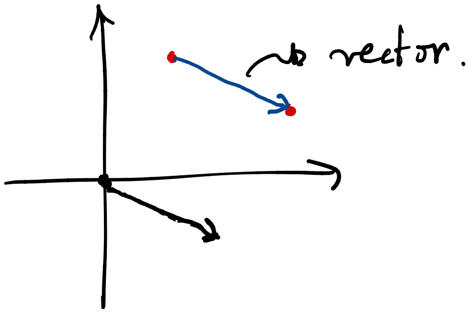
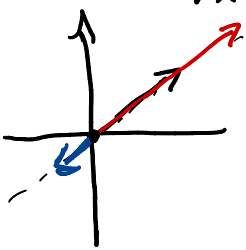
MAT 203 - Lecture 1.

What is a vector?



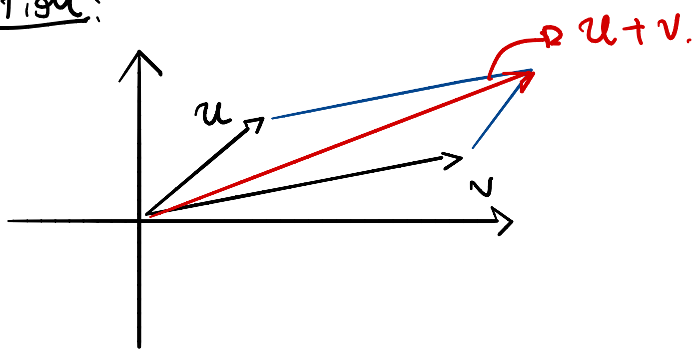
- All our vectors start from the origin.

Vectors carry on some operation from real numbers.

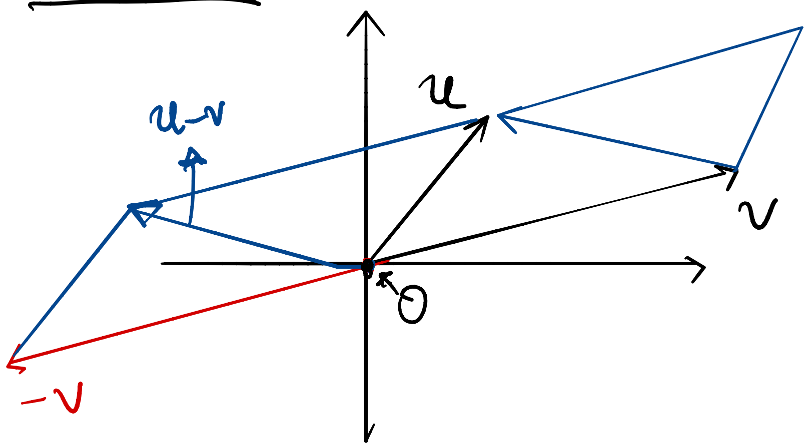


- Scalings preserve direction.
- Scalings can change orientation.

Addition:



Differences: $u-v$?



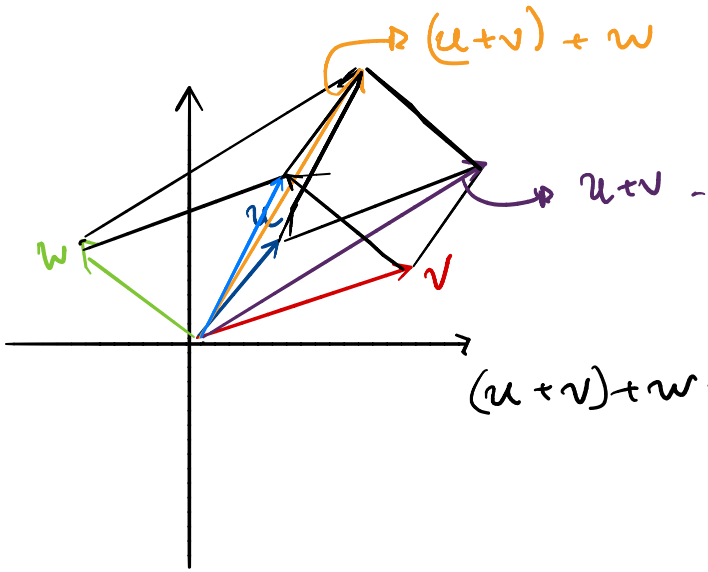
Properties of addition

1) O, the origin, is neutral
$$O+u = u+O = u.$$

2) $u+v = v+u$

3) $u+(v+w) = (u+v)+w$ *

(*) 4) Any non-zero vector has an inverse
$$v+(-v) = O.$$



$(u+v)+w = u+(v+w)$.

Properties of scaling:

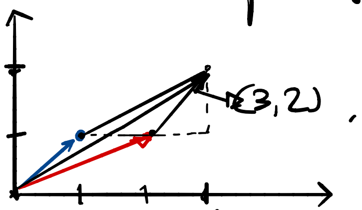
1) $1 \cdot u = u$
2) $\alpha \cdot (\beta u) = (\alpha \cdot \beta) \cdot u$

Distributive laws:

1) $\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$
2) $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$

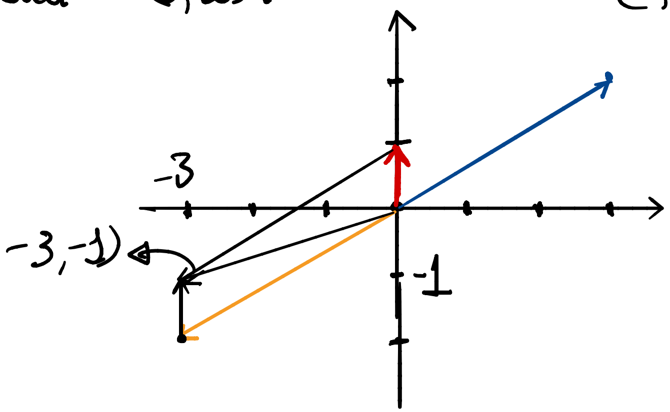
Exercises:

1) Find the sum of vectors $(1,1)$ and $(2,1)$ in the plane.



2) Find the difference between $(0,1)$ and $(3,2)$.

$(0,1)-(3,2)$.

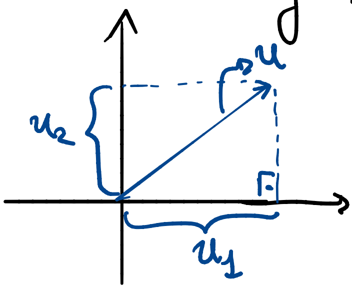


Observation: In Cartesian coordinates, sums work component-wise:

$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1+v_1, u_2+v_2, u_3+v_3)$.

Likewise for scaling us: $\alpha(u_1, u_2, u_3) = (\alpha u_1, \alpha u_2, \alpha u_3)$.

· Norm (or magnitude).



norm

$$\|u\|^2 = u_1^2 + u_2^2$$

$$\|u\| = \sqrt{u_1^2 + u_2^2}$$

In 3D: $\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

<u>Exercise:</u>

3) Find the norm of the vector $u = (3, 4)$.

$$\|u\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

4) Find the norm of the vector $v = (1, 1, 1)$.

$$\|v\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

Nomenclature: if a vector has norm 1, we call it a <u>unit vector</u> or <u>normalized</u>.

Properties of norm:

1) Positive-definite: $\|u\| \geq 0$, and $\|u\| = 0$ if and only if $u = 0$.

2) Scaling dependency: $\|\alpha \cdot u\| = |\alpha| \cdot \|u\|$

Examples:

i) $u = (3, 4)$, $\alpha = 2$: $\alpha \cdot u = (6, 8)$.

$$\|\alpha \cdot u\| = \sqrt{6^2 + 8^2}$$
$$= \sqrt{36 + 64}$$
$$= \sqrt{100}$$
$$= 10$$

ii) $v = (1, 1, 1)$, $\alpha = -3$; $\alpha \cdot v = (-3, -3, -3)$

$$\|\alpha \cdot v\| = \sqrt{(-3)^2 + (-3)^2 + (-3)^2}$$
$$= \sqrt{9 + 9 + 9}$$
$$= \sqrt{27}$$
$$= 3\sqrt{3}$$

3) Triangle inequality:

$\|u + v\| \leq \|u\| + \|v\|$.

Examples:

iii) $u = (1, 2)$, $v = (2, 2)$, $u + v = (3, 4)$

$$\|u\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$
$$\|v\| = \sqrt{2^2 + 2^2} = \sqrt{8}$$

$$\|u + v\| = 5 < \underset{\approx 2.23}{\sqrt{5}} + \underset{\approx 2.83}{\sqrt{8}}$$
$$\approx \boxed{5.06}$$

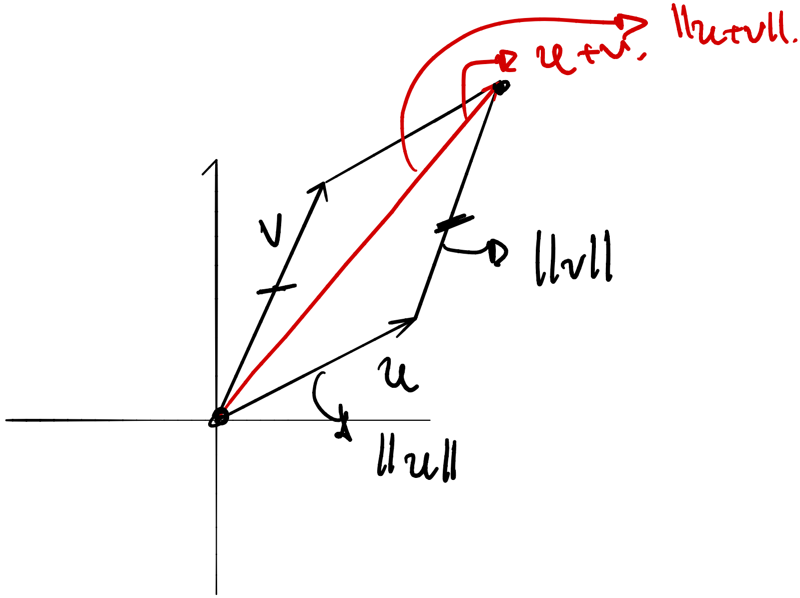
iv) $u = (0, 1)$, $v = (0, 2)$, $u + v = (0, 3)$

- $\|u\| = \sqrt{0^2 + 1^2} = 1$
- $\|v\| = \sqrt{0^2 + 2^2} = 2$
- $\|u + v\| = \sqrt{0^2 + 3^3} = 3.$

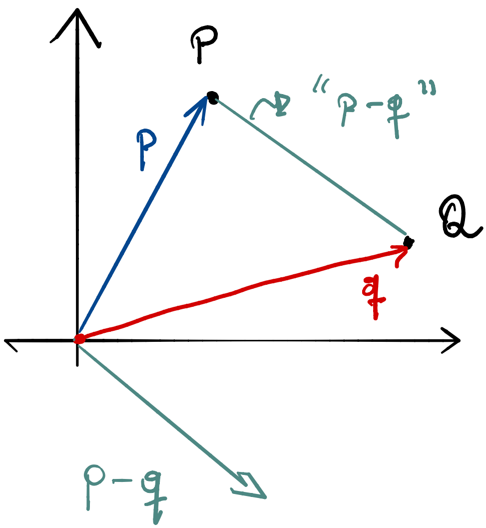
In this case $\|u + v\| = \|u\| + \|v\|$.

v) $u = (0, 1)$, $v = (0, -2)$, $u + v = (0, -1)$

- $\|u\| = \sqrt{0^2 + 1^2} = 1$
- $\|v\| = \sqrt{0^2 + (-2)^2} = 2$
- $\|(u + v)\| = \sqrt{0^2 + (-1)^2} = 1.$



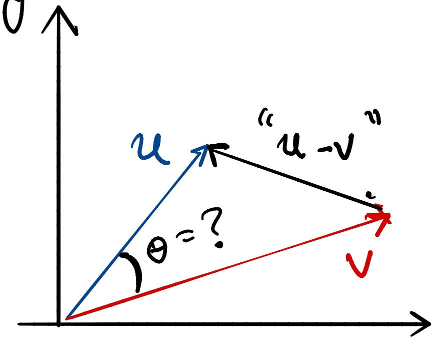
Application: Distace between points.



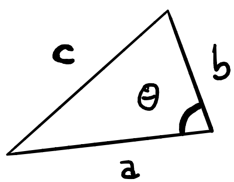
distance between P and Q is the norm of $(p-q)$.

Exercise 5): Find the distance between $(3,2)$ and $(6,6)$. Difference: $(3,2)-(6,6)=(-3,-4)$.

Angle between vectors



Law of cosines:



$$c^2 = a^2 + b^2 - 2\, a\cdot b \cdot \cos\theta$$

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\cdot\|u\|\cdot\|v\|\cdot\cos\theta.$$

Example:

$u = (3,2)$, $v = (4,1)$, $(u-v) = (-1,1)$.

- $\|u\| = \sqrt{3^2+2^2} = \sqrt{13}$
- $\|v\| = \sqrt{4^2+1^2} = \sqrt{17}$
- $\|u-v\| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

So $2 = 13 + 17 - 2\cdot\sqrt{13}\cdot\sqrt{17}\cdot\cos\theta$

$-28 = -2\sqrt{13}\cdot\sqrt{17}\cdot\cos\theta$

$14 = \sqrt{13}\cdot\sqrt{17}\cdot\cos\theta \longrightarrow 14 \cong 14.87\cos\theta$

$\cos\theta \cong 0.94.$

The dot product.

$$(3,2)\cdot(4,1) = 3\cdot 4 + 2\cdot 1$$
$$= 14$$

Dot product

$$\boxed{u\cdot v = \|u\|\cdot\|v\|\cdot\cos\theta}$$

<u>Exercise 6</u>: Check whether vectors $(1,2)$ and $(-4,5)$ are perpendicular.

Answer is no, otherwise dot product would be 0!

Exercise 7: Find the angle between $(1,1)$ and $(0,1)$.

u $\qquad$ v

$$u \cdot v = \|u\| \cdot \|v\| \cdot \cos\theta.$$

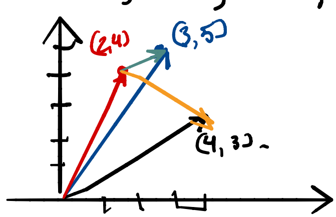
- $\|u\| = \sqrt{1^2 + 1^2} = \sqrt{2}$
- $\|v\| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1.$
- $u \cdot v = (1,1) \cdot (0,1)$
  $= 1 \cdot 0 + 1 \cdot 1$
  $= 1.$

$$1 = \sqrt{2} \cdot 1 \cdot \cos(\theta)$$

$$\frac{1}{\sqrt{2}} = \cos(\theta) \to \frac{\sqrt{2}}{2} = \cos(\theta) \to \theta = \frac{\pi}{4}$$

Application:

Check whether points $(2,4)$, $(3,5)$, $(4,3)$ are collinear.



Points: $(2,4)$, $(3,5)$, $(4,3)$.

Create two difference vectors:

$u = (3,5) - (2,4) = (1,1)$.

$v = (4,3) - (2,4) = (2,-1)$

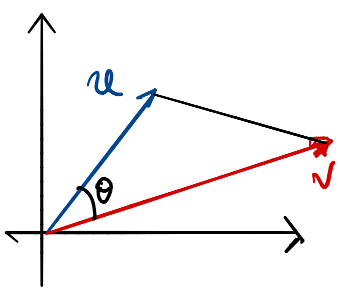
The angles between parallel vectors are either 0 or 180. → 0 and π.

$$u \cdot v = \|u\| \cdot \|v\| \cdot \cos(\theta).$$

- $\|u\| = \sqrt{1^2 + 1^2} = \sqrt{2}$
- $\|v\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$
- $u \cdot v = 1 \cdot 2 + 1 \cdot (-1) = 1.$

$1 = \sqrt{2} \cdot \sqrt{5} \cdot \cos(\theta)$ → neither 1 nor -1, vectors are not parallel. → points are not aligned.

Computing areas of triangles



What is the area of the triangle spanned by u and v?

$$A = \frac{\|u\| \cdot \|v\| \cdot \sin\theta}{2}.$$

Example:

Vectors: $u = (2,4)$, $v = (5,1)$.

$\|u\| = \sqrt{2^2+4^2} = \sqrt{4+16} = \sqrt{20}$

$\|v\| = \sqrt{5^2+1^2} = \sqrt{26}$

$u \cdot v = \|u\| \cdot \|v\| \cdot \cos(\theta)$

$(2 \cdot 5 + 4 \cdot 1) = \sqrt{20} \cdot \sqrt{26} \cdot \cos(\theta)$

$14 = \sqrt{20} \cdot \sqrt{26} \cdot \cos\theta.$

$\frac{14}{\sqrt{20} \cdot \sqrt{26}} = \cos(\theta)$

Fundamental Theorem of Trigonometry:

$\sin^2\theta + \cos^2\theta = 1.$

$\sin^2\theta + \left(\frac{14}{\sqrt{20} \cdot \sqrt{26}}\right)^2 = 1.$

$\sin^2\theta = 1 - \frac{14^2}{20 \cdot 26}$

$\sin^2\theta = \frac{20 \cdot 26 - 14^2}{20 \cdot 26}.$

$\sin^2\theta = \frac{520 - 196}{20 \cdot 26} \rightarrow \sin\theta = \frac{\sqrt{324}}{\sqrt{20} \cdot \sqrt{26}}$

$$\sin\theta = \frac{\sqrt{324}}{\sqrt{20}\cdot\sqrt{26}}$$

$$\sqrt{20}\cdot\sqrt{26}\sin(\theta) = \sqrt{324}.$$

$$\|u\|\cdot\|v\|\sin(\theta) = \sqrt{324} = 18$$

$$A = \frac{\|u\|\cdot\|v\|\sin(\theta)}{2}$$

$$A = \frac{18}{2} = 9.$$

$$\boxed{A = \frac{\sqrt{\|u\|^2\cdot\|v\|^2 - (u\cdot v)^2}}{2}}$$

Properties of dot product.

1) Positive-definiteness:

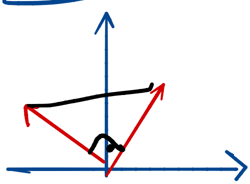
$$u\cdot u = \|u\|^2 \geq 0$$

with equality if and only if $u = 0$.

2) $(\alpha u)\cdot v = \alpha\cdot(u\cdot v)$
3) $(u+v)\cdot w = u\cdot w + v\cdot w.$
4) $u\cdot v = v\cdot u$

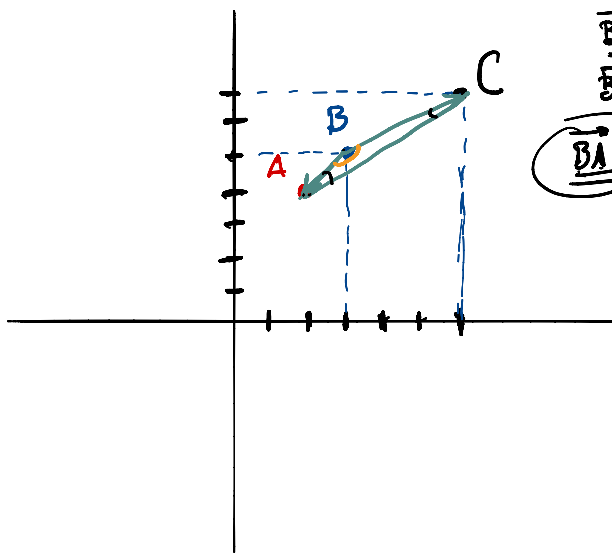
Exercise 8: Check whether the triangle spanned by vectors $(3,4)$, $(-4,3)$ is a right triangle.

Solution



$$(3,4)\cdot(-4,3) = 3\cdot(-4)+4\cdot 3$$
$$= -12+12$$
$$= 0.$$

Exercise 9: Let $A=(2,4)$, $B=(3,5)$, $C=(6,7)$. Is ΔABC acute? $A-B$



$\overrightarrow{BA} = (-1,-1)$

$\overrightarrow{BC} = (3,2)$.

$$\overrightarrow{BA}\cdot\overrightarrow{BC} = (-1)\cdot 3+(-1)\cdot 2$$
$$= -3-2$$
$$= -5.$$

$\hat{B}$ is obtuse!