3. Axially Loaded Bars

In this section we examine the stress and strain in bars subject to axial loading in more detail. The bar is the simplest structure we can analyze allowing us to build some methodology and understanding. Bars are also relatively straightforward to perform tests on allowing our math models to be validated and expanded. In addition, many aerospace components contain bar like structures under axial loading.

Axial Stresses

Example 3.1 A circular bar is made with two different diameters as indicated in Figure 3.1. The diameter of part A is 2 in and that of B is 4 in. Given the external forces indicated compute the normal stress on a plane perpendicular to the bar’s axis in each part.

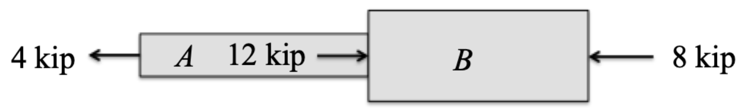


Figure 3.1 A loaded stepped bar of circular cross section.

Solution: Passing a cut plane through part A yields the free-body diagram illustrated in Figure 3.2a. Summing forces yields simply that $P = 4$ kip. Recalling the area of a circle is $A = \pi r^2 = \pi\ (1)^2 = \pi$, the normal stress becomes:

$$\sigma = \frac{P_A}{A} = \frac{4000}{\pi} = 1273 \text{ psi}$$

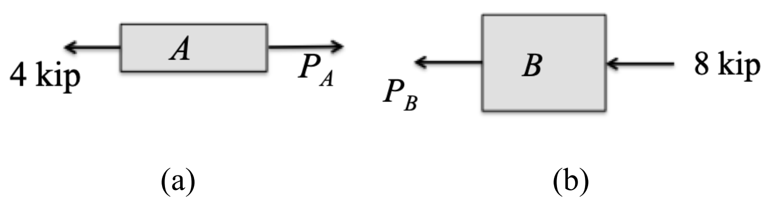


Figure 3.2 (a) A section of segment A of Figure 3.1 made by passing a cut plane perpendicular to the axis of the bar through A. (b) A section of segment B of the bar in Figure 3.1 made by passing a cut through B.

Next consider a perpendicular cut through part B as illustrated in Figure 3.2b. From Figure 3.1 the axial *external* load is compressive with 12 -4 = 8 kip acting on each end. The free-body diagram of Figure 3.2b yields: $P_B = -\ 8$ kip (-8000 psi). The radius of part B is $r = 2$ in., so the area becomes:

$$A = \pi\ r^2 = 4\pi$$

Thus, the normal stress in part B is:

$$\sigma = \frac{P_B}{A} = \frac{-8000}{4\pi} = -637 \text{ psi}$$

Example 3.2 Recall Example 1.4 and the structure of Figure 1.9, repeated below in Figure 3.3, which is used to support a rescue basket on the side of a helicopter. With the forces in each member determined in Example 1.4 in terms of the external force F, compute the normal stresses in members CD and BD that have a square cross section of 0.1 m on a side. Assume the device needs to support F = 1,000 N.

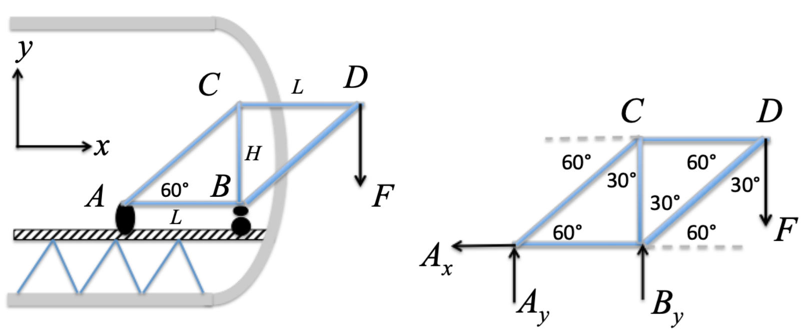


Figure 3.3 A truss model of a truss used to hold a rescue basket off of a helicopter and its whole body, free-body diagram.

Solution: First consider joint D and recall its separated free-body diagram of Figure 1.10 resulting in the following forces

$$F_{BD} = \frac{-2}{\sqrt{3}} F = \frac{-2000}{\sqrt{3}} = -1{,}154.7 \text{ N}, \quad F_{CD} = \frac{-1}{\sqrt{3}} F = -577.3 \text{ N}$$

The normal stress in these two force members then becomes ($\sigma = P/A$):

$$_{BD} = \frac{F_{BD}}{(0.1)^2} = -115{,}470 \text{ N/m}^2, \quad \sigma_{CD} = \frac{F_{CD}}{(0.1)^2} = -57{,}735 \text{ N/m}^2$$

The stress in the other members can be found in a similar fashion.

Angular Stress The stress distribution across all cut plane angles is needed to understand fatigue and failure of structures. Consider again an axially loaded bar (load F) and consider the geometry of the cut plane as the angle of the plane changes from the perpendicular to the long axis as illustrated in Figure 3.4. Note that for this simple structure the cross-sectional area determines the stress for a fixed applied force F.

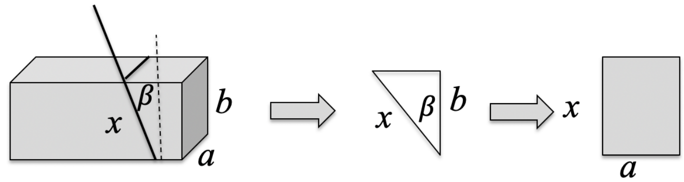


Figure 3.4 The geometry of the area as the angle of the cut plane changes.

The area of the angular cut is $A_\beta = xa$ where x is found from the trigonometry indicated on the right in Figure 3.4:

$$b = x\cos\beta \Rightarrow x = \frac{b}{\cos\beta} \Rightarrow A_\beta = \frac{ab}{\cos\beta}$$

Also note that at $\beta = 0$, the shear stress is zero and all the stress is normal to the surface. However as β increases shear stress develops.

To see this more clearly examine the triangular wedge of Figure 3.5 and determine the normal and shear stress in terms of the angle of the cut. Let σ denote the normal stress associated with a cut perpendicular to the long axis of the bar and β denote the angle of the cut. Let F be the applied force. The vector F is resolved into two components one perpendicular to the cut (normal force) and one parallel to it (shear force) as indicated in Figure 3.5 b. From the vector diagram, the force normal to the cut plane is $N = F\cos\beta$. Remembering that the normal stress, σ, on the face in Figure 3.5 a is related to the area $A = ab$, by $\sigma = F/A$, the normal stress on the cut surface becomes

$$\sigma_\beta = \frac{N}{A_\beta} = \frac{\sigma A\cos\beta}{A/\cos\beta} = \sigma\cos^2\beta$$

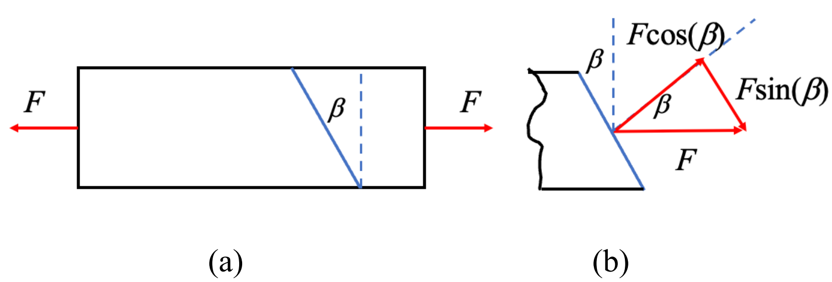


Figure 3.5 (a) The cut plane and (b) the vector diagram.

Next consider the shear force $S = -F\sin\beta$, so that the shear, τ_β, that develops on the cut plane becomes

$$\tau_\beta = \frac{-F\sin\beta}{A/\cos\beta} = \frac{-\sigma A\sin\beta}{A/\cos\beta} = -\sigma\sin\beta\cos\beta$$

Note that the stress normal to the cut plane, σ_β, is just σ if the cut plane is perpendicular, ($\beta = 0$), and reduces as β increases.

These relationships between the angle β and the shear and normal stress for a prismatic bar can be used to determine the maximum values they take on. Since the normal stress is proportional to $\cos^2\beta$, the maximum value occurs at β = 0 and is thus $\sigma_{max} = P/A$. Differentiating the expression for shear stress with respect to β and setting it equal to zero reveals its maximum and minimum values:

$$\frac{d}{d\beta}(\tau_\beta) = \sigma \frac{d}{d\beta}(\sin\beta\cos\beta) = \sigma\left(\sin^2\beta - \cos^2\beta\right) = 0$$

The term in parenthesis is zero for values of β = 45° and 135° resulting in: $\tau_{\beta\max} = |\sigma / 2|$.

Example 3.3 Two structures are glued together along the 50° plane as indicated in Figure 3.6. The glued together piece is subject to a 200 N tensile load. What stresses will the glue have to withstand? The cross section of the piece is 10 mm by 50 mm.

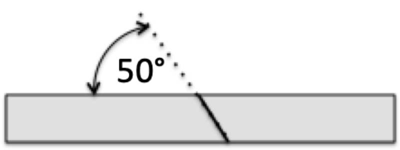


Figure 3.6 Two pieces glued together at a 50° angle. The cross-sectional area is 10 x 50 mm^2.

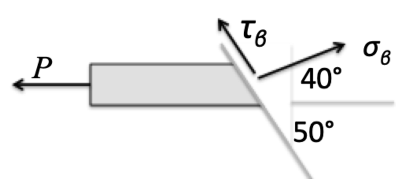
Solution: The normal stress perpendicular to the bar's axis is

$$\sigma = \frac{P}{A} = \frac{200\text{ N}}{(0.01)(0.05)\text{m}^2} = 400{,}000\text{ Pa}$$

Using the relations developed above for normal stress at an angle and recognizing the angle orthogonal to the glue plane from Figure 3.7 yields:

$$\sigma_\beta = \sigma\cos^2\beta = 400{,}000\cos^2 40° = 235{,}000\text{ Pa}$$

Figure 3.7 The angular positions of the normal and shears stress relative to the glue plane in Figure 3.6.



Computing the shear stress yields:

$$\tau_\beta = -\sigma\sin\beta\cos\beta = -400{,}000\sin 40°\cos 40° = -197{,}000\text{ Pa}$$

The minus sign indicates that the shear stress is opposite of the assumed direction so pointing downwards in the figure. The glue used must be able to accommodate these two stresses, neither of which is as large as if the parts were glued together at 0°

Axial Strain Recall from Figure 2.12 that the strain in a bar pulled on by a force is $\varepsilon = \Delta L/L$ where L is the length of the bar and ΔL is the amount the bar elongates under the action of a force. ΔL is often denoted in short form by $\delta = \Delta L$. Also recall that for structures made of linear, isotropic material (which means its material properties are constant in each direction), the structure behaves according to Hooke's law, which states that normal stress and strain are proportional and related linearly by the simple relation:

$$\sigma = E\varepsilon.$$

The constant of proportionality E is called modulus of elasticity, or elastic modulus. Combining the stress produced by a force P, $\sigma = P/A$, where A is the cross-sectional area, with the stress-strain relationship, $\sigma = E\varepsilon$, and elongation relationship $\varepsilon = \delta/L$, results in

$$\delta = \varepsilon L = \frac{\sigma}{E}L = \frac{P}{AE}L = \frac{PL}{EA}$$

This expression relates the elongation that a bar under a force P will experience to its material property, E, its length, L, and its cross-sectional area, A. This allows both a prediction of a structures behavior and a guide to designing bars where the elongation is an important consideration.

Poisson's Ratio and Lateral Strain So far in treating axially loaded bars we have ignored what happens in the lateral or transverse direction. Basically, if a bar is stretched to increase its longitudinal length then simultaneously it's lateral dimension shrinks. Likewise, if the bar is compressed and its length decreases and its width increases. This is a topic taken up later and is mentioned here to insure you are aware of this. The parameter that quantifies this phenomenon is called Poisson's Ratio and is the ratio of the lateral strain to its axial strain. The effect is illustrated in Figure 3.8.

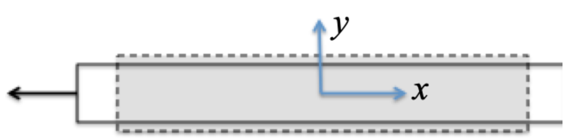
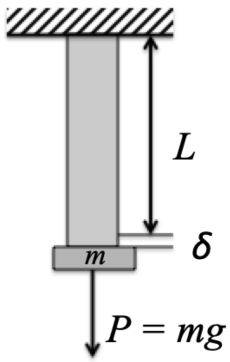
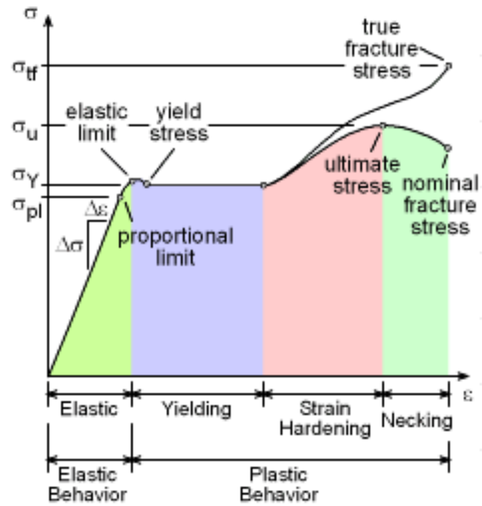


Figure 3.8 As the bar is pulled in the x direction (axial) it lengthens but narrows in the y direction (lateral), called the Poisson Effect.

Measurement All formulas need to be validated by some type of measurement. The stress-strain relationship can be measured by both simple experiments and to a high degree of accuracy with more sophisticated measurement instruments. Measurement of material properties forms an entire engineering discipline by itself. Here we just describe a simple experiment. The experiment suggested in Figure 3.9 can be used to plot applied force versus elongation by adding known mass to the end to increase force and using a rule to measure the elongation. Such plots can be made for a variety of different sample prismatic bars of different materials.



(a)



(b)

Figure 3.9 (a) A crude experiment to determine the relationship between force and deflection and hence using $\varepsilon = \delta/L$ and $\alpha = P/A$ produce plots (b) of stress versus strain.

Note that the green shaded part in Figure 3.9b shows that the equation $\sigma = E\varepsilon$ holds only up to a certain value of strain. The value of the slope of that line is the modulus E. As the force and deflection increase the equation no longer holds and other theories have to be used to describe the relationship between stress and strain. This is listed as the "proportional limit" in the figure. Other events happen as the stress and strain are increased and eventually the bar will break. All the different situations listed in Figure 3.9b are discussed in later courses. Here we will focus on the linear relationship for the most part.

Example 3.4 The experiment listed in Figure 3.9a can also be used to calculate the value of the modulus. In that regard consider a prismatic bar of length 200 mm and circular cross section of 10 mm diameter. Suppose a 16kN axial force is applied and produces a deflection of 0.6 mm. What is the value of elastic modulus?

Solution: The cross-sectional area is $A = \pi r^2 = \pi(0.01/2)^2$, so the normal stress is

$$\sigma = \frac{P}{A} = \frac{16000}{\pi\left(0.01/2\right)^2} = 203.7 \text{ MP}$$

The axial strain is

$$\varepsilon = \frac{\delta}{L} = \frac{0.6}{200} = 0.003$$

From the linear stress strain relationship, the modulus is

$$E = \frac{\sigma}{\varepsilon} = \frac{203.7\times10^6}{0.003} = 67.9 \text{ GPa}$$

To increase accuracy several different forces and deflections can be measured to obtain more data to curve fit and to put measurement bounds on the value of E.

Elongation in Nonuniform Bars Notice that for the same applied force to a different material (i.e., different modulus and/or cross-sectional area) the strain and hence elongation will be different. Hence when building a structure using different materials it is very important to account for the difference in elongations in order not to introduce additional stress. Figure 3.10 illustrates three structures put together of three different lengths, cross sectional areas and modulus. The elongation of the total structure is just the sum:

$$\delta = \delta_1 + \delta_2 + \delta_3$$

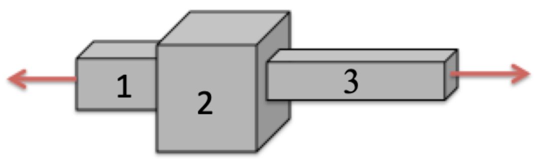


Figure 3.10 An axial prismatic bar made of three parts each with a different modulus, cross sectional area and length.

Example 3.5 A structure is machined out of 6061 aluminum (E =68.9 GPa) in the shape of the axial bar shown in Figure 3.10. Given the following values calculate the elongation of the part subject to an axial load of 400 N: A_1 = 100 mm^2, L_1 = 0.2 m, A_2 = 200 mm^2, L_2 = 0.3 m, A_3 = 50 mm^2, and L_3 = 0.4 m.

Solution: This is a straightforward use of the formula suggested above Figure 3.10:

$$\delta = \left(\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}\right)\frac{P}{E} =$$

$$\left(\frac{0.2\ \text{m}}{0.0001\ \text{m}^2} + \frac{0.3\ \text{m}}{0.0002\ \text{m}^2} + \frac{0.4\ \text{m}}{0.00005\ \text{m}^2}\right)\frac{400\ \text{N}}{68{,}900{,}000{,}000\ \text{N/m}^2} = 6.7 \times 10^{-5}\ \text{m}$$

This is very small and likely negligible in many applications. To get a 6.7 mm of elongation the force would need to be 40,000 N.

Here the subscripts correspond to the length, modulus and cross-sectional area of each part. In general, if there are n segments then the total elongation is the sum of each component:

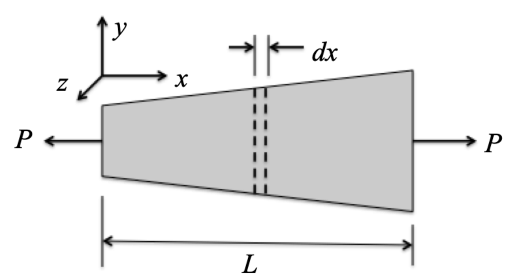
$$\delta = \sum_{i=1}^{n} \frac{PL_i}{E_i A_i}$$

Another common configuration is if the cross-sectional area changes. An example of a tapered bar is given in Figure 3.11 where the perpendicular cross-sectional area varies along the length of the bar. In this case the total elongation is found by integrating over the length of the bar:

$$\delta = \int_0^L \frac{P}{A(x)E}dx$$

Example 3.6 (From Geer 5th Ed.) Determine the elongation of the tapered bar of circular cross section illustrated in Figure 3.11 in terms of the diameter at each end point, the applied load *P*, its length *L* and its elastic modulus *E*.

Figure 3.11 A tapered beam with variable cross section under axial loading.



Solution: The formula for elongation, δ, is straightforward. What is tricky is to determine the cross-sectional area. Here an example is given for computing the area function for a tapered shape by extending the shape to be triangular and using the formula for similar triangles. This is illustrated in Figure 3.12.

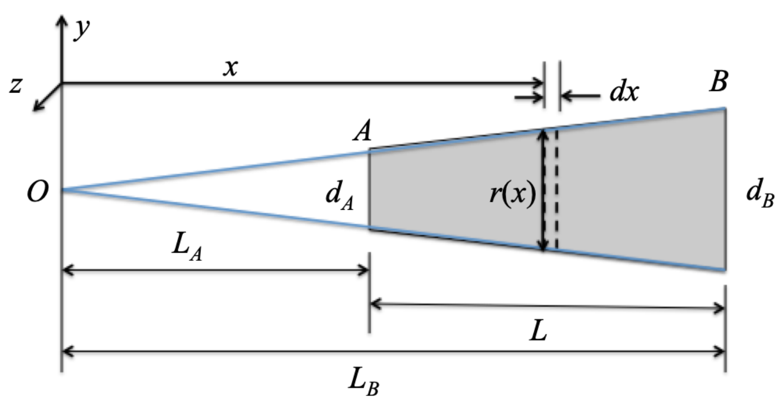


Figure 3.12 The tapered bar of Figure 3.11 extended to form two similar triangles: *OA* and *OB*. The quantities d_a and d_b are the diameters at points *A* and *B* and *r*(*x*) is the diameter at a distance *x* from the origin.

From the relationship for similar triangles the following ratio holds:

$$\frac{L_A}{L_B} = \frac{d_A}{d_B}$$

Again, using similar triangles and the notation in Figure 3.12 yields

$$\frac{r(x)}{d_A} = \frac{x}{L_A} \Rightarrow r(x) = \frac{d_A x}{L_A}$$

This resulting value for the diameter as a function of *x* allows the cross-sectional area to be written as

$$A(x) = \pi\left(\frac{r(x)}{2}\right)^2 = \frac{\pi d_A^2 x^2}{4L_A^2}$$

The area function *A*(*x*) can now be placed in the integral expression for elongation resulting in:

$$\delta = \frac{P}{E}\int_{L_A}^{L_B} \frac{1}{A(x)}dx = \frac{4PL_A^2}{\pi E d_A^2}\int_{L_A}^{L_B}\frac{1}{x^2}dx = \frac{4PL_A^2}{\pi E d_A^2}\left[\frac{-1}{x}\right]_{L_A}^{L_B} = \frac{4PL_A^2}{\pi E d_A^2}\left[\frac{1}{L_A} - \frac{1}{L_B}\right]$$

$$= \frac{4PL_A^2}{\pi E d_A^2}\left(\frac{L}{L_A L_B}\right) = \frac{4PL}{\pi E d_A^2}\left(\frac{L_A}{L_B}\right) = \frac{4PL}{\pi E d_A^2}\left(\frac{d_A}{d_B}\right) = \frac{4PL}{\pi E d_A d_B}$$

In simplifying the last term in the first line recall that $L = L_B - L_A$.

Statically Indeterminate Structures Recall the prior discussion that statically indeterminate refers the situation when there are more unknown parameters then equations, so a solution cannot be found. For axial loaded structures the elongation relationship can be used to provide an additional equation to supplement the equilibrium equations to allow a solution. This is illustrated in the following example.

Example 3.7 The bar in Figure 3.13 consists of two segments of different lengths and cross-sectional area but of the same material (i.e. same modulus). The bar is fixed at both ends and subject to an axial force *F* applied at the boundary between the two segments as indicated. Determine the reaction forces at the fixed ends.

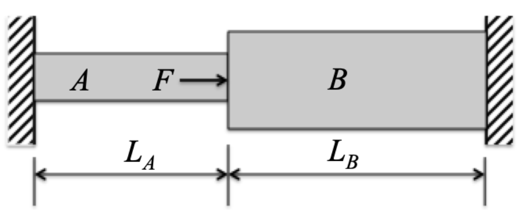


Figure 3.13 An axial loaded bar fixed at both ends consisting of two segments with different lengths and cross-sectional area.

Solution: A free-body diagram of the bar is given in Figure 3.14. Note because the external force *F* is applied on the interface between segments *A* and *B* the reaction forces at the ends of *A* and *B* are different. The sum of forces yields

$$F - P_A + P_B = 0$$

Thus, the equilibrium equation has two unknowns but summing forces and moments only produces just one equation.

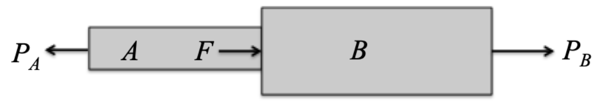


Figure 3.14 Free-body diagram of the structure in Figure 3.13

However, if we examine the elongation of each segment, denoted δ_A and δ_B, they must add to zero because the structure is constrained not to move. Thus $\delta_A + \delta_B = 0$ becomes an additional equation that can be used to make the system statically determinate. This type of equation is called a *compatibility condition*. Recall the expressions for elongation of each part:

$$\delta_A = \frac{P_A L_A}{EA_A}, \quad \delta_B = \frac{P_B L_B}{EA_B}$$

So, the compatibility condition becomes:

$$\frac{P_A L_A}{EA_A} + \frac{P_B L_B}{EA_B} = 0$$

Combining this last expression with the equilibrium equation $F - P_A + P_B = 0$ yields two equations in two unknowns, which can be solved to yield

$$P_A = \frac{F}{1 + \frac{A_B L_A}{L_B A_A}}, \qquad P_A = \frac{-F}{1 + \frac{A_A L_B}{L_A A_B}}$$

Note that the result turns out to be independent of the elastic modulus. With P_A and P_B known the normal stress in each bar is known:

$$\sigma_A = \frac{P_A}{A_A}, \qquad \sigma_B = \frac{P_B}{A_B}$$

The solution above depended on three elements: the equilibrium equation, the compatibility equation and the elongation equations in each part. This approach is used in a number of more complicated structural problems as well.

Flexibility and Stiffness Methods There are two approaches to solving statically indeterminate problems, which for simple cases as analyzed here are really the same. However, in more complex problems the two approaches make a difference. Here they are just introduced so you are aware of them. Both methods are based on the same set of equations and they are introduced here just in terms of the simple case of an axially loaded bar of Figure 3.13. The equations are

The Force-Deformation Relationships: $\begin{cases} \delta_A = \dfrac{P_A L_A}{EA_A} \\ \delta_B = \dfrac{P_B L_B}{EA_B} \end{cases}$

The Compatibility Condition: $\delta_A + \delta_B = 0$

The Equilibrium Equation(s): $F - P_A + P_B = 0$

The Flexibility Method: This approach writes the force-deformation equations in terms of the constant coefficient $f = L/EA$, called a *flexibility coefficient*, for each part:

$$\delta_A = \frac{P_A L_A}{EA_A} = f_A P_A, \text{ and } \delta_B = \frac{P_B L_B}{EA_B} = f_B P_B$$

Substitution of these expressions into the compatibility equation yields:

$$f_A P_A + f_B P_B = 0$$

This last expression combined with the equilibrium equation yields two equations in the two unknown forces P_A and P_A. Once solved P_A and P_B are used to determine the deformations δ_A and δ_B. This method is also referred to as the *force method.*

The Stiffness Method: This approach starts out by writing the force-deformation equations in terms of the constant $k = EA/L$, called the *stiffness* of the structure. Specifically, for this two-component bar:

$$P_A = \frac{EA_A}{L_A}\delta_A = k_A \delta_A \quad \text{and} \quad P_B = \frac{EA_B}{L_B}\delta_B = k_B \delta_B$$

The forces in this stiffness formulation are substituted into the equilibrium equation to obtain

$$F - k_A \delta_A + k_B \delta_B = 0$$

This last expression combined with the compatibility equation results in two equations in the two unknowns δ_A and δ_B. Once these deformations are determined returning to the force- deformation equation yield the values of the forces P_A and P_B. This approach is sometimes called the *displacement method*. Using the stiffness as the constant rather than the flexibility is more common in dynamics problems, especially those involving springs.

Example 3.8 Suppose the bar in Figure 3.13 does not quite reach to the wall on the right as pictured in Figure 3.15 and is then subject to an axial force of 160 kip applied at the mid-section as indicated. Compute the normal stresses in parts A and B, given that the modulus is E = 12 x 10^6 psi, part A has dimensions L_A = 10 in with a circular cross section of 2-in diameter and part B has dimensions L_B = 8 in with a circular cross section of 4-in diameter. The distance b before the force is applied is b = 0.02 in.

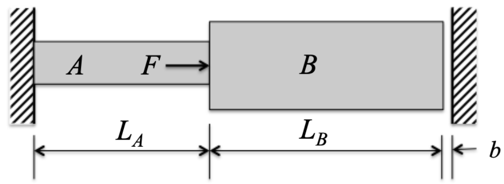


Figure 3.15 A bar with applied force almost touching the wall on the right before the force is applied. This issue is does it touch the wall, making the structure statically indeterminate or not.

Solution: In this situation there are two cases to consider. The first is if the 160-kip force does not elongate the bar far enough to hit the wall. The second is if the 160-kip force does cause it to hit the wall. In the first case of not hitting the wall the free-body diagram is given in Figure 3.16.

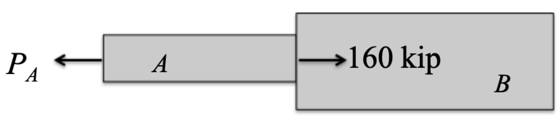


Figure 3.16 The free-body diagram for the case that the 160-kip force does cause it to hit the wall.

Summing forces yields simply that P_A = 160 kip = 160,000 lb. There is no axial force acting on part B if it is not touching the wall on the right. Thus, the deflection is

$$\delta_A = \frac{L_A}{EA_A}P_A = \frac{10}{\left(12\times10^6\right)\left(\pi 2^2/4\right)}(160{,}000) = 0.0424 \text{ in.}$$

This value of δ_A = 0.424 in. is larger than b = 0.02 in. so the bar hits the wall once the force is applied and the system becomes statically indeterminate (two unknowns, P_A and P_B) and only one equation of equilibrium).

Because the deflection predicts that this will hit the wall the free-body diagram changes to that of Figure 3.17. From the fee-body diagram of Figure 3.17 the equation

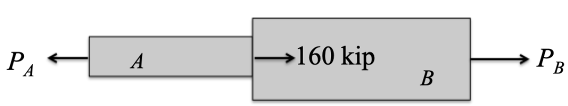


Figure 3.17 The free-body diagram for the case that the applied force causes the bar to connect to the wall on the right.

of *equilibrium* becomes:

$$-P_A + P_B + 160{,}000 = 0$$

The *force-deformation* relationships in this case are:

$$\delta_A = \frac{P_A L_A}{EA_A} \text{ and } \delta_B = \frac{P_B L_B}{EA_B}$$

The *compatibility* equation in this case is (this accounts for the gap):

$$\delta_A + \delta_B = b \text{ so that } \frac{L_A}{EA_A} P_A + \frac{L_B}{EA_B} P_B = 0.02$$

This last equation along with the equilibrium equation yields two equations in the two unknown reaction forces P_A and P_B. Solving these by simple algebraic substitution yields

$$P_A = 89{,}500 \text{ lb. and } P_B = -70{,}500 \text{ lb.}$$

With the reaction forces known the normal stresses become

$$\sigma_A = \frac{P_A}{A_A} = \frac{89{,}500}{\pi(2)^2/4} = 28{,}490 \text{ psi, and } \sigma_B = \frac{P_B}{A_B} = \frac{-70{,}500}{\pi(4)^2/4} = -5{,}610 \text{ psi}$$

While the equations in Example 3.8 are easily solved it is instructive to examine the solution by using a matrix inversion because there are only two variables so the computations can be done by hand. Since there are two variables and two equations the matrix involved is of dimension 2 x 2. First recall how to symbolically compute the inverse of a 2 x 2 matrix. Let the matrix D that we want to invert have generic values:

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The *determinant* of D is the scalar $|D| = ad - bc$. As long as the determinant is not zero, the inverse of D is

$$D^{-1} = \frac{1}{|D|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Multiplying the inverse matrix times a vector of forces $\mathbf{F} = [f_1 \;\; f_2]^T$ yields:

$$\begin{bmatrix} P_A \\ P_B \end{bmatrix} = D^{-1}\mathbf{F} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} df_1 - bf_2 \\ -cf_1 + af_2 \end{bmatrix}$$

The above is the symbolic solution to two equations in two unknowns written in the form $D\mathbf{R} = \mathbf{F}$ and allows a symbolic solution or numerical solution of Example 3.8 for the case where the structure hits the wall on the right. Rearranging the two equations in matrix/vector form in terms of the values given in that problem the two coupled equations of deformation and equilibrium can be written as:

$$\begin{bmatrix} \dfrac{L_A}{EA_A} & \dfrac{L_B}{EA_B} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \end{bmatrix} = \begin{bmatrix} 0.02 \\ -160{,}000 \end{bmatrix}$$

Solving this by multiplying by the inverse of the coefficient matrix D yields

$$\begin{bmatrix} P_A \\ P_B \end{bmatrix} = \frac{1}{\dfrac{L_A}{EA_A} + \dfrac{L_B}{EA_B}} \begin{bmatrix} 0.02 - \dfrac{L_B}{EA_B}(-160{,}000) \\ 0.02 + \dfrac{L_A}{EA_A}(-160{,}000) \end{bmatrix}$$

Reading of the values of P_A and P_B from this last expression yields:

$$P_A = \left(\frac{L_A}{EA_A} + \frac{L_B}{EA_B} \right)^{-1} \left(0.02 + \frac{L_B}{EA_B}(160{,}000) \right)$$
$$P_B = \left(\frac{L_A}{EA_A} + \frac{L_B}{EA_B} \right)^{-1} \left(0.02 - \frac{L_A}{EA_A}(160{,}000) \right)$$

These two analytical expressions for P_A and P_B allow us to perform parametric studies to understand how various materials (E) and dimensions (L and A) affect the forces and hence strains in the bar. This is easily done in a code such as MATLAB.

Nonprismatic Bars and Distributed Loads Consider a bar with varying cross-sectional area in the x direction so that the area is a function of x: $A(x)$. If $A(x)$ changes gradually, i.e. if $d[A(x)]/dx$ is small, the distribution of normal stress in a perpendicular plane can be approximated by

$$\sigma(x) = \frac{P}{A(x)}$$

which is the equation for a prismatic bar except that the area changes with x and hence so does the normal stress. We had some previous examples of this earlier when we treated tapered bars. Now the question is how do we determine the change in the bar's length in this variable area case?

Recall from the earlier discussion of the deformation and modulus of elasticity that for a bar

$$\delta = \varepsilon L = \frac{\sigma}{E} L = \frac{P}{AE} L = \frac{PL}{EA}$$

If we apply this formula to an infinitesimal element dx which stretches to dx' when the bar is loaded, we get:

$$\delta = dx' - dx = \frac{Pdx}{EA(x)}$$

If we integrate this expression across the length of the bar, we get that the total change in length of the bar is

$$\delta = L' - L = \int_0^L \frac{P}{EA(x)} dx$$

The integration allows and adding up of all the changes of length of each small element in the bar.

Example 3.9 Consider the 2-meter-long tapered bar of Figure 3.18 subject to forces at each end of 20 MN. The modulus of elasticity of the material is E = 120 GPa. The cross-sectional area is described by

$$A(x) = 0.03 + 0.008x^2 \text{ m}^2$$

(a) Calculate the normal stress in the bar at x = 1 m and (b) the resulting change in length of the bar.

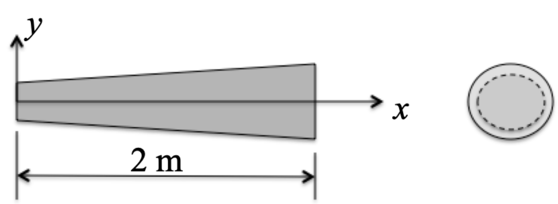


Figure 3.18 A tapered bar subject to a 20 MN force at each end.

Solution: (a) The cross-sectional area at x =1 m is $A(1) = 0.03 + 0.008(1)^2 = 0.038 \text{ m}^2$. Thus, the normal stress at x =1 m is

$$\sigma(1) = \frac{P}{A(1)} = \frac{20 \times 10^6}{0.038} = 526 \text{ MN/m}^2$$

(b) The deflection was given prior as

$$\delta = \int_0^L \frac{P}{EA(x)} dx$$

Substitution of the indicated values yields:

$$\delta = \int_0^L \frac{P}{EA(x)} dx = \int_0^2 \frac{(20 \times 10^6) dx}{(120 \times 10^9)(0.03 + 0.008x^2)} = 8.62 \text{ mm}$$

Distributed Axial Loads: A Rocket standing on a launch pad or a column in a building or the launch pad structure itself will experience and distributed axial load due to gravity. Here we start by consider the simple bar of Figure 3.19 in order to get a handle on how to solve such problems. The distributed force is modeled at the infinitesimal scale by a function q such that the force on each infinitesimal element dx is the product qdx.

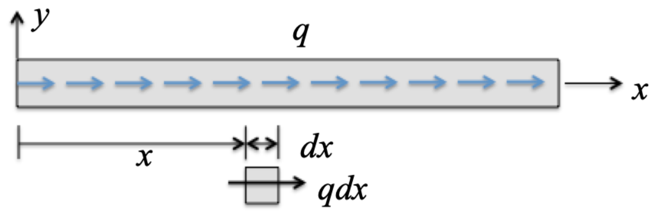


Figure 3.19 A symbolic representation of a distributed axial load indicating that the force on the infinitesimal element dx is the product qdx. Here q is a force per unit length.

Example 3.10 Consider the cantilevered bar in Figure 3.20 and determine the change in length due to the distributed load indicated.

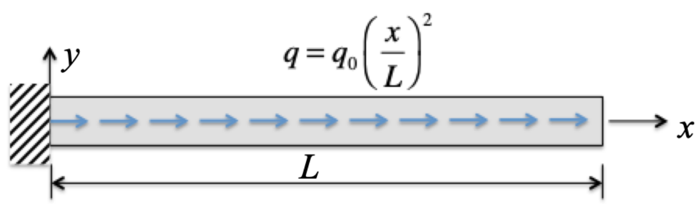


Figure 3.20 A cantilever bar with distributed axial load $q(x)$.

Solution: The approach is to pass a cut plan through some point along the x-axis and separate out the remaining piece in order to determine the internal axial force as a function of x as shown in Figure 3.21.

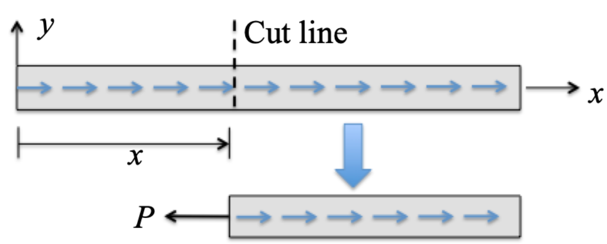


Figure 3.21 The free-body diagram of a cut out a section of the bar, indicating the internal force P that results in order to keep the section in equilibrium.

Summing forces in the x direction on the cut-out section yields:

$$-P + \int_x^L q_0\left(\frac{x}{L}\right)^2 dx = 0 \Rightarrow P = \int_x^L q_0\left(\frac{x}{L}\right)^2 dx$$

$$\Rightarrow P = \frac{q_0}{3}\left(L - \frac{x^3}{L^2}\right)$$

This last expression determines how the distributed axial force causes the internal force to decrease along the x-axis as x increases. At the free end, $x = L$, $P(x) = 0$.

With $P(x)$ known, the bar's change in length can be determined. Recall the element of length dx in Figure 3.19. The change in length of the element dx is

$$\delta_{\text{element}} = \frac{Pdx}{EA} = \frac{q_0}{3EA}\left(L - \frac{x^3}{L^2}\right)dx$$

Integrating this last expression across the length of the bar yields that the bar will change length by

$$\delta == \int_0^L \frac{q_0}{3EA}\left(L - \frac{x^3}{L^2}\right)dx = \frac{q_0 L^2}{4EA}$$

Example 3.11 A mounting stud for an aircraft engine (propeller) is partially embedded in a surrounding material and partially out, illustrated in Figure 3.22. The segment inside the material produces a distributed force along the stud (modeled as a bar) when subject to a force of $F = 10^5$ N at the tip. The distributed force results from the friction caused by being embedded in the metal fixture. Modeling the distributed force per unit area as q, determine the value of q, given the modulus of the steel cylindrical stud is 190 MPa with radius $r = 0.1$ m. (a) Calculate the maximum normal stress and (b) the change in length of the stud.

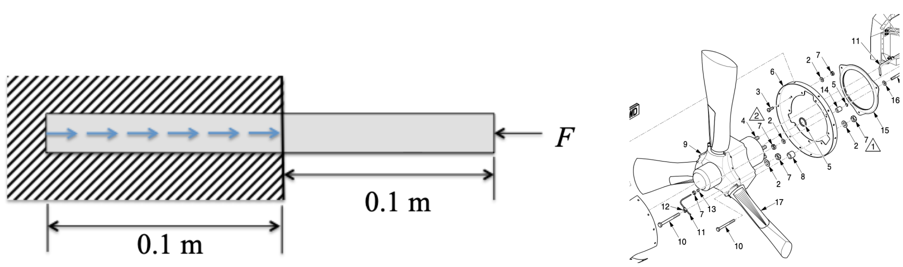


Figure 3.22 Model of a mounting stud (left side) as a bar (not to scale) embedded into a face plate. The many studs in a propeller engine mount (right side).

Solution: From the equilibrium of the entire stud the sum of the forces yields

$$\int_0^{0.1} q\,dx - F = 0 \Rightarrow q\,x\big|_0^{0.1} - F = 0 \Rightarrow q = \frac{10^5}{0.1} = 10^6 \text{ N/m}$$

a) To determine the internal axial force in the stud, extract a section of it out of the embedded section as shown in Figure 3.23, form a free-body diagram and sum forces in the x direction.

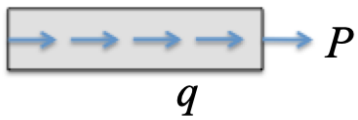


Figure 3.23 A removed section and its free-body diagram.

The sum of forces is:

$$\int_0^x q\,dx + P = 0$$

Integrating this the above from x = 0 to x = 0.1, yields the maximum compressive force is

$$P = -q\,x\Big|_0^{0.1} = -10^5 \text{ N} \Rightarrow P = -10^5 \text{ N}$$

Thus, the maximum compressive normal stress in the stud is

$$\sigma = \frac{P}{A} = \frac{-10^5}{\pi(0.1)^2} = -3{,}183{,}098.9 \text{ N/m}^2$$

b) In order to determine the change in the length of the stud it has to be divided into two segments, one outside the mounting and one embedded in the mounting. This is much like the case considered earlier of a stepped bar with discrete forces applied. In order to compute the total δ we treat each segment separately and add the two displacements. First compute the δ for the segment external to the fixture:

$$\delta_{\text{external}} = \frac{\text{max load} \times \text{Length}}{EA} = \frac{PL}{EA} = \frac{-10^5(0.1)}{190{,}000{,}000\pi(0.1)^2} = -0.0016 \text{ m}$$

Next compute the δ for the embedded segment:

$$\delta_{\text{embed}} = \int_0^{0.1} \frac{-10^6}{EA} x\,dx = \frac{-10^6}{EA}\frac{(0.1)^2}{2} = \frac{-10^5(0.1)}{2 \times 190{,}000{,}000\pi(0.1)^2} = -0.0008 \text{ m}$$

Adding the two changes in length yields that the stud changes length by

$$\delta_{\text{embed}} + \delta_{\text{external}} = -0.0016 \text{ m} - 0.0008 \text{ m} = -0.0024 \text{ m}$$

Likely within tolerance of the engine mounting.

Example 3.12 Consider the distributed load of a bar caused by its own weight. Such a consideration arises in aerospace with a rocket on its launch pad or in civil structures with support columns. Figure 3.24 shows such a bar along with the drawings need to analyze the effects of the distributed weight force. The unloaded bar has a length of *L*, cross sectional area *A*, weight density γ and modulus of elasticity *E*. Calculate the change and length and total length due to the weight when mounted on the ground.

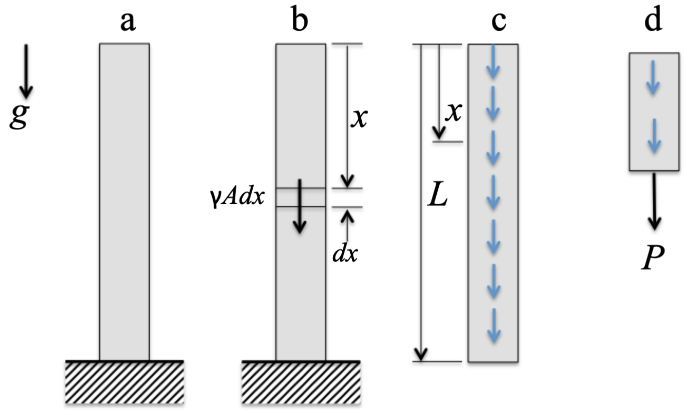


Figure 3.24 (a) A bar loaded by its own weight. (b) The axial force on a differential element of length, *dx*. (c) Illustration of the distributed force and (d) the removed segment free-body diagram for determining the internal axial force.

Solution: Summing forces on the segment in Figure 3.24d to find the force *P* yields:

$$P + \int_0^x \gamma A\,dx = 0 \Rightarrow P = -\int_0^x \gamma A\,dx = -\gamma Ax$$

Examining *P* shows that the internal force increases linearly from 0 at the top to a maximum of $\gamma AL=W$ the weight of the bar at the bottom.

The change in length of the element *dx* (Figure 3.24a) is

$$\delta_{\text{element}} = \frac{Pdx}{EA} = \frac{-\gamma Axdx}{EA}$$

The total change in length is due to gravity becomes

$$\delta = -\int_0^L \frac{\gamma Axdx}{EA} = -\frac{\gamma AL^2}{2EA} = -\frac{WL}{2EA}$$

This the bar will shrink in height and its new height will be (*L*- *WL*/2*EA*).

Thermal Strains Airplanes, rockets and satellites go through large changes in temperatures, which can greatly affect their structural components. There are two main effects caused by varying temperatures. The first is temperature can cause a structure to expand or contract and the second is temperature can cause a change in material properties that a structure is made of. Here we will focus on the effect of contraction and expansion. The Concord expanded about 8.5 inches in flight because of thermal strains.

Basically, a change in temperature will cause a change in strain. This is captured by the expression for thermally induced strain, denoted ε_T, and defined by:

$$\varepsilon_T = \alpha(\Delta T)$$

Here α is a coefficient of thermal expansion (units of 1/degrees), which is a material property, and ΔT is the change in temperature (units of degrees). The coefficient of thermal expansion is often abbreviated as C.T.E. In order to keep track of different kinds of strain we will denote the strain resulting from a mechanical stress we denote it by ε_M. In cases where both types of strains are present the total strain is just the sum

$$\varepsilon = \varepsilon_T + \varepsilon_M$$

Consider the cantilever bar of Figure 3.25 at two different temperatures as indicated. The increase in temperature has caused a corresponding increase in the length of the beam of the value $\delta = \varepsilon_T L = \alpha \Delta T L$.

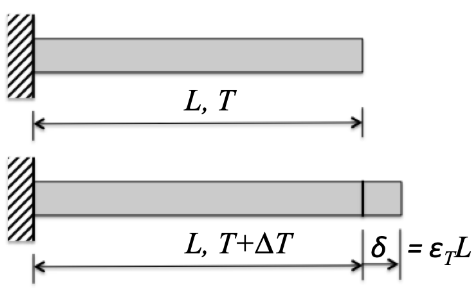


Figure 3.25 (Top) A cantilever bar of length L at temperature T. (Bottom) The same bar at an elevated temperature by an amount ΔT illustrating the resulting increase in length due to the thermal strain.

Next suppose that the same beam is now fixed on the right end as well as the left end. In this case the thermal strain is restricted by the additional fixed end as illustrated in Figure 3.26. Let P denote the compressive force provided by the fixed right end.

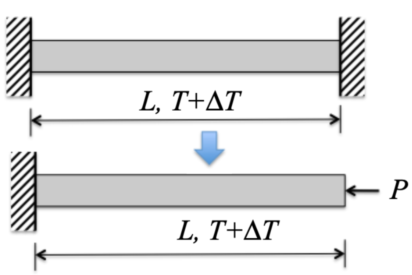


Figure 3.26 Illustration of the compressive force P developed because the constraint at the right side prevents the thermal strain from elongating the bar.

To calculate the force P exerted by the constraint at the right end recall the relationship between elongation and force is

$$\delta = \frac{PL}{EA}$$

In this case the thermally induced elongation is $\delta = \varepsilon_T L = \alpha \Delta T L$. So, the force becomes

$$\delta = \frac{PL}{EA} = \alpha \Delta T L \Rightarrow P = \alpha \Delta T E A$$

This force causes an axial compressive stress to develop of value:

$$\sigma = -\frac{P}{A} = -\alpha \Delta T E$$

For the bar fixed at both ends subject to an increase in temperature what happens to the strain? Because it is fixed the total strain has become zero. Thus

$$\varepsilon = \varepsilon_T + \varepsilon_M = 0 \Rightarrow \varepsilon_M = -\varepsilon_T$$

This implies that the mechanical strain developed in the constrained bar is equal and opposite to the induced thermal strain.

Example 3.13: Two aluminum bars and one iron bar are cantilevered at their left and attached to a crossbar at the write end. Since the iron and aluminum have different coefficient of thermal expansion, they will tend to elongate different amounts causing the internal axial forces to differ. Calculate the normal stresses in the three bars when subjected to a 100° F temperature change. The modulus of Aluminum is $E_{\text{Al}} = 10^7$ psi and its CTE is $\alpha_{\text{Al}} = 13.3 \text{x} 10^{-6}$ °F^{-1} and $E_{\text{Fe}} = 28.5 \text{ x} 10^6$ psi and its CTE is $\alpha_{\text{Fe}} = 6.5 \text{ x} 10^{-6}$ °F^{-1}.

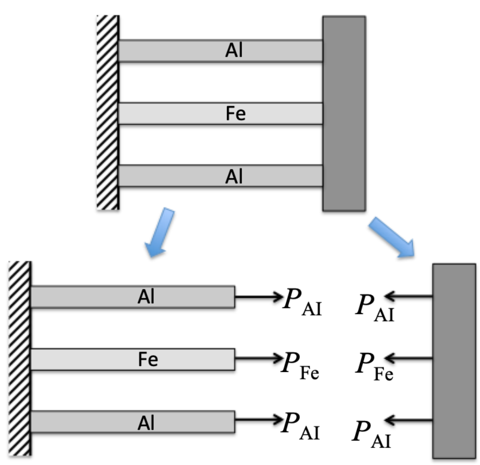


Figure 3.27 (top) Three bars, one of a different material cantilevered but connected by a crossbar at the right end.(bottom) The forces and free-body-diagrams obtained by separating the crossbar. The system is subject to a temperature change of ΔT = 100° F, each bar has a cross sectional area of A = 0.5 in^2 and a starting length of L = 10 in.

Solution: Summing forces on the crossbar yields

$$2P_{\text{Al}} + P_{\text{Fe}} = 0$$

Next consider the deformation relations. The change in length of one of the aluminum bars is

$$\delta_{\text{Al}} = \frac{P_{\text{Al}}L}{EA_{\text{Al}}} + \alpha_{\text{Al}}\Delta TL$$

The change in length of the iron bar is

$$\delta_{\text{Fe}} = \frac{P_{\text{Fe}}L}{EA_{\text{Fe}}} + \alpha_{\text{Fe}}\Delta TL$$

The compatibility relation, forced by both the iron and aluminum bars being constrained by the crossbar is $\delta_{\text{Fe}} = \delta_{\text{Al}}$. Substitution of the deformation equations into the compatibility relation yields that

$$\delta_{\text{Fe}} = \delta_{\text{Al}} \Rightarrow \frac{P_{\text{Fe}}L}{EA_{\text{Fe}}} + \alpha_{\text{Fe}}\Delta TL = \frac{P_{\text{Al}}L}{EA_{\text{Al}}} + \alpha_{\text{Al}}\Delta TL$$

This last expression combined with the equilibrium equations results into equations in the two unknows P_{Al} and P_{Fe}. Solving yields that P_{Al} = - 2000 lb. and P_{Fe} = 4000 lb. The normal stresses become:

$$\sigma_{\text{Al}} = \frac{P_{\text{Al}}}{A} = \frac{-2000}{0.5} = -4000 \text{ psi}$$

$$\sigma_{\text{Fe}} = \frac{P_{\text{Fe}}}{A} = \frac{4000}{0.5} = 8000 \text{ psi}$$

This problem illustrates the importance of understanding that different materials have different CTE values and how that effects stresses when the two materials have to work in concert.

Example 3.14: The walls holding the prismatic bar in Figure 3.28 will safely support loads of up to 30,000 psi. The coefficient of thermal expansion is 8 x 10^{-6} °F^{-1} and the elastic modulus is 28 x 10^6 psi. What is the safest temperature gradient it can be subject too?

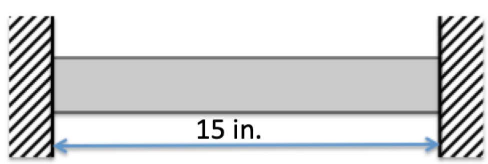


Figure 3.28 A fixed-fixed bar subject to a temperature gradient.

Solution Considering the deflection a temperature gradient of ΔT ($L\alpha\Delta T$) and the deflection due to the compressive force (PL/AE) the total deflection must be zero so that

$$\Delta L = 0 = L\alpha\Delta T - \frac{PL}{EA} \Rightarrow \Delta T = \frac{P/A}{\alpha E} = \frac{\sigma}{\alpha E}$$

Substitution of numbers:

$$\Delta T = \frac{30{,}000 \text{ lb/in}^2}{\left(8\times10^{-6} \ {}^\circ\text{F}^{-1}\right)\left(28\times10^{6} \text{ lb/in}^2\right)} = 134 \ {}^\circ\text{F}$$