

Write out the numerator and denominator in Lekkala2013 θ_i separately.

In[6]:= **numerator** =

$$\text{Sinh}[\theta_1] \text{Sinh}[\theta_2] + \alpha \text{Cosh}[\theta_1] \text{Sinh}[\theta_2] - \lambda \text{Cosh}[\theta_1] \text{Cosh}[\theta_2] + 2\lambda - \lambda^2 \frac{\text{Sinh}[\theta_1]}{\text{Sinh}[\theta_2]}$$

Out[6]= $2\lambda - \lambda \text{Cosh}[\theta_1] \text{Cosh}[\theta_2] - \lambda^2 \text{Csch}[\theta_2] \text{Sinh}[\theta_1] + \alpha \text{Cosh}[\theta_1] \text{Sinh}[\theta_2] + \text{Sinh}[\theta_1] \text{Sinh}[\theta_2]$

In[7]:= **denominator** = $\text{Cosh}[\theta_1] \text{Sinh}[\theta_2] + \alpha \text{Sinh}[\theta_1] \text{Sinh}[\theta_2] - \lambda \text{Sinh}[\theta_1] \text{Cosh}[\theta_2]$

Out[7]= $-\lambda \text{Cosh}[\theta_2] \text{Sinh}[\theta_1] + \text{Cosh}[\theta_1] \text{Sinh}[\theta_2] + \alpha \text{Sinh}[\theta_1] \text{Sinh}[\theta_2]$

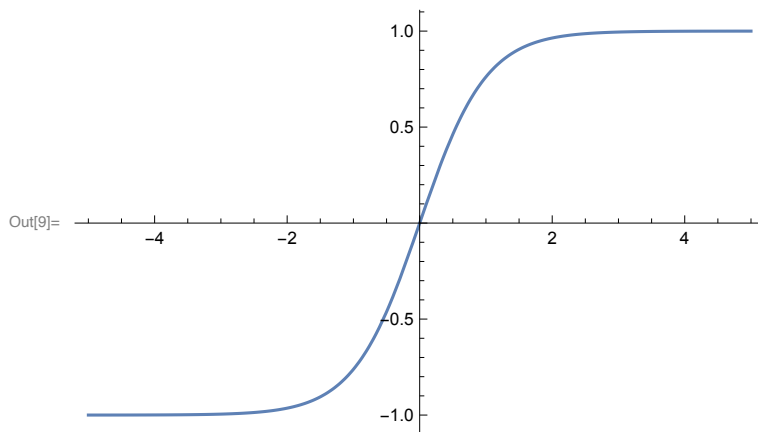
Divide the denominator by $\text{Cosh}[\theta]$ to see if that tames the θ_1 divergence.

In[8]:= $\frac{\text{denominator}}{\text{Cosh}[\theta_1]}$ // Simplify

Out[8]= $\text{Sinh}[\theta_2] - \lambda \text{Cosh}[\theta_2] \text{Tanh}[\theta_1] + \alpha \text{Sinh}[\theta_2] \text{Tanh}[\theta_1]$

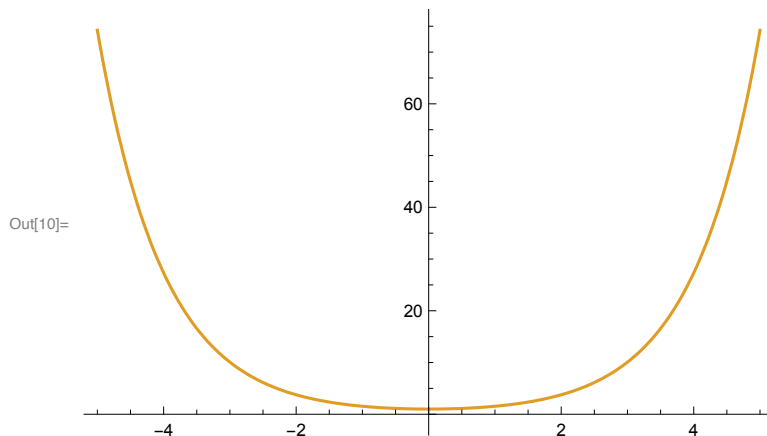
It does ...

In[9]:= **Plot**[{**Tanh**[θ_1]}, { θ_1 , -5, 5}]



... but the θ_2 functions in the denominator still diverge.

In[10]:= **Plot**[{**Sinh**[θ], **Cosh**[θ_2]}, { θ_2 , -5, 5}]



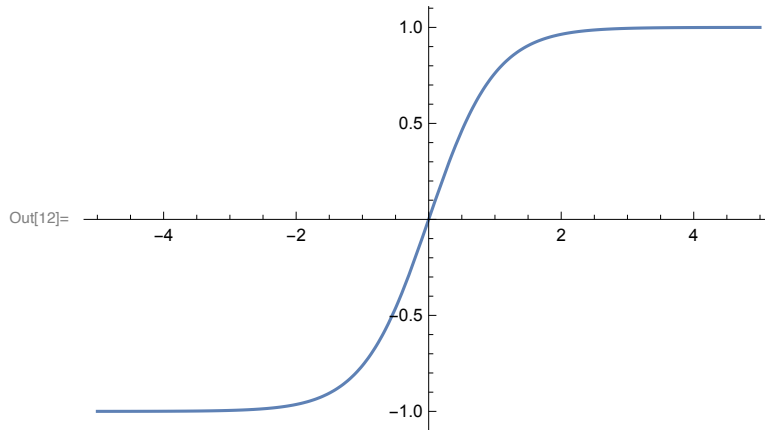
Try dividing the denominator by $\text{Cosh}[\theta_1]\text{Cosh}[\theta_2]$.

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In[11]:= 
$$\frac{\text{denominator}}{\text{Cosh}[\theta_1] \text{Cosh}[\theta_2]} \quad // \text{ Simplify}$$

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Out[11]:=  $\text{Tanh}[\theta_2] + \text{Tanh}[\theta_1] (-\lambda + \alpha \text{Tanh}[\theta_2])$ 
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In[12]:= Plot[Tanh[θ2], {θ2, -5, 5}]
```



Now, by inspection, both the θ_1 and θ_2 functions in the denominator are well-behaved. What about the numerator?

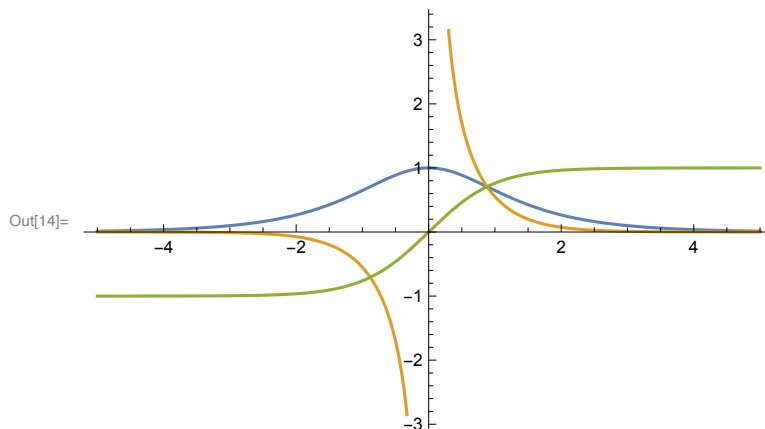
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In[13]:= 
$$\frac{\text{numerator}}{\text{Cosh}[\theta_1] \text{Cosh}[\theta_2]} \quad // \text{ Simplify}$$

```

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Out[13]:=  $-\lambda + 2\lambda \text{Sech}[\theta_1] \text{Sech}[\theta_2] - \lambda^2 \text{Csch}[\theta_2] \text{Sech}[\theta_2] \text{Tanh}[\theta_1] + \alpha \text{Tanh}[\theta_2] + \text{Tanh}[\theta_1] \text{Tanh}[\theta_2]$ 
```

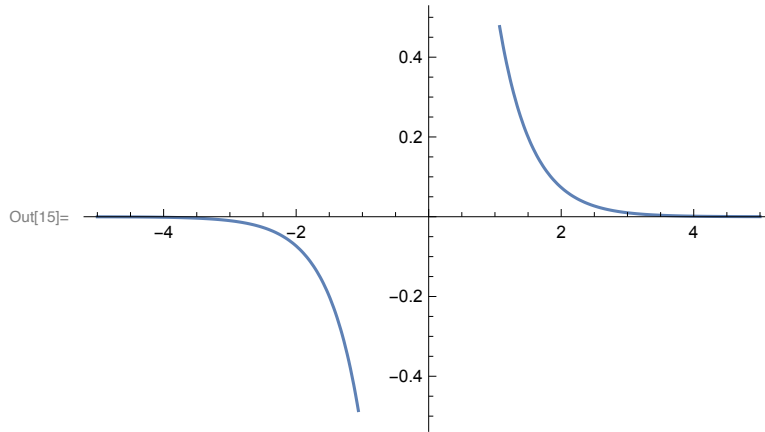
All three θ_2 functions in the numerator are ok at large θ_2 . But one of them diverges at small θ_2 , which is a problem.

```
In[14]:= Plot[{Sech[θ2], Csch[θ2] Sech[θ2], Tanh[θ2]}, {θ2, -5, 5}]
```



This function is the culprit.

In[15]:= **Plot**[**Csch**[θ_2] **Sech**[θ_2], { θ_2 , -5, 5}]



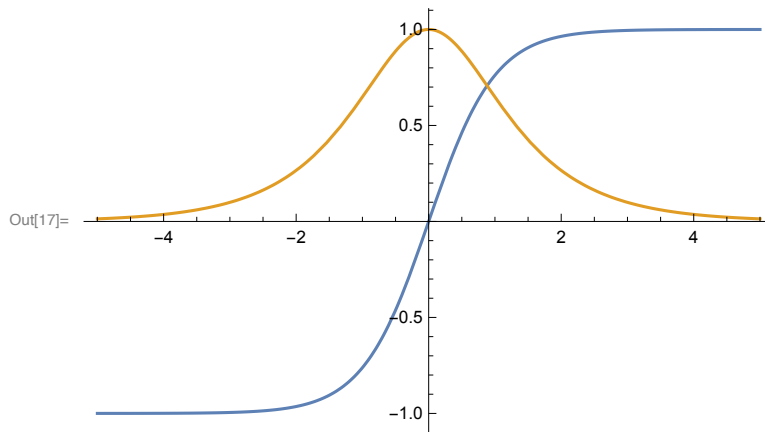
The argument $\theta_2 = \eta h_s$ in the paper, and η does not go to zero. So, in practice, I think the function **Csch**[θ_2] **Sech**[θ_2] should be ok.

In summary, write the big fraction θ_i as follows

$$\text{In[16]:= } \frac{-\lambda + 2 \lambda \text{Sech}[\theta_1] \text{Sech}[\theta_2] - \lambda^2 \text{Csch}[\theta_2] \text{Sech}[\theta_2] \text{Tanh}[\theta_1] + \alpha \text{Tanh}[\theta_2] + \text{Tanh}[\theta_1] \text{Tanh}[\theta_2]}{\text{Tanh}[\theta_2] + \text{Tanh}[\theta_1] (-\lambda + \alpha \text{Tanh}[\theta_2])}$$

;

In[17]:= **Plot**[{**Tanh**[θ_2], **Sech**[θ_2]}, { θ_2 , -5, 5}]



Python does not have **Sech** and **Csch** function. However, note that

In[18]:= **1 / Sech**[θ_1]

Out[18]= **Cosh**[θ_1]

In[19]:= **1 / Csch**[θ_2]

Out[19]= **Sinh**[θ_2]

So let us implement θ_i as (we are leaving out the $\epsilon_s/\epsilon_{\text{eff}}$ prefactor here)

```
In[20]:=  $\theta I = -\lambda \operatorname{Coth}[\theta 2] + \left( \operatorname{Tanh}[\theta 1] \operatorname{Tanh}[\theta 2] + \alpha \operatorname{Tanh}[\theta 2] - \lambda + 2 \lambda / (\operatorname{Cosh}[\theta 1] \operatorname{Cosh}[\theta 2]) - \right.$   

 $\left. \lambda^2 \operatorname{Tanh}[\theta 1] / (\operatorname{Cosh}[\theta 2] \operatorname{Sinh}[\theta 2]) \right) / (\operatorname{Tanh}[\theta 2] + \operatorname{Tanh}[\theta 1] (-\lambda + \alpha \operatorname{Tanh}[\theta 2])) ;$ 
```

Expand the denominator

```
In[26]:=  $\operatorname{Tanh}[\theta 2] + \operatorname{Tanh}[\theta 1] (-\lambda + \alpha \operatorname{Tanh}[\theta 2])$  // Expand
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```
Out[26]=  $-\lambda \operatorname{Tanh}[\theta 1] + \operatorname{Tanh}[\theta 2] + \alpha \operatorname{Tanh}[\theta 1] \operatorname{Tanh}[\theta 2]$ 
```

Check that this expression agrees with what we started with.

```
In[22]:=  $\theta I - \left( -\lambda \operatorname{Coth}[\theta 2] + \frac{\text{numerator}}{\text{denominator}} \right)$  // FullSimplify
```

```
Out[22]= 0
```

This form is well behaved because $\theta 2$ does not go to zero. Check the large h_s limit, where $\theta 1$ and $\theta 2$ go to infinity.

```
In[24]:= Limit[ $\theta I$ , { $\theta 1 \rightarrow \text{Infinity}$ ,  $\theta 2 \rightarrow \text{Infinity}$ }]
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Out[24]=  $1 - \lambda$ 
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