Eigenvalues for $x_1 = y_1 = z_1 = 0$? (3)

$$J := \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$$\det(J - \lambda E) = (-\sigma-\lambda)(-1-\lambda)(-b-\lambda) - \sigma r(-b-\lambda)$$

$$= -(b+\lambda)\left\{\left[\lambda^2 + (1+\sigma)\lambda + \sigma\right] - \sigma r\right\}$$

$$= -(b+\lambda)\left(\lambda^2 + (1+\sigma)\lambda + \sigma(1-r)\right)$$

$$\lambda_{1/2} = -\frac{\sigma+1}{2} \pm \frac{1}{2}\sqrt{(\sigma+1)^2 + 4\sigma(r-1)}$$

$$\lambda_3 = -b \qquad \longrightarrow$$

for $r < 1$: $\lambda_{1,2,3} < 0$

$r = 1$: $\lambda_1 = 0$, $\lambda_2 = -(\sigma+1)$, $\lambda_3 = -b$

$r = 0$: $\lambda_{1,2} = -\frac{\sigma+1}{2} \pm \frac{1}{2}(\sigma-1)$: $\lambda_1 = -1$, $\lambda_2 = -\sigma$, $\lambda_3 = -b$

For $r > 1$: $\lambda_1 > 0$ unstable

Stability for x_2, y_2, z_2:

$$\begin{pmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b-\lambda \end{pmatrix}$$

$$P(\lambda) = (-\sigma-\lambda)\cdot\left[(1+\lambda)(b+\lambda) + b(r-1)\right]$$

$$\mp\sigma \left[1\cdot(-b-\lambda) + b(r-1)\right]$$

$$= -(+\sigma+\lambda)\left[\lambda^2 + (b+1)\lambda + b + b(r-1)\right)$$

$$\mp\sigma\left(\mp b \mp \lambda \mp b(r-1)\right]$$

$$= -\Big(\lambda^3 + \lambda^2(\sigma+b+1) + \lambda\Big[\sigma(b+1) \underset{\boxed{\mp\sigma+b}}{+b(r-1)}\Big]$$

$$-\Big(+\sigma b(r-1) \underset{+\sigma b}{\mp \sigma b} + 2\sigma b(r-1)\Big)\Big)$$

$$= -\left(\lambda^3 + (1+b+\sigma)\lambda^2 + b(r+\sigma)\lambda + 2\sigma b(r-1)\right)$$

For $r > 1$ all coeff's > 0

- Third order polynomial has one real root λ_0 ; $\lambda_0 < 0$!

$$\boxed{\begin{array}{l} ax^3+bx^2+cx+d=0 \\ \text{Discriminant} \\ \Delta = b^2c^2 - 4ac^3 \\ -4b^3d - 27a^2d^2 \\ +18abcd \end{array}}$$

- All roots are real < 0 for $1 < r < r_1 = 1.34561\ldots$

- For $r > r_1$ we get two complex conjugate roots $\lambda_r + i\lambda_i$, $\lambda_r - i\lambda_i$

$$\boxed{\begin{array}{l} \text{If } \Delta > 0 \text{ 3 real roots} \\ \text{If } \Delta < 0 \text{ 1 real, 2 compl. conj.} \end{array}}$$

- Characteristic Polynomials

$$P(\lambda) = (\underline{\underline{\lambda}} - \underline{\underline{\lambda_0}})(\underline{\underline{\lambda}} - \underline{\underline{\lambda_r}} - i\lambda_i)(\underline{\underline{\lambda}} - \underline{\underline{\lambda_r}} + i\lambda_i)$$

$$= \cancel{\lambda^3 + (\lambda - \lambda_0)(\lambda_r^2 + \lambda_i^2)}$$

$$\cancel{+ \lambda_0(\lambda^2 - 2\lambda\lambda_r)}$$

$$= (\underline{\underline{\lambda}} - \underline{\underline{\lambda_0}})(\underline{\underline{\lambda}}^2 - 2\underline{\underline{\lambda}}\lambda_r + \lambda_r^2 \underline{\underline{+ \lambda_i^2}})$$

$$= \lambda^3 + \lambda^2(-\lambda_0 - 2\lambda_r) + \lambda\left((\lambda_r^2 + \lambda_i^2) + 2\lambda_0\lambda_r\right)$$

$$- \lambda_0(\lambda_r^2 + \lambda_i^2)$$

$$\Rightarrow \quad 1 + b + \sigma = -\lambda_0 - 2\lambda_r$$

$$b(r+\sigma) = \lambda_r^2 + \lambda_i^2 + 2\lambda_0\lambda_r$$

$$2\sigma b(r-1) = -\lambda_0\left(\lambda_r^2 + \lambda_i^2\right)$$

Transition to instability if λ_r becomes positive; so set $\lambda_r = 0 \Rightarrow$

$$2\sigma \cancel{b}(r-1) = \underbrace{(1+b+\sigma)}_{-\lambda_0} \cancel{b} \underbrace{(r+\sigma)}_{\lambda_i^2}$$

$$2\sigma r - 2\sigma = r + br + \sigma r + \sigma + \sigma b + \sigma^2$$

$$r(\sigma - b - 1) = \sigma(3 + b + \sigma)$$

$$r = r_{crit} = \frac{\sigma(3+b+\sigma)}{\sigma - b - 1}$$

$\boxed{r = r_{crit}\,!}$ For $r < r_{crit}$:

$\cancel{2\sigma b(r-1) < (-\lambda_0 - 2\lambda_r)\cdot\lambda_i^2}$

$\cancel{\frac{2\sigma b(r-1)}{\lambda_i^2} < -\lambda_0 - 2\lambda_r}$

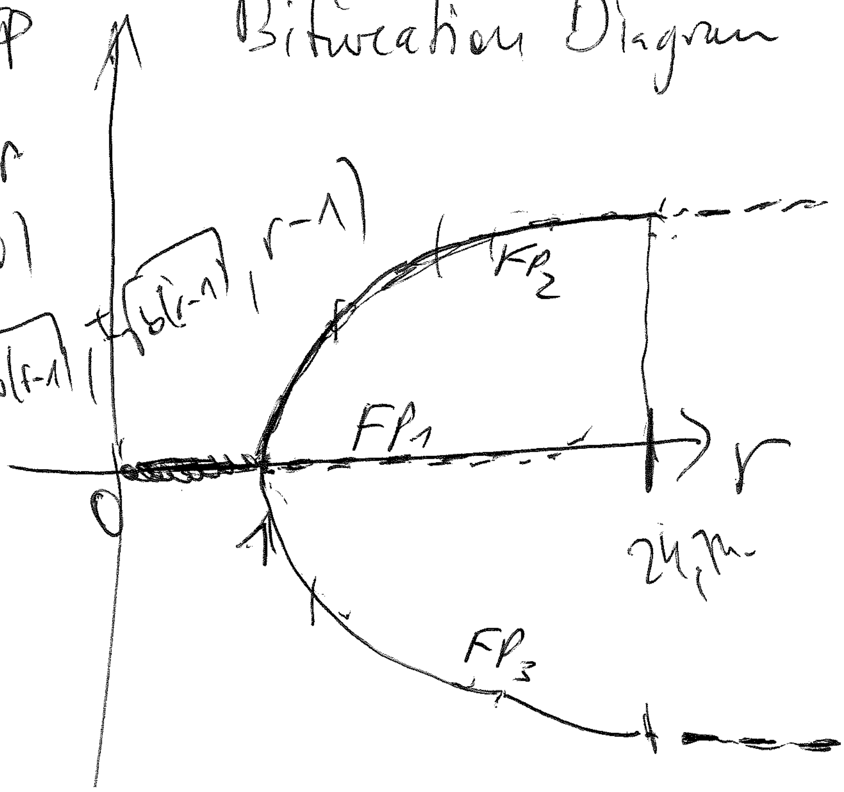
Bifurcation Diagram

FP

x-comp

$FP_1 = (0, 0, 0)$

$FP_{2/3} = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$



— stable

--- unstable