

Computational Physics - Exercise 8

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June 19, 2019

1 Fixed Points of the Lorenz dynamical System

The Lorenz attractor problem is given by the following coupled set of differential equations

$$\dot{x} = -\sigma(x - y) \quad (1)$$

$$\dot{y} = rx - y - xz \quad (2)$$

$$\dot{z} = xy - bz \quad (3)$$

The fixed points for this problem are $\lambda_1 = (0, 0, 0)$ for all r and $\lambda_{2,3} = (\pm a_0, \pm a_0, r - 1)$ with $a_0 = \sqrt{b(r - 1)}$ for $r > 1$.

In this exercise we want to examine the stability of $\lambda_{2,3}$ by the Jacobian taken at the fixed points, and then looking for its eigenvalues by means of finding the zero points of the following characteristic polynomial:

$$P(\lambda) = \lambda^3 + (1 + b + \sigma)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r - 1) \quad (4)$$

We first plot $P(\lambda)$ as a function of λ . For that we use $\sigma = 10, b = 8/3$:

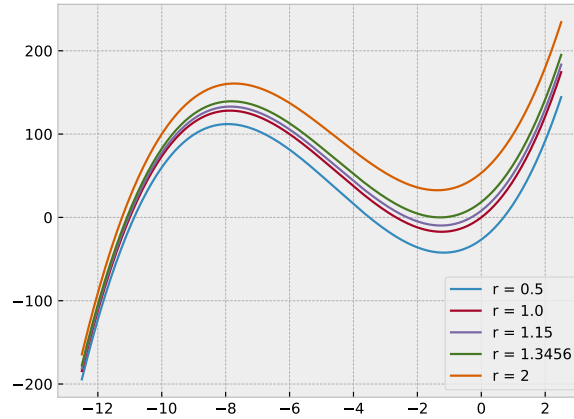


Figure 1: Characteristic Polynomial of the Lorenz attractor

To check if the points are stable, one has to examine the eigenvalues of the Jacobian matrix at the stationary point. If an eigenvalue has a positive real part, then the point is unstable. If all real parts are negative the point is stable. That means:

For $r < 1$	All Solutions unstable, because there is always at least one $\lambda > 0$
For $1 < r < 1.3456$	all $\lambda < 0$ and the Solution is stable
For $r > 1.3456$	two of the three roots vanish, and the remaining root is < 0 , so in theory, those solutions are stable too.

Note, that this is only correct for the characteristic Polynomial. Later we will see that the Fix Point $(0, 0, 0)$ at $r < 1$ is indeed stable, since the Jacobian has a different

shape, and hence, there is a different polynomial. In the $r < 1$ case the only fixpoint is $x = y = z = 0$ and all eigenvalues of the resulting Jacobimatrix are negative (shown in lecture at 12.6). As a result, the simulations for $r < 0$ reach the stable point of $x = y = z = 0$ in Figure (3).

Next we determine the (complex) roots for different values of r :

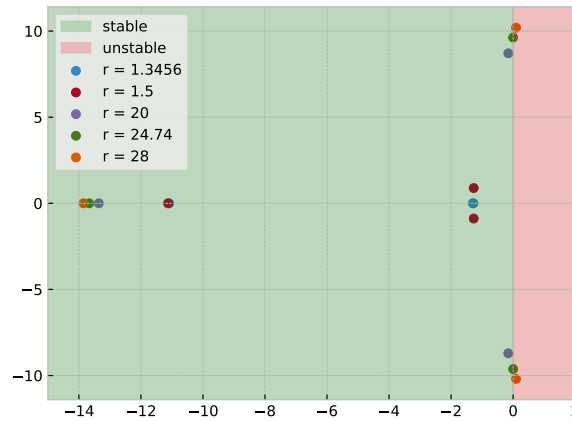


Figure 2: (Complex) roots of the Characteristic Polynomial

For $1 < r < r_{\text{crit},1} = 1.3456$, all Fixpoints are < 0 and real, and are hence stable. For $r_{\text{crit},1} < r < r_{\text{crit},2} = 24.74$, the real parts of the fixed points are still < 0 and the Fixpoints (including the complex conjugate ones) are hence stable. For $r > r_{\text{crit},2}$, the real part of the complex conjugate roots becomes > 0 . The Fixed points are hence unstable.

2 The Lorenz attractor

We solve the Lorenz equations numerically with `rk4`, for the values $r = 0.5, 1.15, 1.3456, 24$ and 28 . For that we use the previously integrated Runge-Kutta-4 algorithm (see Exercise 2). All we need to do is to integrate the coupled Lorenz equations:

```
def f(y0,x0): # y0 array that consists of [x,y,z]
    deriv = np.array([
        - sig*(y0[0] - y0[1]),
        r*y0[0] - y0[1] - y0[0]*y0[2],
        y0[0]*y0[1] - b*y0[2]])
    return deriv
```

Using `rk4`, we plot the trajectories for all r each with Starting point C_+ and C_- :

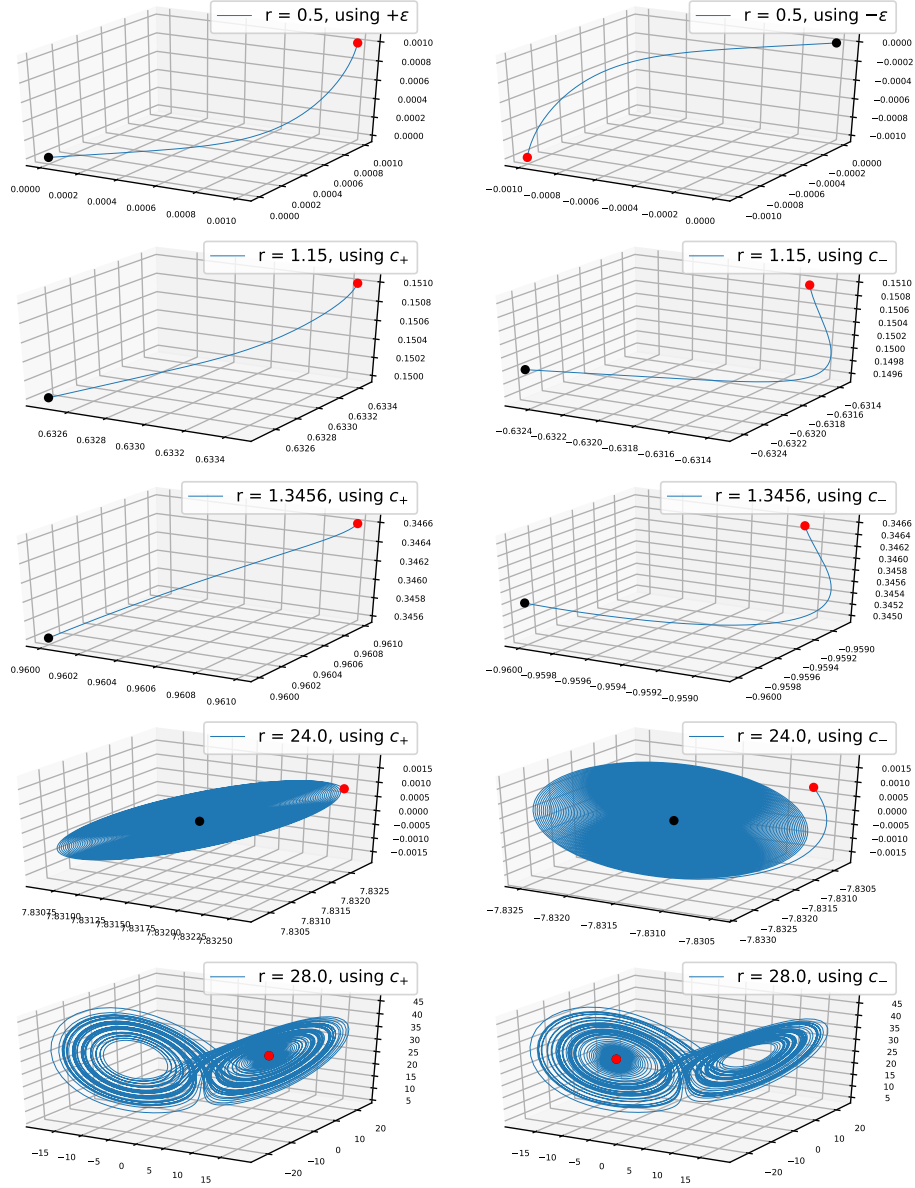


Figure 3: Determining Fixed points of the Lorenz dynamical System (the red dot describes the starting point, the black dot the fix point)

As discussed before, the fixed points for $r < 1$ are stable. The same app The same applies to all other points until we hit the value $r_{\text{crit},2} \approx 24.74$. Here, there is an oscillation around the fixed point that goes on for a while. For large enough t , a second "disk" emerges, and the oscillation changes back and forth between the two.

Finally, we take a closer look at the case $r = 27$. We determine the sequence z_k , where z_k is a local maximum in z on the solution curve after k Periods. For that, we write a small block of code, that extracts all local maxima from the solution curve by a simple algorithm, and writes them into a one-dimensional array:

```
def checkz(array):
    initial = array[0][2]
    up = True
    maxima = ([])
    for n in range(array.shape[0]-1):
        if up == True:
            if array[n][2] > array[n+1][2]:
                up = False
                maxima.append(array[n][2])
        if up == False:
            if array[n][2] < array[n+1][2]:
                up = True
    return maxima
```

Then we plot z_{k+1} as a function of z_k . For small z , we see a linear relationship, to which we create a fit and determine its slope:

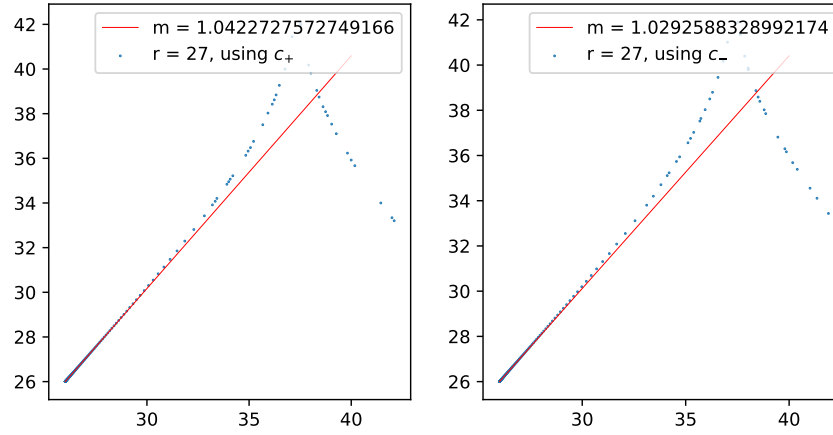


Figure 4: Determining the slope of z_{k+1}/z_k

As seen in Figure 4, $m > 1$. The theory of discrete maps says, that there is no periodic solution if $|m| > 1$. For $r = 27 > 24.74$ that coincides with our previous assumption.