We then again have: $\dot{\rho} + \vec{\nabla}\vec{j} = 0$.

Thus, for the Klein-Gordon-Equation one gets a different definition of the probability density!

Insertion of $\phi = N e^{i(\vec{p}\vec{x} - Et)}$ yields:

$$\rho = 2E_{\pm} |N^2| \longrightarrow \underline{\text{negative}} \text{ probability density! Problem!}$$
$$\vec{j} = 2\vec{p}\,|N^2|$$

Normalization

This negativ probability density is an inherent problem of the Klein-Gordon equation; this problem is not present for the Schödinger equation, as here: $\rho = |N^2|$ with $\psi = N e^{i(\vec{p}\vec{x} - Et)}$; $N = 1/\sqrt{V}$.

C. Dirac Equation

won't work

Idea: Avoid negative probabilities (and energies) using a linearized ansatz...

$$\text{Ansatz: } \hat{H}\psi = (\vec{\alpha}\hat{\vec{p}} + \beta m)\psi \quad \text{or}$$
$$i\frac{\partial}{\partial t}\psi = (-i\vec{\alpha}\vec{\nabla} + \beta m)\psi$$

Could have been only mathematical construction w/o physical meaning, but...

where α, β are determined requiring that also Klein-Gordon equation, i.e. $\hat{H}^2\psi = (\hat{\vec{p}}^2 + m^2)\psi$ is satisfied.

$$\rightarrow \alpha_i^2 = \beta^2 = \mathbb{1}\ ; \quad \alpha_i\alpha_k + \alpha_k\alpha_i = 0 \quad \text{for } i \neq k$$
$$\alpha_i\beta + \beta\alpha_i = 0$$

As α_i and β do not commute these relations cannot be satisfied by numbers $\longrightarrow$ 4x4 matrices.

i.e:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad i = 1,2,3 \quad ; \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

with σ_i = Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Covariant form of Dirac equation:

[Multiplication with β]

$\gamma^0 = \beta$

$$i\gamma^0 \frac{\partial}{\partial t}\psi + i\vec{\gamma}\vec{\nabla}\psi - m\psi = 0 \quad \text{with } \gamma^\mu = (\beta, \beta\vec{\alpha})$$

$$\rightarrow \boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0} \quad \text{Dirac equation}$$

Remark on notation: Def.: $\gamma^\mu a_\mu = \not{a}$
$\rightarrow (i\not{\partial} - m)\psi = 0$

with $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$, $\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$, $\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$

and:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\gamma^{0\dagger} = \gamma^0 ; \quad \gamma^{k\dagger} = -\gamma^k ; \quad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 ; \ldots$$

Reminder: $A^\dagger = (A^*)^T$

- Solution: Four-component spinors $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$
- (Book) Describes free relativistic spin-½ particles; operator: $\hat{S} = \frac{1}{2}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$
- (2) Again one gets solutions with negative energies
- (1) But: probability density always positive.

[M. Thomson, Ch. 4.4]
$\hat{S}^2\psi = \frac{3}{4}\psi$

Continuity equation:

See M. Thomson, Ch. 4.3

Use: $i\partial_0\psi + i\alpha_\mu\partial_\mu\psi - m\beta\psi = 0 \quad (*)$

As we now have spinors for ψ complex conjugates have to be replaced by hermitian conjugates: $\psi^* \mapsto \psi^\dagger = (\psi^*)^T$.

$$\rightarrow -i\partial_0\psi^\dagger - i\partial_\mu\psi^\dagger\alpha_\mu^\dagger - m\psi^\dagger\beta^\dagger = 0 \quad (**)$$

$\psi^\dagger(*) - (**)\psi$ yields:

hermitian!

$$i\psi^\dagger\partial_0\psi + i(\partial_0\psi^\dagger)\psi + \left(i\psi^\dagger\alpha_\mu\partial_\mu\psi + i(\partial_\mu\psi^\dagger)\alpha_\mu^\dagger\psi\right) = 0$$

Use: $\alpha_i^\dagger = \alpha_i$, $\beta^\dagger = \beta$

Factor "i" cancels $\sim$

$$\partial_0(\psi^\dagger\psi) + \vec{\nabla}\cdot(\psi^\dagger\vec{\alpha}\psi) = 0 \quad \text{(continuity eq.)}$$

Thus: $\rho = \psi^\dagger\psi$; $\vec{j} = \psi^\dagger\vec{\alpha}\psi$ with $\rho = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 > 0$.

$$\boxed{\begin{aligned} j^\mu = (\rho, \vec{j}) &= \psi^\dagger\gamma^0\gamma^\mu\psi \\ &= \bar{\psi}\gamma^\mu\psi \end{aligned}}$$

with $\alpha_i = \gamma^0\gamma^i$, $(\gamma^0)^2 = \mathbb{1}$

Dirac 4-vector current or (covariant current)

adjoint spinor

using: $\boxed{\bar{\psi} = \psi^\dagger\gamma^0}$ Adjoint Spinor!

Solutions to the Dirac equation:

All four components ψ_i satisfy the Klein-Gordon equation; this was the original intuition of the Ansatz ...

i.e.: $(\Box + m^2)\psi_i = 0$ for $i = 1, 2 . 4$

From: $(\gamma^0\partial_0 + \ \gamma^3\partial_3)(\gamma^0\partial_0 + \ \gamma^3\partial_3)$ and $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$

Proof: $$0 = \gamma^\nu\partial_\nu(i\gamma^\mu\partial_\mu - m)\psi = i\tfrac{1}{2}\underbrace{(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)}_{2g^{\mu\nu}}\partial_\nu\partial_\mu\psi - m\gamma^\nu\partial_\nu\psi$$
$$= i g^{\mu\nu}\partial_\nu\partial_\mu + i m^2\psi$$
$$= i(\partial^\mu\partial_\mu + m^2)\psi \ .$$

Hence for the solutions of the Dirac-equation, we can make the following Ansatz:

4-vectors! Thomson $u(p) = u(E, \vec{p})$

$$\text{(i)}\ \psi = u(p)\, e^{-ipx} \quad \text{and} \quad \text{(ii)}\ \psi = v(p)\, e^{ipx}$$

Spinor × plane wave solution

where $u(p)$, $v(p)$ are four-component spinors independent of x, i.e. time and space coordinates.

Insertion of (i) into Dirac-equation yields:

$i\gamma^\mu\partial_\mu e^{-ipx} = -(i)^2\gamma^\mu p_\mu = \gamma^\mu p_\mu$

$$\rightsquigarrow \quad (i\gamma^\mu\partial_\mu - m)\, u(p)\, e^{-ipx} = 0$$
$$(\gamma^\mu p_\mu - m)\, u(p) = 0$$

or in short-hand notation: $(\not{p} - m)u = 0$.

Furthermore, we use:

$$\gamma^\mu p_\mu = E\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma}\vec{p} \\ -\vec{\sigma}\vec{p} & 0 \end{pmatrix} \quad \text{and} \quad u(p) = \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ represent the Pauli-matrices and $u_A(p)$, $u_B(p)$ are two-component spinors.

Remark: Two further states for $E = -m$!

Now consider particle at rest; i.e.: $\vec{p} = 0$, $E = m$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2m\mathbb{1} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0 \longrightarrow u(p) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

with the eigenvalue $E = m$ for $\psi = u(p)\, e^{-ipx}$

with $\vec{p} \neq 0$: $\sqrt{m^2 + \vec{p}^2}$.

[Energy operator: $\gamma^0 p_0$; eigenvalue equation: $i\gamma^0\partial_0 u e^{-ipx} = m u e^{-ipx}$]

Similar with $\psi = v(p)\, e^{+ipx}$; $v(p) = \begin{pmatrix} v_A(p) \\ v_B(p) \end{pmatrix}$ ..

$$\left(i\gamma^\mu \partial_\mu - m\right) v(p)\, e^{+ipx} = 0$$

$$\left(\gamma^\mu p_\mu + m\right) v(p) = 0$$

or again in short : $(\not{p} + m)\, v = 0$

Again considering particle at rest, i.e. $\vec{p} = 0$, $E = m$:

$$\begin{pmatrix} 0 & 2m\,\mathbb{1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_A \\ v_B \end{pmatrix} = 0 \longrightarrow v(p) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Attention: sign!

... again with eigenvalue $E = +m$ for $\psi = v(p)\, e^{ipx}$

See Thomson 4.7.4

[here, we need to use diff. operator; eigenvalue eq.: $-i\gamma^0\partial_0 v e^{+ipx} = +m v e^{+ipx}$]

Remark: The Dirac-equation has 4 independent solutions. These can be formulated in different ways; only using the ansatz $u(p)\, e^{-ipx}$ yields two extra solution with neg. energy eigenvalues $E = -m$ which are difficult to interprete; using only $v(p)\, e^{ipx}$ the same happens.

Using two solutions of the form $u e^{-ipx}$ and two of the form $v e^{+ipx}$ gets rid of the negative energy eigenvalues. However we will have to see how to handle/interprete the different signs in the exponent of the exponential.

Anyhow,
First consider now $\vec{p} \neq 0$, ...

Using :

$$\left[E \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma}\vec{p} \\ -\vec{\sigma}\vec{p} & 0 \end{pmatrix} - m \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \right] u = 0$$

i.e. $(\not{p} - m)\, u = 0$

yields ...

$$\begin{pmatrix} (E-m)\mathbb{1} & -\vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -(E+m)\mathbb{1} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0 \longrightarrow u_A = \frac{\vec{\sigma}\vec{p}}{E-m}\, u_B \quad \& \quad u_B = \frac{\vec{\sigma}\vec{p}}{E+m}\, u_A$$

Hence, ... by choosing one of the two-component spinors u_A or u_B the other one is fully defined

both fullfilled at the same time as $E^2 = \vec{p}^2 + m^2$ ... try!

Choosing simplest expressions for $u_A = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \ldots$

PARTICLE SPINORS

$$u_1(p) = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u_2(p) = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

spin ↑ for $\vec{p} = (0,0,p_z)$

spin ↓ for $\vec{p} = (0,0,p_z)$

dep. on chosen normalization

N_1, N_2: if $\psi^\dagger\psi = 2E$ $N_1 = N_2 = \sqrt{E+m}$.

important: H_0 and S_z do not commute

The fact that u_1 and u_2 are spin ½ states with spin ↑, ↓ for $\vec{p} = (0,0,p_z)$ can be shown by calculating $\hat{S}_z u_{1,2} \ldots$ with $\hat{S}_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix}\sigma_z & 0 \\ 0 & \sigma_z\end{pmatrix} \ldots$

[compare Thomson 4.8 and 4.4]
[see below]

In a completely analogue way using $(\not{p}+m)v = 0$ one obtains ...

ANTIPARTICLE SPINORS

$$v_1(p) = N_1 \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2(p) = N_2 \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

spin ↑ for $\vec{p} = (0,0,p_z)$

spin ↓ for $\vec{p} = (0,0,p_z)$

again with $N_1 = N_2 = \sqrt{E+m}$ for Lorentz-inv. normalization.

Remark: These solutions correspond to solutions one would get using $E<0$ when deriving $u(p) \ldots$

Advantage of this choice: Don't need to remember whether energy E is positive or negative ...

Interpretation of negative energies and/or different signs in the exponential of $\psi = u e^{-ipx}$ and $\psi = v e^{ipx}$?!

A. Dirac Interpretation.

Vacuum = See of occupied negative energy levels

Due to Pauli principle filled negative energy levels have no influence as long as all are occupied.

(SLIDE)

→ Prediction of antiparticles as hole in the sea of neg. energy states