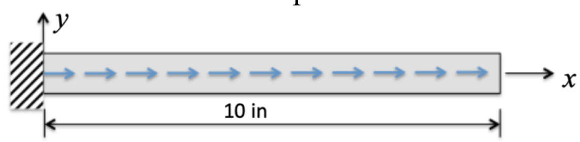
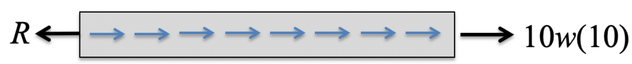
Distributed Loads

Example 3.15 Consider the distributed load of $w(x) = 50$ lb/in acting along the 10 in. bar shown in the figure. What is the reaction force at the clamped end?



Solution: The free-body-diagram is



The resultant distributed force is $w(x)(10) = (50 \text{ lb/in})(10 \text{ in}) = 500$ lb. Summing forces is then just

$$R - 500 = 0 \text{ or } R = 500 \text{ lb.}$$

Example 3.16 Again consider the 10 in bar above only this time let $w(x) = 50(100 - x^{2)}$ lb/in and compute the reaction force at the clamped end.

Solution: The free-body diagram is basically the same but this time to get the total force acting due to the distributed load the load function, $w(x)$, needs to be integrated along x to get the force it generates:

$$\sum F = 0 = -R + \int_0^{10} (100 - x^2)dx$$

$$\sum F = 0 \Rightarrow R = 50\int_0^{10}\left(100 - x^2\right)dx$$

Evaluating the integral yields:

$$R = 50\left(100x - \frac{x^3}{3}\right)\Bigg|_0^{10} = 50\left(100 \bullet 10 - \frac{10^3}{3}\right) = 33.3 \times 10^3 \text{ lb}$$

Here we revisit the results of applying a distributed load to a bar, but this time we concern ourselves with what happens inside the bar. Consider a bar of variable modulus, $E(x)$, and area, $A(x)$, as described in Figure 3.29a subject to a distributed load function $w(x)$. Let $u(x)$ denote the internal displacement at a point x from the left end of a differential element Δx as shown in Figure 3.29b and compute the strain $\varepsilon(x)$ as Δx approaches zero.

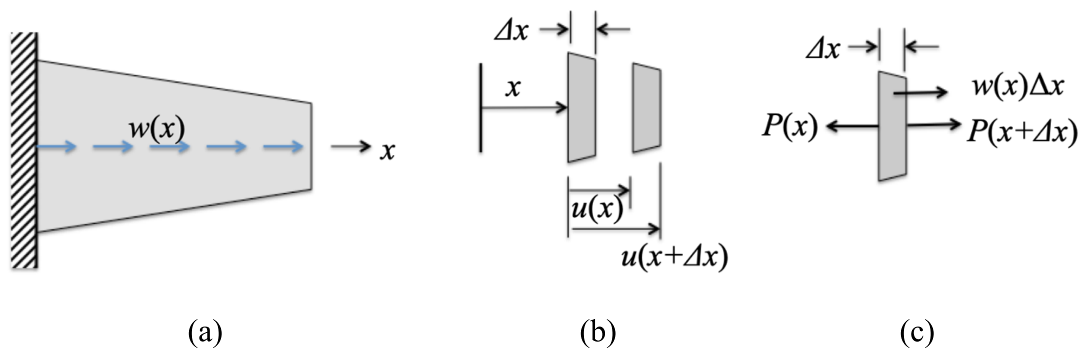


Figure 3.29 (a) A bar subject to an axial distributed load $w(x)$ in units of force per unit length. (b) Details of the change in displacement, $u(x)$, across an infinitesimal element of the bar. (c) a free-body-diagram showing the forces acting on the infinitesimal element Δx.

Recalling that strain is defined as a change in length divided by the original length, the strain in the infinitesimal element Δx is:

$$\frac{u(x+\Delta x)-u(x)}{\Delta x}$$

Recalling from calculus that in the limit that Δx tends to zero, the result is an expression for the strain at the point x:

$$\varepsilon(x)=\lim_{\Delta x\to 0}\frac{u(x+\Delta x)-u(x)}{\Delta x}=\frac{d\left[u(x)\right]}{dx}$$

This is called the *strain-displacement relationship* and describes how the strain is related to displacement along the bar.

Next consider the free-body-diagram of the element Δx illustrated in Figure 3.29c. Summing forces along x yields:

$$P(x+\Delta x)-P(x)+w(x)\Delta x=0$$

$$\Rightarrow w(x)=-\frac{P(x+\Delta x)-P(x)}{\Delta x}$$

Taking the limit of this last expression yields:

$$w(x)=-\lim_{\Delta x\to 0}\frac{P(x+\Delta x)-P(x)}{\Delta x}=-\frac{dP(x)}{dx}$$

This expression relates the applied force to the value of the normal force in the bar.

Recall the definition of normal stress

$$\sigma(x)=\frac{P(x)}{A(x)}=E(x)\varepsilon(x)$$

Multiplying this out and substituting in the strain-displacement relationship for $\varepsilon(x)$ results in:

$$P(x) = E(x)A(x)\varepsilon(x) \Rightarrow P(x) = E(x)A(x)\frac{du}{dx}$$

This is a first order differential equation relating the normal strain to the derivative of the local displacement. Taking the derivative of this last expression with respect to x and recalling that $w = - dP/dx$ yields:

$$w(x) = -\frac{d}{dx}\left(E(x)A(x)\frac{du}{dx}\right)$$

This is a second order differential equation relating the applied distributed force to the local deflection, $u(x)$. This expression holds for all x between 0 and L. In order to solve for $w(x)$, one effectively integrates twice which introduces two constants of integration. These constants are determined by enforcing two boundary conditions, one at $u(0)$ and one at $u(L)$. Likewise, the equation relating the internal normal force to the local deflection is a first order differential equation and it requires one constant of integration, i.e., one boundary condition at $u(0)$. There are other possibilities for boundary conditions and advanced ways to derive them, but these are beyond the current level of discussion.

For the distributed load there are the following possibilities for boundary conditions depending on the configuration. Either the displacement or its slope can be zero.

$$u(0) = 0, \text{or } \frac{du(0)}{dx} = 0$$

and

$$u(L) = 0, \text{or } \frac{du(L)}{dx} = 0$$

where the derivative is taken before substitution of $x = L$ into the function. Consider a bar cantilevered on the left end (i.e. at $x = 0$). Because the bar is fixed at $x = 0$, there cannot be any displacement, so the boundary condition becomes $u(0) = 0$. At the free end there is no point force so that $P(L) = 0$, provided $A(L) = E(L) \neq 0$. Then from the strain-displacement relationship this implies that the boundary condition at the free end must be zero because there is no strain at the free end:

$$\frac{du(L)}{dx} = 0$$

provided neither $A(x)$ or $E(x)$ are zero. Because of the this, two boundary conditions for a cantilever bar are

$$u(0) = 0 \text{ and } \frac{du(L)}{dx} = 0$$

For a bar fixed at both ends the two boundary conditions are

$$u(0) = u(L) = 0$$

Example 3.17 The 2-meter long, prismatic bar illustrated in Figure 3.30 has a constant cross-sectional area, $A(x) = 0.03$ m^2, and constant modulus $E(x) = 200 \times 10^9$ Pa. It is subject to a distributed axial load of $w(x) = 12(1 + 0.4x)$ MN/m. Calculate (a) the axial force P as a function of x, and (b) the bar's resulting change in length.

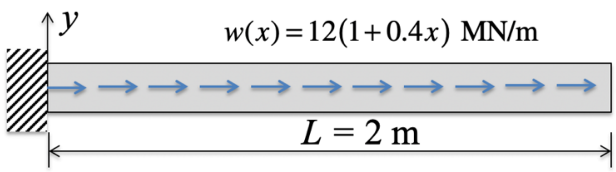


Figure 3.30 A cantilevered bar subject to the distributed force given with constant cross-sectional area and modulus.

Solution: To get to the axial force the equation relating the applied distributed force to the local deflection needs to be integrated and solved for du/dx. Substitution of the given function for $w(x)$ into the equation relating the applied distributed force to the local deflection yields:

$$w(x) = -\frac{d}{dx}\left(E(x)A(x)\frac{du}{dx}\right) \Rightarrow -EA\frac{d}{dx}\left(\frac{du}{dx}\right) = 12(1+0.4x)$$

Integrating both sides of this equation using indefinite integration yields:

$$-\int EA\frac{d}{dx}\left(\frac{du}{dx}\right)dx = \int 12(1+0.4x)dx$$

Evaluating the integrals yields:

$$-12\left(x+0.2x^2\right)+C_1 = EA\frac{du}{dx}$$

where C_1 is the yet to be determined unknown constant of integration. Integrating this again yields:

$$-12\int\left[\left(x+0.2x^2\right)+C_1\right]dx = EA\int\frac{du}{dx}$$

Evaluating the integrals and solving for $u(x)$ yields:

$$u(x) = \frac{1}{EA}\left[-12\left(\frac{1}{2}x^2+\frac{0.2}{3}x^3\right)+C_1x+C_2\right]$$

where C_2 is the yet to be determined unknown constant of integration. Next the boundary conditions are applied to determine the two constants of integration. Applying $u(0) = 0$ to this last expression yields: $C_2 = 0$. Recall $du(L)/dx = 0$ at the free end. Thus, applying the boundary condition at $x = L$ yields:

$$\left[-12\left(L+0.2L^2\right)+C_1\right] = EAu'(L) = 0$$

$$\Rightarrow C_1 = 12\left(L+0.2L^2\right) = 33.6$$

where u' is shorthand for du/dx. Thus, the expression for the displacement along the bar is:

$$u(x) = \frac{1}{EA}\left[-12\left(\frac{1}{2}x^2 + \frac{0.2}{3}x^3\right) + 33.6x\right]$$

The normal force along the bar is:

$$P(x) = -12\left(x + 0.2x^2\right) + 33.6$$

To find the total elongation, compute $u(2)$:

$$u(2) = \frac{1}{EA}\left[-12\left(\frac{1}{2}2^2 + \frac{0.2}{3}2^3\right) + 33.6 \bullet 2\right] = 6.133 \text{ mm}$$

What happens to the boundary conditions if there is a point force at the free end? This will change the boundary condition at $x = L$. Consider the cantilevered bar of Figure 3.31 subject to both a distributed force, $w(x)$, and a point force, F, at the free end. Consider and infinitesimal slice just inside the right end as illustrated in Figure 3.31b. Summing forces in the x direction yields:

$$\sum_x f = 0 \Rightarrow F + w(L)dx - P(L) = 0$$
$$\lim_{dx \to 0}\left[F + w(L)dx - P(L)\right] = 0 \Rightarrow P(L) = F$$

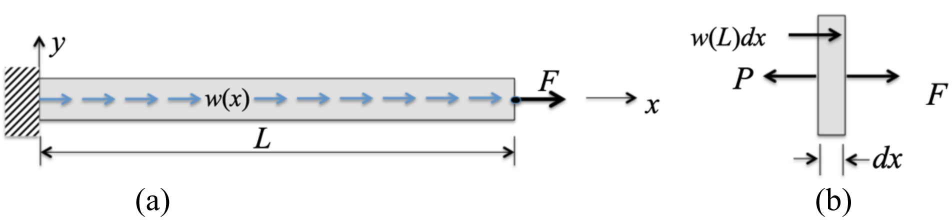


Figure 3.31 (a) A cantilever bar with a point force applied to its tip and a distributed force. (b) a free-body-diagram of the differential element near the tip.

At the right end the boundary condition is still $u(0) = 0$, and now the second boundary condition becomes $P(L) = F$. Should the point force be applied in the negative x direction then the right end boundary condition becomes $P(L) = - F$.

Example 3.18: Consider the tapered, non-prismatic bar of Figure 3.32, fixed at one end and with a point force of $F = 10$ kN at the free end. The thickness is $b = 0.02$ m, the modulus is constant throughout at $E = 200$ GPa and the area is variable defined by the height function

$$h(x) = h_1 + \frac{h_2 - h_1}{L}x$$

where $h_1 = 0.3$ m, $h_2 = 0.1$ m and $L = 1$ m. Calculate the total elongation.

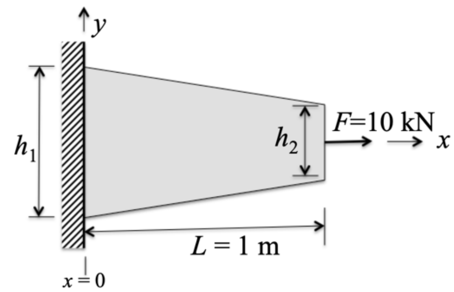


Figure 3.32 A tapered bar subject to a point force.

Solution: Substitution of the given values the area function becomes

$$A(x) = 0.02\left(0.3 + \frac{0.1 - 0.3}{1}x\right) = 0.02(0.3 - 0.2x)$$

Recall that

$$w(x) = 0 = \frac{d}{dx}\left[E(x)A(x)\frac{du}{dx}\right]$$

where $u(x)$ is the internal displacement. Integrating each side of this expression using an indefinite integration yields:

$$C_1 = EA(x)\frac{du}{dx} \Rightarrow du = \frac{C_1}{EA(x)}dx$$

Integrating this last expression again results in:

$$u(x) = \frac{C_1}{E}\int\frac{dx}{A(x)} \Rightarrow \frac{C_1}{E}\int\frac{dx}{0.02(0.3 - 0.2x)} = \frac{C_1}{E}[-250\ln(1.5 - x)] + C_2$$

There are now two constants of integration that need to be evaluated by using the boundary conditions. Note that at the end point $x = L$, the boundary condition is (recall prior discussion) $P(L) = F$. However, $P(L) = EA(L)du/dx$, so that

$$C_1 = EA(L)\frac{du}{dx} = P(L) = F = 10 \text{ kN}$$

The fixed boundary condition is $u(0) = 0$, so that:

$$u(0) = 0 = \frac{-250(10{,}000)}{2 \times 10^9}\ln(1.5) + C_2 \Rightarrow C_2 = 5.07 \times 10^{-6}$$

Thus

$$u(x) = -1.25 \times 10^{-5}\ln(1.5 - x) + 5.07 \times 10^{-6} \text{ m}$$

The deflection is just $\delta = u(L) - u(0) = u(L)$ and:

$$\delta = -1.25 \times 10^{-5} \ln(1.5 - 1) + 5.07 \times 10^{-6} = 0.0137 \text{ mm}$$

The stress is also easily calculated now by:

$$\sigma(x) = \frac{P(x)}{A(x)} = \frac{C_1}{A(x)} = \frac{F}{A(x)} = \frac{10 \text{ kN}}{0.02(0.3 - 0.2x)}$$

Example 3.19: Recall Example 3.12 of the bar standing up (see Figure 3.24 and 3.33) with a distributed load caused by gravity and this time solve it using the approach if setting up the displacement as a differential equation. This time use a mass density instead of a weight density as before and model the weight as a distributed load. Given the modulus *E,* the length *L,* the area *A* and the mass density ρ, compute the elongation (computed before), the displacement along the bar as well as the maximum stress.

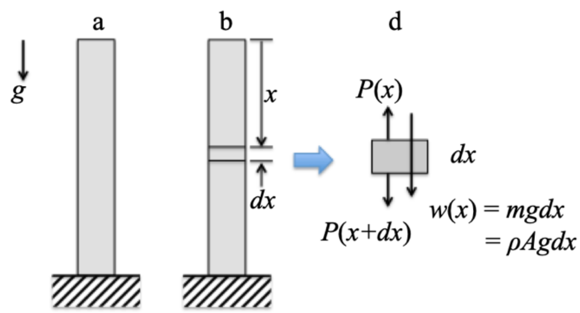


Figure 3.33 (a) A bar subject to a distributed gravitational force. (b) An infinitesimal element of the bar. (c) the free-body-diagram of the infinitesimal element showing the distributed force per unit length.

Solution: Recall that

$$w(x) = -\frac{d}{dx}\left(E(x)A(x)\frac{du}{dx} \right)$$

The distributed force in this case is gravity so this becomes

$$w(x) = -\frac{d}{dx}\left(EA\frac{du}{dx} \right) = \rho A g$$

Integrating yields:

$$EA\frac{du}{dx} = -\rho A g x + C_1$$

Integrating again yields:

$$EAu(x) = -\frac{\rho A g x^2}{2} + C_1 x + C_2$$

Next apply the boundary conditions to evaluate the constants of integration. This are $u'(0) = 0$ and $u(L) = 0$. The derivative of $u(x)$ evaluated at $x = 0$ is

$$\left.\frac{du}{dx}\right|_0 = -\frac{\rho Ag(0)}{EA} + \frac{C_1}{EA} = 0 \Rightarrow C_1 = 0$$

Evaluating $u(L) = 0$ yields:

$$u(L) = -\frac{\rho AgL^2}{2EA} + C_2 = 0 \Rightarrow C_2 = \frac{\rho AgL^2}{2EA}$$

Thus, the final expression for the displacement along the bar is

$$u(x) = -\frac{\rho gx^2}{2E} + \frac{\rho gL^2}{2E}$$

Thus. the deflection at the top ($x = 0$) is just:

$$u(0) = \frac{\rho gL^2}{2E}$$

This is the amount the bar shrinks due to its own weight.

The stress is

$$\sigma(x) = E\frac{du}{dx} = -\rho gx$$

So, the stress is zero at the top and maximum at the bottom (pointing up) of value $\sigma(x) = -\rho AgL$.

Next consider the same bar but fixed at both top and bottom. This of course will be statically indeterminate. However, using the method of computing the deflection through the bar does not require the use of a compatibility equations.

Example 3.20: Compute the maximum displacement and stress in the bar of Figure 3.34, under a gravitational load.

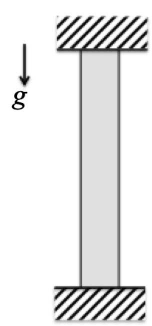


Figure 3.34 A fixed-fixed bar of modulus E, length L, area A and mass density ρ. This configuration is statically indeterminate, but the method of computing the deflection still works without introducing compatibility conditions.

Solution: Again, start with the differential equation for deflection equal to the load function as before acting on the differential element dx:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = -\rho Ag$$

Integrating and dividing through by *EA* yields:

$$EA\frac{du}{dx} = -\rho Agx + C_1 \Rightarrow \frac{du}{dx} = \frac{-\rho Agx}{EA} + \frac{C_1}{EA}$$

Integrating again yields

$$u(x) == \frac{-\rho Agx^2}{2EA} + \frac{C_1}{EA}x + C_2$$

In this case the boundary conditions are $u(0) = u(L) = 0$. The boundary condition at 0 yields:

$$u(0) = \frac{-\rho Ag0^2}{2EA} + \frac{C_1}{EA}0 + C_2 = 0 \Rightarrow C_2 = 0$$

The boundary condition at *L* yields:

$$u(L) = \frac{-\rho AgL^2}{2EA} + \frac{C_1}{EA}L = 0 \Rightarrow C_1 = \frac{\rho AgL}{2}$$

Rewriting the solution for $u(x)$ with the constants evaluated yields:

$$u(x) = \frac{\rho g}{2E}x(L - x)$$

The stress function becomes:

$$\sigma(x) = E\frac{du}{dx} = \rho g\left(\frac{L}{2} - x\right)$$

Examining this last expression shows that the stress starts out at the top ($x = 0$) with a maximum value of $\rho gL/2$ in tension, hits zero at $x = L/2$ and continues down the length of the bar in compression ending with the value $-\rho gL/2$.

On the other hand, the maximum value of the deflection occurs a $u' = 0$, which occurs at $x = L/2$. The value of $u_{\max}$ is

$$u_{\max}(x) = \frac{\rho gL^2}{8E}$$

The value of the deflection is of course zero at the end points

---

Example 3.21: This problem examines how to compute the internal deflection when a bar is subject to both a distributed force (gravity in this case) and a point force attached along its interior (at the midpoint of the bar in this case) as described in Figure 3.35. Calculate the displacement and stress functions internal to the bar symbolically in terms of the cross-sectional area, A, the modulus E, the length L, the density per unit area ρ and the acceleration due to gravity g.

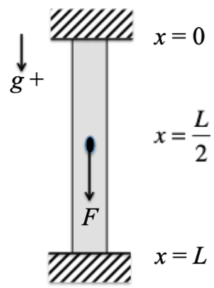


Figure 3.35 The bar of Figure 3.34 subject to a point force midway along its length. The bar has cross-sectional area, A, modulus E, length L, density per unit area ρ and is subject to the force, F at its midpoint and the distributed force of gravity mg.

Solution: The approach here is to divide the bar into two parts, above and below the point of application of the applied force F. Solve for $u(x)$ in each region and use *matching conditions* at the point of application of the external force. The two differential equations are

$$
\begin{aligned}
&\frac{d}{dx}\left(EA\frac{du_1}{dx}\right) = -\rho A g, \quad 0 \le x \le \frac{L}{2} \\
&\frac{d}{dx}\left(EA\frac{du_2}{dx}\right) = -\rho A g, \quad \frac{L}{2} \le x \le L
\end{aligned}
$$

Here $u_1(x)$ is the displacement from the top to the midpoint and $u_2(x)$ is the displacement from the midpoint to the bottom of the bar. Integrating each twice as before but realizing that each will have different constants of integration (4 total) yields:

$$
\left.
\begin{aligned}
u_1(x) &= -\frac{\rho g x^2}{2E} + \frac{C_1}{EA}x + C_2 \\
\sigma_1(x) &= \frac{P_1(x)}{A} = -\rho g x + C_1
\end{aligned}
\right\}, \quad 0 \le x \le \frac{L}{2}
$$

$$
\left.
\begin{aligned}
u_2(x) &= -\frac{\rho g x^2}{2E} + \frac{C_3}{EA}x + C_4 \\
\sigma_2(x) &= \frac{P_2(x)}{A} = -\rho g x + C_3
\end{aligned}
\right\}, \quad \frac{L}{2} \le x \le L
$$

To evaluate the constants of integration we have the two end conditions $u_1(0) = 0$ and $u_2(L) = 0$. But this only gives two conditions. We need two more. First, we enforce continuity of $u(x)$ at the point $L/2$: $u_1(L/2) = u_2(L/2)$. Next examine a force balance at $L/2$. A free-body diagram of the element dx located at the point $x = L/2$ is given in Figure 3.35

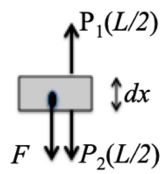


Figure 3.35 A free-body-diagram of the element dx located at the point $x = L/2$.

Summing the forces in the x direction in Figure 3.35 yields the fourth condition used to determine the constants of integration (remember + is down):

$$P_2(L/2) - P_1(L/2) = -F$$

Using these four conditions and solving for the 4 constants of integration yields:

$$u_1(0) = 0 \Rightarrow -\frac{\rho g 0^2}{2E} + \frac{C_1}{EA}0 + C_2 \Rightarrow \underline{C_2 = 0}$$

$$u_2(L) = 0 \Rightarrow \underline{-\frac{\rho g L^2}{2E} + \frac{C_3}{2E}L + C_4 = 0}$$

$$u_1(L/2) = u_2(L/2) \Rightarrow -\frac{\rho g}{2E}\left(\frac{L}{2}\right)^2 + \frac{C_1}{EA}\left(\frac{L}{2}\right) = -\frac{\rho g}{2E}\left(\frac{L}{2}\right)^2 + \frac{C_3}{EA}\left(\frac{L}{2}\right)^2 + C_4$$

$$\underline{\Rightarrow C_4 = \frac{L}{2EA}\left(C_1 - C_3\right)}$$

$$P_2(L/2) - P_1(L/2) = -F \Rightarrow -\rho g A\left(\frac{L}{2}\right) + C_3 + \rho g A\left(\frac{L}{2}\right) - C_1 = -F$$

$$\Rightarrow \underline{C_3 - C_1 = -F}$$

The four underlined expressions yield the four equations in the four unknowns. The last two together yields:

$$C_4 = \frac{FL}{2EA}$$

With this value of C_4 known the condition at $u_2(L) = 0$ determines C_3 to be

$$C_3 = \frac{1}{2}\left(\rho g A L - F\right)$$

Last, the expression $C_3 - C_1 = -F$ with the previous relation yields:

$$C_1 = \frac{1}{2}\left(\rho g A L + F\right)$$

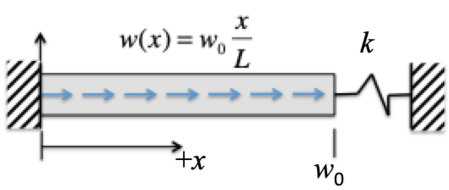
Thus, the expression for the displacement and stress becomes

$$\left.\begin{aligned} u_1(x) &= -\frac{\rho g x^2}{2E} + \frac{\rho g A L + F}{2EA}x \\ \sigma_1(x) &= \frac{P_1(x)}{A} = -\rho g x + \frac{\rho g A L + F}{2A} \end{aligned}\right\},\quad 0 \le x \le \frac{L}{2}$$

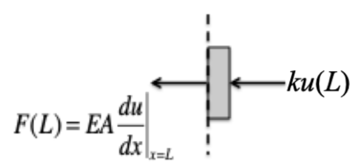
$$\left.\begin{aligned} u_2(x) &= -\frac{\rho g x^2}{2E} + \frac{\rho g A L - F}{2EA}x \\ \sigma_2(x) &= \frac{P_2(x)}{A} = -\rho g x + \frac{\rho g A L - F}{2A} \end{aligned}\right\},\quad \frac{L}{2} \le x \le L$$

This next example illustrates how to incorporate a spring at the boundary, often encountered in dynamics problems and a distributed force at the same time. The point is how to include and attached system at a boundary. A spring develops a forced kx where the constant k is the *stiffness* as defined earlier and x is the displacement of the end of the spring.

Example 3.22: The bar-spring system of Figure 3.36 subject to a distributed force of $w(x) = w_0 x/L$, where w_0 is a given constant. Compute the deflection at the end of the bar, $u(L)$, in terms of the symbolic constants cross-sectional area, A, modulus E, length L, w_0 and the stiffness k.



(a)



(b)

Figure 3.36 (a) Schematic of a bar attached to a spring subject to a distributed load. (b) The free-body-diagram of the infinitesimal element at the tip of the bar assume that $w(L)dx$ is negligible.

Solution: The differential equation for the internal displacement is

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = -w(x) = -w_0\frac{x}{L}$$

Integrating this expression yields:

$$EA\frac{du}{dx} = -\frac{w_0}{2L}x^2 + C_1$$

Integrating again yields:

$$EAu(x) = -\frac{w_0}{6L}x^3 + C_1 x + C_2$$

The boundary condition at the clamped end, $u(0) = 0$ yields $C_2 = 0$. Summing forces on the free-body-diagram of infinitesimal element at the tip given in Figure 3.36b yields:

$$EA\left.\frac{du}{dx}\right|_{x=L} + ku(L) = 0$$

Using the expression above for the first derivative of *u* this becomes:

$$-ku(L) = \frac{-w_0}{2L}L^2 + C_1$$

Using the second integral evaluated at *L* yields:

$$u(L) = \frac{k}{EA}\frac{w_0}{6L}L^3 - \frac{kL}{EA}C_1 = -\frac{w_0 L}{2} + C_1$$

Solving for C_1 and simplifying yields:

$$C_1 = \frac{w_0 L}{2}\left[\frac{1 + {}^{kL}\!/_{3EA}}{1 + {}^{kL}\!/_{EA}}\right]$$

So that $u(x)$ becomes:

$$u(x) = \frac{w_0 L^2}{2EA}\left[-\frac{1}{3}\left(\frac{x}{L}\right)^3 + \frac{1 + {}^{\bar{k}}\!/_{3}}{1 + \bar{k}}\left(\frac{x}{L}\right)\right], \text{ where } \bar{k} = \frac{kL}{EA}$$

The trick here and important concept in using the differential equation method is to properly identify the boundary conditions.