Math 115 - Team Homework Assignment #4, Winter 2021

- **Due Date:** Thursday, March 25 or Friday, March 26, 2021 (Your instructor will tell you the exact time.)
- It is important that you try these problems **before your first meeting** with your team. This helps your group work more efficiently during your meeting. But don't be discouraged if you can't solve most of the problems on your own – this is why you have a team.
- The team homework roles are **Manager, Reporter, Editor**, and **Clarifier**. When a team takes their roles seriously, group work is more efficient! For details, see the Student Guide at `http://www.math.lsa.umich.edu/courses/sg/Math115StudentGuide.html#h.s42qebn9jbt`.
- Make sure **everyone** on your team is involved and that no one feels excluded. If you notice someone is quiet, actively encourage them to contribute.
- Ask your teammates to explain their reasoning if you don't understand it. Use this as an opportunity to learn from them! Also keep in mind that one of the best ways to deepen your own understanding is to explain it to someone else.
- Make sure that every step of your solution is **thoroughly explained** and **justified**. Another Math 115 student, who did not understand how to solve this question before, should understand it after reading your solution.

---

1. Consider the function $g(x) = \sqrt{2}\cos(x)$. In this problem, it is helpful to remember the *double-angle identity* $\sin(2x) = 2\cos(x)\sin(x)$. It is also helpful to calculate decimal approximations of $\sqrt{2}$ and $\frac{\pi}{2}$.

   (a) i. Find a formula for the distance between the origin $(0, 0)$ and the point $(x, g(x))$.

   ii. Let $D(x)$ be the square of this distance. Explain why $D(x)$ always has its minima and maxima at the same x-values that minimize and maximize the distance itself.

   Please note that you are never asked to find the exact x- or y-values of the minima of $D(x)$, and indeed it is *not possible* to do so with any formula. You are encouraged to draw graphs (especially the graph of $g(x)$, but also others!) to help solve these problems.

   (b) Explain why the global minimum of $D(x)$ must occur on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then answer the following:

   i. How many critical points does $D(x)$ have on this interval?

   ii. What are the local maximal values of $D(x)$ on this interval?

   iii. How many points on the graph of $g(x)$ are closest to the origin?

   (c) Now let $h(x) = \frac{1}{2}g(x)$. How many points on the graph of $h(x)$ are closest to the origin?

2. In this problem, we consider an elementary logistic model of infection.

   Suppose the fictional town of Brookvale has a population of 10,000 people, and a disease breaks out. Let $f(t)$ denote the total number of people, t days after January 1, who have ever been infected. For our elementary model, we make some simplifying[1] assumptions:

   - We suppose that a person who has been infected always remains contagious.
   - We suppose that every infected person comes into contact with some proportion k of the non-infected population each day.
   - We suppose that, with every contact, the probability that an infected person infects a non-infected person is some fixed number p.

   Consider the t^{th} day after January 1. How many people will be infected on the $(t+1)^{\text{th}}$ day? Our assumptions tell us that the number of *newly infected* people on the $(t+1)^{\text{th}}$ day is approximately equal to

$$\#(\text{contagious people after } t \text{ days}) \cdot \#(\text{non-infected contacts per contagious person}) \cdot (\text{probability contact leads to infection})$$
$$= f(t) \cdot k(10{,}000 - f(t)) \cdot p$$

   Recall the way we give practical interpretations of equations with derivatives. Observe that our sentence about newly infected people is a practical interpretation for the equation

$$f'(t) = f(t) \cdot k(10{,}000 - f(t)) \cdot p$$

   We call this the *logistic differential equation.*

   (a) Consider the function

$$f(t) = \frac{10{,}000}{1 + e^{-10{,}000kpt}}$$

   Recall that k and p are constant numbers. Find $f'(t)$, and show that the logistic differential equation is true for this function.[2] We call $f(t)$ the *logistic function.*

   Let us suppose that Brookvale Medical Center is *overwhelmed* if there is ever a time t where $f'(t) > 700$. This means, approximately, that there are more than 700 new infections per day.

   (b) Suppose that $k = 0.004$. This means that a person comes into contact with about 40 people per day. Suppose also that $p = 0.012$, which means that each contact has a roughly 1.2% chance of infection. Find the global maximum value of the function $f'(t)$. Is Brookvale Medical Center ever overwhelmed?

   (c) Suppose instead that the citizens of Brookvale wear personal protective equipment, so that $p = 0.008$, which means that each contact has a roughly 0.8% chance of infection. Suppose that k is still 0.004. Now find the global maximum value of $f'(t)$. Is Brookvale Medical Center ever overwhelmed?[3]

   (d) Now suppose instead that the citizens of Brookvale also implement social distancing measures, so that $k = 0.003$. This means that a person comes into contact with about 30 people per day. They still wear personal protective equipment, so $p = 0.008$. Now find the global maximum value of $f'(t)$. Is Brookvale Medical Center ever overwhelmed?

[1] It is a matter of fact that these assumptions are not *so* unrealistic that they make the model useless, but it is worthwhile to take a moment to try to think of a better set of assumptions, and how they might change the mathematical model. To paraphrase a famous adage, all mathematical models are *wrong*, but some are *useful*.

[2] **HINT:** In this part and in the following ones, the numbers involved might obfuscate the solution. You are invited to first pretend that $f(t)$ is instead the basic logistic function $\frac{1}{1+e^{-t}}$. Notice that the $f(t)$ that we are using differs from the basic one only by horizontal and vertical stretches. Similarly, $f'(t)$ differs from $\frac{d}{dt}\frac{1}{1+e^{-t}}$ only by horizontal and vertical stretches. What does stretching do to extrema?

[3] You may notice, if you graph the function, that our changed numbers imply that the outbreak began on a different date. To have a mathematical model with the same outbreak date, we would simply need to perform a horizontal shift to offset this. A horizontal shift will not change the global maximum value, so for convenience we do not bother.