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MATH 310.1002: Homework 12
Due. Mon, November 23

1. Use regular partitions P_n and Theorem 5.1.8 to prove that $f(x) = x$ is integrable on $[a,b]$ and to evaluate

$$\int_a^b x\,\mathrm{d}x.$$

Proof. Define the partition P_n as $P_n = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$. Notice that this divides $[a,b]$ into n subintervals with length $\Delta x = \frac{b-a}{n}$. Also note that $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = x_{k-1}$ and $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k\} = x_k$ since f is strictly increasing on $\mathbb{R}$. Denote $x_k = a + k\Delta x$. Then

$$\begin{aligned} U(f,P_n) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} x_k \cdot \frac{b-a}{n} \\ &= \sum_{k=1}^{n} \left(a + k\cdot\frac{b-a}{n}\right)\frac{b-a}{n} \\ &= \sum_{k=1}^{n} \frac{a(b-a)}{n} + \frac{k(b-a)^2}{n^2} \end{aligned}$$

$$\begin{aligned} L(f,P_n) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} x_{k-1} \cdot \frac{b-a}{n} \\ &= \sum_{k=1}^{n} \left(a + (k-1)\cdot\frac{b-a}{n}\right)\frac{b-a}{n} \\ &= \sum_{k=1}^{n} \frac{a(b-a)}{n} + \frac{(k-1)(b-a)^2}{n^2} \end{aligned}$$

So f is integrable on $[a,b]$ and

$$\int_a^b x\,\mathrm{d}x = \frac{b^2-a^2}{2}.$$

□

2. Use the fact that every nondegenerate interval contains both rational and irrational numbers to prove that the function f given below is not integrable on $[0,1]$.

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$$

Proof. Let $P = \{0 = x_0, x_1, \ldots, x_n = 1\}$ be a partition of $[0,1]$. Consider the interval $[x_{k-1}, x_k]$ for $k \in \{1, 2, \ldots, n\}$. Since $[x_{k-1}, x_k] \cap \mathbb{Q} \neq \varnothing$, then $\sup\{f(x) : x \in [x_{k-1}, x_k]\} = 1 = M_k$ and since $[x_{k-1}, x_k] \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \varnothing$, then $\inf\{f(x) : x \in [x_{k-1}, x_k]\} = 0 = m_k$. So it follows that

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} 0(x_k - x_{k-1}) \\ &= 0. \\ \implies & \underline{\int} f = \sup L(f,P) = 0 \end{aligned}$$

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) = 1 - 0 = 1. \\ \implies & \overline{\int} f = \inf U(f,P) = 1. \end{aligned}$$

Since the upper integral of f does not equal the lower integral of f, f is not integrable on $[0,1]$. □

3. Use regular partitions P_n and Theorem 5.1.8 to prove that $g(x) = x^2$ is integrable on $[2,5]$ and evaluate

$$\int_2^5 x^2\,\mathrm{d}x.$$

Proof. Define the partition P_n to be $P_n = \{2 = x_0, x_1, x_2, \ldots, x_n = 5\}$. This divides the interval $[2,5]$ into n subintervals, each with length $\Delta x = \frac{5-2}{n} = \frac{3}{n}$. Consider the interval $[x_{k-1}, x_k]$ for $k \in \{1, 2, 3, \ldots, n\}$. Then $M_k = \sup\{x^2 : x \in [x_{k-1}, x_k]\} = 25$ and $m_k = \inf\{x^2 : x \in [x_{k-1}, k]\} = 4$ and thus

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = 4\sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= 4\Big[(x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1})\Big] \\ &= 4(5-2) = 12. \end{aligned}$$

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = 25\sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= 25\Big[(x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1})\Big] \\ &= 25(5-2) = 75. \end{aligned}$$

□

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 3x$ and the partition $P = \{0, 1, 2, 3, 4, 5, 6\}$ of the interval $[0, 6]$. Evaluate $L(f, P)$ and $U(f, P)$.
Answer:

$$m_1 = \inf\{x^2 - 3x : x \in [0,1]\} = -2 \qquad m_2 = \inf\{x^2 - 3x : x \in [1,2]\} = -2.25 \qquad m_3 = \inf\{x^2 - 3x : x \in [2,3]\} = -2$$
$$m_4 = \inf\{x^2 - 3x : x \in [3,4]\} = 0 \qquad m_5 = \inf\{x^2 - 3x : x \in [4,5]\} = 4 \qquad m_6 = \inf\{x^2 - 3x : x \in [5,6]\} = 10$$

$$M_1 = \sup\{x^2 - 3x : x \in [0,1]\} = 0 \qquad M_2 = \sup\{x^2 - 3x : x \in [1,2]\} = -2 \qquad M_3 = \sup\{x^2 - 3x : x \in [2,3]\} = 0$$
$$M_4 = \sup\{x^2 - 3x : x \in [3,4]\} = 4 \qquad M_5 = \sup\{x^2 - 3x : x \in [4,5]\} = 10 \qquad M_6 = \sup\{x^2 - 3x : x \in [5,6]\} = 18$$

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = 1(-2 + 2.25 - 2 + 0 + 4 + 10) = 12.25$$
$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = 1(0 - 2 + 0 + 4 + 10 + 18) = 20$$