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MATH 310.1002: Homework 6
Due. Wed, October 7

1. Let $\{a_n\}_n$ be a sequence. Suppose that there exists $M > 0$ and $r \in (0,1)$ such that $|a_n - a_{n+1}| < Mr^n$ for all $n \in \mathbb{N}$. Prove that $\{a_n\}_n$ converges. (*Hint:* First prove that $\{a_n\}_n$ is a Cauchy sequence.)
Answer: We wish to prove that $\{a_n\}_n$ is Cauchy, i.e. that for any $\varepsilon > 0$, there is a N_ε such that $|a_n - a_m| \leq \varepsilon$ for all $n, m \geq N_\varepsilon$. It is given that $|a_n - a_{n+1}| < Mr^n$, so then

$$\begin{aligned}
&n \to n+1 : |a_{n+1} - a_{n+2}| < Mr^{n+1} \\
&n \to n+2 : |a_{n+2} - a_{n+3}| < Mr^{n+2} \\
&\vdots \\
&n \to n+k-1 : |a_{n+k-1} - a_{n+k}| < Mr^{n+k-1},
\end{aligned}$$

for all $n \in \mathbb{N}$. Adding the above inequalities gives

$$|a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + |a_{n+2} - a_{n+3}| + \cdots + |a_{n+k-1} - a_{n+k}| < Mr^n + \cdots Mr^{n+k-1}.$$

By the triangle inequality, we get the following:

$$\begin{aligned}
|a_n - a_{n+k}| &= |(a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) + (a_{n+2} - a_{n+3}) + \cdots + (a_{n+k-1} - a_{n+k})| \\
&\leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + |a_{n+2} - a_{n+3}| + \cdots + |a_{n+k-1} - a_{n+k}| \\
&< Mr^n(1 + r + r^2 + \cdots + r^{k-1}) = Mr^n \frac{1 - r^k}{1 - r} < \frac{M}{1-r} r^n.
\end{aligned}$$

Solving the previous inequality for n gives

$$\begin{aligned}
r^n &< \frac{\varepsilon(1-r)}{M} \\
n \ln r &< \ln \frac{\varepsilon(1-r)}{M} \\
n &> \frac{\ln \frac{\varepsilon(1-r)}{M}}{\ln r}.
\end{aligned}$$

Let $N_\varepsilon = \left\lfloor \frac{\ln \frac{\varepsilon(1-r)}{M}}{\ln r} \right\rfloor + 1$. Then for every $n, m \geq N_\varepsilon$, we have $|a_n - a_m| < \varepsilon$.

2. Let $\{a_n\}_n$ be a sequence such that $|a_n - a_{n+1}| \to 0$. Must $\{a_n\}_n$ converge? If so, prove it, and if not, find a counterexample.
Answer: No, $\{a_n\}_n$ does not have to converge.

Proof. To prove that this sequence does not have to converge, we will provide a counterexample. Let $a_{n+1} - a_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$, which does converge to 0. Then we have the following:

$$\begin{aligned}
a_n - a_{n-1} &= \frac{1}{n} \\
a_{n-1} - a_{n-2} &= \frac{1}{n-1} \\
a_{n-2} - a_{n-3} &= \frac{1}{n-2} \\
&\vdots \\
a_2 - a_1 &= \frac{1}{2}.
\end{aligned}$$

Adding together the above equations gives $a_n - a_1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}$. Solving for a_n then gives $a_n = a_1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}$. Observe that this is a harmonic series that diverges to infinity, so $\{a_n\}_n$ does not converge.

$\square$

3. Prove that the sequence $\{a_n\}_n$ given below is not a Cauchy sequence.

$$a_n = (-1)^n \frac{n+1}{3n}.$$

4. Find three convergent subsequences of $\{a_n\}_n$ with distinct limits and find these limits. Is $\{a_n\}_n$ Cauchy?

$$a_n = \frac{1}{n} + \cos\frac{n\pi}{3}.$$

Answer:

$$\begin{aligned} a_{6n} &= \frac{1}{6n} + \cos\frac{6n\pi}{3} = \frac{1}{6n} + \cos(2n\pi) = \frac{1}{6n} + 1 \to 1 \\ a_{6n+1} &= \frac{1}{6n+1} + \cos\frac{(6n+1)\pi}{3} = \frac{1}{6n+1} + \cos\Big(2n\pi + \frac{\pi}{3}\Big) = \frac{1}{6n+1} + \frac{1}{2} \to \frac{1}{2} \\ a_{6n+2} &= \frac{1}{6n+2} + \cos\frac{(6n+2)\pi}{3} = \frac{1}{6n+2} + \cos\Big(\frac{2\pi}{3}\Big) = \frac{1}{6n+2} - \frac{1}{2} \to -\frac{1}{2} \end{aligned}$$

Because the subsequences of $\{a_n\}_n$ all approach different values as $n \to \infty$, $\{a_n\}_n$ is not Cauchy.

5. In each case find $\limsup_n a_n$ and $\liminf_n a_n$.

(a) $a_n = 1 + (-1)^n \frac{2n+3}{n}$
Answer: We will consider the 2 subsequences a_{2n} and a_{2n+1}.

$$\begin{aligned} a_{2n} &= 1 + (-1)^{2n}\Big(\frac{2(2n)+3}{2n}\Big) = 1 + 1\Big(\frac{4n+3}{2n}\Big) \implies \lim\Big[1 + 1\Big(\frac{4n+3}{2n}\Big)\Big] = 1 + 2 = 3 \\ a_{2n+1} &= 1 + (-1)^{2n+1}\Big(\frac{2(2n+1)+3}{2n+1}\Big) = 1 - 1\Big(\frac{4n+5}{2n}\Big) \implies \lim\Big[1 - 1\Big(\frac{4n+5}{2n}\Big)\Big] = 1 - 2 = -1. \end{aligned}$$

So, $\limsup_n a_n = \lim a_{2n} = 3$ and $\liminf_n a_n = \lim a_{2n+1} = -1$.

(b) $a_n = \cos\frac{n\pi}{3}$
Answer:

$$a_1 = \cos\frac{\pi}{3} = \frac{1}{2},\ a_2 = \cos\frac{2\pi}{3} = -\frac{1}{2},\ a_3 = \cos\pi = -1,\ a_4 = -\frac{1}{2},\ a_5 = \frac{1}{2},\ a_6 = 1$$

$$\begin{aligned} \liminf x_n &= \sup_n \inf x_k = \sup_n \inf\Big\{-1, -\frac{1}{2}, \frac{1}{2}, 1\Big\} = \sup_n\{-1\} = -1 \\ \limsup x_n &= \inf_n \sup x_k = \inf_n \sup\Big\{-1, -\frac{1}{2}, \frac{1}{2}, 1\Big\} = \inf_n\{1\} = 1 \end{aligned}$$

(c) $a_n = \frac{((-1)^n - 2)^n}{2^n}$
Answer: We will consider the 2 subsequences a_{2n} and a_{2n+1}.

$$\begin{aligned} a_{2n} &= \frac{[(-1)^{2n} - 2]^{2n}}{2^{2n}} = \frac{(1-2)^{2n}}{2^{2n}} = \frac{(-1)^{2n}}{2^{2n}} = \frac{1}{2^{2n}} \\ a_{2n+1} &= \frac{[(-1)^{2n+1} - 2]^{2n+1}}{2^{2n+1}} = \frac{(-1-2)^{2n+1}}{2^{2n+1}} = \frac{(-3)^{2n+1}}{2^{2n+1}} = -\Big(\frac{3}{2}\Big)^{2n+1}. \end{aligned}$$

$\lim a_{2n} = \lim \frac{1}{2^{2n}} = 0$ and $\lim a_{2n+1} = \lim\Big[-\Big(\frac{3}{2}\Big)^{2n+1}\Big] = -\infty$. So,
$\limsup_n a_n = \lim a_{2n} = 0$ and $\liminf_n a_n = \lim a_{2n+1} = -\infty$.