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MATH 331.1001
Scribing Week 8
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Lecture 19

Theorem (19.1)**.** Let $H \trianglelefteq G$ be a normal subgroup.

1. The set of left cosets G/H is a group with binary operation $xH * yH = xyH$ and identity element $e_{G/H} := e_G H = H$.
2. The group structure from 1. makes $\pi : G \to G/H$, $\pi(g) := gH$ a surjective group homomorphism.

Proof.

1. The main point is to check that the binary operation is well-defined (since all group axioms will follow immediately from those on G). Suppose $x'H = xH$ and $y'H = yH$. WTS $x'y'H = xyH$. The first two equalities imply $\exists h, \tilde{h} \in H$ such that $x' = xh$ and $y' = yh$. WTS $\exists h'$ such that $x'y' = xyh$. Consider $x'y' = xhy\tilde{h} = xehy\tilde{h} = xyy^{-1}hy\tilde{h}$. Since $H \trianglelefteq G$, $ghg^{-1} \in H$, where $g := y^{-1}$. Therefore $\exists h' \in H$ such that $y^{-1}hy = h'$ which implies that the RHS$= xyh'\tilde{h} \in xyH$. Thus $x'y' \in xyH$.
2. This is straightforward: $\pi(xy) = xyH = xH * yH = \pi(x) * \pi(y)$. This is surjective vacuously.

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Basic Examples of Quotient Groups

Remark. Groups of the form G/H are called **Quotient/Factor Groups**.

Example. Let $G = S_3$, $H = \langle \sigma \rangle$. Note $G/H = \{H, \tau H\}$ is a group of order $[G : H] = 2$. So $G/H \cong \mathbb{Z}/2$ by theorem 15.2.

Proposition (19.2)**.** Let $\varphi : G \to G'$ be a group homomorphism. Then $\ker \varphi$ is a normal subgroup of G.

Remark. Proposition 19.2 implies that if $H \leq G$ is a subgroup and there exists a function $\varphi : G \to G'$ such that $H = \ker \varphi$, then $H \trianglelefteq G$.

Proposition (19.3)**.** Let $H \trianglelefteq G$ be a normal subgroup. Then the kernel of $\pi : G \to G/H$ is H.

Lecture 20

Notation: Let G be abelian, $H \leq G$ a subgroup. If we write G additively, then we write cosets of H as $a + H = aH$. We write the group operation in G/H as $x + H + y + H := (x + y) + H$.
i.e. $\mathbb{Z}/n$: $k + n\mathbb{Z}$ and for $\mathbb{Q}/\mathbb{Z}$: $\frac{a}{b} + \mathbb{Z}$.

Example of Analyzing G/H via Proposition 19.2/19.3

- Consider $\det : GL_2(\mathbb{R}) \to \mathbb{R}^\times$.
  $\ker(\det) = \{A \in GL_2 | \det A = 1\} := SL_2(\mathbb{R})$. Proposition 19.2 implies $\ker(\det) \trianglelefteq GL_2$.
  Note: SL_2 is not abelian.
- We will be analyzing $GL_2/SL_2 := \{ASL_2 | A \in SL_2\}$.
- We will also analyze left cosets:

Lemma (20.1)**.** Let $H \leq G$ be a subgroup. Then $xH = yH$ iff $x^{-1}y \in H$.

Proof. This will be #1 on PS 5. He tricked us! ∎

Observations:

1. By the above Lemma, $ASL_2 = BSL_2$ iff $A^{-1}B \in SL_2$ iff $\det(A^{-1}B) = 1$ iff $\det A^{-1} \det B = 1$ iff $\det A = \det B$.

2. $\forall A \in GL_2$, $ASL_2 = \begin{bmatrix} \det A & 0 \\ 0 & 1 \end{bmatrix} SL_2$ which implies that as a set, $GL_2/SL_2 := \left\{ \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} SL_2 \middle| r \in \mathbb{R} \setminus \{0\} \right\}$.

3. Note: $\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} SL_2 = \begin{bmatrix} r' & 0 \\ 0 & 1 \end{bmatrix} SL_2$ iff $r = r'$.

But what about the group operation? By definition,
$\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} SL_2 * \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} SL_2 = \begin{bmatrix} rs & 0 \\ 0 & 1 \end{bmatrix} SL_2 = \begin{bmatrix} sr & 0 \\ 0 & 1 \end{bmatrix} SL_2 = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} SL_2 * \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} SL_2$. Therefore, GL_2/SL_2 is abelian!
This is AMAZINGLY important because G and H are non-abelian, thus G, H being non-abelian does NOT imply that G/H is non-abelian.

Observation 1 implies that we have a well-defined function $\overline{\det} : GL_2/SL_2 \to \mathbb{R} \setminus \{0\}$, $\overline{\det}\left(\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} SL_2 \right) := r = \det\left(\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} \right)$.
$\overline{\det}$ is a group homomorphism. Also, it is surjective since $\det : GL_2 \to \mathbb{R}^\times$ is surjective. It is also injective by Observation 3. Therefore, $\overline{\det}$ is a group isomorphism and thus $GL_2/SL_2 \cong \mathbb{R}^\times$.

1st Isomorphism for Groups

Theorem (20.2)**.** Let $\varphi : G \to G'$ be a group homomorphism. Then the function $\overline{\varphi} : G/\ker\varphi \to \mathrm{im}\varphi$, $\overline{\varphi}(x\ker\varphi) := \varphi(x)$ is a group isomorphism.

Proof. (See Paulin). ■

Example. Let $\varphi : \mathbb{Z} \to S_3$, $\varphi(k) := \sigma^k$ be a group homomorphism such that $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Hence, $\mathrm{im}\varphi = \langle \sigma \rangle = \{e, \sigma, \sigma^2\}$. Then $\ker\varphi = \{k \in \mathbb{Z} | \sigma^k = e\}$. Lemma 14.3 says if $\sigma^k = e$, then $|\sigma| \big| k$ which implies $k \in 3\mathbb{Z}$. By Theorem 20.2, $\mathbb{Z}/3 \cong \langle \sigma \rangle$ on $\overline{\varphi}$.

Q: 2 subgroups from $\varphi : G \to H$: $\ker\varphi \leq G$ and $\mathrm{im}\varphi \leq H$. We've already seen that $\ker\varphi$ is always normal, but what about $\mathrm{im}\varphi$ in H??