MAT 203 - Summer I 2020
July 2, 2020

Homework 4 solutions

Exercise 1 Evaluate the triple iterated integral

$$\int_0^1 \int_0^{1+\sqrt{y}} \int_0^{xy} y \, dz \, dx \, dy.$$

Solution:

$$\begin{aligned}
\int_0^1 \int_0^{1+\sqrt{y}} \int_0^{xy} y \, dz \, dx \, dy &= \int_0^1 \int_0^{1+\sqrt{y}} \left[yz \Big|_{z=0}^{z=xy} \right] dx \, dy \\
&= \int_0^1 \int_0^{1+\sqrt{y}} xy^2 \, dx \, dy \\
&= \int_0^1 \left[\frac{x^2y^2}{2} \Big|_{x=0}^{x=1+\sqrt{y}} \right] dy \\
&= \int_0^1 \frac{(1+\sqrt{y})^2 y^2}{2} \, dy \\
&= \int_0^1 \frac{(1+2\sqrt{y}+y)y^2}{2} \, dy \\
&= \int_0^1 \frac{y^2}{2} + y^{\frac{5}{2}} + \frac{y^3}{2} \, dy \\
&= \left[\frac{y^3}{6} + \frac{2}{7} y^{\frac{7}{2}} + \frac{y^4}{8} \Big|_{y=0}^{y=1} \right] \\
&= \frac{1}{6} + \frac{2}{7} + \frac{1}{8} \\
&= \frac{97}{168}
\end{aligned}$$

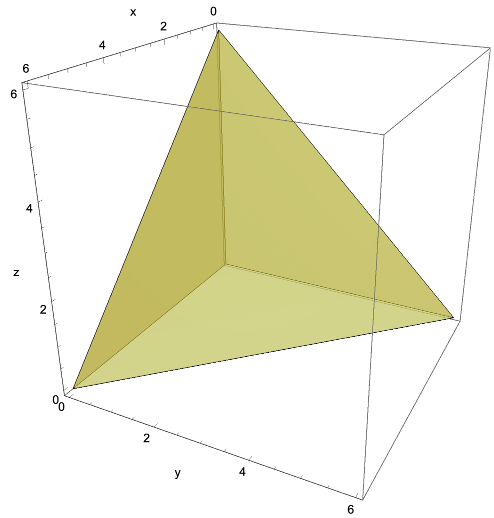
Exercise 2 Sketch the solid whose volume is given by the iterated integral. Rewrite the integral using the order

$$dydxdz$$

(do not evaluate the integral in either order).

$$\int_0^6 \int_0^{6-x} \int_0^{6-x-y} dzdydx$$

Solution: The region of integration is the tetrahedron sketched below.



We seek to describe volume of the solid by means of integration with respect to the order $dy\,dx\,dz$. That is, we consider z as an independent variable, x as depending upon z, and y depending on both x and z. The range of values that z can attain within this solid is $0 \leq z \leq 6$. For a fixed value of z, the lower bounds for x and y are constants, $x_{\text{lower}} = 0$, $y_{\text{lower}} = 0$. The upper bound for x is attained when y is at its minimum, $y = 0$. By means of the equation of the plane bounding the tetrahedron,

$$x_{\text{upper}} + 0 + z = 6 \Rightarrow x_{\text{upper}} = 6 - z.$$

For fixed x and z, the upper limit of integration relative to y can be found analogously,

$$x + y_{\text{upper}} + z = 6 \Rightarrow y_{\text{upper}} = 6 - x - z.$$

The integral can thus be rewritten as

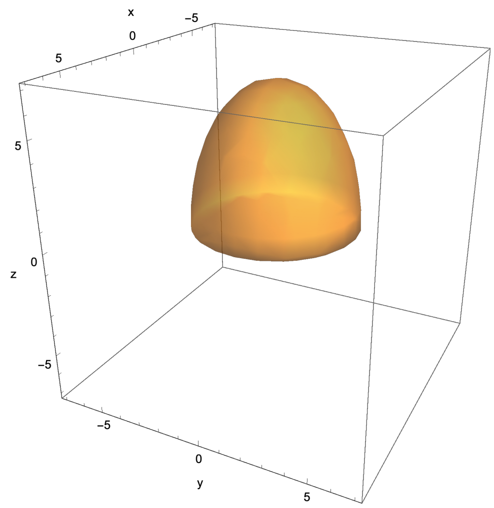
$$\int_0^6 \int_0^{6-z} \int_0^{6-x-z} dy\,dx\,dz.$$

Exercise 3 Use cylindrical coordinates to find the volume of the solid bounded above by

$$3x^2 + 3y^2 + z^2 = 45,$$

and below by the xy-plane.

Solution: Below is a plot of the region of integration.



We will use the angular variable θ as independent variable, $0 \leq \theta \leq 2\pi$. We also choose z as our intermediate variable of integration. As it turns out, the range of values for 0 does not depend on the angle θ, $0 \leq z \leq \sqrt{45} = 3\sqrt{5}$. Finally, we turn to the axial radius r, whose range of values depends upon the height z. At its maximum, r is linked to z by the equation of the ellipsoid,

$$3r_{\text{upper}}^2 + z^2 = 45 \Rightarrow r_{upper} = \sqrt{\frac{45 - z^2}{3}}$$

The volume can be computed in cylindrical coordiantes as follows:

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{3\sqrt{5}} \int_0^{\sqrt{\frac{45-z^2}{3}}} r\,dr\,dz\,d\theta \\
&= \int_0^{2\pi} \int_0^{3\sqrt{5}} \left[\frac{r^2}{2} \Big|_{r=0}^{r=\sqrt{\frac{45-z^2}{3}}} \right] dz\,d\theta \\
&= \int_0^{2\pi} \int_0^{3\sqrt{5}} \frac{45 - z^2}{6}\,dz\,d\theta \\
&= \int_0^{2\pi} \left[\frac{15z}{2} - \frac{z^3}{18} \Big|_{z=0}^{z=3\sqrt{5}} \right] d\theta \\
&= \int_0^{2\pi} 15\sqrt{5}\,d\theta \\
&= 30\pi\sqrt{5}
\end{aligned}$$

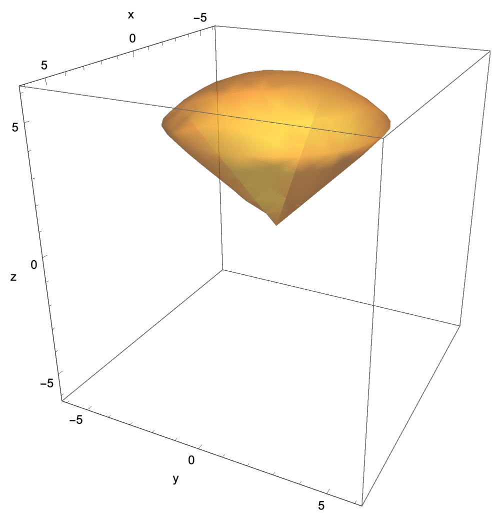
Exercise 4 Use spherical coordinates to find the volume of the solid bounded above by

$$x^2 + y^2 + z^2 = 36,$$

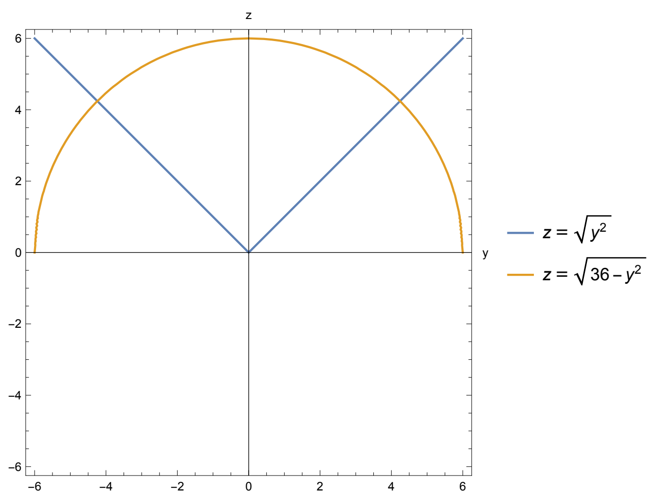
and below by

$$z = \sqrt{x^2 + y^2}.$$

Solution: Below is a plot of the region of integration.



We will choose the radius (distance to origin) as the independent variable, identifying the bounds as $r_{\text{lower}} = 0$ and $r_{\text{upper}} = 6$ (the radius of the sphere). The bounds for longitude angle θ are independent of r, $\theta_{\text{lower}} = 0, \theta_{\text{upper}} = 2\pi$. To understand the range of values for the latitude angle we project this plot onto the zy-plane ($x = 0$). Below is a projection of the bounding surfaces.



The intersection points occur when

$$y^2 = 36 - y^2 \Rightarrow y^2 = 18,$$

thus the corresponding value of z is $z = 3\sqrt{2}$. Using the relation between Cartesian and spherical coordinates, we find the angle of intersection

$$\begin{aligned}
z &= r\cos(\phi) \\
3\sqrt{2} &= 6\cos(\phi) \\
\frac{\sqrt{2}}{2} &= \cos(\phi) \\
\frac{\pi}{4} &= \phi.
\end{aligned}$$

Thus we compute the volume in spherical coordinates as

$$\begin{aligned}
V &= \int_0^6 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} r^2 \sin(\phi)\, d\phi\, d\theta\, dr \\
&= \int_0^6 \int_0^{2\pi} \left[-r^2 \cos(\phi) \Big|_{\phi=0}^{\phi=\frac{\pi}{4}} \right] d\theta\, dr \\
&= \int_0^6 \int_0^{2\pi} \frac{r^2(2-\sqrt{2})}{2}\, d\theta\, dr \\
&= \int_0^6 r^2(2-\sqrt{2})\pi\, dr \\
&= \left[\frac{r^3(2-\sqrt{2})\pi}{3} \Big|_{r=0}^{r=6} \right] \\
&= 72(2-\sqrt{2})\pi.
\end{aligned}$$

Exercise 5 Use the change of variables

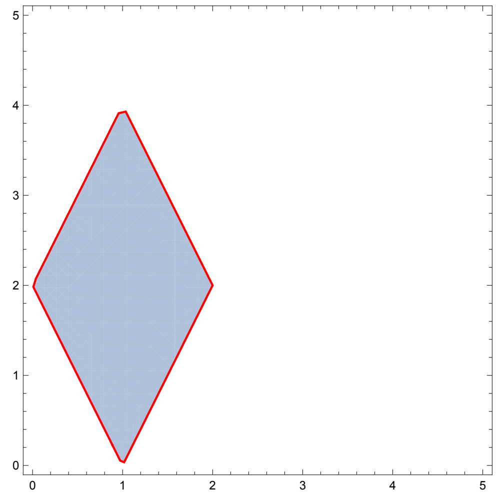
$$x = \frac{u+v}{4}, y = \frac{v-u}{2}$$

to evaluate the integral

$$\int\int_R 16xy\, dA,$$

where R is the parallelogram with vertices $(0, 2), (1, 0), (1, 4), (2, 2)$.

Solution: Below is a plot of the region of integration.



The boundary curves are described by the following equations,

$$\begin{aligned} 2x - y &= -2, \\ 2x - y &= 2, \\ 2x + y &= 2, \\ 2x + y &= 6. \end{aligned}$$

The equations relating x, y to u, v may be rewritten as

$$u = 2x - y, \;\; v = 2x + y,$$

whereas the relation between the area elements is

$$dA = dx\,dy = \frac{1}{4} du\,dv.$$

We may thus calculate the integral in terms of the new coordinates as

$$\begin{aligned} \int\int_R 16xy\,dA &= \int_2^6 \int_{-2}^2 16\left(\frac{u+v}{4}\right)\left(\frac{v-u}{2}\right)\frac{1}{4}\,du\,dv \\ &= \int_2^6 \int_{-2}^2 \frac{v^2-u^2}{2}\,du\,dv \\ &= \int_2^6 \left[\frac{uv^2}{2} - \frac{u^3}{6}\Big|_{u=-2}^{u=2}\right] dv \\ &= \int_2^6 -\frac{8}{3} + 2v^2\,dv \\ &= \left[-\frac{8v}{3} + \frac{2v^3}{3}\Big|_{v=2}^{v=6}\right] \\ &= 128. \end{aligned}$$

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Exercise 6 Use the change of variables

$$x = u, y = \frac{v}{u}$$

to evaluate the integral

$$\int\int_R \frac{x}{1 + x^2y^2} dA$$

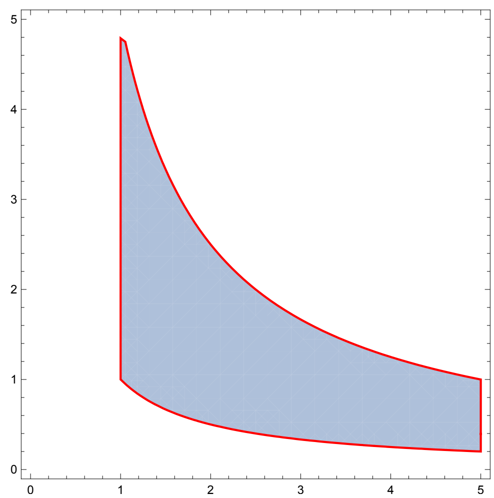
where

$$R$$

is the region bounded by the curves

$$x = 1, x = 5, xy = 1, xy = 5.$$

Solution: Below is a plot of the region of integration.



The relations between differentials (in the region where $u \neq 0$) are

$$\begin{aligned} dx &= du \\ dy &= -\frac{v}{u^2} du + \frac{1}{u} dv, \end{aligned}$$

therefore the area element can be expressed as

$$dA = dy\, dy = \frac{1}{u} du\, dv.$$

The integral is thus

$$
\begin{aligned}
\int\int_R \frac{x}{1+x^2y^2} dA &= \int_1^5 \int_1^5 \left(\frac{u}{1+v^2}\right)\left(\frac{1}{u}\right)\, du\, dv \\
&= \int_1^5 \int_1^5 \left(\frac{1}{1+v^2}\right)\, du\, dv \\
&= \int_1^5 \left[\frac{u}{1+v^2}\bigg|_{u=1}^{u=5}\right]\, dv \\
&= \int_1^5 \frac{4}{1+v^2}\, dv \\
&= \left[4\arctan(v)\bigg|_{v=1}^{v=5}\right] \\
&= 4\arctan(5) - \pi.
\end{aligned}
$$