LARGEST SMITH NUMBER

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ABSTRACT. We find large Smith numbers by explicitly calculating digit sums through several methods relying on computer programs. This paper explicitly constructs a Smith number with 1 094 654 215 464 digits, exceeding previous record of 32 066 910 digits.

CONTENTS

1. INTRODUCTION

A Smith number is defined by A. Wilanksy as “a composite number the sum of whose digits is the sum of all digits of all its prime factors”[5].

2. NOTATION AND BASIC FACTS

The following notation and basic facts are taken from Patrick Costello [3]. For any positive integer n, let $S(n)$ denote the sum of the digits of n. For any positive integer n, let $S_p(n)$ denote the sum of digits of the prime factorization of n. For example, $S(12) = 1 + 2 = 3$ and $S_p(12) = S_p(2 \cdot 2 \cdot 3) = 2 + 2 + 3 = 7$.

3. AN UPDATE ON COSTELLO 2002

Patrick Costello was able to construct a $32{,}066{,}910$ digit Smith number by using the known prime repunit R_{1031} and Chris Caldwell's large palindromic prime $M = 10^{28572} + 8 \cdot 10^{14286} + 1$. I will briefly go through the similar steps as Costello with more recently verified primes to construct a new largest Smith number.

Firstly, two facts are necessary

Fact 1 (Lewis [4])**.** If you multiply $9R_n$ by any natural number less than $9R_n$, then the digit sum is $9n$, i.e., $S(M \cdot 9R_n) = 9M = S(9R_n)$ when $M < 9R_n$.

Fact 2 (Wayland, Oltikar [6])**.** If $S(u) > S_p(u)$ and $S(u) = S_p(u) \pmod 7$, then $10^k \cdot u$ is a Smith number, where $k = \frac{S(u)-S_p(u)}{7}$.

Date: July 5, 2022.

Chris Caldwell's list of large proven primes[7] lists the prime $M = 3\cdot 10^{665829}+1$. It additionally lists the large prime repunit R_{49081} discovered and proven by Paul Underwood[8]. Notice that M is not palindromic and requires different method for bounding coefficients than Costello's 2002 paper.

For a power t, the term M^t can be represented as a sum of coefficients multiplied by powers of 10^{665829}.

$$M^t = \sum_{k=0}^{t} c_k 10^{665829k}, \quad c_{t,k} = \binom{t}{k} 3^k$$

When t is fixed, I will drop the t for ease of notation: $c_k = c_{t,k}$.

Theorem 3.1. *Fix $t > 0$ and let $k(t) = \lceil \frac{3t-1}{4} \rceil$, then $c_k \leq c_{k(t)}$ for $0 \leq k \leq t$.*

Proof. Define the ratio of coefficients $b_k = \frac{c_{k+1}}{c_k}$, then

$$\begin{aligned} b_k &= \frac{c_{k+1}}{c_k} \\ &= \frac{\binom{t}{k+1}3^{k+1}}{\binom{t}{k}3^k} \\ &= \frac{3\frac{1}{(k+1)!(t-(k+1))!}}{\frac{1}{k!(t-k)!}} \\ b_k &= \frac{3(t-k)}{k+1} \end{aligned}$$

Notice that the b_k ratios are decreasing with $b_{t-1} < 1 < b_0$. Then $k(t) = \lceil \frac{3t-1}{4} \rceil$ is the minimal integer so that $b_{k(t)} \leq 1$. Since these b_k are ratios of c_k, $c_{k(t)}$ is the largest coefficient. □

Theorem 3.2. *Fix $t \leq 81525$ and suppose $N = 9R_{49081}M^t$, then for all $0 \leq k \leq t$, $c_{t,k} < 9R_{49081}$ and $9R_{49081}c_{t,k} < 10^{665829}$.*

Proof. Fix t, k as in the theorem, then by Theorem 3.1,

$$\begin{aligned} c_{t,k} &\leq c_{t,k(t)} \\ &= \binom{t}{k(t)} 3^{k(t)} \end{aligned}$$

It's clear that the function $t \mapsto \binom{t}{k(t)}3^{k(t)}$ is an increasing function. By explicit calculation,

$$c_{t,k(t)} \leq c_{81525,k(81525)} < 9R_{49081} < c_{81526,k(81526)}$$

Since $(9R_{49081})^2 < 10^{665829}$, then $9R_{49081}c_{t,k} < 10^{665829}$ as well.

□

Now suppose $N = 9R_{49081}M^t$ for a power $t \leq 81525$. We know each coeffient $c_k < 9R_{49081}$ and $9R_{49081}c_k < 10^{665829}$. The latter constraint means the digit sum of N is the sum of the digit sums of $9R_{49081}c_k$. The first constraint allows us to apply fact 1 to prove the digit sum of $9R_{49081}c_k$ is $9 \cdot 49081$. Since k varies from $0 \leq k \leq t$, then

$$S(N) = (t+1) \cdot 9 \cdot 49081$$

The prime factorization of N is simply $3 \cdot 3 \cdot R_{49081} \cdot M^t$. Keeping in mind $S(M) = 4$, then

$$S_p(N) = 3 + 3 + 49081 + 4t$$

Note that $S(N) > S_p(N)$ and

$$\begin{aligned} S(N) - S_p(N) &= (t+1) \cdot 9 \cdot 49081 - (3 + 3 + 49081 + 4t) \\ &= 441725t + 392642 \\ &= 4t + 5 \pmod 7 \\ &= 4(t+3) \pmod 7 \end{aligned}$$

Fix $t = 81519$, then $t \equiv 4 \pmod 7$ and $t \le 81525$. By above, $S(N) - S_p(N) \equiv 0 \pmod 7$. Calculate k

$$k = \frac{S(N) - S_p(N)}{7} = \frac{441725t + 392642}{7} = 5144196131$$

then Fact 2 to proves $10^k N$ is a Smith number. The explicit description for this Smith number is

$$\begin{aligned} 10^k N &= (3 \cdot 10^{665829} + 1)^t \cdot 9R_{49081} \cdot 10^k \\ &= (3 \cdot 10^{665829} + 1)^{81519} \cdot (10^{49081} - 1) \cdot 10^{5144196131} \end{aligned}$$

This Smith number has 59 421 998 358 digits.

4. Best by adding coefficients

The previous section relied on **Fact 1** to greatly simplify calculation of the coefficient digit sums. This caused the coefficient $t = 81519$ to be limited by the size of a provably-prime repunit R_{49081}. With modern processors, it is possible to explicitly calculate each coefficient and the resulting digit sum, allowing us to discard the R_{49081} from the product and achieve far larger t powers.

For selecting a power t of M^t while planning to explicitly calculate each $S(c_{t,k})$, we only need to worry about two constraints. Firstly, it is necessary that $S(M^t) = S_p(M^t) \pmod 7$ to apply **Fact 2**. Secondly, t will need to be small enough that each coefficient $c_{t,k} < 10^{665829}$ so that the coefficients can be separated in the digit representation. By Theorem 3.1, it's sufficient to show that only $c_{t,k(t)} < 10^{665829}$.

After a somewhat quick search for a bound on the second constraint, I found that fixing $t_0 = 1105923$ yields

$$c_{t_0,k(t_0)} = 8.5967... \cdot 10^{665828} < 10^{665829}$$

This is maximal as

$$10^{665829} < c_{t_0+1,k(t_0+1)} = 3.4387... \cdot 10^{665829}$$

As long as $t \le t_0$, then the second constraint is satisfied. Since we are not relying on **Fact1**, the digit sum of M^t is much less predictable. Therefor, finding a $t \le t_0$ so that $S(M^t) = S_p(M^t) \pmod 7$ simply involves checking $t_0, t_0 - 1, t_0 - 2, ...$ until one of these values happen to satisfy the congruence.

By a lucky $\frac{1}{7}$ chance, it turns out $S(M^{t_0}) = S_p(M^{t_0}) \pmod 7$. This took 4-ish hours to calculate using all cores on an AMD Ryzen 7 5800X 8-Core Processor.

$$\begin{aligned} S(M^{t_0}) &= 2508098743612 \\ S_p(M^{t_0}) &= (3+1)\cdot 1105923 = 4423692 \\ S(M^{t_0}) - S_p(M^{t_0}) &= 2508094319920 \end{aligned}$$

Let $k = \frac{S(M^{t_0}) - S_p(M^{t_0})}{7} = 358299188560$, then finally let $N = M^{t_0}10^k$. N is a Smith number with explicit form

$$N = (3 \cdot 10^{665829} + 1)^{1105923} \cdot 10^{358299188560}$$

This Smith number has 1 094 654 215 464 digits.

5. Implementation & Afterword

The large-integer calculations throughout the paper were done using the Multiple Precision Integers and Rationals Library (MPIR). The outputs of these calculations can be reproduced using associated code at `https://github.com/MarlonTri/smith-calc`.

For future work, an even larger t power can be achieved. Section 4 relied on each keeping coefficients small enough that they could be separated in the digit sum. However, it wouldn't be too much more computationally expensive to allow coefficients to get larger than 10^{665829} and then summing together adjacent coefficients before calculating digit sums.

References

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