Write out the numerator and denominator in Lekkala2013 θ_l separately.

In[6]:=
```
numerator =
 Sinh[θ1] Sinh[θ2] + α Cosh[θ1] Sinh[θ2] - λ Cosh[θ1] Cosh[θ2] + 2 λ - λ^2 Sinh[θ1]/Sinh[θ2]
```

Out[6]=
```
2 λ - λ Cosh[θ1] Cosh[θ2] - λ^2 Csch[θ2] Sinh[θ1] + α Cosh[θ1] Sinh[θ2] + Sinh[θ1] Sinh[θ2]
```

In[7]:=
```
denominator = Cosh[θ1] Sinh[θ2] + α Sinh[θ1] Sinh[θ2] - λ Sinh[θ1] Cosh[θ2]
```

Out[7]=
```
-λ Cosh[θ2] Sinh[θ1] + Cosh[θ1] Sinh[θ2] + α Sinh[θ1] Sinh[θ2]
```

Divide the denominator by Cosh[θ] to see if that tames the θ1 divergence.

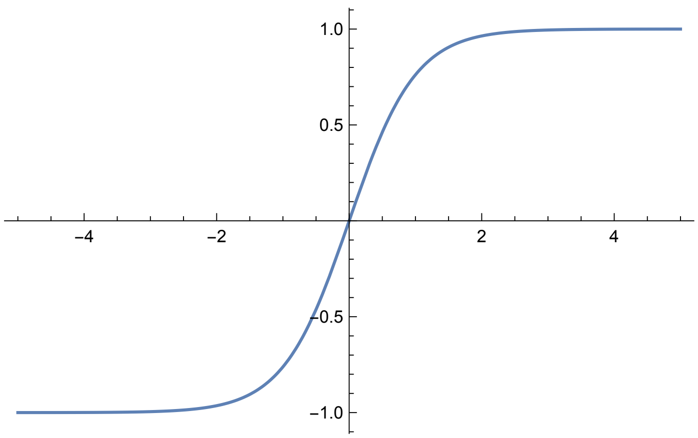
In[8]:=
```
denominator/Cosh[θ1] // Simplify
```

Out[8]=
```
Sinh[θ2] - λ Cosh[θ2] Tanh[θ1] + α Sinh[θ2] Tanh[θ1]
```

It does ...

In[9]:=
```
Plot[{Tanh[θ1]}, {θ1, -5, 5}]
```

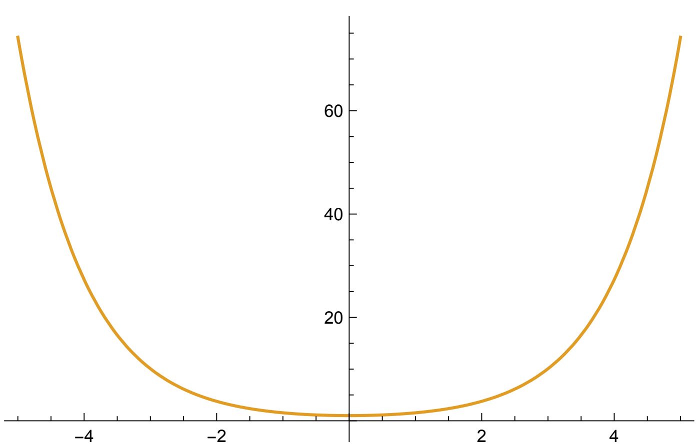
Out[9]=



... but the θ2 functions in the denominator still diverge.

In[10]:=
```
Plot[{Sinh[θ], Cosh[θ2]}, {θ2, -5, 5}]
```

Out[10]=



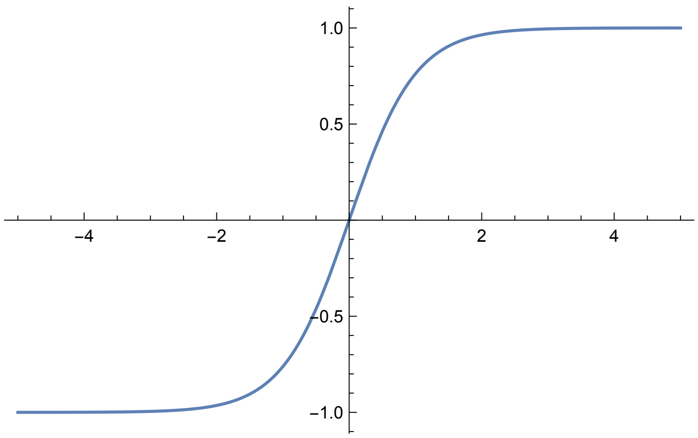
Try dividing the denominator by Cosh[θ1]Cosh[θ2].

In[11]:=
```
denominator / (Cosh[θ1] Cosh[θ2]) // Simplify
```

Out[11]=
```
Tanh[θ2] + Tanh[θ1] (-λ + α Tanh[θ2])
```

In[12]:=
```
Plot[Tanh[θ2], {θ2, -5, 5}]
```

Out[12]=



Now, by inspection, both the θ1 and θ2 functions in the denominator are well-behaved. What about the numerator?

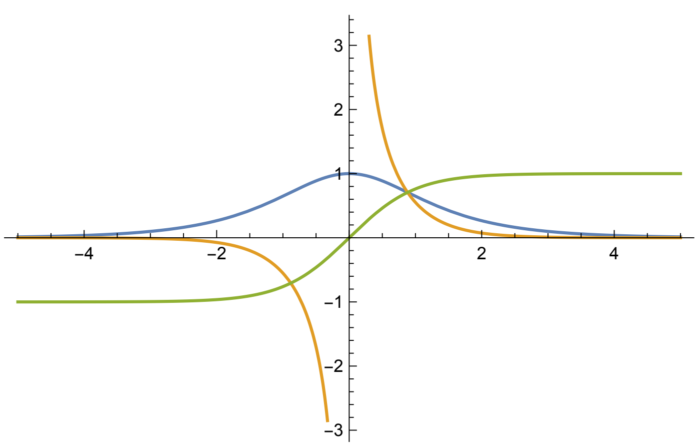
In[13]:=
```
numerator / (Cosh[θ1] Cosh[θ2]) // Simplify
```

Out[13]=
$$-\lambda + 2\lambda\,\mathrm{Sech}[\theta 1]\,\mathrm{Sech}[\theta 2] - \lambda^2\,\mathrm{Csch}[\theta 2]\,\mathrm{Sech}[\theta 2]\,\mathrm{Tanh}[\theta 1] + \alpha\,\mathrm{Tanh}[\theta 2] + \mathrm{Tanh}[\theta 1]\,\mathrm{Tanh}[\theta 2]$$

All three θ2 functions in the numerator are ok at large θ2. But one of them diverges at small θ2, which is a problem.

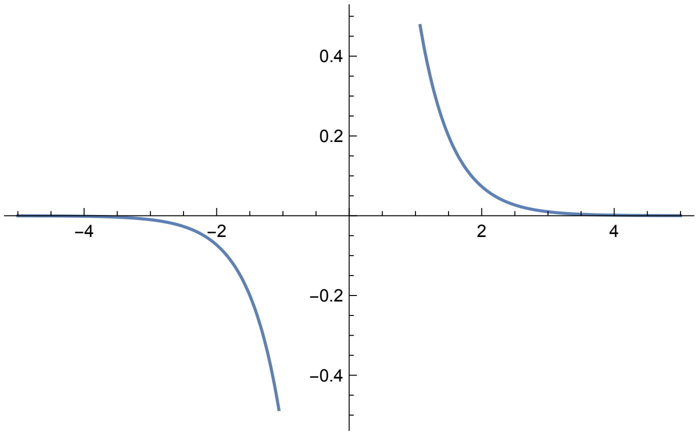
In[14]:=
```
Plot[{Sech[θ2], Csch[θ2] Sech[θ2], Tanh[θ2]}, {θ2, -5, 5}]
```

Out[14]=



This function is the culprit.

In[15]:=
```
Plot[Csch[θ2] Sech[θ2], {θ2, -5, 5}]
```

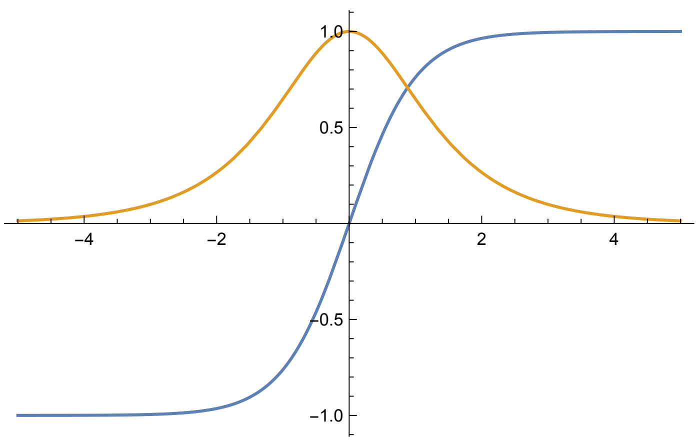
Out[15]=


The argument $\theta 2 = \eta\, h_s$ in the paper, and η does not go to zero. So, in practice, I think the function Csch[θ2] Sech[θ2] should be ok.

In summary, write the big fraction θ_l as follows

In[16]:=
$$\frac{-\lambda + 2\,\lambda\,\mathrm{Sech}[\theta 1]\,\mathrm{Sech}[\theta 2] - \lambda^2\,\mathrm{Csch}[\theta 2]\,\mathrm{Sech}[\theta 2]\,\mathrm{Tanh}[\theta 1] + \alpha\,\mathrm{Tanh}[\theta 2] + \mathrm{Tanh}[\theta 1]\,\mathrm{Tanh}[\theta 2]}{\mathrm{Tanh}[\theta 2] + \mathrm{Tanh}[\theta 1]\,(-\lambda + \alpha\,\mathrm{Tanh}[\theta 2])}$$
;

In[17]:=
```
Plot[{Tanh[θ2], Sech[θ2]}, {θ2, -5, 5}]
```

Out[17]=


Python does not have Sech and Csch function. However, note that

In[18]:=
```
1 / Sech[θ1]
```

Out[18]=
```
Cosh[θ1]
```

In[19]:=
```
1 / Csch[θ2]
```

Out[19]=
```
Sinh[θ2]
```

So let us implement θ_l as (we are leaving out the $\epsilon_s / \epsilon_{eff}$ prefactor here)

In[20]:=
```
θI = -λ Coth[θ2] + (Tanh[θ1] Tanh[θ2] + α Tanh[θ2] - λ + 2 λ / (Cosh[θ1] Cosh[θ2]) -
    λ^2 Tanh[θ1] / (Cosh[θ2] Sinh[θ2])) / (Tanh[θ2] + Tanh[θ1] (-λ + α Tanh[θ2])) ;
```

Check that this expression agrees with what we started with.

In[22]:=
$$\theta I - \left(-\lambda\,\mathrm{Coth}[\theta 2] + \frac{\mathrm{numerator}}{\mathrm{denominator}}\right) \text{ // FullSimplify}$$

Out[22]= 0

This form is well behaved because $\theta 2$ does not go to zero. Check the large h_s limit, where $\theta 1$ and $\theta 2$ go to infinity.

In[24]:=
```
Limit[θI, {θ1 → Infinity, θ2 → Infinity}]
```

Out[24]= $1 - \lambda$