STAT 461.1002: Homework 6
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27 October, 2020

Section 3.6 Problems

1. 3.6.5. Use Theorem 3.6.1 (Let W be any random variable, discrete or continuous, having mean μ and for which $E(W^2)$ is finite. Then $\mathrm{Var}(W) = \sigma^2 = E(W^2) - \mu^2$) to find the variance of the random variable Y, where $f_Y(y) = 3(1-y)^2$, $0 < y < 1$.
Answer:
$$\mu = \int_0^1 y \cdot f_Y(y)\,\mathrm{d}y = \int_0^1 3y(1-y)^2\,\mathrm{d}y = 3\int_0^1 y^3 - 2y^2 + y\,\mathrm{d}y = 3\left(\frac{y^4}{4} - \frac{2y^3}{3} + \frac{y^2}{2}\right)\Big|_0^1 = \frac{1}{4}.$$
$$E[Y^2] = \int_0^1 y^2 \cdot f_Y(y)\,\mathrm{d}y = \int_0^1 3y^2(1-y)^2\,\mathrm{d}y = 3\int_0^1 y^4 - 2y^3 + y^2\,\mathrm{d}y = 3\left(\frac{y^5}{5} - \frac{y^4}{2} + \frac{y^3}{3}\right)\Big|_0^1 = \frac{1}{10}.$$
So, $\mathrm{Var}(Y) = E[Y^2] - \mu^2 = \frac{1}{10} - \frac{1}{16} = \frac{3}{80}$.

2. **(461 only)** 3.6.11. Suppose that Y is an exponential random variable, so $f_Y(y) = \lambda e^{-\lambda y}$, $y \geq 0$. Show that the variance of Y is $1/\lambda^2$.
Answer:
We first find μ using integration by parts:
$$\mu = \int_0^\infty y\lambda e^{-\lambda y}\,\mathrm{d}y = -ye^{-\lambda y}\Big|_0^\infty - \int_0^\infty -\frac{e^{-\lambda y}}{\lambda}\,\mathrm{d}y.$$
We now use L'Hospital's Rule to evaluate $\lim_{y\to\infty} -ye^{-\lambda y} = \lim \frac{-y}{e^{\lambda y}} = \lim \frac{-1}{\lambda e^{\lambda y}} = \frac{-1}{\infty} = 0$. So,
$$\mu = \int_0^\infty \frac{e^{-\lambda y}}{\lambda}\,\mathrm{d}y = \frac{1}{\lambda}\left(\frac{e^{-\lambda y}}{\lambda}\right)\Big|_0^\infty = \frac{1}{\lambda}.$$
We now compute $E[Y^2]$ using integration by parts:
$$E[Y^2] = \int_0^\infty y^2\lambda e^{-\lambda y}\,\mathrm{d}y = -y^2e^{-\lambda y}\Big|_0^\infty - \int_0^\infty -2ye^{-\lambda y}\,\mathrm{d}y = \int_0^\infty 2ye^{-\lambda y}\,\mathrm{d}y = \frac{2}{\lambda^2}.$$
So, $\mathrm{Var}(Y) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$.

3. Let Y have PDF $f_Y(y) = 3(1-y)^2$, $0 < y < 1$. Let $W = -5Y + 12$. Find the variance and standard deviation of W.
Answer:
$\mathrm{Var}(W) = \mathrm{Var}(-5Y + 12) = 25\mathrm{Var}(Y)$. Since we already found $\mathrm{Var}(Y) = \frac{3}{80}$, we can easily find that $\mathrm{Var}(W) = 25 \cdot \frac{3}{80} = \frac{15}{16}$.

Section 3.7 Problems

4. 3.7.1. If $p_{X,Y}(x,y) = cxy$ at the points $(1,1)$, $(2,1)$, $(2,2)$ and $(3,1)$, and equals 0 elsewhere, find c.
Answer:
$p_{X,Y}(1,1) = c$, $p_{X,Y}(2,1) = 2c$, $p_{X,Y}(2,2) = 4c$, and $p_{X,Y}(3,1) = 3c$. Since $p_{X,Y}$ must sum to 1, then $10c = 1 \implies c = \frac{1}{10}$.

5. **(461 only)** 3.7.2. Let X and Y be two continuous random variables defined over the unit square. What does c equal if $f_{X,Y}(x,y) = c(x^2 + y^2)$?
Answer:
We use the fact that integrating over the entire unit square should equal 1:
$$\int_0^1\int_0^1 c(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x = c\int_0^1 \left(x^2y + \frac{y^3}{3}\right)\Big|_0^1\,\mathrm{d}x = c\int_0^1 x^2 + \frac{1}{3}\,\mathrm{d}x = c\left(\frac{x^3+x}{3}\right)\Big|_0^1 = \frac{2c}{3} = 1$$
$$\implies c = \frac{2}{3}.$$

6. 3.7.8. Consider the experiment of tossing a fair coin three times. Let X denote the number of heads on the last flip, and let Y denote the total number of heads on the three flips. Find $p_{X,Y}(x,y)$.
Answer:
The Sample Space $S = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H)(T,H,T),(T,T,H),(T,T,T)\}$. So then the discrete pdf $p_{X,Y}(x,y)$ can be displayed as follows, where the rows represent X = # of heads on the last flip and the columns represent Y = # of heads total:

	$Y=0$	$Y=1$	$Y=2$	$Y=3$
$X=0$	$P(0,0)=1/8$	$P(0,1)=2/8$	$P(0,2)=1/8$	$P(0,3)=0$
$X=1$	$P(1,0)=0$	$P(1,1)=1/8$	$P(1,2)=2/8$	$P(1,3)=1/8$

7. 3.7.11. Let X and Y have the joint pdf
$$f_{X,Y}(x,y) = 2e^{-(x+y)}, \quad 0<x<y, \quad 0<y.$$
Find $P(Y<3X)$.
Answer:
$$P(Y<3X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\infty}\int_x^{3x} 2e^{-x}\cdot e^{-y}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\infty} 2e^{-x}(-e^{-y})\Big|_x^{3x}\,\mathrm{d}x$$
$$= \int_0^{\infty} -2e^{-4x}+2e^{-2x}\,\mathrm{d}x = -\frac{e^{-4x}}{2} - e^{-2x}\Big|_0^{\infty} = \frac{1}{2}.$$

8. 3.7.19. - Part (b) only. Find $f_X(x)$ and $f_Y(y)$: $f_{X,Y}(x,y) = \frac{3}{2}y^2$, $0\le x\le 2$, $0\le y\le 1$.
Answer:
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y = \int_0^1 \frac{3}{2}y^2\,\mathrm{d}y = \frac{3}{2}\left(\frac{y^3}{3}\right)\Big|_0^1 = \frac{3}{6} = \frac{1}{2}.$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x = \int_0^2 \frac{3}{2}y^2\,\mathrm{d}x = \frac{3}{2}(y^2x)\Big|_0^2 = \frac{3y^2}{2}\cdot 2 = 3y^2.$$

9. 3.7.26. An urn contains twelve chips—four red, three black, and five white. A sample of size 4 is to be drawn without replacement. Let X denote the number of white chips in the sample, Y the number of red. Find $F_{X,Y}(1,2)$.
Answer:
We first note that the order in which we draw the chips does not matter, so we can say that there are $\binom{12}{4} = 495$ ways for 4 chips to be drawn. We then note that there are $\binom{4}{y}$ ways to get y red chips from the 4 total red chips, and $\binom{5}{x}$ ways to pick x white chips from the 5 total white chips. For the remaining black chips, we can figure out that there are $\binom{3}{4-(x+y)}$ ways for them to be chosen (since the sample size is 4 and $x+y$ are the amount of chips already chosen). So then
$$p_{X,Y}(x,y) = \frac{\binom{5}{x}\binom{4}{y}\binom{3}{4-(x+y)}}{495}$$
To find $F_{X,Y}(1,2)$, we need to sum all of the probabilities of the different random variables $X\le 1$ and $Y\le 2$ (However, we disregard the $P(X=0,Y=0)$ case otherwise there would not be enough chips in the sample size)
$$F_{X,Y}(1,2) = P(X=0,Y=1)+P(X=0,Y=2)+P(X=1,Y=0)+P(X=1,Y=1)+P(X=1,Y=2)$$
$$= \frac{\binom{5}{0}\binom{4}{1}\binom{3}{3}}{495} + \frac{\binom{5}{0}\binom{4}{2}\binom{3}{2}}{495} + \frac{\binom{5}{1}\binom{4}{0}\binom{3}{3}}{495} + \frac{\binom{5}{1}\binom{4}{1}\binom{3}{2}}{495} + \frac{\binom{5}{1}\binom{4}{2}\binom{3}{1}}{495}$$
$$= \frac{(1\cdot 4\cdot 1)+(1\cdot 6\cdot 3)+(5\cdot 1\cdot 1)+(5\cdot 4\cdot 3)+(5\cdot 6\cdot 3)}{495} \approx 0.358.$$

10. 3.7.28. - Part (a) only. Find $F_{X,Y}(x,y)$ given that $f_{X,Y}(x,y) = \frac{1}{2}$, $0\le x\le y\le 2$.
Answer:
$$F_{X,Y}(u,v) = \int_{-\infty}^{u}\int_{-\infty}^{v} f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^u\int_x^v \frac{1}{2}\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\int_0^u y\Big|_x^v\,\mathrm{d}x = \frac{1}{2}\int_0^u v-x\,\mathrm{d}x = \frac{1}{2}\left(vx-\frac{x^2}{2}\right)\Big|_0^u = \frac{uv}{2}-\frac{u^2}{4}.$$

11. 3.7.38. Two fair dice are tossed. Let X denote the number appearing on the first die and Y the number on the second. Show that X and Y are independent.
Answer:
First note that the probablity for any number rolled is 1/6, i.e. $P_X(x) = P_Y(y) = \frac{1}{6}$, $\forall X, Y$. Let each row be the random variable X and each column be the random variable Y.

	$Y=1$	$Y=2$	$Y=3$	$Y=4$	$Y=5$	$Y=6$
$X=1$	2	3	4	5	6	7
$X=2$	3	4	5	6	7	8
$X=3$	4	5	6	7	8	9
$X=4$	5	6	7	8	9	10
$X=5$	6	7	8	9	10	11
$X=6$	7	8	9	10	11	12

Then by the table, $P_{X,Y}(x,y) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P_X(x) \cdot P_Y(y)$. So X and Y are independent.

12. 3.7.43. Suppose that random variables X and Y are independent with marginal pdfs $f_X(x) = 2x$, $0 \le x \le 1$, and $f_Y(y) = 3y^2$, $0 \le y \le 1$. Find $P(Y < X)$.
Answer:
We first find the joint pdf $f_{X,Y}(x,y)$ since X and Y are independent: $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = 2x \cdot 3y^2 = 6xy^2$. Now we integrate:

$$P(Y < X) = \int_0^\infty \int_0^\infty f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^x 6xy^2\,\mathrm{d}y\,\mathrm{d}x = 2\int_0^1 xy^3\Big|_0^x \mathrm{d}x = 2\int_0^1 x^4\,\mathrm{d}x = 2\left(\frac{x^5}{5}\right)\Big|_0^1 = \frac{2}{5}.$$

13. **(461 only)** 3.7.44. Find the joint cdf of the independent random variables X and Y, where $f_X(x) = \frac{x}{2}$, $0 \le x \le 2$, and $f_Y(y) = 2y$, $0 \le y \le 1$.
Answer:
We repeat the same process in #12: $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = \frac{x}{2} \cdot 2y = xy$. Now we integrate:

$$F_{X,Y}(u,v) = \int_{-\infty}^u \int_{-\infty}^v f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^u \int_0^v xy\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\int_0^u xy^2\Big|_0^v \mathrm{d}x = \frac{1}{2}\int_0^u xv^2\,\mathrm{d}x = \frac{1}{4}(x^2v^2)\Big|_0^u = \frac{u^2v^2}{4}.$$