Dillan Marroquin
MATH 310.1002: Homework 8
Due. Wed, October 21

1. Use the definition to prove that the function h given by $h(x) = \frac{3}{x}$ for $x \neq 0$ is uniformly continuous on $[\frac{1}{2}, \infty)$.

   Proof. Let $\varepsilon > 0$. We wish to show that $|h(x) - h(a)| < \varepsilon$, or in other words, that $|\frac{3}{x} - \frac{3}{a}| < \varepsilon$. Observe that we can write the previous inequality as $\frac{3|x-a|}{xa} < \varepsilon$. Take $\delta = \frac{\varepsilon}{3}$. Then for every $x, a \in [\frac{1}{2}, \infty)$ with $|x - a| < \delta$, we have that $|h(x) - h(a)| = \frac{3|x-a|}{xa} \leq 3|x - a| < 3\delta = \varepsilon$. So h is uniformly continuous on $[\frac{1}{2}, \infty)$.

   □

2. Prove that the function g given by $g(x) = \cos\frac{\pi}{x}$ for $x \neq 0$ is not uniformly continuous on $(0, 1]$.

   Proof. We will prove this by contradiction. Assume that g is actually uniformly continuous on $(0, 1]$. Then for $\varepsilon = \frac{1}{2}$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < \frac{1}{2}$.
   Consider the sequences $x_n = \frac{1}{2n}$ and $a_n = \frac{1}{2n+1/2}$. Since $\lim|x_n - a_n| = 0$, we know there must exist an $n \in \mathbb{N}$ such that $|x_n - a_n| < \delta$. However $g(x_n) = \cos\frac{\pi}{1/2n} = \cos 2\pi n = 1$ and $g(a_n) = \cos\frac{\pi}{1/2n + 1/2} = 0$. Therefore $|g(x_n) - g(a_n)| = 1 > \frac{1}{2}$. This is a contradiction.

   □

3. Let f be a function with $D_f = \mathbb{R}$. Prove that f is uniformly continuous if there exist positive constants $K, r > 0$ such that for all $x, y \in \mathbb{R}$,
$$|f(x) - f(y)| \leq K|x - y|^r.$$

   Proof. Let $\varepsilon > 0$. We must find a $\delta > 0$ such that $|f(x) - f(y) < \varepsilon|$ whenever $|x - y| < \delta$. Let $K|x - y|^r < \varepsilon$. Then $|x - y|^r < \frac{\varepsilon}{K}$ and $|x - y| < \left(\frac{\varepsilon}{K}\right)^{\frac{1}{r}}$. So, we can choose $\delta = \left(\frac{\varepsilon}{K}\right)^{\frac{1}{r}}$. This finishes the proof.

   □

4. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on the intervals $[1, \infty)$ and $[0, 1]$. Is f uniformly continuous on the interval $[0, \infty)$.

   Proof. Let $\varepsilon > 0$. We wish to show that $|h(x) - h(a)| < \varepsilon$, or that $|\sqrt{x} - \sqrt{a}| < \varepsilon$. Observe that this is equivalent to $\frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \varepsilon$ and that $\frac{|x-a|}{\sqrt{x}+\sqrt{a}} < |x - a|$. If we choose $\delta = \varepsilon$, then we get that $|x - a| < \delta$ implies $|h(x) - h(a)| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < |x - a| < \delta = \varepsilon$. So then f is clearly uniformly continuous on $[1, \infty)$.
   To prove that f is uniform continuous on $[0, 1]$, we determine if f has a continuous extension to $\overline{I}$. Observe that $\overline{I} = [0, 1]$ and that f is clearly continuous over this interval, so then f is uniform continuous on $[0, 1]$.
   Because f has been proven to be uniform continuous on all points on its domain, f is uniform continuous on $[0, \infty)$.

   □

5. Prove that the sequence of functions $\frac{1}{x^2+n}$ converges uniformly on $\mathbb{R}$.

   Proof. Denote $f_n(x) = \frac{1}{x^2+n}$ and let $\varepsilon > 0$. We wish to show that $|f_n(x) - f(x)| < \varepsilon$, i.e. $\left|\frac{1}{x^2+n}\right| < \varepsilon$. Since $n \in \mathbb{N}$ and $x \in \mathbb{R}$, observe that $\left|\frac{1}{x^2+n}\right| = \frac{1}{x^2+n} \leq \frac{1}{n}$. Let $N = \lfloor\frac{1}{\varepsilon}\rfloor + 1$. For any $n \geq N$, we have that $|f_n(x) - f(x)| < \frac{1}{n} < \varepsilon$ whenever $n \geq N$.

   □

6. Prove that the sequence $\frac{\sin nx}{n}$ converges uniformly on $[0, 1]$.

   Proof. Let $\varepsilon > 0$ and observe that $\lim\frac{\sin nx}{n} = \lim\frac{-1}{n} = \lim\frac{1}{n} = 0$. so the sequence $\frac{\sin nx}{n} = f_n$ converges pointwise to $f(x) = 0$.
   We now want to prove that f_n converges uniformly to $f(x) = 0$. Observe that $|f_n(x) - f(x)| = |\frac{\sin nx}{n} - 0| < \frac{1}{n}$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Since $\lim\frac{1}{n} = 0$, then by *Theorem 3.4.6*, f_n is uniformly convergent.

   □

7. Prove that the sequence $\sin\frac{x}{n}$ converges to 0 pointwise on $\mathbb{R}$ but not uniformly.

Proof. First, we will prove that this sequence converges to 0 pointwise. Let $x \in \mathbb{R}$. Then $\lim_n \sin \frac{x}{n} = \sin \lim_n \frac{x}{n} = \sin 0 = 0$. So $\sin \frac{x}{n}$ converges to 0 pointwise.
We will now prove by contradiction that this sequence does not converge uniformly. Denote $f_n = \sin \frac{x}{n}$. We assume that $f_n \to 0 - f(x)$ and let $\varepsilon = \frac{1}{2}$. Then by definition, there must exist some $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon = \frac{1}{2}$ for every $n \geq N$ and for all $x \in \mathbb{R}$. This is equivalent to $|\sin \frac{x}{n} - 0| < \frac{1}{2}$. Taking $x = \frac{\pi n}{2}$ gives $1 < \varepsilon = \frac{1}{2}$. Contradiction. $\square$