FINAL EXAM STUDY GUIDE

THEOREMS:

Upper/Lower Riemann Integrals: $\overline{\int} f = \inf\{U(f,P) : P \text{ partition of } [a,b]\}$, and $\underline{\int} f = \sup\{L(f,P) : P \text{ partition of } [a,b]\}$

Theorem: Let $f : [a,b] \to \mathbb{R}$. If f is monotone, then f is integrable.

Theorem: Let $f : [a,b] \to \mathbb{R}$. If f is continuous, then f is integrable.

Mean Value Theorem for Integrals: Let $f : [a,b] \to \mathbb{R}$ be continuous. Then $\exists c \in (a,b)$ such that $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

Fundamental Theorem of Calculus: If f has an antiderivative F, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$. Let $f : [a,b] \to \mathbb{R}$ be continuous. Then $F(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$ is differentiable and $F' = f$. Then $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.

Improper Integrability: $f : [a,b) \to \mathbb{R}$ is improperly integrable if 1) f is Riemann integrable on any interval $[a,c] \subset [a,b)$ and if 2) $\lim_{c\to b^-} \int_a^c f(x)\,\mathrm{d}x$ exists and is finite.

STUDY GUIDE:

A1: Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -2$ for $x \in \mathbb{Q}$ and $f(x) = 3$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ is NOT integrable on $[0,1]$.

Proof. Let $P = \{0 = x_0, x_1, x_2, \ldots, x_n = 1\}$ and $[x_{k-1}, x_k]$ for $k \in \{1, 2, \ldots, n\}$. Since $[x_{k-1}, x_k] \cap \mathbb{R} \setminus \mathbb{Q} \neq \emptyset$ and $[x_{k-1}, x_k] \cap \mathbb{Q} \neq \emptyset$,

$$\inf\{f(x) : x \in [x_{k-1}, x_k]\} = -2 = m_k \qquad \sup\{f(x) : x \in [x_{k-1}, x_k]\} = 3 = M_k.$$

$$\text{So, } L(f,P) = \sum_{k=1}^n m_k(x_{k-1} - x_k) \qquad U(f,P) = \sum_{k=1}^n M_k(x_{k-2} - x_k)$$

$$= -2[(x_0 - x_1) + (x_1 - x_2) + \cdots + (x_{n-1} - x_n] = -2(x_0 - _n) = -2 \qquad = \cdots = -3.$$

Therefore, the upper and lower integrals will not be equal, so it is not integrable on $[0,1]$. ☐

B1: $f : \mathbb{R} \to \mathbb{R}, f(x) = \cos x$ with partition $P = \{\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi\}$ of $[\frac{\pi}{2}, \pi]$. Evaluate $L(f,P)$ and $U(f,P)$.

Answer: $f(\pi/2) = 0, \quad f(2\pi/3) = -\frac{1}{2}, \quad f(3\pi/4) = -\frac{\sqrt{2}}{2}, \quad f(5\pi/6) = -\frac{\sqrt{3}}{2}, \quad f(\pi) = -1$

$$\begin{aligned} m_1 &= \inf\{f(x) : x \in [\frac{\pi}{2}, \frac{2\pi}{3}]\} = -\frac{1}{2} & M_1 &= \sup\{f(x) : x \in [\frac{2\pi}{3}, \frac{3\pi}{4}]\} = 0 \\ m_2 &= -\frac{\sqrt{2}}{2}, m_3 = -\frac{\sqrt{3}}{2}, m_4 = -1 & M_2 &= -\frac{1}{2}, M_3 = -\frac{\sqrt{2}}{2}, M_4 = -\frac{\sqrt{3}}{2} \\ L(f,P) &= m_1(\frac{2\pi}{3} - \frac{\pi}{2}) + \cdots + m_4(\pi - \frac{5\pi}{6}) & U(f,P) &= M_1(\frac{2\pi}{3} - \frac{\pi}{2}) + \cdots + M_4(\pi - \frac{5\pi}{6}) \\ &= \cdots = \frac{\pi(-6 - \sqrt{2} - \sqrt{3})}{24} & &= \cdots = -\frac{\pi(1 + \sqrt{2} + 2\sqrt{3})}{24} \end{aligned}$$

C2: Show $h(x) = \int -e^{-} x^{e^x} \frac{1}{t^4+1}\,\mathrm{d}t$ is differentiable on $\mathbb{R}$ and find $h(0), h'(x), h'(0)$.

Proof. Define $k : \mathbb{R} \to \mathbb{R}, k(t) = \frac{1}{t^4+1}$. This is continuous, thus integrable on $[a,b]$. By the FTC, k has an antiderivative

$K:\mathbb{R}\to\mathbb{R}$, $K'(t)=k(t)$. Then $h(x)=K(e^x)-K(e^{-x})$ is also differentiable by composition.

$$h(0)=\int_1^1 \frac{1}{t^4+1}\,\mathrm{d}t=0$$
$$h'(x)=K'(e^x)e^x-K'(e^{-x})(-e^{-x})=\frac{e^x+e^{-x}}{e^{4x}+1}$$
$$h'(0)=\frac{1}{2}+\frac{1}{2}=1.$$

□

D3: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin x-4x$ for $x\le 0$, $f(x)=3\ln(1-x)$ for $0<x<1$, e^{-x} for $x\ge 1$. Does $\lim_{x\to 1^-} f(x)$ exist? Determine $f'(0)$, $f'(1)$ or prove it does not exist. Prove f is integrable on $[-\pi,0]$ and prove f is integrable on $[-\pi,\frac{1}{2}]$. Then prove the improper integral $\int_1^\infty f^2(x)\,\mathrm{d}x$ is convergent and evaluate.
Answer: $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} 3\ln(1-x)=-\infty$. Since f is not continuous at $x=1$, f is not differentiable at $x=1$.

$$f_l'(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{\sin x-4x}{x}=-3 \qquad f_r'(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{3\ln(1-x)-0}{x}=-3$$
$$\therefore f \text{ is differentiable @ } x=1$$

On $[-\pi,0]$, $f(x)=\sin x-4x$ is continuous, thus integrable over this interval. Since f is integrable on $[-\pi,0]\cup[0,\frac{1}{2}]$, it is integrable on $[-\pi,\frac{1}{2}]$.
$\int_1^\infty f^2(x)\,\mathrm{d}x=\int_1^\infty (e^{-x})^2\,\mathrm{d}x$. e^{-2x} is continuous on any $[1,c]$, $c\in(1,\infty)$ $\implies$ it is integrable. Also, $\lim_{c\to\infty}\int_1^c e^{-2x}\,\mathrm{d}x=\lim_{c\to\infty} -\frac{1}{2}e^{-2x}\Big|_1^c=\frac{1}{2e^2}$, so this exists.

E2: Prove $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^{-\frac{1}{x^2}}$ for $x\ne 0$ and $f(0)=0$ is differentiable. Is $g(x)=e^{-\frac{1}{x^2}}\ln x$ improperly integrable on $(0,1)$?
Answer: On the intervals $(-\infty,0)\cup(0,\infty)$, f is differentiable by composition and $f'(x)=e^{-\frac{1}{x^2}}\frac{2}{x^3}$.
$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\frac{\frac{1}{x}}{e^{\frac{1}{x^2}}}$. Using L.R. we find that $f'(0)=0$.
For g, there is a singularity at $x=0$: $\lim_{x\to 0^+}\frac{\ln x}{e^{\frac{1}{x^2}}}=0$ (L.H.). So g can be extended by continuity to $\overline{g}:[0,\infty)\to\mathbb{R}$, So $\int_0^1 g(x)\,\mathrm{d}x=\int_0^1\overline{g}(x)\,\mathrm{d}x$. So this is improperly integrable on $(0,1)$.

F1: $F:\mathbb{R}\mathbb{R}$, $F(x)=x^2\sin(\frac{1}{x^2})$ for $x\ne 0$ and $F(0)=0$. Prove F is differentiable, determine F', then determine if F is integrable on $[-1,1]$ and if F' is integrable on $[2,7]$ and $[-1,1]$.
Answer: F is differentiable on $(-\infty,0)\cup(0,\infty)$ by composition. To see if differentiable at $x=0$, we evaluate $\lim_{x\to 0}\frac{F(x)-F(0)}{x-0}=0$.
$F'(x)=2x\sin(\frac{1}{x^2})-\frac{2}{x}\cos(\frac{1}{x^2})$ for $x\ne 0$ and 0 otherwise.
F being differentiable on $\mathbb{R}\implies F$ is continuous on $[-1,1]\implies F$ is integrable on $[-1,1]$.
F' being differentiable on $[2,7]$ by composition $\implies F'$ is continuous on $[2,7]\implies F'$ is integrable on $[2,7]$.
For $x_n=\frac{1}{\sqrt{2n\pi}}$, $F'(x_n)=2x_n\sin(\frac{1}{x_n^2})-\frac{2}{x_n}\cos(\frac{1}{x_n^2})=\cdots=-2\sqrt{2n\pi}$. So, $\lim_{n\to\infty}F'(x_n)=-\infty$. So F' is not bounded on $[-1,1]$ and thus not integrable.