Dillan Marroquin
MATH 310.1002: Homework 9
Due. Wed, October 28

1. Consider $f : \mathbb{R} \to \mathbb{R}$ to be a function.

   (a) Write what it means by definition that $\lim_{x\to a^-} f(x) = \infty$.
   Answer: $\lim_{x\to a^-} f(x) = \infty$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $f(x) < \varepsilon$ whenever $x \in I \setminus \{a\}$ and $\delta < x < a$.

   (b) Write what it means by definition that $\lim_{x\to -\infty} f(x) = \infty$.
   Answer: $\lim_{x\to -\infty} f(x) = \infty$ if for every $M > 0$, there is an $m > 0$ such that $f(x) > M$ whenever $x < m$.

   (c) Write what it means by definition that $\lim_{x\to \infty} f(x) = 1$.
   Answer: $\lim_{x\to \infty} f(x) = 1$ if for every $\varepsilon > 0$, there is an $m > 0$ such that $|f(x) - 1| < \varepsilon$ whenever $m < x$.

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{\sqrt{x^6+3}}{x^3-1}$ for $x < 1$ and $f(x) = \frac{x-2}{x^3+\sin \pi x}$ for $x \geq 1$.

   (a) Determine the limit $\lim_{x\to -\infty} f(x)$.
   Answer:

$$\lim_{x\to -\infty} \frac{\sqrt{x^6+3}}{x^3-1} = \lim_{x\to -\infty} \frac{\sqrt{x^6\left(1+\frac{3}{x^6}\right)}}{x^3\left(1-\frac{1}{x^3}\right)} = \lim_{x\to -\infty} \frac{\sqrt{1+\frac{3}{x^6}}}{1-\frac{1}{x^3}} = \frac{\sqrt{1}}{1} = 1.$$

   (b) Determine the limit $\lim_{x\to -1} f(x)$.
   Answer:

$$\lim_{x\to -1} \frac{\sqrt{x^6+3}}{x^3-1} = \frac{\sqrt{1+3}}{-1-1} = -1$$

   (c) Is f continuous at $x = -1$? Justify your answer.
   Answer: Yes. f is continuous at $x = -1$ because $\lim_{x\to -1} f(x) = -1 = f(-1)$.

   (d) Determine the limits $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.
   Answer:

$$\lim_{x\to 1^-} \frac{\sqrt{x^6+3}}{x^3-1} = \frac{\sqrt{1+3}}{0^-} = -\infty.$$
$$\lim_{x\to 1^+} \frac{x-2}{x^3+\sin \pi x} = \frac{1-2}{1+\sin \pi} = -1.$$

   (e) Is f continuous at $x = 1$? Justify your answer.
   Answer: No. Since $\lim_{x\to 1^-} f(x) = -\infty \neq -1 = \lim_{x\to 1^+} f(x)$, then $\lim_{x\to 1} f(x)$ does not exist and thus is not continuous at $x = 1$.

   (f) Evaluate $\lim_{x\to \infty} f(x)$. Mention the theorem you are using and provide all the details.
   Answer:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x-2}{x^3+\sin \pi x}.$$

   Now we apply the Squeeze Theorem to the denominator of this expression. Observe that $-1 \leq \sin \pi x \leq 1$ implies $x^3 - 1 \leq x^3 + \sin \pi x \leq x^3 + 1$ and that $\lim x^3 - 1 = \lim x^3 + 1 = \infty$. We may rewrite our initial limit:

$$\lim_{x\to\infty} \frac{x-2}{x^3+\sin \pi x} = \lim_{x\to\infty} \frac{x-2}{x^3-1} = \lim_{x\to\infty} \frac{\frac{1}{x^2}-\frac{2}{x^3}}{1-\frac{1}{x^3}} = \frac{0}{1} = 0.$$

3. Discuss the existence and the value of the limit $\lim\limits_{x\to 1} \frac{x-b}{(x-1)^2}$ for different values of the parameter $b \in \mathbb{R}$.

Answer: We note that $\frac{x-b}{(x-1)^2}$ is continuous everywhere except when $(x-1)^2 = 0$, so we will consider values of b that effect the denominator.

Case 1: For $b = 1$, we have $\lim\limits_{x\to 1} \frac{1}{x-1}$. This limit does not exist.

Case 2: For any $b > 1$, we have $\lim\limits_{x\to 1} \frac{x-b}{(x-1)^2}$. We now consider the left and right-sided limits to see if the limit exists.

$$\lim_{x\to 1^-} \frac{x-b}{(x-1)^2} = \frac{1-b}{0^-} = \infty$$
$$\lim_{x\to 1^+} \frac{x-b}{(x-1)^2} = \frac{1-b}{0^+} = \infty,$$

so $\lim\limits_{x\to 1} \frac{x-b}{(x-1)^2} = \infty$ for all $b < 1$.

Case 3: For any $b < 1$, we have $\lim\limits_{x\to 1} \frac{x-b}{(x-1)^2}$. We now consider the left and right-sided limits to see if the limit exists.

$$\lim_{x\to 1^-} \frac{x-b}{(x-1)^2} = \frac{1-b}{0^-} = -\infty$$
$$\lim_{x\to 1^+} \frac{x-b}{(x-1)^2} = \frac{1-b}{0^+} = -\infty,$$

so $\lim\limits_{x\to 1} \frac{x-b}{(x-1)^2} = -\infty$ for all $b > 1$.

4. Find the value(s) of the parameter $a \in \mathbb{R}$ for which the limit $\lim\limits_{x\to 1} \frac{x-a}{x^2+2x-3}$ exists.

Answer: We first factor the denominator: $\lim\limits_{x\to 1} \frac{x-a}{x^2+2x-3} = \lim\limits_{x\to 1} \frac{x-a}{(x+3)(x-1)}$. We now observe that evaluating the limit directly at this point would cause the denominator to be 0 and therefore not exist. If we choose $a = 1$, we can avoid this: $\lim\limits_{x\to 1} \frac{x-(1)}{(x+3)(x-1)} = \lim\limits_{x\to 1} \frac{1}{x+3} = \frac{1}{4}$. So choosing $a = 1$ will allow for this limit at 1 to exist.

5. Evaluate the limits $\lim\limits_{x\to\infty} \frac{x^3-x^2+1}{2x^2+5}$ and $\lim\limits_{x\to-\infty} \frac{(3x+1)^2}{(2x-1)(x+2)}$.

Answer:

$$\begin{aligned}\lim \frac{x^3-x^2+1}{2x^2+5} &= \lim \frac{x^2(x-1+\frac{1}{x^2})}{x^2(2+\frac{5}{x^2})} \\ &= \lim \frac{x-2+\frac{1}{x^2}}{2+\frac{5}{x^2}} = \frac{\infty}{2} = \infty\end{aligned}$$

$$\begin{aligned}\lim \frac{(3x+1)^2}{(2x-1)(x+2)} &= \lim \frac{9x^2+6x+1}{2x^2+3x-2} \\ &= \lim \frac{x^2(9+\frac{6}{x}+\frac{1}{x^2})}{x^2(2+\frac{3}{x}-\frac{2}{x^2})} = \frac{9}{2}\end{aligned}$$

6. Does the limit $\lim\limits_{x\to 0} \frac{|\sin x|}{x}$ exist? Justify your answer.

Answer: This limit does not exist. To show this, we take the one-sided limits as x approaches 0.

$$\lim_{x\to 0^-} \frac{|\sin x|}{x} = -1 \qquad \lim_{x\to 0^+} \frac{|\sin x|}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

Since $\lim\limits_{x\to 0^-} \frac{|\sin x|}{x} \neq \lim\limits_{x\to 0^+} \frac{|\sin x|}{x}$, the limit does not exist.