

# LARGEST SMITH NUMBER

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ABSTRACT. We find large Smith numbers by explicitly calculating digit sums through several methods relying on computer programs. This paper explicitly constructs a Smith number with 59 421 998 357 digits, exceeding previous record of 32 066 910 digits.

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## 1. INTRODUCTION

A Smith number is defined by A. Wilansky as “a composite number the sum of whose digits is the sum of all digits of all its prime factors”[5].

## 2. NOTATION AND BASIC FACTS

The following notation and basic facts are taken from Patrick Costello [3]. For any positive integer  $n$ , let  $S(n)$  denote the sum of the digits of  $n$ . For any positive integer  $n$ , let  $S_p(n)$  denote the sum of digits of the prime factorization of  $n$ . For example,  $S(12) = 1 + 2 = 3$  and  $S_p(12) = S_p(2 \cdot 2 \cdot 3) = 2 + 2 + 3 = 7$ .

## 3. AN UPDATE ON COSTELLO 2002

Patrick Costello was able to construct a 32,066,910 digit Smith number by using the known prime repunit  $R_{1031}$  and Chris Caldwell’s large palindromic prime  $M = 10^{28572} + 8 \cdot 10^{14286} + 1$ . I will briefly go through the similar steps as Costello with more recently verified primes to construct a new largest Smith number.

Firstly, two facts are necessary

**Fact 1** (Lewis [??]). If you multiply  $9R_n$  by any natural number less than  $9R_n$ , then the digit sum is  $9n$ , i.e.,  $S(M \cdot 9R_n) = 9M = S(9R_n)$  when  $M < 9R_n$ .

**Fact 2** (Wayland, Oltikar [??]). If  $S(u) > S_p(u)$  and  $S(u) = S_p(u) \pmod{7}$ , then  $10^k \cdot u$  is a Smith number, where  $k = \frac{S(u) - S_p(u)}{7}$ .

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Chris Caldwell's list of large proven primes[?] lists the prime  $M = 3 \cdot 10^{665829} + 1$ . It additionally lists the large prime repunit  $R_{49081}$  discovered and proven by Paul Underwood[?]. Notice that  $M$  is not palindromic and requires different method for bounding coefficients than Costello's 2002 paper.

For a power  $t$ , the term  $M^t$  can be represented as a sum of coefficients multiplied by powers of  $10^{665829}$ .

$$M^t = \sum_{k=0}^t c_k 10^{665829k}, \quad c_{t,k} = \binom{t}{k} 3^k$$

When  $t$  is fixed, I will drop the  $t$  for ease of notation:  $c_k = c_{t,k}$ .

**Theorem 3.1.** Fix  $t > 0$  and let  $k(t) = \lceil \frac{3t-1}{4} \rceil$ , then  $c_k \leq c_{k(t)}$  for  $0 \leq k \leq t$ .

*Proof.* Define the ratio of coefficients  $b_k = \frac{c_{k+1}}{c_k}$ , then

$$\begin{aligned} b_k &= \frac{c_{k+1}}{c_k} \\ &= \frac{\binom{t}{k+1} 3^{k+1}}{\binom{t}{k} 3^k} \\ &= \frac{3 \frac{1}{(k+1)!(t-(k+1))!}}{\frac{1}{k!(t-k)!}} \\ b_k &= \frac{3(t-k)}{k+1} \end{aligned}$$

Notice that the  $b_k$  ratios are decreasing with  $b_{t-1} < 1 < b_0$ . Then  $k(t) = \lceil \frac{3t-1}{4} \rceil$  is the minimal integer so that  $b_{k(t)} \leq 1$ . Since these  $b_k$  are ratios of  $c_k$ ,  $c_{k(t)}$  is the largest coefficient.  $\square$

**Theorem 3.2.** Fix  $t \leq 81525$  and suppose  $N = 9R_{49081}M^t$ , then for all  $0 \leq k \leq t$ ,  $c_{t,k} < 9R_{49081}$  and  $9R_{49081}c_{t,k} < 10^{665829}$ .

*Proof.* Fix  $t, k$  as in the theorem, then by Theorem 3.1,

$$\begin{aligned} c_{t,k} &\leq c_{t,k(t)} \\ &= \binom{t}{k(t)} 3^{k(t)} \end{aligned}$$

Define that last line as  $f(t)$ , which is an increasing function. By explicit calculation,

$$f(t) \leq f(81525) < 9R_{49081} < f(81526)$$

Since  $(9R_{49081})^2 < 10^{665829}$ , then  $9R_{49081}c_{t,k} < 10^{665829}$  as well.  $\square$

Now suppose  $N = 9R_{49081}M^t$  for a power  $t \leq 81525$ . We know each coefficient  $c_k < 9R_{49081}$  and  $9R_{49081}c_k < 10^{665829}$ . The latter constraint means the digit sum of  $N$  is the sum of the digit sums of  $9R_{49081}c_k$ . The first constraint allows us to apply fact 1 to prove the digit sum of  $9R_{49081}c_k$  is  $9 \cdot 49081$ . Since  $k$  varies from  $0 \leq k \leq t$ , then

$$S(N) = (t+1) \cdot 9 \cdot 49081$$

The prime factorization of  $N$  is simply  $3 \cdot 3 \cdot R_{49081} \cdot M^t$ . Keeping in mind  $S(M) = 4$ , then

$$S_p(N) = 3 + 3 + 49081 + 4t$$

Note that  $S(N) > S_p(N)$  and

$$\begin{aligned} S(N) - S_p(N) &= (t+1) \cdot 9 \cdot 49081 - (3+3+49081+4t) \\ &= 441725t + 392642 \\ &= 4t + 5 \pmod{7} \\ &= 4(t+3) \pmod{7} \end{aligned}$$

Fix  $t = 81519$ , then  $t \equiv 4 \pmod{7}$  and  $t \leq 81525$ . By above,  $S(N) - S_p(N) \equiv 0 \pmod{7}$ . Calculate  $k$

$$k = \frac{S(N) - S_p(N)}{7} = \frac{441725t + 392642}{7} = 5144196131$$

then Fact 2 to proves  $10^k N$  is a Smith number. The explicit description for this Smith number is

$$\begin{aligned} 10^k N &= (3 \cdot 10^{665829} + 1)^t \cdot 9R_{49081} \cdot 10^k \\ &= (3 \cdot 10^{665829} + 1)^{81519} \cdot (10^{49081} - 1) \cdot 10^{5144196131} \end{aligned}$$

This Smith number has 59 421 998 358 digits.

#### 4. BEST BY ADDING COEFFICIENTS

The previous section relied on **Fact 1** to greatly simplify calculation of the coefficient digit sums. This caused the coefficient  $t = 81519$  to be limited by the size of a provably-prime repunit  $R_{49081}$ . With modern processors, it is possible to explicitly calculate each coefficient and the resulting digit sum, allowing us to discard the  $R_{49081}$  from the product and achieve far larger  $t$  powers.

For selecting a power  $t$  of  $M^t$  while planning to explicitly calculate each  $S(c_{t,k})$ , we only need to worry about two constraints. Firstly, it is necessary that  $S(M^t) = S_p(M^t) \pmod{7}$  to apply **Fact 2**. Secondly,  $t$  will need to be small enough that each coefficient  $c_{t,k} < 10^{665829}$  so that the coefficients can be separated in the digit representation. By Theorem 3.1, it's sufficient to show that only  $c_{t,k(t)} < 10^{665829}$ .

After a somewhat quick search for a bound on the second constraint, I found that fixing  $t_0 = 1105923$  yields

$$c_{t_0, k(t_0)} = 8.5967... \cdot 10^{665828} < 10^{665829}$$

This is maximal as

$$10^{665829} < c_{t_0+1, k(t_0+1)} = 3.4387... \cdot 10^{665829}$$

As long as  $t \leq t_0$ , then the second constraint is satisfied. Since we are not relying on **Fact1**, the digit sum of  $M^t$  is much less predictable. Therefor, finding a  $t \leq t_0$  so that  $S(M^t) = S_p(M^t) \pmod{7}$  simply involves checking  $t_0, t_0 - 1, t_0 - 2, \dots$  until one of these values happen to satisfy the congruence.

By a lucky  $\frac{1}{7}$  chance, it turns out  $S(M^{t_0}) = S_p(M^{t_0}) \pmod{7}$ . This took 4-ish hours to calculate using all cores on an AMD Ryzen 7 5800X 8-Core Processor.

$$\begin{aligned} S(M^{t_0}) &= 2508098743612 \\ S_p(M^{t_0}) &= (3+1) \cdot 1105923 = 4423692 \\ S(M^{t_0}) - S_p(M^{t_0}) &= 2508094319920 \end{aligned}$$

Let  $k = \frac{S(M^{t_0}) - S_p(M^{t_0})}{7} = 358299188560$ , then finally let  $N = M^{t_0}10^k$ .  $N$  is a Smith number with explicit form

$$N = (3 \cdot 10^{665829} + 1)^{1105923} \cdot 10^{358299188560}$$

This Smith number  $N$  has 1 094 654 215 464 digits.

## 5. BEST BY OVERLAPPING COEFFICIENTS

### REFERENCES

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