Homework 4

Exercise 1 For each of the power series below: determine their radius of convergence; check for convergence at the endpoints of the interval of convergence. (Hint: read examples 4 and 5 on section 8.5 of the textbook).

(a)

$$\sum_{n=1}^{\infty} \sqrt{n} x^n$$

(b)

$$\sum_{n=1}^{\infty} \frac{(2n\,!)}{2^n} x^n$$

(c)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} (x-3)^n$$

(d)

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$$

(Hint: for part (d), rewrite the expression so that the factor involving x becomes $(x-a)^n$, for some number $a \in \mathbb{R}$.)

Exercise 2 Find power series representations of the following functions nearby the given points. Compute the radius of convergence of these power series. You may use any technique studied in class (operations with known power series or direct application of the Taylor theorem). You do not need to prove that the power series represents the function on its interval of convergence.

(a) The function $f(x) = \frac{1}{x+1}$, at the point $a = 1$.

(b) The function $f(x) = e^x + e^{-x}$, at the point $a = 0$.

(c) The function $f(x) = \ln(1 + x^2)$, at the point $a = 1$.

(d) The function $f(x) = x^2 \sin(x)$, at the point $a = 0$.

Exercise 3 This exercise is to show one can use the Taylor theorem to compute the values of certain series. Consider the series

$$\sum_{i=1}^{\infty} \frac{(-1)^n}{n}.$$

In a previous section, we saw that this series was convergent. To compute its value, do the following items.

(a) Compute the Taylor series of the function $f(x) = ln(x)$ at the point $a = 1$. (Do not simply write down the series - I want to see the computations of the coefficients).

(b) Compute the radius of convergence of this Taylor series, and check if 1 is in its interval of convergence.

(c) Use the results obtained in parts (a) and (b) to compute the value of the sum.

Remark: You may assume that the function is represented by its Taylor series in its interval of convergence.

Exercise 4 Solve the following initial value problems:

(a) $\frac{dy}{dx} + x^2 y = x^2$, with $y(0) = 1$.

(b) $\frac{dy}{dx} = yx^4$, with $y(0) = 4$.

(c) $\frac{dy}{dx} + \sin(x)y = e^{\cos x}$, with $y(0) = 1$.

(d) $\frac{dy}{dx} = 2x(1 - x)$, with $y(0) = 10$.