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MATH 310.1002: Homework 13
Due. Mon, December 7

1. Suppose that g is continuous and nonnegative on $[a,b]$ and that $g(a) > 0$. Show that g is integrable on $[a,b]$ and

$$\int_a^b g(x)\,\mathrm{d}x > 0.$$

Proof. Let $\varepsilon = \frac{g(a)}{2}$. Since g is continuous at $x = a$, then there must exist some $\delta > 0$ such that $|g(x) - g(a)| < \varepsilon$ whenever $|x - a| < \delta$. Observe that we have $\frac{g(a)}{2} \leq g(x)$, for all $x \in [a, a+\delta]$ and

$$\int_a^{a+\delta} \frac{g(a)}{2}\,\mathrm{d}x \leq \int_a^{a+\delta} g(x)\,\mathrm{d}x.$$

It then follows that

$$\int_a^b g(x)\,\mathrm{d}x = \int_a^{a+\delta} g(x)\,\mathrm{d}x + \int_{a+\delta}^b g(x)\,\mathrm{d}x \geq \frac{\delta g(a)}{2} > 0.$$

This completes the proof. □

2. Evaluate the integral

$$\int_{-2}^{2} (|x+1| + |x|)\,\mathrm{d}x$$

Proof. Answer □

3. For $x \in \mathbb{R}$ define $F(x) = \int_0^x f(t)\,\mathrm{d}t$ where the function f is given by

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 3x^2 & \text{if } 0 \leq x < 1 \\ 2x+1 & \text{if } x \geq 1 \end{cases}$$

Prove that F is differentiable at 1 and 2 but not at 0. Find $F'(1)$. You may assume that f is integrable on any closed bounded interval $[a,b]$.

Proof. proof □

4. Show that the given function g is differentiable on its natural domain and find its derivative.

$$g(x) = \int_{x^3}^{e^x} \cos t^2\,\mathrm{d}t \quad \text{for } x \in \mathbb{R}.$$

Proof. Denote $f : \mathbb{R} \to \mathbb{R}$, $f(t) = \cos t^2\,\mathrm{d}t$ and observe that this function is continuous on $\mathbb{R}$. Let $x \in \mathbb{R}$ and observe that f is continuous on $[x^3, e^x]$ or on $[e^x, x^3]$. Since f is continuous on $\mathbb{R}$, then it has an antiderivative $F : \mathbb{R} \to \mathbb{R}$, $F(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t$ by the Fundamental Theorem of Calculus. Then $g(x) = \int_{x^3}^{e^x} f(t)\,\mathrm{d}t = F(e^x) - F(x^3)$. Since F is differentiable, then g is differentiable by composition. □

5. Suppose that $f : [1,2] \to \mathbb{R}$ is continuous and that $\int_1^2 x^k f(x)\,\mathrm{d}x = 5 + k^2$ for $k = 0,1,2$. Evaluate $\int_1^4 f(\sqrt{x})\,\mathrm{d}x$ and $\int_0^1 x^2 f(x+1)\,\mathrm{d}x$.

Proof. proof □

6. Prove that the function h given below has a minimum value and find it.

$$h(x) = \int_1^x (\ln t)^3\,\mathrm{d}t \quad \text{for all } x > 0.$$

Proof. proof ☐

7. Let f be defined on $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by $f(x) = \tan x$ for all $x \in I$. Given that $f(I) = \mathbb{R}$ and $f'(x) = \sec^2 x$ for all $x \in I$, prove that f has a differentiable inverse defined on $\mathbb{R}$ and

$$f^{-1}(x) = \int_0^x \frac{1}{t^2+1}\,\mathrm{d}t \quad \text{for all } x \in \mathbb{R}.$$

Proof. proof ☐

8. Evaluate the improper integrals

$$\int_1^\infty \frac{1+x}{x^3}\,\mathrm{d}x, \qquad\qquad \int_{-\infty}^0 x^2 e^{x^3}\,\mathrm{d}x.$$

Answer: We first evaluate the left integral by integrating by parts. Let $u = 1+x$, $\mathrm{d}u = 1$, $v = -\frac{1}{2x^2}$, and $\mathrm{d}v = x^{-3}$. Then

$$\begin{aligned}
\int_1^\infty \frac{1+x}{x^3}\,\mathrm{d}x &= (1+x)\left(-\frac{1}{2x^2}\right)\Bigg|_1^\infty - \int_1^\infty \left(-\frac{1}{2x^2}\right)\cdot 1\,\mathrm{d}x \\
&= \frac{-x-1}{2x^2}\Bigg|_1^\infty + \int_1^\infty \frac{1}{2x^2}\,\mathrm{d}x \\
&= \frac{1}{2}\left(-\frac{1}{x} - \frac{1}{x^2}\right)\Bigg|_1^\infty - \left(\frac{1}{2x}\right)\Bigg|_1^\infty \\
&= 1 + \frac{1}{2} = \frac{3}{2}.
\end{aligned}$$

To solve the right integral, we let $u = x^3$ and $\frac{1}{3}\,\mathrm{d}u = x^2\,\mathrm{d}x$.

$$\begin{aligned}
\int_{-\infty}^0 x^2 e^{x^3}\,\mathrm{d}x &= \frac{1}{3}\int_{-\infty}^0 e^u\,\mathrm{d}u \\
&= \frac{1}{3}e^u\Bigg|_{-\infty}^0 = \frac{1}{3}.
\end{aligned}$$

9. Establish which of the following functions are improperly integrable

   (a) $f(x) = \sin x$ on $(0,\infty)$.

   Proof. Let $c,d \in (0,\infty)$ such that $c < d$. Observe that f is continuous on $\mathbb{R}$ and thus is Riemann integrable on $[c,d]$. We now check to see if the limit exists:

   $$\lim_{c\to 0}\lim_{d\to\infty}\int_c^d \sin x\,\mathrm{d}x =$$

   ☐

   (b) $f(x) = \frac{1}{x^2}$ on $[-1,1]$.
   Answer:

   (c) $f(x) = \ln(\sin x)$ on $(0,1)$.
   Answer: