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MATH 331.1001
Scribing Week 10
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Lecture 24

We will begin with a proof of Corollary 23.3 from last lecture...

Proof Cor 23.3. By Proposition 23.2, all σ commute with one another. Thus $\alpha^l = \sigma_1^l \sigma_2^l \cdots \sigma_k^l \ \forall l \geq 1$. As $\{\sigma_i\}$ is disjoint for all i, σ_i^{-1} and $\sigma_{j\neq 1}^l$ are disjoint also. Then $\alpha^m = e$ iff $\sigma_i^m = e \ \forall i = 1,\dots,k$. Lemma 14.2 implies $|\sigma_i| \big| m$. Thus $\alpha^m = e$ iff m is a multiple of $|\sigma_1|, |\sigma_2|, \dots, |\sigma_k|$. By definition of order, $|\alpha|$ must be the smallest such that $\alpha^m = e$, so $|\alpha^m| = e$, so $|\alpha| = \operatorname{lcm}\{|\sigma_i|\}_{i=1}^k$. ∎

Generators of S_n

Here, we are looking for the set of elements of $\{\sigma_1, \dots, \sigma_n\} \subseteq S_n$ such that every element of S_n can be written as $\sigma_1^{k_1}, \dots, \sigma_n^{k_n}$ for $k_1, \dots, k_n \in \mathbb{Z}$.

Proposition (24.1)**.** Let $n \geq 2$ and $\sigma = (i_1 \dots i_k) \in S_n$ be a k-cycle. Then σ can be written as a product of transpositions. In particular, $\sigma = (i_1 \ i_k)(i_1 \ i_{k-1}) \dots (i_1 \ i_2)$ (i.e. $k-1$ transpositions).

Remark (Ex 24.2). Consider $\sigma = (1\ 2\ 3\ 4) \in S_4$. Then $(1\ 4)(1\ 3)(1\ 2) = (1\ 2\ 3\ 4)$. Note that $1 \mapsto 2$, $2 \mapsto 1 \mapsto 3$, and $3 \mapsto 1 \mapsto 4$.
Note: Decompositions into transpositions are note unique! For example, $(1\ 2)(2\ 3)(1\ 2)(3\ 4)(1\ 2) = (1\ 2\ 3\ 4)$ as well.

Theorem (24.3)**.** Every non-identity element of S_n is uniquely (up to rearrangement) a product of disjoint cycles, each of length 2.

This is how we define our cycle notation: $\alpha = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 5 & 6 & 4 \end{smallmatrix}\right) \in S_6 = (1\ 3)(4\ 5\ 6)$.

Corollary (24.4)**.** For all $n \geq 2$, S_n is generated by the set of transpositions $\{(i\,j) \in S_n | 1 \leq i < j \leq n\}$.

Remark. We also know that S_3 is generated by $\sigma = (1\ 2\ 3)$ and $\tau = (2\ 3)$. We can also show that $S_{n>3}$ is generated by $\sigma = (1\ 2\ 3\ \dots\ n-1\ n)(n-1\ n)$.

Lecture 25

Sign of Permutation

Definition (25.1)**.** Let $\sigma \in S_n$. We say σ is **<u>even/odd</u>** iff σ can be written as an even/odd number of transpositions. We write $\operatorname{sgn}(\sigma) := +1$ if σ is even or -1 if σ is odd.

Example. If $\sigma = (i_1\ i_2\ \dots\ i_k) \in S_n$ is a k-cycle, then σ is even if k is odd, or odd if k is even.
Proposition 24.1 implies $\sigma = (i_1\ i_k)(i_1\ i_{k-1}) \cdots (i_1\ i_2)$.

Theorem (25.2)**.** A permutation can't be both odd and even. In particular, $\operatorname{sgn} : S_n \to \{\pm 1\}$ is well-defined.

Evidence for Theorem 25.2: Let $\vec{e_1}, \dots, \vec{e_n}$ be a standard basis of $\mathbb{R}^n$. So $\vec{e_1} = [1\ 0\ 0\ \cdots\ 0], \vec{e_2} = [0\ 1\ 0\ \cdots\ 0], \dots$. To each $\sigma \in S_n \mapsto n \times n$ matrix P_σ:
$P_\sigma := [\vec{e_{\sigma(1)}}, \vec{e_{\sigma(2)}}, \dots, \vec{e_{\sigma(n)}}]$.

Example.
1. $S_2 = \{(1), (1\ 2)\}$. We have $(1) \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $(1\ 2) \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.
2. $\sigma = (1\ 2\ 3) \in S_3 \mapsto P_\sigma = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$. Note that $\det(P_\sigma) = 1 = \operatorname{sgn}(\sigma)$ since σ is a 3-cycle.

<u>Fact:</u> $S_n \to \mathrm{GL}_n(\mathbb{R})$ is a group homomorphism.

If an $n \times n$ matrix $A = [\vec{a_1}\ \vec{a_2} \dots \vec{a_n}]$, then swapping any 2-columns changes the sign of the determinate.

<u>Fact:</u> $\operatorname{sgn}(\sigma) = \det(P_\sigma)$.