5./7.6.2013

Nonlinear Dynamics 4.2, 4.3, Lorenz

Fixed Points, Stability, Deterministic Chaos
- ordinary diff. eqs. – simple?

4.2. Population Dynamics

Malthus (1798)

$$\frac{dN}{dt} = (b-d)N = rN$$

Dimensionless form: $\frac{du}{d\tau} = u \Rightarrow u(\tau) = e^{\tau}$

$\tau = rt$, $u = N/N_0$

$\Rightarrow N(t) = N_0 e^{rt}$ (~ radioactive decay)

Verhulst (1836)

unlimited growth unreal

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

$\dot{N} > 0$ $N < K$

$\dot{N} < 0$ $N > K$

Dimensionless form:

$\tau = rt$, $u = \frac{N}{K}$

$$\frac{du}{d\tau} = u(1-u)$$

Nonlinear!

Fixed Points:

Stationary Points / Solutions:

$\frac{dn}{d\tau} = 0$: Malthus: $n = 0$

Verhulst: $n = 0$, $n = 1$ $(N = K)$

Stability of Fixed Point? $\frac{dn}{d\tau} = f(n)$

Let $n(\tau) = n^* + \delta n(\tau)$; $\delta n / n \ll 1$

with fixed point n^*: $f(n^*) = 0$

$$\frac{dn}{d\tau} = \cancel{\frac{dn^*}{d\tau}} + \frac{d\delta n}{d\tau} = f(n) =$$

$$= f(n^* + \delta n) = f(n^*) + f'(n^*) \cdot \delta n + \mathcal{O}(\delta n^2)$$

$$\Rightarrow \frac{d}{d\tau} \delta n = f'(n^*) \cdot \delta n + \mathcal{O}(\delta n^2)$$

$$\Rightarrow \delta n(\tau) = \delta n(0) \cdot \exp(f'(n^*) \cdot \tau)$$

Stable: $f'(n^*) < 0$

Unstable: $f'(n^*) > 0$ Indefinite: $= 0$

Examples:

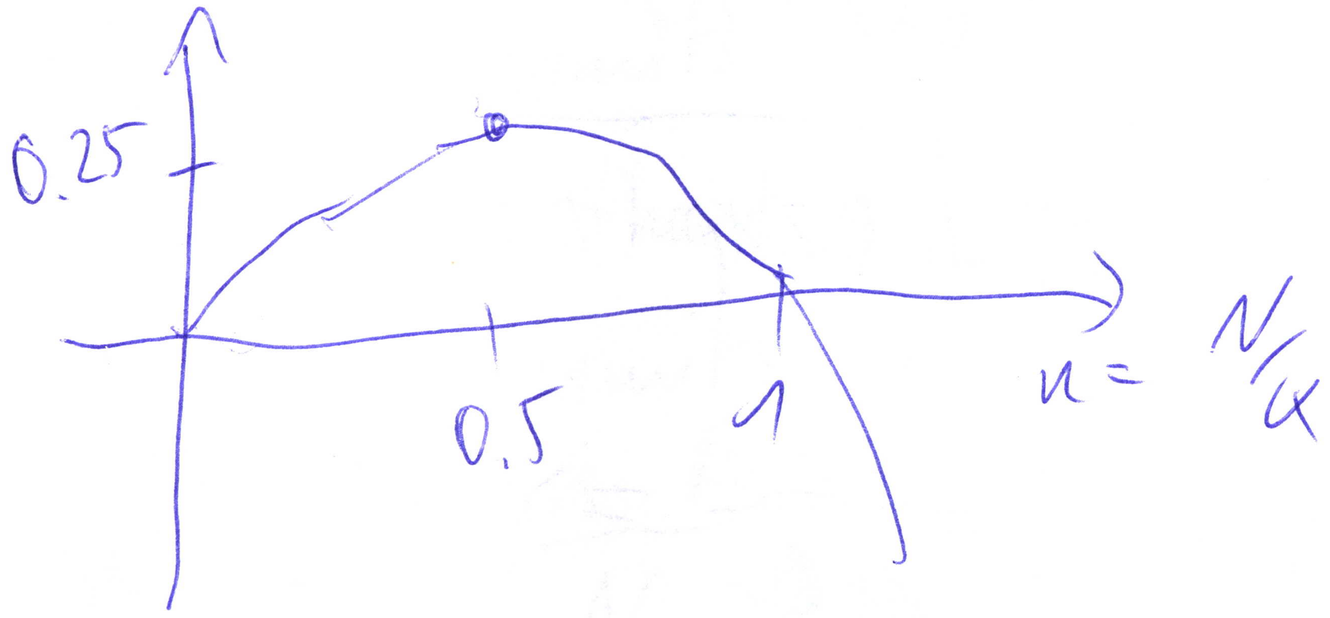
Malthus: $f(n) = \frac{dn}{d\tau} = n$; $f'(n) = 1 > 0$

$\Rightarrow$ fixed point $n^* = 0$ unstable

Verhulst: $f(n) = n(1-n)$; $f'(n) = 1 - 2n$

$\Rightarrow$ fixed point $n^* = 0$ unstable

fixed point $n^* = 1$ stable!



logistic growth!

Tutorial: one more complex ensemble

Direct solution: sep. of variables

$$d\tau = \frac{dn}{n(1-n)} = \frac{dn}{n} + \frac{dn}{(1-n)} = \frac{a(1-n) + bn}{n(1-n)}$$

$$\tau - \tau_0 = \log n - \log n_0 - \log(1-n) + C$$

3a

$$\tau - \tau_0 = \log n - \log(1-n)$$

$$= \log \frac{n}{1-n} + C$$

$$\frac{n}{1-n} \cdot C = \exp(\tau - \tau_0)$$

w.l.g. $\tau_0 = 0$

$$n \cdot C = \exp(\tau) - n \exp(\tau)$$

$$n(C + \exp(\tau)) = \exp(\tau)$$

$$n(\tau) = \frac{\exp(\tau)}{C + \exp(\tau)}$$

$\tau = 0$ $\quad n_0 = \frac{1}{1+C} \Rightarrow 1 + C = \frac{1}{n_0}$

$$C = \frac{1}{n_0} - 1 = \frac{1-n_0}{n_0}$$

$$n(\tau) = \cancel{\frac{2n_0 \exp(\tau)}{1 + \exp(\tau)}} = \cancel{\frac{n_0 e^{\tau}}{n_0 + (1-n_0)e^{\tau}}}$$

$$\cancel{n' = 2n_0 \frac{e^{\tau}}{1+e^{\tau}} - \frac{2n_0 e^{\tau} \cdot e^{\tau}}{(1+e^{\tau})^2}}$$

$$n(\tau) = \frac{n_0 e^{\tau}}{1 + n_0(e^{\tau} - 1)}$$

Interacting Populations

Volterra–Lotka System

$$\frac{dN_1}{dt} = N_1 (a - bN_2)$$

$$\frac{dN_2}{dt} = N_2 (cN_1 - d)$$

N_1 prey
N_2 predator

$a, b, c, d > 0$

N_1 without N_2: Malthusian growth
N_2 limits growth of N_1
N_2 without N_1: „radioactive" decay
N_1 limits decay of N_2

Dimensionless form:

$$u_1 = \frac{c}{d} N_1 \; ; \; u_2 = \frac{b}{a} N_2 \; ; \; \alpha = \frac{d}{a} \; ; \; \tau = at$$

$$\frac{du_1}{d\tau} = u_1 (1 - u_2)$$

$$\frac{du_2}{d\tau} = \alpha u_2 (u_1 - 1)$$

Only <u>one</u> free parameter!

Fixed Points:

(i) $u_1^* = u_2^* = 0$

(ii) $u_1^* = u_2^* = 1$

Stability Analysis for the multi-dim. form:

Formally analogous, but vector form:

$$\frac{d\vec{u}}{d\tau} = \vec{f}(\vec{u}) \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1(u_1,u_2) \\ f_2(u_1,u_2) \end{pmatrix}$$

Let $\vec{u}(\tau) = \vec{u}^* + \vec{v}(\tau)$ with $|\vec{v}|/|\vec{u}| \ll 1$

Fixed Point: $\vec{f}(\vec{u}^*) = 0$

$$\frac{d\vec{u}}{d\tau} = \frac{d\vec{u}^*}{d\tau} + \frac{d\vec{v}}{d\tau} = \vec{f}(\vec{u}^* + \vec{v}(\tau))$$

$$= \vec{f}(\vec{u}^*) + D\vec{f}(\vec{u}^*) \cdot \vec{v} + O(\vec{v}^2)$$

What is $D\vec{f}$? Jacobi-Matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{pmatrix}$$

$$\frac{d\vec{v}}{d\tau} = D\vec{F}(\vec{u}^*) \cdot \vec{v} + \mathcal{O}(v^2)$$

Formal Solution:

$$\vec{v}(\tau) = \exp\left(D\vec{F}(\vec{u}^*) \cdot \tau\right) \cdot \vec{v}(0)$$

What is $\exp(A\tau)$ with Matrix A?

$$\exp(A\tau) = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tau^n$$

Let $\vec{x}_i$ be EV of A with EW λ_i:

$$A\vec{x}_i = \lambda \vec{x}_i \Rightarrow A^n \vec{x}_i = \lambda^n \vec{x}_i \Rightarrow$$

$$\Rightarrow \exp(A\tau)\vec{x}_i = \sum_{n=0}^{\infty} \frac{\lambda_i^n}{n!} \tau^n = \exp(\lambda_i t)\vec{x}_i$$

$\Rightarrow \exp(\lambda_i t)$ is EW of $\exp(A\tau)$

Let $\vec{v}(0) = \sum c_i \vec{x}_i$ with $\vec{x}_i$ EV of A

$$\vec{v}(\tau) = \exp(A\tau) \sum c_i \vec{x}_i = \sum c_i \exp(\lambda_i t) \cdot \vec{x}_i,$$

Stability: all $\lambda_i < 0$

Instability: one $\lambda_i > 0$

Return to Volterra-Lotka:

$$A = \begin{pmatrix} 1-u_2 & -u_1 \\ \alpha u_2 & \alpha(u_1-1) \end{pmatrix}$$

Fixed Points: (i) $u_1 = u_2 = 0$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}$$

EW: $0 = \det(A - \lambda E) = (1-\lambda)(-\alpha-\lambda)$

$\lambda_1 = 1, \lambda_2 = -\alpha$ $\cancel{0 = -\alpha - \lambda + \lambda(\alpha+\lambda)}$

Unstable! $\cancel{= \lambda^2 + (\alpha-1)\lambda - \alpha}$

~~$\lambda_{1/2} =$~~ (ii) $u_1 = u_2 = 1$

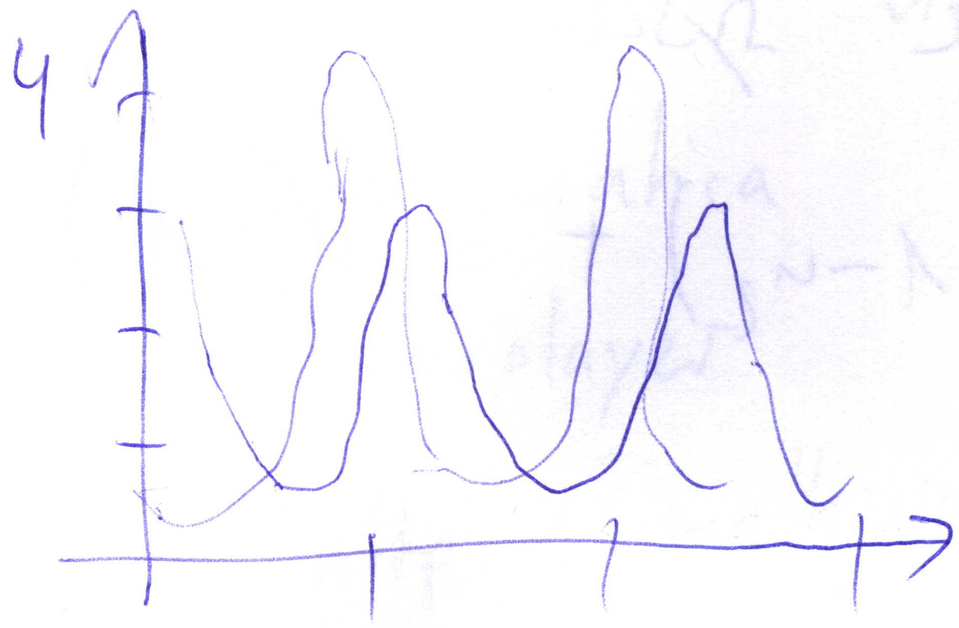
$$A = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix}$$

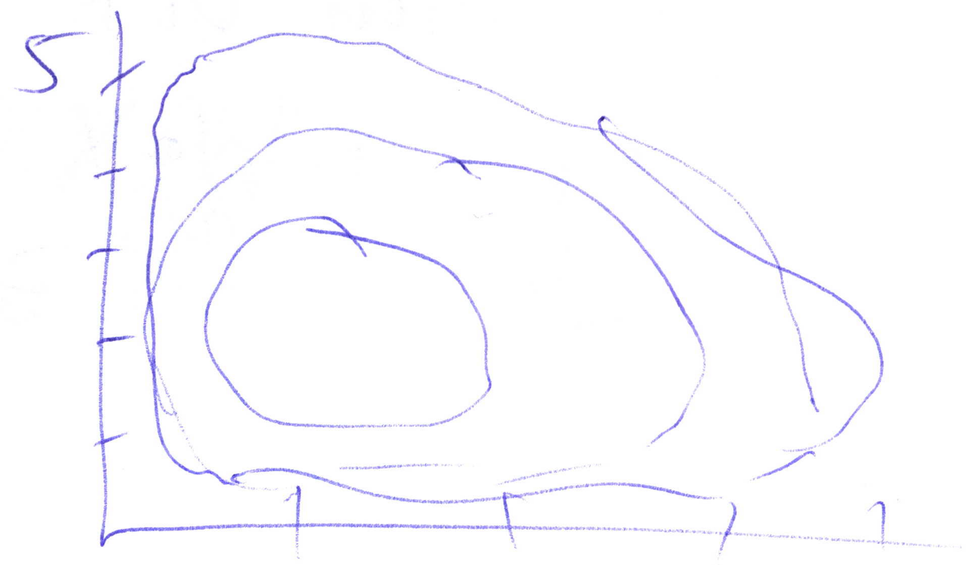
$0 = +\lambda^2 + \alpha \Rightarrow \lambda_{1/2} = \pm i\sqrt{\alpha}$

$\vec{v}(\tau) = c_1 \exp(i\sqrt{a}\tau)\vec{x}_1 + c_2 \exp(-i\sqrt{a}\tau)\vec{x}_2$

Purely oscillatory solutions?

Solution of Volterra-Lotka





$\alpha = 0.5 \quad u_1(0) = 1, \; u_2(0) = 3$

$u_1(0) = 1$

$u_2(0) = 2, 3, 4, 5$

K predator and K prey species

$$\frac{dN_i}{dt} = N_i \left(a_i - \sum_j b_{ij} P_j \right)$$

$$\frac{dP_i}{dt} = P_i \left(\sum_j c_{ij} N_j - d_i \right)$$

Fixed Point $\quad a_i = \sum b_{ij} P_j, \; d_i = \sum c_{ij} N_j$

$$A = \begin{pmatrix} 0 & -N_i^* b_{ij} \\ c_{ij} P_i^* & 0 \end{pmatrix} \Rightarrow$$

marginal or unstable!

Since $\operatorname{tr} A = \sum \lambda_j$

3.2.6. Mathematica

math line (gnuplot?)

mathematica X-Win

mathplayer (free)

http://mathworld.wolfram.com

Competition: eg. Maple

See our elearning2 ...

Some examples:

Solve [x^3 - 10x^2 + 21x - 10 == 0, x]

SHIFT + ENTER : Evaluate

Plot [Sin[x], {x, 0, Pi}]

Integrate [x^n, {x, 0, 1}]

Solve [x^2 + x + 1 == 0, x] // N

(evaluate numbers with limited prec., N[50])

n = 2
Clear [n] } assign / clear values

= assign value

== defines math. equations

%n
Use result of line n

Use X-Window Pull Down Templates!!
Save results as .nb file -> M. Notebook

More complex example:

```
NDSolve[ {u'[t] == u[t] (1 - v[t]),
          v'[t] == 0.5 v[t] (u[t] - 1),
          u[0] == 1, v[0] == 3},
          {u, v}, {t, 0, 40}]
```

2-body
3-body

Script: Interpolating Function, ParametricPlot

Lists + Matrix Operations

M = {{0,1}, {1,2}}

v = {1,4,7,6,5,3}

M = Table[1/(i+j-1), {i,10}, {j,10}]

Evaluate[Eigenvectors[M]] // N

Eigenvalues[M]

eigen = Eigenvalues[M]

v2 = eigen[[2]] get 2nd element

Inverse[M], Norm[M], M.Inverse[M]

M.v, v.v