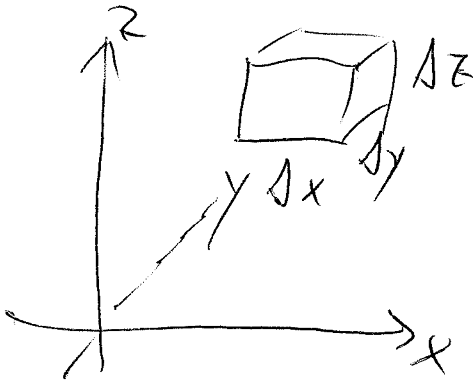
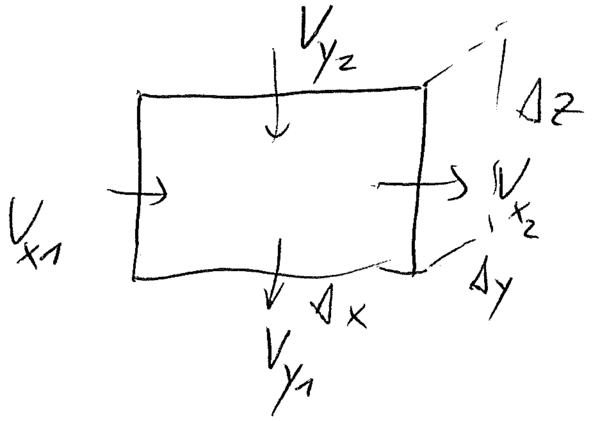
Wed 19.6.19 Lorenz dynamical system

Final Topics: (1) • Contraction



$$V = \Delta x \Delta y \Delta z$$

What happens to volume under Lorenz dynamical equation? (Every point integrated forward in time ...)



$$\begin{aligned}\Delta V = &\ (\overbrace{V_{x2} - V_{x1}}^{\Delta V_x})\,\Delta t\,\Delta y\,\Delta z \\ &+ (\overbrace{V_{y2} - V_{y1}}^{\Delta V_y})\,\Delta t\,\Delta x\,\Delta z \\ &+ (\underbrace{V_{z2} - V_{z1}}_{\Delta V_z})\,\Delta t\,\Delta y\,\Delta x\end{aligned}$$

Divide by $\Delta x \Delta y \Delta z \Delta t$, use $\Delta \log V = \frac{\Delta V}{V}$

$$\frac{\Delta \log V}{\Delta t} = \frac{\Delta V_x}{\Delta x} + \frac{\Delta V_y}{\Delta y} + \frac{\Delta V_z}{\Delta z}$$

$\lim\limits_{\Delta x, \Delta y, \Delta z, \Delta t \to 0}$:

$$\frac{d \log V}{dt} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$v_x = \dot{x},\ v_y = \dot{y},\ v_z = \dot{z}$

$$= -\sigma - 1 - b < 0!$$

(2) Dimension of the attractor A:

- $\dim A \neq 3$ (volume contraction)
- $\dim A \neq 2$ (curves would intersect, more mathematically: Theorem of Poincaré-Bendixson)
- $\dim A \neq 0$ (no stable FP)
- $\dim A \neq 1$? most difficult to show
  could be limit cycle
  exclude with theory of discrete maps

(3) Theorem of Poincaré-Bendixson

(1D, 2D)

$\dot{\vec{x}} = \vec{f}(\vec{x})$ $\quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

If trajectory $\vec{x}(t)$ remains within closed bounded region D for all $t \geq 0$:

$\Longrightarrow$ $\vec{x}(t)$

(i) is a closed (periodic) orbit (~~limit~~ cycle)

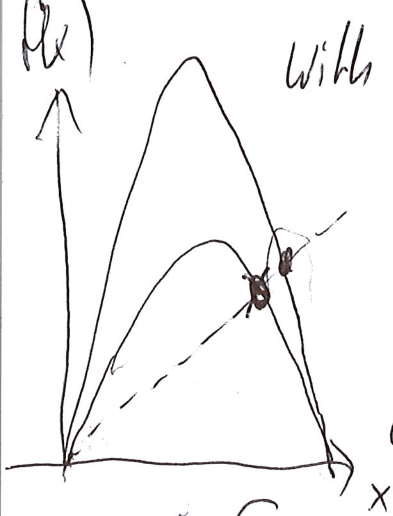
(ii) approaches a closed orbit (limit cycle)

(iii) approaches a fixed point

NB: if there is no stable fixed point, then only (limit) cycle remains.

(4) Short Excursion Theory of discrete maps

~~Pfutte~~ • Set $x_0, x_1, \ldots x_n, x_{n+1}, \ldots$

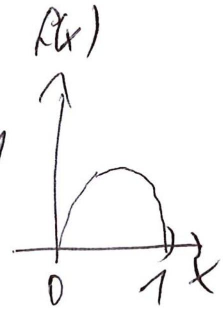


with $x_{n+1} = f(x_n)$

• Fixed Points and stability

$$x^* = f(x^*)$$

• Stability if $|f'(x^*)| < 1$



Example: Logistic Map, $c > 0$

$$x_{n+1} = c\,x_n\,(1-x_n) = f(x_n)$$

$$(1-2x)\cdot c = c\,(1-x) - cx = f'(x)$$

Fixed Point: $x = 1-\frac{1}{c}$, $f'(x) = 2-c$,

Stable for: $1 < c < 3$

Iterated Map: $x_{n+1} = f^{(p)}(x_n) = f(f(f\ldots(f(x_n))$

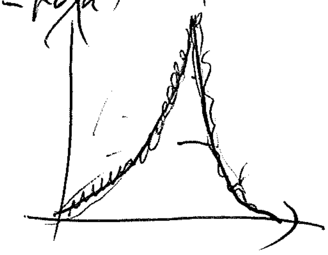
Fixed Point of $f^{(p)}$ is p-periodic point of f

$$\ldots \underbrace{x_n, x_{n+1}, \ldots x_{n+p-1}}_{x_n = x_{n+p}} \ldots$$

Stability of FP of $f^{(p)}$: $|f^{(p)\prime}(x_{n+p})| = |f'(x_{n+p})\cdots f'(x_n)|$

Also $x_{n+1}, x_{n+2}, \ldots$ are FP of $f^{(p)}$, all stable or all unstable

Now look at $z_{k+1} = f(z_k)$ for Lorenz dynamical system as discrete map



Let us assume the Lorenz attractor has a stable p-periodic orbit. Then the discrete map $z_{k+p} = f^{(p)}(z_k)$ must have a stable fixed point.
However, this is impossible, because $|f'(z_k)| > 1$ everywhere!
And therefore also

$$|f^{p\prime}(z_k)| = \prod_{i=1}^{p} |f'(z_{k+i})| > 1!!$$

Therefore $\dim A \neq 1$!
Caveat: the map (the line) $f(z)$ is not steady! Also fractal ...