LARGEST SMITH NUMBER

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Abstract. We find large Smith numbers by explicitly calculating digit sums through several methods relying on computer programs. This paper explicitly constructs a Smith number with 59,421,998,357 digits, exceeding previous record of 32,066,910 digits.

Contents

1. Introduction

A Smith number is defined by A. Wilanksy as "a composite number the sum of whose digits is the sum of all digits of all its prime factors"[5].

2. Notation and Basic Facts

The following notation and basic facts are taken from Patrick Costello [3]. For any positive integer n, let $S(n)$ denote the sum of the digits of n. For any positive integer n, let $S_p(n)$ denote the sum of digits of the prime factorization of n. For example, $S(12) = 1 + 2 = 3$ and $S_p(12) = S_p(2 \cdot 2 \cdot 3) = 2 + 2 + 3 = 7$.

3. An update on Costello 2002

Patrick Costello was able to construct a $32,066,910$ digit Smith number by using the known prime repunit R_{1031} and Chris Caldwell's large palindromic prime $M = 10^{28572} + 8 \cdot 10^{14286} + 1$. I will briefly go through the similar steps as Costello with more recently verified primes to construct a new largest Smith number.

Firstly, two facts are necessary

Fact 1 (Lewis [??])**.** If you multiply $9R_n$ by any natural number less than $9R_n$, then the digit sum is $9n$, i.e., $S(M \cdot 9R_n) = 9M = S(9R_n)$ when $M < 9R_n$.

Fact 2 (Wayland, Oltikar [??])**.** If $S(u) > S_p(u)$ and $S(u) = S_p(u) \pmod 7$, then $10^k \cdot u$ is a Smith number, where $k = \frac{S(u) - S_p(u)}{7}$.

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Chris Caldwell's list of large proven primes[?] lists the prime $M = 3\cdot 10^{665829}+1$. It additionally lists the large prime repunit R_{49081} discovered and proven by Paul Underwood[??]. Notice that M is not palindromic and requires different method for bounding coefficients than Costello's 2002 paper.

For a power t, the term M^t can be represented as a sum of coefficients multiplied by powers of 10^{665829}.

$$M^t = \sum_{k=0}^{t} c_k 10^{665829k}, \quad c_{t,k} = \binom{t}{k} 3^k$$

When t is fixed, I will drop the t for ease of notation: $c_k = c_{t,k}$.

Theorem 3.1. *Fix $t > 0$ and let $k(t) = \lceil \frac{3t-1}{4} \rceil$, then $c_k \leq c_{k(t)}$ for $0 \leq k \leq t$.*

Proof. Define the ratio of coefficients $b_k = \frac{c_{k+1}}{c_k}$, then

$$\begin{aligned} b_k &= \frac{c_{k+1}}{c_k} \\ &= \frac{\binom{t}{k+1}3^{k+1}}{\binom{t}{k}3^k} \\ &= \frac{3\frac{1}{(k+1)!(t-(k+1))!}}{\frac{1}{k!(t-k)!}} \\ b_k &= \frac{3(t-k)}{k+1} \end{aligned}$$

Notice that the b_k ratios are decreasing with $b_{t-1} < 1 < b_0$. Then $k(t) = \lceil \frac{3t-1}{4} \rceil$ is the minimal integer so that $b_{k(t)} \leq 1$. Since these b_k are ratios of c_k, $c_{k(t)}$ is the largest coefficient. □

Theorem 3.2. *Fix $t \leq 81525$ and suppose $N = 9R_{49081}M^t$, then for all $0 \leq k \leq t$, $c_{t,k} < 9R_{49081}$ and $9R_{49081}c_{t,k} < 10^{665829}$.*

Proof. Fix t, k as in the theorem, then by Theorem 3.1,

$$\begin{aligned} c_{t,k} &\leq c_{t,k(t)} \\ &= \binom{t}{k(t)} 3^{k(t)} \end{aligned}$$

Define that last line as $f(t)$, which is an increasing function. By explicit calculation,

$$f(t) \leq f(81525) < 9R_{49081} < f(81526)$$

Since $(9R_{49081})^2 < 10^{665829}$, then $9R_{49081}c_{t,k} < 10^{665829}$ as well.

□

Now suppose $N = 9R_{49081}M^t$ for a power $t \leq 81525$. We know each coeffient $c_k < 9R_{49081}$ and $9R_{49081}c_k < 10^{665829}$. The latter constraint means the digit sum of N is the sum of the digit sums of $9R_{49081}c_k$. The first constraint allows us to apply fact 1 to prove the digit sum of $9R_{49081}c_k$ is $9 \cdot 49081$. Since k varies from $0 \leq k \leq t$, then

$$S(N) = (t+1) \cdot 9 \cdot 49081$$

The prime factorization of N is simply $3 \cdot 3 \cdot R_{49081} \cdot M^t$. Keeping in mind $S(M) = 4$, then

$$S_p(N) = 3 + 3 + 49081 + 4t$$

Note that $S(N) > S_p(N)$ and

$$\begin{aligned} S(N) - S_p(N) &= (t+1) \cdot 9 \cdot 49081 - (3 + 3 + 49081 + 4t) \\ &= 441725t + 392642 \\ &= 4t + 5 \pmod 7 \\ &= 4(t+3) \pmod 7 \end{aligned}$$

Fix $t = 81519$, then $t \equiv 4 \pmod 7$ and $t \leq 81525$. By above, $S(N) - S_p(N) \equiv 0 \pmod 7$. Calculate k

$$k = \frac{S(N) - S_p(N)}{7} = \frac{441725t + 392642}{7} = 5144196131$$

then Fact 2 to proves $10^k N$ is a Smith number. The explicit description for this Smith number is

$$\begin{aligned} 10^k N &= (3 \cdot 10^{665829} + 1)^t \cdot 9R_{49081} \cdot 10^k \\ &= (3 \cdot 10^{665829} + 1)^{81519} \cdot (10^{49081} - 1) \cdot 10^{5144196131} \end{aligned}$$

This Smith number has 59,421,998,358 digits.

4. Best by adding coefficients

The previous section relied on **Fact 1** to greatly simplify calculation of the coefficient digit sums. This caused the coefficient $t = 81519$ to be limited by the size of a provably-prime repunit R_{49081}. With modern processor performances, it is possible to explicitly calculate each coefficient and the resulting digit sum without relying on multiplying by a repunit. For selecting a power t of M^t while planning to explicitly calculate each $S(c_{t,k})$, we only need to worry about two constraints. Firstly, $S(M^t) - S_p(M^t)$ needs to be divisible by 7 to apply **Fact 2**. Secondly, t will need to be small enough that $c_{t,k(t)} < 10^{665829}$ so that the coefficients can be separated in the digit representation.

After a somewhat quick search for a bound on the second constraint, I found that fixing $t_0 = 1105923$ yields

$$\begin{aligned} c_{t_0,k(t_0)} &= 8.5967... \cdot 10^{665828} \\ c_{t_0+1,k(t_0+1)} &= 3.4387... \cdot 10^{665829} \end{aligned}$$

As long as $t \leq t_0$, then the second constraint is satisfied. Since we are not relying on **Fact1**, the digit sum of M^t is much less predictable. Therefor, finding a $t \leq t_0$ so that $S(M^t) - S_p(M^t) = 0 \pmod 7$ simply involves checking $t_0, t_0 - 1, t_0 - 2, ...$ until one of these values happen to satisfy the congruence.

By a lucky $\frac{1}{7}$ chance, it turns out $S(M^{t_0}) - S_p(M^t) = 0 \pmod 7$. This took 4-ish hours to calculate.

$$\begin{aligned} S(M^{t_0}) &= \\ S_p(M^{t_0}) &= (3+1) \cdot 1105923 = 4423692 \\ S(M^{t_0}) - S_p(M^t) &= XXX \end{aligned}$$

Let $k = \frac{XXX}{7}$, then finally let $N = M^{t_0} 10^k$. N is a Smith number with explicit form

$$N = (3 \cdot 10^{665829} + 1)^{1105923} \cdot 10^{XXX/7}$$

5. Best by overlapping coefficients

References

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[4] Kathy Sharon Lewis, *Smith numbers: an infinite subset of N*, Eastern Kentucky University, 1994.

[5] A. Wilansky, *Smith Numbers*, The Two-Year College Mathematics Journal **13** (1982), no. 1, 21-21, DOI 10.1080/00494925.1982.11972573.