MATH 310.1002: Homework 1

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1. For any $t > 0$ denote $A_t = (-t, 2+t)$.

   (a) Determine $A_3 \setminus A_1$.
   Answer:
   $$A_3 \setminus A_1 = (-3,5) \setminus (-1,3) = (-3,-1] \cup [3,5)$$

   (b) Determine
   $$\bigcap_{n=1}^{10} A_{1/n}$$

   Answer:
   $$\bigcap_{n=1}^{10} A_{1/n} = (-1,3) \cap (-\tfrac{1}{2}, 2\tfrac{1}{2}) \cap (-\tfrac{1}{3}, 2\tfrac{1}{3}) \cdots \cap (-\tfrac{1}{10}, 2\tfrac{1}{10})$$
   $$= (-\tfrac{1}{10}, 2\tfrac{1}{10})$$

   (c) Prove that
   $$\bigcap_{t>0} A_t = [0,2]$$

   Proof. To prove this, we must prove that each set is a subset of the other.

   We will begin by showing $\bigcap_{t>0} A_t \subset [0,2]$. Let's assume that $x \in \mathbb{R}$ and that $x \notin [0,2]$, so for example, we'll choose $x > 2$. Now let $t_0 = \frac{x-2}{2}$. We now have $x \in (-t_0, 2+t_0)$, which is a contradiction, so $\bigcap_{t>0} A_t \subset [0,2]$.

   Now we will show that $[0,2] \subset \bigcap_{t>0} A_t$. Let $y \in [0,2]$. Then $y \in (-t, 2+t)$ for all $t > 0$, so $y \in \bigcap_{t>0} A_t$. $\square$

2. Let $f : A \to B$ and $g : B \to C$ be functions. Prove

   (a) If $g \circ f$ is onto, then g is onto.

   Proof. We will denote $g \circ f$ as h. Suppose h is surjective. We wish to prove that g is also surjective, or in other words, that there exists a $b \in B$ such that $g(b) = c$. By definition of h, there exists some $a \in A$ such that $h(a) = c$. This means that $h(f(a)) = c$. If we take $b = f(a)$, then $b \in B$ and $g(b) = c$, thus g is surjective. $\square$

   (b) If $g \circ f$ is one-to-one, then f is one-to-one.

   Proof. We will denote $g \circ f$ as h. Suppose h is injective. To prove that f is also injective, we will show that for all $a, a' \in A$, $f(a) = f(a')$ implies $a = a'$. By definition of h, we know that for all $d, d' \in h$, $h(d) = h(d')$ implies $d = d'$. $\square$

3. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and the sets $X = (-1,4)$, $Y = [1,4]$.

   (a) Determine $f^{-1}(X)$ and $f^{-1}(Y)$.
   Answer:
   $$f^{-1}(X) = \varnothing$$
   $$f^{-1}(Y) = [-2,0) \cup (0,2]$$

(b) Determine $f(f^{-1}(X))$ and $f(f^{-1}(Y))$.
Answer:

$$f(f^{-1}(X) = f(\varnothing) = \varnothing$$
$$f(f^{-1}(Y) = f([-2,0) \cup (0,2]) = [1,4]$$

4. Prove that the function $f : D \to C$ is onto if and only if for every subset $X \subset C$ we have $f(f^{-1}(X)) = X$.

Proof. Let us first prove that if f is surjective, then $f(f^{-1}(X)) = X$ for every subset $X \subset C$.
Let $y \in f(f^{-1}(X))$. Then $y = f(x)$ for some $x \in f^{-1}(X)$. So by definition we have that $x \in f^{-1}(X)$ if and only if $f(x) \in X$. Now let $x \in X$. Then since f is surjective, $x = f(d)$ for some $d \in D$ and by definition, $d \in f^{-1}(X)$. So $d \in f(f^{-1}(X))$.

Now we will prove that if $f(f^{-1}(X)) = X$ for every subset $X \subset C$, then f is surjective.
Assume that $c \in C$ and consider the set $X = \{c\}$. Knowing that $f(f^{-1}(X)) = X$, then there must be some $a \in f^{-1}(X)$ such that $f(a) = c$. Thus f is surjective.

□

5. Prove that for all $n \in \mathbb{N}$,
$$1 + 3 + \cdots + (2n-1) = n^2.$$

Proof. We will prove by induction. First, observe that we can write $1 + 3 + \cdots + (2n-1)$ as $\sum_{i=1}^{n} 2n - 1$. For the base case, let $n = 1$. Then $1 = 1^2$ and the base case holds. Now let's assume that $\sum_{i=1}^{n} 2n - 1 = n^2$ for $n \in \mathbb{N}$. Then,
$$\left(\sum_{i=1}^{n} 2n - 1\right) + 2n - 1 = (n+1)^2.$$
Substituting n^2 in for $\sum_{i=1}^{n} 2n - 1 = n^2$ gives $n^2 + 2n + 1 = (n+1)^2$ which is of course true.

□

6. Let $x_1, x_2, x_3, \ldots$ be a sequence of numbers defined recursively by
$$x_1 = 0 \quad \text{and} \quad x_{n+1} = \frac{1 + x_n}{2}.$$
Prove that $x_n < x_{n+1}$ for all $n \in \mathbb{N}$. Can you find a formula for x_n?

Proof. For our base case, let us choose $n = 1$. Then, $x_2 = \frac{1+x_1}{2} = \frac{1+0}{2} = \frac{1}{2}$. Observe that $0 < \frac{1}{2}$, so the base case holds. Now we may assume that $x_n < x_{n+1}$ for all $n \in \mathbb{N}$. We wish to prove that $x_{n+1} < x_{n+2}$ Notice that we can write x_{n+2} recursively in terms of x_{n+1} and ultimately achieve $x_{n+2} = \frac{1+x_{n+1}}{2}$.
We can now use our previous assumption that $x_n < x_{n+1}$ to safely assume that
$$\frac{1 + x_n}{2} < \frac{1 + x_{n+1}}{2}$$
$$x_{n+1} < x_{n+2}$$
which thus ends our proof.

□

7. **Bonus problem.** Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = 2a_n + \sqrt{3a_n^2 - 2}$, for any $n \in \mathbb{N}$. Prove that all the terms of the sequence are positive integers.

Proof. To prove this, we will first rewrite the equation and solve for -2:
$$a_{n+1} - 2a_n = \sqrt{3a_n^2 - 2}$$
$$(a_{n+1} - 2a_n)^2 = 3a_n^2 - 2$$
$$a_{n+1}^2 - 4a_{n+1}a_n + 4a_n^2 = 3a_n^2 - 2$$
$$a_{n+1}^2 - 4a_{n+1}a_n + a_n^2 = -2$$
(Unsure where to proceed from here)): □