Solid Mechanics and Aerospace Structures, an Introduction

August 3, 2022

0. Introduction and Preliminaries

We are concerned with the very basic models of the things that hold rockets, satellites and airplanes together: the *structure*. Structure is derived from the Latin word meaning a fitting together. In aerospace its largely what you see when you look at any aerospace system: rocket, airplane, satellite, launch tower etc. However, there are many internal parts that are structures as well such as the spar of an aircraft or the fuel tank inside a rocket. This course is intended to give you the very basic theories and formulations to understand how to design and analyze a structure. Academically, this course is a combination of statics and strength of materials and is an outgrowth of Newton's basic laws, in particular the sum of all forces acting on an object that is not accelerating must be zero.

Units One interesting aspect of aerospace structures (versus say civil structures) is that the weight (actually mass) of an aerospace structure must be as minimal as possible because range, fuel economy and load capability all decrease with increasing structural mass. As mass is such a crucial quantity and its general use in earth bound systems is generally confused with weight it's important to understand the units used in calculations of aerospace structures. There are two common systems: they are SI units (System International, or sometimes called the MKS system) which is mass (symbol is m) based and the US Customary (or also called English system) which is weight based (symbol is W). Being able to convert between these two systems and being able to think in terms of both systems is extremely useful. While you likely had this drilled into you in high school and college physics courses and lovely conversion tables and interactive sites are available on the web, we will briefly go over the two systems. The goal here is to give you a feeling or intuition for numerical values in these two systems. At the surface of the earth the two systems of units are related by the constant g the acceleration due to gravity simply by $W = mg$. However, as we take flight and leave the surface of the earth W and g change while m remains constant.

The following comparisons are intended to help your intuition in using units:

Mass of a Boeing 737: Operating empty **41,145kg** (**90,710lb**), max takeoff **70,535kg** (155,500lb).

The author's mass: sadly 175 lb = 79.379 kg.

Mass of the MQ-1 Predator is **512 kg** empty, and 1,043 kg Maximum Take Off Weight (MTOW) or about 1,128 lbs dry and about 2,300 lbs.

Mass of Falcon-1 is 61,000 lbs (or 27,569 kg).

Since you may be dealing with space flight it's important to recognize the difference between mass and weight. If you have not done so before use your phones tonight to Google several other common values (length, volume, area etc.) of some aerospace structures in both units. The idea is to have some intuition about units, so when you make a calculation, you are not orders of magnitude off or missing a term. When solving problems using equations you should always check to see that the units are consistent.

Example 0.1 A cube sat weighs on average about 2.91 lbs. at sea level and orbits in LEO (Low Earth Orbit, about 2,000 km). Compute the mass (in kg) of the satellite and its weight at sea level and again in orbit.

Solution: Convert 2.91 lbs. to Newtons by using the conversion (1 lb =4.48 N)

$$\frac{4.48\text{ N}}{1\text{ lb}} = 1 \Rightarrow W = 2.91\text{ lb} = 2.91\text{ lb}\frac{4.48\text{ N}}{1\text{ lb}} = (2.91)(4.48)\text{ N} = 13.06\text{ N}$$

To compute the mass use $F = mg$ where g =9.81 m/s^2 is the acceleration due to gravity at sea level. Thus:

$$m = \frac{F}{g} = \frac{13.06\text{ N}}{9.81\ {}^{m}\!/_{s^2}} = 1.33\text{ kg}$$

In space the gravitational pull on a mass is less than that on the surface of the earth (recall from your physics class). The weight of an object a distance r from the center of the earth to the center of the object is given by

$$W = mg\left(\frac{R_E}{r}\right)^2$$

Here R_E = 6,370 km is the radius of the earth (r = 6,370 +2,000 km).

$$W = (1.33\text{ kg})\left(9.81\,{}^{m}\!/_{s^2}\right)\left(\frac{6{,}370\text{ km}}{(6{,}370+2{,}000)\text{ km}}\right)^2 = 7.56\text{ N}$$

A substantial reduction in weight from sea level while the mass remains constant. Thus, when working on aerospace structures, it is best to think and work in terms of mass.

Dimensions: Dimensions are also useful for checking one's theories and calculations and are distinct from units. Every equation needs to be dimensionally consistent across the equal sign and between plus and minus signs. The basic dimensions are length, denote by L, mass, denoted by M, time, denoted by T and temperature, denoted by θ. A unit is a particular value of a dimension. For instance, both meters and inches have dimensions of length and are a specific measure of the amount of dimension. Thus, the are the same dimension but are different units. All other dimensions are defined in terms of these four basic dimensions: T, M, L, θ. For example, the dimension for force is given the name newton in the SI system, denoted N, is defined as ML/T^2 and has units of $N = kg*m/s^2$. Tables of dimensions and units are available in most engineering and physics textbooks and are readably assessable on the internet.

Structures of Interest in Aerospace: In this introductory treatment the focus is on three basic types of structures, which capture the essence of how more realistic, complex structures behave when loaded with aerodynamic, thrust and gravitational forces. The basic elements are:

1. A slender bar subject to end forces resulting in elongation or compression
2. A slender shaft subject to a torque resulting in twisting
3. A slender beam subject to perpendicular forces resulting in bending

As one moves closer to aerospace applications these effects can couple so that a single structure may experience all three of these deflections (discussed in later courses). The basic elements are illustrated in Figure 0.1.

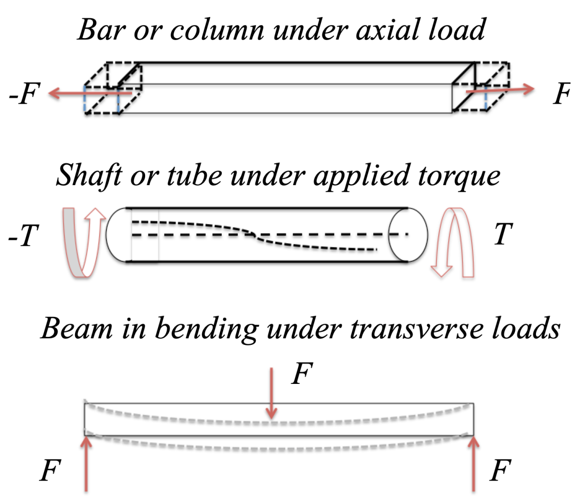

Figure 0.1 The three basic types of loads and an illustration of the type of deflection they cause. Forces applied at the ends of a structure can compress or elongate it. Applied torques will twist the structure and forces applied perpendicular to the structure will cause it to bend. Shells will be covered in 315.

Axial loads occur in landing gear, various supports and rocket fuselage structures. Torque loads also show up in airplane and rocket fuselage components. Bending of beams occurs because of aerodynamic loads on wings, which can also apply torques to the wing. The coupling of bending and torsion in wings is often used to explain flutter in fixed wing aircraft.

Recall from physics that a twisting motion or rotation is caused by a force acting at a distance called a moment. Moments and forces are vectors having both a magnitude and a direction. Thus, the rules for manipulating vectors permeate the discussions for aerospace structures. Forces can be classified in several different ways as externally applied loads, forces of restraint and internal forces. At times two equal forces applied in opposite directions but separated by a distance may act on a structure. Such an arrangement results in a moment called a *couple*. Couples are unique because they do not depend on the points of application of the forces, only the distance between the forces and the force's magnitude.

Vectors and Matrices (a review, recall your Physics and Math classes)
A matrix is a rectangular array of numbers or symbols organized in rows and columns subject to a special arithmetic. A vector is a matrix consisting of a single row or column. (Samples of both are given in Example 0.2.) Together along with various rules for manipulation they form a powerful way to solve a linear system of equations and a systematic way to represent forces and moments. They are used extensively in almost all branches of aerospace engineering. In addition, the main code used in your studies here is MATLAB, which is based on matrix manipulations.

Vectors have a special use as they are directly used to represent physical forces in three dimensions. They are also used to express displacement, velocity, acceleration and forces in the study of flight dynamics and aerodynamics. As illustrated in Figure 0.2, vectors have both a magnitude and direction. While in many cases use is restricted to two dimensions the most useful case is to represent them in 3-space using a rectangular coordinate system of Figure 0.2. The standard way to keep track of the rectangular coordinates of a 3D vector is to use the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ each of which have a magnitude of one and lie along the x, y, and z directions respectively. The unit vectors in rectangular coordinates are:

$$\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A general vector **A** then is written as:

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

as illustrated in Figure 0.2. A vector can also be expressed as a column of numbers by:

$$\mathbf{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

In Matlab this column vector is written as `A= [Ax; Ay; Az].` A matrix is any rectangular array of numbers, a vector being an example of a 3 x 1 matrix. With the exception of vectors, we are mostly concerned with square matrices used to solve systems of equations as illustrated in Example 0.2. The key rules used in the following are how to invert a matrix and how to multiply a matrix times a vector. Both are trivial commands in Matlab.

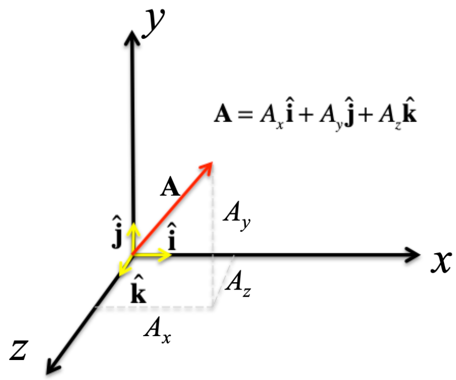

Figure 0.2 Representation of a three-dimensional vector using rectangular coordinates and unit vectors. Other coordinate systems are also possible corresponding to a particular problem's geometry. A_x, A_y and A_z are called the *components* of the vector **A**.

The magnitude, or length of the vector **A** is computed by the Pythagorean theorem to be:

$$||\mathbf{A}|| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Once vectors are expressed in component form various useful operations can be defined. Suppose there are two forces acting on a structure defined by the two vectors **A** and **B**. The resulting behavior of the structure will depend on the sum of these two forces defined by simply adding like components:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}}$$

A scalar times a vector is defined by just multiplying the scalar times each element in the vector. Let α be a scalar and **A** a vector then $\alpha\mathbf{A}$ becomes:

$$\alpha\mathbf{A} = \begin{bmatrix} \alpha A_x \\ \alpha A_y \\ \alpha A_z \end{bmatrix}$$

Unlike scalars (single numbers), multiplying vectors is a bit more involved. In fact, there are three useful ways to define the product of two vectors

1. The scalar product (or dot product, or inner product) so called because it results in a scalar.
2. The vector product (or cross product) so called because it results in a new vector.
3. The outer product, which results in a matrix and is used in vibration analysis not discussed here.

The scalar product of two vectors **A** and **B**, denoted $\mathbf{A} \bullet \mathbf{B}$, is defined by:

$$\mathbf{A}\bullet\mathbf{B} = ||\mathbf{A}||||\mathbf{B}||\cos\theta = A_x B_x + A_y B_y + A_z B_z$$

where θ is the angle between the two vectors. Note if θ=90° the scalar product is zero and the vectors are orthogonal (perpendicular) to each other. For example:

$$\hat{\mathbf{i}}\bullet\hat{\mathbf{j}} = (1)(0) + (0)(1) + (0)(0) = 0$$

Also note that the components of a vector are determined by the dot product by:

$$A_x = \mathbf{A}\bullet\hat{\mathbf{i}},\ A_y = \mathbf{A}\bullet\hat{\mathbf{j}},\ \text{and}\ A_z = \mathbf{A}\bullet\hat{\mathbf{k}}$$

When treated as an array the dot product is written at $\mathbf{A}^T\mathbf{B}$ where the superscript T indicates the transpose of the vector defined by the row:

$$\mathbf{A}^T = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix}$$

The computation is the same as defined above:

$$\mathbf{A}^T\mathbf{B} = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix}\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = A_x B_x + A_y B_y + A_z B_z$$

A scalar α times a vector **A** is defined by multiplying each element of **A** by α:

$$\alpha\mathbf{A} = \begin{bmatrix} \alpha A_x \\ \alpha A_y \\ \alpha A_z \end{bmatrix}$$

The cross product of two vectors results in a third vector that is perpendicular to the plane made by the two vectors and has magnitude:

$$||\mathbf{A}\bullet\mathbf{B}|| = ||\mathbf{A}||||\mathbf{B}||\sin\theta$$

where θ is the angle between the two vectors. The resulting vector is

$$\mathbf{A}\times\mathbf{B} = \left(A_y B_z - A_z B_y\right)\hat{\mathbf{i}} - \left(A_x B_z - A_z B_x\right)\hat{\mathbf{j}} + \left(A_x B_y - A_y B_x\right)\hat{\mathbf{k}} = \begin{bmatrix} A_y B_z - A_z B_y \\ -\left(A_x B_z - A_z B_x\right) \\ A_x B_y - A_y B_x \end{bmatrix}$$

Note that the dot product commutes, i.e., $\mathbf{A}\bullet\mathbf{B} = \mathbf{B}\bullet\mathbf{A}$, but the cross product is not, i.e., $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$. The cross product obeys the "right hand rule". In mechanics the cross product is used to calculate and define a moment (or torque) about a point. Let the vector **r** denote the vector extending from the point of interest to the point of application of a force **F**. The resulting moment **M** is:

$$\mathbf{M} = \mathrm{r}\times\mathbf{F}$$

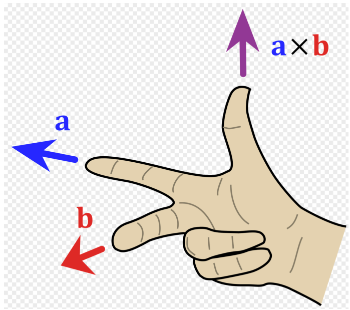

Figure 0.3 Illustration of the right-hand rule used to get a sense of the direction of a moment. However, the rules of the cross product give the result directly.

A matrix is any rectangular array of numbers, a vector being an example of a 3 x 1 matrix. With the exception of vectors, we are mostly concerned with square matrices used to solve systems of equations as illustrated in Example 0.2. Rules for manipulating matrices are given in Appendix A. The key rules used in the following are how to invert a matrix and how to multiply a matrix times a vector. Both are trivial commands in Matlab.

Sample Matlab Commands

(also see https://www.mathworks.com/help/matlab/getting-started-with-matlab.html)

```
% Define two vectors A, B
>> A=[1;2;3]
A =
     1
     2
     3
>> B=[1;-2;0]
B =
     1
    -2
     0
>> C=A-B, D=A+B % difference ,
sum
C =
     0
     4
     3
D =
     2
     0
     3

% compute the magnitude of A
>> a=norm(A)
a =
    3.7417
>>r=dot(A,B) % dot product
r =
    -3
% Cross product AxB & BxA
>>s=cross(A,B), s1=cross(B,A)
s =
     6
     3
    -4
s1 =
    -6
    -3
     4
```

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1. Introduction to Statics

From Newton's Laws (usually in a chapter called *Equilibrium of Rigid Bodies* in your Physics Text): The concept of equilibrium is defined as all forces and moments acting on a structure adding to zero. That means the object is not accelerating. Note this does not mean that velocity, **v**, is zero. Here bold type means a vector or matrix and italic means a scalar. Equilibrium implies that the sum of all forces acting on the structure must be the zero vector:

$$\sum_{i=1}^{n} \mathbf{F}_i = m\mathbf{a} = 0$$

If the object has the possibility of rotating, then the sum of the moments about a point must also be zero (zero rotational acceleration) if the object is to remain in equilibrium:

$$\sum_{i=1}^{n} \mathbf{M}_i = I\omega = 0$$

One must identify all the forces acting on a solid object and assuming the object is in equilibrium ($\mathbf{a}$=ω=0), chose a coordinate system, and then sum the forces providing a mathematical model that can help you design, build and fly. If the object is three-dimensional, then the above two vector equations result in 6 scalar equations in the components of the various forces and moments. Note that any text on *Statics* will present statics in great detail, more detail than we require for understanding aero structures, but makes a good reference with lots of examples should you get stuck on the homework.

Free Body Diagrams (FBD): FBDs are simple drawings of a structure of interest showing all the forces, moments and couples acting on it isolated from its surroundings along with a frame of reference. FBDs are introduced in physics texts. The idea is to ensure that all the relevant forces and moments are accounted for, representing the forces and the moments as vectors. To isolate a body, it must be separated from its connection points to the rest of the system and these external forces must be accounted for. For two-dimensional structures there are four common types of connections listed here with the forces and moments they support. One of the most common supports is a built in or fixed support, which causes reactions of a force and a moment. A pinned support only results in a force and a roller type support only supports a reaction force perpendicular to the connection. A simple support also generates a perpendicular reaction force. These are illustrated in Figure 1.4.

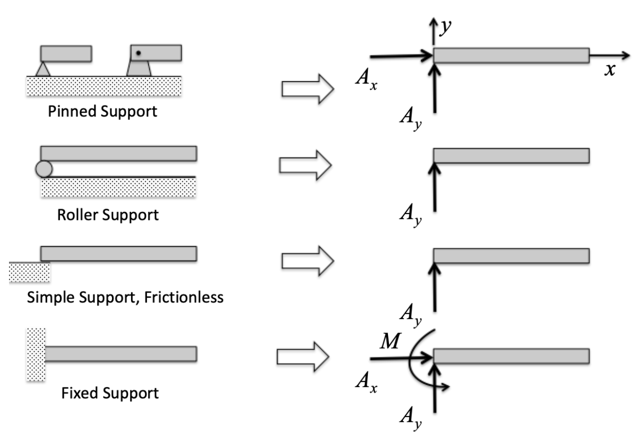


Figure 1.4 Schematics of connections and their corresponding reactions.

The reactions for three-dimensional structures are a bit more complicated and will be discussed when the need arises.

Example 1.2: A common structural configuration is a cantilevered beam (a structure fixed at one end and free at the other). Sometimes it is used as a *very* simple example of a wing. Keep in mind that all models are wrong, or at best, approximations. Some models are more useful than others. Here we take the simplest possible model to get an idea of the forces involved. Consider the cantilever beam model of a wing in Figure 1.5 and suppose the flow arpund the "wing" causes a couple and a force as indicated in the drawing. Make a FBD and use Newton's laws to compute the value of the reaction forces that the body of the aircraft would experience. Note the numbers and values are made up to be simple just for illustration of the technique.

The process for solving for the external reaction forces and moments is:

1. Identify what is given and what is being asked for
2. Draw a FBD sketching the isolate object, labeling points and drawing external forces and moments.
3. Sum forces in the 3 orthogonal directions (x, y, and z)
4. Sum the moments about a point (treat counterclockwise as positive)
5. Solve the resulting equations.

Note that the choice of counterclockwise to be a positive moment is consistent with the choice of the $x-y$ coordinate system as the corresponding moment vector would be in the positive z direction should the problem be three dimensional according to the righthand rule.

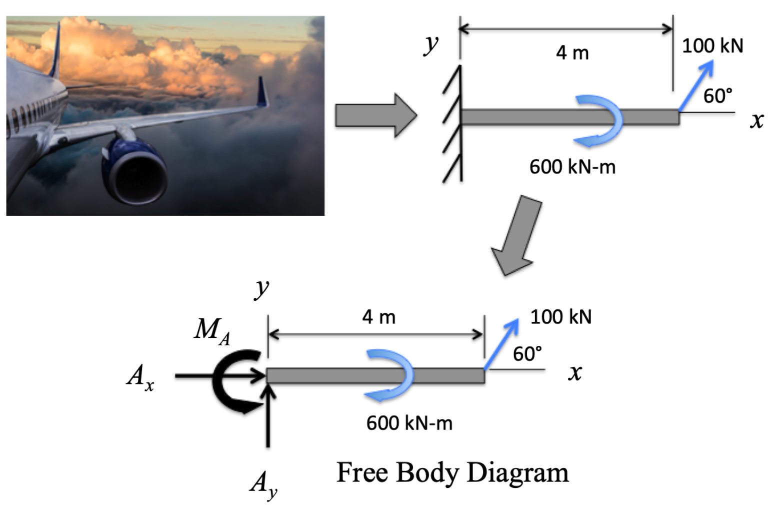


Figure 1.5 Simple model of a wing as a cantilevered beam and it's free body diagram.

Solution: The 100 kN force and 600 kN-m couple are given. We want to compute the reaction forces at the connection end. The FBD is labeled and sketched in Figure 1.5. The procedure is to sum the forces and the moments. The external force at the fixed end is already decomposed into forces along the x and y directions. However, the applied force at the free end needs to be decomposed into x and y components using simple trigonometry so that the sum of forces split into the x and y directions become

$$\sum F_x = A_x + 100\cos 60° = 0$$
$$\sum F_y = A_y + 100\sin 60° = 0$$

And the sum of the moments about the point at the fixed end becomes

$$\sum M = M_A - 600 + 4(100\sin 60°) = 0$$

The result of applying the equilibrium condition yields 3 equations in 3 unknowns having solution: A_x = -50 kN, A_y =-86.6 kN and M_A =253.6 kN-m. For homework, make sure you can solve these equations and get the same answer. If these were real values of the applied load, then this would give designers some idea of how to design the connection of the wing to the fuselage.

Couples: If two equal but opposite forces act on a structure separated by a perpendicular distance, they produce a turning moment called a couple. This is illustrated in Figure 1.6. The effect on the structure does not matter where the couple is applied along the structure (as long as it is not at a fixed point). If the two force vectors act along the same line, then no couple results.

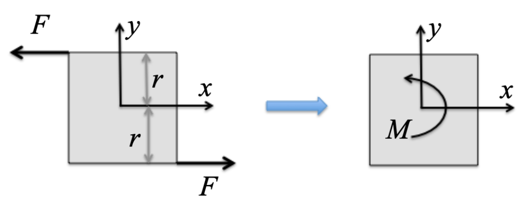


Figure 1.6 The concept of a couple, two equal and opposite forces separated by a distance and not along the same line of action line.

In the Example 1.2 an external couple of 600 kN-m was applied to the structure. This could have been the result of two equal and opposite 600 kN forces applied to the structure. To see that this is equivalent consider the alternate FBD in Figure 1.7 where instead of the resulting couple the two forces causing the couple are given. Clearly these two forces cancel in the equation summing forces in the y – direction. It is interesting to see how this effects the moment equation. The three equations of equilibrium become

$$\sum F_x = A_x + 100\cos 60° = 0$$
$$\sum F_y = A_y + 100\sin 60° + 600 - 600 = 0$$
$$\sum M = M_A + 4(100\sin 60°)\underbrace{+600(1) - 600(2)}_{= -600} = 0$$

These are exactly the same as those determined in Example 1.2 by treating the two equal and opposite forces as a couple with no reference to a location other than acting on the body.

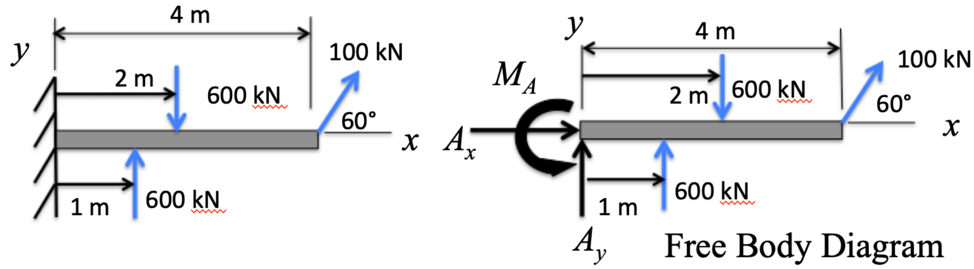


Figure 1.7 The same as Figure 1.5 but illustrating two forces making up the couple.

Trusses: When more than one structural member is present in the object of interest the FBD becomes a bit more complicated. Structures made up of only two force members are called *trusses*. A two-force member has only two forces acting on it along the same line of action and no moments (couples). If the two-force member is in equilibrium the forces must be along the same line of action and equal and opposite in direction. If they were not along the same line a moment would be created and it would no longer be a two-force member.

There are two approaches to analyzing the forces in a truss: the method of joints and the method of cuts. Later in the text we will examine the deformation within a member and there the concept of cuts is more useful. In the method of joints, each joint is isolated from the rest of the structure and analyzed using a FBD and the resulting equations of equilibrium. Working through the truss one eventually finds enough equations to determine all the relevant forces and moments. Examples of this approach are plentiful in books on *Statics*.

To understand the "cut" approach, an example is used. The basic concept assumes that if a structure is in equilibrium, then any piece we pull out of the structure must still be in equilibrium. Basically, the method of cuts involves pulling a member out of its truss, making a FBD and enforcing the equilibrium equations. The procedure for determining internal forces uses the "cut" concept. This will again be used later to understand internal stress in aerospace structures. The approach is:

1. Identify what forces and geometry are given and what forces are to be determined
2. Draw the global free-body diagram
3. Find the reaction forces by summing forces and moments
4. Cut out part of the structure and draw the FBD of the cut-out part

   a. Note this is possible because the entire system is in equilibrium so each part must be.
   b. Include external and internal forces
5. Sum moments and forced on the cut-out structure to determine the internal forces

Example 1.3: Consider the seat bracket of Figure 1.8. This is again an over simplified sample to show the method. The bracket is relatively crude and is for an experimental aircraft such as the one in FXB Atrium. The idea is to figure out the forces applied to the back panel so that a material can be chosen that is strong enough yet lightweight. Thus, we need to find the forces at points *A* and *E*. Assume the pilot has a mass of 70 kg, the connection at *A* is pinned and the connection at *E* is a roller.

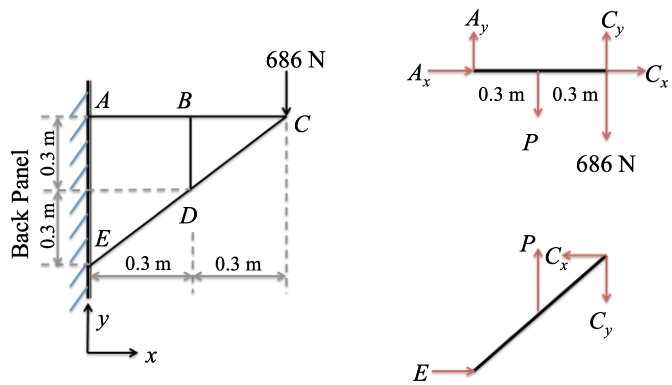


Figure 1.8 (Left) Diagram of a seat bracket hold in 70 kg pilot. (Top right) FBD of member *AC* assuming a pinned connection at *A*. (Bottom right) FBD of member *EC*, noting that member *BD* is a two-force member and the connection at *E* is a roller.

Solution: Summing the forces and the moments about point *A* on member *AC* yields:

$$\sum F_x = A_x + C_x = 0$$
$$\sum F_y = A_y + C_y - P - 686\text{ N} = 0$$
$$\sum M_A = -0.3P + 0.6C_y - (0.6)\bullet 686\text{ N} = 0$$

Next summing the forces and the moments about point *E* on member *EC* yields:

$$\sum F_x = E - C_x = 0$$
$$\sum F_y = P - C_y = 0$$
$$\sum M_E = 0.3P + 0.6C_x - 0.6C_y = 0$$

These two FBDs result in 6 equations in 6 unknowns, which can be solved in a couple of different ways. The simplest method is to write the 6 equations into one matrix equation, the unknown forces as one vector and the given forces in another vector and the matrix made up of the coefficients in the equations of equilibrium. Thus writing the equilibrium equations in matric form yields

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0.6 & 0 & -0.3 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0.6 & -0.6 & 0 & 0.3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} A_x \\ A_y \\ C_x \\ C_y \\ E \\ P \end{bmatrix}}_{\mathbf{R}} = \underbrace{\begin{bmatrix} 0 \\ 686 \\ 0.6 \bullet 686 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{F}} \text{N}$$

This representation of the equations of equilibrium is a matrix equation of the form **DR**=**F**. Here the vector **R** contains all of the unknown reactions. Treating the first row of the matrix as a vector and take its dot product with the vector **R** set equal to the first element of **F**, results in the first equation, etc. Multiplying the first row of the matrix times the vector of unknows and setting this equal to the first element of the applied force vector yields the first equation of the 6: $A_x + A_y = 0$. The solution of the matrix equation is of the form $\mathbf{R} = \mathbf{D}^{-1}\mathbf{F}$, where $\mathbf{D}^{-1}$ indicates the inverse of the matrix **D**, easily solved in Matlab.

The solution can also be obtained by substitution of various solutions of one equation into another. For example, the 5th equation yields that $C_y = P$, then the third equation yields the value for C_y and both can be used in the 2nd equation to compute A_y, etc. The solution is A_x =-686 N, A_y = 686 N, $P = C_y$ = 1372 N, $E = C_x$ = 686 N.

The good news about putting the equations in to Matrix form and solving using matrix inversion in Matlab means you can easily change the load (pilot weight) to set limits on the design by simply changing the values in the vector **F**.

In Matlab the solution is (typing in the command window):

```
>> % enter the matrix of coefficients D
>> D=[1 0 1 0 0 0;0 1 0 1 0 -1;0 0 0 0.6 0 -0.3;0 0 -1 0 1 0;0 0 0 -1 0
1;0 0 0.6 -0.6 0 0.3]
D =
    1.0000         0    1.0000         0         0         0
         0    1.0000         0    1.0000         0   -1.0000
         0         0         0    0.6000         0   -0.3000
         0         0   -1.0000         0    1.0000         0
         0         0         0   -1.0000         0    1.0000
         0         0    0.6000   -0.6000         0    0.3000
>> % enter the vector of loads F
>> F=[0;686;0.6*686;0;0;0]
F =
         0
  686.0000
  411.6000
```

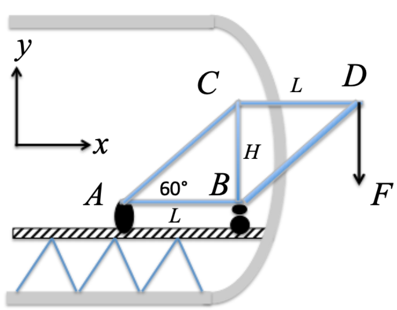
```
          0
          0
          0
>> % compute the solution using the matrix inverse
>> X =inv(D)*F
X =
        -686
         686
         686
        1372
         686
        1372
```
While its might be annoying to type out the matrix **D**, once saved the reaction to all sorts of other loads can be determined by merely changing the value of the load vector **F**.

Example 1.4: Consider the structure of Figure 1.9, which is used to support the rescue basket of the side of a helicopter. Determine the forces in each member. State if each member is in compression or tension.

Solution: First note the geometry is a parallelogram divided into two right triangles. Also note that the angles are all 30°, 60° and 90°. These angles all have exact values:

$$\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}, \quad \sin 30° = \cos 60° = \frac{1}{2}$$

While tempting to use a calculator for these it is more accurate to use these exact values. The free-body diagram of the whole structure is given on the right in Figure 1.8. The free-body diagram of each joint is given in Figure 1.9.



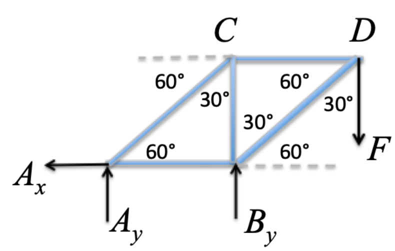


Figure 1.9 A model of a truss used to hold a rescue basket off a helicopter and its whole body, free-body diagram. The truss is pinned at point *A* and point *B* is a roller support.

First consider the whole body FBD in Figure 1.9 and determine the reaction forces at points *A* and *B*. Summing forces yields

$$\sum_x F = -A_x = 0$$

$$\sum_y F = A_y + B_y - F = 0$$

Summing the moments about point A yields

$$\sum_A M = LB_y - 2LF = 0$$

Thus, we have three equations in the three unknown reaction forces resulting in

$$A_x = 0,\ B_y = 2F \text{ and } A_y = -F$$

Next consider each joint and its FBD as given in Figure 1.10. Each member is a two-force member so the FBD of the member itself will not yield any new information. However, they are illustrated in Figure 1.10 to help understand if they are in tension or compression. At the outset they are assumed to all be in tension (hence the directions indicated in the joint FBDs).

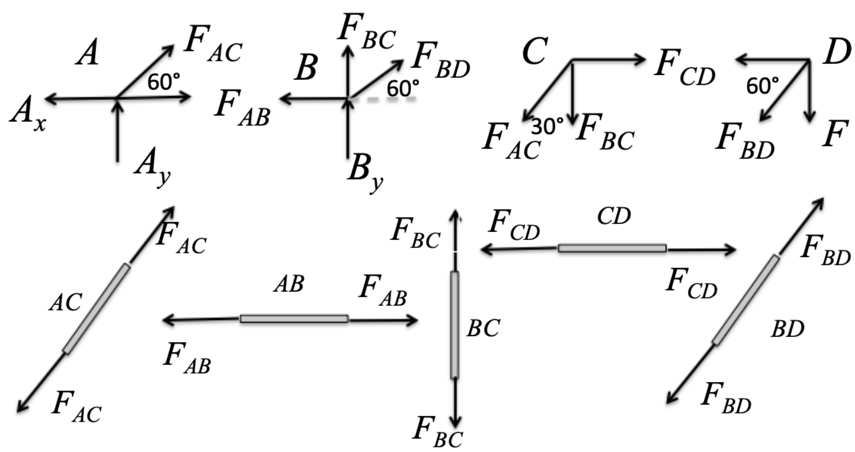


Figure 1.10 (Top) Free-body diagrams of each joint. (Bottom) FBD of the two force members. Note that that in the FBD the force in each member is assumed to be tension and is equal and opposite to the correspond force in the joint its connected to. If the calculation yields a negative force, then it means that member was in compression instead.

Because the two-force member AB must be in equilibrium and is assumed to be in tension, the directions of the force F_{AB} at point A and point B are as indicated required by equilibrium. Summing the forces at point A yields:

$$\sum_x F = -A_x + F_{AB} + F_{AC}\cos 60° = 0$$

$$\sum_y F = A_y + F_{AC}\sin 60° = 0$$

This is a system of two equations in two unknown reaction forces. Solving these yields:

$$F_{AB} = -\frac{1}{\sqrt{3}}F, \quad F_{AC} = \frac{2}{\sqrt{3}}F$$

Next sum the forces at joint B to get

$$\sum_x F = \frac{1}{\sqrt{3}} F + F_{BD} \cos 60° = 0 \Rightarrow F_{BD} = -\frac{2}{\sqrt{3}} F$$

$$\sum_y F = B_y + F_{BC} + F_{BD} \sin 60° = 0 \Rightarrow F_{BC} = -F$$

Summing the forces at joint C yields:

$$\sum_x F = F_{CD} - F_{AC} \cos 60° = 0 \Rightarrow F_{CD} = \frac{1}{\sqrt{3}} F$$

$$\sum_y F = -F_{BC} - F_{AC} \cos 30° = 0 \Rightarrow F = F$$

Note that the forces in the y direction in this case are all known, so this equation provides a check on the prior calculations.
Summing forces at joint D yields:

$$\sum_x F = -F_{CD} - F_{BD} \cos 60° = 0 \Rightarrow F = F$$

$$\sum_y F = -F - F_{BD} \cos 30° = 0 \Rightarrow F = F$$

Again, all the forces are known so this set of equations is used to provide an additional check. All the internal forces are now known. Based on their signs, member AB is in compression, member AC is in tension, member BC is in compression, member BD is in compression and member CD is in compression. Why? For example, consider member AB. In the drawing the forces on AB are drawn pointing out, indicating tension. However, when the value of the force is determined it is negative indicating that the arrows in the drawing should go the other way, thus member AB is in compression.

Statically Indeterminate Structures

Recall Example 1.2 and note that with the connections given, there were 3 equations of equilibrium in 3 unknowns so that a solution exists. Such a system is called *statically determinate.* However, if another support is added to the right end, say a simple support as indicated in Figure 1.11, the situation changes. There are still only three equations of static equilibrium, but now there is another unknown force, B, bringing the total number of unknowns to 4. Such as system cannot be solved for the 4 unknown forces and is called *statically indeterminate.* To be able to solve a statically indeterminate system additional information is needed. An additional equation can be obtained by considering the deformation of the structure. Such internal deformations will be discussed later in the text.

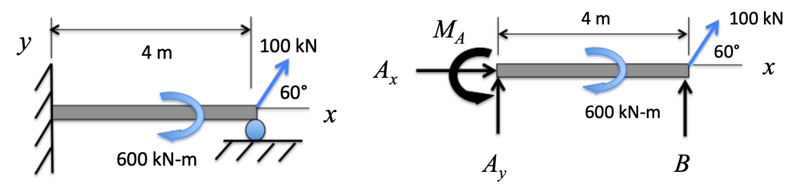


Figure 1.11 The system of Example 1.2 with and added roller support at the right end along with its free-body diagram.

The Centers

The idea of a center plays a critical role in many aerospace discussions. The basic idea is that of *center of gravity*, which is the point at which you can balance a stick horizontally. The center of gravity is simply that point on a body at which the entire weight may be considered as concentrated. If supported at the center of gravity the body remains in equilibrium, or balanced. Because airplanes and rockets are often moving in different gravitational fields the center of mass is more useful to consider then the center of gravity. The concept of center of mass is defined in your physics text in terms of a system of two particles in a gravitational field as a point C located a distance x_{cm} from some a specified origin, O, defined by

$$x_{\mathrm{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

which is sort of a mass weighted mean of the two distances x_1 and x_2. This expression defines the location of the center of mass and is illustrated in the following example.

Example 1.5: Consider the massless beam supported only at a single point with a simple frictionless connection as illustrated in Figure 1.12. Show that the moment about the center of mass with gravity pointing down is zero corresponding to the intuitive notion that the stick will not rotate about the center of mass.

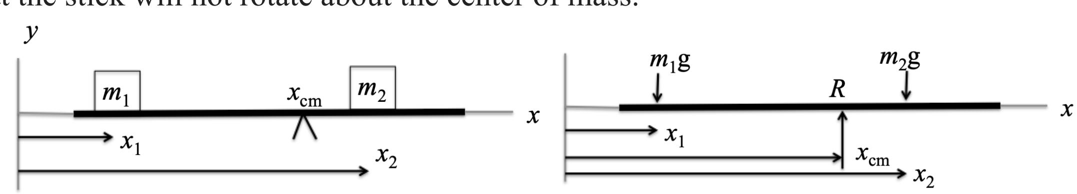


Figure 1.12 A massless beam connecting two masses simply supported at its center of mass (left) and the corresponding free body diagram (right).

Solution: Taking moments about x_{cm} and setting them to zero yields (note the reaction force does not contribute to the moments about the center of mass):

$$\sum M = (x_{cm} - x_1)m_1 g - (x_2 - x_{cm})m_2 g = 0$$

$$\Rightarrow x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Which is the formula for the center of mass and that point is in equilibrium. Near the earth's surface the center of mass and center of gravity are the same (note the units are length).

The concept of center of mass can also be formulated by recalling the idea of a moment. Consider a long skinny bar as illustrated in Figure 1.12. The *center of mass* of the bar can be found by summing moments and dividing by the total mass. Let *dm* denote an infinitesimal amount of mass of the bar so its moment would be *xdm*. Let ρ denote is density in mass per unit length so that $dm = \rho dx$. The total mass is denoted $M = \rho L$, where L is the total length of the bar. The center of mass of the bar is defined as (sum of the moments divided by the total mass):

$$x_{\text{cm}} = \frac{\int_0^L x\,dm}{M} = \frac{\rho\int_0^L x\,dx}{M} = \frac{1}{\rho L}\frac{\rho}{2}x^2\Big|_0^L = \frac{L}{2}$$

which agrees with intuition provided the density is constant. Near the earth's surface and in the case of constant density of the bar, the center of gravity, center of mass and *centroid* are all the same.

Centroids A centroid of a shape defines is geometric center. Similar to the calculation for the center of mass, the centroid of a line is defined as

$$x = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

The integral, $\int_0^L x\,dL$, is called the *first moment of a line.* Centroids of various geometric quantities are found by computing the ratio

$$\frac{\text{total moments}}{\text{total length, area or volume}}.$$

Area Centroid: Consider the moment of an infinitesimal element of area, *dA*. The centroid, or geometric center of the area is defined by its location in a rectangular coordinate system to be

$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA}, \qquad \bar{y} = \frac{\int_A y\,dA}{\int_A dA}$$

To find the centroid when the upper boundary of the shape is a function of the general form $y = f(x)$ and lower boundary as the x axis between points a and b then the calculations become:

$$\bar{x} = \frac{\int_a^b x \cdot f(x)\,dx}{\int_A dA}, \quad \bar{y} = \frac{\int_c^d y \cdot f(y)\,dy}{\int_A dA}$$

where the shape extends from point c to point d along the y axis. Given $y = f(x)$, it is sometimes a bit tricky to compute the inverse relationship $x = f(y)$ needed to calculate the y coordinate of the centroid. An alternate approach to computing the y coordinate is to consider the moment in the x direction and integrate along x instead of y. In this case the y coordinate becomes the integral of the product of the midpoint of the element $[f(x)/2]$ and the element $[f(x)]$ divided by the area:

$$\bar{y} = \frac{\int_a^b \frac{f(x)}{2} \cdot f(x)\,dx}{\int_A dA}$$

Volume Centroid: Consider the moment of an infinitesimal element of area, dV. The centroid, or geometric center of the volume is defined to be

$$\bar{x} = \frac{\int_V x\,dV}{\int_V dV}, \quad \bar{y} = \frac{\int_V y\,dV}{\int_V dV}, \quad \bar{z} = \frac{\int_V z\,dV}{\int_V dV}$$

The integrals over areas and volumes can be tricky to compute. Numerous examples are given in calculus books and can be found on various web sites.

Example 1.6 Compute the centroid of a triangle of length b and height h defined by the curve

$$y(x) = \frac{h}{b}x$$

This function is just the equation of a line of slope h/b. The key to computing centroids is in understanding the geometry and using it to pick a suitable differential element of area. Figure 1.13 shows the slice of infinitesimal area used here.

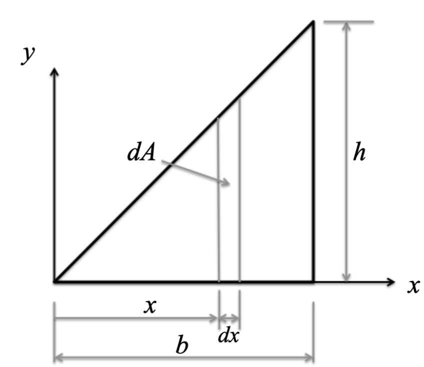


Figure 1.13 Diagram for computing the centroid of a triangular shape defined by a straight line of slope (h/b) fixed at the origin.

Solution: Define the infinitesimal area dA given symbolically in Figure 1.13. The x component of the centroid is

$$\bar{x} = \frac{\int_a^b x \cdot y(x)\,dx}{\int_A dA} = \frac{\int_0^b x \cdot \frac{h}{b} x\,dx}{\int_0^b \frac{h}{b} x\,dx} = \frac{\frac{h}{b}\left[\frac{x^3}{3}\right]_0^b}{\frac{h}{b}\left[\frac{x^2}{2}\right]_0^b} = \frac{2}{3}b$$

The alternate approach is used to compute the y component of the centroid allowing integration over x. This requires computing the total moments about the y axis by integrating across x performed by taking the midpoint of the element of area, $f(x)/2$, times the distance from the y axis, $f(x)$:

$$\bar{y} = \frac{\int_a^b \frac{f(x)}{2} \cdot f(x)\,dx}{\int_A dA} = \frac{\int_0^b \frac{1}{2}\left(\frac{h}{b}x\right)\left(\frac{h}{b}x\right)dx}{\int_0^b \frac{h}{b} x\,dx} = \frac{\frac{1}{2}\left(\frac{h}{b}\right)^2\left[\frac{x^3}{3}\right]_0^b}{\frac{h}{b}\left[\frac{x^2}{2}\right]_0^b} = \frac{1}{3}h$$

Thus, the centroid is at the point [(2/3)b, (1/3)h].

Computing centroids for complicated shapes occurs in many places in aerospace structures. A list of centroids can be found at

https://en.wikipedia.org/wiki/List_of_centroids

The calculation of centroids is also useful in computing other important centers in aerospace. A significant force on an aircraft in flight is the force due to pressure on the wings. It is known that changing the angle of attack (usually denoted by α) means that the pressure distribution above and below the wing is changing. The distribution of pressure on a surface can be represented by a single force acting at a single point called the *center of pressure.* The center of pressure is denoted by C_p and is calculated by

$$C_p(\alpha) = \frac{\int_{wing} xP(x)\,dx}{\int_{wing} P(x)\,dx}$$

Where $P(x)$ is the variation of pressure around the wing (a distributed force) and x is the distance from the leading edge of the wing. This is the formula for the calculation of the equivalent point force due to distributed forces acting on a rigid body discussed in the next section. Note that the center of pressure above the wing and the center of pressure below the wing will differ as the angle of attack changes. That means that the two resultant pressure forces will act some distance apart causing a *pitching moment* to develop on the wing resulting in a rotational motion on the wing. Figure 1.14 illustrates the resultant center of pressure in relationship to the center of mass for a stable flight.

These two forces cause a pitching moment to develop. The center of pressure changes with the angle of attack. If the center of pressure moves behind the center of mass, the direction of the moment changes and instability results.

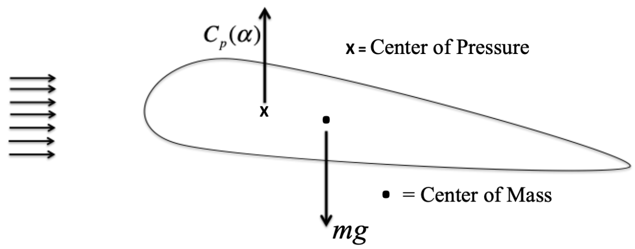


Figure 1.14 The center of pressure and center of mass

While the two centers of pressure change with angle of attack, there is a point on the airfoil where the pitching moment is zero. The *aerodynamic center* is the point at which the *pitching moment does not change with angle of attack* and at which the total lift force is considered to act. For low speed, thin airfoils typical for UAVs, the location of the aerodynamic center is located along the chord line about one quarter of the distance from the leading edge (called the *quarter chord point*). The aerodynamic center is the axis around which the airfoil moment acts and is also the point at which the net lift force is assumed to act. The aerodynamic center is illustrated in Figure 1.15. As discussed later, the aerodynamic center is key in understanding aircraft stability properties.

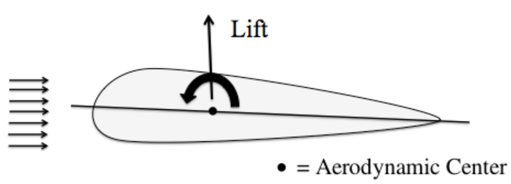


Figure 1.15 The aerodynamic center is located on the chord line one quarter of the distance from the leading edge for most thin airfoils in many cases.

Distributed Loads

Many of the loads on aerospace structures are not forces acting at a point but rather are distributed along the surface of the structure. The pressure on wing and the aerodynamic loads on rockets and aircraft are examples of distributed loads. One way to model a distributed load is to define a function called a *loading curve*. Figure 1.16 illustrates a loading curve acting in a plane. Because the load is distributed the calculus notation of infinitesimal elements is useful. At each infinitesimal element dx along the x axis the force exerted on the surface is $w(x)dx$ where $w(x)$ is the function defining the loading curve with units of force/length (such as N/m).

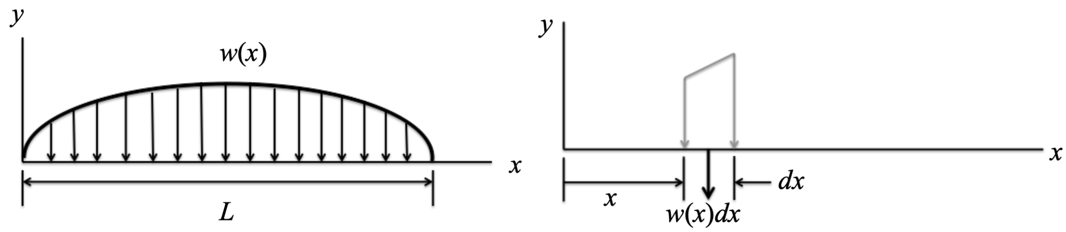


Figure 1.16 (Left) A force $w(x)$ distributed along the x axis and (right) the force acting on each infinitesimal element.

The total force applied to a beam along the x - axis between the points 0 and L is (remember calculus, this is the area under the curve):

$$F = \int_0^L w(x)\,dx\,,$$

and the total clockwise moment about the origin is

$$M = \int_0^L w(x)x\,dx\,.$$

Distributed loads can be represented by a single equivalent force acting at a point along its length for the purpose of computing reaction forces by using the concept of a centroid. Recall that for a line along the x-axis, the centroid is given by:

$$\overline{x} = \frac{\int_L x\,dA}{\int_L dA}$$

The concept is that the point force F applied at the centroid is equivalent to the effect of the distributed force defined by $w(x)$.

Example 1.7: Consider the cantilever beam of Figure 1.17 (a simplified model of a wing spar) with the distributed load indicated and compute the reaction forces at the fixed end by first integration of the distributed load and then by using the equivalent point load.

Solution: The load is simple in this case $w(x)$ =100 N/m. So that the integrals become simply

$$F(x) = \int_0^4 100\,dx = 100x\big|_0^4 = 400 \text{ N}$$

$$M(x) = \int_0^4 100x\,dx = 100\left(\frac{1}{2}x^2\right)\Bigg|_0^4 = 800 \text{ Nm}$$

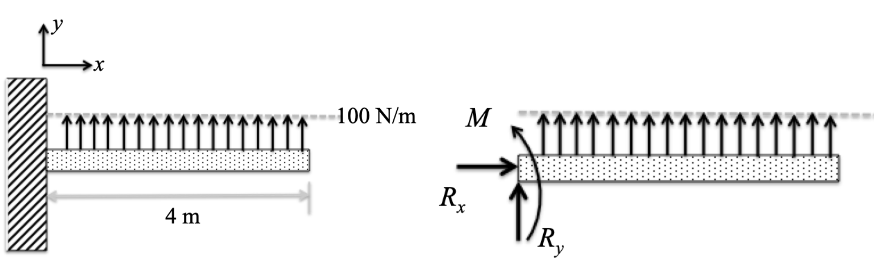


Figure 1.17 A cantilever beam (left) and its free body diagram (right).

Summing forces in the x direction yields $R_x = 0$. Summing forces in the y direction yields

$$\sum_y F = R_y + F = 0 \Rightarrow R_y = -400 \text{ N}$$

Summing moments about the left end yields

$$\sum_R M = M + 800 = 0 \Rightarrow M = -800 \text{ Nm}$$

Next solve the same problem using the point force approach, that is, first model the distributed force as an equivalent point force acting at the centroid. Summing forces in the x direction again yields $R_x = 0$. Summing forces in the y direction again yields R_y = -400 N, since

$$F_{\text{Resultant}} = \int_0^4 100\,dx = 100(x)\Big|_0^4 = 400 \text{ N}$$

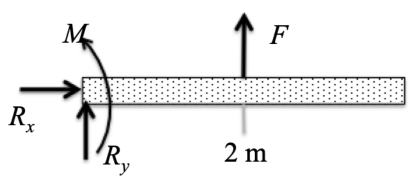


Figure 1.18 Free body diagram of the beam in Figure 1.17 with the distributed force modeled as a point force applied at the centroid.

Summing the moments about the left end yields

$$\sum_R M = M + 2 \cdot 400 = 0 \Rightarrow M = -800 \text{ Nm}$$

This is of course the same result as the first method giving confidence that the two methods are equivalent. While this is an extremely simple example it does provide a check on the concept.

Example 1.8: Consider the beam of Figure 1.19. The maximum of the loading curve at B is 100 N/m. Use the equivalent point load approach to compute the reaction forces at A and B.

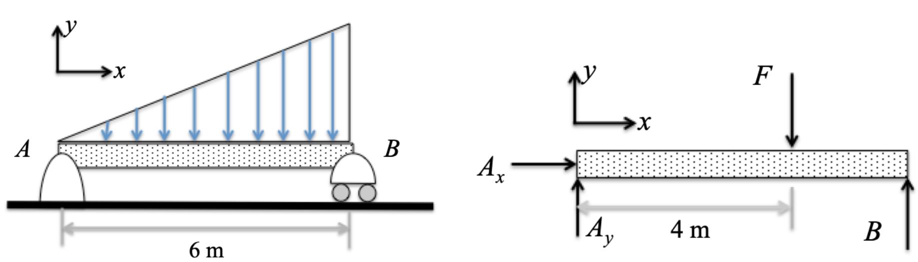


Figure 1.19 (Left) beam pinned at A and supported by a roller at B subject to a triangular distributed load. (Right). The free body diagram of the beam considering the distributed loading curve modeled as a point load.

Solution: The x component of the centroid of the triangle is computed in Example 1.5 to be

$$\bar{x} = \frac{2}{3}L = \frac{2}{3}6 = 4 \text{ m}$$

Computing the equivalent force for the loading curve by the area rule yields:

$$F_{\text{eq}} = \frac{1}{2}bh = \frac{1}{2}Lh = \frac{1}{2}(6 \text{ m})(100 \text{ N/m}) = 300 \text{ N}$$

Here the base of the triangle is: $b = L = 6$ m, and the height of the triangle is $h = 100$ N/m.

Alternately, the equation for the loading curve is a line of slope h/b:

$$f(x) = \frac{h}{b}x \Rightarrow F_{\text{eq}} = \int_0^6 f(x)dx = \int_0^6 \frac{h}{b}xdx = \frac{h}{b}\frac{1}{2}x^2\bigg|_0^6 = 300 \text{ N}$$

in agreement with the area calculation. Next consider computing the equivalent moment by using the area calculation:

$$M_{\text{eq}} = F_{\text{eq}} \cdot \bar{x} = 300 \text{ N} \cdot 4 \text{ m} = 1200 \text{ Nm}$$

Computing the moment by integration of the loading curve yields

$$M_{\text{eq}} = \int_0^6 f(x)xdx = \int_0^6 \frac{h}{b}x^2dx = \frac{h}{b}\frac{1}{3}x^3\bigg|_0^6 = \frac{100}{6}\frac{1}{3}6^3 = 1200 \text{ Nm}$$

in agreement with the area calculation.

Taking the moments about point A yields:

$$\sum_A M = 6B - 4F_{\text{eq}} = 0 \Rightarrow B = \frac{4}{6}F_{\text{eq}} = \frac{2}{3}300 = 200 \text{ N}$$

Summing forces in the x direction yields $A_x = 0$. Summing forces in the y direction yields

$$\sum_y F = A_y + B - F_{\text{eq}} = 0 \Rightarrow A_y = 100 \text{ N}$$

Example 1.9 A crude model of a wing and winglet is given in Figure 1.20. Consider the "wing" to be a beam cantilevered at the right end into the fuselage and having a mass of 270 kg, which is already modeled as acting as a point force 3 m from the fuselage. The distributed force on the winglet is 30 N/m and the maximum of the load curve on the wing is 300 N/m. Compute the reaction forces at R.

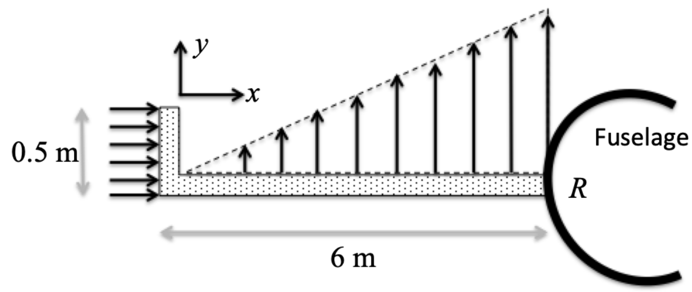


Figure 1.20 A crude model of a wing with winglet (vertical beam) with maximum loads of 30 N/m and 300 N/m and a mass of 270 kg.

Solution: The force on the vertical component is using the area method is just

$$F_2 = (0.5 \text{ m})(30 \text{ N/m}) = 15 \text{ N}$$

The y coordinate of the centroid is just 0.25 m. The equivalent force on the horizontal section is computed using the area of a triangle to be

$$F_1 = \frac{1}{2}bh = \frac{1}{2}(6 \text{ m})(300 \text{ N/m}) = 900 \text{ N}$$

The x coordinate of the centroid is just (see Example 1.17)

$$\bar{x} = \frac{2}{3}L = \frac{2}{3}6 = 4 \text{ m}$$

illustrated as 2 m from point R as this is where moments will be taken. Figure 1.20 gives the free body diagram. Summing Forces in the x direction yields

$$\sum_x F = F_2 + R_x = 0 \Rightarrow R_x = -15 \text{ N}$$

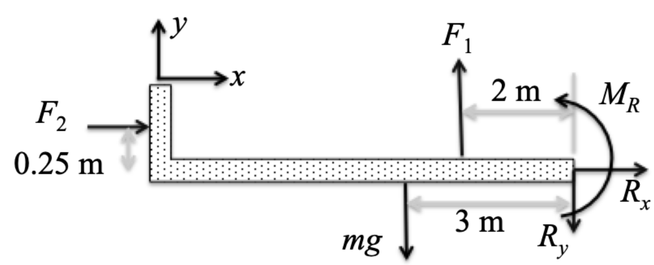


Figure 1.21 Free body diagram of the system in Figure 1.20.

Summing forces in the y direction yields:

$$\sum_y F = F_1 - mg - R_y = 0 \Rightarrow R_y = \underbrace{900 \text{ N}}_{F_1} - \underbrace{(270\text{kg})(9.8\text{m/s}^2)}_{mg} = -1{,}746 \text{ N}$$

Summing the moments about point R yields:

$$\sum_R M = M - F_1(2) + mg(3) - F_2(0.25) = 0 \Rightarrow M = -6{,}134.25 \text{ Nm}$$