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MATH 440.1001
Scribing Week 4
Due. 22 February 2021

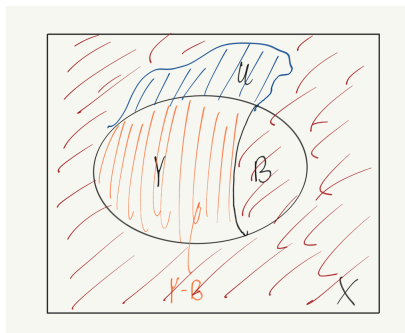
Review of Last Lecture

Theorem (9.5)**.** Let $A \subseteq X$ be a subset of a topological space X. Then $\overline{A} = A \cup L(A)$.

Lecture 10

Proposition (10.1)**.** Let X be a space, $Y \subseteq X$ a subspace. Then $B \subseteq Y$ is closed in Y if and only if there exists a closed subset $A \subseteq X$ such that $B = Y \cap A$.

10.1. ($\Longrightarrow$) Suppose $B \subseteq Y$ is closed. Recall that the subspace topology on Y, $W \subseteq Y$, is open if and only if $W = Y \cap U$ for some open $U \subseteq X$:



($\Longleftarrow$) Suppose $B = Y \cap A$, $A \cap X$ closed. We want to show B is closed in Y, that is, $Y - B$ is open in Y. To do this, we work backwards from the picture above. $\square$

Closure is a Relative Notion...

Ex (10.2)**.** Find $\overline{A}$.

1. Let $A = [-1, 0) \subseteq \mathbb{R}$. What is $\overline{A}$?
   $\overline{A} = A \cap L(A) = [-1, 0]$ as a subset of $\mathbb{R}$.

2. Let $Y = [-1, 0] \cup (0, 2] \subseteq \mathbb{R}$. Give Y the subspace topology. Now $A = [-1, 0) \subseteq Y$. What is $\overline{A}$?
   Note: $A = Y \cap [-1, 0]$. Prop 10.1 implies A is a closed subset of Y which implies $\overline{A} = A$ as a subset of Y.

Definition (10.3)**.** Let X be a space, $A \subseteq X$. The **interior** of A, $\mathring{A} = \mathrm{Int}(A)$, is the union of all open subsets contained in A.
The **boundary** of A is the subset $\partial A := \overline{A} - \mathring{A}$.

Remark (10.4). Facts about $\mathring{A}$ and ∂A.

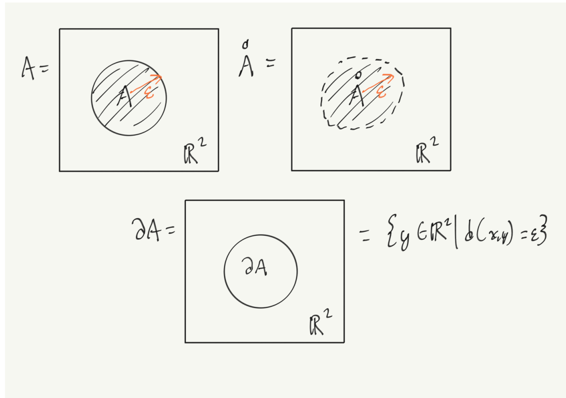
1. $\mathring{A}$ is open in X and is the "largest" open subset of X contained in A.

2. $\partial A \subseteq X$ is a closed subset of X: $\partial A = \overline{A} \cap (X - \mathring{A})$.

Ex (10.5)**.** A few examples involving $\mathring{A}$ and ∂A.

1. $X = \mathbb{R}$, $A = (0, 1)$.
   $\mathring{A} = A$, $\overline{A} = [0, 1]$, $\partial A = \overline{A} - \mathring{A} = \{0, 1\}$.

2. $X = \mathbb{R}^2$, $A = \overline{B}_x(\varepsilon)$.



Proposition (10.6)**.** Let X be a space. Let $A \subseteq X$, $x \in X$. Then $x \in \mathring{A}$ if and only if there is an open neighborhood $V \subseteq X$ of x such that $V \subseteq A$.

Proposition (10.7)**.** Let X be a space, $A \subseteq X$, $x \in X$. Then $x \in \partial A$ if and only if for every open neighborhood $U \subseteq X$ of x, we have $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$.

Lecture 11

Important Use of Closure

- Recall from Real Analysis: $\forall x \in \mathbb{R}$ and $\forall \varepsilon > 0$, $\exists q \in \mathbb{Q}$ such that $r - \varepsilon < q < r + \varepsilon$.
- Important consequence of this: Every continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniquely determined by its value on rationals, i.e. if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and $\forall q \in \mathbb{Q}$ where $f(q) = g(q)$, then $f = g$.

Definition (11.1)**.** Let X be a topological space. A subset $A \subseteq X$ is **dense** if and only if $\underline{\overline{A} = \mathbf{X}}$.
We say X is **separable** if and only if X contains a countable dense subset, i.e. $\exists A \subseteq X$ such that $\textbf{card}(A) = \textbf{card}(\mathbb{N})$ and $\overline{A} = X$.

Main Examples

- $\mathbb{Q} \subseteq \mathbb{R}$ is dense and countable in Euclidean space.
- $\mathbb{Q}^n \subseteq \mathbb{R}^n$ is dense and countable. Therefore $(\mathbb{R}^n, \mathcal{T}_{\text{Euclid}})$ is separable.
- $(\mathbb{R}, \mathcal{T}_{\text{cof}})$. Let $U \subseteq \mathbb{R}$ be open and non-empty. By definition of $\mathcal{T}_{\text{cof}}$, $\mathbb{R} - U$ is a finite subset of $\mathbb{R}$.
  $\overline{U} = \mathbb{R} \implies U$ is dense in $\mathbb{R}$.

Section M13: Basis for Topologies

- In this section, we will generalize open balls in Euclidean space.
- Recall how we characterized open subsets of $\mathbb{R}^n$: $U \subseteq \mathbb{R}^n$ is open if and only if U = union of open balls.

Definition (11.2)**.** If X is a set, a **basis for a topology** on X is a collection $\mathcal{B}$ of subsets of X such that

1. For each $x \in X$, there is at least 1 basis element $B \in \mathcal{B}$ such that $x \in B$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$:

Key Idea: If $\mathcal{B}$ is a basis for a topology on a set X, then the topology generated by $\mathcal{B}$, $\mathcal{T}_{\mathcal{B}}$, is as follows:

$$U \in \mathcal{T}_{\mathcal{B}} \text{ is open if and only if } \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \text{ and } B \subseteq U. \, (X, \mathcal{T}_{\mathcal{B}})$$

Proposition (11.3)**.** $A \subseteq X$ is dense if and only if $A \cap U \neq \emptyset$ when U is any non-empty open subset of X.