STAT 461.1002: Homework 7
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Section 3.8 Problems

1. 3.8.3 part (a) Let X and Y be two independent random variables. Given the marginal pdfs shown below, find the pdf of $X+Y$. In each case, check to see if $X+Y$ belongs to the same family of pdfs as do X and Y.
$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ and $p_Y(k) = e^{-\mu}\frac{\mu^k}{k!}$, $k = 0, 1, 2, \ldots$.
Answer: We directly apply Theorem 3.8.3, where $W = X+Y$.

$$\begin{aligned} p_W(w) &= \sum_{\text{all x}} p_X(x) \cdot p_Y(w-x) \\ &= \sum_{\text{all x}} e^{\lambda}\frac{\lambda^x}{x!} \cdot e^{-\mu}\frac{\mu^{w-k}}{(w-k)!} \\ &= e^{-\lambda-\mu}\sum_{\text{all x}} \frac{\lambda^x \mu^{w-k}}{x!(w-k)!} \\ &= \frac{e^{-\lambda-\mu}}{w!}(\lambda+\mu)^w = \frac{e^{-\lambda-\mu}(\lambda+\mu)^w}{w!} \end{aligned}$$

We note that $W = X+Y$ belongs to the same family of pdfs as X and Y.

2. 3.8.7 Let Y be a continuous nonnegative random variable. Show that $W = Y^2$ has pdf $f_W(w) = \frac{1}{2\sqrt{w}}f_Y(\sqrt{w})$. (*Hint:* First find $F_W(w)$.)
Answer: Observe that we can express $F_W(w)$ differently: $F_W(w) = P(W \leq w) = P(Y^2 \leq w) = P(Y \leq \sqrt{w}) = F_Y(\sqrt{w})$, so then $F_W(w) = F_Y(\sqrt{w})$.
Now we recall that $f_W(w) = \frac{\mathrm{d}}{\mathrm{d}w}F_W(w) = \frac{\mathrm{d}}{\mathrm{d}w}F_Y(\sqrt{w}) = \frac{1}{2\sqrt{w}}f_Y(\sqrt{w})$. So then $W = Y^2$ has pdf $f_W(w) = \frac{1}{2\sqrt{w}}f_Y(\sqrt{w})$

3. 3.8.8 Let Y be a uniform random variable over the interval $[0,1]$. Find the pdf of $W = Y^2$.
Answer: We discovered in problem 2 that the pdf of a continuous nonnegative random variable $W = Y^2$ is $f_W(w) = \frac{1}{2\sqrt{w}}f_Y(\sqrt{w})$. So then $f_W(w) = \frac{1}{2\sqrt{w}}(0) + \frac{1}{2\sqrt{w}}(1) = \frac{1}{2\sqrt{w}}$, $0 \leq w \leq 1$.

4. 3.8.13 Suppose that X and Y are two independent random variables, where $f_X(x) = xe^{-x}$, $x \geq 0$, and $f_Y(y) = e^{-y}$, $y \geq 0$. Find the pdf of Y/X.
Answer: We directly apply Theorem 3.8.4 to find the pdf, where $W = Y/X$:

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx)\,\mathrm{d}x \\ &= \int_0^{\infty} |x| x e^{-x} \cdot e^{-wx}\,\mathrm{d}x = \int_0^{\infty} x^2 e^{-x(1+w)}\,\mathrm{d}x \end{aligned}$$

Now we integrate by parts with $u = x^2$, $\mathrm{d}u = 2x$, $v = -\frac{e^{-x(1+w)}}{1+w}$, $\mathrm{d}v = e^{-x(1+w)}$.

$$\begin{aligned} f_W(w) = \int_0^{\infty} x^2 e^{-x(1+w)}\,\mathrm{d}x &= \frac{-x^2 e^{-x(1+w)}}{1+w} - \int_0^{\infty} \frac{-2xe^{-x(1+w)}}{1+w}\,\mathrm{d}x \\ &= \frac{-x^2 e^{-x(1+w)}}{1+w}\bigg|_0^{\infty} - \frac{2}{1+w}\int_0^{\infty} xe^{-x(1+w)}\,\mathrm{d}x \\ &= \frac{2}{1+w}\int_0^{\infty} xe^{-x(1+w)}\,\mathrm{d}x \end{aligned}$$

Again, we integrate by parts with $u = x$, $\mathrm{d}u = 1$, $v = -\frac{e^{-x(1+w)}}{1+w}$, $\mathrm{d}v = e^{-x(1+w)}$.

$$\begin{aligned} f_W(w) &= \frac{2}{1+w}\left(-\frac{xe^{-x(1+w)}}{1+w}\bigg|_0^{\infty} - \int_0^{\infty} -\frac{e^{-x(1+w)}}{1+w}\,\mathrm{d}x \right) \\ &= \frac{2}{(1+w)^2}\int_0^{\infty} e^{-x(1+w)}\,\mathrm{d}x = -\frac{2}{(1+w)^2}\left(\frac{e^{-x(1+w)}}{1+w}\right)\bigg|_0^{\infty} = \frac{2}{(1+w)^3}, \text{ for all } w \geq 0. \end{aligned}$$

Section 3.9 Problems

5. 3.9.5 Suppose that X_i is a random variable for which $E(X_i) = \mu \neq 0$, $i = 1, 2, \dots, n$. Under what conditions will the following be true?

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \mu$$

Answer: We first rewrite the expression on the left-hand side of the above equation using the properties of expected value:

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E[X_i] = \sum_{i=1}^{n} a_i \mu = \mu \sum_{i=1}^{n} a_i$$

We now make the observation that in order for the initial equation to be true, $\sum_{i=1}^{n} a_i = 1$.

6. **(461 only)** 3.9.2 Suppose that $f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}$, $0 \le x$, $0 \le y$. Find $E(X+Y)$.
Answer: First, we find the marginal PDFs of X and Y:

$$f_X(x) = \int_0^\infty \lambda^2 e^{-\lambda(x+y)}\,\mathrm{d}y = \lambda^2 e^{-\lambda x} \int_0^\infty e^{-\lambda y}\,\mathrm{d}y = \lambda^2 e^{-\lambda x}\left(-\frac{e^{-\lambda y}}{\lambda}\right)\Bigg|_0^\infty = \lambda^2 e^{-\lambda x}\left(0 + \frac{1}{\lambda}\right) = \lambda e^{-\lambda x}, \forall x \ge 0$$
$$f_Y(y) = \int_0^\infty \lambda^2 e^{-\lambda(x+y)}\,\mathrm{d}x = \lambda^2 e^{-\lambda y} \int_0^\infty e^{-\lambda x}\,\mathrm{d}x = \cdots = \lambda e^{-\lambda y}, \forall y \ge 0$$

We now compute $E[X]$ and $E[Y]$:

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x}\,\mathrm{d}x = -\lambda^2 x e^{-\lambda x}\Big|_0^\infty - \int_0^\infty -\lambda^2 e^{-\lambda x}\,\mathrm{d}x = 0 - \left(\frac{1}{\lambda e^{\lambda x}}\right)\Bigg|_0^\infty = \frac{1}{\lambda}$$
$$E[Y] = \int_0^\infty y\lambda e^{-\lambda y}\,\mathrm{d}y = E[X] = \frac{1}{\lambda}$$
$$\therefore\ E[X+Y] = 2\frac{1}{\lambda} = \frac{2}{\lambda}.$$

7. 3.9.14 Show that $\mathrm{Cov}(aX+b, cY+d) = ac\mathrm{Cov}(X,Y)$ for any constants a, b, c, and d.
Answer:

$$\begin{aligned}
\mathrm{Cov}(aX+b, cY+d) &= E[(aX+b)(cY+d)] - E[aX+b]\cdot E[cY+d] \\
&= E[acXY + adX + bcY + bd] - E[aX+b]\cdot E[cY+d] \\
&= E[acXY] + E[adX] + E[bcY] + E[bd] - E[aX+b]\cdot E[cY+d] \\
&= acE[XY] + adE[X] + bcE[Y] + bd - (aE[X]+b)(cE[Y]+d) \\
&= acE[XY] + adE[X] + bcE[Y] + bd - acE[X]E[Y] - adE[X] - bcE[Y] - bd \\
&= acE[XY] - acE[X]E[Y] \\
&= ac(E[XY] - E[X]E[Y]) \\
&= ac\mathrm{Cov}(X,Y).
\end{aligned}$$

8. 3.9.18 Suppose that $f_{X,Y}(x,y) = \frac{2}{3}(x+2y)$, $0 \le x \le 1$, $0 \le y \le 1$. Find $\mathrm{Var}(X+Y)$.
Answer: Okay. This one's kinda lengthy. $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$. So we will find $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $2\mathrm{Cov}(X,Y)$ seperately:

$$f_X(x) = \frac{2}{3}\int_0^1 x + 2y\,\mathrm{d}y = \frac{2}{3}(xy + y^2)\Big|_0^1 = \frac{2}{3}(x+1), \forall x \in [0,1]$$
$$E[X] = \frac{2}{3}\int_0^1 x^2 + x\,\mathrm{d}x = \frac{2}{3}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Bigg|_0^1 = \frac{2}{3}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{9}$$
$$E[X^2] = \frac{2}{3}\int_0^1 x^3 + x^2\,\mathrm{d}x =$$
$$\mathrm{Var}(X) = \frac{7}{18} - \frac{25}{81} = \frac{13}{162}$$

$$f_Y(y) = \frac{2}{3}\int_0^1 x + 2y\,\mathrm{d}x = \frac{2}{3}\left(\frac{x^2}{2} + 2xy\right)\Bigg|_0^1 = \frac{2}{3}\left(\frac{1}{2} + 2y\right), \forall y \in [0,1$$
$$E[Y] = \frac{2}{3}\int_0^1 \frac{y}{2} + 2y^2\,\mathrm{d}y = \frac{2}{3}\left(\frac{y^2}{4} + \frac{2y^3}{3}\right)\Bigg|_0^1 = \frac{11}{18}$$
$$E[Y^2] = \frac{2}{3}\int_0^1 \frac{y^2}{2} + 2y^3\,\mathrm{d}y = \frac{2}{3}\left(\frac{y^3}{6} + \frac{y^4}{2}\right)\Bigg|_0^1 = \frac{4}{9}$$
$$\mathrm{Var}(Y) = \frac{4}{9} - \frac{121}{324} = \frac{23}{324}$$

$$E[XY] = \int_0^1 \int_0^1 \frac{2x^2y + 4xy^2}{3}\, \mathrm{d}y\, \mathrm{d}x = \int_0^1 \left(\frac{x^2y^2}{3} + \frac{4xy^3}{9} \right)\Bigg|_0^1 \mathrm{d}x = \int_0^1 \frac{x^2}{3} + \frac{4x}{9}\, \mathrm{d}x = \frac{x^3 + 2x^2}{9}\Bigg|_0^1 = \frac{1}{3}$$

$$2\mathrm{Cov}(X,Y) = 2\left[\frac{1}{3} - \left(\frac{5}{9} \cdot \frac{11}{18}\right)\right] = -\frac{1}{81}$$

$$\mathrm{Var}(X+Y) = \frac{13}{162} + \frac{23}{324} - \frac{1}{81} = \frac{5}{36}. \qquad \text{Wow.}$$

9. 3.9.21 A *Poisson random variable* has pdf $p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$ and $\lambda > 0$. Also, $E(X) = \lambda$. Suppose the Poisson random variable U is the number of calls for technical assistance received by a computer company during the firm's nine normal workday hours, with the average number of calls per hour equal 7.0. Also suppose each call costs the company \$50. Let V be a Poisson random variable representing the number of calls for technical assistance received during a day's remaining fifteen hours. Assume the average number of calls per hour is four for that time period and that each such call costs the company \$60. Find the expected cost and the variance of the cost associated with the calls received during a twenty-four-hour day.
   Answer: It is given that $E[U_i] = 7.0$, where $i = 1, 2, \ldots, 9$ is the i-th hour in the workday. It is also given that $E[V_j] = 4$, where $j = 1, 2, \ldots, 15$ is the j-th hour in the remaining 15 hours. If we set W to be the sum of $50 \cdot U_i$ for each i and $60 \cdot V_j$ for each j. So then we must find $E[W]$ and $\mathrm{Var}(W)$...

$$\begin{aligned} E[W] &= E\left[50\sum_{i=1}^{9} U_i + 60\sum_{j=1}^{15} V_j\right] = 50\sum_{i=1}^{9} E[U_i] + 60\sum_{j=1}^{15} E[V_j] \\ &= 50\sum_{i=1}^{9} 7 + 60\sum_{j=1}^{15} 4 = 50(9)(7) + 60(15)(4) = \$6750 \\ \mathrm{Var}(W) &= 50^2(9)(7) + 60^2(15)(4) = \$373,500 \end{aligned}$$

Optional R Basics

10. Set up R and start working through basic exercises (see R setup and R exercises on Canvas)
    Answer: