MATH 310.1002: Homework 3
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1. Find $\sup_A f$ and $\inf_A f$ in each of the following cases. In each case determine whether the function has a maximum or a minimum on A.

   (a) $f(x) = x(3 - x)$, $A = (-1, 3)$.
   Answer: First, we acknowledge that $f(A) = (-4, \frac{9}{4}]$. Then the $\sup_A f = \frac{9}{4}$ and $\inf_A f = -4$. The function has a maximum on A at $\frac{9}{4}$, however there is no minimum.

   (b) $f(x) = \dfrac{x+3}{x-1}$, $A = (1, 5]$.
   Answer: Again, we acknowledge that $f(A) = [2, \infty]$. Then the $\sup_A f = \infty$ and $\inf_A f = 5$. The minimum of this function on A is 5, however it has no maximum.

2. Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and $t \in T$. Prove that

$$\sup S \leq \inf T.$$

   Proof. To prove this, we first observe that since S is a nonempty subset of $\mathbb{R}$, then there exists an $M \in S$ such that $M \geq s$ for all $s \in S$. So, $\sup S = M$. Next, observe that since T is also a nonempty subset of $\mathbb{R}$, then there exists an $m \in T$ such that $m \leq t$ for all $t \in T$. So $\inf T = m$.
   Since $s \leq t$ for all $s \in S$ and $t \in T$, then $M \leq m$ and $\sup S \leq \inf T$.

   □

3. Let A and B be two nonempty sets of real numbers. Prove that

$$\inf(A \cup B) = \min\{\inf A, \inf B\}.$$

   Proof. We first note that since both A and B are nonempty subsets of $\mathbb{R}$, then there exists an $m \in A$ and an $n \in B$ such that $m \leq x$ for every $x \in A$ and $n \leq y$ for all $y \in B$. This implies that $\inf A = m$ and $\inf B = n$. If we were to take $\min\{\inf A, \inf B\}$, then 2 cases arise.

   Case 1: Suppose that $m < n$. Then, $\min\{m, n\} = m$ and so $\inf(A \cup B) = \min\{\inf A, \inf B\}$. A similar case holds for $n < m$.
   Case 2: Suppose that $m = n$. Then $\min\{m, n\} = m = n$ and $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

   In either case, $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

   □

4. Solve the equation $|2 - x^2| = 1$.
   Answer: This absolute value expression creates two equations, which we will solve separately:

$$\begin{aligned} 2 - x^2 &= 1 \\ -x^2 &= -1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned} \qquad\qquad \begin{aligned} 2 - x^2 &= -1 \\ -x^2 &= -3 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

   So $x = -\sqrt{3}, -1, 1, \sqrt{3}$.

5. Solve the equation $|2 - x^2| \geq 1$.
   Answer: This absolute value inequality creates two inequalities which we will solve separately:

$$\begin{aligned} 2 - x^2 &\geq 1 \\ -x^2 &\geq -1 \\ x^2 &\leq 1 \end{aligned} \qquad\qquad \begin{aligned} 2 - x^2 &\leq -1 \\ -x^2 &\leq -3 \\ x^2 &\geq 3 \end{aligned}$$

After evaluating, we will see that $x \leq -\sqrt{3}$ or $x \geq -1$.

6. Let x, y be real numbers such that $|x-3| < \dfrac{1}{2}$ and $|3-y| < \dfrac{1}{2}$. Prove that $|x-y| < 1$.

   Proof. First, let us solve for x in the first inequality:

$$-\frac{1}{2} < x - 3 < \frac{1}{2}$$
$$\frac{5}{2} < x < \frac{7}{2}.$$

   Now let us solve for y:

$$-\frac{1}{2} < 3 - y < \frac{1}{2}$$
$$\frac{5}{2} < y < \frac{7}{2}.$$

   We now see that $x, y \in (\frac{5}{2}, \frac{7}{2})$ and that the $\sup(\frac{5}{2}, \frac{7}{2}) = \frac{7}{2}$ and also that $\inf(\frac{5}{2}, \frac{7}{2}) = \frac{5}{2}$. Since $\left|\frac{7}{2} - \frac{5}{2}\right| = 1$ and $\frac{5}{2}, \frac{7}{2} \notin (\frac{5}{2}, \frac{7}{2})$, then $|x-y| < 1$.

   $\square$

7. Guess the following limit and prove that your answer is correct by using the definition (Def. 2.1.4):

$$\lim_{n\to\infty} \frac{3n+2}{2n-1}.$$

   Proof. Let $\epsilon > 0$. Then we must show that there exists a δ such that $n > \delta$ implies $\left|\frac{3n+2}{2n-1} - \frac{3}{2}\right| < \epsilon$. Rewriting the expression in absolute values gives us

$$\left|\tfrac{3n+2}{2n-1} - \tfrac{3}{2}\right| = \left|\tfrac{6n-6n+4+3}{4n-2}\right| = \tfrac{7}{4n-2} < \tfrac{7}{4n}.$$

   So, $\left|\frac{3n+2}{2n-1} - \frac{3}{2}\right| < \epsilon$ when $\frac{7}{4n} < \epsilon$, or whenever $n > \frac{7}{4\epsilon}$. Thus, we may choose $\delta = \frac{7}{4\epsilon}$.

   $\square$

8. Use the definition (Def. 2.1.4) to prove that

$$\lim_{n\to\infty} \left[\sqrt{n^2+1} - n\right] = 0.$$

   Proof. Let $\epsilon > 0$ be arbitrary. We want to show that there exists a $\delta \in \mathbb{R}$ where if $0 < \delta < n$, then $\left|(\sqrt{n^2+1} - n) - 0\right| < \epsilon$. First, let us rewrite the expression on the left of the inequality:

$$\left|(\sqrt{n^2+1} - n)\right| = \left|(\sqrt{n^2+1} - n) \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}\right| = \left|\frac{1}{\sqrt{n^2+1}+n}\right| = \frac{1}{\sqrt{n^2+1}+n}.$$

   We can now say that the last expression will always be smaller than $\frac{1}{2n}$, so $|(\sqrt{n^2+1} - n) - 0| < \epsilon$ whenever $\epsilon > \frac{1}{2n}$. Thus, $n > \frac{1}{2\epsilon}$ and we may choose $\delta = \frac{1}{2\epsilon}$.

   $\square$