Chapter 8: Functions on Euclidean Space

Definition. $F_n \xrightarrow{\text{pw}} F$ if $\forall \varepsilon > 0$, $\exists N = N(x)$ with $\|F_n(x) - F(x)\| < \varepsilon$ for $n \geq N$.

Definition. $F_n \xrightarrow{u} F$ if $\forall \varepsilon > 0$, $\exists N$ w/ $\|F_n(x) - F(x)\| < \varepsilon$ $\forall n \geq N$ and $\forall x \in D$.

Theorem. Let $F_n, F : D \to \mathbb{R}^q$, $D \subseteq \mathbb{R}^p$. If $\|F_n(x) - F(x)\| \leq b_n$ $\forall x \in D$ and $b_n \to 0$ and $\{b_n\} \subseteq \mathbb{R}$, then $F_n \xrightarrow{u} F$.

Theorem. If $F_n \xrightarrow{u} F$ and F_n is continuous, then F is continuous.

Theorem (W.M. Test)**.** If $\|F_k(x)\| \leq M_k$ for some $M_k > 0$ $\forall x \in D$ with $\sum_{k=1}^{\infty} M_k$ convergent $< \infty$. Then $\sum_{k=1}^{\infty} F_k$ is uniformly convergent.

Definition. $L : \mathbb{R}^p \to \mathbb{R}^q$ is **linear** if $L(x+y) = L(x) + L(y)$ and $L(cx) = cL(x)$ $\forall x, y \in \mathbb{R}^p$ and $\forall c \in \mathbb{R}$.

Chapter 9: Differentiation in Several Variables

Definition. $\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$.

Definition. We say F is **differentiable** at a iff $\exists$ linear function such that $\lim_{h \to 0} \frac{F(a+h) - F(a) - L(h)}{\|h\|} = 0$.

Theorem (Chain Rule)**.** $d(F \circ G)(a) = dF(G(a))dG(a)$.

Definition. Level surfaces are surfaces where $f(x,y,z) = c$, e.g. $S = f(x,y,z) = 1$ for function $f(x,y,z) = x^2 + y^2 - z^2$. To find tangent plane to S at (a,b,c), calculate $\nabla f(a,b,c) = (a', b', c')$. Then tangent plane to S is $a'(x-a) + b'(y-b) + c'(z-c) = 0$.

Theorem (Taylor Formula)**.** $f(a) + \mathrm{d}f(a)h + \frac{1}{2!}\mathrm{d}^2 f(a)h^2 + \cdots + \frac{1}{n!}\mathrm{d}^n f(a)h^n$.

Theorem. A critical point is a point a in the domain such that $\mathrm{d}f(a) = 0$. Plug in these CPs to $\Delta = f_{x^2} f_{y^2} - f_{xy}^2$. If $\Delta < 0$, saddle point. If $\Delta > 0$, take f_{x^2} at those CPs. If $f_{x^2} > 0$, local min. If $f_{x^2} < 0$, local max.

Theorem (Inverse Function Theorem)**.** $\mathrm{d}F^{-1}(F(a)) = \mathrm{d}F(a)^{-1}$.

Theorem. $\mathrm{d}G(a) = -\left(\frac{\partial F}{\partial y}(a,b)\right)^{-1} \frac{\partial F}{\partial x}(a,b)$.