Particle Physics – Exercise Sheet 3 – WS 2020/21

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discussed on Friday 27[th] November in the exercise groups

3.1 Free particle spinors

The two ($i = 1, 2$) free particle solutions to the Dirac equation are given by $\psi_i = e^{+i(\vec{p}\cdot\vec{x}-Et)}u_i(p)$ and the two free antiparticle solutions are given by $\psi_i = e^{-i(\vec{p}\cdot\vec{x}-Et)}v_i(p)$ and the spinors u_i and v_i are given by:

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \quad \text{and} \quad v_1 = N\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix},$$

where $N = \sqrt{E+m}$. Let the subscripts A and B denote the upper and lower two-component column vectors of the spinors:

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_A \\ v_B \end{pmatrix}.$$

a) It was shown in the lecture that solving the Dirac equation using above spinor representation leads to coupled equations for e.g. u_A in terms of u_B (with $\vec{\sigma}$ being the Pauli matrices):
$$u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\,u_B \quad \text{and} \quad u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,u_A$$
Using above equations, show that E and $\vec{p}$ must obey the Einstein energy-momentum relation. **Hint:** Substitute the solution to one of the equations into the other.

b) Show for u_1 that in the non-relativistic limit, where $\beta \equiv v/c \ll 1$, the lower components (u_B) are smaller than the upper ones (u_A) by a factor $\approx v/c$. **Hint:** Choose a direction of propagation (for example $\vec{p}$ along z) and compare the two norms $|u_A|$ and $|u_B|$.

3.2 Particles, antiparticles and charge conjugation

In the Pauli-Dirac representation, the charge conjugation operator $\hat{C}$, which transforms a particle wave function ψ into its charge-conjugate wave function ψ_c, is given by

$$\hat{C}\psi = \psi_c = i\gamma^2\psi^* \quad \text{with} \qquad \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}.$$

a) Find the charge conjugates of $\psi_1 = e^{+i(\vec{p}\cdot\vec{x}-Et)}u_1(p)$ and $\psi_2 = e^{+i(\vec{p}\cdot\vec{x}-Et)}u_2(p)$ with u_1, u_2 given in exercise 3.1. Interpret the result.

b) The Dirac equation for an electron with charge $q = -e$ in the presence of an electromagnetic field $A^\mu = (\phi, \boldsymbol{A})$ is given by
$$\gamma^\mu(\partial_\mu - ieA_\mu)\psi + im\psi = 0.$$
Calculate the corresponding equation after charge-conjugation and interpret the result.

3.3 Chirality and Helicity

The helicity is defined as the spin projection on the direction of the momentum vector and the helicity operator for a spin-$\frac{1}{2}$ fermion in the four dimensional spinor space is defined as

$$\hat{H} = \frac{\vec{\Sigma}\vec{p}}{2p} = \frac{1}{2p}\begin{pmatrix} \vec{\sigma}\vec{p} & 0 \\ 0 & \vec{\sigma}\vec{p} \end{pmatrix}.$$

a) Show that the states

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \\ k\cos\frac{\theta}{2} \\ k\sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}, \quad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\phi} \\ k\sin\frac{\theta}{2} \\ -k\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix},$$

with (p, θ, ϕ) being the spherical coordinates of the momentum vector $\vec{p}$ and $k = \frac{p}{E+m}$, are eigenstates of the helicity operator. **Hint:** Express $\vec{p}$ using the spherical coordinates and show that

$$\frac{\vec{\sigma}\vec{p}}{2p} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & \cos\theta \end{pmatrix}$$

b) The left- and right-handed chiral component u_L and u_R of a spinor u are given by the projection operators

$$P_L = \frac{1-\gamma^5}{2}, \qquad P_R = \frac{1+\gamma^5}{2},$$
$$u_L = P_L u, \qquad u_R = P_R u.$$

Contrary to the helicity states, the chiral states are Lorentz invariant. Use the matrix representation of the projection operators and calculate the chiral projection of the positive helicity spinor, i.e. $P_L u_\uparrow$ and $P_R u_\uparrow$. Discuss the limit $\frac{p}{E+m} \to 1$.

The matrix γ^5 is defined as follows:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

3.4 Feynman Rules and Diagrams

Consider the following electromagnetic scattering processes

- $e^-(p) + e^+(k) \to e^-(p') + e^+(k')$ (Bhabha Scattering)
- $e^-(p) + e^-(k) \to e^-(p'] + e^-(k')$ (Møller Scattering)

where p, k, p', k' are the 4-momenta of the particles.

a) Draw the possible tree-level Feynman diagrams of the two processes.
b) Write down the scattering amplitude $\mathcal{M}$ for each diagram following the Feynman rules.