Introduction to Computational Physics SS2019

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Exercise 2 from May 1, 2019
Return by noon of May 10, 2019

1 4^{th} Order Runge-Kutta Method (RK4)

Get acquainted with the RK4 algorithm. We provide the `rk4` subroutine from the Numerical Recipes library in `python`, `C` or `F77` on our lecture webpage (elearning). You can find other implementations on the web. As a first test solve a simple differential equation (exponential decay):

$$\dot{x} = -rx \tag{1.1}$$

with $r = 1$ and $x(0) = x(t_0) = 1$. The test whether your program code is working well should be that you reproduce the correct gradient of the solution in the semi-logarithmic plot. Investigate the accuracy of the integration by comparing to the analytical solution, and by varying the step size over various orders of magnitude.

2 Three-Body Problem (HOMEWORK)

Adapt your Runge-Kutta-4 integrator to the gravitational 3-body problem. Simplify the system by setting the gravitational constant to $G = 1$ and reduce the dimensionality of the problem by only considering motions in a plane. You obtain 12 coupled ordinary differential equations of first order which you can convert into the standard form $y' = f(y,t)$. Integrate the system forward in time starting from various initial congurations as given below.

(a) In a first step, set the masses of all three bodies to $m_1 = m_2 = m_3 = 1$ and select the following initial conditions for $y(0)$:

$$\begin{aligned}
(y_1, y_2) &= -0.97000436 \quad 0.24308753 \\
(y_3, y_4) &= -0.46620368 \quad -0.43236573 \\
(y_5, y_6) &= 0.97000436 \quad -0.24308753 \\
(y_7, y_8) &= -0.46620368 \quad -0.43236573 \\
(y_9, y_{10}) &= 0.0 \quad 0.0 \\
(y_{11}, y_{12}) &= 0.93240737 \quad 0.86473146
\end{aligned}$$

Here, y_{1+4i} and y_{2+4i} are the initial coordinates and y_{3+4i} and y_{4+4i} the initial velocities for the objects $i = 0$, 1 and 2. Try to integrate with a step size h between 0.01 and 0.001 and plot the result. (10 points)

(b) Now consider a different problem. Choose the masses of the three bodies to be $m_1 = 3, m_2 = 4$ and $m_3 = 5$, and place them at the corners of a right triangle (one angle is 90^o) with edge lengths of $\ell_1 = 3$, $\ell_2 = 4$ and $\ell_3 = 5$, such that m_1 is opposite to the edge ℓ_1, m_2 opposite to ℓ_2, and m_3 opposite to ℓ_3. Set the initial velocities to zero. This is the Meissel-Burrau problem. We recommend to place the origin of your coordinate system into the center of mass of the system.

Use the Runge-Kutta-4 integrator to follow the time evolution of the system until it dissolves. Record the points in time when two bodies have minimum separation and store this data in a file. Investigate the behavior of the system for different integration steps h, starting with $h = 0.1$. How small does h need to be in order to obtain reliable estimates for the time of the first five closest encounters. Plot for different step sizes h: *(i)* the trajectories of the three bodies in the orbital plane, *(ii)* the mutual distances of the three bodies in logarithmic scaling as well as *(iii)* the error of the total energy of the system in logarithmic scaling as function of time (linear). (10 points)

(c) Voluntary problem for further study: Use the same initial configuration as in (b), but now add an initial velocity of $v = 0.1$ to the most massive particle ($m_3 = 5$) in the direction towards the body $m_2 = 4$. Study again the trajectories of the three bodies for the same values of h as before.

Can you find other interesting variations of the initial conditions to find "entertaining" solutions? The best may be presented in the tutorial or the lecture.