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MATH 310.1002: Homework 7
Due. Wed, October 14

1. Let f given by $f(x) = x^3$ for all $x \in \mathbb{R}$. Use the *definition* to prove that f is continuous at 2.

   Proof. Let $\varepsilon > 0$. We want to show that $|x^3 - 8| < \varepsilon$. We expand the left side of the inequality and solve for $|x-2|$.

   $$|(x-2)(x^2+2x+4)| < \varepsilon$$
   $$|x-2| < \frac{\varepsilon}{|x^2+2x+4|}.$$

   Choose $\delta < 1$. Then $|x-2| < \delta$ means $1 < 2-\delta < x < 2+\delta < 3$ and so $7 < x^2+2x+4 < 19$. So then $\frac{\varepsilon}{19} < \frac{\varepsilon}{|x^2+2x+4|} < \frac{\varepsilon}{7}$. Take $\delta = \min\{1, \frac{\varepsilon}{19}\}$. Then $|x-2| < \delta$ implies $|x^3 - 8| < \varepsilon$ and so f is continuous at 2.

   □

2. Prove that the function g defined below is not continuous at 0 but is continuous everywhere else.

   $$g(x) = \begin{cases} \cos\frac{1}{x} & \text{if } x \neq 0. \\ 0 & \text{if } x = 0. \end{cases}$$

   Proof. First we will assume, to the contrary, that g is continuous at 0. Then we want to show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|g(x) - g(0)| < \varepsilon$ when $|x - 0| < \delta$. Since we are only concerned about the continuity at $x = 0$, $g(x) = 0$ according to the rules of g. However, $|g(x) - g(0)| = |0 - 0| = 0$, so there exists no such $\varepsilon > |g(x) - g(0)|$. Contradiction.
   To prove g is continuous everywhere else, we only need to prove that $\cos\frac{1}{x}$ is continuous. Because $\cos\frac{1}{x}$ is a composition of continuous functions, g is continuous everywhere besides $x = 0$.

   □

3. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x - 2$ if $x \in \mathbb{Q}$ and $f(x) = x^2$ if $x \notin \mathbb{Q}$. Determine the points where f is continuous.

   Proof. We will let $x = a$ be a continuous point on $f(x)$. We know that there exists a sequence of rational numbers $(x_n)_n$ where $\lim_n x_n = a$. Then $\lim_n f(x_n) = \lim_n 3x_n - 2 = 3a - 2$.
   We also know that there exists a sequence of irrational numbers $(y_n)_n$ such that $\lim_n y_n = a$. Then $\lim_n f(y_n) = \lim_n y_n^2 = a^2$.
   For $f(x)$ to be continuous at $x = a$, we want to have $a^2 = 3a - 2$. Solving this equation gives $a \in \{1, 2\}$ and thus f is not continuous on $\mathbb{R} \setminus \{1, 2\}$ and continuous everywhere else.

   □

4. Use the Intermediate Value Theorem to prove that $x^3 = x^2 + 2x + 3$ for some $x \in (1, 3)$.

   Proof. Let's consider the function $g : [1, 3] \to \mathbb{R}$, $g(x) = x^3 - x^2 - 2x - 3$. Observe that this is a continuous function and that $g(1) = -5$ and $g(3) = 9$. Then by the Intermediate Value Theorem, there is an $x \in (1, 3)$ where $g(x) = 0$ and where $x^3 = x^2 + 2x + 3$.

   □

5. Let f be a continuous function with domain $D_f = [a, b]$ and suppose that $f(a) < f(b) < f(c)$ for some $c \in (a, b)$. Prove that f is not one-to-one.

   Proof. First let $y \in (f(b), f(c))$. Then there is an $x_1 \in (c, b)$ such that $f(x_1) = y$. Observe that $y \in (f(a), f(c))$, so then there exists $x_2 \in (a, c)$ such that $f(x_2) = y$. However $x_2 < c < x_1$, so then $x_1 \neq x_2$. Since $f(x_1) = f(x_2)$ but $x_1 \neq x_2$, f is not one-to-one.

   □

6. Let $f : [a, b] \to [a, b]$ be a continuous function. Prove there is $x \in [a, b]$ such that $f(x) = x$.

Proof. Consider the function $g : [a,b] \to \mathbb{R}$, $g(x) = f(x) - x$. We know that this function is continuous and that $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$. So then by the Intermediate Value Theorem, there must exist an $x \in [a,b]$ where $g(x) = 0$ such that $f(x) = x$.

$\square$