MATH 310.1002: Homework 4
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I. **Using the properties of the limits determine**

1.
$$\lim_{n\to\infty}\frac{\cos(n^2)}{n}$$

Answer: We will apply the Squeeze Theorem. Observe that $-1 \le \cos n^2 \le 1$, so then $-\frac{1}{n} \le \frac{\cos n^2}{n} \le \frac{1}{n}$ and

$$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n} = 0.$$

So,

$$\lim_{n\to\infty}\frac{\cos(n^2)}{n} = 0.$$

2.
$$\lim_{n\to\infty}\frac{n^2+1}{2n-3}$$

Answer:
$$\lim_{n\to\infty}\frac{n^2+1}{2n-3} = \lim_{n\to\infty}\frac{n(n+1/n)}{n(2-3/n)} = \lim_{n\to\infty}\frac{n+1/n}{2-3/n} = \lim_{n\to\infty}\frac{n}{2} = \infty$$

3.
$$\lim_{n\to\infty}\frac{6n(\sqrt{n}+1)}{(2\sqrt{n}-1)^3}$$

Answer:

$$\lim_{n\to\infty}\frac{6n(\sqrt{n}+1)}{(2\sqrt{n}-1)^3} = \lim_{n\to\infty}\frac{6n(\sqrt{n}+1)}{-12n+8n\sqrt{n}+6\sqrt{n}-1} = \lim_{n\to\infty}\frac{6n\sqrt{n}\left(1+\frac{1}{\sqrt{n}}\right)}{n\sqrt{n}\left(-\frac{12}{\sqrt{n}}+8+\frac{6}{n}-\frac{1}{n\sqrt{n}}\right)}$$

$$= \lim_{n\to\infty}\frac{6+\frac{6}{\sqrt{n}}}{-\frac{12}{\sqrt{n}}+8+\frac{6}{n}-\frac{1}{n\sqrt{n}}} = \frac{6}{8} = \frac{3}{4}.$$

4.
$$\lim_{n\to\infty}\left(\sqrt{n^2+5n+1}-\sqrt{n^2-n+1}\right)$$

Answer:

$$\begin{aligned}
&= \lim_{n\to\infty}\left(\sqrt{n^2+5n+1}-\sqrt{n^2-n+1}\right)\cdot\frac{\sqrt{n^2+5n+1}+\sqrt{n^2-n+1}}{\sqrt{n^2+5n+1}+\sqrt{n^2-n+1}}\\
&= \lim_{n\to\infty}\frac{n^2+5n+1-n^2+n-1}{\sqrt{n^2+5n+1}+\sqrt{n^2-n+1}}\\
&= \lim_{n\to\infty}\frac{6n}{\sqrt{n^2+5n+1}+\sqrt{n^2-n+1}}\\
&= \lim_{n\to\infty}\frac{1/n}{1/n}\cdot\frac{6n}{\sqrt{n^2+5n+1}+\sqrt{n^2-n+1}}\\
&= \lim_{n\to\infty}\frac{6}{\sqrt{1+5/n+1/n^2}+\sqrt{1-1/n+1/n^2}}\\
&= 3.
\end{aligned}$$

5.
$$\lim_{n\to\infty} \sqrt[n]{n!}$$

Answer: We will solve using Squeeze Theorem. Denote $\sqrt[n]{n} = 1 + x_n$ so that $n = (1 + x_n)^n$. Now observe that $x_n > 0$ implies that $\sqrt[n]{n} - 1 > 0$. Then it follows that

$$n = 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \cdots > \frac{n(n-1)}{2}x_n^2$$
$$\frac{2n}{n(n-1)} > x_n^2$$
$$\sqrt{\frac{2}{n-1}} > x_n > 0.$$

Then by the Squeeze Theorem, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = \lim_{n\to\infty} 0 = 0$. We now know $\lim_{n\to\infty} \sqrt[n]{n} = 1$ by consequence. It now suffices to find a sequence $(y_n)_n$ such that $\sqrt[n]{n!} > y_n$. Take $y_n = \sqrt{n}$. Since $\lim_{n\to\infty} \sqrt{n} = \infty$, by the Squeeze Theorem, $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$ as well.

6.
$$\lim_{n\to\infty} \sqrt[n]{2^n + 3^n + 5^n}$$

Answer: We will solve using Squeeze Theorem. First, we observe that $\sqrt[n]{5^n} \leq \sqrt[n]{2^n + 3^n + 5^n}$ and that $\lim_{n\to\infty} \sqrt[n]{5^n} = 5$. Next, we observe that $\sqrt[n]{5^n + 5^n + 5^n} > \sqrt[n]{2^n + 3^n + 5^n}$ and that $\lim_{n\to\infty} \sqrt[n]{5^n + 5^n + 5^n} = \lim_{n\to\infty} 5\sqrt[n]{3} = 5 \cdot 1 = 5$.
Thus, $\lim_{n\to\infty} \sqrt[n]{2^n + 3^n + 5^n} = 5$ by the Squeeze Theorem.

7.
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin k}{n^2 + k}$$

Answer: Looks to be 0.

8.
$$\lim_{n\to\infty} (n \sin n - n\sqrt{n})$$

Answer: We will solve using Squeeze Theorem. Observe that $-1 \leq \sin n \leq 1$. Subtracting $\sqrt{n}$ and multiplying by n on all sides gives $n(-1 - \sqrt{n}) \leq n(\sin n - \sqrt{n}) \leq n(1 - \sqrt{n})$. Now we observe that $\lim_{n\to\infty} n(-1 - \sqrt{n}) = \infty(-\infty) = -\infty$ and that $lim_{n\to\infty} n(1 - \sqrt{n}) = \infty(-\infty) = -\infty$.
Therefore by the Squeeze Theorem, $\lim_{n\to\infty}(n \sin n - n\sqrt{n}) = -\infty$.

II. Determine the values of $a, b \in \mathbb{R}$ for which

$$\lim_{n\to\infty} \left(\sqrt{an^2 + bn + 5} - 2n\right) = 3$$

Answer:

$$\sqrt{an^2 + bn + 5} - 2n = \frac{\sqrt{an^2 + bn + 5}^2 - (2n)^2}{\sqrt{an^2 + bn + 5} + 2n} = \frac{(a-4)n^2 + bn + 5}{n\left[s\sqrt{a + \frac{b}{n} + \frac{5}{n^2}} + 2\right]} = \frac{(a-4)n + b + \frac{5}{n}}{\sqrt{a + \frac{b}{n} + \frac{5}{n^2}} + 2}.$$

Observe that if $a - 4 \neq 0$, the limit is either ∞ or $-\infty$. This is a contradiction, so $a - 4 = 0$ and thus $a = 4$. Now we may solve for b:

$$\lim_{n\to\infty} \frac{b + \frac{5}{n}}{\sqrt{4 + \frac{b}{n} + \frac{5}{n^2}} + 2} = \frac{b}{4} = 3.$$

Thus, $b = 12$ and $a = 4$.

III. Give an example of two sequences $(x_n)_n$, $(x_y)_n$ such that $\lim x_n = 1$, $\lim y_n = \infty$, and the $(x_n^{y_n})_n$ is not convergent.

Answer: Let us choose $y_n = n$ and $x_n = \sqrt[n]{\frac{1}{n} + 1}$. Then $(x_n^{y_n})_n$ diverges.