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MATH 310.1002: Homework 10
Due. Mon, November 2

1. Use the *definition* of derivative to find $f'(2)$ where

$$f(x) = \frac{1}{x^2} \quad \text{for } x \neq 0$$

Answer:

$$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{\frac{1}{x^2}-\frac{1}{4}}{x-2} = \lim_{x\to 2} \frac{\frac{-(x^2-4)}{4x^2}}{x-2} = \lim_{x\to 2} \frac{\frac{-(x+2)(x-2)}{4x^2}}{x-2} = \lim_{x\to 2} \frac{-(x+2)}{4x^2} = \frac{-4}{16} = -\frac{1}{4}$$

2. Let $g(x) = |x-3|$ for $x \in \mathbb{R}$. Prove that g is not differentiable at $x = 3$.
Answer: Observe that $g'(x) = \frac{x-3}{|x-3|}$. We will prove that the left-sided limit of $g'(3)$ does not equal the right-sided limit of $g'(3)$, i.e. $g_l'(3) \neq g_r'(3)$:

$$g_l'(3) = \lim_{x\to 3^-} \frac{x-3}{|x-3|} = \lim_{x\to 3^-} \frac{x-3}{-(x-3)} = -1 \qquad g_r'(3) = \lim_{x\to 3^+} \frac{x-3}{|x-3|} = \lim_{x\to 3^+} \frac{x-3}{x-3} = 1.$$

Because $-1 = g_l'(3) \neq g_r'(3) = 1$, g is not differentiable at $x = 3$.

3. Let $h(x) = x^2|x|$. Prove that h is differentiable on $\mathbb{R}$ and determine $h'(x)$.
Answer: We first recognize that this function can be rewritten piecewise due to the absolute value:

$$h(x) = \begin{cases} -x^3, & x < 0 \\ x^3, & x \geq 0 \end{cases}.$$

Now we can find the derivatives piecewise:

$$h'(x) = \begin{cases} -3x^2, & x < 0 \\ 3x^2, & x \geq 0 \end{cases}.$$

4. Let a, b be real numbers and the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x+1$ for $x < 1$ and $g(x) = x^2 + ax + b$ for $x \geq 1$. Determine for which values of a, b the function g is differentiable on $\mathbb{R}$ and for these values of a, b evaluate $g'(x)$.
Answer: By observation, g is differentiable everywhere except possibly at $x = 1$, so we must determine what values of a, b make g continuous at $x = 1$. To do this, we take the left and right-sided limits as x approaches that point and see when these both equal $g(1) = a + b$: $\lim_{x\to 1^+} 2x+1 = 2+1 = 3$ and $\lim_{x\to 1^-} x^2 + ax + b = 1 + a + b$. Therefore, we choose $g(1) = 3$ and so $b = 2 - a$.
Notice that $g_r'(x) = 2x + a,\ x \geq 1$ and $g_l'(x) = 2,\ x < 1$ and that $g_r'(1) = 2 + a$ and $g_l'(1) = 2$. Then we know that $2 = 2 + a$ and so $a = 0$. Therefore, $a = 0$, $b = 2$.

5. Let $k(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x}$ and $l(x) = x^{3x}$. Evaluate $k'(x)$ and $l'(x)$.
Answer: We first rewrite $l(x)$ to be $l(x) = e^{\ln x^{3x}} = e^{3x\ln x}$. Now we differentiate both functions:

$$\begin{aligned} k'(x) &= \tfrac{\mathrm{d}}{\mathrm{d}x} \arctan x + \tfrac{\mathrm{d}}{\mathrm{d}x} \arctan \tfrac{1}{x} \\ &= \frac{1}{1+x^2} + \left(\frac{1}{1+(\frac{1}{x})^2} \cdot -\frac{1}{x^2} \right) \\ &= \frac{1}{1+x^2} - \frac{1}{1+x^2} \\ &= 0. \end{aligned}$$

$$\begin{aligned} l'(x) &= e^{3x\ln x}(3\ln x + 3x \cdot \frac{1}{x}) \\ &= xe^{3x}(3\ln x + 3) \\ &= 3xe^{3x}(\ln x + 1). \end{aligned}$$