Computational Physics - Exercise 8

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1 Fixed Points of the Lorenz dynamical System

The Lorenz attractor problem is given by the following coupled set of differential equations

$$\dot{x} = -\sigma(x - y) \tag{1}$$
$$\dot{y} = rx - y - xz \tag{2}$$
$$\dot{z} = xy - bz \tag{3}$$

The fixed points for this problem are $\lambda_1 = (0, 0, 0)$ for all r and $\lambda_{2,3} = (\pm a_0, \pm a_0, r - 1)$ with $a_0 = \sqrt{b(r-1)}$ for $r > 1$.
In this exercise we want to examine the stability of $\lambda_{2,3}$ by the Jacobian taken at the fixed points, and then looking for its eigenvalues by means of finding the zero points of the following characteristic polynomial:

$$P(\lambda) = \lambda^3 + (1 + b + \sigma)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r - 1) \tag{4}$$

We first plot $P(\lambda)$ as a function of λ. For that we use $\sigma = 10, b = 8/3$:

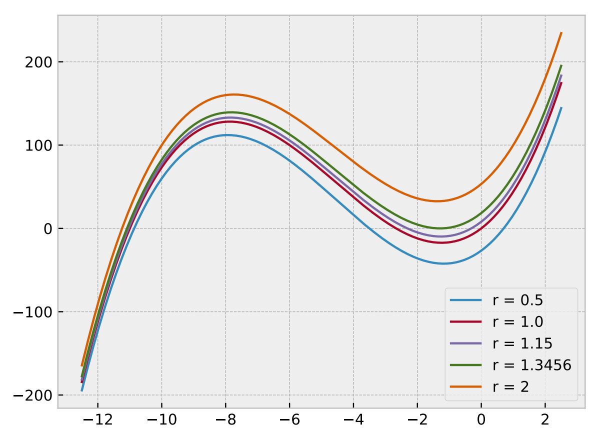

Figure 1: Characteristic Polynomial of the Lorenz attractor

To check if the points are stable, one has to examine the eigenvalues of the Jacobimatrix at the stationary point. If an eigenvalue has a positive real part, then the point is unstable. If all realparts are negative the point is stable. That means:

For $r < 1$	All Solutions unstable, because there is always at least one $\lambda > 0$
For $1 < r < 1.3456$	all $\lambda < 0$ and the Solution is stable
For $r > 1.3456$	two of the three roots vanish, and the remaining root is < 0, so in theory, those solutions are stable too.

Note, that this is only correct for the characteristic Polynomial. Later we will see that the Fix Point $(0, 0, 0)$ at $r < 1$ is indeed stable, since the Jacobian has a different shape, and hence, there is a different polynomial. In the $r < 1$ case the only fixpoint is $x = y = z = 0$ and all eigenvalues of the resulting Jacobimatrix are negative (shown in lecture at 12.6). As a result, the simulations for r < 0 reach the stable point of $x = y = z = 0$ in Figure (3).

Next we determine the (complex) roots for different values of r:

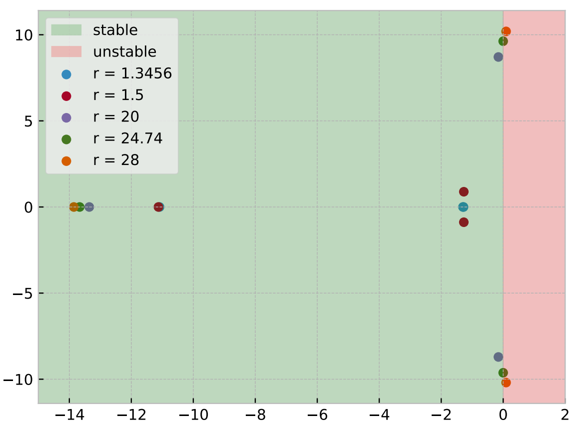

Figure 2: (Complex) roots of the Characteristic Polynomial

For $1 < r < r_{\text{crit},1} = 1.3456$, all roots are < 0 and real, and the fixpoints are hence stable.
For $r_{\text{crit},1} < r < r_{\text{crit},2} = 24.74$, the real parts of the roots are still < 0 and the Fixpoints (including the complex conjugate ones) are hence stable.
For $r > r_{\text{crit},2}$, the real part of the complex conjugate roots becomes > 0. The Fixed points are hence unstable.

2 The Lorenz attractor

We solve the Lorenz equations numerically with `rk4`, for the values $r = 0.5, 1.15, 1.3456, 24$ and 28. For that we use the previously integrated Runge-Kutta-4 algorithm (see Exercise 2). All we need to do is to integrate the coupled Lorenz equations:

```
def f(y0,x0): # y0 array that consists of [x,y,z]
    deriv = np.array([
    - sig*(y0[0] - y0[1]),
    r*y0[0] - y0[1] - y0[0]*y0[2],
    y0[0]*y0[1] - b*y0[2]])

    return deriv
```

Using `rk4`, we plot the trajectories for all r each with Starting point C_+ and C_-:

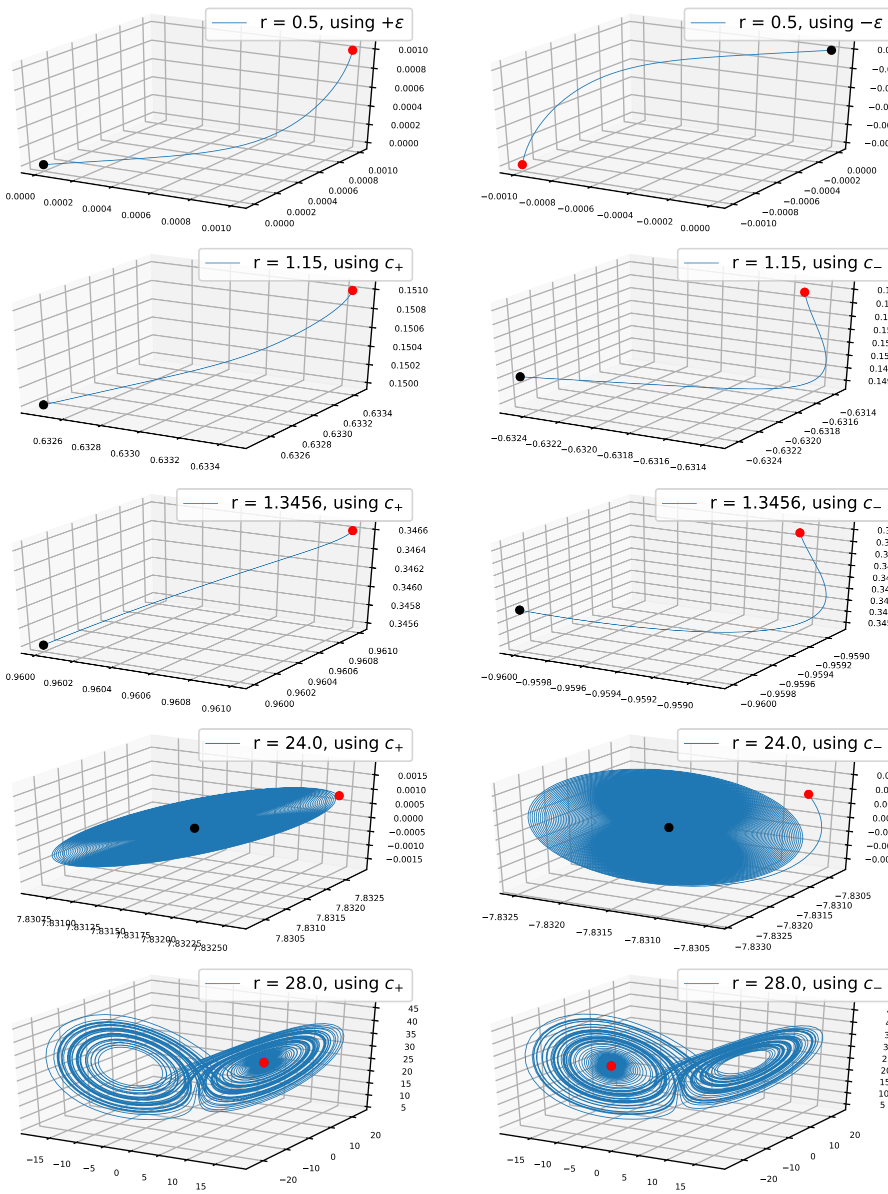

Figure 3: Determining Fixed points of the Lorenz dynamical System (the red dot describes the starting point, the black dot the fix point)

As discussed before, the fixed point for all $r < 1$ is stable. The same applies to all other points until we hit the value $r_{\text{crit},2} \approx 24.74$. Here, there is an oscillation around the fixed point that goes on for a while. For large enough t, a second "disk" emerges, and the oscillation changes back and forth between the two.

Finally, we take a closer look at the case $r = 27$. We determine the sequence z_k, where z_k is a local maximum in z on the solution curve after k Periods. For that, we write a small block of code, that extracts all local maxima from the solution curve by a simple algorithm, and writes them into a one-dimensional array:

```
def checkz(array):
    initial = array[0][2]
    up = True
    maxima = ([])
    for n in range(array.shape[0]-1):
        if up == True:
            if array[n][2] > array[n+1][2]:
                up = False
                maxima.append(array[n][2])
        if up == False:
            if array[n][2] < array[n+1][2]:
                up = True
    return maxima
```

Then we plot z_{k+1} as a function of z_k. For small z, we see a linear relationship, to which we create a fit and determine its slope:

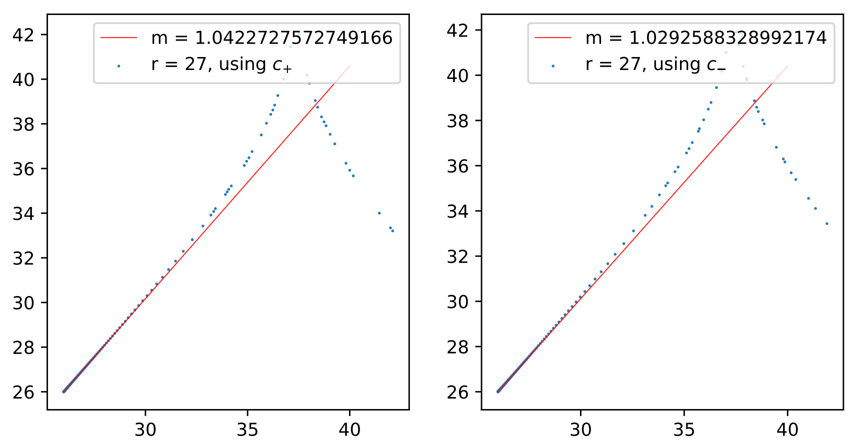

Figure 4: Determining the slope of z_{k+1}/z_k

As seen in Figure 4, $m > 1$. The theory of discrete maps says, that there is no periodic solution if $|m| > 1$. For $r = 27 > 24.74$ that coincides with our previous assumption.