Write out the numerator and denominator in Lekkala2013 θ_l separately.

In[7]:=
```
numerator = Sinh[θ1] Sinh[θ2] - λ Cosh[θ1] Cosh[θ2] + 2 λ - λ^2 Sinh[θ1]/Sinh[θ2]
```

$$\text{numerator} = \text{Sinh}[\theta1]\,\text{Sinh}[\theta2] - \lambda\,\text{Cosh}[\theta1]\,\text{Cosh}[\theta2] + 2\lambda - \lambda^2 \frac{\text{Sinh}[\theta1]}{\text{Sinh}[\theta2]}$$

Out[7]= $2\lambda - \lambda\,\text{Cosh}[\theta1]\,\text{Cosh}[\theta2] - \lambda^2\,\text{Csch}[\theta2]\,\text{Sinh}[\theta1] + \text{Sinh}[\theta1]\,\text{Sinh}[\theta2]$

In[8]:=
```
denominator = Cosh[θ1] Sinh[θ2] - λ Sinh[θ1] Cosh[θ2]
```

Out[8]= $-\lambda\,\text{Cosh}[\theta2]\,\text{Sinh}[\theta1] + \text{Cosh}[\theta1]\,\text{Sinh}[\theta2]$

At large value of the arguments $\theta1$ and $\theta2$, both numerator and denominator diverge. However, the ratio seems to remain finite or goes to zero. Try to rewrite the ratio in a way that avoids the problem. Divide the numerator by Cosh[θ] to see if that tames the $\theta1$ divergence.

In[52]:=

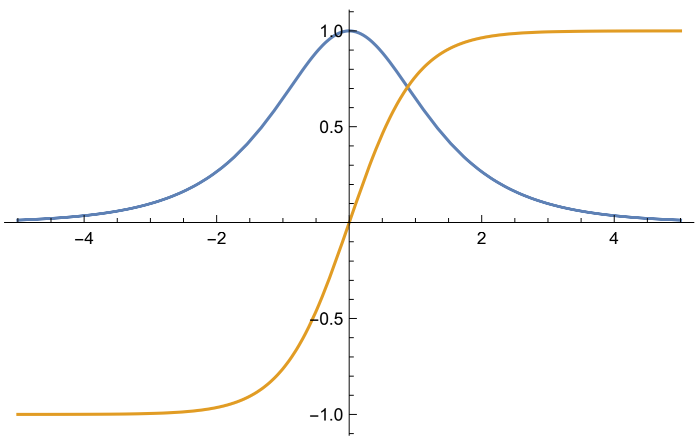
$$\frac{\text{Numerator[ans]}}{\text{Cosh}[\theta1]}\ \text{// Simplify}$$

Out[52]= $-\lambda\,\text{Coth}[\theta2] + 2\lambda\,\text{Csch}[\theta2]\,\text{Sech}[\theta1] + \text{Tanh}[\theta1] - \lambda^2\,\text{Csch}[\theta2]^2\,\text{Tanh}[\theta1]$

After the division, check that the remaining functions of $\theta1$ do not blow up .

In[54]:=
```
Plot[{Sech[θ1], Tanh[θ1]}, {θ1, -5, 5}]
```

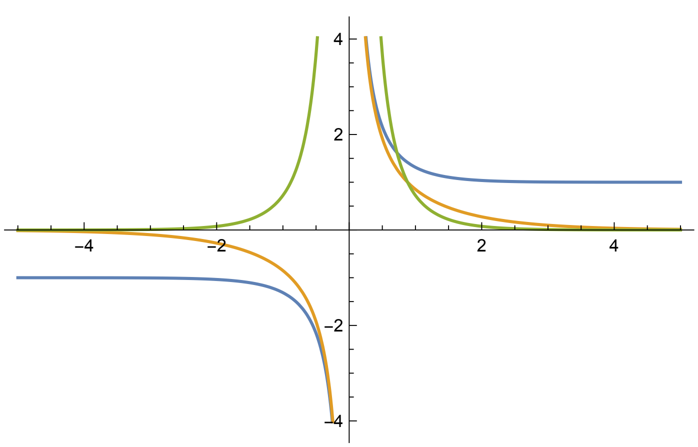
Out[54]=



After the division, the functions of $\theta2$ still blow up .

In[58]:=
```
Plot[{Coth[θ2], Csch[θ2], Csch[θ2]^2}, {θ2, -5, 5}]
```

Out[58]=



Divide the denominator by Cosh[θ] to see if that tames the θ1 divergence.

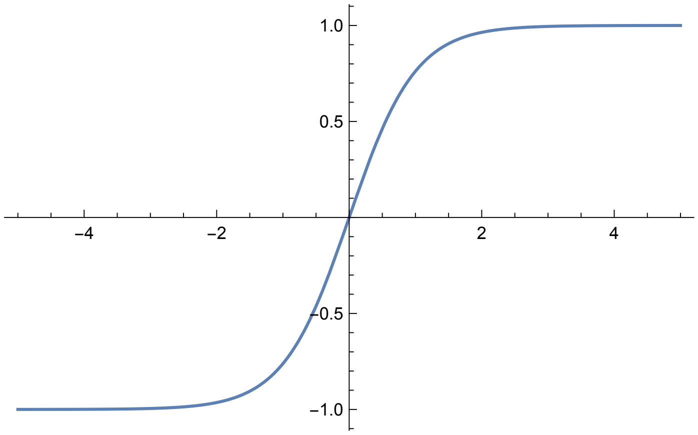
In[53]:= $\frac{\texttt{Denominator[ans]}}{\texttt{Cosh[}\theta\texttt{1]}}$ `// Simplify`

Out[53]= $1 - \lambda\,\mathrm{Coth}[\theta 2]\,\mathrm{Tanh}[\theta 1]$

It does ...

In[56]:=
```
Plot[{Tanh[θ1]}, {θ1, -5, 5}]
```

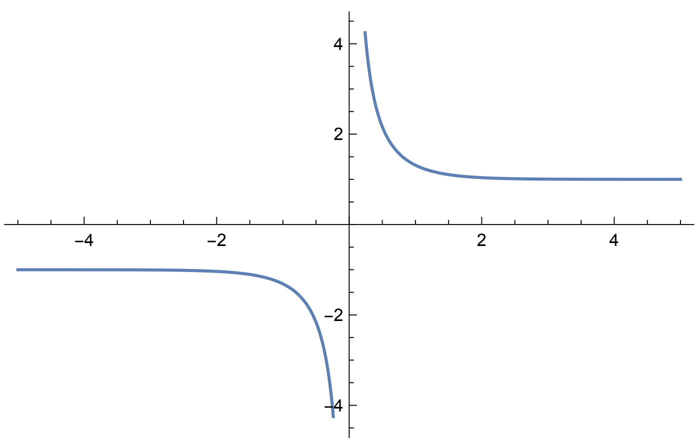
Out[56]=



... but the θ2 functions in the denominator still diverge.

In[60]:=
```
Plot[Coth[θ2], {θ2, -5, 5}]
```

Out[60]=



Try dividing the denominator by Cosh[θ1]Coth[θ2].

$\frac{\texttt{Denominator[ans]}}{\texttt{Cosh[}\theta\texttt{1] Coth[}\theta\texttt{2]}}$ `// Simplify`

Out[93]= $-\lambda\,\mathrm{Tanh}[\theta 1] + \mathrm{Tanh}[\theta 2]$

Now, by inspection, both the θ1 and θ2 functions in the denominator are well-behaved. What about the numerator?

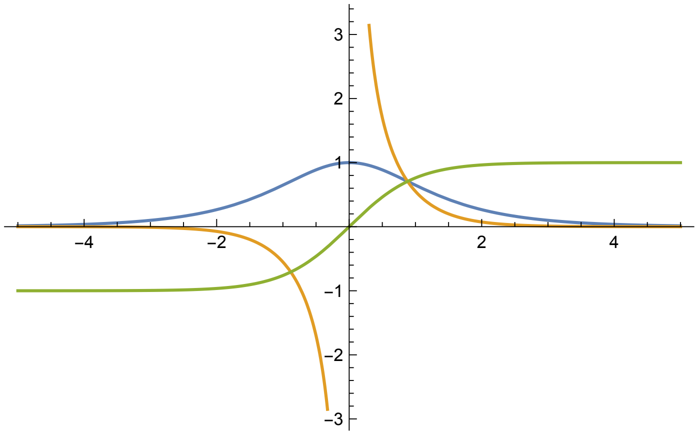
In[95]:= $\frac{\text{Numerator[ans]}}{\text{Cosh}[\theta 1]\ \text{Coth}[\theta 2]}$ // Simplify

Out[95]= $-\lambda + 2\,\lambda\,\text{Sech}[\theta 1]\ \text{Sech}[\theta 2] - \lambda^2\,\text{Csch}[\theta 2]\ \text{Sech}[\theta 2]\ \text{Tanh}[\theta 1] + \text{Tanh}[\theta 1]\ \text{Tanh}[\theta 2]$

All three $\theta 2$ functions in the numerator are ok at large $\theta 2$. But one of them diverges at small $\theta 2$, which is a problem.

In[96]:=
```
Plot[{Sech[θ2], Csch[θ2] Sech[θ2], Tanh[θ2]}, {θ2, -5, 5}]
```

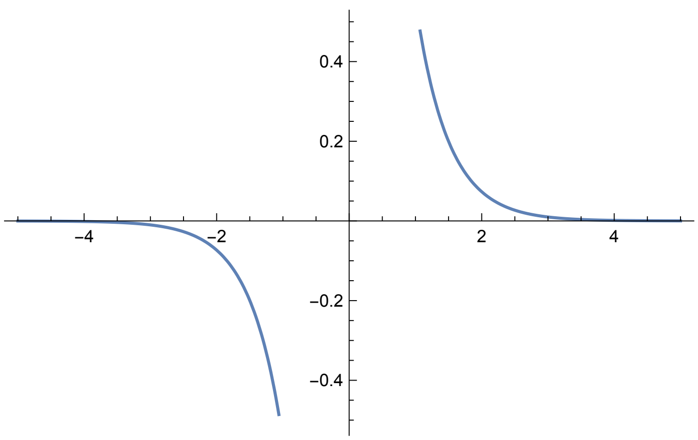
Out[96]=



This function is the culprit.

In[97]:=
```
Plot[Csch[θ2] Sech[θ2], {θ2, -5, 5}]
```

Out[97]=

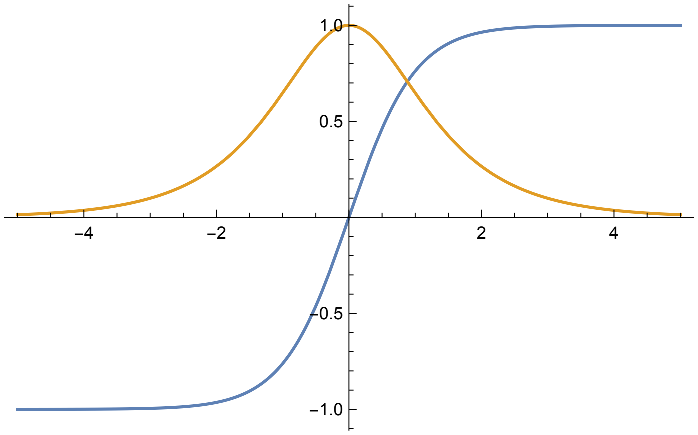


The argument $\theta 2 = \eta\, h_s$ in the paper, and η does not go to zero. So, in practice, I think the function Csch[$\theta 2$] Sech[$\theta 2$] should be ok.

In summary, write the big fraction θ_I as follows

In[105]:= $\frac{\text{Tanh}[\theta 1]\ \text{Tanh}[\theta 2] - \lambda + 2\,\lambda\,\text{Sech}[\theta 1]\ \text{Sech}[\theta 2] - \lambda^2\,\text{Tanh}[\theta 1]\ \text{Csch}[\theta 2]\ \text{Sech}[\theta 2]}{\text{Tanh}[\theta 2] - \lambda\,\text{Tanh}[\theta 1]}$;

In[107]:= `Plot[{Tanh[θ2], Sech[θ2]}, {θ2, -5, 5}]`

Out[107]=


Python does not have Sech and Csch function. However, note that

In[116]:= `1 / Sech[θ1]`

Out[116]= `Cosh[θ1]`

In[117]:= `1 / Csch[θ2]`

Out[117]= `Sinh[θ2]`

So we can implement the big fraction θ_I as

In[120]:= `θI =`

$$\frac{\mathrm{Tanh}[\theta1]\ \mathrm{Tanh}[\theta2] - \lambda + 2\lambda\ /\ (\mathrm{Cosh}[\theta1]\ \mathrm{Cosh}[\theta2]) - \lambda^2\ \mathrm{Tanh}[\theta1]\ /\ (\mathrm{Cosh}[\theta2]\ \mathrm{Sinh}[\theta2])}{\mathrm{Tanh}[\theta2] - \lambda\ \mathrm{Tanh}[\theta1]};$$

Check that this expression agrees with what we started with.

In[123]:= $\theta I - \frac{\mathrm{numerator}}{\mathrm{denominator}}$ `// FullSimplify`

Out[123]= `0`