Partial Differential Equations

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INTRODUCTION

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables.

PARTIAL DIFFERENTIAL EQUATION (P.D.E.)

Definition. An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a *partial differential equation.*

For examples of partial differential equations we list the following:

$$\partial z / \partial x + \partial z / \partial y = z + xy \qquad \ldots (1) \qquad (\partial z / \partial x)^2 + \partial^3 z / \partial y^3 = 2x(\partial z / \partial x) \qquad \ldots (2)$$

$$z(\partial z / \partial x) + \partial z / \partial y = x \qquad \ldots (3) \qquad \partial u / \partial x + \partial u / \partial y + \partial u / \partial z = xyz \qquad \ldots (4)$$

$$\partial^2 z / \partial x^2 = (1 + \partial z / \partial y)^{1/2} \qquad \ldots (5) \qquad y\left\{(\partial z / \partial x)^2 + (\partial z / \partial y)^2\right\} = z(\partial z / \partial y) \qquad \ldots (6)$$

ORDER OF A PARTIAL DIFFERENTIAL EQUATION

Definition.

The order of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

The above equations (1), (3), (4) and (6) are of the first order, (5) is of the second order and (2) is of the third order.

DEGREE OF A PARTIAL DIFFERENTIAL EQUATION

The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, i.e., made free from radicals and fractions so far as derivatives are concerned.

The above equations (1), (2), (3) and (4) are of first degree while equations (5) and (6) are of second degree.

LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definitions.

A partial differential equation is said to be linear if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a non-linear partial differential equation.

The above equations (1) and (4) are linear while equations (2), (3), (5) and (6) are non-linear.

NOTATIONS

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations

$p = \partial z / \partial x, \qquad q = \partial z / \partial y, \qquad r = \partial^2 z / \partial x^2, \qquad s = \partial^2 z / \partial x \partial y \qquad \text{and} \qquad t = \partial^2 z / \partial y^2$

In case there are n independent variables, we take them to be $x_1, x_2, \ldots, x_n$ and z is then regarded as the dependent variable. In this case we use the following notations :

$p_1 = \partial z / \partial x_1, \qquad p_2 = \partial z / \partial x_2, \qquad p_3 = \partial z / \partial x_3, \qquad \text{and} \qquad p_n = \partial z / \partial x_n.$

Sometimes the partial differntiations are also denoted by making use of suffixes. Thus we write $u_x = \partial u / \partial x, \quad u_y = \partial u / \partial y, \quad u_{xx} = \partial^2 u / \partial x^2, \quad u_{xy} = \partial^2 u / \partial x \partial y$ and so on.

Classification of first order partial differential equations into linear, semi-linear, quasi-linear and non-linear equations with examples.

Linear equation. *A first order equation* $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form $P(x, y)\, p + Q(x, y)\, q = R(x, y)\, z + S(x, y)$.

For examples, $yx^2p + xy^2q = xyz + x^2y^3$ and $p + q = z + xy$

are both first order linear partial differential equations.

Semi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e. if the given equation is of the form* $P(x, y)\, p + Q(x, y)\, q = R(x, y, z)$

For examples, $xyp + x^2yq = x^2y^2z^2$ and $yp + xq = (x^2z^2/y^2)$

are both first order semi-linear partial differential equations.

Quasi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form*

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

For examples, $x^2zp + y^2zp = xy$ and $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$

are first order quasi-linear partial differential equations.

Non-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *which does not come under the above three types, in known as a non-liner equation.*

For examples, $p^2 + q^2 = 1$, $p\, q = z$ and $x^2 p^2 + y^2 q^2 = z^2$

are all non-linear partial differential equations.

Origin/Formation of partial differential equations.

We shall now examine the interesting question of how partial differential equations arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

Rule I. Derivation of a partial differential equation by the elimination of arbitrary constants.

Consider an equation $F(x, y, z, a, b) = 0$, ...(1)

where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get

$\partial F / \partial x + p(\partial F / \partial z) = 0$ and $\partial F / \partial y + q(\partial F / \partial z) = 0$ ..(2)

Eliminating two constants a and b from three equations of (1) and (2), we shall obtain an equation of the form

$f(x, y, z, p, q) = 0$, ...(3)

which is partial differential equation of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

Working rule for solving problems:

For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constantsa, b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations

$F(x, y, z, a, b) = 0$, $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$. The equation free from a and b will be the required partial differential equation. Three situations may arise:

Situation I.

When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one

For example, consider $$\mathrm{z} = \mathrm{ax} + \mathrm{y} \ldots \ldots\ldots \ldots \ldots . (1)$$

Where a is the only arbitrary constant and $\boldsymbol{x}, \boldsymbol{y}$ are two independent variables.

Differentiating (1) partially w.r.t. 'x', we get $\frac{\partial z}{\partial x} = a \ldots \ldots\ldots \ldots\ldots (2)$

Differentiating (1) partially w.r.t. 'y', we get $\frac{\partial z}{\partial y} = 1 \ldots \ldots\ldots \ldots\ldots (3)$

Eliminating a between (1) and (2) yields $z = x\left(\frac{\partial z}{\partial x}\right) + y \ldots \ldots \ldots\ldots (4)$

Since (3) does not contain arbitrary constant, so (3) is also partial differential under consideration. Thus, we get two partial differential equations (3) and (4).

Situation II.

When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.

Example: *Eliminate a and b from* $az + b = a^2x + y$ ... (1)

Differentiating (1) partially w.r.t 'x' and 'y', we have

$a(\partial z / \partial x) = a^2$ ... (2) $a(\partial z / \partial y) = 1$ ... (3)

Eliminating a from (2) and (3), we have $(\partial z / \partial x)(\partial z / \partial y) = 1$,

which is the unique partial differential equation of order one.

Situation III.

When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one

Example: *Eliminate a, b and c from* $z = ax + by + cxy$ ... (1)

Differentiating (1) partially w.r.t., 'x' and 'y', we have

$\partial z / \partial x = a + c\,y$ ... (2) $\partial z / \partial y = b + c\,x$ ... (3)

From (2) and (3), $\partial^2 z / \partial x^2 = 0,$ $\partial^2 z / \partial y^2 = 0$ ... (4)

and $\partial^2 z / \partial x \partial y = c$ ... (5)

Now, (2) and (3) $\Rightarrow$ $x(\partial z / \partial x) = ax + cxy$ and $y(\partial z / \partial y) = by + cxy$

$\therefore$ $x(\partial z / \partial x) + y(\partial z / \partial y) = ax + by + cxy + cxy$

or $x(\partial z / \partial x) + y(\partial z / \partial y) = z + xy(\partial^2 z / \partial x \partial y)$, using (1) and (5) ... (6)

Thus, we get three partial differential equations given by (4) and (6), which are all of order two.

SOLVED EXAMPLES BASED ON RULE I

Ex. 1. *Find a partial differential equation by eliminating a and b from $z = ax + by + a^2 + b^2$.*

Sol. Given $$z = ax + by + a^2 + b^2. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = a \quad \text{and} \quad \partial z/\partial y = b.$$

Substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain,

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\partial x)^2 + (\partial z/\partial y)^2,$$

which is the required partial differential equation.

Ex. 2. *Eliminate arbitrary constants a and b from $z = (x - a)^2 + (y - b)^2$ to form the partial differential equation.*

Sol. Given $$z = (x - a)^2 + (y - b)^2. \qquad ...(1)$$

Differentiating (1) partially with respect to a and b, we get

$$\partial z/\partial x = 2(x - a) \quad \text{and} \quad \partial z/\partial y = 2(y - b).$$

Squatring and adding these equations, we have

$$(\partial z/\partial x)^2 + (\partial z/\partial y)^2 = 4(x - a)^2 + 4(y - b)^2 = 4\,[(x - a)^2 + (y - b)^2]$$

or $$(\partial z/\partial x)^2 + (dz/\partial y)^2 = 4z, \text{ using (1).}$$

Ex. 3. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations :*

(*a*) $z = a(x + y) + b$. (*b*) $z = ax + by + ab$.

Sol. (*a*) Given $z = a(x + y) + b$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = a$.

Eliminating a between these, we get $\partial z/\partial x = \partial z/\partial y$,

which is the required partial differential equation.

(*b*) Given $z = ax + by + ab$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = b$ ....(2)

Substituting the values of a and b from (2) in (1), we get

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/dx)(\partial z/\partial y),$$

which is the required partial differential equation.

Ex. 4. *Eliminate a and b from* $z = axe^y + (1/2) \times a^2e^{2y} + b$.

Sol. Given $z = axe^y + (1/2) \times a^2e^y + b$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = ae^y \quad \text{...(2)}$$

and $$\partial z/\partial y = axe^y + a^2e^{2y} = x(ae^y) + (ae^y)^2. \quad \text{...(3)}$$

Substituting the value of ae^y from (2) in (3), we get $\partial z/\partial y = x(\partial z/\partial x) + (\partial z/\partial x)^2$.

Ex. 5. *Form the differential equation by eliminating a and b from* $z = (x^2 + a)(y^2 + b)$.

Sol. Given $z = (x^2 + a)(y^2 + b)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = 2x(y^2 + b)$ or $(y^2 + b) = (1/2x) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = 2y(x^2 + a)$ or $(x^2 + a) = (1/2y) \times (\partial z/\partial y)$. ...(3)

Substituting the values of $(y^2 + b)$ and $(x^2 + a)$ from (2) and (3) in (1) gives

$z = (1/2y) \times (\partial z/\partial y) \times (1/2x) \times (\partial z/\partial x)$ or $4xyz = (\partial z/\partial x)(\partial z/\partial y)$,

which the required partial differential equation.

Ex. 6. *Form differential equation by eliminating constants A and p from* $z = A e^{pt} \sin px$.

Sol. Given $z = A e^{pt} \sin px$. ...(1)

Differentiating (1) partially with respect to x and t, we get

$\partial z/\partial x = Ap\, e^{pt} \cos px$ ...(2) $\partial z/\partial t = Ap\, e^{pt} \sin px$. ...(3)

Differentiating (2) and (3) partially with respect to x and t respectively gives

$\partial^2 z/\partial x^2 = -Ap^2 e^{pt} \sin px$. ...(4) $\partial^2 z/\partial t^2 = Ap^2 e^{pt} \sin px$. ...(5)

Adding (4) and (5), $\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0$,

which is the required partial differential equation.

Rule II. Derivation of partial differential equation by the elimination of arbitrary function ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z.

Proof. Given $$\phi(u, v) = 0. \qquad ...(1)$$

We treat z as dependent variable and x and y as independent variables so that

$\partial z/\partial x = p$, $\quad \partial z/\partial y = q$, $\quad \partial y/\partial x = 0$ and $\quad \partial x/\partial y = 0$.

Differentiating (1) partially with respect to x, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

or $$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = 0$$

or $$\frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = -\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right). \qquad ...(3)$$

Similarly, differentiating (1) partially w.r.t. 'y', we get

$$\frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = -\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right) \qquad ...(4)$$

Eliminating ϕ with the help of (3) and (4), we get

$$\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right) = \left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)$$

or $$\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)$$

or $$Pp + Qq = R, \qquad ...(5)$$

where $P = \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z}-\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}$, $\quad Q = \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}$, $\quad R = \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}-\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}$.

Thus we obtain a linear partial differntial equation of first order and of first degree in p and q.

Note. If the given equation between x, y, z contains two arbitrary functions, then in general, their elimination gives rise to equations of higher order.

SOLVED EXAMPLES BASED ON RULE II:

Ex. 1. *Form a partial differential equation by eliminating the arbitrary function f from the equation* $x + y + z = f(x^2 + y^2 + z^2)$.

Sol. Given $$x + y + z = f(x^2 + y^2 + z^2). \quad ...(1)$$

Differentiating partially w.r.t. 'x' and 'y', (1) gives

$$1 + p = f'(x^2 + y^2 + z^2).(2x + 2zp). \quad ...(2)$$

and $$1 + q = f'(x^2 + y^2 + z^2).(2y + 2zq). \quad ...(3)$$

Eliminating $f'(x^2 + y^2 + z^2)$ from (2) and (3), we obtain

$(1 + p)/(2x + 2zp) = (1 + q)/(2y + 2zq)$ or $(1 + p)(y + zq) = (1 + q)(x + zp)$

or $(y - z)p + (z - x)q = x - y$, which is the required partial differential equations.

Ex. 2. *Eliminate the arbitrary functions f and F from* $y = f(x - at) + F(x + at)$.

Sol. Given $$y = f(x - at) + F(x + at). \quad ...(1)$$

From (1), $$\partial y/\partial x = f'(x - at) + F'(x + at)$$

and hence $$\partial^2 y/\partial x^2 = f''(x - at) + F''(x + at). \quad ...(2)$$

Also, $$\partial y/\partial t = f'(x - at) . (-a) + F'(x + at) . (a)$$

and hence $$\partial^2 y/\partial t^2 = f''(x - at) . (-a)^2 + F''(x + at) . (a)^2$$

or $$\partial^2 y/\partial t^2 = a^2[f''(x - at) + F''(x + at)]. \quad ...(3)$$

Then, (2) and (3) $\Rightarrow$ $\partial^2 y/\partial t^2 = a^2(\partial^2 y/\partial x^2)$.

Ex. 3. *Eliminate arbitrary function f from*

(*i*) $z = f(x^2 - y^2)$.

(*ii*) $z = f(x^2 + y^2)$.

Sol. (*i*) Given $\qquad z = f(x^2 - y^2)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(x^2 - y^2) \times 2x$ so that $f'(x^2 - y^2) = (1/2x) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = f'(x^2 - y^2) \times (-2y)$ so that $f'(x^2 - y^2) = -(1/2y) \times (\partial z/\partial y)$. ...(3)

Eliminating $f'(x^2 - y^2)$ between (2) and (3), we have

$$\frac{1}{2x}\frac{\partial z}{\partial x} = -\frac{1}{2y}\frac{\partial z}{\partial y} \qquad \text{or} \qquad y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$

(*ii*) Proceed as in part (1). **Ans.** $y(\partial z/\partial x) - x(\partial z/\partial y) = 0$

Ex. 4. *Form a partial differential equation by eliminating the function f from*

(*i*) $z = f(y/x)$. (*ii*) $z = x^n f(y/x)$.

Sol. Given $\qquad z = f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(y/x) \times (-y/x^2)$ or $f'(y/x) = -(x^2/y) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = f'(y/x) \times (1/x)$ or $f'(y/x) = x(\partial z/\partial y)$. ...(3)

Eliminating $f'(y/x)$ between (2) and (3), we have

$$-\frac{x^2}{y}\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y} \qquad \text{or} \qquad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

which is the required partial differential equation.

(*ii*) Given $\qquad z = x^n f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = n\,x^{n-1} f(y/x) + x^n f'(y/x) \times (-y/x^2) \qquad ...(2)$$

and $$\partial z/\partial y = x^n f'(y/x) \times (1/x). \qquad ...(3)$$

Multiplying both sides of (2) by x, we have $$x(\partial z/\partial x) = n\,x^n f(y/x) - yx^{n-1} f'(y/x). \qquad ...(4)$$

Multiplying both sides of (3) by y, we have $$y(\partial z/\partial y) = y\,x^{n-1} f'(y/x). \qquad ...(5)$$

Adding (4) and (5), $$x(\partial z/\partial x) + y(\partial z/\partial y) = n\,x^n f(y/x)$$

or $$x(\partial z/\partial x) + y(\partial z/\partial y) = nz, \text{ by (1)}$$

Ex. 5. *Form partial differential eqn. by eliminating the function f from $z = e^{ax+by} f(ax - by)$.*

Sol. Given $$z = e^{ax+by} f(ax - by). \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = e^{ax+by}\, a\, f'(ax - by) + a\, e^{ax+by} f(ax - by) \qquad ...(2)$$

and $$\partial z/\partial y = e^{ax+by} \{-b\, f'(ax - by)\} + b\, e^{ax+by} f(ax - by). \qquad ...(3)$$

Multiplying (2) by b and (3) by a and adding, we get

$$b(\partial z/\partial x) + a(\partial z/\partial y) = 2ab\, e^{ax+by} f(ax - by) \quad \text{or} \quad b(\partial z/\partial x) + a(\partial z/\partial y) = 2abz, \text{ by (1)}$$

Ex. 7 . *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$.

Sol. Given $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0.$ ...(1)

Let $u = x^2 + y^2 + z^2$ and $v = z^2 - 2xy$. ...(2)

Then, (1) becomes $\phi(u, v) = 0.$ ...(3)

Differentiating (3) partially w.r.t. 'x', we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0, \quad \text{...(4)}$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Now, from (2), we have

$\partial u/\partial x = 2x,\quad \partial u/\partial y = 2y,\quad \partial u/\partial z = 2z,\quad \partial v/\partial x = -2y,\quad \partial v/\partial y = -2x,\quad \partial v/\partial z = 2z.$ ...(5)

Using (5), (4) reduces to $(\partial\phi/\partial u)(2x + 2pz) + (\partial\phi/\partial v)(-2y + 2pz) = 0$

or $(x + pz)(\partial\phi/\partial u) = (y - pz)(\partial\phi/\partial v).$ ...(6)

Again, differentiating (3) partially w.r.t. 'y', we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial y}\right)=0$$

or $(\partial\phi/\partial u)(2y + 2qz) + (\partial\phi/\partial v)(-2x + 2qz) = 0$, by (5)

or $(y + qz)(\partial\phi/\partial u) = (x - qz)(\partial\phi/\partial v).$ ...(7)

Dividing (6) by (7), $(x + pz)/(y + qz) = (y - pz)/(x - qz)$

or $pz(y + x) - qz(y + x) = y^2 - x^2$ or $(p - q)z = y - x.$

Equations solvable by direct integration (Partial Differential Equations of order Two with Variable Coefficients)

INTRODUCTION:

In the present chapter, we propose to discuss partial differential equations of order two with variable coefficients. An equation is said to be of order two, if it involves at least one of the differential coefficients $r\ (= \partial^2 z/\partial x^2)$, $s\ (= \partial^2 z/\partial x\partial y)$, $t\ (= \partial^2 z/\partial y^2)$, but none of higher order ; the quantities p and q may also enter into the equation. Thus, the general form of a second order partial differential equation is

$$f(x, y, z, p, q, r, s, t) = 0. \qquad ...(1)$$

The most general linear partial differential equation of order two in two independent variables x and y with variable coefficients is of the form

$$Rr + Ss + Tt + Pp + Qq + Zz = F, \qquad ...(2)$$

where R, S, T, P, Q, Z, F are functions of x and y only and not all R, S, T are zero.

Ex 1. *Solve the following partial differential equations:*

(*i*) $r = 6x$. (*ii*) $ar = xy$

(*iii*) $r = x^2 e^y$ (*iv*) $r = 2y^2$

(*v*) $r = \sin(xy)$

Sol. (*i*) Given equation can be written as $\partial^2 z/\partial x^2 = 6x$. ...(1)

Integrating (1) with respect to 'x', $\partial z/\partial x = 3x^2 + \phi_1(y)$, ...(2)

where $\phi_1(y)$ is an arbitrary function of y.

Integrating (2) with respect to 'x', $z = x^3 + x\phi_1(y) + \phi_2(y)$,

where $\phi_2(y)$ is an arbitrary function of y.

(ii) Given equation can be written as $\partial^2 z/\partial x^2 = (1/a) \times xy$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial x = (y/a) \times (x^2/2) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = (y/6a) \times x^3 + x\phi_1(y) + \phi_2(y)$,

which is the required general solution, ϕ_1, ϕ_2 being arbitrary functions.

(iii) Try yourself. **Ans.** $z = (e^y/12) \times x^4 + x\phi_1(y) + \phi_2(y)$.

(iv) Try yourself. **Ans.** $z = x^2y^2 + x\phi_1(y) + \phi_2(y)$.

(v) Given equation can be written as $\partial^2 z/\partial x^2 = \sin(xy)$. ...(1)

Integrating (1) w.r.t. 'x'. $\partial z/\partial x = -(1/y) \times \cos(xy) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = -(1/y^2) \times \sin(xy) + x\phi_1(y) + \phi_2(y)$,

which is the required general solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 2. *Solve* (i) $t = \sin(xy)$ (ii) $t = x^2 \cos(xy)$.

Sol.(i) Given equation can be written as $\partial^2 z/\partial y^2 = \sin(xy)$. ...(1)

Integrating (1) w.r.t. 'y', $\partial z/\partial y = -(1/x) \times \cos(xy) + \phi_1(x)$. ...(2)

Integrating (2) w.r.t., 'y', $z = -(1/x^2) \times \sin(xy) + y\phi_1(x) + \phi_2(x)$,

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

(ii) Given equation can be written as $\partial^2 z/\partial y^2 = x^2\cos(xy)$. ...(1)

Integrating (1) w.r.t. 'y', $\partial z/\partial y = x\sin(xy) + \phi_1(x)$. ...(2)

Integrating (2) w.r.t. 'y', $z = -\cos(xy) + y\,\phi_1(x) + \phi_2(x)$,

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 3. *Solve the following partial differential equations:*

(*i*) $xys = 1$ (*ii*) $xy^2 s = 1 - 2x^2 y$

(*iii*) $\log s = x + y$ (*iv*) $s = x - y$

(*v*) $s = x^2 - y^2$ (*vi*) $x^2 s = \sin y$

(*vii*) $s = (x/y) + a$ (*viii*) $s = 0$.

Sol. (*i*) Re-written the given equation, $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{1}{xy}. \qquad ...(1)$$

Integrating (1) w.r.t. 'x', $\partial z/\partial y = (1/y) \times \log x + \phi_1(y)$.

Integrating (2) w.r.t. 'y', $z = \log x \log y + \int \phi_1(y) dy + \psi_2(x)$

or $z = \log x \log y + \psi_1(y) + \psi_2(x)$, taking $\psi_1(y) = \int \phi_1(y) dy$.

which is the required general solution, ψ_1, ψ_2 being arbitrary functions.

(*ii*) Given equation is $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{1}{xy^2} - \frac{2x}{y}. \qquad ...(1)$$

Integrating (1) w.r.t. 'x', $\partial z/\partial y = (1/y^2) \times \log x - (x^2/y) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = -(1/y) \times \log x - x^2 \log y + \int \phi_1(y) dy + \psi_2(x)$

or $z = -(1/y) \times \log x - x^2 \log y + \psi_1(y) + \psi_2(x)$, taking $\psi_1(y) = \int \phi_1(y) dy$.

which is the required general solution, ψ_1, ψ_2 being arbitrary functions.

(*iii*) The given equation $\log s = x + y$ can be rewritten as

$s = e^{x+y}$ or $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = e^x \cdot e^y. \qquad ...(1)$$

Integrating (1) w.r.t. 'x', $\partial z/\partial y = e^x e^y + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = e^x e^y + \int \phi_1(y)dy + \psi_2(x)$

or $z = e^{x+y} + \psi_1(y) + \psi_2(x)$, where $\psi_1(y) = \int \phi_1(y)dy$, ψ_1, ψ_2 being arbitrary functions

(*iv*) Try yourself. **Ans.** $z = (1/2) \times (x^2 y - xy^2) + \psi_1(y) + \psi_2(x)$.

(*v*) Try yourself. **Ans.** $z = (1/3) \times (x^3 y - xy^3) + \psi_1(y) + \psi_2(x)$.

(*vi*) Given equation can be written as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\sin y}{x^2}$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = -(1/x) \times \sin y + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = (1/x) \cos y + \int \phi_1(y)dy + \psi_2(x)$

or $z = (1/x) \cos y + \psi_1(y) + \psi_2(x)$, where $\psi_1(y) = \int \phi_1(y)dy$, ψ_1, ψ_2 being arbitrary functions

(*vii*) Try yourself. **Ans.** $z = (1/2) \times x^2 \log y + axy + \psi_1(y) + \psi_2(x)$.

(*viii*) Try yourself. **Ans.** $z = \psi_1(y) + \psi_2(x)$.

Ex. 4. *Solve* (*i*) $xr = p$ (*ii*) $rx = (n-1)p$.

Sol.(*i*) Given equation can be rewritten as

$$x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial^2 z/\partial x^2}{\partial z/\partial x} = \frac{1}{x}.$$

Integrating, $\log(\partial z/\partial x) = \log x + \log \phi_1(y)$ or $\partial z/\partial x = x\phi_1(y)$.

Integrating it w.r.t. x, $z = (x^2/2) \times \phi_1(y) + \phi_2(y)$, where $\phi_1(y)$ and $\phi_2(y)$ are arbitrary functions.

(*ii*) Given $x\dfrac{\partial^2 z}{\partial x^2} = (n-1)\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial^2 z/\partial x^2}{\partial z/\partial x} = \dfrac{n-1}{x}$.

Integrating, $\log(\partial z/\partial x) = (n-1)\log x + \log \phi_1(y)$ or $\partial z/\partial x = x^{n-1}\phi_1(y)$.

Integrating it, $z = (x^n/n) \times \phi_1(y) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 5. *Solve* (*i*) $xr + 2p = 0$ (*ii*) $2yq + y^2 t = 1$.

Sol.(*i*) The given equation can be rewritten as

$$x\frac{\partial p}{\partial x} + 2p = 0 \quad \text{or} \quad x^2\frac{\partial p}{\partial x} + 2xp = 0 \quad \text{or} \quad \frac{\partial}{\partial x}(x^2 p) = 0. \qquad ...(1)$$

Integrating (1) w.r.t. 'x', $x^2 p = \phi_1(y)$ or $p = \partial z/\partial x = (1/x^2) \times \phi_1(y)$.

Integrating it w.r.t. 'x', $z = -(1/x) \times \phi_1(y) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions.

(*ii*) The given equation can be rewritten as

$$2yq + y^2\frac{\partial q}{\partial y} = 1 \quad \text{or} \quad \frac{\partial}{\partial y}(y^2 q) = 0. \qquad ...(1)$$

Integrating (1) w.r.t. 'y', $y^2 q = \phi_1(x)$ or $q = \partial z/\partial y = (1/y^2) \times \phi_1(x)$.

Integrating it, $z = -(1/y) \times \phi_1(x) + \phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 6. *Solve* $xs + q = 4x + 2y + 2$.

Sol. The given equation can be re-written as

$$x\frac{\partial p}{\partial y} + \frac{\partial z}{\partial y} = 4x + 2y + 2 \quad \text{or} \quad \frac{\partial}{\partial y}(xp + z) = 4x + 2y + 2.$$

Integrating it w.r.t. 'y', $xp + z = 4xy + y^2 + 2y + \phi_1(x)$

or $$x\frac{\partial z}{\partial x} + z = 4xy + y^2 + 2y + \phi_1(x) \quad \text{or} \quad \frac{\partial}{\partial x}(xz) = 4xy + y^2 + 2y + \phi_1(x).$$

Integrating it w.r.t. 'x', $xz = 2x^2 y + xy^2 + 2xy + \int \phi_1(x)dx + \psi_2(y)$

or $\therefore$ Required solution is $xz = 2x^2 y + xy^2 + 2xy + \psi_1(x) + \psi_2(y)$, where $\psi_1(x) = \int \phi_1(x)dx$.

Ex. 7. *Solve* $ys + p = \cos(x+y) - y\sin(x+y)$.

Sol. The given equation can be rewritten as

$$y\frac{\partial q}{\partial x} + \frac{\partial z}{\partial x} = \cos(x+y) - y\sin(x+y) \quad \text{or} \quad \frac{\partial}{\partial x}(yq+z) = \cos(x+y) - y\sin(x+y).$$

Integrating it w.r.t. 'x', $\quad yq + z = \sin(x+y) + y\cos(x+y) + \phi_1(y)$.

or $$y\frac{\partial z}{\partial y} + z = \sin(x+y) + y\cos(x+y) + \phi_1(y) \quad \text{or} \quad \frac{\partial(yz)}{\partial y} = \sin(x+y) + y\cos(x+y) + \phi_1(y).$$

Integrating it w.r.t. 'y', $$yz = \int \sin(x+y)dy + \int y\cos(x+y)\,dy + \int \phi_1(y)\,dy + \psi_2(y)$$

or $$yz = \int \sin(x+y)dy + y\sin(x+y) - \int \sin(x+y)\,dy + \psi_1(y) + \psi_2(y)$$

[Integrating by parts and taking $\psi_1(y) = \int \phi_1(y)\,dy$]

Required solution is $yz = y\sin(x+y) + \psi_1(y) + \psi_2(y)$, ψ_1, ψ_2 being arbitrary functions.

Linear Partial differential equations of order one:

Quasi-linear equation. *A first order partial differential equation $f(x, y, z, p, q) = 0$ is known as quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form*

$$P(x, y, z)\,p + Q(x, y, z)\,q = R(x, y, z)$$

For examples, $x^2zp + y^2zp = xy$ and $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy$ are first order quasi-linear partial differential equations.

LAGRANGE'S EQUATION

A quasi–linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z. Such a partial differential equation is known as *Lagrange equation.*

For Example $xyp + yzq = zx$ is a Lagrange equation.

Lagrange's method of solving Pp + Qq = R, when P, Q and R are functions of x, y, z:

Working Rule for solving $Pp + Qq = R$ by Lagrange's method.

Step 1. Put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R. \qquad ...(1)$$

Step 2. Write down Lagrange's auxiliary equations for (1) namely,

$$(dx)/P = (dy)/Q = (dz)/R \qquad ...(2)$$

Step 3. Solve (2) by using the well known methods (refer Art. 2.5, 2.7, 2.9 and 2.11). Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step 4. The general solution (or integral) of (1) is then written in one of the following three equivalent forms :

$\phi(u, v) = 0$, $\quad u = \phi(v) \quad$ or $\quad v = \phi(u)$, ϕ being an arbitrary function.

Type 1 based on Rule I for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Suppose that one of the variables is either absent or cancels out from any two fractions of given equations (1). Then an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of given equations (1).

Ex. 1. *Solve* $(y^2z/x)p + xzq = y^2$.

Sol. Given $$(y^2z/x)p + xzq = y^2. \quad ...(1)$$

The Lagrange's auxiliary equations for (1) are $$\frac{dx}{(y^2z/x)} = \frac{dy}{xz} = \frac{dz}{y^2}. \quad ..(2)$$

Taking the first two fractions of (2), we have

$$x^2zdx = y^2zdy \quad \text{or} \quad 3x^2dx - 3y^2dy = 0, \quad ...(3)$$

Integrating (3), $x^3 - y^3 = c_1$, c_1 being an arbitrary constant ...(4)

Next, taking the first and the last fractions of (2), we get

$$xy^2dx = y^2zdz \quad \text{or} \quad 2xdx - 2zdz = 0. \quad ...(5)$$

Integrating (5), $x^2 - z^2 = c_2$, c_2 being an arbitrary constant ...(6)

From (4) and (6), the required general integral is

$\phi(x^3 - y^3, x^2 - z^2) = 0$, ϕ being an arbitrary function.

Ex. 3. *Solve* $p \tan x + q \tan y = \tan z$.

Sol. Given $$(\tan x)p + (\tan y)q = \tan z. \quad ...(1)$$

The Lagrange's auxiliary equations for (1) are $$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}. \quad ...(2)$$

Taking the first two fractions of (2), $\cot x\, dx - \cot y\, dy = 0$.

Integrating, $\log \sin x - \log \sin y = \log c_1$ or $(\sin x)/(\sin y) = c_1$. ...(3)

Taking the last two fractions of (2), $\cot y\, dy - \cot z\, dz = 0$.

Integrating, $\log \sin y - \log \sin z = \log c_2$ or $(\sin y)/(\sin z) = c_2$. ...(4)

From (3) and (4), the required general solution is

$$\sin x/\sin y = \phi(\sin y/\sin z), \ \phi \text{ being an arbitrary function.}$$

Type 2 based on Rule II for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Suppose that one integral of (1) is known by using rule I explained in Art 2.5 and suppose also that another integral cannot be obtained by using rule I of Art. 2.5. Then one integral known to us is used to find another integral as shown in the following solved examples. Note that in the second integral, the constant of integration of first integral should be removed later on.

Ex. 1. *Solve* $z(z^2 + xy)(px - qy) = x^4$.

Sol. Given $$xz(z^2 + xy)p - yz(z^2 + xy)q = x^4. \quad ...(1)$$

The Lagrange's subsidiary equations for (1) are $$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}. \quad ...(2)$$

Cancelling $z(z^2 + xy)$, the first two fractions give

$$(1/x)\,dx = -(1/y)\,dy \quad \text{or} \quad (1/x)\,dx + (1/y)\,dy = 0. \quad ...(3)$$

Integrating (3), $\log x + \log y = \log c_1$ or $$xy = c_1. \quad ...(4)$$

Using (4), from (2) we get $$\frac{dx}{xz(z^2+c_1)} = \frac{dz}{x^4}$$

or $$x^3 dx = z(z^2 + c_1)dz \quad \text{or} \quad x^3 dx - (z^3 + c_1 z)dz = 0. \quad ...(5)$$

Integrating (5), $x^4/4 - z^4/4 - (c_1 z^2)/2 = c_2/4$ or $x^4 - z^4 - 2c_1 z^2 = c_2$

or $$x^4 - z^4 - 2xy\, z^2 = c_2, \text{ using (4)} \quad ...(6)$$

From (4) and (6), the required general integral is

$$\phi(xy,\ x^4 - z^4 - 2xy\, z^2) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $xzp + yzq = xy$.

Sol. Given $xzp + yzq = xy$. ...(1)

The Lagrange's subsidiary equations for (1) are $\dfrac{dx}{xz} = \dfrac{dy}{yz} = \dfrac{dz}{xy}$. ...(2)

Taking the first two fractions of (2), $(1/x)dx - (1/y)dy = 0$ ...(3)

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(4)

From (4), $x = c_1 y$. Hence, from second and third fractions of (2), we get

$(1/yz)dy = (1/c_1 y^2)dz$ or $2c_1 y\, dy - 2z\, dz = 0$. ...(5)

Integrating (5), $c_1 y^2 - z^2 = c_2$ or $xy - z^2 = c_2$, using (4). ...(6)

From (4) and (6), the required solution is $\phi(xy - z^2, x/y) = 0$, ϕ being an arbitrary function.

Type 3 based on Rule III for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction in (1) will be equal to $(P_1 dx + Q_1 dy + R_1 dz)/(P_1 P + Q_1 Q + R_1 R)$. ...(2)

If $P_1 P + Q_1 Q + R_1 R = 0$, then we know that the numerator of (2) is also zero. This gives $P_1 dx + Q_1 dy + R_1 dz = 0$ which can be integrated to give $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. As a special case, these can be constants also. Sometimes only one integral is possible by use of multipliers. In such cases second integral should be obtained by using rule I of Art. 2.5 or rule II of Art. 2.7 as the case may be.

Ex.1. *Solve* $\{(b-c)/a\}yzp + \{(c-a)/b\}zxq = \{(a-b)/c\}xy$.

Sol. Given $\{(b-c)/a\}yzp + \{(c-a)/b\}zxq = \{(a-b)/c\}xy$. ...(1)

The Lagrange's subsidiary equations of (1) are $\dfrac{a\,dx}{(b-c)yz} = \dfrac{b\,dy}{(c-a)zx} = \dfrac{c\,dz}{(a-b)xy}$. ...(2)

Choosing x, y, z as multipliers, each fraction for (2)

$$= \frac{a\,xdx + by\,dy + cz\,dz}{xyz[(b-c)+(c-a)+(a-b)]} = \frac{ax\,dx + by\,dy + cz\,dz}{0}.$$

$\therefore\ ax\,dx + by\,dy + cz\,dz = 0$ or $2axdx + 2bydy + 2czdz = 0$.

Integrating, $ax^2 + by^2 + cz^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Again, choosing ax, by, cz as multipliers, each fraction of (2)

$$= \frac{a^2xdx + b^2ydy + c^2\,zdz}{xyz[a(b-c)+b(c-a)+c(a-b)]} = \frac{a^2xdx + b^2ydy + c^2\,zdz}{0}.$$

$\therefore\ a^2xdx + b^2ydy + c^2zdz = 0$ or $2a^2xdx + 2b^2ydy + 2c^2zdz = 0$.

Integrating, $a^2x^2 + b^2y^2 + c^2z^2 = c_2$, c_2 being an arbitrary constant. ...(4)

From (3) and (4), the required general solution is given by

$\phi\,(ax^2 + by^2 + cz^2,\ a^2x^2 + b^2y^2 + c^2z^2) = 0$, where ϕ is an arbitrary function.

Ex. 2. *Solve* $z(x+y)p + z(x-y)q = x^2 + y^2$.

Sol. Given $$z(x+y)p + z(x-y)q = x^2 + y^2. \quad \text{...(1)}$$

The Langrange's subsidiary equations for (1) are $$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}. \quad \text{...(2)}$$

Choosing $x, -y, -z$, as multipliers, each fraction

$$= \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2-y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}.$$

$\therefore \quad x\,dx - y\,dy - z\,dz$ or $2x\,dx - 2y\,dy - 2z\,dz = 0$.

Integrating, $x^2 - y^2 - z^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Again, choosing $y, x, -z$ as multipliers, each fraction

$$= \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}.$$

$\therefore \quad y\,dx + x\,dy - z\,dz = 0$ or $2d(xy) - 2zdz = 0$.

Integrating, $2xy - z^2 = c_2$, c2 being an arbitrary constant.

From (3) and (4), the required general solution is given by

$$\phi\,(x^2 - y^2 - z^2,\ 2xy - z^2) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $(mz - ny)p + (nx - lz)q = ly - mx$.

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}. \qquad \text{...(1)}$$

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = \frac{xdx+ydy+zdz}{0}$$

$\therefore\ xdx + ydy + zdz = 0$ or $2xdx + 2ydy + 2zdz = 0$

Integrating, $x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(2)

Again, choosing l, m, n as multipliers, each fraction of (1)

$$= \frac{ldx+mdy+n\,dz}{l(mx-ny)+m(nx-lz)+n(ly-mx)} = \frac{ldx+mdy+n\,dz}{0}.$$

$\therefore\ ldx + mdy + n\,dz = 0$ so that $l\,x + m\,y + n\,z = c_2$. ...(3)

From (2) and (3), the required general solution is given by

$$\phi\,(x^2 + y^2 + z^2,\ l\,x + m\,y + n\,z) = 0,\ \phi \text{ being an arbitrary function.}$$

Type 4 based on Rule IV for solving ***(dx)/P = (dy)/Q = (dz)/R.*** ...(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction of (1) will be equal to $(P_1 dx + Q_1 dy + R_1 dz)/(P_1 P + Q_1 Q + R_1 R)$. ...(2)

Suppose the numerator of (2) is exact differential of the denominator of (2). Then (2) can be combined with a suitable fraction in (1) to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$(P_2 dx + Q_2 dy + R_2 dz)/(P_2 P + Q_2 Q + R_2 R) \qquad ...(3)$$

is such that its numerrator is exact differential of denominator. Fractions (2) and (3) are then combined to given an integral. This method may be repeated in some problems to get another integral. Sometimes only one integral is possible by using the above rule IV. In such cases second integral should be obtained by using rule 1 of Art. 2.5 or rule 2 of Art. 2.7 or rule 3 of Art. 2.9.

Ex. 1. *Solve* $(y + z)p + (z + x)q = x + y$.

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$. ...(1)

Choosing 1, –1, 0 as multipliers, each fraction of (1) = $\dfrac{dx - dy}{(y+z)-(z+x)} = \dfrac{d(x-y)}{-(x-y)}$. ...(2)

Again, choosing 0, 1, –1 as multipliers, each fraction of (1) $= \dfrac{dy - dz}{(z+x)-(x+y)} = \dfrac{d(y-z)}{-(y-z)}$. ...(3)

Finally, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(y+z)+(z+x)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)}. \qquad ...(4)$$

$$(2), (3) \text{ and } (4) \Rightarrow \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)}. \qquad ...(5)$$

Taking the first two fractions of (5), $\dfrac{d(x-y)}{x-y} = \dfrac{d(y-z)}{y-z}$.

Integrating, $\log(x-y) = \log(y-z) + \log c_1$, c_1 being an arbitrary constant.

or $\log\{(x-y)/(y-z)\} = \log c_1$ or $(x-y)/(y-z) = c_1$. ...(6)

Taking the first and the third fractions of (5), $2\dfrac{d(x-y)}{(x-y)} + \dfrac{d(x+y+z)}{x+y+z} = 0$

Integrating, $2\log(x-y) + \log(x+y+z) = \log c_2$ or $(x-y)^2(x+y+z) = c_2$. ...(7)

From (6) and (7), the required general solution is

$$\phi[(x-y)^2(x+y+z), (x-y)/(y-z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}. \qquad ...(1)$$

Taking the first two fractions of (1), $x^2dx = -y^2dy$ or $3x^2dx + 3y^2dy = 0$.

Integrating, $x^3 + y^3 = c_1$, c_1 being an arbitrary as constant. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx-dy}{y^2(x-y)+x^2(x-y)} = \frac{dx-dy}{(x-y)(x^2+y^2)}. \qquad ...(3)$$

Combining the third fraction of (1) with fraction (3), we get

$$\frac{dx - dy}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)} \qquad \text{or} \qquad \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0.$$

Integrating, $\log(x-y) - \log z = \log c_2$ or $(x-y)/z = c_2$. ...(4)

From (3) and (4), solution is $\phi(x^3+y^3, (x-y)/z) = 0$, ϕ being an arbitrary function.

Ex. 3. *Solve* $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}. \qquad \text{...(1)}$$

Taking the last two fractions of (1), we have

$(1/y)dy = (1/z)dz$ so that $(1/y)dy - (1/z)dz = 0$.

Integrating, $\log y - \log z = \log c_1$ or $y/z = c_1$. ...(2)

Choosing x, y, z as mnultipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}. \qquad \text{...(3)}$$

Combining the third fraction of (1) with fraction (3), we have

$$\frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)} = \frac{dz}{-2xz} \qquad \text{or} \qquad \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} - \frac{dz}{z} = 0.$$

Integrating, $\log(x^2+y^2+z^2) - \log z = \log c_2$ or $(x^2+y^2+z^2)/z = c_2$. ... (4)

From (2) and (4) solution is $\phi(y/z, (x^2+y^2+z^2)/z) = 0$, ϕ being an arbitrary function.

Applications of Partial Differential Equations

INTRODUCTION:

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a *boundary value problem.*

METHOD OF SEPARATION OF VARIABLES :

It involves a solution which breaks up into a product of functions each of which contains only one of the variables. The following example explains this method :

Example *Solve (by the method of separation of variables) :*

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$

Solution. Assume the trial solution $z = X(x)Y(y)$ ...(*i*)

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \qquad \text{where } X' = \frac{dX}{dx},\ Y' = \frac{dY}{dy} \text{ etc.}$$

Separating the variables, we get $\dfrac{X'' - 2X'}{X} = -\dfrac{Y'}{Y}$ ...(*ii*)

Since x and y are independent variables, therefore, (*ii*) can only be true if each side is equal to the same constant, a (say).

$$\therefore \qquad \frac{X'' - 2X'}{X} = a, \text{ i.e. } X'' - 2X' - aX = 0 \qquad \ldots(iii)$$

and
$$-Y'/Y = a, \quad \text{i.e.,} \quad Y' + aY = 0 \qquad \ldots(iv)$$

To solve the ordinary linear equation (*iii*), the auxiliary equation is

$$m^2 - 2m - a = 0, \quad \text{whence } m = 1 \sqrt{(1 + a)}.$$

$\therefore$ the solution of (*iii*) is $X = c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}$

and the solution of (*iv*) is $Y = c_3 e^{-ay}$.

Substituting these values of X and Y in (*i*), we get

$$z = \{c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}\} \,.\, c_3 e^{-ay}$$

i.e.,
$$z = \{k_1 e^{[1 + \sqrt{(1+a)}]x} + k_2 e^{[1 - \sqrt{(1+a)}]x}\}\, e^{-ay}$$

which is the required complete solution.

Example 18.2. *Using the method of separation of variables, solve* $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ *where* $u(x, 0) = 6e^{-3x}$.

Solution. Assume the solution $u(x, t) = X(x)T(t)$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad (X' - X)\,T = 2XT'$$

or
$$\frac{X' - X}{2X} = \frac{T'}{T} = k \text{ (say)}$$

$$\therefore \qquad X' - X - 2kX = 0 \quad \text{or} \quad \frac{X'}{X} = 1 + 2k \quad \ldots(i) \quad \text{and} \quad \frac{T'}{T} = k \qquad \ldots(ii)$$

Solving (*i*), $\log X = (1 + 2k)x + \log c$ or $X = ce^{(1+2k)x}$

From (*ii*), $\log T = kt + \log c'$ or $T = c'e^{kt}$

Thus $u(x, t) = XT = cc'\, e^{(1+2k)x}\, e^{kt}$ ...(*iii*)

Now $6e^{-3x} = u(x, 0) = cc'\, e^{(1+2k)x}$

$\therefore$ $cc' = 6$ and $1 + 2k = -3$ or $k = -2$

Substituting these values in (*iii*), we get

$$u = 6e^{-3x}\, e^{-2t} \quad \text{i.e.,} \quad u = 6e^{-(3x + 2t)} \text{ which is the required solution.}$$

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1). **Solve the one dimensional wave equation** $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ **where** $c^2 = \frac{T}{m}$.

Solution:

Assume that a solution of (1) is of the form

$z = X(x)T(t)$ where X is a function of x and T is a function of t only.

Then $$\frac{\partial^2 y}{\partial t^2} = X \,.\, T'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X'' \,.\, T$$

Substituting these in (1), we get $XT'' = c^2X''T$ *i.e.*, $\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$ ...(2)

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations :

$$\frac{d^2X}{dx^2} - kX = 0 \quad ...(3) \quad \text{and} \quad \frac{d^2T}{dt^2} - kc^2T = 0 \quad ...(4)$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say $X = c_1e^{px} + c_2e^{-px}$; $T = c_3e^{cpt} + c_4e^{-cpt}$.

(*ii*) *When k is negative* and $= -p^2$ say $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(*iii*) *When k is zero*. $X = c_9x + c_{10}$; $T = c_{11}t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt}) \quad ...(5)$$
$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad ...(6)$$
$$y = (c_9x + c_{10})(c_{11}t + c_{12}) \quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), *i.e.*, of the form

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{...(8)}$$

is the only suitable solution of the *wave equation*.

---

2). Solve the one dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ **.**

Solution:

Assume that a solution of (1) is of the form

$$u(x, t) = X(x) \,.\, T(t)$$

where X is a function of x alone and T is a function of t only.

Substituting this in (1), we get

$$XT' = c^2X''T, \text{ i.e., } X''/X = T'/c^2T \quad \text{...(2)}$$

Clearly the left side of (2) is a function of x only and the right side is a function of t alone. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{...(3)} \qquad \text{and} \qquad \frac{dT}{dt} - kc^2T = 0 \quad \text{...(4)}$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$X = c_1 e^{px} + c_2 e^{-px},\ T = c_3 e^{c^2 p^2 t} ;$$

(*ii*) *When k is negative* and $= -p^2$, say :

$$X = c_4 \cos px + c_5 \sin px,\ T = c_6 e^{-c^2 p^2 t} ;$$

(*iii*) *When k is zero :*

$$X = c_7 x + c_8,\ T = c_9.$$

Thus the various possible solutions of the heat-equation (1) are

$$u = (c_1 e^{px} + c_2 e^{-px})\, c_3 e^{c^2 p^2 t} \qquad ...(5)$$

$$u = (c_4 \cos px + c_5 \sin px)\, c_6 e^{-c^2 p^2 t} \qquad ...(6)$$

$$u = (c_7 x + c_8) c_9 \qquad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, *i.e.*, u is to decrease with the increase of time t. Accordingly, the solution given by (6), *i.e.*, of the form

$$u = (C_1 \cos px + C_2 \sin px)\, e^{-c^2 p^2 t} \qquad ...(8)$$

is the only suitable solution of the heat equation.

Example 18.10. *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *with boundary conditions* $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$ *and* $u(1, t) = 0$, *where* $0 < x < 1$, $t > 0$.

Solution. The solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ...(*i*)

is $$u(x, t) = (c_1 \cos px + c_2 \sin px)\, e^{-p^2 t} \qquad ...(ii)$$

When $x = 0$, $u(0, t) = c_1 e^{-p^2 t} = 0$ *i.e.*, $c_1 = 0$.

$\therefore$ (*ii*) becomes $$u(x, t) = c_2 \sin px e^{-p^2 t} \qquad ...(iii)$$

When $x = 1$, $u(1, t) = c_2 \sin p \,.\, e^{-p^2 t} = 0$ or $\sin p = 0$

i.e., $p = n\pi$.

$\therefore$ (*iii*) reduces to $u(x, t) = b_n e^{-(n\pi)^2 t} \sin n\pi x$ where $b_n = c_2$

Thus the general solution of (*i*) is $u(x, t) = \sum b_n e^{-n^2\pi^2 t} \sin n\pi x$ ...(*iv*)

When $t = 0$, $3 \sin n\pi x = u(0, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x$

Comparing both sides, $b_n = 3$

Hence from (*iv*), the desired solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x.$$

3). Solve the two dimensional Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution:

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = 0 \qquad \text{...(1)}$$

Let $u = X(x)Y(y)$ be a solution of (1).

Substituting it in (1), we get $\frac{d^2X}{dx^2}Y + X\frac{\partial^2 Y}{\partial y^2} = 0$

or separating the variables, $\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$ ...(2)

Since x and y are independent variables, (2) can hold good only if each side of (2) is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \text{ and } \frac{d^2Y}{dy^2} + kY = 0.$$

Solving these equations, we get

(*i*) *When k is positive* and is equal to p^2, say

$$X = c_1e^{px} + c_2e^{-px},\ Y = c_3 \cos py + c_4 \sin py$$

(*ii*) *When k is negative*, and is equal to $-p^2$, say

$$X = c_5 \cos px + c_6 \sin px,\ Y = c_7e^{py} + c_8e^{-py}$$

(*iii*) *When k is zero* ; $X = c_9x + c_{10}$, $Y = c_{11}y + c_{12}$.

Thus the various possible solutions of (1) are

$$u = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py) \qquad \text{...(3)}$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7e^{py} + c_8e^{-py}) \qquad \text{...(4)}$$

$$u = (c_9x + c_{10})(c_{11}y + c_{12}) \qquad \text{...(5)}$$

Of these we take that solution which is consistent with the given boundary conditions.