STAT 461.1002: Homework 8
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19 November, 2020

Section 3.11 Problems

1. 3.11.4. Five cards are dealt from a standard poker deck. Let X be the number of aces received, and Y the number of kings. Compute $P(X = 2|Y = 2)$.
   Answer: We first recognize that $p_Y(y)$ and $P_{X,Y}(x,y)$ both follow hypergeometric distributions.
   Let w = # of kings in a 52 deck = 4, N = # of cards in a standard deck = 52, n = # of cards dealt = 5. Then

$$p_Y(2) = \frac{\binom{4}{y=2}\binom{48}{5-(y=2)}}{\binom{52}{5}} = \frac{6\cdot 17296}{2598960} = \frac{2162}{54145}$$

$$p_{X,Y}(2,2) = \frac{\binom{4}{x=2}\binom{4}{y=2}\binom{44}{5-(2)-(2)}}{\binom{52}{5}} = \frac{6^2(44)}{2598960} = \frac{33}{54145}$$

$$\therefore P(X=2|Y=2) = \frac{p_{X,Y}(2,2)}{p_Y(2)} = \frac{33/54145}{2162/54145} = \frac{33}{2162} \approx 0.0153.$$

2. 3.11.5. Given that two discrete random variables X and Y follow the joint pdf $p_{X,Y}(x,y) = k(x+y)$, for $x = 1,2,3$ and $y = 1,2,3$,
   (a) Find k.
   Answer: We use the fact that the sum of all values of x and y in our pdf must equal 1:

$$1 = \sum_{x=1}^{3}\sum_{y=1}^{3} k(x+y) = k\sum_{x=1}^{3}(x+1)+(x+2)+(x+3) = k\sum_{x=1}^{3} 3x+6$$
$$= k[(3+6)+(6+6)+(9+6)] = 36k \implies k = \frac{1}{36}.$$

   (b) Evaluate $p_{Y|x}(1)$ for all values of x for which $p_X(x) > 0$.
   Answer: We use the fact that the marginal pdf of x is equal to the sum of all probabilities of y. So,

$$p_X(x) = \sum_{y=1}^{3}\frac{1}{36}(x+y) = \frac{1}{36}[(x+1)+(x+2)+(x+3)] = \frac{1}{36}(3x+6) = \frac{x+2}{12}.$$

$$\therefore p_{Y|x}(1) = \frac{p_{X,Y}(x,1)}{p_X(x)} = \frac{\frac{x+1}{36}}{\frac{x+2}{12}} = \frac{x+1}{3x+6} \qquad \text{Now plug in } x = 1,2,3 \text{ separately\ldots}$$

$$p_{Y|1}(1) = \frac{(1)+1}{3(1)+6} = \frac{2}{9} \qquad p_{Y|2}(1) = \frac{(2)+1}{3(2)+6} = \frac{1}{4} \qquad p_{Y|3}(1) = \frac{(3)+1}{3(3)} = \frac{2}{9}.$$

3. 3.11.9. Let X and Y be independent Poisson random variables where $p_x(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ and $p_Y(k) = e^{-\mu}\frac{\mu^k}{k!}$ for $k = 0,1,\ldots$. Show that the conditional pdf of X given that $X+Y = n$ is binomial with parameters n and $\frac{\lambda}{\lambda+\mu}$. (Hint: See Question 3.8.3.)
   Answer: We want to show that $P(X = k|X+Y)$, the conditional pdf of X given $X+Y = n$, is binomial. We use the fact that X and Y are independent and the definition of conditional probability to assist us.

$$P(X=k|X+Y) = \frac{P(X=k)\cdot P(X+Y)}{P(X+Y)} = \frac{P(X=k)\cdot P(X+Y)}{\sum_{i=0}^{n}P(X=i)\cdot P(Y=n-i)}$$
$$= \frac{\frac{e^{-\lambda}\lambda^k}{k!}\cdot\frac{e^{-\mu}\mu^{n-k}}{(n-k)!}}{\sum_{i=0}^{n}\frac{e^{-\lambda}\lambda^i}{i!}\cdot\frac{e^{-\mu}\mu^{n-i}}{(n-i)!}} = \frac{\frac{\lambda^k\mu^{n-k}}{k!(n-k)!}}{\frac{1}{n!}\sum_{i=0}^{n}\frac{n!\lambda^i\mu^{n-i}}{i!(n-i)!}} = \cdots = \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k}$$

   Observe that this is binomial, with parameters $\left(n, \frac{\lambda}{\lambda+\mu}\right)$.

4. 3.11.16. Suppose that X and Y are distributed according to the joint pdf

$$f_{X,Y}(x,y) = \frac{2}{5} \cdot (2x+3y), \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

Find

(a) $f_X(x)$.
Answer:

$$f_X(x) = \int_0^1 f_{X,Y}(x,y)\,\mathrm{d}y = \frac{2}{5}\int_0^1 2x+3y\,\mathrm{d}y = \frac{2}{5}\left(2xy + \frac{3y^2}{2}\right)\Bigg|_0^1 = \frac{4x+3}{5}.$$

(b) $f_{Y|x}(y)$.
Answer:

$$f_{Y|x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{4x+6y}{5}}{\frac{4x+3}{5}} = \frac{4x+6y}{4x+3}.$$

(c) $P\left(\frac{1}{4} \le Y \le \frac{3}{4} | X = \frac{1}{2}\right)$.
Answer:

$$P(1/4 \le Y \le 3/4 \mid X = 1/2) = \int_{1/4}^{3/4} f_{Y|1/2}(y)\,\mathrm{d}y = \int_{1/4}^{3/4} \frac{4(\frac{1}{2})+6y}{4(\frac{1}{2})+3}\,\mathrm{d}y = \frac{2}{5}\int_{1/4}^{3/4} 1+3y\,\mathrm{d}y = \frac{2}{5}\left(y+\frac{3y^2}{2}\right)\Bigg|_{1/4}^{3/4}$$
$$= \cdots = \frac{1}{2}.$$

(d) $E(Y|x)$
Answer:

$$E[Y|x] = \int_0^1 y \cdot f_{Y|x}(y)\,\mathrm{d}y = \int_0^1 \frac{4xy+6y^2}{4x+3}\,\mathrm{d}y = \frac{2}{4x+3}\int_0^1 3y^2+2xy\,\mathrm{d}y = \frac{2}{4x+3}\left(y^3+xy^2\right)\Bigg|_0^1$$
$$= \cdots = \frac{2x+2}{4x+3}.$$

5. **(461 only)** 3.11.17. If X and Y have the joint pdf

$$f_{X,Y}(x,y) = 2, \quad 0 \le x < y \le 1$$

find $P\left(0 < X < \frac{1}{2} | Y = \frac{3}{4}\right)$.
Answer: We first find $f_Y(y)$, then use this to compute $f_{X|y}(x)$.

$$f_Y(y) = \int_0^y 2\,\mathrm{d}x = 2x\Big|_0^y = 2y.$$
$$f_{X|y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2y} = \frac{1}{y}.$$
$$P(0 < X < 1/2 | Y = 3/4) = \int_0^{\frac{1}{2}} f_{X|3/4}(x)\,\mathrm{d}x = \int_0^{\frac{1}{2}} \frac{4}{3}\,\mathrm{d}x = \frac{4x}{3}\Bigg|_0^{\frac{1}{2}} = \frac{2}{3}.$$

Section 3.12 Problems

6. 3.12.3. (Hint: See Examples in section 3.12) Find $E[e^{3X}]$ if X is a binomial random variable with $n = 10$ and $p = \frac{1}{3}$.
Answer: We apply the formula for the MGF of a binomial distribution, where $n = 10$, $p = 1/3$.

$$M_X(t) = (1-p+pe^t)^n = \left(1 - \frac{1}{3} + \frac{1}{3}e^t\right)^{10} = \left(\frac{2+e^t}{3}\right)^{10}$$
$$M_X(3) = \left(\frac{2+e^3}{3}\right)^{10} \approx 467{,}591{,}999.$$

7. 3.12.4. (Hint: Carefully compare to textbook example 3.12.1) Find the moment-generating function for the discrete random variable X whose probability function is given by

$$p_X(k) = \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right), \quad k = 0, 1, 2, \ldots$$

Answer: We apply the definition of a MGF:

$$\begin{aligned}
M_X(t) = E[e^{tw}] &= \sum_{\text{all } k} e^{tk} p_W(k) = \sum_{k=0}^{\infty} e^{kt} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right) = \frac{1}{4} \sum_{k=0}^{\infty} \left(\frac{3e^t}{4}\right)^k \\
&= \frac{1}{4}\left(\frac{1}{1 - \frac{3e^t}{4}}\right) \qquad \text{(By properties of a geometric series, where } r = \frac{3e^t}{4}\text{)} \\
&= \frac{1}{4 - 3e^t}.
\end{aligned}$$

8. 3.12.8. Let Y be a continuous random variable with $f_Y(y) = ye^{-y}$, $0 \le y$. Show that $M_Y(t) = \frac{1}{(1-t)^2}$.
Answer:

$$M_Y(t) = \int_{-\infty}^{\infty} e^{ty} y e^{-y} \, \mathrm{d}y = \int_0^{\infty} y e^{y(t-1)} \, \mathrm{d}y$$

Now we integrate by parts with $u = y$, $\mathrm{d}u = 1$, $v = \frac{\exp y(t-1)}{t-1}$, $\mathrm{d}v = e^{y(t-1)}$.

$$\begin{aligned}
\int_0^{\infty} y e^{y(t-1)} \, \mathrm{d}y &= \frac{ye^{y(t-1)}}{t-1} - \int_0^{\infty} \frac{e^{y(t-1)}}{t-1} \, \mathrm{d}y = -\frac{1}{t-1} \int_0^{\infty} e^{y(t-1)} \, \mathrm{d}y \\
&= -\frac{1}{t-1} \left(\frac{e^{y(t-1)}}{t-1}\right)\Bigg|_0^{\infty} = \frac{1}{(t-1)^2}.
\end{aligned}$$

9. 3.12.12. What is $E(Y^4)$ if the random variable Y has the moment-generating function $M_Y(t) = (1 - \alpha t)^{-k}$?
Answer: We must find the 4th derivative of the MGF and then evaluate this at $t = 0$.

$$\begin{aligned}
M_Y'(t) &= -k(1-\alpha t)^{-k-1}(-\alpha) = \alpha k (1-\alpha t)^{-k-1} \\
M_Y''(t) &= (\alpha k)(-k-1)(1-\alpha t)^{-k-2}(-\alpha) = (\alpha^2 k^2 + \alpha^2 k)(1-\alpha t)^{-k-2} \\
M_Y^{(3)}(t) &= (-k-2)(\alpha^2 k^2 + \alpha^2 k)(1-\alpha t)^{-k-3}(-\alpha) = (\alpha^3 k^3 + 3\alpha^3 k^2 + 2\alpha^3 k)(1-\alpha t)^{-k-3} \\
M_Y^{(4)}(t) &= (-k-3)(\alpha^3 k^3 + 3\alpha^3 k^2 + 2\alpha^3 k)(1-\alpha t)^{-k-4}(-\alpha) = (\alpha^4 k^4 + 6\alpha^4 k^3 + 11\alpha^4 k^2 + 6\alpha^4 k)(1-\alpha t)^{-k-4} \\
M_Y^{(4)}(0) &= (\alpha^4 k^4 + 6\alpha^4 k^3 + 11\alpha^4 k^2 + 6\alpha^4 k) 1^{-k-4} = E[Y^4].
\end{aligned}$$

10. 3.12.20. Calculate $P(X \le 2)$ if $M_X(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^5$.
Answer: We first note that this MGF resembles that of a binomial random variable where $M_X(t) = (1 - p - pe^t)^n$. In fact, if we let $n = 5$ and $p = 3/4$, we can calculate this probability. We will calculate $P(X = 0)$, $P(X = 1)$, and $P(X = 2)$ separately.

$$\begin{aligned}
&\text{Note: } p_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \\
&P(X = 0) = \frac{5!}{0!(5-0)!}\left(\frac{3}{4}\right)^0 \cdot \left(1 - \frac{3}{4}\right)^{5-0} \approx 0.001 \\
&P(X = 1) = \frac{5!}{1!(5-1)!}\left(\frac{3}{4}\right)^1 \cdot \left(1 - \frac{3}{4}\right)^{5-1} \approx 0.015 \\
&P(X = 2) = \frac{5!}{2!(5-2)!} \approx \left(\frac{3}{4}\right)^2 \cdot \left(1 - \frac{3}{4}\right)^{5-2} \approx 0.088 \\
&\therefore P(X \le 2) \approx 0.001 + 0.015 + 0.088 = 0.104.
\end{aligned}$$