2. Mechanics of Deformable Bodies *Stress and Strain*

In this chapter we introduce to the concept of stress and strain by considering the simplest structure and modeling forces internal to the structure resulting from the application of external forces. A fundamental concept in the analysis of deformable structures is based on the idea of equilibrium. The concept is that if a structure is in equilibrium, then its parts must all be in equilibrium. In particular, if an object is in equilibrium dividing it into two pieces means that each piece is still in equilibrium. This process will reveal the forces internal to the structure as illustrated in the following section.

Stress If a structure is in equilibrium is separated into two pieces, the remaining pieces would still be in equilibrium. To visualize this, consider the structure on the left in Figure 2.1 that is in equilibrium when acted on the forces shown. If the right side of the cut plane is removed both pieces must stay in equilibrium so we conclude that an internal force must be generated indicated on the right in Figure 2.1. For the remaining piece to be in equilibrium the internal force must balance the forces $\mathbf{F}_1$ and $\mathbf{F}_2$, as indicated in the figure on the right.

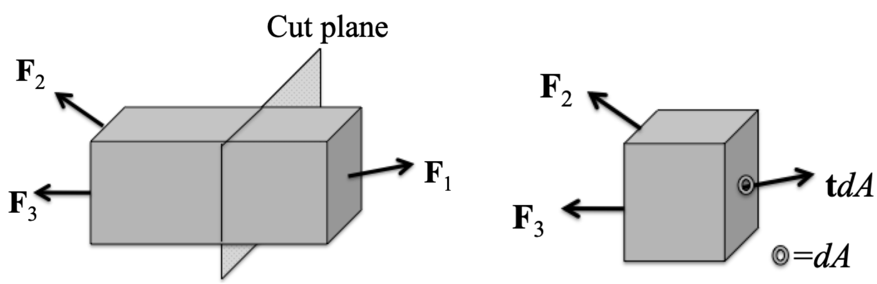


Figure 2.1 A structure in equilibrium (left) and a piece separated from it (right) illustrating the internal forces the must result to keep the piece in equilibrium.

The study of solid-state physics and quantum mechanics reveals that these internal forces result from chemical bonds and atomic forces. To examine each of these forces would be an overwhelming task and until recently[1] there has been no available modeling that would result in useful engineering calculations. Thus, the approach of continuum mechanics is used, which treats the problem of internal forces as *distributed loads*. This basic approach to modeling an internal force is to consider a force per unit area $\mathbf{t}$, called the *traction* vector, defined over an infinitesimal area dA as illustrated on the right side in Figure 2.1. The local, internal balancing force is $\mathbf{t}dA$. Note that $\mathbf{t}$ is a vector having both magnitude and direction. To obtain the average force acting across the entire cut plane integrate across the area of the cut to get:

[1] Increased computational power has resulted in molecular dynamics simulations able to capture atomic level motions to predict macro scale forces in some cases

$$\mathbf{t}_{\text{av}} = \frac{1}{A}\int_A \mathbf{t}\,dA$$

Rearranging the expression for $\mathbf{t}_{\text{av}}$ yields that the average total internal force exerted on the area A is given by the integral

$$\int_A \mathbf{t}\,dA = \mathbf{t}_{\text{av}}A$$

Since the traction is in fact a vector it can be separated into two components, one perpendicular (i.e., normal) to the cut plane and one parallel to it as illustrated in Figure 2.2. The components of the local force $\mathbf{t}dA$ have physical significance. The component parallel to the cut surface, denoted τ, is called the *shear stress* and σ is called the *normal stress*. The corresponding *shear force*, $dF= \tau dA$, describes the internal force caused by external forces applied parallel to the cut plane. The *normal force*, σdA describes the internal forces directed normal to the plane.

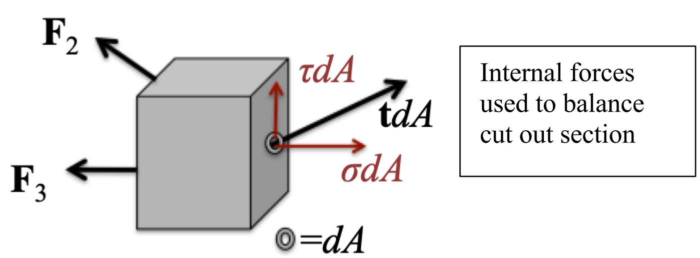


Figure 2.2 Representation of the shear and normal force components of the resulting internal force.

The units of $\mathbf{t}$, τ and σ are force/area expressed in Pascals (Pa) = N/m^2, or in US Customary units, lb/in^2 or psi. These same definitions of traction, shear and normal stress are used in the study of fluids (AERO 225) as well. The normal stress here is taken to be positive if it points away from the volume and negative if it points towards the cut plane. (However, in AERO 225 the normal stress on a volume of gas at rest is considered positive if it points into the plane because it is the pressure. If the fluid is at rest the shear force is zero. So, try not to confuse the two sign conventions.)

The average components of shear stress, $\tau_{\text{av}}dA$, and normal stress, $\sigma_{\text{av}}dA$, can be determined on any plane if all the external forces are known. This is important because it gives insight into how a particular structure will behave, fail, or fatigue under known loads.

One might ask at this point how do you integrate a vector: the answer is term by term. For example:

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \Rightarrow \int_A \mathbf{t}\, dA = \begin{bmatrix} \int_A t_x\, dA \\ \int_A t_y\, dA \\ \int_A t_z\, dA \end{bmatrix}$$

Next, several examples are given to help understand how to compute shear and normal stresses and to get accustom to working with the various forces.

Note that structures that have a uniform cross-sectional area throughout their length are called *prismatic*. This is a mechanics use of the word prismatic and like much technical jargon the definition is different than the common use of the word, which refers to having the form of a prism.

Example 2.1 Consider the bar illustrated in Figure 2.3 with cross section area that is uniform throughout its length. Compute the average normal stress in terms of the applied external load F that is applied along the axial direction.

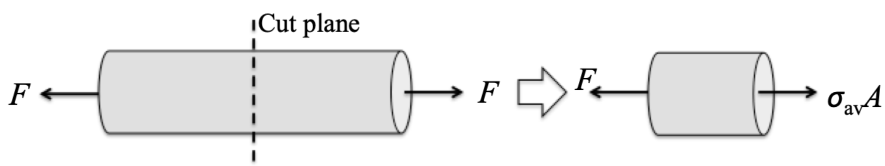


Figure 2.3 A bar subject to equal and opposite external forces F and the free-body diagram of a cut section of the bar.

Solution: The free-body diagram in this case is one-dimensional, horizontal direction. Summing forces in that direction yields that the average normal stress is just (using vectors pointing to the right as positive)

$$\sigma_{\text{av}} A - F = 0 \Rightarrow \sigma_{\text{av}} = \frac{F}{A}$$

Thus, the average normal stress in this case is just proportional to the external load. Since the area, A, does not change for the location of any cut plane, this relationship holds for any cut as long as it's perpendicular to the central axis of the bar. Since there is no external force component in the perpendicular direction the shear stress is zero. Unlike forces whose sign convention is with respect to a coordinate system, the sign convention for stress is with respect to the structure (i.e., in or out).

Stress is a scalar and the normal force related to the stress is a vector. Since it's a scalar a sign convention is needed. If the normal stress σ turns out to be a positive number than the resulting normal force points away from the structure causing it to stretch (elongate). In this case we consider the structure to be in *tension*. Conversely if the $\sigma < 0$, the normal force acts inward towards the body causing it to be in *compression*. This is consistent with our earlier discussion of compression and tension for two force members in a truss. Note that the shape of the cross-sectional area does not change the axial loads as long as it is the same shape throughout (i.e., it if is prismatic). Thus, all shapes with the same cross-sectional area will carry the same axial load.

Factor of Safety: The calculations so far are for average normal stress, which does not inform designers directly of when failure will occur. The nature of the distribution of stress can cause the maximum stress to be much higher than the average stress. For example, if the plane is changed so will the value of the stress change, as we will see later in the course. Once the maximum stress is determined a safety factor is defined by

$$SF = \frac{\sigma_{\text{allowed}}}{\sigma_{\text{applied}}} \text{ and } \frac{\tau_{\text{allowed}}}{\tau_{\text{applied}}}$$

This number varies by discipline, for civil structures it lies between 4 and 6, for mechanical systems it lies between 2 and 3, whereas for aerospace structures it lies between 1.1 and 1.5. The safety factors of aerospace structures are typically smaller because stress calculations undergo more analysis and testing in order to conserve weight, which generally increases with larger values of *SF*.

Example 2.2 Consider the frame in Figure 2.4 that supports a 10 kN force. Determine the average normal and shear stresses acting on the cut plane indicated. Note that each bar is a two-force member. When finished, compare the resulting normal force on the plane to that of the normal force on a cut plane perpendicular to the bar's axis.

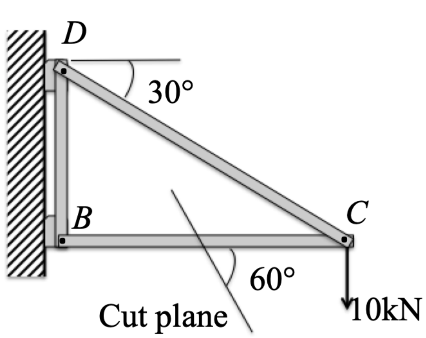


Figure 2.4 A truss fixture supporting a mass causing 10 kN force. Each bar is a solid cylindrical metal of diameter 80 mm pinned together.

Solution: First determine the axial forces on bar BC by using the method of joints and isolating the joint at C. The free-body diagram of joint C along with the coordinate system is given in Figure 2.5.

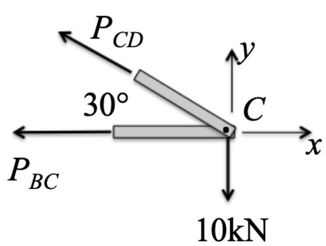


Figure 2.5 A free-body diagram of the joint at C, using the coordinate system indicated. This is employing the method of joints introduced earlier.

Summing forces in the x and y directions yields:

$$\sum_y F = P_{CD}\sin 30° - 10 = 0 \Rightarrow P_{CD} = 20 \text{ kN}$$

$$\sum_x F = -P_{BC} - P_{CD}\cos 30° = 0 \Rightarrow P_{BC} = -17.3 \text{ kN}$$

The minus sign on means the direction of the arrow in Figure 2.5 is wrong. Thus, the arrow really points to the right and the bar of interest, P_{BC}, is subject to a compressive load of 17.3 kN, as indicated in Figure 2.6.

Because P_{BC} is a two-force member we can separate that member and obtain the free-body diagram given in Figure 2.6 illustrating its compressive nature.

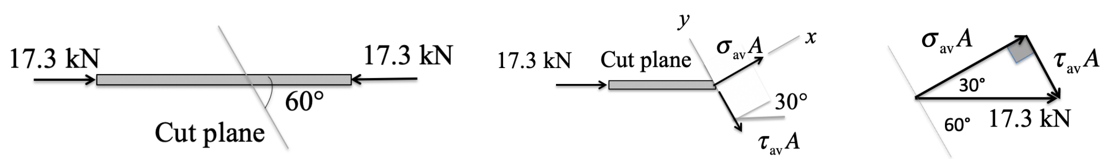


Figure 2.6 (Left) Member P_{BC} isolated and indicating it compressive nature. (Middle) a free-body diagram of the cut-out section and (Right) the corresponding vector diagram.

Working with the free-body diagram and vector diagram in middle of Figure 2.6 and using the coordinate system shown (rotated to allow separation of shear and normal forces), summing forces yields:

$$\sum_x F = \sigma_{av}A + 17.3\cos 30° = 0 \Rightarrow \sigma_{av}A = -15 \text{ kN}$$

$$\sum_y F = -\tau_{av}A - 17.3\sin 30° = 0 \Rightarrow \tau_{av}A = -8.66 \text{ kN}$$

Next the area A needs to be computed which is not the cross section of the cylinder but rather the 60° slice across the cylinder of radius 0.04 m. Recall the area of a circle is πr^2 so that the area of the 60° slice is an ellipse with area $A = \pi\, r\, (r/\sin 60°)$, so that:

$$A = \frac{\pi(0.04)^2}{\sin 60°} = 0.00580 \text{ m}^2$$

Computing the average normal stress to the cut plane yields $\sigma_{\text{av}} = -2.58 \text{ MPa}$ and the average shear along the plane is $\tau_{\text{av}} = -1.49 \text{ MPa}$.

If the plane is perpendicular to the axis of the bar, then the average shear stress is zero and according to Example 2.1 the average normal (stress) force is 17.3 kN, much larger that than that of the 60° cut. On the other hand, the average shear force is much less, i.e., zero.

Example 2.3 In the prior example, the average normal stress is shown to change depending on the angle of the cut plane. In this example, the nature of the change in normal stress due to changing size of a component is presented (a non-prismatic bar). A tapered structure such as the one in Figure 2.7 can be found in many aircraft and rocket structures. For example, the fixed landing gear strut on a Cessna 182 is a tapered spring steel structure that also acts as a shock absorber as well as a support for the wheel and tire. Calculate the normal stress at cuts 1 and 2.

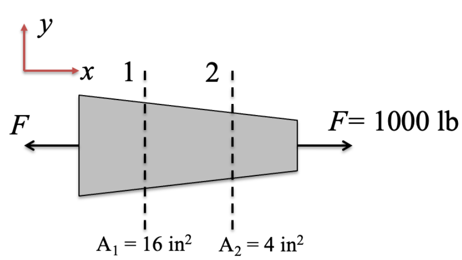


Figure 2.7 A tapered structure with rectangular cross section subject to a 10,000 lbs. tensile force. The cross-sectional area at cut 1 is $A_1 = 16$ in^2 and $A_2 = 4$ in^2 at cut 2.

Solution: Since this is a two-force member and the cuts are perpendicular to the axis of the structure the force on each cut will just be 10,000 lb per Example 2.1. Thus, the normal stress at cut 1 is

$$\sigma_{\text{av}} = \frac{F}{A_1} = \frac{10{,}000}{16} = 625 \text{ psi}$$

The normal stress at cut 2 is

$$\sigma_{\text{av}} = \frac{F}{A_2} = \frac{10{,}000}{4} = 2{,}500 \text{ psi}$$

Thus, there is a pretty large change in average stress along the length of a tapered structure. Note that if you knew the equation for the area of the taper as a function of the distance along the axis one could write a simple formula for the change in normal stress in the bar. Also note that this is an over simplified model of a landing gear strut but does give an idea of how the stresses change along the length of a non-prismatic bar.

Example 2.4 In many airports and on aircraft carriers there are jet blast deflectors to block jet exhaust and redirect the high-energy exhaust from a jet engine to prevent damage and injury. A simple model of such a structure is given in Figure 2.8 along with a photo of such a structure. Compute the compressive stress on a support *BC* (called a shore) if the blast is modeled as a distributed force from the jet engine as a triangular distribution with zero force at the bottom and $9\text{x}10^5$ N/m at the top. Assume all the connections are pinned and the triangle made by the shore and blast shield is equilateral of 1 meter on a side, and the blast shield itself is 2 meters long. Assume the shore has a cross-sectional area of 0.1 m^2.

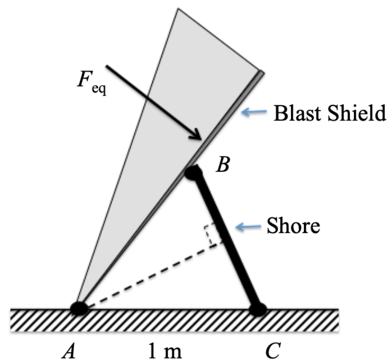


Figure 2.8 (Left) the blast shield on an aircraft carrier and (right) a crude model of the shield system.

Solution: The geometry here is straight forward. First compute the magnitude of the force distribution and is location based on the calculations of Example 1.5. The centroid of the triangular distributed force is

$$\bar{x} = \frac{2}{3}L = \frac{2}{3}2 = \frac{4}{3} \text{ m}$$

The equivalent force for a triangular distribution is

$$F_{\text{eq}} = \frac{1}{2}bh = \frac{1}{2}(2\text{m})(9\text{x}10^5\,\text{N/m}) = 9\text{x}10^5\,\text{N}$$

Next take the moments around point *A* and recall that the shore is a two-force member. Designating the force in the shore as P_{BC} the moment equation becomes:

$$\sum_A M = -\left(9\times10^5 \text{ N/m}\right)\left(\frac{4}{3}\text{ m}\right) + P_{BC}(1\text{ m})\cos 30° = 0$$

$$\Rightarrow P_{BC} = 1.39\times10^6 \text{ N}$$

The average normal stress in the shore becomes

$$\sigma_{\text{av}} = \frac{P_{BC}}{A} = \frac{1.39\times10^6}{(0.1)^2} = 1.385\times10^8 \text{ N/m}^2 = 138.5 \text{ MPa}$$

Note that this example is only an approximation of blast shield modeling. The real item is more complicated including the distributed force, but the idea to determine stresses in the shore is similar.

Example 2.5 This example examines the shear stress in a pin, a very common connection in many aerospace examples. For example, landing gear is full of pin type connections. Figure 2.9 illustrates the pin connection in a truss structure for the truss of Example 2.2. Assuming the pin has a 20 mm radius calculate the average shear stress on the pin at connection *C*.

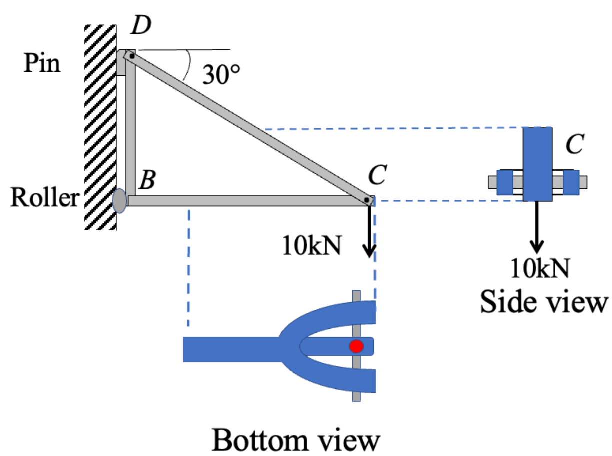


Figure 2.9 Various views of the pin connect detail at *C*.

Solution: The idea here is to pass cut planes through both sides of the pin as illustrated in Figure 2.10 on either side of the member *BC*. From the solution of Example 2.2 the compressive force acting on member *BC* is 17.3 N. The bottom half of Figure 2.10 constitutes a free-body diagram of the isolated member BC showing the forces exerted on the pin.

Summing forces along the direction of the long axis of the *BC* yields:

$$17.3 - 2\tau_{\text{av}} A = 0$$

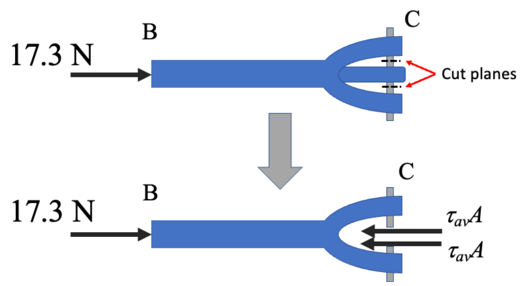


Figure 2.10 Member *BC* and with the cut planes isolating it from member *DC*.

Thus $\tau_{av}A$ =8.66 kN. Since the area $A = \pi(0.02)^2 = 0.00126$ m^2, the average shear stress on the pin is $\tau_{av} = 6.89$ MPa.

Example 2.6 So far, the stress distributions have been constant along the cut plane. This example examines the situation when the stress distribution is not constant. A non-uniform force being applied to the structure can cause such a distribution. Figure 2.11 illustrates a part with its stress distribution. Compute the maximum normal stress, the total normal force and the average normal stress.

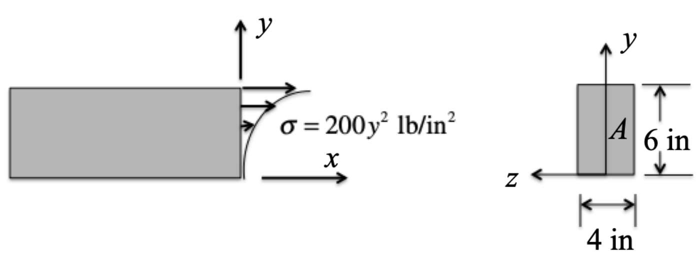
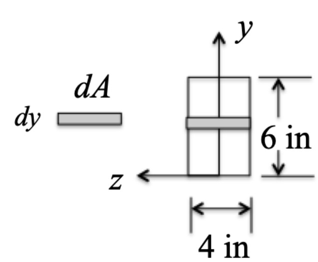


Figure 2.11 A separated section of a rectangular bar, its stress distribution and cross-sectional area.

Solution: The maximum normal stress occurs at $y = 6$ in. or

$$\sigma_{max} = (200)(6)^2 = 7200 \text{ lb/in}^2$$

Figure 2.12 The cross-sectional area illustrating the element of integration required to determine the total force.



Using the strip element indicated in Figure 2.12 integrate the element dA to determine the total normal force,

$$F_{\text{normal}} = \int_A \sigma\, dA = \int_0^6 (200y^2)(4dy) = 800\left[\frac{y^3}{3}\right]_0^6 = 57{,}600 \text{ lb}$$

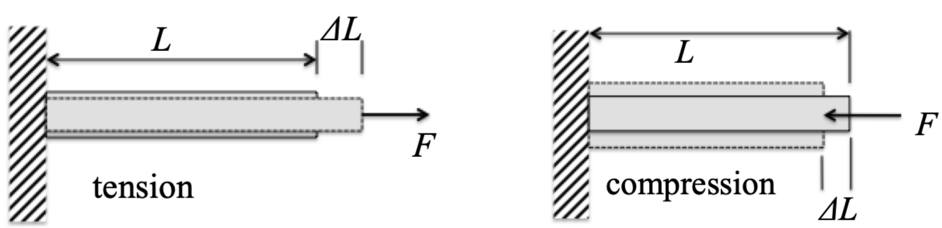
The average stress is

$$\sigma_{\text{av}} = \frac{F}{A} = \frac{1}{A}\int_A \sigma\, dA = \frac{1}{(4)(6)} 57{,}600 = 2400 \text{ lb/in}^2$$

In this case, note that the average normal stress is much less than the maximum stress. This underscores the idea that average stress cannot always be used to determine failure in design calculations.

Strain Stress is a way to quantify the internal forces in structure. In a like way, strain is a method of quantifying the internal displacements in a structure. Like stress, there are two types of strain: *normal strain* and *shear strain*. We start with a simple case of examining the result of external forces acting on a simple structure in one direction and develop a formula to quantify how the structure changes shape as the result of external forces. Consider a cantilevered, prismatic bar of length L subject to a force applied along its longitudinal axis as indicated in Figure 2.12. If the force acts away from the beam (a) in the figure, then the bar will elongate a distance ΔL from its original or references state L. The normal strain denoted ε is defined as the ratio:

$$\varepsilon = \frac{\Delta L}{L}$$



(a) $\varepsilon > 0$ width shrinks (b) $\varepsilon < 0$ width increases

Figure 2.12 (a) The change in length of a bar due to a force causing tension and (b) the change in length of a bar causing compression. The solid lines represent the original shape, and the dashed lines represent the deformed shape.

Strain is the ratio of two displacements and is dimensionless (no units). In case (a) in Figure 2.12 the member is in tension and ε is considered to be positive. In case (b) the member is in compression and the strain is considered to be negative ($\varepsilon < 0$).

In more complicated structures the strain is a very local quantity and varies depending on location and orientation (although it is not a vector). A shear strain is also possible. To grasp the concept of shear strain, consider the infinitesimal rectangular element in Figure 2.13 and its deformed state by applying forces that tend to shear the element. The shear strain is the angular displacement γ (in radians). The shear strain is defined as a measure of the change in the angle between two lines originally perpendicular to each other in the on-deformed state, and then after the element is deformed.

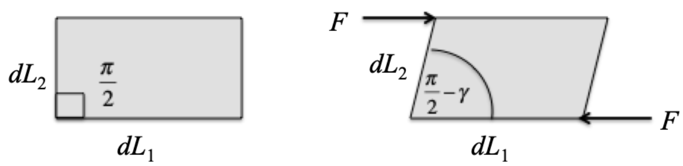


Figure 2.13 An infinitesimal element in the shape of a rectangle sheared into a parallelogram used to define the shear strain γ in radians. (left side, undeformed; right side, deformed)

Strain in Complex Structures Next consider generalizing the shear and normal strains to more complex structures by examining an arbitrary three-dimensional structure of odd shape such as the one illustrated in Figure 2.14. Consider a small line in the material of length *dL*. As the structure deforms the line will change to a new length *dL′*. The extensional strain in this case is defined as the change in length divided by the original length:

$$\varepsilon = \frac{dL' - dL}{dL}$$

(a) (b)

Figure 2.14 An infinitesimal 3D material with shown a line between two points both before deformation (a) and after deforming (b) used to define extensional strain.

Similar to the case of the cantilevered bar in Figure 2.12, the strain is positive if the material is stretched and negative if it is compressed. The value of ε depends on where the line is and may vary from one place to another in the structure.

If the value of ε is known at a given point and direction in the structure then the new length, *dL′*, is just

$$dL' = (1+\varepsilon)dL$$

Integrating yields that

$$L' = L + \int_L \varepsilon dL$$

where the integration is carried out over the length of the line. The change in length caused by the strain, denoted δ, is

$$\delta = L' - L = \int_L \varepsilon dL$$

If the strain is a constant value along the length L, then $\delta = \varepsilon L$. The units of δ are that of length and δ is referred to as the elongation associated with the strain.

Example 2.7 Consider the bar in Figure 2.12 to be 2 m long in its on-deformed state. (a) Assume that the strain is constant throughout and that an external force elongates the bar to 2.04 m and compute the normal strain. (b) If the strain is not constant but follows the relationship $\varepsilon = ax/L$, compute the value of the constant a if the deformed length of the bar is 2.01 m.

Solution:

(a) In this case $\Delta L = 0.04$ m so $\varepsilon = \Delta L/L = 0.04/2 = 0.02$.

(b) In this case the bar's change in length is $\delta = 0.1$ so that

$$\delta = 0.01 = \int_L \varepsilon dL \Rightarrow 0.01 = \int_0^2 \frac{ax}{2}dx = \frac{a}{2}\left[\frac{x^2}{2}\right]_0^2 = a$$

Thus, the value of the constant is $a = 0.01$.

The Relation Between Stress and Strain Assuming the material a structure is made of is linear and isotropic (which means its material properties are constant in each direction), the materials behave according to Hooke's law which states that normal stress and strain are proportional and related linearly by the simple relation: $\sigma = E\varepsilon$. The constant of proportionality E goes by several names: modulus of elasticity, elastic modulus and Young's modulus. Likewise, the shear stress and shear strain are related by $\tau = G\gamma$, where G is referred to as the shear modulus. Both of these stress-strain relationships are only true for small strains. At some point all models are wrong in the sense that they do not predict behavior to the precision necessary for a particular application. Thus, we will eventually expand the prior work to include more advanced situations and more complex structures.

Some examples of these material values are given in the following:

Metal	Young's Modulus (E) (GPa)	Density (ρ) (kg m-3)
Structural Steel	190	7900
Aluminum Alloy	69	2700
Titanium Alloy	116	4500

See the following web site for more detail:
https://www.bestech.com.au/wp-content/uploads/Modulus-of-Elasticity.pdf