MAT 514 - Lecture 1

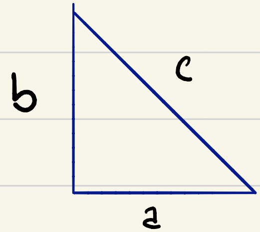
- Expectations from Calculus:
  - Parametrized curves, their velocity
  - Line integrals
  - Double integrals, Fubini's Theorem
  - ★ Green's theorem

Complex numbers

- Natural numbers (counting numbers): $1, 2, 3, \ldots$
- Integer numbers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
- Rational numbers: $\frac{2}{3}, \frac{1}{2}, \frac{-4}{5}, \frac{17}{248}, \ldots$

From geometry: Pythagoras' theorem states that sides of a right triangle are related by

$$a^2 + b^2 = c^2$$



If a right triangle has "legs" measuring 1 and 1, its hypothenuse is not rational!

Irrational numbers: $\sqrt{2}, \sqrt{5}, \sqrt[3]{7}, \pi, e \ldots$

Real numbers include all of the above.

Yet we can't solve

$$x^2 = -1$$

on the set of real numbers. Real numbers form an ordered field, with the property:

If $a > 0$ and $b \geq c$, then $ab \geq ac$.
Alternatively, if $a < 0$, $b \geq c$, then $ab \leq ac$.

Therefore: if $a > 0$, then

$$a \cdot a > a \cdot 0$$

$$a^2 > 0.$$

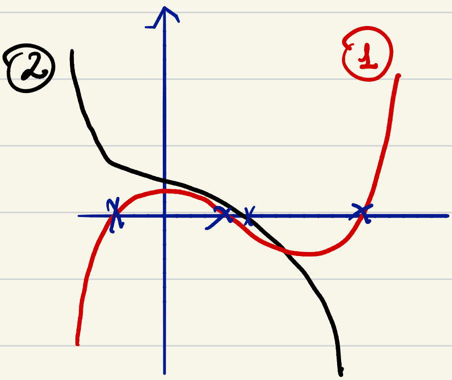
Else, if $a < 0$, then $0 > a$, so

$$a \cdot 0 < a \cdot a.$$

$$0 < a^2.$$

Finally, if $a = 0$, then $a^2 = 0$.

In the early development of Calculus, it was known that all cubic equations have a solution.



$p(x) = ax^3 + bx^2 + cx + d.$

Two possibilities for graph:

1) $\lim\limits_{x \to -\infty} p(x) = -\infty$

$\lim\limits_{x \to +\infty} p(x) = +\infty.$

2) $\lim\limits_{x \to -\infty} p(x) = +\infty$, $\lim\limits_{x \to +\infty} p(x) = -\infty$

By the Intermediate Value Theorem, p has a real root.

The Italian School (del Ferro, Cardano, Ferrara...) developed tools to tackle cubics, involving radicals. Operating with them led to expressions such as

$$\sqrt{n}$$

where n is a negative number. However, if you perform the same algebraic operations you would with real number, then the actual root predicted by the formula is real.

<u>Notation:</u> $i = \sqrt{-1}$, as a symbol (no algebra yet).

<u>Definition:</u> A complex number is an ordered pair of real numbers, $z = (a, b)$, represented as

$$z = a + ib.$$

The components a, b are called <u>real</u> and <u>imaginary</u> parts of z.

<u>Operations</u>:

1) Addition works componentwise:

$$(a, b) + (c, d) = (a+c, b+d) \text{ or}$$

if $z = a + ib$, $w = c + id$ then

$$z + w = (a+c) + i(b+d).$$

2) Multiplication follows a distributive rule:

$$(a, b) \cdot (c, d) = (ac - bd,\ ad + bc)$$

or, if

$$z = a + ib$$
$$w = c + id,$$

then

$$\begin{aligned} z \cdot w &= (a + ib)(c + id) \\ &= ac + i(ad) + i(bc) + (ib)(id) \\ &= ac + i(ad) + i(bc) + i^2 bd \\ &= ac + i(ad) + i(bc) - bd. \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

3) Complex zero: $0 = 0 + 0i$

Complex one: $1 = 1 + 0i$.

Complex zero and complex one fulfill the roles of neutral elements with respect to addition and multiplication, respectively.

4) Additive inverses (negatives):

if $z = a+bi$ then $-z = -a - bi$ and
$z + (-z) = 0$.

5) Multiplicative inverses: if $z = a+bi$, and $z \neq 0$, then

$$z^{-1} = \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$$

satisfies

$$z \cdot z^{-1} = 1.$$

6) Distributive law
- addition is distributive over multiplication

$$(z+w)\cdot u = zu + wu$$

Examples

A) $(3+2i)\cdot(1-i) = 3\cdot 1 - 3\cdot i + 2i - 2i^2$
$= 3 - 3i + 2i - 2\cdot(-1)$
$= (3+2) + (-3+2)i$

$= 5 - i.$

B) Finding inverses.

I wish to find a multiplicative inverse for

$z = 1 + 2i.$

That is, we need $z^{-1} = a + bi$ so that:

$$z \cdot z^{-1} = 1.$$

$$(1+2i)(a+bi) = 1$$

$$(a + bi + 2ai + 2bi^2) = 1.$$

$$(a - 2b) + (b+2a)i = 1 + 0i$$

$$\begin{cases} a - 2b = 1 \\ b + 2a = 0 \end{cases} \xrightarrow{\cdot 2} \begin{cases} a - \cancel{2b} = 1 \\ \cancel{2b} + 4a = 0 \end{cases} \oplus$$

$$5a = 1$$

$$a = \frac{1}{5}.$$

From the second equation: $b + 2 \cdot \frac{1}{5} = 0 \Rightarrow b = -\frac{4}{5}.$

$$\boxed{z^{-1} = \frac{1}{5} - \frac{4}{5}i}$$

New concept: the conjugate of a complex number

$$z = a + bi.$$

is

$$\bar{z} = a - bi.$$

Propreties:

1) $\overline{(\bar{z})} = z$

2) $\overline{(z+w)} = \bar{z} + \bar{w}$

3) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

<u>Norm of a complex number</u>

If $z = a + bi$ then its norm is

$$|z| = \sqrt{a^2 + b^2}$$

With these tools,

$$z^{-1} = \frac{\bar{z}}{|z|^2},$$

or equivalently:

$$z \cdot \bar{z} = |z|^2.$$

Example

e)

$$z = 1 + 3i$$

$$\bar{z} = 1 - 3i$$

$$\begin{aligned} z \cdot \bar{z} &= (1 + 3i)(1 - 3i) \\ &= (1 - \cancel{3i} + \cancel{3i} - 9i^2) \\ &= 1 + 9 \\ &= 10. \end{aligned}$$

$$\begin{aligned} |z| &= \sqrt{1^2 + 3^2} \\ &= \sqrt{1 + 9} \\ &= \sqrt{10}. \end{aligned}$$

Properties of norm

1) $|z| \geq 0$, and the only number with norm 0 is 0 itself.

2) $|\bar{z}| = |z|$, compatibility with conjugation

3) $|z \cdot w| = |z| \cdot |w|$, compatibility with multiplication.

4) $|z + w| \leq |z| + |w|$, triangle inequality.

Example 1: Given

$$z = 1 + 2i$$
$$w = 3 + 4i$$

The sum is

$$z + w = (1 + 2i) + (3 + 4i)$$
$$= 4 + 6i$$

The norms are:

$$|z| = \sqrt{1^2 + 2^2} = \sqrt{5}; \quad |w| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

- $|z + w| = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52}.$

Note that:

$$\sqrt{52} \cong 7.21$$

while

$$5 + \sqrt{5} \cong 7.24.$$

In this case, as in general, the sum of norms is greater than norm of the sum