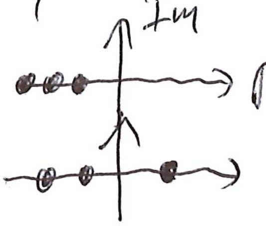
Fr 14.6.19 Lorenz dynamical system (1)

FP_1: $(0,0,0)$; char. Polyn. $P(\lambda) = (b+\lambda)\left[\lambda^2 + (1+\sigma)\lambda + \sigma(1-r)\right]$

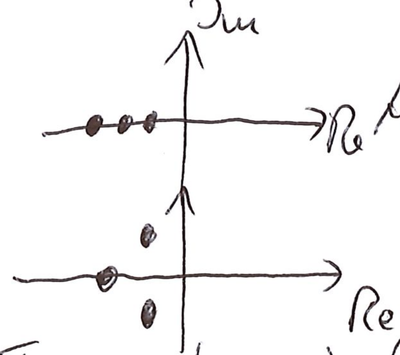


$0 \leq r < 1$ All $\lambda_{1,2,3} < 0$ stable

$r > 1$ One $\lambda_{1,2,3} > 0$ unstable

$FP_{2/3}$: $\left(\pm\sqrt{b(r-1)}; \pm\sqrt{b(r-1)}; r-1\right)$ Only $r > 1$

char. Polyn. $-P(\lambda) = \lambda^3 + (1+b+\sigma)\lambda^2 + b(r+\sigma)\lambda + 2\sigma b(r-1)$

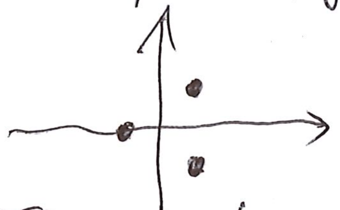


$1 < r < r_{crit1} = 1.346\ldots$ All $\lambda_{123} < 0$ stable

$r_{crit1} < r < r_{crit2} = 24.74\ldots$ All $Re(\lambda_{123}) < 0$

Two complex conjug. λ stable

$r > r_{crit2}$



$Re(\lambda) > 0$ for comp. conj.

both $FP_{2/3}$ get unstable!

char. Polyn. ansatz: (Theorem of Vieta):

$$-P(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_r - i\lambda_i)(\lambda - \lambda_r + i\lambda_i)$$

with three EW $\lambda_1 = \lambda_0$; $\lambda_2 = \lambda_r + i\lambda_i$; $\lambda_3 = \lambda_r - i\lambda_i$

$$-P(\lambda) = \lambda^3 + \lambda^2(-\lambda_0 - 2\lambda_r) + \lambda\left[\lambda_r^2 + \lambda_i^2 + 2\lambda_0\lambda_r\right] - \lambda_0(\lambda_r^2 + \lambda_i^2)$$

Equate both forms of $P(\lambda)$; comp. coeff:

$$1+b+\sigma = -\lambda_0 - 2\lambda_r$$
$$b(r+\sigma) = \lambda_r^2 + \lambda_i^2 + 2\lambda_0\lambda_r$$
$$2\sigma b(r-1) = -\lambda_0(\lambda_r^2+\lambda_i^2)$$

To find r_{crit2} we look for $\lambda_r = 0$!

$$2\sigma b(r-1) = \underbrace{(1+b+\sigma)}_{-\lambda_0}\underbrace{b(r+\sigma)}_{\lambda_i^2}$$

$$2\sigma r - 2\sigma = r + br + \sigma r + \sigma + b\sigma + \sigma^2$$

$$r(\sigma - b - 1) = \sigma(3+b+\sigma)$$

$$r = r_{crit2} = \frac{\sigma(3+b+\sigma)}{\sigma - b - 1} \sim 24.74\ldots$$

For $r < r_{crit2}$ $FP_{2/3}$ stable

$r > r_{crit2}$ $FP_{2/3}$ unstable

Volume contraction:

$$\frac{\partial}{\partial x}\dot{x} + \frac{\partial}{\partial y}\dot{y} + \frac{\partial}{\partial z}\dot{z} = -\sigma - 1 - b < 0$$

(Trace of Jacobian)

Globally contracting! div (velocity) < 0
$\dot{x}, \dot{y}, \dot{z}$

(7)

Definition of the „Attractor" A
(trajectory $\vec{x}(t)$ approaches A and remains on it)

- if $\vec{x}_0 = \vec{x}(t_0) \in A$, $\vec{x}(t) \in A$ for all $t > t_0$
- $\underbrace{\text{in some open environment of } A}$ if $\vec{x}_0 = \vec{x}(t_0)$ starts, $\lim_{t\to\infty} \vec{x}(t) \in A$

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stable FP and limit cycles are attractors

Why is the Lorenz Attractor strange?

For $r > r_{crit}$: all FP unstable

dim $A \neq 0$ (no point)

dim $A \neq 3$ (Volume contraction)

dim $A \neq 2$ (curves would intersect, also Poincaré-Bendixson contradicts)

dim $A \neq 1$? most difficult to show
could be limit cycle
exclude only => discrete maps