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MATH 331.1001
Scribing Week 7
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Lecture 16

Proposition (16.1)**.** Let φ be a group homomorphism.

1. $\mathrm{im}\varphi$ is a subgroup of H.
2. $\ker\varphi$ is a subgroup of G.

Proof. (1.) Use Proposition 8.3, which tells how to find a subgroup, and Proposition 7.1 (which states that for $\varphi : G \to H$, $\varphi(e_G) = e_H$ and $\varphi(x^{-1}) = \varphi(x)^{-1}\,\forall x \in G$). Since $\varphi(e_G) = e_H$, we have $e_H \in \mathrm{im}\varphi \neq \emptyset$ and so $\mathrm{im}\varphi$ is not empty. Let $a, b \in \mathrm{im}\varphi$. We want to show that $ab^{-1} \in \mathrm{im}\varphi$. By definition of $\mathrm{im}\varphi$, $\exists x, y \in G$ such that $\varphi(x) = a$ and $\varphi(y) = b$. Then $ab^{-1} = \varphi(x)\varphi(y)^{-1} = \varphi(x)\varphi(y^{-1}) = \varphi(xy^{-1})$ by definition of group homomorphism. Thus, $ab^{-1} \in \mathrm{im}\varphi$.
(2.) We will use the same previous propositions. By Proposition 7.1, $\varphi(e_G) = e_H$ which implies that $e_H \in \ker\varphi \neq \emptyset$. Let $x, y \in \ker\varphi$. We want to show $xy^{-1} \in \ker\varphi$. Note that $\varphi(y) = e_H$ which implies $\varphi(y)^{-1} = e_H$. Then by definition of group homomorphism, $\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1})$, and since $x \in \ker\varphi$, $\varphi(x)\varphi(y^{-1}) = e_H\varphi(y^{-1}) = e_H^2 = e_H$. Thus $\varphi(xy^{-1}) = e_H \in \ker\varphi$. ∎

Lecture 17

Proposition (17.1)**.** Let $\varphi : G \to H$ be a group homomorphism. Then φ is injective iff $\ker\varphi = \{e_G\}$, i.e. iff $\ker\varphi$ is the trivial subgroup.

Proof. Suppose φ is injective. We want to show $x = e_G$. Let $x \in \ker\varphi$. Then $\varphi(x) = e_H$ by definition, and by Proposition 7.1, $\varphi(e_G) = e_H$. Since φ is injective, $\varphi(x) = e_H$ and $\varphi(e_G) = e_H$ implies $e_G = x$ and thus $\ker\varphi = \{e_G\}$.
Conversely, suppose $\ker\varphi = \{e_G\}$. Assume $\varphi(x) = \varphi(y)$. Then $\varphi(x)\varphi(y)^{-1} = e_H$. By Proposition 7.1,
$\varphi(x)\varphi(y)^{-1} = \varphi(x)\varphi(y^{-1})$ and by definition of group homomorphism $= \varphi(xy^{-1}) = e_H$. Thus $xy^{-1} \in \ker\varphi$, but since $\ker\varphi = \{e_G\}$, $xy^{-1} = e_G$ and by multiplying each side by y on the right, we obtain $x = y$ as desired. ∎

Corollary (17.2)**.** Let $\varphi : G \to H$ be a group homomorphism. Then φ is a group isomorphism iff $\ker\varphi = \{e_G\}$ and $\mathrm{im}\varphi = H$.

Normal Subgroups

(Which, by the way, the term "normal" sucks!)

Idea: Recall given a subgroup $H \leq G$, we can define an equivalence relation on G, $x \sim_H y$, iff $x^{-1}y \in H$. The equivalence classes are the left cosets of H: $[x] = xH := \{xh | h \in H\}$. We denote the set of lefts cosets as $G/H = \{xH | x \in G\}$. Consider the groups $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$.

Remark (17.3). In the above groups, G induces a group operation (and identity) on G/H such that the function $\pi : G \to G/H$, $x \mapsto [x] = xH$ is a group homomorphism!

Example. Let $G = S_3$, $H = \langle\sigma\rangle$, $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$. Note that $|G/H| = [G : H] = 2$ by Lagrange's Theorem.
Then $G/H = \{eH, \tau H\} = \{H, \tau H\}$ for $\tau = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right)$. The goal is to put a group structure on G/H as in Remark 17.3. That is, we want $xH \cdot yH \stackrel{?}{=} xyH$ and $e_{G/H} \stackrel{?}{=} e_{S_3}H = H$. This works! Verify by hand: e.g. $\tau H \cdot \tau H = \tau^2 H = eH = H$. $\tau H = \{\tau, \tau\sigma, \tau\sigma^2\} = \tau\sigma H$.

Example. Let $G = S_3$, $H = \langle\tau\rangle$. Then $G/H = \{H, \sigma H, \sigma^2 H\}$ (we can verify this is correct by hand). Again, we want to define a group operation on G/H. BUUUT it does not work!

Lecture 18

Example (Non-Example)**.** (Continuation from last lecture) Let $G = S_3$, $H = \langle \tau \rangle$, where $\tau = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right)$ and $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$. Then $G/H = \{eH = H, \sigma H, \sigma^2 H\}$ and $\sigma H = \{\sigma e = \sigma, \sigma\tau\}$.
If G/H is indeed a group, we must have that $\sigma H * \tau H = \sigma H$ and $\tau H * \sigma H = \sigma H$ because $\tau H = H$ is our identity element.

1. By definition of *, $\sigma H * \tau H = \sigma\tau H = \{\sigma\tau, \sigma\tau \circ \tau = \sigma\} = \sigma H$. Therefore, 1. is true.
   Note: $xH \cap yH \neq \emptyset$ implies $xH = yH$ because left cosets are equivalence classes of an equivalence relation.
2. $\tau H * \sigma H = \tau\sigma H = \{\tau\sigma, \tau\sigma\tau\}$ by definition of *. But $\tau\sigma \neq \sigma$ and $\tau\sigma \neq \sigma\tau$, therefore $\tau\sigma H \neq \sigma H$ and thus we conclude $G/H = S_3\langle\tau\rangle$ is not a group.

But what went wrong?? We will see that this happened because $\langle\tau\rangle$ is not a normal subgroup.

Definition (18.1)**.** A subgroup $H \leq G$ is **normal** iff for all $g \in G$, the set $gHg^{-1} := \{ghg^{-1} | h \in H\}$ is equal to H. We write $G \trianglelefteq H$

Remark. If $H \trianglelefteq G$, then

1. For all $g \in G$ and for all $h \in H$, $ghg^{-1} \in H$. i.e. $\exists h' \in H$ such that $ghg^{-1} = h'$ (since $gHg^{-1} \subseteq H$), but in general $h' \neq h$.
2. Let $h \in H$. Then $\forall g \in G, \exists h' \in H$ such that $h = gh'g^{-1}$ (since $H \subseteq gHg^{-1}$.

Proposition (18.2 USEFUL)**.** Let $H \leq G$ be a subgroup. Assume $\forall g \in G$, $\forall h \in H$, we have $ghg^{-1} \in H$. Then

1. $\forall g \in G$, $gHg^{-1} \leq H$ and
2. $\forall g \in G$, $H \leq gHg^{-1}$.

i.e. H is normal.

1st Examples/Non-Examples

Proposition (18.3)**.** Let G be abelian. Then every subgroup of G is normal.

Proof. Let H be a subgroup, $g \in G$, and let $h \in H$. Since G is abelian, $ghg^{-1} = hgg^{-1}$. Therefore, $ghg^{-1} = h \in H$. ∎

Proposition (18.4)**.** A subgroup $H \leq G$ is normal iff $\forall x \in G$, $xH = Hx$.

Corollary (18.5)**.** If $H \leq G$ is a subgroup and $[G : H] = 2$, then H is normal.

Example. $H = \langle\sigma\rangle$ is normal in $G = S_3$ since $[S_3 : H] = 2$.

Example (Non-Example)**.** $H := \langle\tau\rangle$ is not normal in $G = S_3$. Observe that $\sigma\tau\sigma^{-1} \notin H$.