Computational Physics - Exercise 8

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1 Fixed Points of the Lorenz dynamical System

The Lorenz attractor problem is given by the following coupled set of differential equations

$$\dot{x} = -\sigma(x - y) \tag{1}$$
$$\dot{y} = rx - y - xz \tag{2}$$
$$\dot{z} = xy - bz \tag{3}$$

The fixed points for this problem are $\lambda_1 = (0, 0, 0)$ for all r and $\lambda_{2,3} = (\pm a_0, \pm a_0, r - 1)$ with $a_0 = \sqrt{b(r - 1)}$ for $r > 1$.
In this exercise we want to examine the stability of $\lambda_{2,3}$ by the Jacobian taken at the fixed points, and then looking for its eigenvalues by means of finding the zero points of the following characteristic polynomial:

$$P(\lambda) = \lambda^3 + (1 + b + \sigma)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r - 1) \tag{4}$$

We first plot $P(\lambda)$ as a function of λ. For that we use $\sigma = 10, b = 8/3$:

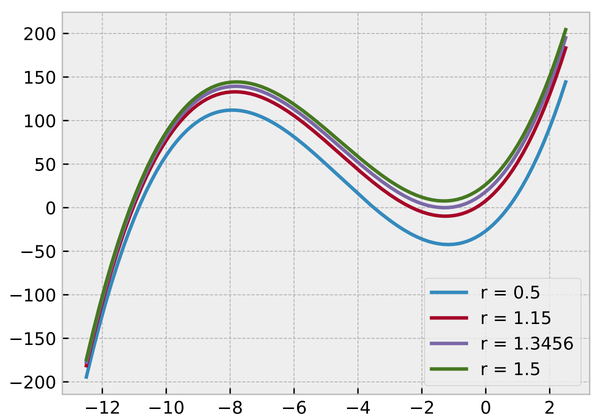

Figure 1: Characteristic Polynomial of the Lorenz attractor

To check if the points are stable, one has to examine the area around the point. Lies the point in an upwards slope, it is unstable. Otherwise, it is stable. In case of a saddle point, no definite answer can be made.
That means the only stable points we have, are the ones for values of $r < 1.3456$. For larger r two of the 3 stationary points vanish.

Next we determine the (complex) roots for different values of r:

r	Re(x)	Im(x)
1.3456	-11.09	0
1.3456	-1.3	0
1.3456	-1.28	0
1.5	-11.13	0
1.5	-1.27	0.88
1.5	-1.27	-0.88
24	-13.62	0
24	-0.02	9.49
24	-0.02	-9.49
28	-13.85	0
28	0.09	10.19
28	0.09	-10.19

Table 1: (Complex) roots of the Characteristic Polynomial

For $r < r_{\text{crit},1} = 1.3456$, all Fixpoints are < 0 and real, and are hence stable.
For $r_{\text{crit},1} < r < r_{\text{crit},2} = 24.74$, the real parts of the fixed points are still < 0 and the Fixpoints (including the complex conjugate ones) are hence stable.
For $r > r_{\text{crit},2}$, the real part of the complex conjugate solutions becomes > 0. The Fixed points are hence unstable.

2 The Lorenz attractor

We solve the Lorenz equations numerically with `rk4`, for the values $r = 0.5, 1.15, 1.3456, 24$ and 28. For that we use the previously integrated Runge-Kutta-4 algorithm (see Exercise 2). All we need to do is to integrate the coupled Lorenz equations:

```
def f(y0,x0): # y0 array that consists of [x,y,z]
    deriv = np.array([
    - sig*(y0[0] - y0[1]),
    r*y0[0] - y0[1] - y0[0]*y0[2],
    y0[0]*y0[1] - b*y0[2]])

    return deriv
```

Using `rk4`, we plot the trajectories for all r each with Starting point C_+ and C_-:

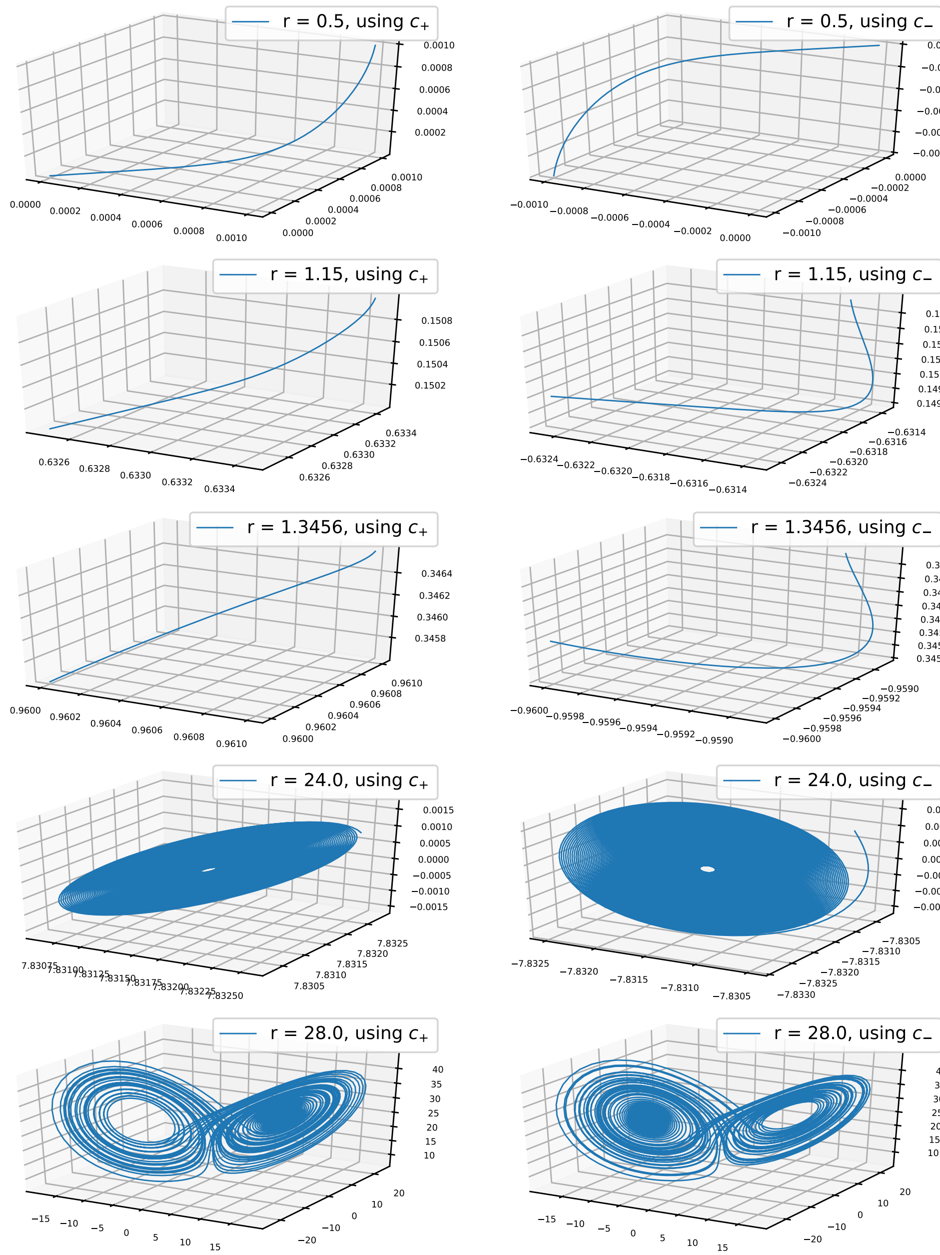

Figure 2: Determining Fixed points of the Lorenz dynamical System

As discussed before, the fixed points for