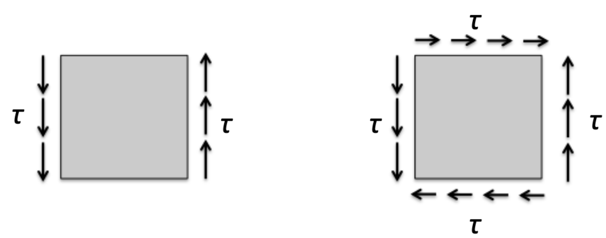
Torsion

State of Stress In this section we look at the static behavior of shafts, which are twisted or in torsion. This is different from rotation, which falls under dynamics. Unlike prismatic bars, which are dominated by normal stress and strain, shaft torsion is dominated by shear stress and strain. Recall that with a bar we could load it in such away that uniform normal stress occurred without and introduction of shear stress (unless we looked at a non normal plane). With a shaft we will consider cases where the structure is loaded is such away that only a pure state of shear stress results on particular planes in the material. The situation is illustrated in Figure 4.1 and is called a state of *pure shear stress.*



(a) (b)

Figure 4.1 (a) a cube of material not in equilibrium because two surfaces have no balancing shear stress and (b) a cube of material in equilibrium because the indicated shear stresses cancel.

Next consider taking a slice out of the material plane in Figure 4.1b as illustrated in Figure 4.2. Similar to the case for normal stress in a bar, the slice still remains in equilibrium as indicated in Figure 4.2 and no normal stress arises.

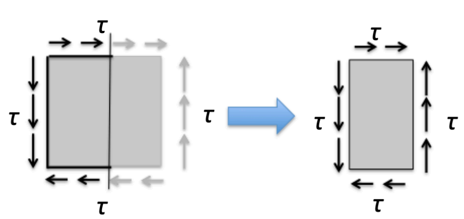


Figure 4.2 Planes parallel to the faces of the cube in Figure 4.1b are subjected to the same uniform shear stress.

Shear Modulus The state of pure shear stress on the cube of Figure 4.1b will cause a shear strain, γ, as illustrated in Figure 4.3.

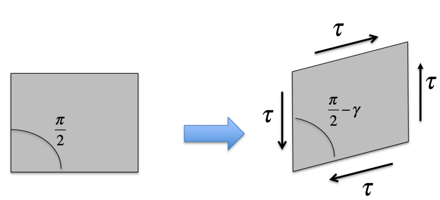


Figure 4.3 Shear strain γ resulting from the shear stress applied to the cube.

As was mentioned earlier the relationship for many materials between shear stress and shear strain can be determined experimentally to be $\tau = G\gamma$ where G is called the shear modulus or sometimes the modulus of rigidity. The units are the same as τ, force/area (N/m^2), because γ is a dimensionless measure (radians). Later in the course we will learn that the shear modulus and elastic modulus are related by Poisson's ratio, ν, such that:

$$G = \frac{E}{2(1+\nu)}$$

Recall that ν is the ratio of strain in the transverse direction to strain in the axial direction.

Stresses in Oblique Planes: Just like in the case of axially loaded bars where shear developed in oblique planes, if we look at non-orthogonal cut plane of the cube normal stresses will develop. Figure 4.4 illustrates this for a cut plane at an arbitrary angle θ.

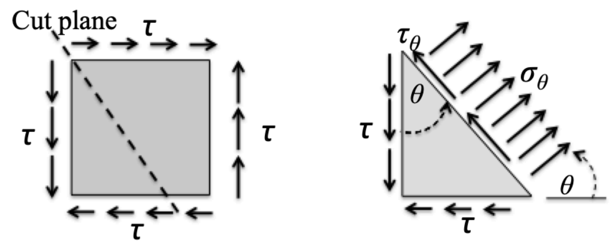


Figure 4.4 Passing a plane at an angle θ, and the resulting free-body-diagram.

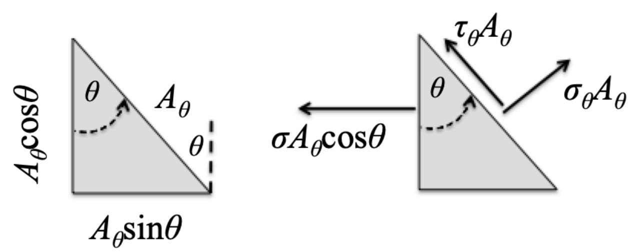


Figure 4.5 (a) Areas of the faces resulting from the cut. (b) The forces on the resulting free-body-diagram. Here the notation is the same as in Figure 3.5.

Summing forces in Figure 4.5b along the σ_θ direction yields:

$$\sigma_\theta A_\theta - \left(\tau A_\theta \cos\theta\right)\sin\theta + \left(\tau A_\theta \sin\theta\right)\cos\theta$$

Dividing by the area and rearranging yields that the normal stress acting on the cut plane is:

$$\sigma_\theta = 2\tau \sin\theta \cos\theta$$

Summing forces in the τ_θ direction yields:

$$\tau_\theta A_\theta - \left(\tau A_\theta \cos\theta\right)\cos\theta - \left(\tau A_\theta \sin\theta\right)\sin\theta = 0$$

Rearranging and dividing by the area yields that the shear stress acting on the cut plane is

$$\tau_\theta = \tau\left(\cos^2\theta - \sin^2\theta\right)$$

Examination of the shear and normal stress shows that the maximum values all have the value of τ the applied shear.

Bars with Circular Cross Section: Torsion refers to twisting of a shaft about its center axis. Aerospace examples include the shafts in propellers and in jet engines, a rocket's structure and containment structures (nacelles). We approach developing formulations for modeling twist by first examining the kinematics (geometry), then examining shear strain (material constitutive relations), then shear stress (using equilibrium) and finally arriving at torque. This is opposite of our approach to bars under compression or tension where we started with the force, then normal stress (σ), then strain (ε) and finally deformation (δ).

Figure 4.6 illustrates the convention used to indicate that a twisting motion is applied via a torque on opposite ends of a shaft. The double arrowhead is used as a short way to indicate torque applied to twist around the central axis.

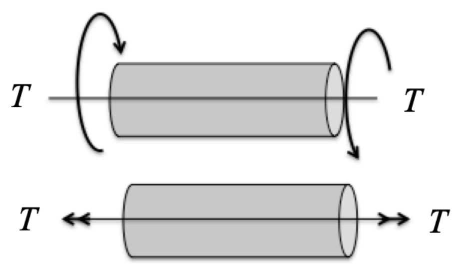


Figure 4.6 Two equivalent ways to represent a torque applied to twist a shaft. The signs of the arrows are both positive by using the right hand rule with the thumb pointing out of the surface.

Next consider a the prismatic solid circular shaft illustrated in Figure 4.7, fixed at one end with a torque T applied to the other end. With the left end fixed, a torque applied at the right end will cause a rotation ϕ about the center axis. Here we ignore warping and assume that there are no extensional strains in the axial or circumferential directions. Note that if the cross section were square or rectangular the applied torque would cause warping (a topic for the follow on course AERO 315).

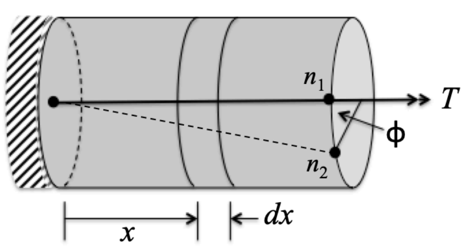


Figure 4.7 A schematic of a cantilevered shaft subject to a torque. Note that the arc length determined by the distance along the circumference between the points n_1 and n_2 grows from zero at the root ($x = 0$) to increasingly larger values as x increases.

Next consider the infinitesimal slice of length dx indicated in Figure 4.7 with a relative rotation of $d\phi$. The shear stain undergone by that slice is described in Figure 4.8 by examining four points on the surface of the shaft.

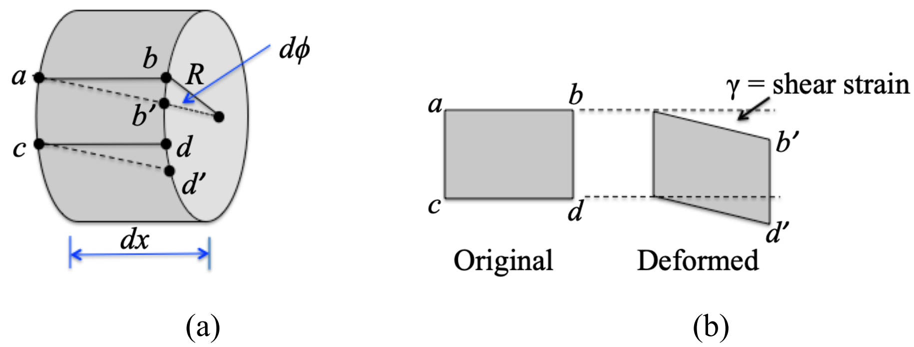


Figure 4.8 (a) The infinitesimal slice of Figure 4.7 of length dx with relative rotation of one face by the angle $d\phi$. (b) The surface element in unloaded (original) and deformed state illustrating the shear strain γ.

Examining Figure 4.8b, since we are assuming that no normal stress or strain results from the applied twist the arc lengths bd and $b'd'$ must remain the same. Using vertical bars to indicate the distance yields $|bd| = |b'd'|$. Examining the angle γ from the Figure 4.8b reveals that:

$$\tan\gamma = \frac{|bb'|}{|ab|} \Rightarrow |bb'| = |ab|\gamma \text{ for small } \gamma$$

Note from Figure 4.8a that $|bb'| = Rd\phi$ and that $|ab| = dx$, so that:

$$\gamma = R\frac{d\phi}{dx}$$

This last expression is the shear strain on the surface of a cylindrical bar under torsion.

For linearly elastic (small γ), isotropic materials the stress-strain relationship is $\tau = G\gamma$, which is Hook's law for shear. Substitution of the above value for γ into Hooke's law yields:

$$\tau = GR\frac{d\phi}{dx}$$

This holds at the surface of the cylindrical bar. For some interior point defined by $0 \le r \le R$ the strain and Hooke' law become respectively:

$$\gamma = r\frac{d\phi}{dx}, \text{ and } \tau = Gr\frac{d\phi}{dx}$$

Next consider examining forces in a cut plane perpendicular to the bar's axis subject to a uniform circumferential stress τ. Every element around the circumference is acted on by the same state of shear stress. This is illustrated in Figure 4.9. Here τ is zero (minimum) at the center and maximum at the outer surface. At various points along the radius the shear at a point grows in value proportional to the distance, r, from the center:

$$\tau = \frac{r}{R}\tau_{\max}$$

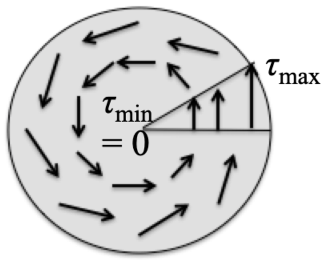


Figure 4.9 A cross section of the cylindrical bar of Figure 4.7 illustrating how the shear stress changes radially and circumferentially. Every element around the circumference (same *r*) is acted in by the same state of shear stress.

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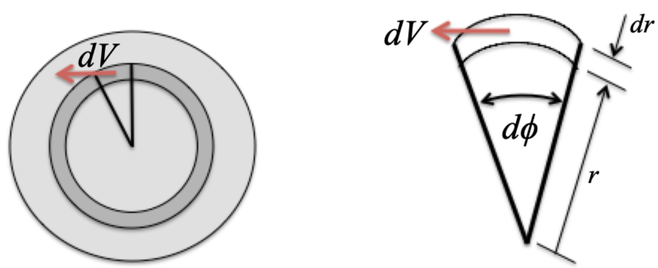
Internal Torque Resultant: The resultant of the symmetric stress distribution in the planer slice illustrated above must equal the internal torque, denoted T_{int}. In order to compute the internal torque, consider the geometry of a differential annular element described in Figure 4.10. Let V denote the shear force and dV denote the shear force on the differential element so that $dV = \tau dA$. Here A is the area of the annular element, which from Figure 4.10b is

$$dA = rdrd\phi$$



(a) (b)

Figure 4.10 (a) An annular element of the cross section a distance *r* from the center. (b)

The geometry of the differential element used to determine the relationship between shear force and shear.

The differential moment T_{int} about the central axis is

$$dT_{\text{int}} = rdV = r\tau dA$$

Integrating and recalling from Hook's law that

$$\tau = GR\frac{d\phi}{dx}$$

the moment T_{int} about the central axis is

$$T_{\text{int}} = \int dT_{innt} = \int_r r\tau\, dA = \int_A Gr^2 \frac{d\phi}{dx} dA = G\frac{d\phi}{dx}\int_A r^2 dA$$

The integral left in the above expression is the polar moment of inertia J:

$$J = \int_A r^2 dA$$

The polar moment of inertia has units of L^4 and is a geometric property of the cross section. Using this shorter notation the interior torque becomes:

$$T_{\text{int}} = GJ\frac{d\phi}{dx}$$

This last expression is similar to the expression for the normal shear force $P = EA(du/dx)$. The quantity GJ is called the *torsional rigidity* and is constant depending only on the material and geometry. This last expression for T_{int} can be integrated to obtain and expression for the bar's angle of twist:

$$T_{\text{int}} = GJ\frac{d\phi}{dx} \Rightarrow T_{\text{int}}\int_0^L dx = GJ\int_0^\phi d\phi \Rightarrow T_{\text{int}}L = GJ\phi \Rightarrow \underline{\phi = \frac{T_{\text{int}}L}{GJ}}$$

An expression for the shear stress in terms of interior torque is

$$\tau = Gr\frac{d\phi}{dx} \Rightarrow T_{\text{int}} = \frac{\tau}{r}J \Rightarrow \underline{\tau = \frac{rT_{\text{int}}}{J}}$$

These last two underlined expressions are the torsional shear equivalent of the elongation δ ($= PL/\text{EA}$) and the normal stress σ.

Next consider again the polar moment of inertia, J. There are two cases of interest. First is when the shaft is a solid mass of radius R and the second is when the shaft is hollow forming a shell with inside radius R_{i} and outside radius R_{o}. For the solid case the calculation for J becomes:

$$J_{\text{solid}} = \int_A r^2 dA = \int_0^{2\pi}\int_0^R r^2 r dr d\phi = \frac{\pi}{2}R^4$$

For the shell case, the calculation for J becomes:

$$J_{\text{shell}} = \int_A r^2 dA = \int_0^{2\pi}\int_{R_{\text{i}}}^{R_{\text{o}}} r^2 r dr d\phi = \frac{\pi}{2}\left(R_{\text{o}}^4 - R_{\text{i}}^4\right)$$

Shells are common in many aerospace structures such as rocket housings and aircraft fuselages. More complete investigations of shells forms the topic of AERO 315.

Sign conventions for torque are based on the right hand rule, positive if the "thumb" points away from the surface and negative if it points into the surface. A positive torque implies a counter clockwise rotation and hence the corresponding shear stress is positive.

Example 4.1: For the shaft illustrated in Figure 4.11a calculate the twist angle at the end $\phi(L)$. Assume the torsional rigidity, GJ, is constant with respect to x, and the point torque T is constant at $x = L$.

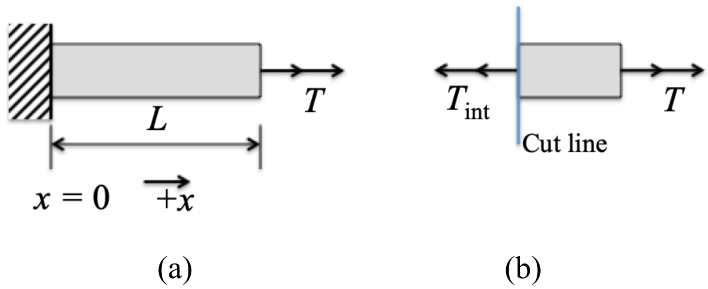


Figure 4.11 (a) A cantilevered shaft of length L with constant torque T applied at its free end. (b) A free-body diagram of a cut section.

Solution: Consider the cut section of Figure 4.11b and its free-body diagram. Summing forces along x reveals that $T_{\text{int}} = T$. Thus

$$T_{\text{int}} = T = GJ\frac{d\phi}{dx} \Rightarrow T\int dx = GJ\int d\phi$$
$$\Rightarrow Tx + C_1 = GJ\phi(x)$$

The boundary condition at the fixed end implies $\phi(0) = 0$ so that $C_1 = 0$ and

$$\phi(x) = \frac{Tx}{GJ} \Rightarrow \phi(L) = \phi_{\text{end}} = \frac{TL}{GJ}$$

This is the rotation at the end of the shaft for uniform torsion. Note that the units for the rotation are radians:

$$\frac{[\mathrm{F}\cdot\mathrm{L}]\cdot[\mathrm{L}]}{\left[\frac{\mathrm{F}}{\mathrm{L}^2}\right]\left[\mathrm{L}^4\right]}$$

Example 4.2: Compute the full rotation of the shaft of Figure 4.12 and the maximum shear stress given the constants D, T, L, and G.

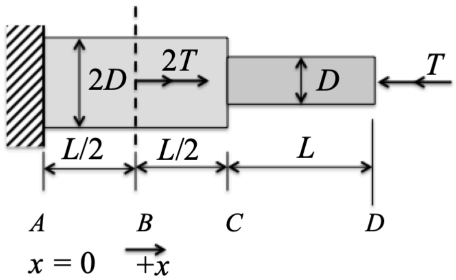


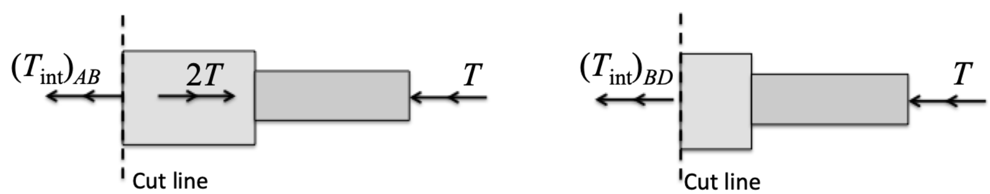
Figure 4.12 A stepped shaft with two external torques applied, one at point B and one at the tip, point D.

Solution: Because of the step in diameter and the additional torque applied mid point (B) on the first part of the shaft the problem is broken up into 3 parts and a little accounting is in order along with some additional notation. Let the double subscripts refer to quantities in the region defined by the subscripts so that J_{AC} is the polar moment of inertia for section AC, and J_{CD} is the polar moment of inertia for section CD. Likewise define ϕ_{AB} as the twist in section AB, ϕ_{BC} as the twist in section BC, and ϕ_{CD} as the twist in section CD. Computing the relative polar moments yields:

$$J_{AC} = \frac{\pi}{2}\left(\frac{2D}{2}\right)^4 = \frac{\pi}{2}D^4 \quad \text{and } J_{CD} = \frac{\pi}{2}\left(\frac{D}{2}\right)^4 = \frac{\pi}{32}D^4$$

Note that $J_{AC} = 16J_{CD}$.

In order to calculate the internal torque in section AB make a cut to the left of the torque applied at point B as illustrated in Figure 4.12a



(a) (b)

Figure 4.12 (a) The free-body diagram of the section AB. (b) The free-body diagram of the section BD

From the free-body diagram in Figure 4.12a the sum of the moments results in

$$\sum M = 0 \Rightarrow 2T - T - \left(T_{\text{int}}\right)_{AB} = 0 \Rightarrow \underline{\left(T_{\text{int}}\right)_{AB} = T}$$

From the free-body diagram in Figure 4.12b the sum of the moments results in

$$\sum M = 0 \Rightarrow -T - \left(T_{\text{int}}\right)_{BD} = 0 \Rightarrow \underline{\left(T_{\text{int}}\right)_{BD} = -T}$$

The constitutive equation, $\phi = TL/GJ$, applied to each of the three segments yields:

$$\phi_{AB} = \frac{\left(T_{\text{int}}\right)_{AB}\left(L/2\right)}{GJ_{AC}} = \frac{TL}{32GJ}, \quad \phi_{Bc} = \frac{\left(T_{\text{int}}\right)_{BD}\left(L/2\right)}{GJ_{AC}} = \frac{-TL}{32GJ} \text{ and } \phi_{CD} = \frac{\left(T_{\text{int}}\right)_{BD} L}{GJ_{CD}} = \frac{-TL}{GJ}$$

In the previous calculation we used the short hand $J = J_{CD}$ and note that $J_{AC} = 16J_{CD}$.

Next consider the compatibility conditions for angular twist:

$$\phi_{AD} = \phi_{AB} + \phi_{BC} + \phi_{CD} = \phi_D - \phi_A = \phi_D, \text{ where } \phi_D = \phi(0) = 0$$

Substitution of the deflection values computed above the total rotation is

$$\phi_D = \phi(L) = \frac{TL}{32GJ} - \frac{TL}{32GJ} - \frac{TL}{GJ} = -\frac{TL}{GJ} = \frac{-32TL}{\pi D^4 G}$$

The negative sign indicates that the rotation is clockwise as viewed from the free end, looking towards the fixed end.

To find the maximum value of the shear stress base only on magnitudes consider the maximum value in each segment then compare them. Recalling that $\tau = T_{\text{int}} r/J$, the shear stress in each segment becomes:

$$\left(\tau_{AB}\right)_{\max} = \frac{T\left(2D/2\right)}{\frac{\pi}{2}D^4} = \frac{2T}{\pi D^3}, \quad \left(\tau_{BC}\right)_{\max} = \frac{|-T|\left(2D/2\right)}{\frac{\pi}{2}D^4} = \frac{2T}{\pi D^3},$$

$$\text{and } \left(\tau_{CD}\right)_{\max} = \frac{|-T|\left(D/2\right)}{\frac{\pi}{32}D^4} = \frac{16T}{\pi D^3}$$

Thus the maximum value of the shear stress occurs at the outer surface of the shaft between points C and D with value

$$\tau_{\max} = \frac{16T}{\pi D^3}$$

About Signs Note that the directions of rotations can be a bit confusing. It's important to be consistent. When you make a cut you can use either side and you will get the same interior torque, but realize that the direction of ϕ (clockwise or counter clockwise) is relative to the *non* cut end of the segment. The best way not to make a mistake is to always consider looking at $x = 0$ when determining the direction. This puts all the shear strains and twist angles in the same frame of reference and you can let the direction be determined by the sign of T_{int}, using the convention that clockwise is negative and counter clockwise is positive.

Example 4.3: Compute the angular twist at point B and the maximum shear stress for the statically indeterminate system of Figure 4.13a, given the constants d, T, L and G.

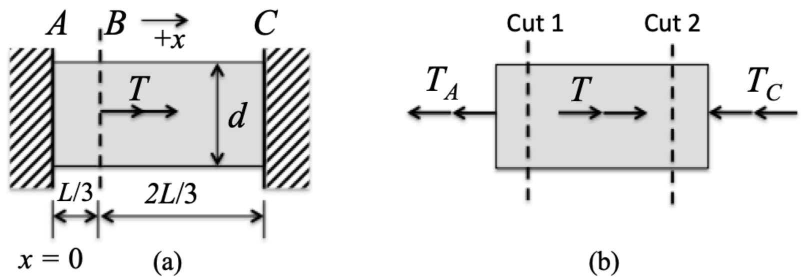


Figure 4.13 (a) A statically indeterminate shaft with an external torque applied. (b) The free-body-diagram of the shaft.

From the free-body-diagram in Figure 4.12b the sum of moments yields

$$\sum M = 0 \Rightarrow T - T_A - T_C = 0$$

which is indeterminate and requires the cuts suggested in the figure. Figure 4.14 illustrates the free-body-diagrams of the two cut sections:

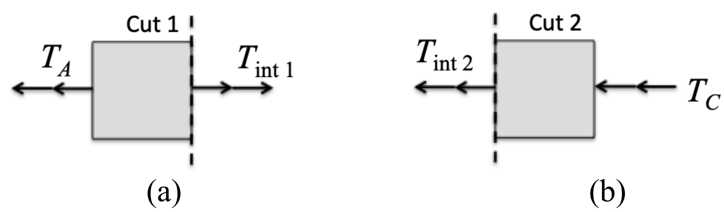


Figure 4.14 (a) the FBD of cut 1. (b) The FBD of cut 2.

From cut one: $T_{\text{int 1}} = T_A$ so the constitutive relation for the angular twist becomes:

$$\phi_{AB} = \frac{T_A \left(L/3 \right)}{GJ}$$

This is the relative rotation of point B with respect to point A. Likewise from cut 2: $T_{\text{int 2}} = - T_C$ so the constitutive relation for the angular twist becomes:

$$\phi_{BC} = \frac{-T_C \left(2L/3 \right)}{GJ}$$

This is the relative rotation of point C with respect to point B. The compatibility equation is that the sum of these two must be zero because the shaft is fixed at both ends:

$$\phi_{AB} + \phi_{BC} = 0 \Rightarrow \frac{L}{GJ}\left(\frac{1}{3}T_A - \frac{2}{3}T_C \right) = 0 \Rightarrow T_A = 2T_C$$

This combined with the equilibrium equations gives us two equations in two unknowns. Solving yields:

$$\underline{T_A = \frac{2}{3}T, \text{ and } T_C = \frac{1}{3}T}$$

Thus, the maximum torque is T_A so the maximum shear stress is (J is computed in Example 4. 2):

$$\tau_{\max} = \frac{T_{nax} r}{J} = \frac{2}{3}T\frac{d}{2}\frac{1}{\left(\pi/32\right)d^4} = \underline{\frac{32}{3}\frac{T}{\pi d^3}}$$

The value of the angular twist at point B is:

$$\phi_B = \phi\left(L/3\right) = \frac{T_A\left(L/3\right)}{GJ} = \frac{2}{3}T\frac{L}{3GJ} = \underline{\frac{2}{9}\frac{TL}{GJ}}$$

Note there is a discontinuity in the torque at $x = 1/3$ where the external torque is applied. The angular twist starts at zero and ends at zero reaching its maximum value at $x = 1/3$.

Non-uniform Torsion and Distributed Loads: Consider the sketch in Figure 4.15a illustrating a distributed torque, denoted by $c(x)$ applied to a tapered cylindrical shaft. Similar to the approach taken for normal stress, examine the differential element of Figure 4.15b to determine the relationship between the distributed torque and the equation for the angular twist.

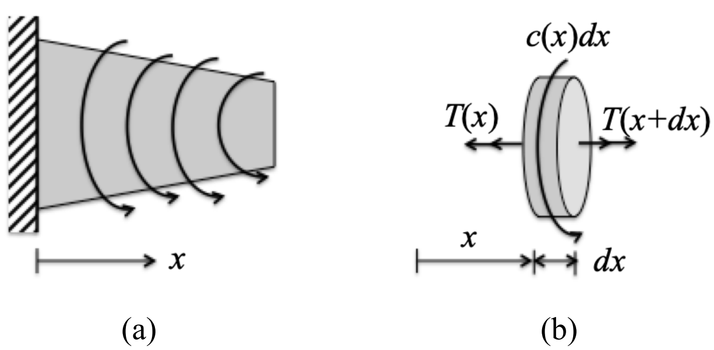


Figure 4.15 (a) A tapered shaft subject to a distributed load, $c(x)$. (b) A free-body-diagram of an infinitesimal element of the shaft in part (a).

Summing the moments on the free-body-diagram of the infinitesimal element of the shaft of Figure 4.15b yields:

$$\sum M = 0 \Rightarrow T(x+dx) + c(x)dx - T(x) = 0$$

Rearranging and taking the limit yields:

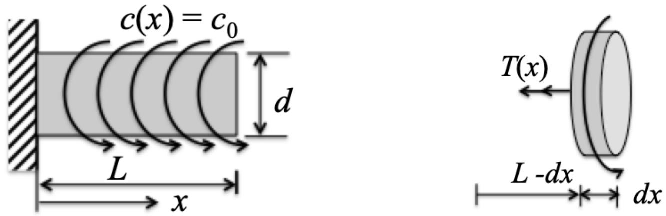
$$\lim_{dx\to 0}\left(\frac{T(x+dx)-T(x)}{dx}\right)=-c(x)\Rightarrow\frac{dT}{dx}=-c(x)$$

where $T(x)$ is the internal torque. Recall that the internal torque is related to the derivative of the angle of twist, $T(x) = GJ(d\phi/dx)$ and combining this with the above results in the differential equation for the angular twist:

$$\frac{d}{dx}\left[G(x)J(x)\frac{d\phi}{dx}\right]=-c(x)$$

This last expression holds on the interval $0 \leq x \leq L$, since T is the internal torque. As in the case for normal strain, this is a second order differential equation requiring two boundary conditions in order to determine the two constants of integration.

Example 4.4: For the cantilevered bar of length L, shear modulus G, diameter d illustrated in Figure 4.16a determine the value of the angular twist at $x = L$ and the maximum shear stress for the distributed load case $c(x) = c_o$, a constant, in terms of L, G, d and c_o.



(a) (b)

Figure 4.16 (a) A cantilevered shaft subject to a distributed load. (b) The moments acting on the differential element at the free end of the bar.

Solution: For a constant distributed torque the differentia equation for the angular twist is

$$\frac{d}{dx}\left[G(x)J(x)\frac{d\phi}{dx}\right]=-c_o$$

Integrating this twice yields:

$$GJ\phi(x)=-\frac{1}{2}c_o x^2+C_1x+C_2,\ \text{where } C_1 \text{ and } C_2 \text{ are constants of integration}$$

At the fixed end the angular twist is zero providing the boundary condition $\phi(0) = 0$. At the free end summing moments on the differential element in Figure 4.16b yields that $T(L) = 0$. Applying the boundary condition $\phi(0) = 0$ implies $C_2 = 0$. Recalling that $T(x) = GJ(d\phi/dx)$, the second boundary condition becomes

$$T(L)=GJ\left.\frac{d\phi}{dx}\right|_{x=L}=\left[-c_0L+C_1\right]=0\Rightarrow C_1=c_0L$$

Substitution of the constants of integration into the solution yields the expression for the angular twist:

$$\phi(x)\big|_{x=L} = \frac{1}{GJ}\left[-\frac{1}{2}c_0x^2 + c_0Lx\right]\Bigg|_{x=L} \Rightarrow \phi(L) = \frac{c_0L^2}{GJ}\left[\frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

The angular twist is zero at the origin and grows with the shape of an inverted parabola so the maximum value of the angular twist is

$$\varphi(L) = \frac{c_0L^2}{2GJ}$$

The value of the torque is

$$T(x) = GJ\frac{d\phi}{dx} = \left[-c_0x + c_0L\right] = c_0\left(L - x\right)$$

Thus the maximum torque occurs at $x = 0$ and has the value $T_{\max} = T(0) = c_0L$. Thus the maximum shear stress is

$$\tau_{\max} = \frac{T_{\max}\left(d/2\right)}{J} = \frac{c_0Ld}{2\left(\pi/32\right)d^4} \Rightarrow \underline{\tau_{\max} = \frac{16c_0L}{\pi d^3}}$$

This occurs at the outer surface at $x = 0$. The shear stress decreases linearly from $x = 0$ to $x = L$.

Example 4.5: For the fixed-fixed bar of length L, shear modulus G, diameter d illustrated in Figure 4.17 determine the value of the angular twist at $x = L$ and the maximum shear stress for the distributed load case $c(x) = c_o$, a constant, in terms of L, G, d and c_o.

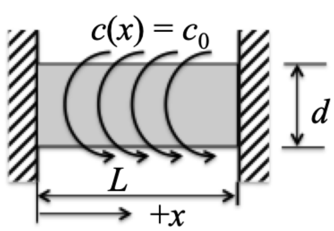


Figure 4.17 A fixed-fixed cylindrical bar subject to a constant distributed load.

Solution: The only difference between this example and Example 4.5 is the right hand boundary condition, which is now fixed so that the two boundary conditions are $\phi(0) = 0$ and $\phi(L) = 0$. The expression for the angular twist remains the same at

$$GJ\phi(x) = -\frac{1}{2}c_o x^2 + C_1x + C_2$$

Here $C_2 = 0$ from the boundary condition, $\phi(0) = 0$. Appling the second boundary condition, $\phi(L) = 0$, yields:

$$GJ\phi(L) = -\frac{1}{2}c_o L^2 + C_1 L = 0 \Rightarrow C_1 = \frac{1}{2}c_o L$$

Thus the value of the angular twist is

$$\phi(x) = \frac{1}{GJ}\left[-\frac{1}{2}c_0 x^2 + \frac{1}{2}c_0 Lx\right] \Rightarrow \phi(x) = \frac{c_0 Lx}{2GJ}\left[1 - \frac{x}{L}\right]$$

This function is quadratic in x starting at zero and ending at zero with. Its maximum value occurs at $d\phi/dx = 0$, or

$$\frac{2GJ}{c_0 L}\frac{d\phi}{dx} = \frac{2GJ}{c_0 L}\frac{d}{dx}\left(Lx - x^2\right) = 0 \Rightarrow L - 2x = 0 \Rightarrow x_{\max} = \frac{L}{2}$$

Thus the value of $\phi_{\max}$ is

$$\phi_{\max} = \phi\left(\frac{L}{2}\right) = \underline{\frac{c_0 L^2}{8GJ}}$$

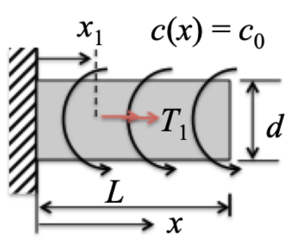
Since

$$T(x) = GJ\frac{d\phi}{dx} = \frac{c_0 L}{2}\left(1 - \frac{2x}{L}\right)$$

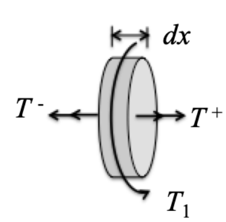
the maximum value occurs at $x = 0$ and has the value $T_{\max} = c_0 L/2$. Note that at $x = L$, T has the value $-c_0 L/2$ so in absolute value $T_{\max} = |c_0 L/2|$. Also note that at $x = L/2$, $T = 0$. $T(x)$ decreases linearly from $x = 0$. The maximum shear stress is then

$$\tau_{\max} = \frac{T_{\max} R}{J} = \frac{c_0\left(L/2\right)\left(d/2\right)}{\left(\pi/32\right)d^4} = \underline{\frac{8c_0}{\pi d^3}}$$

Next consider adding a point torque to a system with a distributed torque as illustrated in Figure 4.18a. The point torque T_1 is applied a distance x_1 from the fixed end. In order to combine the two applied loads the discontinuity at x_1 has to be considered as illustrated in Figure 4.18b.



(a)



(b)

Figure 4.18 (a) A cantilevered cylindrical bar with both a distributed and point torque applied. (b) An infinitesimal cut around the location where the point load is applied.

A key element in solving problems with a point torque applied somewhere along x is to sort out the matching conditions. The solution is broken into two parts and solved for $\phi_1(x)$ in the first region and $\phi_2(x)$ in the second region. Using the free-body diagram of Figure 4.18b the jump in torque is described by summing the moments around dx and forcing the angular twist to be continuous. The sum of the moments yields:

$$T^{+} - T^{-} = -T_1$$

So the matching conditions become

$$T(x_1^{+}) - T\left(x_1^{-}\right) = T_1 \quad \text{and} \quad \phi(x_1^{+}) = \phi(x_1^{-})$$

These conditions along with the boundary conditions can be used to determine the solution for $\phi(x)$.

Summary Here is a table summary of the expressions needed to solve for stress and strain. In addition, boundary conditions are also needed to calculate the constants of integration.

	Axially Loaded Bars	Shaft in Torsion
Stress-strain relationship	$\sigma = E\varepsilon$	$\tau = G\gamma$
Definition of stress	$\sigma = \dfrac{P}{A}$	$\tau = \dfrac{Tr}{J}$
Definition of strain	$\varepsilon = \dfrac{du}{dx}$	$\gamma = r\dfrac{d\phi}{dx}$
Distributed load	$\dfrac{d}{dx}\left(EA\dfrac{du}{dx}\right) = -w(x)$	$\dfrac{d}{dx}\left(GJ\dfrac{d\phi}{dx}\right) = -c(x)$
Deflection with end load	$\delta = \dfrac{PL}{EA}$	$\phi_{\text{end}} = \phi_{AB} = \dfrac{TL}{GJ}$

Here is a list of symbols used and what they stand for compared to like symbols for the axially loaded bar case.

Symbol	Meaning	Analogous to
τ	Shear Stress	σ
γ	Shear Strain	ε
G	Shear Modulus	E
T	External Torque	F
r	Radius	---
J	Polar moment of inertia	A
$c(x)$	Distributed torque	$w(x)$
ϕ	Rotation twist angle	u
$\Phi_{\text{end}} = \Phi_{AB}$	Rotation of shaft	δ