National Higher School Of Mathematics

Arnold's Cat Map: A Study in Chaos and Dynamical Systems

Algebra and Coding

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Contents

1 Introduction

1.1 Background:

Arnold's Cat Map is a mathematical transformation introduced by Vladimir Igorevitch Arnold, a Russian mathematician in 1968 In his book "*Ergodic Problems of Classical Mechanics*"with A. Avez under the section "Classical dynamical systems" . This map is known for its chaotic behavior and has been widely studied in the field of dynamical systems, chaos theory, and fractals. It is particularly interesting because it demonstrates how simple mathematical operations can lead to complex and seemingly random behavior. It finds applications in cryptography, image processing, and information theory. Arnold introduced this transformation as a toral automorphism and used the Figure 1 showed in the next page ,to depict its effect which led to its popular designation as the 'Cat map'.

Before analysing the map and discussing its properties let us intoduce the concept of Dynamical systems and Chaos:

1.2 Dynamical systems:

A dynamical system is a mathematical model used to describe a system that changes over time according to a set of rules. These systems can be represented by equations that define how the state of the system evolves. They are used to model various real-world phenomena. For example:

The motion of planets in the solar system can be modeled as a dynamical system, where the positions and velocities of the planets change over time according to the laws of gravitational attraction.

In biology, the growth of a population of organisms can be described by a dynamical system, with equations representing factors like birth rate, death rate, and carrying capacity of the environment.

Discrete-time dynamical systems:

Here the state of the system evolves at distinct time points, typically at regular intervals, this contrasts with continuous time systems, where the state changes continuously over time.

1.3 Chaos:

Chaos is a phenomenon in dynamic systems where small changes in initial conditions can lead to vastly different outcomes, making long-term predictions difficult. This is observed in systems like the stock market, where minor events can lead to major, unpredictable fluctuations.

Note that there are a lot of results from the field of dynamical systems, chaos theory and number theory which we are going to use throughout this report without providing a proof as the object here is to identify the main properties of this map while getting deep is not our aim Another point is in the rest of this report we are going to use ACM insted of Arnold's cat map

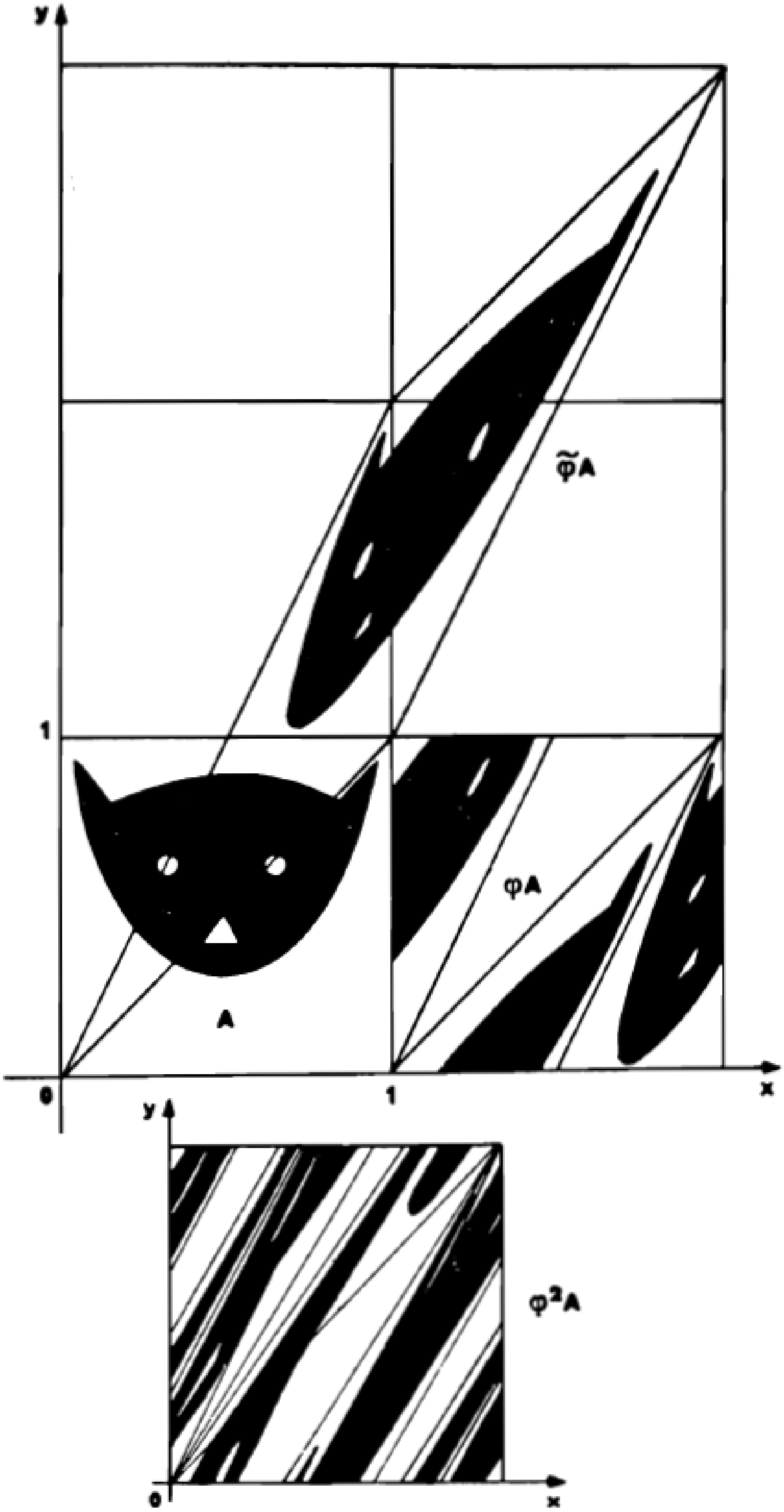


Figure 1: The figure used by Arnold from pg. 6 of V. I. Arnold and A. Avez, Ergodic Problems of Classical Mechanics

2 Main results

2.1 Preliminary:

A unit square is isomorphic to a two-dimensional torus. This means that there is a one-to-one mapping of each point on the unit square to each point on the surface of a torus. Imagine taking a sheet of paper and forming a tube out of it,this is shown in
Figure 2:
(a) Begin with unit square. (b) Identify (glue together) vertical sides. (c) Identify horizontal sides.

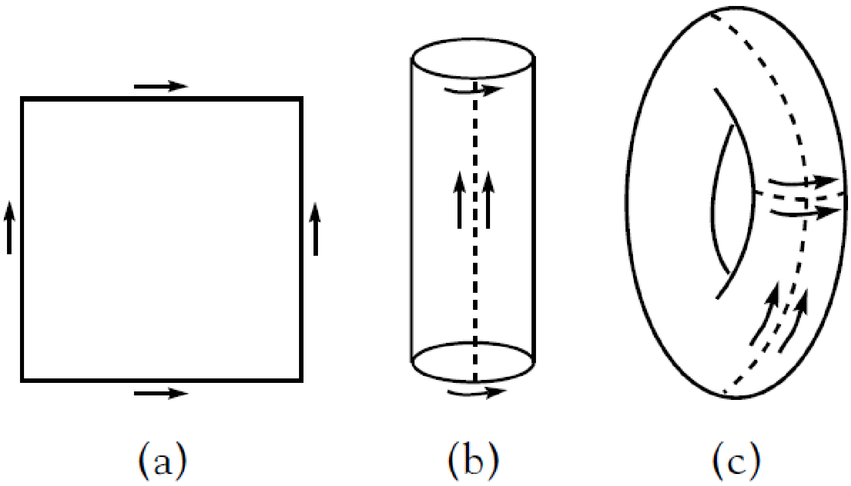


Figure 2: Construction of a torus from a unit square

2.2 Definition:

ACM is an automarphism of the torus $\mathbb{T}^2$ which is described by the following system:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1$$

where x and y are the coordinates of a point on the unit square(equivalently the torus).

This map induces a discrete-time dynamical system in which the evolution is given by iterations of the mapping : $\Gamma_{cat} : \mathbb{T}^2 \to \mathbb{T}^2$ where

$$\Gamma_{cat}\left(\begin{bmatrix} x_n \\ y_n \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \mod 1$$

2.3 The effect of ACM on the unit square:

As can be seen an iteration of ACM is firstly applying the linear transformation associated to the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ followed by taking the output vector modulo 1 to ensure that outputs are still on the unit square.That is it stretches the domain one unit in one direction and two units in the other. Then, the image is reassembled in the square of the original torus by taking the partial images in each unit square and translating them to the corresponding part of the original torus,this is illustrated in Figure 3:

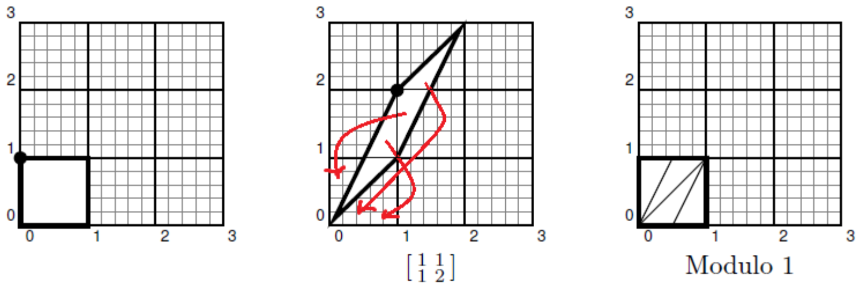


Figure 3: Image of the unit square under ACM

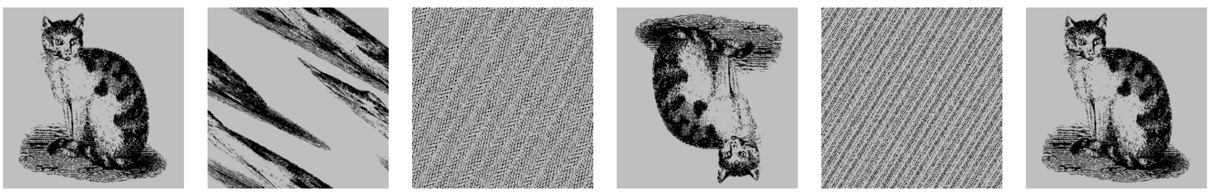
Note that in many other references, the matrix A is given by $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$,while this is a minor change since it will produce a transformation with all the same properties, but we just used the original matrix given by *V.I Arnold*. Expressing the map using the matrix A is quite helpful in studying its properties as we are going to discuss further.

2.4 The effect of ACM on square images:

Consider a square image consisting of N by N pixels, where the coordinates of each pixel are represented by pairs of rational cordinates $(X,Y) = \left(\frac{n}{N};\frac{m}{N}\right)$ where $n,m \in \{1,2\dots N-1\}$. Next,by considering a rescale of the image,we can apply ACM for the pixels with coordinates $(X,Y) = (n,m)$
This makes it possible to work with integers and works with $\mod N$ instead of $\mod 1$ which leads us to define what we call "*Arnold's discrete cat map*":

$$\Gamma_A : {\mathbb{Z}_N}^2 \to {\mathbb{Z}_N}^2 \quad \text{where} \quad \Gamma_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \mod N$$

and $\mathbb{Z}_N = \{1,2\dots N-1\}$.
The effect of Arnold's cat map on a 250 by 250 image is shown in Figure 4,even though the image looks chaotic already after a few iterations, the underlying order among the pixels lets us recover the original image after a certain number of additional iterations. We can also observe that, at one time, the image looks to be turned upside down before the original image appears once again.



(a) $n = 0$ (b) $n = 1$ (c) $n = 10$ (d) $n = 375$ (e) $n = 745$ (f) $n = 750$

Figure 4: The effect of Arnold's cat map on a 250x250 pixels image after n iterations

Transformations appearing in Figure 4 are done using program provided in the Section 5

3 Periodicity:

Firstly,we answer the question: **why ACM is periodic?** The periodicity of ACM results from the fact that the discrete-time dynamical system induced will follow the Poincaré Recurrence Theorem and hence be periodic.

Bellow we provide a simple explanation of this property:
In the case of discrete ACM, let x be an arbitrary point of $\mathbb{Z}_N^2$ and consider the sequence $(x_n)_{n\in N}$ defined on $\mathbb{Z}_N^2$ by:

$$\begin{cases} x_0 = x \\ x_{n+1} = \Gamma_A\left(x_n\right) \end{cases}$$

Once the sequence is defined on a finite set, at a certain index, it has to become periodic. However, if it does not come back to x_0, then, there would exist two points of $\mathbb{Z}_N^2$ with the same image by Γ_A , which is absurd! Thus, we can define an equivalence relation $\mathcal{R}$ on $\mathbb{Z}_N^2$ as follows: $x\mathcal{R}y$, if there exists an integer $n \in \mathbb{N}$ such that, $\Gamma_A^{(n)}(x) = y$, where $\Gamma_A^{(n)}$ is the composition of Γ_A by itself n times. In a graph where the vertices are the elements of $\mathbb{Z}_N^2$ and a directed edge connects every point of $\mathbb{Z}_N^2$ to its image by Γ_A, the equivalence classes of R will construct cycles, where every cycle's cardinal is the minimum number of compositions of Γ_R^2 to return to a point of it.

Now, we explain **why there are no two different points affected to the same position**:
For two vectors $\begin{pmatrix} x_1 \\ y_2 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ to have the same image we must have:

$$\begin{cases} x_1 + y_1 \equiv x_2 + y_2[N] \\ x_1 + 2y_1 \equiv x_2 + 2y_2[N] \end{cases} \Leftrightarrow \begin{cases} x_1 + y_1 \equiv x_2 + y_2[N] \\ y_1 \equiv y_2[N] \end{cases} \Leftrightarrow \begin{cases} x_1 \equiv x_2[N] \\ y_1 \equiv y_2[N] \end{cases}$$

Since $x_1, x_2, y_1, y_2 \in \mathbb{Z}_N^2$, then: $x_1 = x_2$ and $y_1 = y_2$, thus, the map is injective and it maps a set $(\mathbb{Z}_N^2)$ onto itself, so, it is bijective.

3.0.1 Minimal Period

The minimal period of Arnold's discrete cat map is the smallest positive integer n such that $A^n \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{mod} N)$. We denote by $\Pi_A(N)$ the minimal period of Arnold's discrete cat map modulo N.

From Figure 4 we can conclude that $\Pi_A(250) = 750$, since there is no positive integer smaller than $n = 750$ such that the original image reappears.

Although the minimal depends on N ,there is no obvious connection between them as depicted by Figures 5 and 6 bellow, this fact has been the object of many studies and articles where [19] by Wall can be considered as the most prominent.

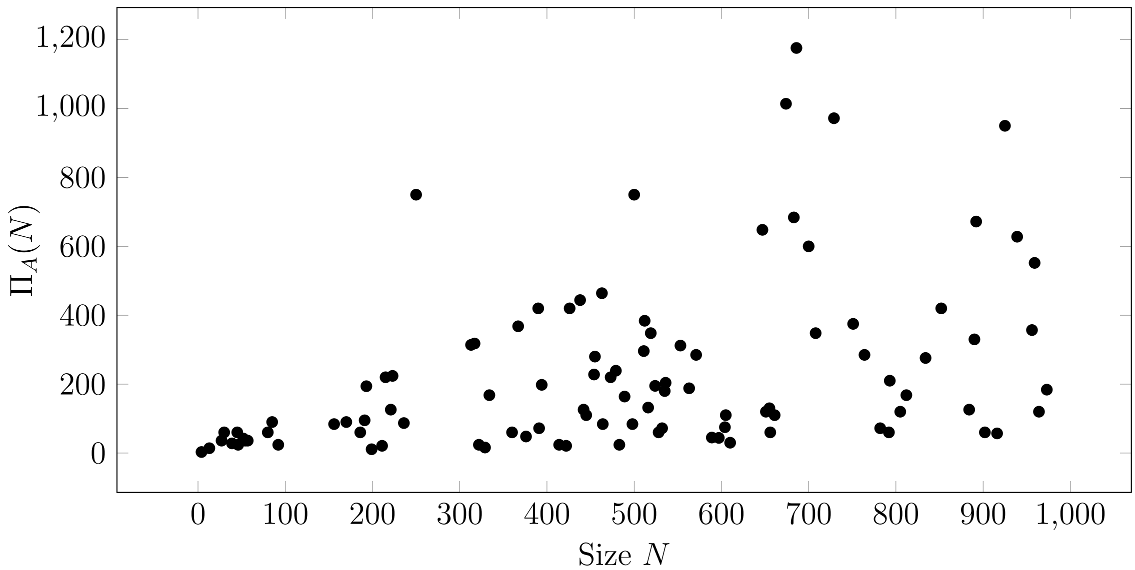


Figure 5: Minimal periods $\Pi_A(N)$of Arnold's cat map

N	$\Pi_A(N)$
64	48
128	96
256	192
512	384
1368	36

Figure 6: Minimum recurrence time for some image sizes

3.0.2 Upper bound for $\Pi_A(N)$ and some special cases:

Theorem:
The upper bound for the minimal period of Arnold's discrete cat map is $3N$.

The proof of this is omitted here but can be found in [3] by Dyson and Falk.
They also proved that for $k = 1, 2, 3, \ldots$

$$\begin{array}{ll} \Psi_A(N) = 3N & \text{when } N = 2 \cdot 5^k, \\ \Psi_A(N) = 2N & \text{when } N = 5^k \text{ or } N = 6 \cdot 5^k, \\ \Psi_A(N) \leq \frac{12}{7}N & \text{for all other } N. \end{array}$$

[1] Besides giving an expression for the upper bound, Dyson and Falk also examines the lower bound for the minimal period of Arnold's cat map.

3.0.3 The connection between ACM and Fibonacci sequence:

The powers of the matrix $F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 \\ F_1 & F_2 \end{bmatrix}$ generate numbers of the Fibonacci sequence:

$$F^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}.$$

[1]A period $\Psi_A(N)$ of Arnold's cat map is an integer k such that $A^k \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\text{mod} N)$

Since F and A have the relationship

$$F^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = A$$

the Fibonacci numbers will also appear when we take powers of the matrix A in the following manner

$$A^n = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n = \begin{bmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{bmatrix}$$

with the first powers of A being

$$A^2 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, A^3 = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}, A^4 = \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix}, A^5 = \begin{bmatrix} 34 & 55 \\ 55 & 89 \end{bmatrix}, \ldots$$

From the definition of the minimal period of Arnold's cat map we know that we are looking for the smallest integer n such that $A^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\bmod N)$ i.e. we must find the smallest n such that $F_{2n-1} \equiv 1 (\bmod N)$ and $F_{2n} \equiv 0 (\bmod N)$. Hence the period of ACM will have a direct connection to the Pisano period[2] of the Fibonacci sequence. From the above-mentioned relation between the matrices F and A follows that the minimal period of ACM will be exactly half the Pisano period for all $N \geq 3$.

4 General Properties:

4.1 Continuity of the map

The cat map Γ_{cat} is a C^∞ diffeomorphism from $\mathbb{T}^2$ to $\mathbb{T}^2$.

Proof. The map $\Gamma : \mathbb{T}^2 \to \mathbb{T}^2$ is continuous if the projections $f_1 : \mathbb{T}^2 \to \mathbb{T}$ and $f_2 : \mathbb{T}^2 \to \mathbb{T}$ are continuous:

$$f_1(x, y) = x + y \bmod 1$$
$$f_2(x, y) = x + 2y \bmod 1$$

Let (x_0, y_0) be a point on the torus, and let $\{(a_k, b_k)\}$ be a sequence converging to the point (x_0, y_0). Let $0 < \epsilon < 1$. Then there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|(x_0, y_0) - (a_n, b_n)\|_2 < \epsilon$$

This implies $|a_n - x_0| < \epsilon$ and $|b_n - y_0| < \epsilon$. If we compare the image of the sequence under the map f_1, and the image of (x_0, y_0), we get

$$\begin{aligned} |f_1(a_k, b_k) - f_1(x_0, y_0)| &= |(a_k + b_k \bmod 1) - (x_0 + y_0 \bmod 1)| \\ &\leq |a_k - x_0 \bmod 1| + |b_k - y_0 \bmod 1| . \end{aligned}$$

For all $n \geq N$, we have that these differences are less than 1 , so the modular 1 has no effect.

$$|a_n - x_0| + |b_n - y_0| < 2\epsilon$$

Therefore the map f_1 is continuous. By very similar calculations, the map f_2 is continuous, and we have that Γ is continuous. Notice also that because f_1 and f_2 are polynomial functions they are C^∞. Since the map is also invertible, Γ_{cat} is a C^∞ diffeomorphism.

[2] The period length of the Fibonacci sequence modulo N, named after Leonardo fibonacci of Pisa

5 Programing Part:

In this part we are going to provide the neccessary python codes used to in this project

5.1 Encoding:

The first program consists of encoding a grayscaled image using ACM,it is a direct application of the transformation :

```
import numpy as np
import cv2

def apply_arnold_cat_map(image_path, iterations):

    image = cv2.imread(image_path, cv2.IMREAD_GRAYSCALE)
    if image is None:
        raise ValueError("Image could not be read. Check the file path.")

    N = image.shape[0]
    encoded_image = np.zeros_like(image)

    for x in range(N):
        for y in range(N):
            x_new, y_new = x, y
            for i in range(iterations):
                x_new, y_new = (x_new + y_new) % N, (x_new +2* y_new) % N
            encoded_image[x_new , y_new] = image[x , y]

    cv2.imshow('Encoded Image', encoded_image)
    cv2.waitKey(0)
    cv2.destroyAllWindows()

    return encoded_image
```

Notes:

There is no much room for optimization here,but instead of performing the ACM transformation iteratively, which is not the most efficient way, especially for a large number of iterations. A more optimal approach would be to calculate the final position of each pixel after a given number of iterations directly and then move the pixel to its new location.

We can encode a colored image by applying the same algorithm to its three layers(RGB).

5.2 Minimal period:

Below is the program used to calculate the minimal period (the minimum recurrence time)of Arnold's Cat Map for a given size of matrix:

```
from numpy import ones as o
from numpy import lcm as l
def Minimal_period(n):

    p = 1
    li = o([n,n])

    for i in range(n):
        for j in range(n):

            if int(li[i][j]) == 1:

                c = 1
                i_1 = i
                j_1 = j

                while ((2*i_1+j_1)%n) != i or ((i_1+j_1)%n) != j:
                    a = i_1
                    i_1 = (2*i_1+j_1)%n
                    j_1 = (a+j_1)%n
                    c = c + 1
                    li[i_1][j_1] = 0

                p = l(p,c)

    return p
```

Main idea:

In order to get back the original image, we need to apply ACM a number of times that is suffcient to get all the pixels back to their initial positions.To do so we calculate the minimal period of each element and take the least common multiple of all periods.

Notes:

This program is optimized since it does not calculate the period of each element,instead, we know that elements on the same orbit will have the same period so we calculate the period of one element from each orbit which reduces the calculations.

Through this program we can get the orbits and the period of elements on each orbit.

5.3 Retrieving:

As disscussed in previous parts,we can retrieve the original image just by keep applying the same algorithm we used to encode, but this time we need more information which are the number of iterations we applied on the original image and the minimal period corresponding to size of the image:

```
def Retrieve(encoded_image_path,iterations):

    N = encoded_image.shape[0]
    period=Minimal_period(N)
```

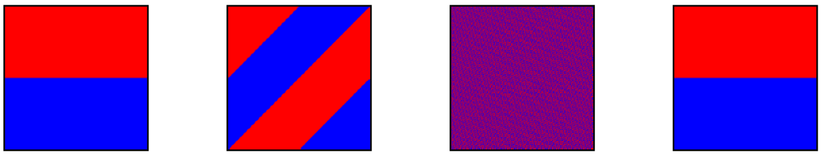
```
    reverse_iterations = period - (iterations % period)

    return apply_arnold_cat_map(encoded_image_path,reverse_iterations)
```

This program uses the two previous programs in order to restore the original image.

6 Conclusion:

Imagine mixing two colors until get a homogeneous mixture and being able to separate them just by continue mixing, that is what Arnold's cat map do.



References

[1] Kathleen T. Alligood, Tim D. Sauer, and James A. Yorke. *Chaos: An Introduction to Dynamical Systems*. Springer, 1996.

[2] V. I. Arnold and A. Avez. *Ergodic Problems of Classical Mechanics*. The Mathematical Physics Monograph Series. 1968.

[3] Freeman J. Dyson and Harold Falk. *Period of a Discrete Cat Mapping*. Taylor and Francis, Ltd, 1992.

[4] Geneva May Collins Hall. "Arnold's Cat Map: An Exposition". MA thesis. University of North Carolina at Chapel Hill, 2022.

[5] Fredrik Svanström. "Properties of a Generalized Arnold's Discrete Cat Map". MA thesis. 2014.