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MATH 331.1001
Scribing Week 12
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Lecture 29

Prop 28.4. 1. Want to show $1_S \in \text{im}\varphi$. By definition of φ, $\varphi(1_R) = 1_S$ which implies $1_S \in \text{im}\varphi$. (The rest of this proof is similar to the proof of Proposition 16.1 for groups.

2. ($\Longrightarrow$) Suppose $\ker\varphi$ is a subring. By definition of subring, $1_R \in \ker\varphi$. Therefore $\varphi(1_R) = 0_S$. On the other hand, $\varphi(1_R) = 1_S$ by definition of ring homomorphism. Let $s \in S$. Then $s = s \cdot 1_S = s \cdot \varphi(1_R) = s \cdot 0_S$. Thus by Proposition 26.3, $s = 0_s$ and therefore $S = \{0_S\}$.
($\Longleftarrow$) Suppose $S = \{0_S\}$. Then $\forall r \in R$, $\varphi(r) = 0_S$. Therefore $\ker\varphi = R$. Every ring is a subring of itself.

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Definition (29.1)**.** Let R be a ring. A subset $I \subseteq R$ is an **<u>ideal</u>** iff

1. I is an additive subgroup of R, i.e. $(I, +, 0_R) \leq (R, +, 0_R)$ and
2. $\forall a \in I$ and $\forall r \in R$, $ra \in I$ and $ar \in I$.

We write $I \trianglelefteq R$.

Examples

1. Let R be a ring. Then $0 = \{0_R\}$ and R are both ideals of R.
2. Let $n \geq 1$. Then $n\mathbb{Z} \trianglelefteq \mathbb{Z}$ is an ideal.
3. Let $R = \mathbb{R}[x]$, let $g \in \mathbb{R}[x]$, and let $I := \{f \in \mathbb{R}[x] : g|f \text{ i.e. } \exists h \in \mathbb{R}[x] \text{ such that } f = gh\}$. Then $I \trianglelefteq \mathbb{R}[x]$.

Lecture 30

Non-Examples of Ideals

1. None of $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.
2. Let R be commutative. Then $R \leq R[x]$ is not an ideal (Axiom 2 does not hold).
3. Let $R = \mathbb{Z}$, $C = 2\mathbb{Z} \cup 3\mathbb{Z}$. In this case, Axiom 2 holds, but Axiom 1 fails since $3(1) + 2(-1) = 1 \notin C$.

Proposition (30.1)**.** Let R be commutative, $I, J \trianglelefteq R$ be ideals.

1. $I \cup J$ is not, in general, an ideal of R. However, $I \cap J \trianglelefteq R$.
2. The subset $I + J := \{a + b \in R | a \in I, b \in J\}$ is an ideal of R.
3. The subset $IJ := \{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n | n \in \mathbb{N}, a_i \in I, b_i \in J\}$ is an ideal of R.

Kernels Revisited

Proposition (30.2)**.** Let $\varphi : R \to S$ be a ring homomorphism. Then...

1. $\ker\varphi \trianglelefteq R$
2. $\ker\varphi = 0$ iff φ is injective.

Proof. 1. Since $\varphi(a + b) = \varphi(a) + \varphi(b)$ $\forall a, b \in R$, φ is an additive group homomorphism. Therefore $\ker\varphi$ is an additive subgroup of $(R, +, 0_R)$ by Proposition 16.2.

2. This follows directly from Proposition 17.1.

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Quotient Rings

Let $I \trianglelefteq R$ be an ideal. Forgot about multiplication for a moment. We know $(I,+) \leq (R,+)$ is a subgroup.
Let $r \in R$. The left cosets of I are of the form $r + I := \{r + a | a \in I\}$. Recall that $R/I := \{r + I | r] inR\}$ is the set of left cosets. This is a group with respect to addition since $(R.+)$ is abelian.

Lecture 31

Generalize Construction of Z/nZ as a Ring

Theorem (31.1 **PROVE ON EXAM!**).

1. The binary operation $\overline{*} : R/I \times R/I \to R/I$, $(r_1 + I)\overline{*}(r_2 + I) := r_1 r_2 + I$ is well-defined.
2. $(R/I, \overline{+}, 0_{R/I}, \overline{*}, 1_{R/I})$ is a ring, where $1_{R/I} := 1_R + I$.
3. The surjective function $\pi : R \to R/I$, $\pi(r) := r + I$ is a ring homomorphism.

Proof.

1. Suppose $r_1 + I = r_1' + I$ and $r_2 + I = r_2' + I$ (1). Want to show $r_1 r_2 + I = r_1' r_2' + I$. Since $r_1 \in r_1 + I$, $r_2 \in r_2 + I$, (1) implies $\exists a_1, a_2 \in I$ such that $r_1 = r_1' + a_1$, $r_2 = r_2' + a_2$. Therefore $r_1 r_2 = (r_1' + a_1)(r_2' + a_2) = r_1' r_2' + r_1' a_2 + a_1 r_2' + a_1 a_2$. By Axiom 2 of Definition 29.1 of ideal, $r_1' a_2$, $a_1 r_2'$, $a_1 a_2 \in I$. Axiom 1 implies $r_1 r_2 - r_1' r_2' \in I$. Therefore by PS5 #1, $r_1 r_2 + I = r_1' r_2' + I$.
2. Straightforward. Skip because this is L O N G.
3. Check the axioms of ring homomorphism:
   (a) $\pi(r_1 + r_2) = (r_1 + r_2) + I = (r_1 + I)\overline{+}(r_2 + I) = \pi(r_1)\overline{+}\pi(r_2)$.
   (b) $\pi(r_1 r_2) = r_1 r_2 + I = (r_1 + I)\overline{*}(r_2 + I) = \pi(r_1)\overline{*}\pi(r_2)$.
   (c) $\pi(1_R) = 1_R + I = 1_{R/I}$.

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Example (Non-Example)**.** (Replace ideal with a subring in R/I). Consider: $\mathbb{Z} \leq \mathbb{Q}$ and the quotient group $\mathbb{Q}/\mathbb{Z}$ $(q_1 + \mathbb{Z})\overline{+}(q_2 + \mathbb{Z}) = (q_1 + q_2) + \mathbb{Z}$. Then $(\frac{1}{2} + \mathbb{Z})\overline{+}(\frac{1}{3} + \mathbb{Z}) = \frac{1}{6} + \mathbb{Z}$. Note that $\frac{1}{2} + \mathbb{Z} = \frac{3}{2} + \mathbb{Z}$. Then $(\frac{3}{2} + \mathbb{Z})\overline{+}(\frac{1}{3} + \mathbb{Z}) = \frac{1}{2} + \mathbb{Z} \neq \frac{1}{6} + \mathbb{Z}$.

1st Isomorphism Theorem for Rings

Theorem (31.2)**.** let $\varphi : R \to S$ be a ring homomorphism. Then the function $\overline{\varphi} : R/\ker\varphi \to \mathrm{im}\varphi$, $\overline{\varphi}(r + \ker\varphi) := \varphi(r)$ is a well-defined ring isomorphism.