# Lorenz System - Mathematical Description (A)

Very short summary — see other links for details

Density $\rho(x,z,t)$ Gravity $F=(0,0,-g)$

Pressure $p(x,z,t)$

Temperature $T(x,z,t)$

Velocities $u_x(x,z,t)=u$ ; $u_z(x,z,t)=w$

Basic equations of hydrodynamics:

- Continuity — mass equation
- Navier Stokes — momentum ; viscosity parameter $\eta$
- Heat Transport — energy ; conductivity parameter $\chi$

Approximations:

$$\rho = \rho_0 \left(1-\alpha\,(T-T_0)\right)$$ linear $\rho$-$T$-relation

Flow Potential Function $\psi(x,z,t)$ with

$$u=-\frac{\partial\psi}{\partial z} \; ; \; w=\frac{\partial\psi}{\partial x}$$

Temperature Deviation Function $\Theta(x,z,t)$ with

$$T(x,z,t)=T_0+\Delta T\left(1-\frac{z}{h}\right)+\Theta(x,z,t)$$

(B)

Fourier Series for $\psi_1, \theta_1$

lowest order ansatz:

$$\psi(x,z,t) = \frac{\chi(1+a^2)\sqrt{2}}{a} X(t) \sin\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right)$$

$$\theta(x,z,t) = \frac{\Delta T}{\pi}\frac{Ra_{cr}}{Ra}\left[\sqrt{2}\,Y(t)\cos\left(\frac{\pi a}{h}x\right)\sin\left(\frac{\pi}{h}z\right) - Z(t)\sin\left(\frac{2\pi}{h}z\right)\right]$$

$$\dot{X} = -\sigma X + \sigma Y \qquad \sigma = \frac{\nu}{\chi}$$

$$\dot{Y} = rX - Y - XZ \qquad b = \frac{4}{1+a^2}$$

$$\dot{Z} = -bZ + XY \qquad r = \frac{Ra}{Ra_{cr}}$$

$$Ra = \frac{\alpha g h^3 \Delta T}{\chi\nu} \qquad Ra_{cr} = \frac{\pi^4(1+a^2)^3}{a^2}$$

# Dynamics of the Lorenz System



$$\dot{X} = -\sigma X + \sigma Y$$
$$\dot{Y} = rX - Y - XZ$$
$$\dot{z} = -bz + xy$$

---

Volume contraction

$$\frac{\partial}{\partial x}\dot{x} + \frac{\partial}{\partial y}\dot{y} + \frac{\partial}{\partial z}\dot{z} = -\sigma - 1 - b < 0$$

Trace of Jacobian

---

$$\frac{d \log V}{dt} = -\sigma - 1 - b \Rightarrow$$

$$V(t) = V_0 \exp\{-(\sigma+1+b) \cdot t$$

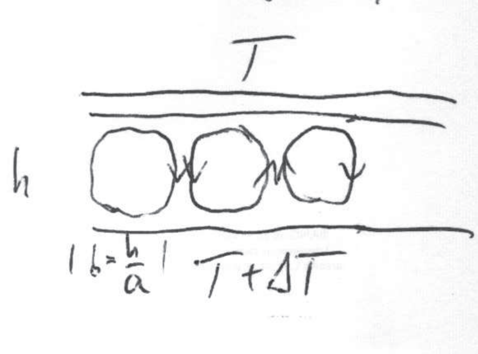
global attractor!

---

$\sigma$ = Viscosity / conductivity   $b = 4/_{1+a^2}$ geometry

$r = {Ra}/{Ra_{cr}} \propto \Delta T$

x: velocity, z, y: temperature

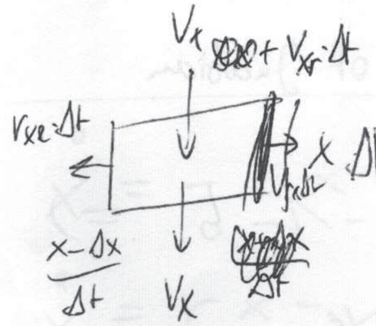


$$\Delta V = \Delta x \Delta y \Delta z$$

$$\cancel{dV = yz\,dx + zx\,dy + yx\,dz}$$

$$\cancel{\Delta V = \Delta y \Delta z\,(V_{x1} - V_{x2}) \cdot \Delta t}$$

$$\cancel{\frac{\Delta V}{\Delta t} = \Delta y \Delta z \frac{\Delta x}{\Delta t} + \Delta z \Delta x \frac{\Delta y}{\Delta t} + \Delta x \Delta y \frac{\Delta z}{\Delta t}}$$




$$\Delta V = \Delta y \Delta z\,(V_{x1} - V_{x2}) \cdot \Delta t \qquad :V$$

$$+ \Delta z \Delta x\,(V_{y1} - V_{y2}) \cdot \Delta t \qquad :\Delta t$$

$$+ \Delta y \Delta x\,(V_{z1} - V_{z2}) \cdot \Delta t$$

$$\frac{\Delta \log V}{\Delta t} = \frac{\Delta V_x}{\Delta x} + \frac{\Delta V_y}{\Delta y} + \frac{\Delta V_z}{\Delta z}$$

$$=$$

Fixed points: $\dot{x} = \dot{y} = \dot{z} = 0$

(i) $x_1 = y_1 = z_1 = 0$

(ii) $\dot{x} = 0 \Rightarrow x = y$

$\dot{z} = 0 \Rightarrow bz = x^2 \Rightarrow z = \frac{x^2}{b}$

$\dot{y} = 0 \Rightarrow (r-1)x = x^3/b \; (= xz)$

$\Rightarrow x^2 = b(r-1) \; ; \; x_{2/3} = \pm\sqrt{b(r-1)}$

$y_{2/3} = \pm\sqrt{b(r-1)}$

$z_{2/3} = r-1$

Stability of fixed points:

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$