second entry is non-zero. On this set, we consider the following relation,

$$\mathbb{Q} = \{((a, b), (c, d)) \in A \times A|\ ad = bc\}.$$

Show that this relation is an equivalence relation, that is:

(a) it is reflexive: $(a, b) \in A$ is related to itself.

(b) it is symmetric: if $((a, b), (c, d)) \in Q$, then $((c, d), (a, b)) \in Q$.

(c) it is transitive: given $((a, b), (c, d)) \in Q$, and $((c, d), (e, f)) \in Q$, then $((a, b), (e, f))$.
Furthermore, describe the equivalence classes of this relation.

Exercise 4 In each of the problems below, you are given a set A and a collection of subsets. Determine if this collection is a partition. Explain your reasoning.

(a) $A = \mathbb{N}$, $\mathcal{P} = \{\{0\}, \{n \in \mathbb{N}|\ n \text{ is even}\}, \{n \in \mathbb{N}|\ n \text{ is a prime number}\}\}$.

(b) $A = \mathbb{N}$, $\mathcal{P} = \{\{0, 1\}, \{n \in \mathbb{N}|\ n \text{ has a prime factor}\}\}$.

Exercise 5 Let $A = \{a, b, c\}$. Give an example of a relation on A that is

(a) antisymmetric and symmetric.

(b) antisymmetric, reflexive, and not symmetric.

(c) antisymmetric, not reflexive, and not symmetric.

(d) symmetric and not antisymmetric.

(e) not symmetric and not antisymmetric.

(f) irreflexive and not symmetric.

(g) irreflexive and not antisymmetric.

(h) antisymmetric, not reflexive, and not irreflexive.

(i) transitive, antisymmetric, and irreflexive.

Exercise 6 Define the relation R on $\mathbb{N}$ by the following property: a is related to b by R if there exists a non-negative integer k such that $b = 2^k a$. Show that this is a partial ordering. Does this relation have the comparability property?

Exercise 7 Consider a partially ordered set A, with order relation R. Assume that $C \subset B \subset A$. Determine whether the statements below are true or false.

(a) Every upper bound for C is an upper bound for B.

(b) Every upper bound for B is an upper bound for C.

Exercise 8 Consider an order relation on the set $A = \{a, b, c, d, e, f, g, h\}$, given by the following properties:

- $g \leq f$
- $h \leq f$
- $h \leq d$
- $f \leq c$
- $f \leq b$
- $e \leq b$
- $e \leq a$.

Construct its Hasse diagram. In addition, find the following bounds:

(a) all upper bounds for the set $\{b, f, g, h\}$.

(b) all lower bounds for the set $\{a, d\}$.

(c) the supremum, if it exists, for the set $\{e, g, h\}$.

(d) the infimum, if it exists, for the set $\{b, f, g\}$.

(e) the smallest element, if it exists, for the set $\{b, c, d\}$.