3.2

$$C\psi = \psi_c = i\gamma^2\psi^* \qquad \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

a) $\psi_1 = e^{i(px-Et)} u_1(p)$ $\qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\psi_2 = e^{i(px-Et)} u_2(p)$ $\qquad \gamma^2 = \begin{pmatrix} 0&0&0&-i \\ 0&0&+i&0 \\ 0&+i&0&0 \\ -i&0&0&0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$C\psi_1 = \psi_{c_1}$

$$= i\gamma^2\psi_1^* = i\begin{pmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&i&0&0 \\ -i&0&0&0 \end{pmatrix} e^{i(px-Et)} N \begin{pmatrix} 1 \\ 0 \\ p_z/N^2 \\ p_x+ip_y/N^2 \end{pmatrix}$$

$$= iNe^{i(px-Et)} \begin{pmatrix} -ip_x \mp p_y/N^2 \\ ip_z/N^2 \\ 0 \\ -i \end{pmatrix} = \begin{pmatrix} p_x+ip_y/N^2 \\ -p_z/N^2 \\ 0 \\ 1 \end{pmatrix} = e^{-i(px-Et)} v_1(p)$$

$= \overline{\psi_1}$

$C\psi_2 = \psi_{c_2}$

$$= i\gamma^2\psi_2^* = i\begin{pmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&i&0&0 \\ -i&0&0&0 \end{pmatrix} e^{-i(px-Et)} N \begin{pmatrix} 0 \\ 1 \\ p_x+ip_y/N^2 \\ -p_z/N^2 \end{pmatrix}$$

$$= Nie^{-i(px-Et)} \begin{pmatrix} ip_z/N^2 \\ ip_x - p_y/N^2 \\ i \\ 0 \end{pmatrix} = Ne^{-i(px-Et)} \overbrace{\begin{pmatrix} -p_z/N^2 \\ -p_x - ip_y/N^2 \\ -1 \\ 0 \end{pmatrix}}^{-v_2(p)}$$

$= -\overline{\psi_2}$

The Conjugation operator transforms a particle wave function to its anti-particle wave function. (for u_2, a phase shift of $180°$ occurs)

b) $A^\mu = (\phi, \vec{A})$

$$\gamma^\mu(\partial_\mu - ieA_\mu)\psi + im\psi = 0$$

$$(\Leftrightarrow \gamma^\mu\partial_\mu\psi - \gamma^\mu ieA_\mu\psi + im\psi = 0)$$

$\gamma_\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$

$\gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$

$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$

$$\gamma^0\gamma^2 = \begin{pmatrix} \mathbb{1} & \\ & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = -\gamma^0\gamma^2$$

$$\gamma^1\gamma^2 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} = -i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = -\gamma^2\gamma^1$$

$$\gamma^2\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_2\sigma_2 & 0 \\ 0 & -\sigma_2\sigma_2 \end{pmatrix} = -\begin{pmatrix} \mathbb{1} & \\ & \mathbb{1} \end{pmatrix} \; *$$

$$\gamma^3\gamma^2 = +i\begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix} = -\gamma^3\gamma^2$$

$(\gamma^\mu)^* = \gamma^\mu$ for $\mu = 0, 1, 3$ and $(\gamma^2)^* = -\gamma^2$

$\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0$ $\qquad \hat{C} = i\gamma^2\psi^*$

First conjugate and multiply $i\gamma^2$

$\Rightarrow i\gamma^2(\gamma^\mu)^*(\partial_\mu + ieA_\mu)\psi - imi\gamma^2\psi = 0$

$\underbrace{i\gamma^2(\gamma^\mu)^*(\partial_\mu + ieA_\mu)}\psi - im\underbrace{i\gamma^2\psi^*}_{\psi'} = 0$

$= i\gamma^2(\gamma^0)^*(\partial_0 + ieA_0) + i\gamma^2(\gamma^1)(\partial_1 + ieA_1) + i\gamma^2(\gamma^2)^*\ldots + i\gamma^2(\gamma^3)^*\ldots$

$= i\gamma^2\gamma^0\ldots + i\gamma^2\gamma^1\ldots - i\gamma^2\gamma^2\ldots + i\gamma^2\gamma^3\ldots$

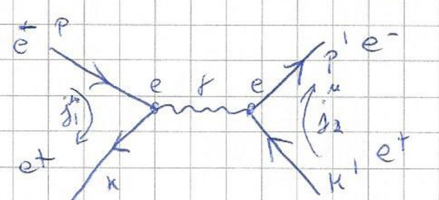
$= -i\gamma^0\gamma^2\ldots - i\gamma^1\gamma^2\ldots - i\gamma^2\gamma^2\ldots - i\gamma^3\gamma^2\ldots$

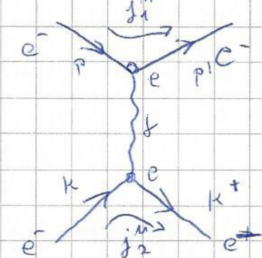
$= \gamma^\mu(\partial_\mu + ieA_\mu)(-i)\gamma^2 \Rightarrow -\gamma^\mu(\partial_\mu + ieA_\mu)\underbrace{i\gamma^2\psi}_{\psi'} - im\psi' = 0$

$\Rightarrow \gamma^\mu(\partial_\mu + ieA_\mu)\psi' + im\psi' = 0$

We see the same equation but for $\psi \to \psi'$ and $e \to -e$
Same particle mass, but different charge $\to$ Antiparticle

3.a $e^-(p) + e^+(k) \to e^-(p') + e^+(k')$ | $e^-(p) + e^-(k) \to e^-(p')e^-(k')$





b) $\mathcal{M}_{fi} = \int j^\mu_{(1)} \frac{-g_{\mu\nu}}{q^2} j^\nu_{(2)}\, d^4x$

$j_1^\mu = \bar{v}(k)[ie\gamma^\mu]u(p)$ $\qquad \gamma: \frac{-ig_{\mu\nu}}{q^2}$

$j_2^\mu = \bar{u}(p')[ie\gamma^\mu]v(k')$ $\qquad e: e$

$\Rightarrow \mathcal{M}_{fi} = \bar{v}(k)[ie\gamma^\mu]u(p)\frac{-ig_{\mu\nu}}{q^2}\bar{u}(p')[ie\gamma^\mu]v(k')$

$\mathcal{M}_{fi} = \int j^\mu_{(1)} \frac{-g_{\mu\nu}}{q^2} j_2^\mu\, d^4x$

$j_1^\mu = \bar{u}(p')\gamma^\mu u(p)\cdot e$ $\qquad \gamma: \frac{-ig_{\mu\nu}}{q^2}$

$j_2^\mu = \bar{u}(k')\gamma^\mu u(k)\cdot e$ $\qquad e: e$

$\Rightarrow \mathcal{M}_{fi} = \bar{u}_e(p')\gamma^\mu u_e(p)\frac{-ig_{\mu\nu}e^2}{q^2}\bar{u}_e(k')\gamma^\nu u_e(k)$

Incoming particle	:	$u(p)$
Outgoing Particle	:	$\bar{u}(p)$
Incoming Antiparticle	:	$v(p)$
Outgoing Antiparticle	:	$\bar{v}(p)$

3.1 Free particle spinors

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \quad \text{wit} \quad u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\, u_B \;, u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_A$$

a) $u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m} \cdot \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_A$

$$\Leftrightarrow (E^2 - m^2)\, u_A = (\vec{\sigma}\cdot\vec{p})^2\, u_A$$

$\sigma_i \sigma_j = -\sigma_j \sigma_i \quad i \neq j$

$$= (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3)^2\, u_A$$

$$= \Big[\sigma_1^2 p_1^2 + \sigma_2^2 p_2^2 + \sigma_3^3 p_3^2 + p_1 p_2 (\sigma_1\sigma_2 + \sigma_2\sigma_1) + p_1 p_3 (\sigma_1\sigma_3 + \sigma_3\sigma_1) + p_2 p_3 (\sigma_2\sigma_3 + \sigma_3\sigma_2)\Big] \cdot u_A$$

$$= \mathbb{1}(p_1^2 + p_2^2 + p_3^2) \cdot u_A$$

$$\Leftrightarrow (E^2 - m^2)\, u_A = \vec{p}^{\,2} \cdot u_A$$

choose $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow E^2 - m^2 = \vec{p}^{\,2} \Leftrightarrow E^2 = m^2 + \vec{p}^{\,2}$

b) $\beta \ll 1$ choose $\vec{p} = (0, 0, p_z)$

zz: $\frac{|u_B|}{|u_A|} \approx \beta$ $\left(u_1 = N \begin{pmatrix} u_A \\ u_B \end{pmatrix} \right.$ choose $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|u_B| = \left| \begin{pmatrix} \frac{p_z}{E+m} \\ 0 \end{pmatrix} \right| = \frac{|\vec{p}|}{E+m} = \frac{\gamma m \beta}{\gamma m + m} = \frac{\gamma\beta}{\gamma + 1} \approx \beta$$

$\Rightarrow$ with $|u_A| = 1 \Rightarrow \frac{|u_B|}{|u_A|} \approx \beta = \frac{v}{c}$

3.3 Chirality and Helicity

$$\hat{H} = \frac{\vec{\Sigma}\vec{p}}{2p} = \frac{1}{2p}\begin{pmatrix} \vec{\sigma}\vec{p} & 0 \\ 0 & \vec{\sigma}\vec{p} \end{pmatrix} \quad , \quad \frac{\vec{\sigma}\vec{p}}{2p} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\varphi - i\sin\varphi) \\ \sin\theta(\cos\varphi + i\sin\varphi) & -\cos\theta \end{pmatrix}$$

a)

$$= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}$$

Show that:

$$u_\uparrow = N\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \\ k\cos(\theta/2) \\ k\sin(\theta/2)e^{i\varphi} \end{pmatrix} \text{ and } u_\downarrow = N\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\varphi} \\ k\sin\frac{\theta}{2} \\ -k\cos\frac{\theta}{2}e^{i\varphi} \end{pmatrix} \quad \text{with } N = \sqrt{E+m}$$

are eigenstates of $\hat{H}$

$$\cancel{\hat{H}u_\uparrow = \sqrt{E+m}\,\hat{H}}$$

$$\hat{H}u_\uparrow = \frac{N}{2p}\begin{pmatrix} \vec{\sigma}\vec{p} & 0 \\ 0 & \vec{\sigma}\vec{p} \end{pmatrix}\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \\ k\cos(\theta/2) \\ k\sin(\theta/2)e^{i\varphi} \end{pmatrix}$$

$$(\hat{H}u_\uparrow)_0 = \frac{N}{2}\left[\cos(\theta)\cos(\theta/2) + \sin(\theta)\sin(\theta/2)\right]$$

$$(\hat{H}u_\uparrow)_1 = \frac{N}{2}\left[e^{i\varphi}\left(\cos(\theta/2)\sin(\theta) - \cos(\theta)\sin(\theta/2)\right)\right]$$

$$(\hat{H}u_\uparrow)_2 = \frac{N}{2}\left[k\left(\cos(\theta/2)\cos(\theta) + \sin(\theta/2)\sin(\theta)\right)\right]$$

$$(\hat{H}u_\uparrow)_3 = \frac{N}{2}\left[ke^{i\varphi}\left(\cos(\theta/2)\sin(\theta) - \cos(\theta)\sin(\theta/2)\right)\right]$$

with $\cos(\theta) = \cos\left(\frac{\theta}{2}\right)^2 - \sin\left(\frac{\theta}{2}\right)^2$

and $\sin(\theta) = 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$

$$\Rightarrow \cos(\theta)\cos(\theta/2) + \sin(\theta)\sin(\theta/2) = \cos\left(\frac{\theta}{2}\right)$$

and $\cos\left(\frac{\theta}{2}\right)\sin(\theta) - \cos(\theta)\sin(\theta/2) = \sin\left(\frac{\theta}{2}\right)$

$$\Rightarrow \hat{H}u_\uparrow = \frac{N}{2}\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \\ k\cos(\theta/2) \\ k\sin(\theta/2)e^{i\varphi} \end{pmatrix} = \frac{1}{2}u_\uparrow$$

3.3 Chirality and Helicity

b) $P_L = \frac{1-\gamma^5}{2}$, $P_R = \frac{1+\gamma^5}{2}$

$u_L = P_L u \qquad u_R = P_R u$

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$u_\uparrow = N \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \\ k\cos\frac{\theta}{2} \\ k\sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix}$$

$$P_L \cdot u_\uparrow = N \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \\ k\cos\frac{\theta}{2} \\ k\sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1-k)\cos\frac{\theta}{2} \\ \frac{1}{2}(1-k)\sin\frac{\theta}{2} e^{i\varphi} \\ \frac{1}{2}(k-1)\cos\frac{\theta}{2} \\ \frac{1}{2}(k-1)\sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix}$$

$$P_R \cdot u_\uparrow = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \\ k\cos\frac{\theta}{2} \\ k\sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1+k)\cos\frac{\theta}{2} \\ \frac{1}{2}(1+k)\sin\frac{\theta}{2} e^{i\varphi} \\ \frac{1}{2}(1+k)\cos\frac{\theta}{2} \\ \frac{1}{2}(1+k)\sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix} = \frac{1+k}{2} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \\ \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix}$$