Definitions

Definition (Norm). Given $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the **norm** of x is defined by $\|x\| := (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$.

Definition (Metric). A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$.
2. $d(x,y) = d(y,x)\ \forall x, y \in X$.
3. $d(x,z) \leq d(x,y) + d(y,z)$, $\forall x, y, z \in X$.

A **metric space** (X, d) is a set X equipped with a metric.

Definition (Euclidean Metric). The **Euclidean Metric** d on $\mathbb{R}^n$ is defined by
$d(\vec{x}, \vec{y}) := \|\vec{x} - \vec{y}\| = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2]^{1/2}$

Definition (Open/Closed Balls). Let $\vec{x} \in \mathbb{R}^n$ and $\varepsilon > 0$. The **open ball** of radius ε centered at $\vec{x}$ with respect to the metric d_p is the subset $B^p_{\vec{x}}(\varepsilon) := \{\vec{y} \in \mathbb{R}^n | d_p(\vec{x}, \vec{y}) < \varepsilon\}$.
Respectively, the **closed ball** is the subset $\overline{B}^p_{\vec{x}}(\varepsilon) := \{\vec{y} \in \mathbb{R}^n | d_p(\vec{x}, \vec{y}) \leq \varepsilon\}$.

Definition (3.1 Continuity). A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at $\vec{a} \in \mathbb{R}^n$** with respect to the metric d_p if and only if $\forall \varepsilon > 0, \exists \delta > 0$ such that $d_p(\vec{a}, \vec{x}) < \delta \implies d_p(f(\vec{a}), f(\vec{x})) < \varepsilon$.
We say f is **continuous** with respect to the metric d_p if and only if f is continuous $\forall \vec{a} \in \mathbb{R}^n$.

Definition (3.3 Converging Sequence). Let S be a set. A **sequence** in S is a function $\sigma : \mathbb{N} \to S$.
A sequence $\{\vec{x_k} \subseteq \mathbb{R}^n\}$ **converges to $\vec{a} \in \mathbb{R}^n$** with respect to the metric d_p if and only if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $d_p(\vec{x_k}, \vec{a}) < \varepsilon\ \forall k \geq N$.

Definition (4.1 Closed/Open Subsets). Let $A \subseteq \mathbb{R}^n$. A is **closed in $\mathbb{R}^n$** if and only if $\forall$ convergent sequences $\{x_k\} \subseteq A$, we have $\lim_{k\to\infty} x_k \in A$.
A is **open in $\mathbb{R}^n$** if and only if $\mathbb{R}^n - A$ is closed.

Definition (6.1 Topological Space). A **topological space** is a pair $(X, \mathcal{T})$ consisting of a set X and a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ of subsets of X satisfying the following axioms:

1. $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
2. If $\{U_i\}_{i\in I}$ is a collection of subsets of X and $\forall i \in I$ and $U_i \in \mathcal{T}$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
3. If $\{U_1, U_2, \dots, U_n\}$ is a finite collection of subsets of X such that $U_i \in \mathcal{T}\ \forall i = 1 \dots n$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

Definition (6.2). Let $(X, \mathcal{T})$ be a topological space.

1. A subset $U \subseteq X$ is **open** if and only if $U \in \mathcal{T}$.
2. A subset $A \subseteq X$ is **closed** if and only if $X - A \in \mathcal{T}$ is open.
3. Elements $x \in X$ are **points** of X.
4. If $x \in X$ and $U \in \mathcal{T}$ such that $x \in U$, then U is a **neighborhood** of x.

Definition (Different Topologies). Let X be a set. Then

1. **Metric Topology**: Let (X, d) be a metric space. $\mathcal{T}_d := \{U \subseteq X | U = \emptyset$ or $\forall x \in U, \exists \varepsilon > 0$ such that $B_x(\varepsilon) \subseteq U\}$.
2. **Discrete Topology**: $\mathcal{T}_{\text{disc}} := \{U \subseteq X\} = \mathcal{P}(X)\}$, i.e. every subset of X will be open.
3. **Trivial Topology**: $\mathcal{T}_{\text{triv}} := \{X, \emptyset\} \subseteq \mathcal{P}(X)$.
4. **Cofinite Topology** (6.5): $\mathcal{T}_{\text{cof}} := \{U \subseteq X | U = \emptyset$ or $X - U$ is finite$\}$.

5. **Subspace Topology** (7.1): Let $A \subseteq X$. Then $\mathcal{T}_A := \{U \cap A | U \in \mathcal{T}\}$.

Definition (6.4 Comparable)**.** Let X be a set and let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X. We say $\mathcal{T}_1$ and $\mathcal{T}_2$ are **comparable** if and only if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ or vice-versa. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say $\mathcal{T}_1$ is **coarser/smaller** than $\mathcal{T}_2$ and that $\mathcal{T}_2$ is **finer/larger** than $\mathcal{T}_1$.

Definition (8.2 Limit Points)**.** Let X be a topological space and $A \subseteq X$ a subset. A point $y \in X$ is a **limit point** of A if and only if for every open subset $U \subseteq X$ containing y, $A \cap (U - \{y\}) \neq \emptyset$. Define $L(A) := \{y \in X | y \text{ is a limit point of } A\}$.

Definition (9.2 Closure)**.** Let X be a topological space, $A \subseteq X$ a subset. The **closure** of A in X, $\overline{A}$, is the intersection of all closed subsets of X containing A, that is, $\overline{A} := \bigcap B$ such that $B \subseteq X$ is closed and $A \subseteq B$.

Definition (10.3 Interior/Boundary)**.** Let X be a space, $A \subseteq X$ be a subset. The **interior** of A, $\mathring{A}$, is the union of all open subsets contained in A, that is, $\mathring{A} := \bigcup U$ such that $U \subseteq A$ is open.
The **boundary** of A is the subset $\partial A := \overline{A} - \mathring{A}$.

Definition (11.1 Dense/Separable)**.** Let X be a topological space. A subset $A \subseteq X$ is **dense** if and only if $\overline{A} = X$.
X is **separable** if and only if X contains a countable dense subset, that is, $\exists A \subseteq X$ such that $\textbf{card}(A) = \textbf{card}(\mathbb{N})$ and $\overline{A} = X$.

Definition (11.2 Basis)**.** If X is a set, a **basis for a topology** on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

1. For each $x \in X$, there is at least 1 element $B \in \mathcal{B}$ such that $x \in B$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition (14.1 2nd Countability)**.** A topological space $(X, \mathcal{T})$ is **2nd Countable** if and only if there is a countable basis $\mathcal{B} = \{B_i\}_{i \geq 1}^{\infty}$ for the topology $\mathcal{T}$.

Definition (15.1 Continuity)**.** Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. Let $x \in X$. A function $f : X \to Y$ is **continuous at x** if and only if for every open subset $V \subseteq Y$ containing $f(x)$, there is an open subset $U \subseteq X$ such that $x \in U$ and $f(U) \subseteq V$. $F : X \to Y$ is **continuous** if and only if $\forall x \in X$, f is continuous at x.

Definition (17.1 Homeomorphism)**.** A continuous injective and surjective function $f : X \to Y$ between topological spaces is a **homeomorphism** if and only if its set-theoretic inverse $f^{-1} : Y \to X$ is continuous.
Two spaces are **homeomorphic** if and only if there exists a homeomorphism $f : X \to Y$ between them. We write $X \cong Y$.

Definition (18.1 Open/Closed maps)**.** A map $f : X \to Y$ is an **open** (or **closed** map if and only if for each open (or closed) subset $B \subseteq X$, the image $f(B) \subseteq Y$ is open (or closed).

Definition (19.3 Metrizable)**.** A topological space $(X, \mathcal{T})$ is **metrizable** if and only if there is a metric $d : X \times X \to \mathbb{R}$ such that the metric topology $\mathcal{T}_d$ equals $\mathcal{T}$.

Definition (19.5 Topological Property)**.** A property $\underline{P}$ of a topological space is a **topological property** if and only if it is preserved by a homeomorphism. i.e. if $(X, \mathcal{T}_X)$ has a property P and $(X, \mathcal{T}_X) \cong (Y, \mathcal{T}_Y)$, then $(Y, \mathcal{T}_Y)$ also has property P.

Definition (19.7 Converging Sequence)**.** A sequence of points $\{x_n\}_{n \geq 1}^{\infty}$ of a topological space X **converges to a point** $x \in X$ if and only if for every open neighborhood U of x, there is an $N > 0$ such that $x_n \in U \; \forall n > N$.

Definition (19.9 Hausdorff)**.** A space $(X, \mathcal{T})$ is **Hausdorff** if and only if for each pair of distinct points $x \neq y \in X$, there exist open subsets $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition (20.5)**.**

1. X is $\underline{T_0}$ if and only if for any points $x \neq y \in X$, there exist open subsets $U \subseteq X$ such that $x \in U$ and $y \notin U$ OR $y \in U$ and $x \notin U$.
2. X is $\underline{T_1}$ if and only if $\forall x \in X$, $\{x\} \subseteq X$ is closed.

3. X is $\underline{T_2}$ if and only if X is Hausdorff.

4. X is $\underline{T_3}$ if and only if for every closed subset $A \subseteq X$ and $\forall x \in X - A$, there exist open subsets $U_A, U_X \in X$ such that $A \subseteq U_A$ and $x \in U_X$.

5. X is $\underline{T_4}$ if and only if for any pair of disjoint closed subsets $A, B \subseteq X$, there exist disjoint open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

Definition (23.1). An **open cover** of a (,)X is a collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of open subsets of X such that $X = \bigcup_{\alpha \in A} I_\alpha$. If $\mathcal{U}' = \{U_\beta\}_{\beta \in A'} \subseteq \mathcal{U}$ is a subcollection of $\mathcal{U}$ such that $X = \bigcup_{\beta \in A'} U_\beta$, then $\mathcal{U}'$ is a **subcover** of $\mathcal{U}$.

Definition (23.3). A (,)X is **compact** if and only if every open cover of X has a finite subcover.

THEOREMS AND SUCH

Theorem (3.4)**.** $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\vec{a} \in \mathbb{R}^n$ if and only if for any sequence $\{\vec{x_k}\} \subseteq \mathbb{R}^n$ that converges to $\vec{a}$, the sequence $\{\vec{y_k}\} \subseteq \mathbb{R}^m$ where $\vec{y_k} = f(\vec{x_k}$ converges to $f(\vec{a})$. That is, $f(\lim \vec{x_k}) = \lim f(\vec{x_k})$.

Proposition (4.2)**.** A nonempty subset $U \subseteq \mathbb{R}^n$ is open if and only if $\forall x \in U$, $\exists \varepsilon > 0$ such that $B_x(\varepsilon) \subseteq U$.

Proposition (4.5)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$.

1. f is continuous at $x \in \mathbb{R}^n$ if and only if for all open subsets $V \subseteq \mathbb{R}^m$ containing $f(x)$, there is an open subset $U \subseteq \mathbb{R}^n$ containing x such that $U \subseteq f^{-1}(V)$.

2. f is continuous if and only if for every open subset $V \subseteq \mathbb{R}^m$, the subset $U = f^{-1}(V)$ is open in $\mathbb{R}^n$.

Theorem (5.2)**.**

1. $\mathbb{R}^n$ and $\emptyset$ are open subsets of $\mathbb{R}^n$.

2. If $\{U_i\}_{i \in I}$ is a collection of open subsets of $\mathbb{R}^n$, then $\bigcup_{i \in I} U_i$ is an open subset.

3. If $\{U_1, \ldots, U_m\}$ is a finite collection of open subsets of $\mathbb{R}^n$, then $\bigcap\limits_{i=1}^{m} U_i$ is an open subset.

Theorem (8.1)**.** Let X be a topological space. Then

1. X and $\emptyset$ are closed subsets.

2. Finite unions of closed subsets are closed.

3. Arbitrary intersections of closed subsets are closed.

Theorem (8.4)**.** Let X be a topological space and let $A \subseteq X$. Then A is closed if and only if $L(A) \subseteq A$.

Remark (9.1)**.** Useful trick to show a subset $V \subseteq X$ is open:
For each $y \in V$, find an open subset $U_y \subseteq X$ such that $y \in U_y$ and $U_y \subseteq V$. Then this implies that $V = \bigcup\limits_{y \in V} U_y$ by Axiom 2 of a topological space and so V is open.

Theorem (9.5)**.** Let $A \subseteq X$ be a subset of a topological space X. Then $\overline{A} = A \cup L(A)$.

Proposition (10.1)**.** Let X be a space, $Y \subseteq X$ a subspace. Then $B \subseteq Y$ is closed in Y if and only if there is a closed subset $A \subseteq X$ such that $B = Y \cap A$.

Proposition (10.6)**.** Let X be a space, $A \subseteq X$, and $x \in X$. Then $x \in \mathring{A}$ if and only if there is an open neighborhood $V \subseteq X$ of x such that $V \subseteq A$.

Proposition (10.7)**.** Let X be a space, $A \subseteq X$, and $x \in X$. Then $x \in \partial A$ if and only if for every open neighborhood $U \subseteq X$ of x, we have $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$.

Proposition (11.3)**.** The subset $A \subseteq X$ is dense if and only if $A \cap U \neq \emptyset$, where U is any non-empty open subset of X.

Proposition (12.1)**.** Let X be a set, $\mathcal{B}$ be a basis for a topology on X.

1. The collection of subsets $\mathcal{T}_{\mathcal{B}} := \{U \subseteq X | \forall x \in U, \exists B \in \mathcal{B}$ such that $x \in B \subseteq U\}$ is a topology on X generated by $\mathcal{B}$.

2. A subset $U \subseteq (X, \mathcal{T}_{\mathcal{B}})$ is open if and only if U is a union of elements in $\mathcal{B}$.

Proposition (12.2)**.** If $(X, \mathcal{T})$ is a topological space and $\mathcal{B} \subseteq \mathcal{T}$ such that $\forall U \in \mathcal{T}$ and $\forall x \in U$ $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$, then $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$.

Proposition (12.4 Comparison Lemma)**.** Let $\mathcal{T}, \mathcal{T}'$ be topologies on X and let $\mathcal{B}, \mathcal{B}'$ be bases for $\mathcal{T}, \mathcal{T}'$ respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if $\forall x \in X$ and $\forall B \in \mathcal{B}$ containing x, $\exists B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Theorem (14.2)**.** If $(X, \mathcal{T})$ is 2nd countable, then X contains a countable dense subset. That is, $(X, \mathcal{T})$ is separable.

Proposition (14.3)**.** Let $\mathbb{R}_l = (\mathbb{R}, \mathcal{T}_l)$ be the lower limit topology. Then $\mathbb{R}_l$ is separable and $\mathbb{R}_l$ is NOT 2nd countable.

Theorem (14.4)**.** Let (X, d) be a metric space. Let $\mathcal{T}_d$ be the metric topology on X induced by d. If $(X, \mathcal{T}_d)$ is separable, then it is 2nd countable.

Theorem (15.2)**.** A function $f : X \to Y$ is is continuous if and only if for every open subset $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open.

Proposition (15.3)**.** Let $(X, d_X), (Y, d_Y)$ be metric spaces and $\mathcal{T}_{d_X}, \mathcal{T}_{d_Y}$ be their corresponding metric topologies. Then a function $f : X \to Y$ is a map between (X, d_X) and (Y, d_Y) if and only if $\forall a \in X$ and $\forall \varepsilon > 0$, $\exists \delta_a > 0$ such that if $d_X(x, a) < \delta_a$, then $d_Y(f(x), f(a)) < \varepsilon$.

Theorem (16.1)**.** Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and let $f : X \to Y$. Then

1. f is continuous if and only if for every closed subset $B \subseteq Y$, $f^{-1}(B) \subseteq X$ is closed.
2. Let $\mathcal{B}_X, \mathcal{B}_Y$ be bases for $\mathcal{T}_X, \mathcal{T}_Y$ respectively. Then f is continuous if and only if $\forall B' \in \mathcal{B}_Y$ and $\forall x \in f^{-1}(B')$, $\exists B \subseteq \mathcal{B}_X$ such that $x \in B \subseteq f^{-1}(B')$.

Proposition (16.3)**.** If $f : X \to Y$, $g : Y \to Z$ are continuous functions between spaces X, Y and Z, then $g \circ f : X \to Z$ is continuous.

Corollary (16.4 Restriction of Domain)**.** If $f : X \to Y$ is continuous and $A \subseteq X$ is equipped with the subspace topology, then the restriction $f|_A := f \circ i_A : A \to Y$ is continuous, where $i_A(a) := a$.

Proposition (16.6)**.** Let $f : X \to Y$ be continuous, $B \subseteq Y$ be a subspace, and $j_Y : B \to Y$ be the inclusion function. Suppose $f(X) \subseteq B$. Then there exists a unique continuous function $g : X \to B$ such that $f = j_B \circ g$.

Theorem (16.7 "Pasting Lemma")**.** Let X, Y be topological spaces and let $\mathbf{a} = \{A_\alpha\}_{\alpha \in I}$ be a collection of subspaces of X such that $X = \bigcup\limits_{\alpha \in I} A_\alpha$. Suppose $\{f : A_\alpha \to Y\}_{\alpha \in I}$ is a collection of continuous functions such that $f_\alpha|_{A_\alpha \cap A_\beta} = f_\beta|_{A_\alpha \cap A_\beta}$ $\forall \alpha, \beta \in I$. If either

1. $\mathbf{a}$ is a collection of open subspaces of X, or
2. $\mathbf{a}$ is a finite collection of closed subspaces of X,

then there exists a unique continuous function $f : X \to Y$ such that $f|_{A_\alpha} = f_\alpha$ $\forall \alpha \in I$.

Proposition (18.2)**.** Let $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism if and only if f is an open (or closed) map.

Proposition (19.1)**.** If $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a homeomorphism, then it induces a bijection of sets $\mathcal{T}_X \leftrightarrow \mathcal{T}_Y$, i.e. there is a one-to-one correspondence between topologies X and Y where $U \mapsto f(U)$ and $V \mapsto f^{-1}(V)$.

Theorem (19.10)**.** If (X, d) is a metric space, then $(X, \mathcal{T}_d)$ is Hausdorff.

Proposition (20.6)**.**

1. T_1 and T_2 properties are necessary, but NOT sufficient for $(X, \mathcal{T})$ to be metrizable.
2. Separation and countability properties are distinct.

Theorem (20.8)**.** If X is Hausdorff, then every sequence in X converges to at MOST one point.

Definition (21.1)**.** X is **regular** if and only if X is T_0 and T_3.
X is **normal** if and only if X is T_1 and T_4.

Proposition (21.2)**.**

1. $T_1 \implies T_0$
2. $T_2 \implies T_1$
3. T_1 and $T_3 \implies T_2$

4. T_1 and $T_4 \implies T_3$

Theorem (21.3)**.** If $(X, \mathcal{T})$ is a topological space and metrizable, then X is T_i for $i = 0, 1, 2, 3, 4$.

Proposition (22.1)**.** If $A \subseteq X$ is a subspace and X is metrizable, then A is T_i for $i = 0, 1, 2, 3, 4$.

Lemma (22.2)**.** If X is metrizable and $A \subseteq X$ is a subspace, then A (equipped with the subspace topology) is metrizable.

Theorem (22.3)**.**

1. If X is Hausdorff (T_2), then every subspace in X is T_2.

2. If X is T_1 and T_3, then every subspace $A \subseteq X$ is T_1 and T_3.

Remark (23.5). Heine-Borel Theorem: Every closed interval $[a, b] \subseteq \mathbb{R}$ is compact. More generally, every closed and <u>bounded</u> subspace of $\mathbb{R}^n$ is compact. e.g. $S^n \subseteq \mathbb{R}^{n+1}$, $\overline{B}_{\vec{x}}(\varepsilon) \subseteq \mathbb{R}^m$.

Theorem (23.6)**.** Let $f : X \to Y$ be a continuous function between topological spaces X and Y. If X is compact, then the subspace $f(X) \subseteq Y$ is compact.

Remark (23.7). Theorem 23.6 implies that compactness is a topological property. That is, if $f : X \to Y$ is a homeomorphism between spaces X and Y, then X is compact if and only if Y is compact.

Corollary (23.8)**.** $\mathbb{R}$ is not homeomorphic to $[a, b]$ for any $a, b \in R$. Hence $[a, b] \not\cong (a, b)$.