Computational Physics - Exercise 7

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1 Many Species Population Dynamics

Given are 3 predator (P) and 3 prey (N) species, that change according to the following equations:

$$\frac{d\mathbf{N}}{dt} = \mathbf{N} \cdot \left(\mathbf{a} - \widehat{\mathbf{b}} \cdot \mathbf{P}\right), \quad \frac{d\mathbf{P}}{dt} = \mathbf{P} \cdot \left(\widehat{\mathbf{c}} \cdot \mathbf{N} - \mathbf{d}\right) \tag{1}$$

With $\mathbf{a}, \mathbf{d}, \widehat{\mathbf{b}}, \widehat{\mathbf{c}}$ being the following vectors/matrices:

$$\mathbf{a} = \begin{pmatrix} 55 \\ 34 \\ 11 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 85 \\ 9 \\ 35 \end{pmatrix} \tag{2}$$

$$\widehat{\mathbf{b}} = \begin{pmatrix} 20 & 30 & 5 \\ 1 & 3 & 7 \\ 4 & 10 & 20 \end{pmatrix}, \quad \widehat{\mathbf{c}} = \begin{pmatrix} 20 & 30 & 35 \\ 3 & 3 & 3 \\ 7 & 8 & 20 \end{pmatrix} \tag{3}$$

Solving $\frac{d\mathbf{N}/\mathbf{P}}{dt} = 0$ yields the following stationary points:

$$\mathbf{N_1^*} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{N_2^*} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{P_1^*} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{P_2^*} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{4}$$

Calculating the Jacobi-Matrix at the non-trivial stationary point:

$$\widehat{\mathbf{A}} = \begin{pmatrix} \frac{\partial}{\partial \mathbf{N}} \left(\frac{d\mathbf{N}}{dt} \right) \Big|_{\mathbf{N_2^*}, \mathbf{P_2^*}} & \frac{\partial}{\partial \mathbf{P}} \left(\frac{d\mathbf{N}}{dt} \right) \Big|_{\mathbf{N_2^*}, \mathbf{P_2^*}} \\ \frac{\partial}{\partial \mathbf{N}} \left(\frac{d\mathbf{P}}{dt} \right) \Big|_{\mathbf{N_2^*}, \mathbf{P_2^*}} & \frac{\partial}{\partial \mathbf{P}} \left(\frac{d\mathbf{P}}{dt} \right) \Big|_{\mathbf{N_2^*}, \mathbf{P_2^*}} \end{pmatrix} = \begin{pmatrix} \mathbf{a} - \widehat{\mathbf{b}} \cdot \mathbf{P} & -\mathbf{N} \cdot \widehat{\mathbf{b}} \\ \mathbf{P} \cdot \widehat{\mathbf{c}} & \widehat{\mathbf{c}} \cdot \mathbf{n} - \mathbf{d} \end{pmatrix} \tag{5}$$

But with (4) we know that $\mathbf{a} - \widehat{\mathbf{b}} \cdot \mathbf{P} = 0 = \widehat{\mathbf{c}} \cdot \mathbf{N} - \mathbf{d}$. So $\widehat{\mathbf{A}}$ reduces to:

$$\widehat{\mathbf{A}} = \begin{pmatrix} 0 & -\widehat{\mathbf{b}} \\ \widehat{\mathbf{c}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\begin{pmatrix} 20 & 30 & 5 \\ 1 & 3 & 7 \\ 4 & 10 & 20 \end{pmatrix} \\ \begin{pmatrix} 20 & 30 & 35 \\ 3 & 3 & 3 \\ 7 & 8 & 20 \end{pmatrix} & 0 \end{pmatrix} \tag{6}$$

Solving this with numpy function `numpy.linalg.eig` gives us the wanted eigenvalues and eigenvectors:

	Eigenvalue	Eigenvector
	33.63i	(0.54i, 0.54i, 0.87, 0.87, -0.23, 0.23
-	33.63i	(0.09i, -0.09i, -0.14, -0.14, 0.13i, -0.13i)
	7.7i	(0.29i -0.29i, -0.034, -0.34, 0.03, -0.03)
-	7.7i	(0.71, 0.71, -0.17i, 0.17i,-0.79 -0.79)
-	0.39	(0.08, 0.0, -0.15i, 0.15i, 0.54, 0.54)
	0.39	(0.31, 0.31, 0.24i, -0.24i, -0.11, -0.11)

Table 1: Eigenvalues and Eigenvectors of Jacobi Matrix A

Choosing an initial state $\mathbf{n} = \sum_{\mathrm{i}=1}^{6} c_i \mathbf{v}_i$ with $c = (3, 3, 1, 1, 5, 0.1)$

$$\mathbf{n} = (1.16 + 2.2i, 1.16 - 2.2i, 1.85 - 0.9i, 1.85 + 0.9i, 1.63, 2.19)$$

We plot the temporal evolution of all 6 populations:

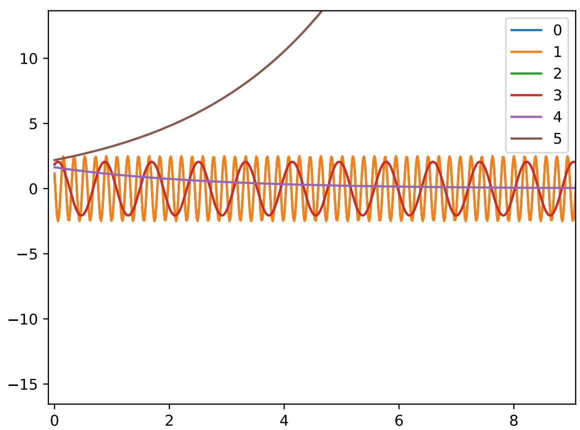

Figure 1: Temporal evolution of the 3 prey (0,1,2) and 3 predator (3,4,5) populations