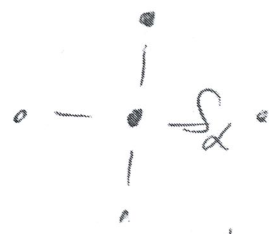
Ising Model

$$H = -B\sum_{\alpha=1}^{N} s_\alpha - J\sum_{\substack{\alpha=1\\ \langle\beta\rangle}}^{N} s_\alpha s_\beta$$

$s_\alpha = \pm 1$



$\langle\beta\rangle$ only nearest neighbour

Spin State $S = \{s_1, \dots, s_N\}$

for $N = 30 \times 30 = 900$ we have $\sim 2^N$ states $\sim 8 \cdot 10^{270}$ crazy number

Thermodynamic Quantities / Statistical Mechanics

- grand canonical ensemble (heat bath, not isolated)

$\sum_{S_i} w(S_i) = 1$ Here!

- probability of state S:
  $w(S) = \exp(-\beta H(S)) / Z$; $\beta = \frac{1}{kT}$
- partition sum (Zustandssumme)
  $$Z = \sum_{S_i} \exp(-\beta H(S_i))$$
  2^N summands

  log = ln = log
- Free Energy $F = -kT \log Z = -\frac{1}{\beta} \log Z$
- Internal Energy $U = kT^2 \frac{\partial \log Z}{\partial T} = -\frac{\partial \log Z}{\partial \beta}$
- Mean Magnetization $M = -\frac{\partial F}{\partial B} = \frac{1}{\beta} \frac{\partial}{\partial B} \log Z$

Compute M and U:

$$M = \frac{1}{\beta Z} \frac{\partial}{\partial B} \sum_{S_i} \exp(-\beta H(S_i))$$

$$= \frac{1}{Z} \cancel{\frac{\partial}{\partial B}} \sum_{S_i} \left(\sum_{\alpha=1}^{N} S_\alpha \right) \exp(-\beta H(S_i))$$

$$= \sum_{S_i} w(S_i) M_i \quad \text{with } M_i = \left(\sum_{\alpha=1}^{N} S_\alpha \right)_i \checkmark$$

$$U = -\frac{1}{Z} \frac{\partial}{\partial \beta} Z = -\frac{1}{Z} \frac{\partial}{\partial \beta} \sum_{S_i} \exp(-\beta H(S_i))$$

$$= \frac{1}{Z} \sum_{S_i} H(S_i) \exp(-\beta H(S_i))$$

$$= \sum_{S_i} w(S_i) H(S_i) \quad \text{with } H(S_i) = U_i$$

internal energy of state.

dimensionless values:

$$\frac{H}{kT} = \beta H; \quad b = \beta B \quad h = \beta H \quad j = \beta J$$

$$h = -b \sum_{\alpha} s_{\alpha} - j \sum_{\alpha, (\beta)} s_{\alpha} s_{\beta}$$

Varying in our experiment:

b: magnetic field

j: temperature (indirect via $j = \frac{J}{kT}$)

Ising Model Mean Field Approx.

First: $J = 0$:

$$w(S)\cdot \mathcal{Z} = \exp(-\beta H(S)) = \exp\left(+\beta B \sum_\alpha s_\alpha\right)$$
$$= \prod_\alpha \exp(\beta B s_\alpha)$$

$$\mathcal{Z} = \sum_{S_i} \exp(-\beta H(S_i)) = \sum_{S_i} \prod_\alpha \exp(\beta B s_\alpha)$$
$$= \left(\exp(\beta B) + \exp(-\beta B)\right)^N$$

Proof by induction: $N = 1$: trivial

$N \to N+1$:

$$\mathcal{Z}_{N+1} = \sum_{S_{i,N+1}} \exp(-\beta H(S_i)) = \mathcal{Z}_N \exp(-\beta B) + \mathcal{Z}_N \exp(+\beta B)$$
$$= \left(\exp(\beta B) + \exp(-\beta B)\right)^{N+1} \checkmark$$

x x

x x

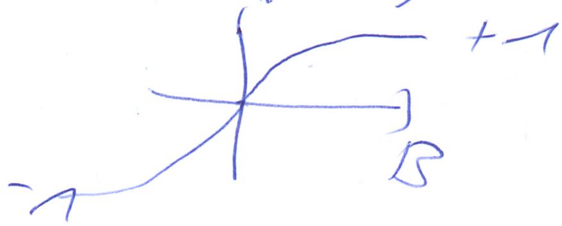
x

$$\log Z = N \cdot \log\left(e^{\beta B} + e^{-\beta B}\right)$$

$$M_{MF} = \frac{1}{\beta} \frac{\partial}{\partial B} \log Z = N \cdot \frac{e^{\beta B} - e^{-\beta B}}{e^{\beta B} + e^{-\beta B}}$$

$$= N \cdot \tanh(\beta B)$$

$$m_{MF} = \tanh(\beta B)$$



Now assume:

$$H = -\sum_\alpha s_\alpha \underbrace{\left(B + J \sum_{\langle\beta\rangle} s_\beta\right)} = -B_{MF} \sum_\alpha s_\alpha$$

with $B_{MF} = B + 4J \langle s \rangle$

$$= B + 4J \tanh(\beta B_{MF})$$

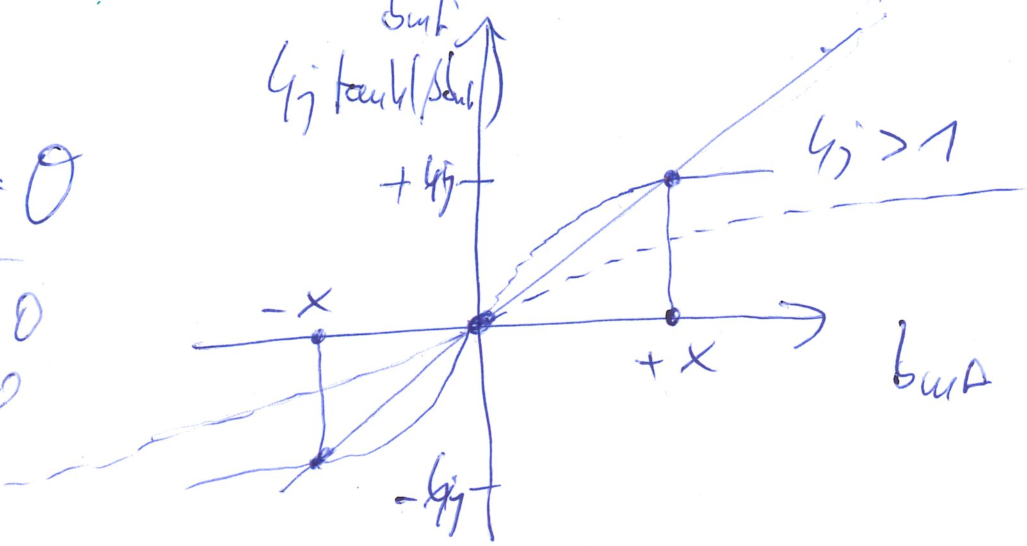
dimensionless

$$b_{MF} = b + 4j \tanh(b_{MF})$$

For each pair of values b, j we get:

1, 2, 3 solutions for b_{MF}

$b = 0$



$b > 0$
$b < 0$

$J_{aix} \sim 0.25$

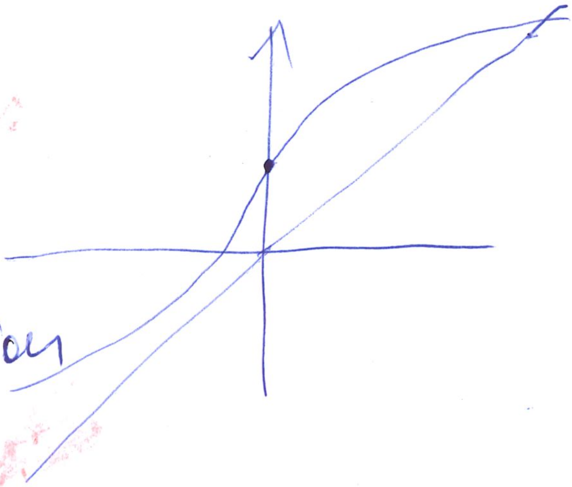
$qJ < 1$: only one solution $b_{mf} = 0$

$\Rightarrow m_{mf} = 0$

$qJ > 1$: three solutions $b_{mf} = 0, +x, -x$

$m_{mf} = \tanh(b_{mf})$ $m_{mf} = 0, \tanh(\cancel{b} \pm \beta x)$

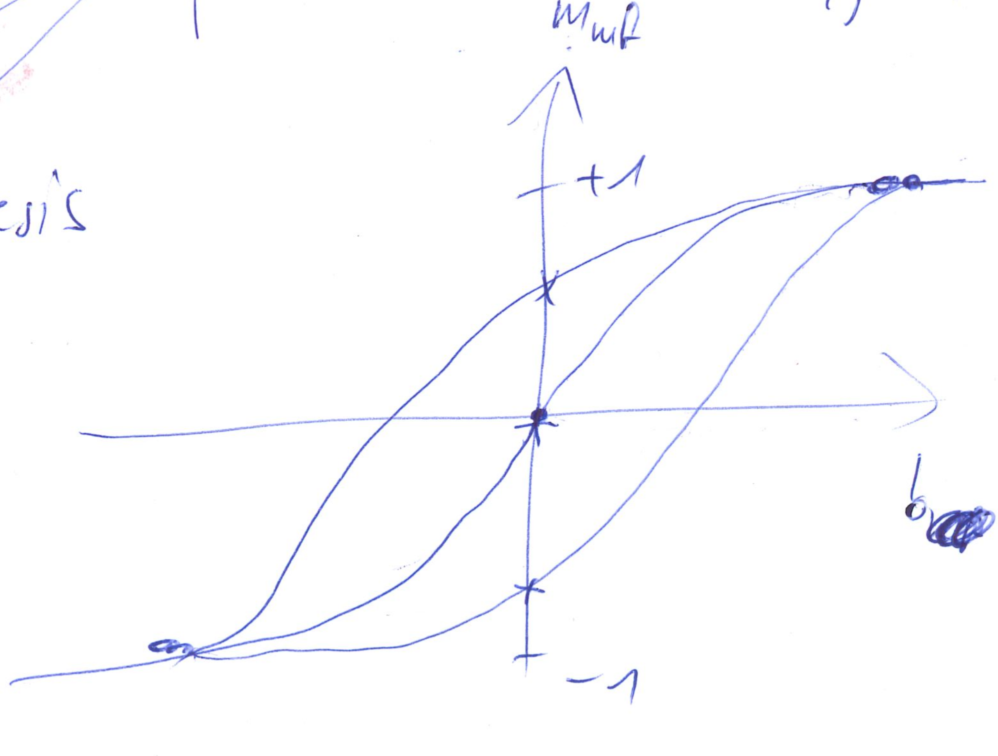
$b > 0$



b large:

only one solution

$qJ > 1$



Finally Hysteresis