STAT 461.1002: Homework 5
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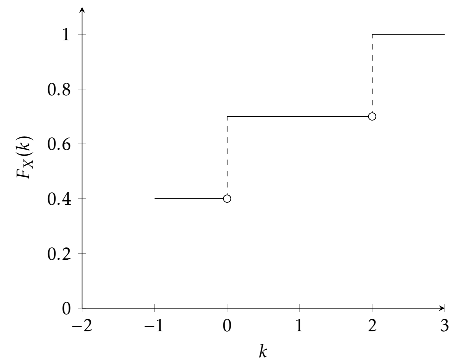
Section 3.3 Problems

1. Consider a random variable X such that $P(X = -1) = 0.4$, $P(X = 0) = 0.3$, and $P(X = 2) = 0.3$. Find the cumulative distribution function (CDF) of X (Hint: make a table) and sketch a graph of the CDF.
Answer:

k	$P_X(k)$ (PMF)	$F_X(k)$ (CDF)
-1	0.4	$P(X \le -1) = 0.4$
0	0.3	$P(X \le 0) = 0.7$
2	0.3	$P(X \le 2) = 1.0$



2. 3.3.11 Suppose X is a binomial random variable with $n = 4$ and $p = \frac{2}{3}$. What is the pdf of $2X + 1$?
Answer: We know $P(X = k) = \binom{4}{k}\left(\frac{2}{3}\right)^4$ for $k = 0, 1, 2, 3, 4$. So $P(Y = k) = P\left(X = \frac{k-1}{2}\right) = \binom{4}{\frac{k-1}{2}}\left(\frac{2}{3}\right)^4$ for $k = 1, 3, 5, 7, 9$.

$$P(Y = 1) = \binom{4}{0}\left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

$$P(Y = 3) = \binom{4}{1}\left(\frac{2}{3}\right)^4 = \frac{64}{81}$$

$$P(Y = 5) = \binom{4}{2}\left(\frac{2}{3}\right)^4 = \frac{32}{27}$$

$$P(Y = 7) = \binom{4}{3}\left(\frac{2}{3}\right)^4 = \frac{64}{81}$$

$$P(Y = 9) = \binom{4}{4}\left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

Section 3.4 Problems

3. 3.4.2 For the random variable Y with pdf $f_Y(y) = \frac{2}{3} + \frac{2}{3}y$, $0 \le y \le 1$, find $P(\frac{3}{4} \le Y \le 1)$.
Answer: Directly apply formula for continuous pdf:

$$\begin{aligned} P(3/4 \le Y \le 1) &= \int_{3/4}^{1} \frac{2}{3}y + \frac{2}{3}\,\mathrm{d}y \\ &= \frac{1}{3}y^2 + \frac{2}{3}y\Big|_{\frac{3}{4}}^{1} = 1 - \left(\frac{9}{48} + \frac{1}{2}\right) = \frac{5}{16}. \end{aligned}$$

4. 3.4.6 Let n be a positive integer. Show that $f_Y(y) = (n+2)(n+1)y^n(1-y)$, $0 \le y \le 1$, is a pdf.
Answer: The easiest way to show that this is a pdf is by seeing if the integral over $\mathbb{R}$ on $f_Y(y) = 1$:

$$\begin{aligned} \int_{-\infty}^{\infty} f(y)\,\mathrm{d}y &= \int_0^1 (n+2)(n+1)y^n(1-y)\,\mathrm{d}y \\ &= (n+2)(n+1)\int_0^1 y^n - y^{1+n}\,\mathrm{d}y \\ &= (n+2)(n+1)\left(\frac{y^{n+1}}{n+1} - \frac{y^{n+2}}{n+2}\Big|_0^1\right) \\ &= (n+2)(n+1)\left(\frac{n+2-n-1}{(n+2)(n+1)}\right) = (n+2)(n+1)\left(\frac{1}{(n+2)(n+1)}\right) = 1 \end{aligned}$$

So, $f_Y(y)$ is a pdf.

5. 3.4.7 Find the cdf for the random variable Y given in Question 3.4.1 ($f_Y(y) = 4y^3$, $0 \le y \le 1$). Calculate $P(0 \le Y \le \frac{1}{2})$ using $F_Y(y)$.
Answer:

$$F_Y(y) = \int_0^y f_Y(y)\,\mathrm{d}y = \int_0^y 4y^3\,\mathrm{d}y = y^4\Big|_0^y = y^4.$$

So, $P(0 \le Y \le 1/2) = (1/2)^4 = 1/16$.

6. **461 only** 3.4.17 Suppose that $f_Y(y)$ is a continuous and symmetric pdf, where *symmetry* is the property that $f_Y(y) = f_Y(-y)$ for all y. Show that $P(-a \le Y \le a) = 2F_Y(a) - 1$.
Answer:

$$\begin{aligned}
P(-a \le Y \le a) &= \int_{-\infty}^{a} f_Y(y)\,\mathrm{d}y + \int_{-a}^{\infty} f_Y(y)\,\mathrm{d}y - 1 \\
&= \int_{-\infty}^{a} f_Y(y)\,\mathrm{d}y + \int_{-\infty}^{a} f_Y(y)\,\mathrm{d}y - 1 && \text{Because } f_Y(y) \text{ is symmetric} \\
&= 2\int_{-\infty}^{a} f_Y(y)\,\mathrm{d}y - 1 && \text{Combined the integrals} \\
&= 2F_Y(a) - 1.
\end{aligned}$$

Section 3.5 Problems

7. Find the expected value of the random variable X defined in Problem 1 (X such that $P(X = -1) = 0.4$, $P(X = 0) = 0.3$, and $P(X = 2) = 0.3$).
Answer: $E[X] = -1(0.4) + 0(0.3) + 2(0.3) = 1/5$

8. 3.5.6 A manufacturer has one hundred memory chips in stock, 4% of which are likely to be defective (based on past experience). A random sample of twenty chips is selected and shipped to a factory that assembles laptops. Let X denote the number of computers that receive faulty memory chips. Find $E(X)$.
Answer: $p = .04$, $n = 20$, so $E[X] = (.04)(20) = 4/5$.

9. 3.5.10 Let the random variable Y have the uniform distribution over $[a, b]$; that is, $f_Y(y) = \frac{1}{b-a}$ for $a \le y \le b$. Find $E(Y)$ using Definition 3.5.1. Also, deduce the value of $E(Y)$, knowing that the expected value is the center of gravity of $f_Y(y)$.
Answer: Apply the definition directly:

$$\begin{aligned}
E[Y] = \mu &= \int_{-\infty}^{\infty} y \cdot f_Y y\,\mathrm{d}y \\
&= \int_a^b y \cdot \frac{1}{b-a}\,\mathrm{d}y = \frac{1}{b-a}\left(\frac{y^2}{2}\Big|_a^b\right) = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{b+a}{2}.
\end{aligned}$$

Alternatively, we know by deduction that the center of gravity is the midpoint of the given interval $[a, b]$. So, $E[Y] = b{+}a/2$

10. 3.5.12 Show that

$$f_Y(y) = \frac{1}{y^2},\ y \ge 1$$

is a valid pdf but that Y does not have a finite expected value.
Answer:

$$P(1 \le Y < \infty) = \int_1^{\infty} \frac{1}{y^2}\,\mathrm{d}y = -\frac{1}{y}\Big|_1^{\infty} = 0 - (-1) = 1.$$

Because $P(1 \le Y < \infty) = 1$, the pdf is valid. However, there is no finite expected value because

$$E[Y] = \int_1^{\infty} y \cdot \frac{1}{y^2}\,\mathrm{d}y = \ln|y|\Big|_1^{\infty} = \infty.$$

11. 3.5.33 Grades on the last Economics 301 exam were not very good. Graphed, their distribution had a shape similar to the pdf

$$f_Y(y) = \frac{1}{5000}(100 - y),\ 0 \le y \le 100$$

As a way of "curving" the results, the professor announces that he will replace each person's grade, Y, with a new grade, $g(Y)$, where $g(Y) = 10\sqrt{Y}$. Will the professor's strategy be successful in raising the class average above 60?
Answer:

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\,\mathrm{d}y = \frac{1}{500}\left(\int_0^{100} 100\sqrt{y} - \sqrt{y^3}\,\mathrm{d}y\right)$$

$$= \frac{1}{500}\left(\frac{200\sqrt{y^3}}{3} - \frac{2\sqrt{y^5}}{5}\Bigg|_0^{100}\right) = \frac{160}{3} \approx 53.\overline{3}.$$

No, this will not raise the average above 60.