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John A. Marohn (jam99@cornell.edu)

Continued from **2024-06-03--jam99--Loring-LDS-signal.nb**.

I have derived a new expression, using pencil and paper, for *ϕ*$rxn that accounts for the amplitude modulation applied in the amplitude-modulated BLDS experiment that we perform in the lab. Use this new expression to look at the BLDS signal at dc, the AM modulation frequency *ω*$am, and 2 *ω*$m.

Define the average interaction energy. Here V$ts is the tip-sample voltage, c is the tip capacitance, and *ϕ* is the reaction potential.

```
In[46]:= Clear[W, V$ts, c, ϕ];
         W[z_, t_] = V$ts[t] × c[0, z] ϕ[z, t]
Out[47]= c[0, z] × V$ts[t] ϕ[z, t]
```

Here *c*[*n*, *z*] represents the nth derivative of the tip-sample capacitance *c*[*0*, *z*]. Every time you take a derivative, you increase *n*.

```
In[48]:= Derivative[0, 1][c][n_, z_] := c[n + 1, z]
```

As a check, take the second derivative of the interaction energy. Observe that I get the expected paper-and-pencil answer.

```
In[49]:= D[W[z, t], {z, 2}] // Simplify
```

Out[49]= $V\$ts[t] \left(c[2, z]\, \phi[z, t] + 2\, c[1, z]\, \phi^{(1,0)}[z, t] + c[0, z]\, \phi^{(2,0)}[z, t]\right)$

Define the reaction field . Here *R*[*k, n, z, ω*] is the frequency-domain response function. The first input *k* indicates the real or imaginary part, with *k* = 0 the real part and *k*=1 the imaginary part. The second input *n* indicates the *n*th derivative. The third and fourth inputs are height and modulation frequency, respectively.

I have changed the sign in front of the Sin[] term to agree with Loring and today's pencil-and-paper calculation.
This change has no effect on the dc or 2 *ω*$m terms.

```
In[50]:= Clear[ϕ$rxn, R, ω$m];
ϕ$rxn[z_, t_] = V$0 c[0, z] (1/2 R[0, 0, z, ω$m] Cos[ω$m t] + 1/2 R[1, 0, z, ω$m] Sin[ω$m t]
      + 1/4 R[0, 0, z, ω$m + ω$am] Cos[(ω$m + ω$am) t] +
      1/4 R[1, 0, z, ω$m + ω$am] Sin[(ω$m + ω$am) t]
      + 1/4 R[0, 0, z, ω$m - ω$am] Cos[(ω$m - ω$am) t] +
      1/4 R[1, 0, z, ω$m - ω$am] Sin[(ω$m - ω$am) t]);
```

Every time you take a derivative of *R* with respect to *z*, increase *n* by one.

```
In[52]:= Derivative[0, 0, 1, 0][R][k_, n_, z_, ω_] := R[k, n + 1, z, ω]
```

Check that we get the expected behavior.

```
In[53]:= D[ϕ$rxn[z, t], z]

Out[53]= V$0 c[1, z] (1/2 Cos[t ω$m] R[0, 0, z, ω$m] + 1/4 Cos[t (-ω$am + ω$m)] R[0, 0, z, -ω$am + ω$m] +
      1/4 Cos[t (ω$am + ω$m)] R[0, 0, z, ω$am + ω$m] + 1/2 R[1, 0, z, ω$m] Sin[t ω$m] +
      1/4 R[1, 0, z, -ω$am + ω$m] Sin[t (-ω$am + ω$m)] +
      1/4 R[1, 0, z, ω$am + ω$m] Sin[t (ω$am + ω$m)]) + V$0 c[0, z]
    (1/2 Cos[t ω$m] R[0, 1, z, ω$m] + 1/4 Cos[t (-ω$am + ω$m)] R[0, 1, z, -ω$am + ω$m] +
      1/4 Cos[t (ω$am + ω$m)] R[0, 1, z, ω$am + ω$m] + 1/2 R[1, 1, z, ω$m] Sin[t ω$m] +
      1/4 R[1, 1, z, -ω$am + ω$m] Sin[t (-ω$am + ω$m)] +
      1/4 R[1, 1, z, ω$am + ω$m] Sin[t (ω$am + ω$m)])
```

Substitute the reaction potential and the oscillating tip-sample voltage into the interaction energy.

In[54]:= `W$new[z_, t_] = W[z, t] /. {ϕ[z, t] → ϕ$rxn[z, t], V$ts[t] → V$0 Cos[ω$m t]}`

Out[54]=
$$V\$0^2\, c[0,z]^2 \,\mathrm{Cos}[t\,\omega\$m] \left(\frac{1}{2}\mathrm{Cos}[t\,\omega\$m]\,R[0,0,z,\omega\$m] + \frac{1}{4}\mathrm{Cos}[t\,(-\omega\$am+\omega\$m)]\,R[0,0,z,-\omega\$am+\omega\$m] + \frac{1}{4}\mathrm{Cos}[t\,(\omega\$am+\omega\$m)]\,R[0,0,z,\omega\$am+\omega\$m] + \frac{1}{2}R[1,0,z,\omega\$m]\,\mathrm{Sin}[t\,\omega\$m] + \frac{1}{4}R[1,0,z,-\omega\$am+\omega\$m]\,\mathrm{Sin}[t\,(-\omega\$am+\omega\$m)] + \frac{1}{4}R[1,0,z,\omega\$am+\omega\$m]\,\mathrm{Sin}[t\,(\omega\$am+\omega\$m)]\right)$$

In[55]:= $\Delta\omega = \frac{w\$c}{2\,k\$c}$ `D[W$new[z, t], {z, 2}] // TrigExpand`

Out[55]=
$$\frac{V\$0^2\,w\$c\,c[1,z]^2\,R[0,0,z,\omega\$m]}{4\,k\$c} + \frac{V\$0^2\,w\$c\,c[0,z]\times c[2,z]\times R[0,0,z,\omega\$m]}{4\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[1,z]^2\,\mathrm{Cos}[t\,\omega\$m]^2\,R[0,0,z,\omega\$m]}{4\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[0,z]\times c[2,z]\,\mathrm{Cos}[t\,\omega\$m]^2\,R[0,0,z,\omega\$m]}{4\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[1,z]^2\,\mathrm{Cos}[t\,\omega\$m]\,\mathrm{Cos}[t\,(-\omega\$am+\omega\$m)]\,R[0,0,z,-\omega\$am+\omega\$m]}{4\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[0,z]\times c[2,z]\,\mathrm{Cos}[t\,\omega\$m]\,\mathrm{Cos}[t\,(-\omega\$am+\omega\$m)]\,R[0,0,z,-\omega\$am+\omega\$m]}{4\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[1,z]^2\,\mathrm{Cos}[t\,\omega\$m]\,\mathrm{Cos}[t\,(\omega\$am+\omega\$m)]\,R[0,0,z,\omega\$am+\omega\$m]}{4\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[0,z]\times c[2,z]\,\mathrm{Cos}[t\,\omega\$m]\,\mathrm{Cos}[t\,(\omega\$am+\omega\$m)]\,R[0,0,z,\omega\$am+\omega\$m]}{4\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[0,z]\times c[1,z]\times R[0,1,z,\omega\$m]}{2\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[0,z]\times c[1,z]\,\mathrm{Cos}[t\,\omega\$m]^2\,R[0,1,z,\omega\$m]}{2\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[0,z]\times c[1,z]\,\mathrm{Cos}[t\,\omega\$m]\,\mathrm{Cos}[t\,(-\omega\$am+\omega\$m)]\,R[0,1,z,-\omega\$am+\omega\$m]}{2\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[0,z]\times c[1,z]\,\mathrm{Cos}[t\,\omega\$m]\,\mathrm{Cos}[t\,(\omega\$am+\omega\$m)]\,R[0,1,z,\omega\$am+\omega\$m]}{2\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[0,z]^2\,R[0,2,z,\omega\$m]}{8\,k\$c} + \frac{V\$0^2\,w\$c\,c[0,z]^2\,\mathrm{Cos}[t\,\omega\$m]^2\,R[0,2,z,\omega\$m]}{8\,k\$c} +$$
$$\frac{V\$0^2\,w\$c\,c[0,z]^2\,\mathrm{Cos}[t\,\omega\$m]\,\mathrm{Cos}[t\,(-\omega\$am+\omega\$m)]\,R[0,2,z,-\omega\$am+\omega\$m]}{8\,k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z]^2\, \mathrm{Cos}[t\, \omega\$m]\, \mathrm{Cos}[t\, (\omega\$am + \omega\$m)]\, R[0,\, 2,\, z,\, \omega\$am + \omega\$m]}{8\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[1,\, z]^2\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 0,\, z,\, \omega\$m]\, \mathrm{Sin}[t\, \omega\$m]}{2\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z] \times c[2,\, z]\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 0,\, z,\, \omega\$m]\, \mathrm{Sin}[t\, \omega\$m]}{2\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z] \times c[1,\, z]\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 1,\, z,\, \omega\$m]\, \mathrm{Sin}[t\, \omega\$m]}{k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z]^2\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 2,\, z,\, \omega\$m]\, \mathrm{Sin}[t\, \omega\$m]}{4\, k\$c} -$$

$$\frac{V\$0^2\, w\$c\, c[1,\, z]^2\, R[0,\, 0,\, z,\, \omega\$m]\, \mathrm{Sin}[t\, \omega\$m]^2}{4\, k\$c} -$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z] \times c[2,\, z] \times R[0,\, 0,\, z,\, \omega\$m]\, \mathrm{Sin}[t\, \omega\$m]^2}{4\, k\$c} -$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z] \times c[1,\, z] \times R[0,\, 1,\, z,\, \omega\$m]\, \mathrm{Sin}[t\, \omega\$m]^2}{2\, k\$c} -$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z]^2\, R[0,\, 2,\, z,\, \omega\$m]\, \mathrm{Sin}[t\, \omega\$m]^2}{8\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[1,\, z]^2\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 0,\, z,\, -\omega\$am + \omega\$m]\, \mathrm{Sin}[t\, (-\omega\$am + \omega\$m)]}{4\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z] \times c[2,\, z]\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 0,\, z,\, -\omega\$am + \omega\$m]\, \mathrm{Sin}[t\, (-\omega\$am + \omega\$m)]}{4\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z] \times c[1,\, z]\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 1,\, z,\, -\omega\$am + \omega\$m]\, \mathrm{Sin}[t\, (-\omega\$am + \omega\$m)]}{2\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z]^2\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 2,\, z,\, -\omega\$am + \omega\$m]\, \mathrm{Sin}[t\, (-\omega\$am + \omega\$m)]}{8\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[1,\, z]^2\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 0,\, z,\, \omega\$am + \omega\$m]\, \mathrm{Sin}[t\, (\omega\$am + \omega\$m)]}{4\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z] \times c[2,\, z]\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 0,\, z,\, \omega\$am + \omega\$m]\, \mathrm{Sin}[t\, (\omega\$am + \omega\$m)]}{4\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z] \times c[1,\, z]\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 1,\, z,\, \omega\$am + \omega\$m]\, \mathrm{Sin}[t\, (\omega\$am + \omega\$m)]}{2\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z]^2\, \mathrm{Cos}[t\, \omega\$m]\, R[1,\, 2,\, z,\, \omega\$am + \omega\$m]\, \mathrm{Sin}[t\, (\omega\$am + \omega\$m)]}{8\, k\$c}$$

To get the dc frequency shift, average over one oscillation cycle. Define a function to average over an oscillation cycle.

In[56]:=
```
cycleAverage[f_] := ω$m/(2 π) Integrate[f, {t, 0, 2 π/ω$m}]
```

Check that averaging 1 over an oscillating cycle returns 1 and check that averaging cosine squared gives 1/2. These checks show that I have the prefactor correct in the above function definition.

In[57]:=
```
{cycleAverage[1], cycleAverage[Cos[ω$m t]^2]}
```

Out[57]= $\left\{1, \frac{1}{2}\right\}$

Because of the am modulation, the Δw$avg result is a mess, so do not compute it.

Obtain the BLDS signal by mimicing lock-in detection at a reference frequency of *ω*$am. Multiply the frequency shift by Cos[*ω*$am *t*] and Sin[*ω*$am *t*] before performing the cycle average.

```
BLDS$X = cycleAverage[Δω Cos[ω$am t]]
```

Out[58]=

$$\frac{1}{2\pi}\omega\$m\left(\frac{V\$0^2\, w\$c\, \omega\$m\, c[1,z]^2\, R[1,0,z,\omega\$m]\, \mathrm{Sin}\left[\frac{\pi\,\omega\$am}{\omega\$m}\right]^2}{-k\$c\,\omega\$am^2+4\,k\$c\,\omega\$m^2}+\right.$$

$$\frac{V\$0^2\, w\$c\, \omega\$m\, c[0,z]\times c[2,z]\times R[1,0,z,\omega\$m]\, \mathrm{Sin}\left[\frac{\pi\,\omega\$am}{\omega\$m}\right]^2}{-k\$c\,\omega\$am^2+4\,k\$c\,\omega\$m^2}-$$

$$\frac{2\,V\$0^2\, w\$c\, \omega\$m\, c[0,z]\times c[1,z]\times R[1,1,z,\omega\$m]\, \mathrm{Sin}\left[\frac{\pi\,\omega\$am}{\omega\$m}\right]^2}{k\$c\,\omega\$am^2-4\,k\$c\,\omega\$m^2}-$$

$$\frac{V\$0^2\, w\$c\, \omega\$m\, c[0,z]^2\, R[1,2,z,\omega\$m]\, \mathrm{Sin}\left[\frac{\pi\,\omega\$am}{\omega\$m}\right]^2}{2\,k\$c\,\omega\$am^2-8\,k\$c\,\omega\$m^2}+$$

$$\frac{V\$0^2\, w\$c\, c[1,z]^2\, R[0,0,z,\omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{4\,k\$c\,\omega\$am}+$$

$$\frac{V\$0^2\, w\$c\, \omega\$m^2\, c[1,z]^2\, R[0,0,z,\omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{2\,k\$c\left(\omega\$am^3-4\,\omega\$am\,\omega\$m^2\right)}+$$

$$\frac{V\$0^2\, w\$c\left(\omega\$am^2-2\,\omega\$m^2\right) c[1,z]^2\, R[0,0,z,\omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{4\,k\$c\left(\omega\$am^3-4\,\omega\$am\,\omega\$m^2\right)}+$$

$$\frac{V\$0^2\, w\$c\, c[0,z]\times c[2,z]\times R[0,0,z,\omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{4\,k\$c\,\omega\$am}+$$

$$\frac{V\$0^2\, w\$c\, \omega\$m^2\, c[0,z]\times c[2,z]\times R[0,0,z,\omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{2\,k\$c\left(\omega\$am^3-4\,\omega\$am\,\omega\$m^2\right)}+$$

$$\frac{V\$0^2\, w\$c\, \left(\omega\$am^2 - 2\,\omega\$m^2\right)\, c[0,\, z] \times c[2,\, z] \times R[0,\, 0,\, z,\, \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{4\, k\$c\, \left(\omega\$am^3 - 4\,\omega\$am\,\omega\$m^2\right)} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z] \times c[1,\, z] \times R[0,\, 1,\, z,\, \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{2\, k\$c\, \omega\$am} +$$

$$\frac{V\$0^2\, w\$c\, \omega\$m^2\, c[0,\, z] \times c[1,\, z] \times R[0,\, 1,\, z,\, \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{k\$c\, \left(\omega\$am^3 - 4\,\omega\$am\,\omega\$m^2\right)} +$$

$$\frac{V\$0^2\, w\$c\, \left(\omega\$am^2 - 2\,\omega\$m^2\right)\, c[0,\, z] \times c[1,\, z] \times R[0,\, 1,\, z,\, \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{2\, k\$c\, \left(\omega\$am^3 - 4\,\omega\$am\,\omega\$m^2\right)} +$$

$$\frac{V\$0^2\, w\$c\, c[0,\, z]^2\, R[0,\, 2,\, z,\, \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{8\, k\$c\, \omega\$am} +$$

$$\frac{V\$0^2\, w\$c\, \omega\$m^2\, c[0,\, z]^2\, R[0,\, 2,\, z,\, \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{4\, k\$c\, \left(\omega\$am^3 - 4\,\omega\$am\,\omega\$m^2\right)} +$$

$$\frac{V\$0^2\, w\$c\, \left(\omega\$am^2 - 2\,\omega\$m^2\right)\, c[0,\, z]^2\, R[0,\, 2,\, z,\, \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]}{8\, k\$c\, \left(\omega\$am^3 - 4\,\omega\$am\,\omega\$m^2\right)} -$$

$$\frac{V\$0^2\, w\$c\, (2\,\omega\$am - \omega\$m)\, c[1,\, z]^2\, R[1,\, 0,\, z,\, -\omega\$am + \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]^2}{16\, k\$c\, \omega\$am\, (\omega\$am - \omega\$m)} -$$

$$\frac{V\$0^2\, w\$c\, (2\,\omega\$am - \omega\$m)\, c[0,\, z] \times c[2,\, z] \times R[1,\, 0,\, z,\, -\omega\$am + \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]^2}{16\, k\$c\, \omega\$am\, (\omega\$am - \omega\$m)} +$$

$$\frac{V\$0^2\, w\$c\, (2\,\omega\$am + \omega\$m)\, c[1,\, z]^2\, R[1,\, 0,\, z,\, \omega\$am + \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]^2}{16\, k\$c\, \omega\$am\, (\omega\$am + \omega\$m)} +$$

$$\frac{V\$0^2\, w\$c\, (2\,\omega\$am + \omega\$m)\, c[0,\, z] \times c[2,\, z] \times R[1,\, 0,\, z,\, \omega\$am + \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]^2}{16\, k\$c\, \omega\$am\, (\omega\$am + \omega\$m)} -$$

$$\frac{V\$0^2\, w\$c\, (2\,\omega\$am - \omega\$m)\, c[0,\, z] \times c[1,\, z] \times R[1,\, 1,\, z,\, -\omega\$am + \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]^2}{8\, k\$c\, \omega\$am\, (\omega\$am - \omega\$m)} +$$

$$\frac{V\$0^2\, w\$c\, (2\,\omega\$am + \omega\$m)\, c[0,\, z] \times c[1,\, z] \times R[1,\, 1,\, z,\, \omega\$am + \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]^2}{8\, k\$c\, \omega\$am\, (\omega\$am + \omega\$m)} -$$

$$\frac{V\$0^2\, w\$c\, (2\,\omega\$am - \omega\$m)\, c[0,\, z]^2\, R[1,\, 2,\, z,\, -\omega\$am + \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]^2}{32\, k\$c\, \omega\$am\, (\omega\$am - \omega\$m)} +$$

$$\frac{V\$0^2\, w\$c\, (2\,\omega\$am + \omega\$m)\, c[0,\, z]^2\, R[1,\, 2,\, z,\, \omega\$am + \omega\$m]\, \mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]^2}{32\, k\$c\, \omega\$am\, (\omega\$am + \omega\$m)} +$$

$$\frac{V\$0^2\, w\$c\, c[1, z]^2\, R[0, 0, z, -\omega\$am + \omega\$m] \left(\frac{4\pi}{\omega\$m} + \frac{(-2\,\omega\$am+\omega\$m)\ \mathrm{Sin}\left[\frac{4\pi\,\omega\$am}{\omega\$m}\right]}{\omega\$am\,(-\omega\$am+\omega\$m)}\right)}{32\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0, z] \times c[2, z] \times R[0, 0, z, -\omega\$am + \omega\$m] \left(\frac{4\pi}{\omega\$m} + \frac{(-2\,\omega\$am+\omega\$m)\ \mathrm{Sin}\left[\frac{4\pi\,\omega\$am}{\omega\$m}\right]}{\omega\$am\,(-\omega\$am+\omega\$m)}\right)}{32\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0, z] \times c[1, z] \times R[0, 1, z, -\omega\$am + \omega\$m] \left(\frac{4\pi}{\omega\$m} + \frac{(-2\,\omega\$am+\omega\$m)\ \mathrm{Sin}\left[\frac{4\pi\,\omega\$am}{\omega\$m}\right]}{\omega\$am\,(-\omega\$am+\omega\$m)}\right)}{16\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0, z]^2\, R[0, 2, z, -\omega\$am + \omega\$m] \left(\frac{4\pi}{\omega\$m} + \frac{(-2\,\omega\$am+\omega\$m)\ \mathrm{Sin}\left[\frac{4\pi\,\omega\$am}{\omega\$m}\right]}{\omega\$am\,(-\omega\$am+\omega\$m)}\right)}{64\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[1, z]^2\, R[0, 0, z, \omega\$am + \omega\$m] \left(\frac{4\pi}{\omega\$m} + \frac{(2\,\omega\$am+\omega\$m)\ \mathrm{Sin}\left[\frac{4\pi\,\omega\$am}{\omega\$m}\right]}{\omega\$am\,(\omega\$am+\omega\$m)}\right)}{32\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0, z] \times c[2, z] \times R[0, 0, z, \omega\$am + \omega\$m] \left(\frac{4\pi}{\omega\$m} + \frac{(2\,\omega\$am+\omega\$m)\ \mathrm{Sin}\left[\frac{4\pi\,\omega\$am}{\omega\$m}\right]}{\omega\$am\,(\omega\$am+\omega\$m)}\right)}{32\, k\$c} +$$

$$\frac{V\$0^2\, w\$c\, c[0, z] \times c[1, z] \times R[0, 1, z, \omega\$am + \omega\$m] \left(\frac{4\pi}{\omega\$m} + \frac{(2\,\omega\$am+\omega\$m)\ \mathrm{Sin}\left[\frac{4\pi\,\omega\$am}{\omega\$m}\right]}{\omega\$am\,(\omega\$am+\omega\$m)}\right)}{16\, k\$c} +$$

$$\left.\frac{V\$0^2\, w\$c\, c[0, z]^2\, R[0, 2, z, \omega\$am + \omega\$m] \left(\frac{4\pi}{\omega\$m} + \frac{(2\,\omega\$am+\omega\$m)\ \mathrm{Sin}\left[\frac{4\pi\,\omega\$am}{\omega\$m}\right]}{\omega\$am\,(\omega\$am+\omega\$m)}\right)}{64\, k\$c}\right)$$

$$\mathbf{BLDS\$Y = factorOut}\left[\frac{V\$0^2\, w\$c}{8\, k\$c}\right]\mathbf{[cycleAverage[\Delta\omega\ Sin[\omega\$am\ t]]]}$$

Out[59]=

$$\frac{1}{128\, k\$c\, \pi} V\$0^2\, w\$c\, \omega\$m$$

$$\left(2\, c[1, z]^2 \left(\frac{1}{\omega\$am^5 - 5\,\omega\$am^3\,\omega\$m^2 + 4\,\omega\$am\,\omega\$m^4}\left(32\,\left(\omega\$am^4 - 3\,\omega\$am^2\,\omega\$m^2 + 2\,\omega\$m^4\right) R[0, 0, z,\right.\right.\right.$$

$$\omega\$m]\ \mathrm{Sin}\left[\frac{\pi\,\omega\$am}{\omega\$m}\right]^2 + 2\,\mathrm{Sin}\left[\frac{2\pi\,\omega\$am}{\omega\$m}\right]\ \left(8\,\omega\$am\,\omega\$m\,\left(\omega\$am^2 - \omega\$m^2\right) R[1, 0, z, \omega\$m] +\right.$$

$$\left(\omega\$am^2 - 4\,\omega\$m^2\right)\ \left(\left(2\,\omega\$am^2 + \omega\$am\,\omega\$m - \omega\$m^2\right) R[0, 0, z, -\omega\$am + \omega\$m] +\right.$$

$$\left(2\,\omega\$\mathrm{am}^2 - \omega\$\mathrm{am}\,\omega\$\mathrm{m} - \omega\$\mathrm{m}^2\right)\,\mathrm{R}[0,0,z,\omega\$\mathrm{am}+\omega\$\mathrm{m}]\Big)\,\mathrm{Sin}\left[\frac{2\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\Big)\Big) -$$

$$\frac{\mathrm{R}[1,0,z,-\omega\$\mathrm{am}+\omega\$\mathrm{m}]\,\left(4\,\pi\,\omega\$\mathrm{am}\,(\omega\$\mathrm{am}-\omega\$\mathrm{m}) + \omega\$\mathrm{m}\,(-2\,\omega\$\mathrm{am}+\omega\$\mathrm{m})\,\mathrm{Sin}\left[\frac{4\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\right)}{\omega\$\mathrm{am}\,(\omega\$\mathrm{am}-\omega\$\mathrm{m})\,\omega\$\mathrm{m}} +$$

$$\left.\frac{\mathrm{R}[1,0,z,\omega\$\mathrm{am}+\omega\$\mathrm{m}]\,\left(4\,\pi\,\omega\$\mathrm{am}\,(\omega\$\mathrm{am}+\omega\$\mathrm{m}) - \omega\$\mathrm{m}\,(2\,\omega\$\mathrm{am}+\omega\$\mathrm{m})\,\mathrm{Sin}\left[\frac{4\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\right)}{\omega\$\mathrm{am}\,\omega\$\mathrm{m}\,(\omega\$\mathrm{am}+\omega\$\mathrm{m})}\right) +$$

$$4\,\mathrm{c}[0,z]\times\mathrm{c}[1,z]\,\left(\frac{1}{\omega\$\mathrm{am}^5 - 5\,\omega\$\mathrm{am}^3\,\omega\$\mathrm{m}^2 + 4\,\omega\$\mathrm{am}\,\omega\$\mathrm{m}^4}\right.$$

$$\Big(32\,\left(\omega\$\mathrm{am}^4 - 3\,\omega\$\mathrm{am}^2\,\omega\$\mathrm{m}^2 + 2\,\omega\$\mathrm{m}^4\right)\,\mathrm{R}[0,1,z,\omega\$\mathrm{m}]\,\mathrm{Sin}\left[\frac{\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]^2 +$$

$$2\,\mathrm{Sin}\left[\frac{2\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\,\Big(8\,\omega\$\mathrm{am}\,\omega\$\mathrm{m}\,\left(\omega\$\mathrm{am}^2 - \omega\$\mathrm{m}^2\right)\,\mathrm{R}[1,1,z,\omega\$\mathrm{m}] +$$

$$\left(\omega\$\mathrm{am}^2 - 4\,\omega\$\mathrm{m}^2\right)\,\Big(\Big(\left(2\,\omega\$\mathrm{am}^2 + \omega\$\mathrm{am}\,\omega\$\mathrm{m} - \omega\$\mathrm{m}^2\right)\,\mathrm{R}[0,1,z,-\omega\$\mathrm{am}+\omega\$\mathrm{m}] +$$

$$\left(2\,\omega\$\mathrm{am}^2 - \omega\$\mathrm{am}\,\omega\$\mathrm{m} - \omega\$\mathrm{m}^2\right)\,\mathrm{R}[0,1,z,\omega\$\mathrm{am}+\omega\$\mathrm{m}]\Big)\,\mathrm{Sin}\left[\frac{2\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\Big)\Big) -$$

$$\frac{\mathrm{R}[1,1,z,-\omega\$\mathrm{am}+\omega\$\mathrm{m}]\,\left(4\,\pi\,\omega\$\mathrm{am}\,(\omega\$\mathrm{am}-\omega\$\mathrm{m}) + \omega\$\mathrm{m}\,(-2\,\omega\$\mathrm{am}+\omega\$\mathrm{m})\,\mathrm{Sin}\left[\frac{4\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\right)}{\omega\$\mathrm{am}\,(\omega\$\mathrm{am}-\omega\$\mathrm{m})\,\omega\$\mathrm{m}} +$$

$$\left.\frac{\mathrm{R}[1,1,z,\omega\$\mathrm{am}+\omega\$\mathrm{m}]\,\left(4\,\pi\,\omega\$\mathrm{am}\,(\omega\$\mathrm{am}+\omega\$\mathrm{m}) - \omega\$\mathrm{m}\,(2\,\omega\$\mathrm{am}+\omega\$\mathrm{m})\,\mathrm{Sin}\left[\frac{4\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\right)}{\omega\$\mathrm{am}\,\omega\$\mathrm{m}\,(\omega\$\mathrm{am}+\omega\$\mathrm{m})}\right) +$$

$$\mathrm{c}[0,z]\,\left(\mathrm{c}[2,z]\,\left(\frac{1}{\omega\$\mathrm{am}^5 - 5\,\omega\$\mathrm{am}^3\,\omega\$\mathrm{m}^2 + 4\,\omega\$\mathrm{am}\,\omega\$\mathrm{m}^4}\right.\right.$$

$$\Big(64\,\left(\omega\$\mathrm{am}^4 - 3\,\omega\$\mathrm{am}^2\,\omega\$\mathrm{m}^2 + 2\,\omega\$\mathrm{m}^4\right)\,\mathrm{R}[0,0,z,\omega\$\mathrm{m}]\,\mathrm{Sin}\left[\frac{\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]^2 +$$

$$4\,\mathrm{Sin}\left[\frac{2\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\,\Big(8\,\omega\$\mathrm{am}\,\omega\$\mathrm{m}\,\left(\omega\$\mathrm{am}^2 - \omega\$\mathrm{m}^2\right)\,\mathrm{R}[1,0,z,\omega\$\mathrm{m}] +$$

$$\left(\omega\$\mathrm{am}^2 - 4\,\omega\$\mathrm{m}^2\right)\,\Big(\Big(\left(2\,\omega\$\mathrm{am}^2 + \omega\$\mathrm{am}\,\omega\$\mathrm{m} - \omega\$\mathrm{m}^2\right)\,\mathrm{R}[0,0,z,-\omega\$\mathrm{am}+\omega\$\mathrm{m}] +$$

$$\left(2\,\omega\$\mathrm{am}^2 - \omega\$\mathrm{am}\,\omega\$\mathrm{m} - \omega\$\mathrm{m}^2\right)\,\mathrm{R}[0,0,z,\omega\$\mathrm{am}+\omega\$\mathrm{m}]\Big)\,\mathrm{Sin}\left[\frac{2\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\Big)\Big) -$$

$$\frac{2\,\mathrm{R}[1,0,z,-\omega\$\mathrm{am}+\omega\$\mathrm{m}]\,\left(4\,\pi\,\omega\$\mathrm{am}\,(\omega\$\mathrm{am}-\omega\$\mathrm{m}) + \omega\$\mathrm{m}\,(-2\,\omega\$\mathrm{am}+\omega\$\mathrm{m})\,\mathrm{Sin}\left[\frac{4\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\right)}{\omega\$\mathrm{am}\,(\omega\$\mathrm{am}-\omega\$\mathrm{m})\,\omega\$\mathrm{m}}$$

$$+$$

$$\left.\frac{2\,\mathrm{R}[1,0,z,\omega\$\mathrm{am}+\omega\$\mathrm{m}]\,\left(4\,\pi\,\omega\$\mathrm{am}\,(\omega\$\mathrm{am}+\omega\$\mathrm{m}) - \omega\$\mathrm{m}\,(2\,\omega\$\mathrm{am}+\omega\$\mathrm{m})\,\mathrm{Sin}\left[\frac{4\,\pi\,\omega\$\mathrm{am}}{\omega\$\mathrm{m}}\right]\right)}{\omega\$\mathrm{am}\,\omega\$\mathrm{m}\,(\omega\$\mathrm{am}+\omega\$\mathrm{m})}\right) +$$

$$c[0,z]\left(\frac{1}{\omega\$am^5-5\,\omega\$am^3\,\omega\$m^2+4\,\omega\$am\,\omega\$m^4}\left(32\left(\omega\$am^4-3\,\omega\$am^2\,\omega\$m^2+2\,\omega\$m^4\right)R[0,2,z,\omega\$m]\,\mathrm{Sin}\left[\frac{\pi\,\omega\$am}{\omega\$m}\right]^2+2\,\mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]\left(8\,\omega\$am\,\omega\$m\left(\omega\$am^2-\omega\$m^2\right)R[1,2,z,\omega\$m]+\left(\omega\$am^2-4\,\omega\$m^2\right)\left(\left(2\,\omega\$am^2+\omega\$am\,\omega\$m-\omega\$m^2\right)R[0,2,z,-\omega\$am+\omega\$m]+\left(2\,\omega\$am^2-\omega\$am\,\omega\$m-\omega\$m^2\right)R[0,2,z,\omega\$am+\omega\$m]\right)\mathrm{Sin}\left[\frac{2\,\pi\,\omega\$am}{\omega\$m}\right]\right)-\frac{R[1,2,z,-\omega\$am+\omega\$m]\left(4\,\pi\,\omega\$am\,(\omega\$am-\omega\$m)+\omega\$m\,(-2\,\omega\$am+\omega\$m)\,\mathrm{Sin}\left[\frac{4\,\pi\,\omega\$am}{\omega\$m}\right]\right)}{\omega\$am\,(\omega\$am-\omega\$m)\,\omega\$m}+\frac{R[1,2,z,\omega\$am+\omega\$m]\left(4\,\pi\,\omega\$am\,(\omega\$am+\omega\$m)-\omega\$m\,(2\,\omega\$am+\omega\$m)\,\mathrm{Sin}\left[\frac{4\,\pi\,\omega\$am}{\omega\$m}\right]\right)}{\omega\$am\,\omega\$m\,(\omega\$am+\omega\$m)}\right)\right)\right)$$

These results are a hot mess. However, we can simpliy the calculated BLDS signals using that ω$am << ω$m. Take the zero-th order Taylor series in ω$am.

In[65]:=
```
BLDS$X$limit = Collect[Series[BLDS$X , {ω$am , 0, 0}] // Normal,
   { R[0, 0, z, ω$m], R[0, 1, z, ω$m], R[0, 2, z, ω$m]}]
```

Out[65]=

$$\frac{\left(2\,V\$0^2\,w\$c\,c[1,z]^2+2\,V\$0^2\,w\$c\,c[0,z]\times c[2,z]\right)R[0,0,z,\omega\$m]}{4\,k\$c}+\frac{V\$0^2\,w\$c\,c[0,z]\times c[1,z]\times R[0,1,z,\omega\$m]}{k\$c}+\frac{V\$0^2\,w\$c\,c[0,z]^2\,R[0,2,z,\omega\$m]}{4\,k\$c}$$

In[66]:=
```
BLDS$Y$limit = Collect[Series[BLDS$Y , {ω$am , 0, 0}] // Normal,
   { R[0, 0, z, ω$m], R[0, 1, z, ω$m], R[0, 2, z, ω$m]}]
```

Out[66]= 0

We can compare the the BLDS$X signal to the Δω$avg signal computed yesterday. The Δω$avg signal is what you get for the time-averaged cantilever when there is no amplitude modulation applied.

In[67]:=

$$\Delta\omega\$avg=\frac{1}{4\,k\$c}\,V\$0^2\,w\$c\left(2\,c[1,z]^2\,R[0,0,z,\omega\$m]+4\,c[0,z]\times c[1,z]\times R[0,1,z,\omega\$m]+c[0,z]\,(2\,c[2,z]\times R[0,0,z,\omega\$m]+c[0,z]\times R[0,2,z,\omega\$m])\right);$$

Take the ratio.

In[68]:=
```
BLDS$X$limit / Δω$avg // FullSimplify
```

Out[68]= 1

In the limit that ω$am << ω$m, the BLDS signal is *exactly the same size* as the time-averaged cantilever frequency.