LARGEST SMITH NUMBER

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Abstract. We find large Smith numbers by explicitly calculating digit sums through several methods relying on computer programs. This paper explicitly constructs a Smith number with 59,421,998,357 digits, exceeding previous record of 32,066,910 digits.

Contents

1. Introduction

A Smith number is defined by A. Wilanksy as "a composite number the sum of whose digits is the sum of all digits of all its prime factors"[5].

2. Notation and Basic Facts

The following notation and basic facts are taken from Patrick Costello [3]. For any positive integer n, let $S(n)$ denote the sum of the digits of n. For any positive integer n, let $S_p(n)$ denote the sum of digits of the prime factorization of n. For example, $S(12) = 1 + 2 = 3$ and $S_p(12) = S_p(2 \cdot 2 \cdot 3) = 2 + 2 + 3 = 7$.

3. An update on Costello 2002

Patrick Costello was able to construct a $32,066,910$ digit Smith number by using the known prime repunit R_{1031} and Chris Caldwell's large palindromic prime $M = 10^{28572} + 8 \cdot 10^{14286} + 1$. I will briefly go through the similar steps as Costello with more recently verified primes to construct a new largest Smith number.

Firstly, two facts are necessary

Fact 1 (Lewis [??])**.** If you multiply $9R_n$ by any natural number less than $9R_n$, then the digit sum is $9n$, i.e., $S(M \cdot 9R_n) = 9M = S(9R_n)$ when $M < 9R_n$.

Fact 2 (Wayland, Oltikar [??])**.** If $S(u) > S_p(u)$ and $S(u) = S_p(u) \pmod 7$, then $10^k \cdot u$ is a Smith number, where $k = \frac{S(u) - S_p(u)}{7}$.

Date: June 9, 2022.

Chris Caldwell's list of large proven primes[?] lists the prime $M = 3\cdot 10^{665829}+1$. It additionally lists the large prime repunit R_{49081} discovered and proven by Paul Underwood[??]. Notice that M is not palindromic and requires different method for bounding coefficients than Costello's 2002 paper.

For a power t, the term M^t can be represented as a sum of coefficients multiplied by powers of 10^{665829}.

$$M^t = \sum_{k=0}^{t} c_k 10^{665829k}, \quad c_k = \binom{t}{k} 3^k$$

Theorem 3.1. *Let $k_0 = \lceil \frac{3t-1}{4} \rceil$, then $c_{k_0} \geq c_k$ for $0 \leq k \leq t$.*

Proof. Define the ratio of coefficients $r_k = \frac{c_{k+1}}{c_k}$, then

$$\begin{aligned} b_k &= \frac{c_{k+1}}{c_k} \\ &= \frac{\binom{t}{k+1}3^{k+1}}{\binom{t}{k}3^k} \\ &= \frac{3\frac{1}{(k+1)!(t-(k+1))!}}{\frac{1}{k!(t-k)!}} \\ &= \frac{3(t-k)}{k+1} \end{aligned}$$

Notice that b_k is a decreasing function with $b_0 > 1$ and $b_{t-1} < 1$. Let $k_0 = \lceil \frac{3t-1}{4} \rceil$, then k_0 is the minimal integer so that $b_{k_0} < 1$. Therefor, c_{k_0} is the coefficient with largest value. □

Theorem 3.2. *Suppose $N = 9R_{49081}M^t$ for a power $t \leq 81525$, then for all $0 \leq k \leq t$, $c_k < 9R_{49081}$ and $9R_{49081}c_k < 10^{665829}$.*

Proof. Let $t_0 = 81525$, then by explicity calculation, $c_k^{t_0} < 9R_{49081}$ while $c_{k_0}^{t_0+1} \geq 9R_{49081}$.

$$\begin{aligned} c_k &\leq c_{k_0} \\ &\leq \binom{t}{k_0} 3^{k_0} \\ &\leq \binom{t}{\lceil \frac{3t-1}{4} \rceil} 3^{\lceil \frac{3t-1}{4} \rceil} \end{aligned}$$

Define that last line as $f(t)$, an increasing function. By explicit calculation,

$$f(81525) < 9R_{49081} < f(81526)$$

□

Now suppose $N = 9R_{49081}M^t$ for a power $t \leq 81525$. We know each coeffient $c_k < 9R_{49081}$ and $9R_{49081}c_k < 10^{665829}$. The latter constraint means the digit sum of N is the sum of the digit sums of $9R_{49081}c_k$. The first constraint allows us to apply fact 1 to prove the digit sum of $9R_{49081}c_k$ is $9 \cdot 49081$. Since k varies from $0 \leq k \leq t$, then

$$S(N) = (t+1) \cdot 9 \cdot 49081$$

The prime factorization of N is simply $3 \cdot 3 \cdot R_{49081} \cdot M^t$. Keeping in mind $S(M) = 4$, then

$$S_p(N) = 3 + 3 + 49081 + 4t$$

Note that $S(N) > S_p(N)$ and

$$\begin{aligned} S(N) - S_p(N) &= (t+1) \cdot 9 \cdot 49081 - (3 + 3 + 49081 + 4t) \\ &= 441725t + 392642 \\ &= 4t + 5 \pmod 7 \\ &= 4(t+3) \pmod 7 \end{aligned}$$

Fix $t = 81519$, then $t \equiv 4 \pmod 7$ and $t \leq 81525$. By above, $S(N) - S_p(N) \equiv 0 \pmod 7$. Calculate k

$$k = \frac{S(N) - S_p(N)}{7} = \frac{441725t + 392642}{7} = 5144196131$$

then Fact 2 to proves $10^k N$ is a Smith number. This Smith number has 59,421,998,357 digits.

4. Best by adding coefficients

$z = 1105923$

5. Best by overlapping coefficients

References

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[4] Kathy Sharon Lewis, *Smith numbers: an infinite subset of N*, Eastern Kentucky University, 1994.

[5] A. Wilansky, *Smith Numbers*, The Two-Year College Mathematics Journal **13** (1982), no. 1, 21-21, DOI 10.1080/00494925.1982.11972573.