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MATH 440.1001
Scribing Week 5
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Lecture 12

Proposition (12.1)**.** Let X be a set, $\mathcal{B}$ be a basis for a topology on X.

1. The collection of subsets $\mathcal{T}_{\mathcal{B}} := \{U \subseteq X | \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}$ is a topology on X generated by $\mathcal{B}$.
2. A subset $U \subseteq (X, \mathcal{T}_{\mathcal{B}})$ is open if and only if U is a union of elements in $\mathcal{B}$.

I will omit the proof of Prop 12.1 which verifies $\mathcal{T}_{\mathcal{B}}$ as a topology on X generated by $\mathcal{B}$.

Proposition (12.2)**.** If $(X, \mathcal{T})$ is a topological space and $\mathcal{B} \subseteq \mathcal{T}$ such that $\forall U \in \mathcal{T}$ and $\forall x \in U$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$, then $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$.

Examples of $\mathcal{T}_{\mathcal{B}}$

Ex (12.3)**.** :

1. The Standard/Euclidean Topology on $\mathbb{R}$.
   This has a basis $\mathcal{B}_{\text{Euc}} := \{(a,b) | a < b \in \mathbb{R}\}$.
2. The Lower Limit Topology $\mathbb{R}_l := \{(\mathbb{R}, \mathcal{T}_l)\}$.
   $\mathcal{B}_l := \{[a,b) | a < b \in \mathbb{R}\}$.
3. The Upper Limit Topology $\mathbb{R}_u := \{(\mathbb{R}, \mathcal{T}_u)\}$.
   $\mathcal{B}_u := \{(a,b] | a < b \in \mathbb{R}\}$.
4. The "K-Topology" $\mathbb{R}_K := (\mathbb{R}, \mathcal{T}_K)$, where $K := \{1/n | n \in \mathbb{N}\}$.
   $\mathcal{B}_K := \mathcal{B}_{\text{Euc}} \cup \{(a,b) - K | a < b \in \mathbb{R}\}$.

Lemma (Comparison Lemma 12.4)**.** Let $\mathcal{T}, \mathcal{T}'$ be topologies on X and let $\mathcal{B}, \mathcal{B}'$ be bases for $\mathcal{T}, \mathcal{T}'$ respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ (i.e. $\mathcal{T}$ is smaller) if and only if $\forall x \in X$ and $\forall B \in \mathcal{B}$ containing x, $\exists B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof. ($\implies$) Let $x \in X$ and $B \in \mathcal{B}$ such that $x \in B$. Since every basis element is an open subset, $B \in \mathcal{T} \subseteq \mathcal{T}'$. Therefore B is an open subset of $(X, \mathcal{T}')$. Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$, Prop. 12.1 implies that $\exists B' \in \mathcal{B}'$ such that $x \in B \subseteq \mathcal{B}$.
($\impliedby$) Let $U \in \mathcal{T}$. We want to show $U \in \mathcal{T}'$. It suffices to show that $\forall x \in U, \exists B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$. Prop. 12.2 says that U is the union of elements in $\mathcal{B}$. This implies that $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$. By hypothesis, this implies $\exists B' \in \mathcal{B}'$ such that $x \in B' \subseteq B \subseteq U$. Thus $B' \subseteq U$ as desired. $\square$

Lecture 13

Back to examples!

- Lower Limit Topology: $\mathbb{R}_l := \{\mathbb{R}, \mathcal{T}_l\}$.
  $\mathcal{B}_l := \{[a,b) | a < b \in \mathbb{R}\}$.
- Claim: $\mathcal{B}_l$ is a basis for a topology. We define $\mathcal{T}_l$ to be generated by the basis.

Recall the axioms for a basis:

1. $\forall x \in X, \exists B \in \mathcal{B}$ such that $x \in B$.

2. $\forall B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Proof. (Axiom 1) Let $x \in \mathbb{R}$. Then $x \in [x, x+1) \in \mathcal{B}_l$.
(Axiom 2) Let $B_1 = [a_1, b_1)$, $B_2 = [a_2, b_2) \in \mathcal{B}_l$ and suppose $x \in [a_1, b_1) \cap [a_2, b_2)$. Let $a_3 = \max\{a_1, a_2\}$, $b_3 = \min\{b_1, b_2\}$. Then $x \in [a_3, b_3) \subseteq [a_1, b_1) \cap [a_2, b_2)$. ☐

Remark. $\mathbb{R} = (\mathbb{R}, \mathcal{T}_{\text{Euc}})$, $\mathcal{B}_{\text{Euc}} := \{(a,b) | a < b \in \mathbb{R}\}$. Compare $\mathcal{T}_{\text{Euc}}$ to $\mathcal{T}_l$ on $\mathbb{R}$.
Claim: $\mathcal{T}_l$ is finer than $\mathcal{T}_{\text{Euc}}$, i.e. $\mathcal{T}_{\text{Euc}} \subseteq \mathcal{T}_l$.

Proof. Use Lem. 12.4 using $\mathcal{B}_{\text{Euc}} = \mathcal{B}$, $\mathcal{B}_l = \mathcal{B}'$. Let $x \in \mathbb{R}$, $B = (a,b) \in \mathcal{B}_{\text{Euc}}$. Let $x \in (a,b)$. We want to show $\exists [c,d) \in \mathcal{B}_l$ such that $x \in [c,d) \subseteq (a,b)$. Let $c = x$, $d = b$. Then $x \in [a,b) \subseteq (a,b)$. ☐

Claim: $\mathcal{T}_l \not\subseteq \mathcal{T}_{\text{Euc}}$.

Proof. We will prove this by contradiction via Lem. 12.4. Consider $B = [a,b) \in \mathcal{B}_l$. Let $x = a$. We want to show $\exists (c,d) \in \mathcal{B}_{\text{Euc}}$ such that $a \in (c,d) \subseteq [a,b)$. If $a \in (c,d)$, then $\exists \varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \subseteq (c,d)$. But $(a-\varepsilon, a+\varepsilon) \not\subseteq [a,b)$. Therefore $(c,d) \not\subseteq [a,b)$. ☐

Lecture 14

Big Ideas:

- 2nd Countable Spaces
- Bases for topologies are NOT unique. Sometimes you can find ”small” ones.

Ex (On $\mathbb{R}$)**.** $\mathcal{B}_{\text{Euc}}$ is a basis for Euclidean topology on $\mathbb{R}$.
On the other hand, consider $\widetilde{\mathcal{B}} := \{(q_1, q_2) | q_1 < q_2 \in \mathbb{Q}\} \subset \mathcal{B}_{\text{Euc}}$.
Claim: $\widetilde{\mathcal{B}}$ is another basis for $\mathcal{T}_{\text{Euc}}$ on $\mathbb{R}$.

I will omit this proof.

Note: There is an injective function $\widetilde{\mathcal{B}} \to \mathbb{Q} \times \mathbb{Q}$ (i.e. an interval $(q_1, q_2) \mapsto (q_1, q_2)$ ordered pair.) $\implies |\widetilde{\mathcal{B}}| \leq |\mathbb{Q} \times \mathbb{Q}| \implies \widetilde{\mathcal{B}}$ is countable.

Definition (14.1)**.** A topological space $(X, \mathcal{T})$ is **2nd countable** if and only if there exists a countable basis $\mathcal{B} = \{B_i\}_{i \geq 1}^{\infty}$ for the topology $\mathcal{T}$.

Theorem (14.2)**.** If $(X, \mathcal{T})$ is 2nd countable, then X contains a countable dense subset, i.e. $(X, \mathcal{T})$ is separable.

Proof. Let $\mathcal{B} = \{B_i\}_{i=1}$ be a countable basis for $(X, \mathcal{T})$. Then $\forall i \in \mathbb{N}$, choose $a_i \in B_i$. Let $D := \{a_1, a_2, \ldots\}$. Clearly $D \subseteq X$ is countable by construction. Claim D is dense in X. Let $U \subseteq X$ be open and $x \in U$. Then by Prop 11.3, $\mathcal{B}$ is a basis which implies $\exists B_i \in \mathcal{B}$ such that $x \in B_i \subseteq U$ which implies $a_i \in U$. This implies $D \cap U \neq \emptyset$. Therefore, D is countable and dense. ☐

Proposition (14.3)**.** Let $\mathbb{R}_l$ be the Lower Limit Topology. Then

1. $\mathbb{R}_l$ is separable
2. $\mathbb{R}_l$ is not 2nd countable

Proof. (1.) MATH 310 Analysis shows that $\mathbb{Q}$ is dense in $\mathbb{R}_l$.
(2.) Let $\mathcal{B}$ be a basis for $\mathbb{R}_l$. We want to show $\mathcal{B}$ is NOT countable. We build a function $f : \mathbb{R} \to \mathcal{B}$ such that $x \mapsto B_x$. Then $x \in B_x \subseteq [x, x+1) \implies |\mathbb{R}| \leq |\mathcal{B}|$. Thus $\mathcal{B}$ is not countable. ☐

Theorem (14.4)**.** Let (X, d) be a metric space and let $\mathcal{T}_d$ be the metric topology on X induced by d. If $(X, \mathcal{T}_d)$ is separable, then it is 2nd countable.