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MATH 310.1002: Homework 11
Due. Mon, November 9

1. Use the Mean Value Theorem to prove that

$$\sqrt{x}-\sqrt{y}<\frac{x-y}{2} \quad \text{if} \quad x>y\geq 1.$$

Proof. Consider the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=2\sqrt{x}-x+y-2\sqrt{y}<0$. Proving the above claim is the same as proving that $f(x)<0$ when $x>y\geq 1$. To prove this, we differentiate $f(x)$ with respect to x and find the critical points: $\frac{\mathrm{d}}{\mathrm{d}x}f(x)=\frac{1}{\sqrt{x}}-1$ and the only critical point is at $x=1$ since $f'(x)=0=\frac{1}{\sqrt{(1)}}-1 \implies x=1$. Choose $x_1=2$ as a test point. Then $f'(2)=\frac{1}{\sqrt{2}}-1<0$ and so the function is decreasing when $x\in(1,\infty)$. By the Mean Value Theorem, the function $f(x)<0$ when $x>y\geq 1$.

□

2. Prove that $e^x<1+x+\frac{x^2}{2}$ for all $x<0$.

Proof. We will prove this by instead proving that $f(x)=\frac{x^2}{2}+x-e^x+1>0$ for all $x<0$. We first take the derivative of f and find its critical points. Observe that f is continuous, so the only critical points are the values of x that make $f'(x)=0$. Observe that $f'(x)=x-e^x+1=0 \iff x=0$. Choosing $x=1$ gives $f'(1)=2-e<0$ and choosing $x=-1$ gives $f'(-1)=-\frac{1}{e}<0$, so f is decreasing on $(-\infty,0)\cup(0,\infty)$. However, since $f(0)=0$ and f is decreasing, then $f(x)>0$ for all $x<0$.

□

3. Show that the function $f(x)=\ln(2x+3)$ is uniformly continuous on $(0,\infty)$.

Proof. First, observe that $|f'(x)|=\left|\frac{2}{2x+3}\right|=\frac{2}{2x+3}$ for $x\in(0,\infty)$. We must find an $M>0\in\mathbb{R}$ with the property that $|f'(x)|\leq M$ for all $x\in(0,\infty)$. To do this, we take the limit as x approaches infinity and as x approaches 0^+.

$$\begin{aligned}\lim_{x\to\infty}f(x)&=\lim_{x\to\infty}\frac{2}{2x+3}\\&=\frac{2}{\infty}=0\end{aligned} \qquad\qquad \begin{aligned}\lim_{x\to0^+}f(x)&=\lim_{x\to0^+}\frac{2}{2x+3}\\&=\frac{2}{2(0)+3}=\frac{2}{3}\end{aligned}$$

Thus, we may choose $M=\frac{2}{3}$. By Theorem 4.3.9, f is uniformly continuous on $(0,\infty)$.

□

4. Use L'Hospital's Rule to evaluate the following limit. Check that all hypotheses are satisfied.

$$\lim_{x\to\infty}x\sin\frac{1}{x}$$

Answer: We will first rewrite this limit to be $\lim_{x\to\infty}x\sin\frac{1}{x}$ and observe that this is differentiable. Since $\frac{1}{x}\neq0$ nor does its derivative equal 0, we may apply L'Hospital's Rule:

$$\begin{aligned}\lim_{x\to\infty}x\sin\frac{1}{x}&=\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\frac{0}{0} \qquad \textit{Indeterminate}\\&\xrightarrow{L.R.}\lim_{x\to\infty}\frac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to\infty}\cos\frac{1}{x}=1.\end{aligned}$$

5. Define $x^x=e^{x\ln x}$ for $x>0$. Prove that $\lim_{x\to0^+}x^x=1$.

Proof. We prove this limit is 1 by finding $\lim_{x\to 0+} \exp(x\ln x) = \exp\left(\lim_{x\to 0^+} x\ln x\right)$. We know that the answer has a base e, so we now work on the inner limit:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{\infty} \qquad \textit{Indeterminate, apply L'Hospital's Rule}$$

$$\xrightarrow{L.R.} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \cdot \frac{x}{x} = \lim_{x\to 0^+} \frac{1}{-\frac{1}{x}} = \frac{1}{-\infty} = 0.$$

Thus, $\lim_{x\to 0^+} x^x = e^0 = 1$.

$\square$