(9)

$$\longrightarrow \quad dn = \frac{d^3\vec{p}}{(2\pi)^3}\cdot V$$ number of states increases with Volume ...

$\Delta N = \frac{(\Delta p)^3}{(2\pi)^3}\cdot V$ — Number of states in mom. space Volume

$$= 4\pi p^2 dp \cdot \frac{V}{(2\pi)^3}$$

But: larger Volume also needs different normalization of wavefunctions to guarantee unitarity of probability density ...

$$\frac{dn}{dp} = \frac{4\pi p^2}{(2\pi)^3}\cdot V$$

Wavefunction normalization:

$$\psi(x,t) = A e^{i(\vec{p}\vec{x}-Et)} \qquad \int \psi^*\psi\, d\vec{x} = \infty$$

Normalization within volume V: $\int_V \psi^*\psi\, d\vec{x} = \rho = 1 \longrightarrow A^2 = 1/V$

auch: siehe e.g. P. Schleper / Fortgeschr.

One V per final state particle ... in p-space ... in $\langle f|\,i\rangle$ ... one extra $1/V$ remains from initial state ... [or several ...] compensated through flux in σ.

$[\delta(p)]^2$ ...

Normalization compensates factor V in phase space as it appears in $|T_{fi}|^2 \sim (1/V)^n$; [n: # particles]

Thus we can choose $V=1$:

$$\frac{dn}{dp} = \frac{4\pi p^2}{(2\pi)^3} \quad \text{and} \quad \rho(E) = \left|\frac{dn}{dE}\right|_{E_i} = \frac{dn}{dp}\cdot\left|\frac{dp}{dE}\right|_{E_i}^{= 1/\beta}$$

with $E^2 = p^2 + m^2$
$2E\,dE = 2p\,dp$
$dp/dE = E/p = 1/\beta$.

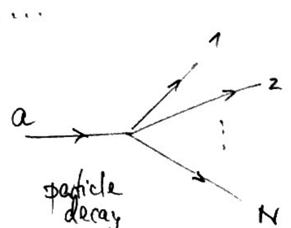
We use dn ...

Remember: $\Gamma_{fi} = 2\pi\int |T_{fi}|^2\delta(E_i - E)\,dn$

For N-particle final state; N-1 indep. momenta ...

$$dn = \prod_{i=1}^{N-1} dn_i = \prod_{i=1}^{N-1}\frac{d^3\vec{p}_i}{(2\pi)^3}$$

$$= (2\pi)^3\cdot\prod_{i=1}^{N}\frac{d^3\vec{p}_i}{(2\pi)^3}\cdot\delta^3\left(\vec{p}_a - \sum_{i=1}^{N}\vec{p}_i\right) \qquad (*)$$



particle decay

Lorentz-invariant phase-space:

Above expression is not Lorentz-invariant as Volume changes by factor $1/\gamma = m/E$, even though normalization of wavefunction would cancel V. more convenient to choose Lorentz-invariant formulation ...

mit obiger Norm.: $\int\psi^*\psi\, d\vec{x} = \frac{1}{V}$

? need to use V/E ...

Choose: $\int_V \psi^*\psi\, d^3\vec{x} = 2E$ ...

Which means that extra factors of 2E enter Matrix element ... $T_{fi} \to M_{fi}$

Lorentz invariant normalization of wave function $A^2 = E/V$

$$\longrightarrow \quad dLIPS = \prod_{i=1}^{N}\frac{d^3\vec{p}_i}{(2\pi)^3\, 2E_i} = (*)/(2\pi)^3$$

Lorentz-inv.

$(2\pi\cdot(2\pi)^3)$

$$\Gamma_{if} = \frac{(2\pi)^4}{2E_a}\cdot\int |M_{fi}|^2\,\delta(E_a - E_1 - E_2)\,\delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\,\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\,\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

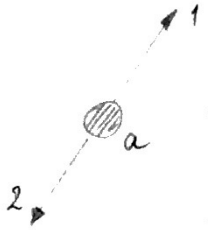
$(*)_{p9}$

not L.I. — from $|M_{fi}|^2$: normalization of initial state — Lorentz invariant

From: $M_{fi} = \langle \psi_1\psi_2|\hat{H}'|\psi_a\rangle \sim \sqrt{2E_1}\cdot\sqrt{2E_2}\cdot\sqrt{2E_a}$

With $|M_{fi}|^2 = 2E_a\, 2E_1\, 2E_2\cdot|T_{fi}|^2$.

Two-body decay:
[phase space!]



10

$E_a = m_a\,,\ \vec{p}_a = 0\,;\ \vec{p}_1 = \vec{p}^{\,*} = -\vec{p}_2$

Process: $a \to 1 + 2$

CMS-Frame

$$\Gamma_{fi} = \underbrace{\frac{1}{8\pi^2 m_a}}_{\text{Canceling all }(2\pi)\text{'s..}} \int \underbrace{|M_{fi}|^2}_{\text{Matrix element (see later)}} \delta(m_a - E_1 - E_2)\,\delta^3(\vec{p}_1 + \vec{p}_2)\,\underbrace{\frac{d^3\vec{p}_1}{2E_1}\cdot\frac{d^3\vec{p}_2}{2E_2}}_{\text{Phase Space}}$$

Using: $E_2^2 = m_2^2 + \vec{p}_1^{\,2}\,,\ d^3\vec{p}_1 = p_1^2\,dp_1 \sin\vartheta\, d\vartheta\, d\varphi = p_1^2\,dp_1\,d\Omega$

$E_1^2 = m_1^2 + \vec{p}_1^{\,2}\,,\ \vec{p}_1 = -\vec{p}_2 \ \ldots$

... and properties of the Dirac delta-functions ..

Requires: $\delta(f(x)) = \left|\frac{df}{dx}\right|^{-1}_{x_0} \delta(x - x_0)$
[see M. Thomson]
[Home work]

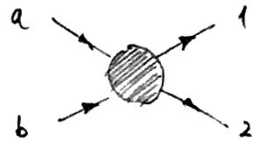
yields:

$$\boxed{\Gamma_{fi} = \frac{p^*}{32\pi^2 m_a^2} \int |M_{fi}|^2\, d\Omega}$$

Valid for all two-body decays

Remark: This is the decay rate in the rest frame of particle a. This is generally quoted as decay width or $\tau = 1/\Gamma$ lifetime. Γ_{fi} in its original definition is however not Lorentz invariant as Lorentz transform changes $m_a \mapsto E_a$ ..

Cross Section
and Lorentz-invariant flux:



$$\sigma = \frac{\Gamma_{fi}}{(v_a + v_b)} = \underbrace{\frac{(2\pi)^4}{4E_aE_b(v_a+v_b)}}_{\text{Lorentz-inv. Flux}} \int \underbrace{|M_{fi}|^2}_{\text{Lorentz-invariant Matrix element}} \delta^4(p_a + p_b - p_1 - p_2)\,\underbrace{\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\cdot\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}}_{\text{Lorentz invariant}}$$

Γ Lorentz-inv. flux:

See M. Thomson

$F = 4E_aE_b(v_a + v_b) = 4E_aE_b\left(\frac{|\vec{p}_a|}{E_a} - \frac{|\vec{p}_b|}{E_b}\right) = 4(E_a|\vec{p}_b| + E_b|\vec{p}_a|)$ [since $\beta = \frac{p}{E}$

$F^2 = 16(E_a^2\vec{p}_b^{\,2} + E_b^2\vec{p}_a^{\,2} + 2E_aE_b|\vec{p}_a||\vec{p}_b|)$

4-vectors: $(p_a \cdot p_b)^2 = (E_aE_b + |\vec{p}_a||\vec{p}_b|)^2 = E_a^2E_b^2 + \vec{p}_a^{\,2}\vec{p}_b^{\,2} + 2E_aE_b|\vec{p}_a||\vec{p}_b|$

$\to F^2 = 16\left[(p_a \cdot p_b)^2 - (E_a^2 - p_a^2)(E_b^2 - p_b^2)\right] \to F = 4\left[\underbrace{(p_a p_b)^2}_{\text{4-vector product squared!!}} - \underbrace{m_a^2 m_b^2}_{\text{masses}}\right]^{1/2}$

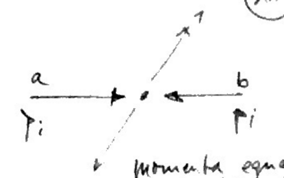
Lorentz invariant.

$E_a^2E_b^2 + \vec{p}_a^{\,2}\vec{p}_b^{\,2} + 2E_aE_b|\vec{p}_a||\vec{p}_b| - E_a^2E_b^2 - \vec{p}_a^{\,2}\vec{p}_b^{\,2} + E_a^2\vec{p}_b^{\,2} + E_b^2\vec{p}_a^{\,2}$

Scattering in CMS-frame:

$4(E_a|\vec{p}_i|+E_b|\vec{p}_i|)$

$$F = 4E_b^* E_a^* \Big(\underbrace{v_a^*}_{p_i^*/E_a^*} + v_b^*\Big) = 4p_i^*(E_a^* + E_b^*) = 4p_i^* \sqrt{s}$$



momenta equal
energies different
$\sqrt{s} = E_a^* + E_b^*$

$$\sigma = \frac{1}{(2\pi)^2} \cdot \frac{1}{4p_i^* \sqrt{s}} \cdot \int |M_{fi}|^2 \delta(\sqrt{s} - E_1 - E_2)\, \delta^3(\vec{p}_1 + \vec{p}_2) \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2}$$

$\frac{(2\pi)^4}{(2\pi)^6}$

$$= \frac{1}{16\pi^2 p_i^* \sqrt{s}} \cdot \frac{p_f^*}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*$$

needs again:
$E_2^2 = m_2^2 + \vec{p}_1^{\,2}$, $d^3 p_1 = p_1^2 dp_1 \sin\vartheta d\vartheta d\varphi$
$= p_1^2 dp_1 d\Omega$ ..

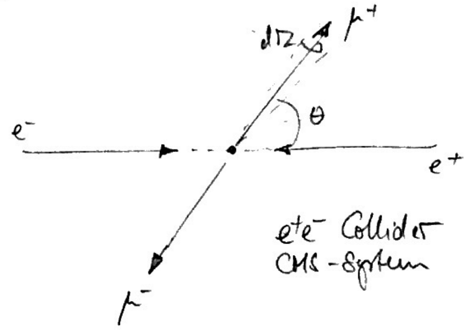
[Homework]

$$\boxed{\sigma = \frac{1}{64\pi^2 s} \cdot \frac{p_f^*}{p_i^*} \int |M_{fi}|^2 d\Omega^*}$$

Total Cross Section for $2 \to 2$ process.

Differential Cross Section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{p_f^*}{p_i^*} |M_{fi}|^2$$



e^+e^- Collider
CMS-System

Also: $\frac{d\sigma}{dE}$, $\frac{d\sigma}{dE\,d\Omega}$ ...

More complicated if to be calculated for fixed target ...
[if interested see M. Thomson 3.5]

IV. Dirac Equation

Reminder: Description of free particles ..

A. Schrödinger Equation.

$$\boxed{i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\vec{\nabla}^2\psi}$$

describes free non-relativistic particles ..

Remember: $\hbar = 1$

Solution for energy $E = p^2/2m$:

$$\psi(\vec{r}, t) = \frac{1}{\sqrt{V}} e^{-i(\vec{p}\vec{x} - Et)}$$

ψ^2 = probability density

$\vec{p} = \hbar\vec{k}$
$E = \hbar\omega$

The Schrödinger equation uses the classical E-p relation $E = p^2/2m$
replacing $E \to i\frac{\partial}{\partial t}$, $\vec{p} \to -i\vec{\nabla}$

Continuity equation:

$$\boxed{\frac{\partial \varrho}{\partial t} + \vec{\nabla}\vec{j} = 0}$$ with $\varrho = \psi^*\psi$ ($|\psi|^2$), $\vec{j} = -\frac{i}{2m}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$

ϱ: prob. density
$\vec{j}$: current density

Proof: A: $i\dot{\psi} + \frac{1}{2m}\nabla^2\psi \;\Big|\cdot(-i\psi^*) \longrightarrow \psi^*\dot{\psi} - \frac{i}{2m}\psi^*\nabla^2\psi = 0$

B: $-i\dot{\psi}^* + \frac{1}{2m}\nabla^2\psi^* \;\Big|\cdot(i\psi) \longrightarrow -\psi\dot{\psi}^* - \frac{i}{2m}\psi\nabla^2\psi^* = 0$

A − B: $\psi^*\dot{\psi} + \dot{\psi}^*\psi - \frac{i}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = 0$

$\longrightarrow \dot{\varrho} + \vec{\nabla}\vec{j} = 0$ ✓

B. Klein-Gordon Equation

$$\boxed{\frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi + m^2\phi = 0}$$ describes free relativistic spin-0 quantum fields

not free particles as we will see.

The Klein-Gordon equation uses $E^2 = p^2 + m^2$
replacing again $E \to i\frac{\partial}{\partial t}$, $\vec{p} \to -i\vec{\nabla}$.

Lorentz-invariant form
using co- and contravariant derivatives:

$$\boxed{\left(\partial^\mu\partial_\mu + m^2\right)\phi = 0}$$ with $\partial^\mu\partial_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

Plane-wave solution:

$$\phi = N \cdot e^{i(\vec{p}\vec{x} - E_\pm t)}$$ with $E_\pm = \pm\sqrt{p^2 + m^2}$

$E<0$ possible

$E<0$: unphysical?
Would be easy way out; but: negative energy values cannot be ignored as the solution would be incomplete without it.

need complete system of eigenfunctions

More problematic ...
Continuity Equation:

$$\nabla^2\phi - \frac{\partial^2}{\partial t^2}\phi - m^2\phi = 0 \qquad \Big|\cdot(-i\phi^*)$$

from left!

$$+ \quad \nabla^2\phi^* - \frac{\partial^2}{\partial t^2}\phi^* - m^2\phi^* = 0 \qquad \Big|\cdot(i\phi)$$

from right!

$$\frac{\partial}{\partial t}\underbrace{\left[i\phi^*\dot{\phi} - i\phi\dot{\phi}^*\right]}_{=:\varrho} + \vec{\nabla}\underbrace{\left[-i\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right)\right]}_{=:\vec{j}} = 0$$

see eg. Halzen/Martin
Thomson