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8 Random Numbers

Principle: Linear congruential generators:

$$J_{n+1} = (a J_n + c) \bmod m$$

General Formula:

$$J_n = (a^n J_0 + c_n) \bmod m$$

with $c_n = c \cdot \sum_{i=0}^{n-1} a^i$

Proof by Induction: $n=1$ ok

$$J_{n+1} = (a[a^n J_0 + c_n] + c) \bmod m$$

$$= (a^{n+1} J_0 + a c_n + c) \bmod m$$

To show: $c_{n+1} = a c_n + c$

$$c_{n+1} = c \cdot \sum_{i=0}^{n} a^i = a c \sum_{i=0}^{n-1} a^i + c \quad \checkmark$$

Max period m; discrete map

Nec. + suff. condition for max period:

- no common prime factors betw. c and m
  $m = m_0 m_1$, $c = c_0 m_1$ $\quad c \bmod m = c_0 m_1 \bmod m_0 m_1 = 0$
- $a-1$ div. by all prime factors of m
  Get $a^n \bmod m$ cover all $1, \dots, m-1$

2

Claim: The Liapounov coefficient of discrete map $J_{n+1} = (a J_n + c) \mod m$ is $\log a > 0 \Rightarrow$ ~~dtt~~ Chaos

one condition for chaos, but periodic!

$$\delta_n = J_n - J_n' = (a^n J_0 + c_n - a^n J_0' - c_n) \mod m$$

$$= a^n (J_0 - J_0') \mod m = a^n \delta_0 \mod m$$

Assume $n <$ Period of LCG $\leq m$

$$\delta_n = \delta_0 \exp(n \lambda_n) = \delta_0 a^n \Rightarrow$$

$$n \lambda_n = n \log a \checkmark$$

Probability Distribution Function $p(x)$

Random Numbers $0 \leq J_i \leq m-1$

Normalize 1: $0 \leq r_i \leq a$ by

$$r_i = J_i / (m-1)$$

Normalize 2:

$$1 = \int_0^a p(x)\,dx = \sum_i x_i \Delta x_i \quad \text{, often } a = 1!$$

(3)

LCG and also ran2.c give equally distributed RN's

- How we can see this?

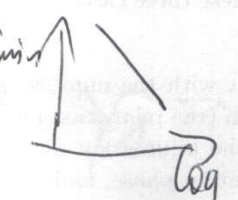
1. Histogram, bins

10 bins
100 RN's
10 per bin

2. Two sequences plot against each other

homog. density

3. Smallest RN r_{min}



$r_{min} \propto \frac{1}{N}$

How to get other distributions?

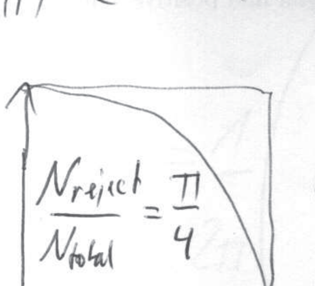
- Rejection Method

$f(x) = bx$

(i) random number r_i, $x_i = a r_i \in [0, a)$

(ii) 2^{nd} random number s_i, if $s_i < f(x_i)$ take

if $s_i > f(x_i)$ reject



Compute π by RN: $r_i \in [0, 1)$

2^{nd} RN: s_i, $s_i < \sqrt{1 - r_i^2}$ take

$s_i >$ reject