Computational Physics - Exercise 1

Maurice Donner & Lukas Häffner

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Exercise 4 - Numerical Simulation of the 2-Body Problem

a)

Every higher order Differential equation can be seperated into a system of first order Differential equations. We are aiming to use this fact to make use of the Euler-forward algorithm later. Our ODE looks like this: `Hallo test >> Test "geht das auch nocoh"`

$$m_1 \cdot \ddot{\mathbf{x}}_1 = \frac{G m_1 m_2}{\mathbf{r}^2}, \quad m_2 \cdot \ddot{\mathbf{x}}_2 = \frac{G m_1 m_2}{\mathbf{r}^2}$$

With $\mathbf{r} = \left(x_b - x_a, y_b - y_a, z_b - z_a\right)$ being the relative position of the two bodies M_a and M_b. The Euler algorithm approximates $x(t + \Delta t) = x(t) + \Delta t \cdot F(x, t)$. The problem now is, that we don't know $F(x, t)$ yet. All we know is, that

$$\dot{\mathbf{x}} = \mathbf{v} \quad \& \quad \dot{\mathbf{v}} = F(x, t)$$

With $F_i(x, t)$ being $G \frac{m_1 m_2}{m_i \mathbf{r}^2}$

Looking only at the x-coordinate, we have to fill the following information into the euler algorithm:

$$\begin{aligned} x(t + \Delta t) &= x(t) + \Delta t \cdot v \\ v(t + \Delta t) &= v(t) + \Delta t \cdot F(x, t) \\ &= v(t) + \Delta t \cdot \tfrac{F_i}{M_i} \end{aligned}$$

```
# Define Force

fx = -G*Ma*Mb/absr3*rx
fy = -G*Ma*Mb/absr3*ry
fz = -G*Ma*Mb/absr3*rz

# Update the quantities of Ma

vxb += fx*dt/Mb
vyb += fy*dt/Mb
vzb += fz*dt/Mb

xb += vxb*dt
yb += vyb*dt
zb += vzb*dt

# Update the quantities of Mb (force acts in the other direction)

vxa += -fx*dt/Ma
vya += -fy*dt/Ma
vza += -fz*dt/Ma

xa += vxa*dt
ya += vya*dt
za += vza*dt

t += dt
```

Let's throw in some starting values! We set $M_a = M_b = G = 1$, $\mathbf{v}_1 = \left(\frac{1}{2}, -\frac{1}{2}, 0\right)$ and $\mathbf{v}_2 = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$. For the time step size we choose $\Delta t = 0.01$ and the simulation time $t_{\text{ sim}} = 100$, which results in

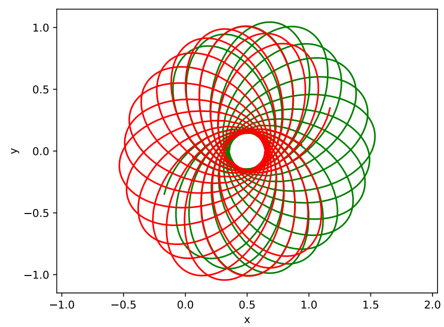

Abbildung 1: Trajectories of two Particles with equal mass, approximated

b)

Now we tackle the question of how to get orbit each other in a perfect circle. The needed starting values can be calculated by setting gravitational acceleration and Zentripetal acceleration equal:

$$\frac{v^2}{R} = G\frac{M_1}{(2R)^2}.$$

The distance between the particles is 1, hence $R = 0.5$, so that

$$v = \sqrt{G\frac{M_1}{4R}} = \frac{1}{\sqrt{2}}$$

This will force the particles onto a perfectly circular orbit:

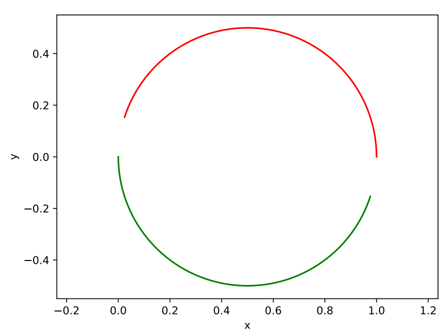

Abbildung 2: Particles orbiting each other in a perfect circle

c)

If we decrease the velocity from b) by a factor $\frac{1}{\sqrt{2}}$ the orbits of the particles will offset in x-Direction, and the perfecty circular orbit will become an ellipse-like shape.

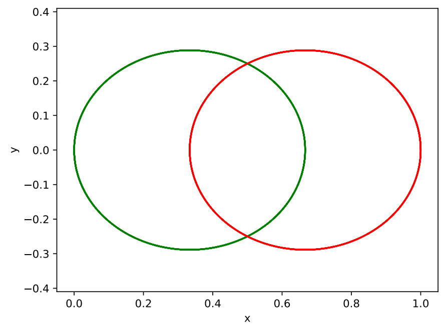

Abbildung 3: Offset orbits due to higher initial speed

d)

If we choose an initial velocity above $\sqrt{2}v_0$, the gravitational pull of the two objects becomes too tiny and the algorithm fails due to rounding .

e)

Decreasing the time step size will result in a smoother curve and a better approximation of the actual solution. The chance of rounding down the Force gets smaller. Increasing the step size will result in bigger divergence and the chance that the gravitational pull being rounded to 0 becomes higher, hence resulting in the particles flying away from each other.

Problem 5 - Error Analysis of Euler Scheme

We use our previous examples and calculate the eccentricity as well as the energy difference between initiation and the different steps.

Δt	v_0	e
10^{-2}	$1/\sqrt{2}$	0.963
10^{-4}	$1/\sqrt{2}$	0.714
10^{-4}	$1/2$	0.991

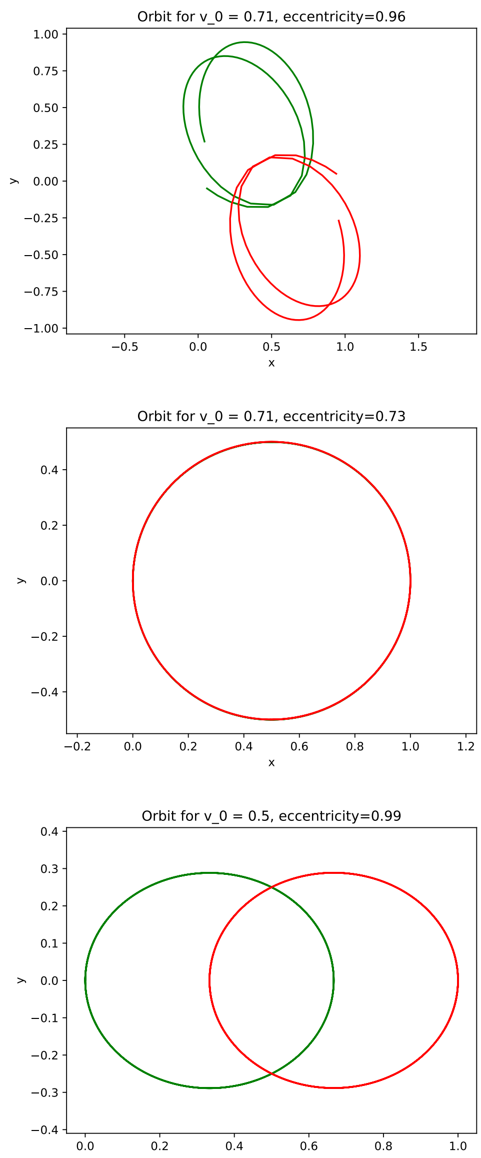

Abbildung 4: Trajectories

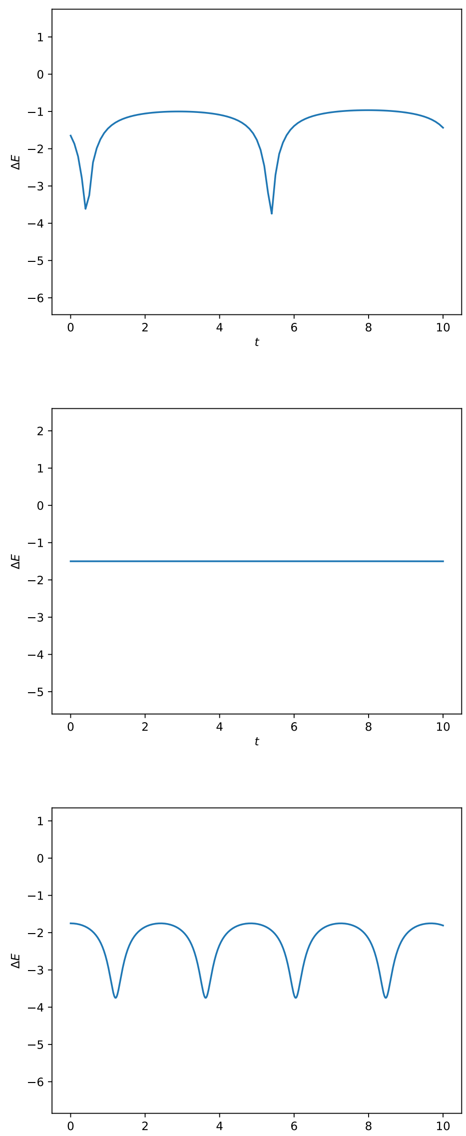

Abbildung 5: Energy differences