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MATH 440.1001
Scribing Week 3
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MATH 440 Problems

Review of Last Lecture

The closed subsets in $(X, \mathcal{T}_{\text{cof}})$ are X and all finite subsets of X.
If X is a finite set, then $\mathcal{T}_{\text{cof}} = \mathcal{T}_{\text{disc}}$.

Lecture 7

Definition (7.1)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be a subset. Let $\mathcal{T}_A := \{U \cap A | U \in \mathcal{T}\}$. Then $(A, \mathcal{T}_A)$ is a topological space and $\mathcal{T}$ is called the **subspace topology** on A.

Definition 7.1 gives many interesting examples of spaces:

1. Graph $(f : \mathbb{R} \to \mathbb{R}) \subseteq \mathbb{R}^2$, i.e. the Euclidean Topology.
2. $S^2 := \{(x, y, z) | x^2 + y^2 + z^2 = 1\}$, i.e. the "2-sphere."

3. Knots in $\mathbb{R}^3$.
4. The Order topology.
5. Lower/Upper Limit topology on $\mathbb{R}$.
6. ...and others.

Lecture 8

Closed Sets/Limit Points in Topological Spaces

Theorem (8.1)**.** Let X be a topological space. Then:

1. $X, \emptyset$ are closed subsets.
2. Finite unions of closed subsets are closed.
3. Arbitrary intersections of closed subsets are closed.

The proof of Theorem 8.1 is a direct consequence of applying DeMorgan's Laws to Definition 6.1.
Recall Definition 4.1 of closed subsets in Euclidean space:

Definition (4.1)**.** Let $A \subseteq \mathbb{R}^n$ be a subset. Then A is **closed** iff every convergent sequence $\{a_k\} \subseteq A$ converges in A, i.e. $\lim\limits_{k \to \infty} a_k \in A$.

We will use this definition to help us understand the following definition:

Definition (8.2)**.** Let X be a topological space where $A \subseteq X$ is a subset. A point $y \in X$ is a **limit point** of A iff $\forall$ open subsets $U \subseteq X$ containing y, $A \cap (U - \{y\}) \neq \emptyset$.
We will define $L(A) := \{y \in X | y \text{ is a limit point of } A\} := A'$

Examples in $\mathbb{R}^1$

1. Define $A := (0, 1]$.
   (a) Show $0 \in L(A)$.
   Let $U \subseteq \mathbb{R}$ be an open subset such that $0 \in U$. Then $\exists \varepsilon > 0$ such that $B_0(\varepsilon) = (-\varepsilon, \varepsilon) \subseteq U$. Therefore $U - \{0\} \supseteq (0, \varepsilon) \implies U - \{0\} \cap A \neq \emptyset$.
   (b) Show $1 \in L(A)$.
   Let $U \subseteq \mathbb{R}$ be an open subset such that $1 \in U$. Then $\exists \varepsilon > 0$ such that $B_1(\varepsilon) \subseteq U$. Then $U - \{1\} \supseteq (1 - \varepsilon, 1) \implies U - \{1\} \cap A \neq \emptyset$.
   (c) We can apply this same strategy to prove that $L(A) = [0, 1]$.

2. Let $A = \{\frac{1}{n} | n \in \mathbb{N}\}$.
   It is obvious that since $\frac{1}{n} \to 0$, then $0 \in L(A)$.
   If $x \neq 0$ and $x \in L(A)$, then $\exists$ a subsequence of $\{\frac{1}{n}\}_{n \geq 0} \to x \neq 0$. This is a contradiction since every convergent subsequence of $\{\frac{1}{n}\}$ converges to 0.
   In particular, $\forall \frac{1}{n} \in A, \exists$ and open subset $U \supseteq \frac{1}{n}$ such that $A \cap U = \{\frac{1}{n}\}$.

Theorem (8.4)**.** Let X be a topological space and let $A \subseteq X$ be a subset. Then A is **closed** iff $L(A) \subseteq A$.

Lecture 9

Remark (8.4). Recall Theorem 8.4 from last lecture. This is a useful trick to show that a subset $V \subseteq X$ is open. $\forall y \in V$, find an open subset $U_y \subseteq X$ such that $y \in U_y$ and $U_y \subseteq V$. Then

$$\begin{aligned} &\implies V = \bigcup_{y \in V} U_y \qquad \textit{By Axiom 2 of topological space} \\ &\implies \text{V is open.} \end{aligned}$$

Definition (9.2)**.** Let X be a topological space and $A \subseteq X$ a subset. Then the **closure** of A in X, $\bar{A}$, is the intersection of all closed subsets of X containing A:

$$\bar{A} := \bigcap_{B \subseteq X \text{ closed},\, A \subseteq B} B$$

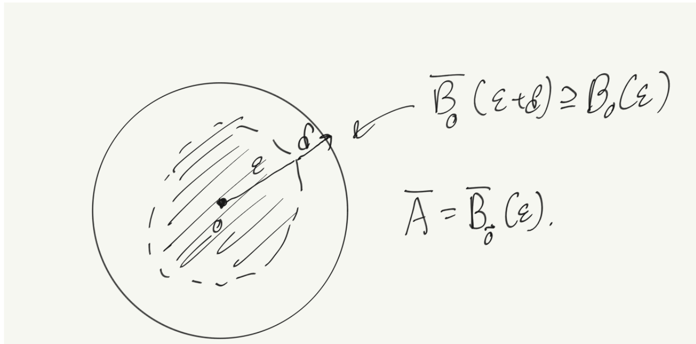
Remark (9.2). There are a few remarks:

1. $\bar{A}$ is closed by #3 in Theorem 8.1.
2. If B is closed and $A \subseteq B$, then $A \subseteq \bar{A} \subseteq B$. Hence $\bar{A}$ is the "smallest" closed subset of X that contains A.
3. If A is closed, then $\bar{A} = A$.

Intuitive Examples of Closure

1. $A = (0, 1)$, $X = \mathbb{R}$.
   Then $\bar{A} = [0, 1] \subseteq [-\frac{1}{n}, 1 + \frac{1}{n}], \forall n \in \mathbb{N} \implies \bar{A} \subseteq \bigcap_{n \in \mathbb{N}} [-\frac{1}{n}, 1 + \frac{1}{n}]$.

2. $A = B_{\vec{0}}(\varepsilon)$, $X = \mathbb{R}^2$.
   Then



3. $A = (0, 1)$, $X = (\mathbb{R}, \mathcal{T}_{\text{disc}})$.
   Then $\bar{A} = A$ since A is already closed.

4. $A = (0, 1)$, $X = (\mathbb{R}, \mathcal{T}_{\text{triv}})$.
   Then $\bar{A} = \mathbb{R}$.

Theorem (9.5)**.** Let $A \subseteq X$ be a subset of the topological space X. Then $\bar{A} = A \cup L(A)$.