5 Engineering Beam Theory

To date we have defined stress and strain for axial loaded bar's and torsional rods. In this section we will examine beams, distinguished from axially loaded bars because we look at loads and deflections perpendicular to the long axis of the beam causing bending. Figure 5.1 illustrates a beam in bending compared to the previously considered axial and torsional deflections. In this case, loads are applied laterally, moments can be supported and the boundary conditions potentially require force summations in two directions and a moment summation. We have considered loads perpendicular to the bar before but not and resulting displacement in the "y" direction (bar deflections being in the x direction). Unfortunately, some texts take the positive y direction as pointing down and we are taking it pointing up as positive.

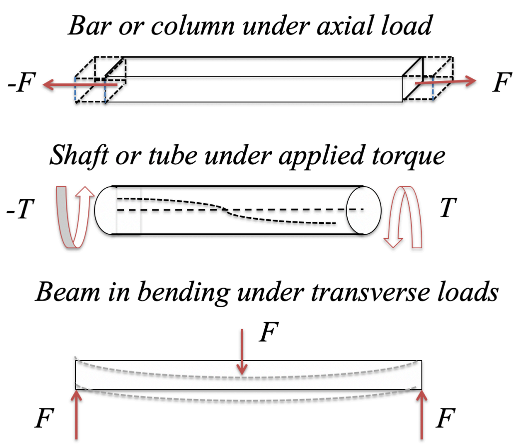


Figure 5.1 An illustration of the differences between bars, shafts and beams and their relative deflections. In this section we examine stresses and strains in beams. In the next section we will look at coupled stresses and strains in 3D.

The classic aerospace example for beam bending is a wing. Actually. all three stresses and strains are in play in a beam and combined analysis is presented in later sections. Recall the various possible boundary conditions and note that this time we are interested in deflections in the y direction. Figure 5.2 summarizes some common boundary conditions.

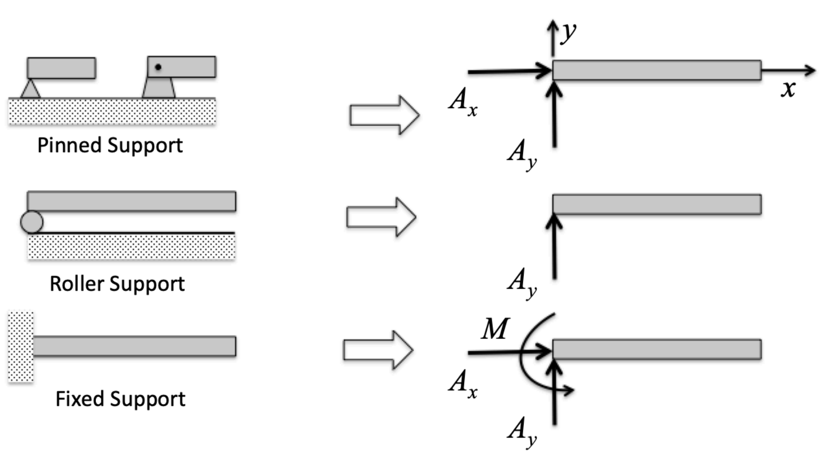
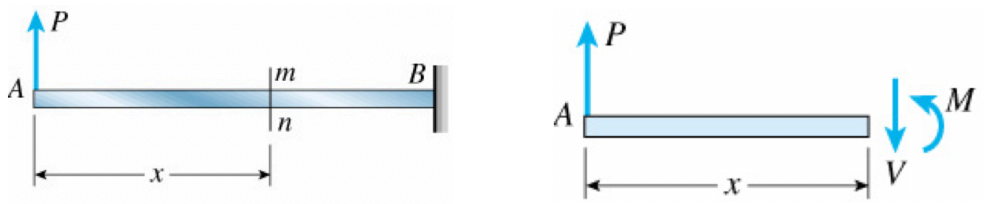


Figure 5.2 Common reaction forces for a beam (as well as a bar).

In beams we consider an internal shear force and a bending moment developing as the result of external forces and moments. Shear forces are denoted by the symbol V. Figure 5.3 illustrates the internal shear force, denoted V, and moment, M, that result as cantilevered beam is subject to a vertical load at P at is tip.



(a) (b)

Figure 5.3 (a) A cantilevered beam subject to a vertical force P. (b) the cut-out segment to the left of the cut line revealing the internal shear force, V, and moment, M.

Taking moments about point the right end in Figure 5.3b yields:

$$\sum M = M - Px = 0 \Rightarrow \underline{M = Px}$$

Summing forces in the y direction yields (taking positive as up):

$$\sum F_y = 0 \Rightarrow P - V = 0 \Rightarrow \underline{V = P}$$

Thus, equilibrium on a cut is enough to determine the internal shear and moment values.

Sign Conventions: First note that for externally applied forces we will take up as positive and down as negative, although this may change for some cases. Next the direction of the shear force and moment depend on how they act on the material. Consider the differential element of at beam depicted in Figure 5.4, which shows the effect on the element as the result of the two shear forces shown. The resulting deformation changes the rectangle into a parallelogram in this orientation is considered positive as indicated on the left and negative if it deforms as indicated on the right.

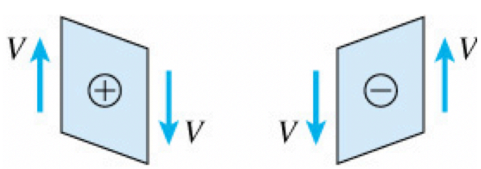


Figure 5.4 The deformation on the left is positive and that on the right is considered to be negative.

Examination of the images in Figure 5.4 implies that shear forces causing a clockwise rotation (left face moving down) are considered positive and shear forces causing a counterclockwise rotation (left face moving up) are considered to be negative.

Next consider a sign convention for dealing with moments as illustrated in Figure 5.5. Examining the figure shows that bending moments causing the top of the element to compress are considered positive, while bending moment causing the bottom to compress are considered to be negative.

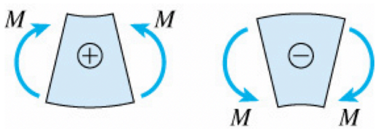


Figure 5.5 The deformation caused by moments considered to be positive for the configuration on the left and negative for the configuration on the right.

The above noted determination of signs depends on how the material deforms and are called *deformation sign conventions.* However, in free-body diagrams we use the standard recognition of direction used in statics. These are called *static sign conventions.* For example, recall Figure 5.3b where V is pointed down but considered to be negative seemingly in contrast to Figure 5.4

Example 5.1 (Figures from Geer) Consider the beam in Figure 5.6 and the two cuts listed there. The beam supports two loads, a force P and couple M_0 acting as shown. Calculate the shear force and bending moment at two different cross sections: (a) a small distance to the left of the midpoint of the beam and (b) a small distance to the right of the midpoint of the beam.

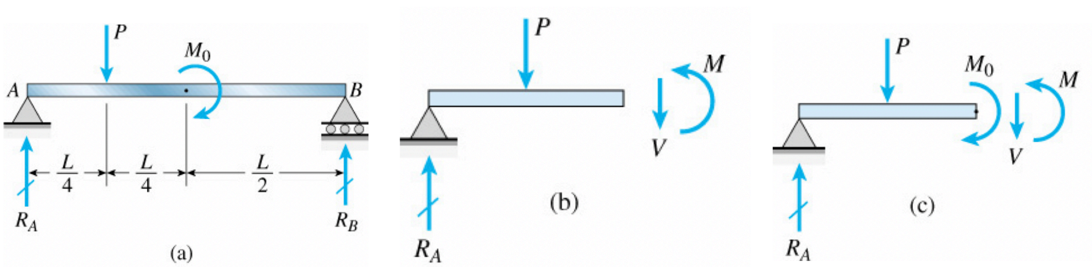


Figure 5.6 (a) A beam loaded by a force and a couple. (b) A cut made just to the left of the midpoint, not including the couple. (c) a cut to the right of the midpoint including the couple (the slash is used to distinguish reactions, from Geer).

Solution: First calculate the reaction forces. Taking moments about first point B and then point A in Figure 5.6a yields:

$$\sum_B M = 0 \Rightarrow\ R_A = \frac{3P}{4} - \frac{M_0}{L} \quad \text{and} \quad \sum_A M = 0 \Rightarrow\ R_B = \frac{P}{4} + \frac{M_0}{L}$$

(a) Examining the FBD of the cut illustrated in Figure 5.6b and summing forces in the y direction yields:

$$\sum_y F = 0 \Rightarrow\ R_A - P - V = 0 \Rightarrow\ V = R_A - P = \underline{-\frac{P}{4} - \frac{M_0}{L}}$$

Summing moments about the point where V is located and considering it to be at $L/2$ from the point A yields:

$$\sum_V M = 0 \Rightarrow \quad -R_A\left(\frac{L}{2}\right)+P\left(\frac{L}{4}\right)+M=0 \quad \Rightarrow \underline{M = \frac{PL}{8}-\frac{M_0}{2}}$$

(b) Next consider taking the cut listed in Figure 5.6c. The difference in this FBD and the previous one is that the couple M_0 shows up in the moment equations. Summing forces and moments yields:

$$V = -\frac{P}{4}-\frac{M_0}{L}, \quad \text{and} \quad M = \frac{PL}{8}+\frac{M_0}{2}$$

NOTE that as one moves along the beam from left to right the shear force does not change, however the moment does, meaning where the couple is applied matters for beams in bending.

Example 5.2: Consider the beam depicted in Figure 5.7 and calculate the internal forces and moment at points B and C (From Bedford).

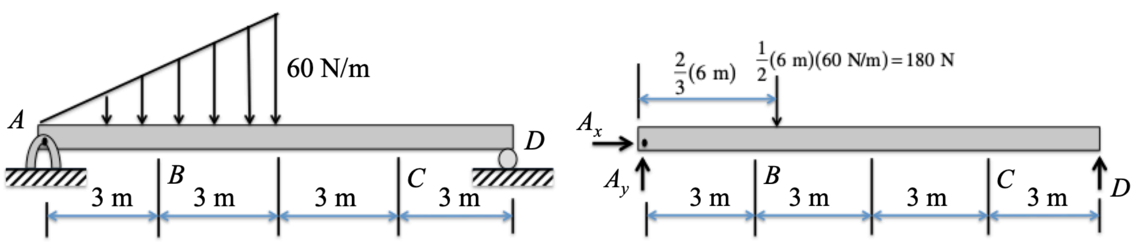


Figure 5.7 (a) A beam subject to a distributed load. (b) The free-body diagram of the beam in part (a).

Solution: First the free-body-diagram of the entire structure is used to determine the reaction forces. The distributed force is represented as a point force as indicated in Figure 5.7b along with the systems free-body diagram using the fact that the load is triangular. The equilibrium equations are (y+ is taken as up):

$$\sum F_x = 0 \Rightarrow \underline{A_x = 0}$$
$$\sum F_y = 0 \Rightarrow -180 + A_y + D = 0$$
$$\sum_A M = 0 \Rightarrow 12D - 4(180) = 0 \Rightarrow \underline{D = 60 \text{ N}}$$

The value of A_y becomes A_y = 120 N. To determine the internal forces and moment at point B, make at cut at point B and consider the segment to the left of point B. Next, recalculate that part of the distributed load that runs along the cut. Since point B is half way along the triangular shaped load the remaining load can be modeled as a point force due to a triangle of height 30 N/m and acting at a point x = (2/3)(3 m) = 2 m. Using the formula for the area of a triangle, the equivalent force placed at 2 m becomes (1/2)(3)(30)= 45 N. This is illustrated in the free body diagram of Figure 5.8a. Summing forces in the x direction yields that P_B = 0, as expected. Summing forces in the y direction yields (y+ is taken as up):

$$\sum F_y = 0 \Rightarrow -V_B - 45 + 120 = 0 \Rightarrow \underline{V_B = 75\text{ N}}$$

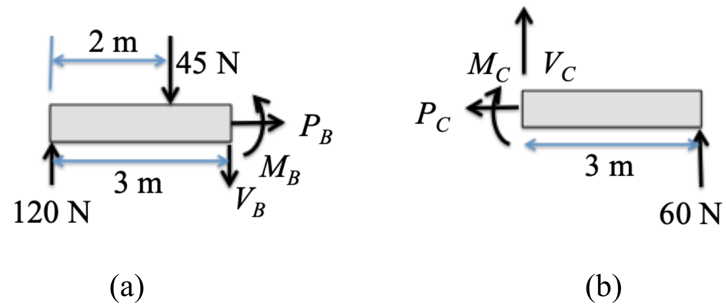


Figure 5.8 (a) The free-body diagram of the segment to the left of the cut at point *B*. (b) The free-body diagram of the segment to the right of the cut at point *C*.

Summing the moments at point *B* yields:

$$\sum_B M = 0 \Rightarrow M_B + (1)45 - 120(3) \Rightarrow \underline{M_B = 315\text{ N-m}}$$

Summing the forces in the y direction yields:

$$\sum_y F = 0 \Rightarrow -V_B - 45 + 120 = 0 \quad \Rightarrow V_B = 75\text{ N}$$

Thus, the internal forces and moment at point *B* are determined. Next consider the computing the forces and moment at point *C*. The relevant free-body diagram is given in figure 5.8b. Again, summing forces in the x direction yields P_C = 0 as expected. Summing forces in the y-direction yields:

$$\sum F_y = 0 \Rightarrow V_C + 60 = 0 \Rightarrow \underline{V_C = -60\text{ N}}$$

Summing moments about point *C* yields:

$$\sum_C M = 0 \Rightarrow -M_C + (3)60 \Rightarrow \underline{M_C = 180\text{ N-m}}$$

It is important to note that the one cannot determine the correct values of the internal moment and shear force if the distributed load is first represented as a point load. Rather the distributed load has to be divided up in the same way the cuts are. For example if the equivalent point force shown in Figure 5.7b were to be used in the free body diagram of Figure 5.8a, it would not appear because it is located to the right of point *B*. Thus summing forces and moments would give incorrect values (V_B =120 N for example) because the distributed load is not present.

Shear Force and Bending Moment Diagrams A key to successful design of beam structures requires knowing how the shear force and moment varies with position along the beam. The following example illustrates the procedure for calculating the internal force and moment as a function of *x*.

Example 5.3: Consider the simply supported beam of Figure 5.9 loaded by a force F two thirds of the way along the beam and compute the shear force and bending moment along the beam.

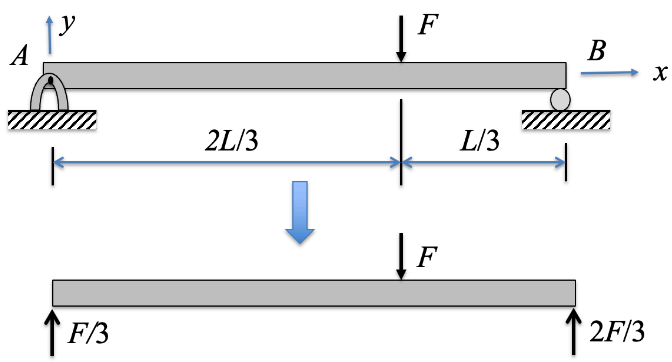


Figure 5.9 (Top) Simply supported beam loaded by a force. (Bottom) The free-body diagram of the beam showing the reaction forces.

Summing forces in the y direction (y+ is taken as up) and taking moments about point A yields:

$$\sum F_y = 0 \Rightarrow F_A + F_B - F = 0$$

$$\sum_A M = 0 \Rightarrow -\frac{2}{3}LF + LF_B = 0$$

$$\Rightarrow F_A = \frac{F}{3}, \quad F_B = \frac{2F}{3}$$

These two equations yield the reaction forces illustrated in Figure 5.9. Next make a cut between the left end and the application of the force as illustrated in Figure 5.10.

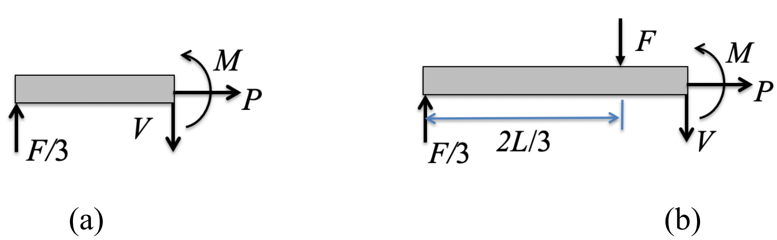


Figure 5.10 (a) A free body diagram of a cut made at an arbitrary position x somewhere between the left end and the application of the force F. (b) A free body diagram of a cut made at an arbitrary position x somewhere between past the point of application of the force F and the left end of the beam.

Summing the forces and moments around the point x in Figure 5.10a yields:

$$\sum F_y = 0 \Rightarrow \frac{F}{3} - V = 0 \Rightarrow \underline{V = \frac{F}{3}}$$

$$\sum_x M = 0 \Rightarrow -\frac{F}{3}x + M = 0 \Rightarrow \underline{M = \frac{Fx}{3}}$$

$$\Rightarrow \left.\begin{array}{c} V = \dfrac{F}{3} \\ M = \dfrac{Fx}{3} \end{array}\right\} \quad 0 \le x \le \frac{2}{3}L$$

Summing the forces and moments around the point x in Figure 5.10b yields:

$$\sum F_y = 0 \Rightarrow \frac{F}{3} - F - V = 0 \Rightarrow \underline{V = -2\frac{F}{3}}$$

$$\sum_x M = 0 \Rightarrow -\frac{F}{3}x + F\left(x - \frac{2L}{3}\right) + M = 0 \Rightarrow \underline{M = \frac{2}{3}F(L-x)}$$

$$\Rightarrow \left.\begin{array}{c} V = -\dfrac{2}{3}F \\ M = \dfrac{2}{3}F(L-x) \end{array}\right\} \quad \frac{2}{3}L \le x \le L$$

These last two sets of equations illustrate how the internal shear force and moment change with the distance, x, along the beam. Note that the shear force takes a step change at the point of application of the external force F as does the moment M which starts to degrease as x passes the point x = 3L/3. This is illustrated in the plots made in Figure 5.11 called the shear force and bending moment diagrams.

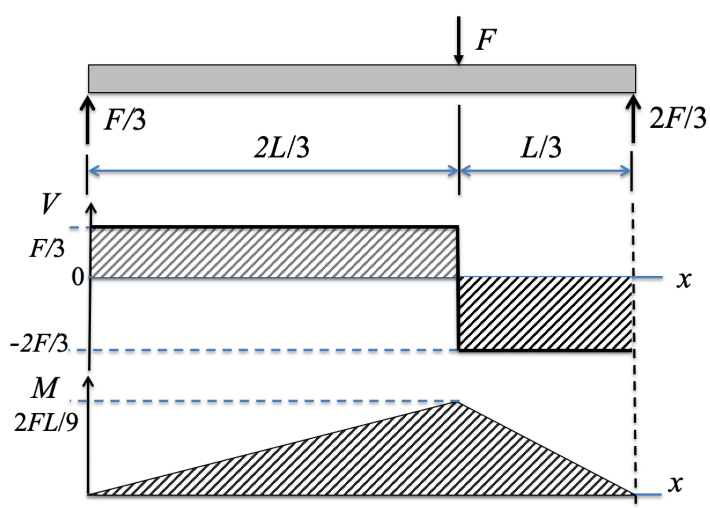


Figure 5.11 Shear force and bending moment diagrams for the beam of Figure 5.9 illustrating how these quantities change along the length of the beam.

This procedure can be repeated for any arrangement of distributed and point forces and moments.

Example 5.4: Calculate and draw the shear and bending moment diagrams for the beam of Figure 5.12.

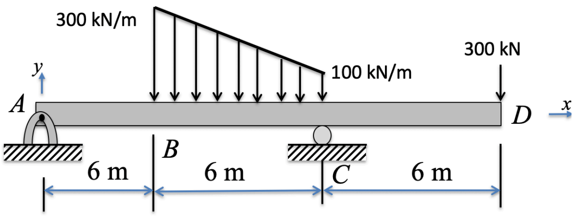


Figure 5.12 A beam subject to both a distributed load and a point force.

Solution: The first task in solving for the shear and bending moment internal to the beam is to take a free-body diagram of the entire structure and compute the two reaction forces at points A and C. This requires first resolving the distributed force into two point forces noting that in this case the distributed force can be treated as the sum of a constant force and a triangular force. The constant force has a magnitude of 100 kN/m. The point of action will be midpoint of the load which is 3 m or 9 m from point A and its value (recall it's the area) is (100 kN/m)(6 m) = 600 kN. After subtracting the 100 kN/m constant load the remaining triangular load has a maximum amplitude of 200 kN/m. The area of the triangle yields the equivalent point load is:

$$\frac{1}{2}(6\text{ m})(200\text{ kN/m}) = 600\text{ kN}$$

The point of action is (2/3)(6 m) from point C or 8 m from point A. With these point force equivalents, the whole body, free-body diagram is given in Figure 5.13.

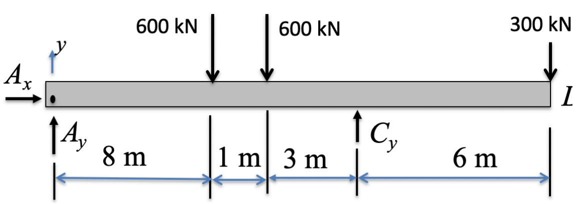


Figure 5.13 The free-body diagram of the structure in Figure 5.12 with the distributed loads modeled as equivalent point forces.

Summing forces in the x direction yields that $A_x = 0$. The remaining equilibrium equations are:

$$\sum_y F = A_y + C_y - 600 - 600 - 300 = 0$$

$$\sum_A M = 12C_y - (8)600 - (9)600 - (18)300 = 0$$

Solving the moment equation yields $C_y = 1300$ kN. Then the force equation yields $A_y = 200$ kN.

Armed with the reaction forces, the internal shear forces and moments are obtained by making various cuts in the structure to expose them. An important issue is how to treat the distributed forces. They cannot be treated as point forces directly as done above in the cuts made to get at the internal shear force, but rather the cut portion of the distributed force must be considered.

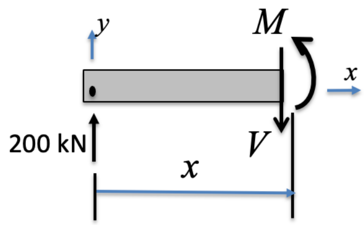


Figure 5.14 A cut between the right end and just before the application of the distributed load for the region $0 < x < 6$.

First consider a cut made between points *A* and just before point *B* as illustrated in Figure 5.14. Summing forces and moments yields:

$$\sum_y F = 200 - V = 0, \quad \sum_V M = M - 200x \Rightarrow$$

$$V = 200 \text{ kN}, \quad M = 200x \text{ kN-m}, \ 0 < x < 6 \text{ m}$$

Next consider the range from x = 6 to x = 12 m. This region is chosen because it includes the distributed force but not the point force, which is already accounted for in the global free-body diagram. The free-body diagram of this cut is illustrated in Figure 5.15. The first step is to figure out how to model the distributed force. Since the distributed force is linear it must have the form $w(x) = ax + b$. The constants a and b are evaluated by knowing the end points: $w(6) = 6a + b = 300$ and $w(12) = 12a + b = 100$ solving these two equations for a and b yields

$$w(x) = -\frac{100}{3}x + 500 \text{ kN/m}$$

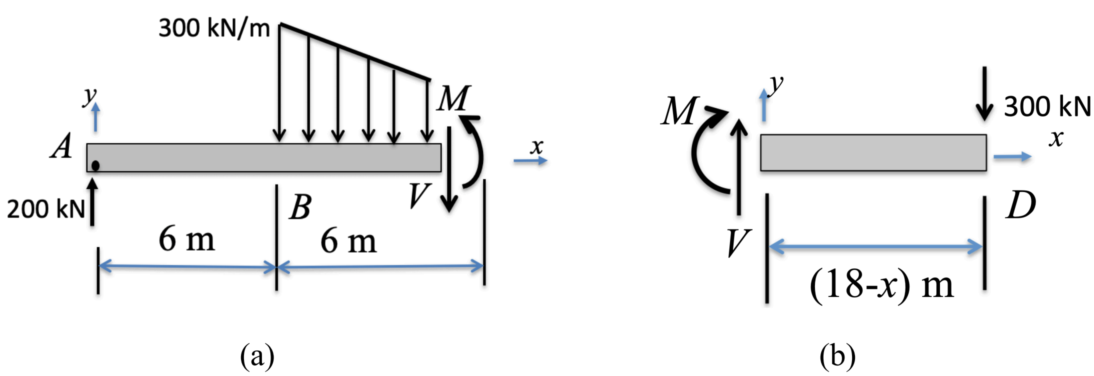


Figure 5.15 (a) A cut between the interval $6 < x < 12$ m. (b) A cut of the remaining part of the beam working from the right end for $12 < x < 18$.

The effect of the distributed load is a downward force defined by the integral

$$F = \int_L w(x)\,dx = \int_6^x \left(-\frac{100}{3}x + 500 \right) dx = -\frac{50}{3}x^2 + 500x - 2400 \text{ kN}$$

and a clockwise moment about point *A* defined by the integral

$$M_A = \int x w(x)\,dx = \int_6^x \left(-\frac{100}{3}x^2 + 500x \right) dx = -\frac{100}{3}x^3 + 250x^2 - 6600 \text{ kN} - \text{m}$$

The equilibrium equations now become

$$\sum_y F = 0 \Rightarrow 200 - V + \frac{50}{3}x^2 - 500x + 2400 = 0$$

$$\sum_A M = 0 \Rightarrow M - Vx + \frac{100}{9}x^3 - 250x^2 + 2600x + 6600 = 0$$

Solving for *V* and *M* yields:

$$\left.\begin{aligned} V &= \frac{50}{3}x^2 - 500x + 2600 \text{ kN} \\ M &= \frac{50}{9}x^3 - 250x^2 + 2600x - 6600 \text{ kN-m} \end{aligned}\right\} \; 6 < x < 12 \text{ m}$$

For the remaining range of values 12 < *x* < 18 consider the cut in Figure 5.15b. The equilibrium equations are

$$\sum_y F = 0 \Rightarrow V - 300 = 0$$

$$\sum_{\text{left end}} M = 0 \Rightarrow -M - 300(18 - x) = 0$$

Solving these yields:

$$\left.\begin{aligned} V &= 300 \text{ kN} \\ M &= 300x - 5400 \text{ kN-m} \end{aligned}\right\} \; 12 < x < 18 \text{ m}$$

The above three regions are now connected to produce the shear and moment diagrams, plotted in Figure 5.16. In summary:

$$V = 200 \text{ kN}, \quad M = 200x \text{ kN-m}, \; 0 < x < 6 \text{ m}$$

$$\left.\begin{aligned} V &= \frac{50}{3}x^2 - 500x + 2600 \text{ kN} \\ M &= \frac{50}{9}x^3 - 250x^2 + 2600x - 6600 \text{ kN-m} \end{aligned}\right\} \; 6 < x < 12 \text{ m}$$

$$\left.\begin{aligned} V &= 300 \text{ kN} \\ M &= 300x - 5400 \text{ kN-m} \end{aligned}\right\} \quad 12 < x < 18 \text{ m}$$

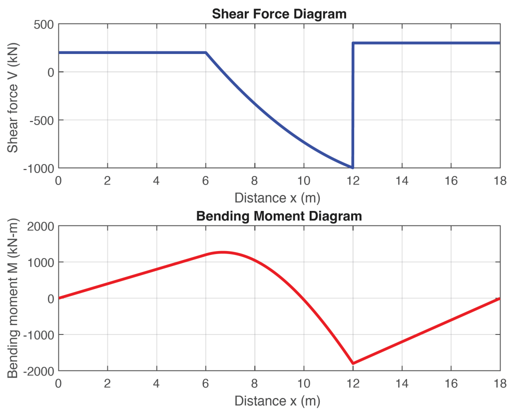


Figure 5.16 Shear and moment diagrams for the structure of Figure 5.12 illustrating how these properties change throughout the beam in relation to the applied distributed and point loads. Note the discontinuities in the shear force (part b) and the passage through zero for both the shear and moment values.

The MATLAB code for generating Figure 5.16 is

```
clc;clear all;close all;
syms x;
V = @(x) piecewise(0<x<6,200, 6<=x<12,(50/3)*x^2-500*x+2600,
12<=x<18, 300);
M = @(x) piecewise(0<x<6,200*x, 6<=x<12,(50/9)*x^3-
250*x^2+2600*x-6600,12<=x<18,300*x-5400);
x = 0:0.01:18;
Vx = subs(V,x);
subplot(2,1,1)
plot(x,Vx,"b","Linewidth",2);
title("Shear Force Diagram")
xlabel("Distance x (m) ")
ylabel("Shear force V (kN) ")
grid on
Mx = subs(M,x);
subplot(2,1,2)
plot(x,Mx,"r","Linewidth",2)
title("Bending Moment Diagram")
xlabel("Distance x (m) ")
ylabel("Bending moment M (kN-m)")
grid on
```

Relationships between Distributed Load, Shear Force and Bending Moment: In the following the relationships between these quantities are derived. One can see from diagrams in Figures 5.11 and 5.16 that the shear force, moment and distributed force are related. The equations developed here tie these relationships to mathematical formulas. Consider an infinitesimal element subjected to a distributed load as illustrated in Figure

5.17. Summing moments and forces will yield the desired relationships for this configuration.

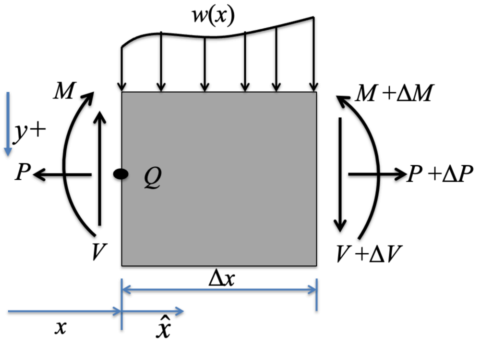


Figure 5.17 A differential element of a bar subject to a distributed load. Note that the axial load, *P*, is also included. The terms *ΔP*, *ΔM* and *ΔV* are the changes in axial force, shear force and bending moment respectively. The distance $\hat{x}$ is the distance from the left edge of the element *Δx*.

Summing the forces in the *x* direction along the element *Δx* yields:

$$\sum F_x = P + \Delta P - P = 0$$

Dividing this expression by *Δx* and taking the limit as *Δx* approaches zero yields that:

$$\frac{dP}{dx} = 0$$

Next sum the forces in the *y* direction:

$$\sum_y F = 0 \Rightarrow -V(x) + V(x+\Delta x) + w(x)\,\Delta x = 0$$

$$\Rightarrow \frac{V(x+\Delta x) - V(x)}{\Delta x} = -w(x) \Rightarrow \underline{\frac{dV}{dx} = -w(x)}$$

If $w(x)$ is pointed up in the negative direction, then plugging say $w(x) = -w_0$ into the expression for dV/dx would yield $dV/dx = w_0$. Summing the moments about point *Q* (counterclockwise moments are positive) and taking the limit as *Δx* approaches zero yields that

$$-M(x) + M(x+\Delta x) - w(x)\,\Delta x\frac{\Delta x}{2} - V(x+\Delta x)\,\Delta x = 0$$

$$\Rightarrow \frac{M(x+\Delta x) - M(x)}{\Delta x} = w(x)\,\frac{\Delta x}{2} + V(x+\Delta x)$$

$$\Rightarrow \underline{\frac{dM}{dx} = V(x)}$$

It is instructive to examine these expressions if the *y* axis is taken as positive up. To that end consider the two cased represent in Figure 5.18. Not the that because counterclockwise is still positive and because $w(x)$ is multiplied by Δx^2 the direction of the $w(x)$ does not matter and the moment equation in both cases is

$$\frac{dM}{dx} = V(x)$$

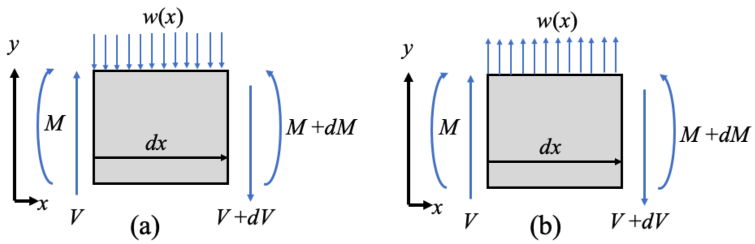


Figure 5.18 (a) The y-axis taken as positive up and the given load is in the negative direction. (b) The y-axis taken as positive up and the given load is in the positive direction.

The equation relating the shear to $w(x)$ is determined by summing forces in Figure 5.18a in the y direction which yields:

$$\sum_y F = -w(x)\,dx + V - (V + dV) = 0 \Rightarrow \frac{dV}{dx} = -w(x)$$

as before. Next, consider the case illustrated in Figure 5.18b, where the load is reversed from that in Figure 5.18a, and applied in the positive direction. Summing forces in Figure 5.18b in the y direction yields:

$$\sum_y F = w(x) + V - (V + dV) = 0 \Rightarrow \frac{dV}{dx} = w(x)$$

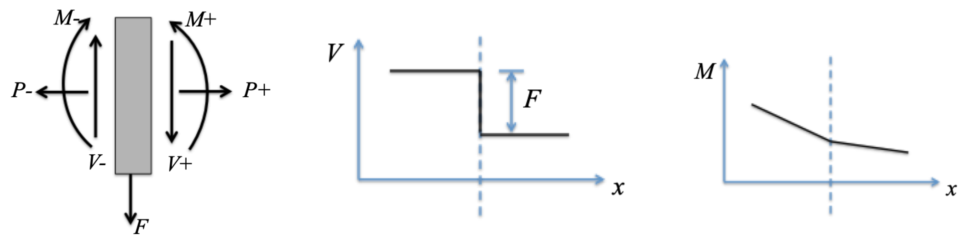
Note that as expected the sign of the load has changed from positive to negative in the relationship between shear and load. The sum of moments remains the same because the load term drops out because of the dx^2 term.

These expressions indicate that the axial force does not depend on x when the beam is only subject to a lateral distributed load. The expression for dV/dx indicates how the shear force changes with the distributed load $w(x)$, and the expression for dM/dx indicates how the bending moment changes with the shear force. The above analysis only applies for a section of a beam subject to a distributed load and must be modified to account for cases where a point force and couple are also applied. This is discussed next.

Consider a small element of the beam +/- around the application of a point force, illustrated in Figure 5.19a, representing a free-body diagram of cut just either side of the force. Summing forces and moments yields:

$$V_- - V_+ = F \quad \text{and} \quad M_- - M_+ = 0$$

These two equations imply that the shear force goes through at step change at the point of application of the force (Figure 5.19b) and the moment varies continuously across the point of application of the force (Figure 5.19c).



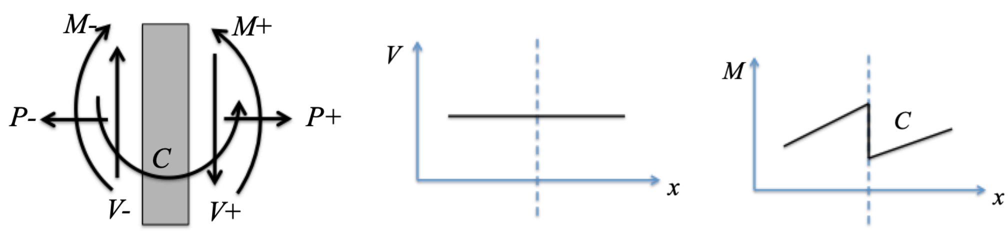
(a) (b) (c)

Figure 5.19 (a) A free body diagram of the region around the application of a point force *F*. (b) The resulting shear diagram where the dashed line is the point of application of *F*. (c) The resulting moment diagram where the dashed line is the point of application of *F*.

Next consider a couple acting on a small element of the beam +/- around the application of the couple, illustrated in Figure 5.20a, representing a free-body diagram of cut just either side of the couple. Summing forces and moments yields:

$$V_- - V_+ = 0 \quad \text{and} \quad M_- - M_+ = -C$$

In this case, the shear force diagram, Figure 5.20b, is continuous and unchanged, while the moment diagram, Figure 5.20c, is discontinuous.



(a) (b) (c)

Figure 5.20 (a) A free body diagram of the region around the application of a couple *C*. (b) The resulting shear diagram where the dashed line is the point of application of the couple *C*. (c) The resulting moment diagram where the dashed line is the point of application of *C*.

Example 5.5: **Shear and Bending Moment by Integration** Consider the beam in Figure 5.21a and use integration of the differential equations for V and M to compute the shear force and bending moments as functions of x. Then use these to plot the shear force and bending moment diagrams.

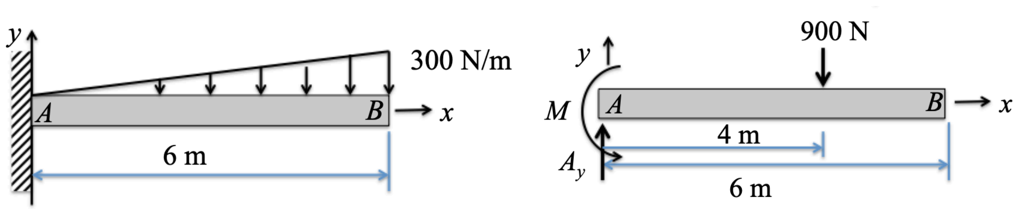


Figure 5.21 (a) A cantilevered beam subject to a distributed load. (b) Its free-body diagram for computing reaction forces and moments.

Solution: The first step is to construct the free-body diagram representing the distributed force as a point force and computing the reaction force and moment at point A. This is shown in Figure 5.21b, with the distributed load represented as a point force. Summing forces in the x - direction yields A_y = 900 N and moments yields M = 3600 N-m.

Next compute the formula for the load $w(x)$. Using the formula for a straight line and evaluating $w(x) = ax + b$ at $x = 0$ and $x = 6$ yields: $w(x) = 50x$. Using the differential form of the shear force:

$$\frac{dV}{dx} = -w(x) \Rightarrow dV = -w(x)dx \Rightarrow \int_{900}^{V} dV = -\int_{0}^{x} 50x\,dx$$

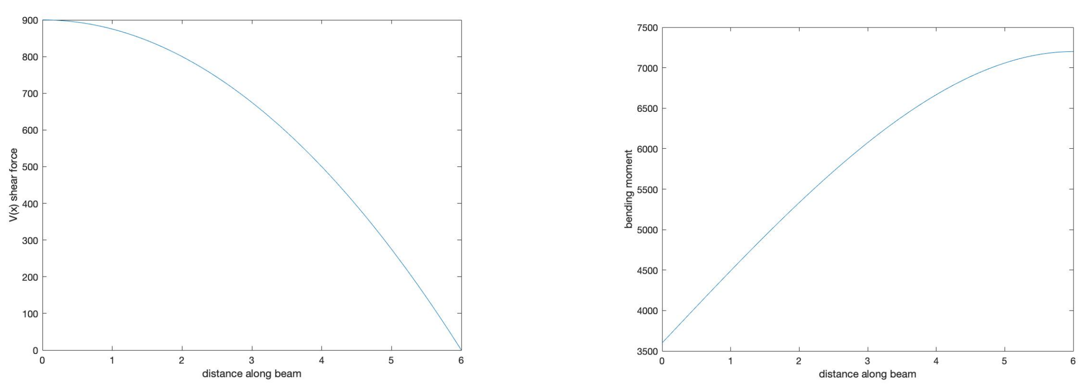
$$\Rightarrow V - 900 = -25x^2 \Rightarrow \underline{V(x) = 900 - 25x^2}$$

Next integrate the moment equation: $dM/dx = V$:

$$\int_{3600}^{M} dM = \int_{0}^{x} V(x)\,dx = \int_{0}^{x} \left(900 - 25x^2\right) dx$$

$$\Rightarrow M - 3600 = 900x - \frac{25}{3}x^3 \Rightarrow \underline{M(x) = 3600 + 900x - \frac{25}{3}x^3}$$

These are plotted in Figure 5.22.



(a) (b)

Figure 5.22 (a) The shear diagram: $V(x)$ vs x. (b) The moment diagram: $M(x)$ vs x. Both plots are for the system of Figure 5.22a.

Matlab codes for Figure 5.22

```
>> x=0:0.01:6;
>> M=-(25/3)*x.^3+900*x+3600;
>> figure
>> plot(x,M)
>> xlabel('distance along beam')
>> ylabel('bending moment')
```

```
>> V=900-25*x.^2;
>> figure
>> plot(x,V)
>> xlabel('distance along beam')
>> ylabel('V(x) shear force')
```

Example 5.6 Consider the beam in Figure 5.23a and use integration of the differential equations for V and M to compute the shear force and bending moments as functions of x. Then use these to plot the shear force and bending moment diagrams. Note that this is a repeat of Example 5.3, except solved by using the integration approach.

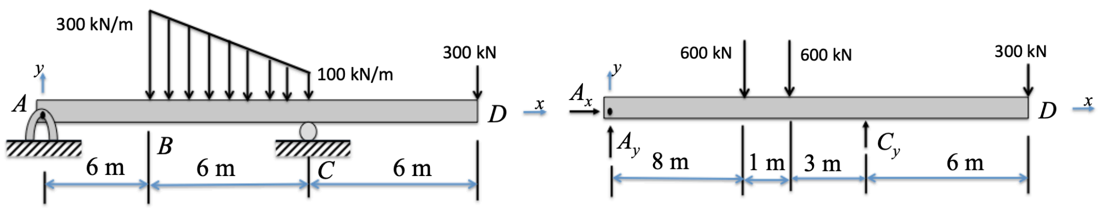


Figure 5.23 (a) A beam subject to both a distributed load and a point force. (b) The whole body, free-body diagram using equivalent point forces to compute the reaction forces at points A and C.

Solution: From Figure 5.23b the reaction forces are (recall Example 5.4) $A_y = 200$ kN and $C_y = 1300$ kN. Also recall that the distributed load is described by

$$w(x) = -\frac{100}{3}x + 500 \text{ kN/m}$$

Shear Force: First consider obtaining the shear force as a function of x using the relationship

$$\frac{dV}{dx} = -w(x)$$

and examine each of the region where there is a change in loading.

For the interval: $0 < x < 6$ m, $w(x) = 0$ so $dV = 0$ so $V(x) =$ constant $= A_y = 200$ kN.

For the interval: 6 m $< x <$ 12 m, where the distributed force is active just shy of the reaction as C, the shear force becomes:

$$\int_{200}^{V} dV = -\int_{6}^{x} w(x)\,dx = -\int_{6}^{x} (-\tfrac{100}{3}x + 500)\,dx$$

$$\Rightarrow V - 200 = \left(\tfrac{100}{6}x^2 - 500x\right)\Big|_6^x = \tfrac{100}{6}x^2 - 500x + 2400$$

$$\Rightarrow V(x) = \tfrac{100}{6}x^2 - 500x + 2600$$

In the interval from: 12 m $< x <$ 18 m the effect of the 1300 kN reaction force at C needs to be considered at the point $x = 12$ m. Adding this to $V(x)$ yields:

$$V(12) = 1300 + \frac{100}{6}(12)^2 - 500(12) + 2600 = 300 \text{ kN}$$

For the rest of the interval noting happens so that $V(x)$ = 300 kN in this interval. Exactly at x = 18 m, however the point force -300 kN acts and $V(18) = 0$.

Bending Moment: For the bending moment integration approach the relationship is

$$\frac{dM(x)}{dx} = V(x)$$

Integrating this is again done segment by segment in the case of lots of external forces following the segments used above for $V(x)$.

For the interval: $0 < x < 6$ m, $V(x)$ = 200 kN so

$$\int_0^M dM = \int_0^x V(x)\,dx = 200\int_0^x dx \Rightarrow \underline{M = 200x \text{ kN=m}}$$

For the interval: 6 m < x < 12 m,

$$V(x) = \tfrac{100}{3}x^2 - 500x + 2600$$

Thus, integrating the moment equation from x = 6 m to and arbitrary point x in the interval yields:

$$\int_{1200}^M dM = \int_6^x V(x)\,dx = \int \left(\tfrac{100}{3}x^2 - 500x + 2600\right)dx$$

$$\Rightarrow \underline{M(x) = \frac{50}{9}x^3 - 250x^2 + 2600x - 6600 \text{ kN-m}}$$

Note that the lower limit of integration on M is the value of M at x = 6, i.e. M =200(6) = 1200 kN-m. Also note that at the point x = 12 m, $M(12)$ = -1800 kN-m. This determines the lower limit on the next integral. So for the interval 12 m < x < 18 m, integrating from x = 12 m to an arbitrary point x yields:

$$\int_{-1800}^M dM = \int_{12}^x V(x)\,dx = \int_{12}^x 300\,dx \Rightarrow \underline{M = 300x - 5400 \text{ kN-m}}$$

Note that at x = 18, $M(18) = 0$.

With the above expressions the shear and moment diagrams are plotted in Figure 5.24 and is in agreement with the solution the same problem by a different method given in Example 5.3

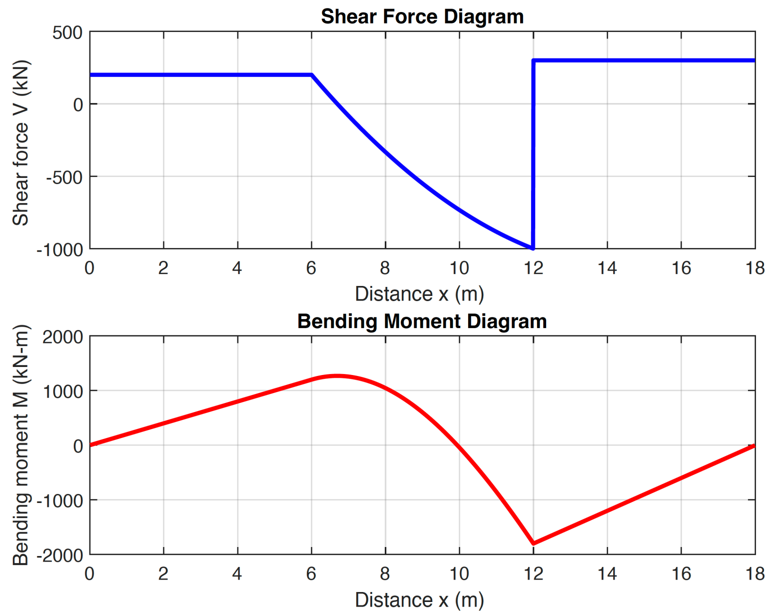


Figure 5.24 The shear and moment diagrams for the system of Figure 5.23a illustrating how these properties change throughout the beam in relationship to the applied distributed and point loads. Note the discontinuity in the shear force plotted in part b and the passage through zero for both the shear and moment values. Note that this is an identical result to that given in Figure 5.16 which solves the same problem via a different method.

The MATLAB code for generating Figure 5.24 follows

```
clc;clear all;close all;
syms x;
V = @(x) piecewise(0<x<6,200, 6<=x<12,(100/6)*x^2-
500*x+2600, 12<=x<18, 300);
M = @(x) piecewise(0<x<6,200*x, 6<=x<12,(50/9)*x^3-
250*x^2+2600*x-6600,12<=x<18,300*x-5400);
x = 0:0.01:18;
Vx = subs(V,x);
subplot(2,1,1)
plot(x,Vx,"b","Linewidth",2);
title("Shear Force Diagram")
xlabel("Distance x (m) ")
ylabel("Shear force V (kN) ")
grid on
Mx = subs(M,x);
subplot(2,1,2)
plot(x,Mx,"r","Linewidth",2)
title("Bending Moment Diagram")
xlabel("Distance x (m) ")
ylabel("Bending moment M (kN-m)")
grid on
```