MATH 310.1002: Homework 2

Dillan Marroquin

September 9, 2020

1. Prove that the equation $x^2 = 3$ has no rational solution.

   Proof. First, observe that $\pm\sqrt{3}$ are the only real solutions to this equation. We must prove that $-\sqrt{3}$ and $\sqrt{3}$ are both irrational. Assume by the contrary that $\sqrt{3}$ and $-\sqrt{3}$ are both rational. Then there are $p, q \in \mathbb{Z}$ such that $\sqrt{3} = \frac{p}{q}$, where $\frac{p}{q}$ is fully reduced. Squaring both sides and solving for p^2 gives $p^2 = 3q^2$, so $3|p^2$. Now we must prove that $3|p$ which is given by 2 cases, both using the contrapositive approach.

   Case 1: Suppose there is a remainder of 1 when dividing p by 3. Then there exists an $a \in \mathbb{Z}$ such that $p = 3a + 1$. Then $p^2 = 9a^2 + 6a + 1 = 3(3a^2 + 2a) + 1$. Since there is a remainder of 1 when dividing p^2 by 3, then $3 \nmid p^2$.

   Case 1: Suppose there is a remainder of 2 when dividing p by 3. Then there exists an $a \in \mathbb{Z}$ such that $p = 3a + 2$. Then $p^2 = 9a^2 + 12a + 3 + 1 = 3(3a^2 + 4a + 1) + 1$. Since there is a remainder of 1 when dividing p^2 by 3, then $3 \nmid p^2$.

   Since $3|p$, we can write that $p = 3k$ for some $k \in \mathbb{Z}$. We now get $\frac{3k}{q} = \sqrt{3}$. After squaring both sides and solving for $9k^2$ we get $9k^2 = 3q^2$. This is a contradiction since this implies that $\frac{p}{q}$ is not in lowest terms. A similar case can be said about $-\sqrt{3}$.

   $\square$

2. Give an example of irrational numbers a, b such that $a + b$ and ab are rational.

   Answer: A conjugate pair satisfies these conditions. I chose $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$.

   Proof. We must prove that there does exist $a, b \notin \mathbb{Q}$ such that $a + b \in \mathbb{Q}$ and $ab \in \mathbb{Q}$. Let $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$. Then $a + b = (1 + \sqrt{2}) + (1 - \sqrt{2}) = 2 \in \mathbb{Q}$. Also, $ab = (1 + \sqrt{2})(1 - \sqrt{2}) = 1 - 2 = -1 \in \mathbb{Q}$.

   $\square$

3. Give an example of irrational numbers a, b such that a^b is rational.

   Answer: A great example would be $e^{\ln 2}$ since these are inverse operations!

   Proof. We must prove that there does exist $a, b \notin \mathbb{Q}$ such that $a^b \in \mathbb{Q}$. Let $a = e$ and $b = \ln 2$. Then $a^b = e^{\ln 2} = 2 \in \mathbb{Q}$.

   $\square$

4. Describe the set of upper bounds of each of the following sets.

   (a) $A = \{3, 1, 0\}$
   Answer: Set of upper bounds of $A = [3, \infty)$.

   (b) $B = \mathbb{N}$
   Answer: The set B has no set of upper bounds.

   (c) $C = \{e^{-x} : x \geq 0\}$
   Answer: Set of upper bounds of $C = [1, \infty)$.

   (d) $D = \{r \in \mathbb{Q} : r^2 < 5\}$
   Answer: Set of upper bounds of $D = [\sqrt{5}, \infty)$.

(e) $E = \{\frac{2n-1}{n} : n \in \mathbb{N}\}$
Answer: Set of upper bounds of $E = [2, \infty)$.

5. For each of the above examples, determine whether the set is bounded above and, if so, find its least upper bound.

   (a) **Answer:** Yes, A is bounded above and $\sup A = 3$.

   (b) **Answer:** No, B is not bounded above. Can be proven using Peano's axioms (in particular **N2**).

   (c) **Answer:** Yes, C is bounded above and $\sup C = 1$.

   (d) **Answer:** Yes, D is bounded above and $\sup D = \sqrt{5}$.

   (e) **Answer:** Yes, E is bounded above and $\sup E = 1$.

6. Show that the following set is bounded above and find its least upper bound.

$$S = \{x : x^2 < 2x + 3\}$$

Proof. First, we will solve the above inequality for 0 to give $x^2 - 2x - 3 < 0$ and then factor to give $(x-3)(x+1) < 0$. For this inequality to be true, $-1 < x < 3$, so $S = (-1, 3)$ and thus $S \subset \mathbb{R}$. Let $m \in \mathbb{R}$ such that $m \geq x$ for every $x \in S$. Then $m \geq 3$, and the set is bounded above.

Answer: $\sup S = 3$.

$\square$

7. Let $a, b \in \mathbb{R}$. Suppose that $a - \frac{1}{n} < b$ for every $n \in \mathbb{N}$. Prove that $a \leq b$.

Proof. Suppose to the contrary that $a > b$. Then, $a - b > 0$. If we rewrite the above inequality as $a - b < \frac{1}{n}$ and then take the reciprocal of both sides, we get $\frac{1}{a-b} > n$ for all $n \in \mathbb{N}$. This is a contradiction since $\frac{1}{a-b}$ would be an upper bound for $\mathbb{N}$, but this is clearly not the case.

$\square$

8. Let A be a nonempty set with least upper bound m. Prove that for every $n \in \mathbb{N}$, there is an $a \in A$ such that

$$m - \frac{1}{n} < a.$$

Proof. For the contrary, assume that there exists some $n \in \mathbb{N}$ such that for every $a \in A$ we have $m - \frac{1}{n} \geq a$. This would mean that $m - \frac{1}{n}$ is an upper bound for A. This is a contradiction since $m - \frac{1}{n} < m$ and is actually a lower bound by definition.

$\square$