Vv285 Honors Mathematics III

Functions of Multiple Variables

Term Project

Date Due: 12:00 PM, Thursday, the 2nd of August 2018

General Information

The goal of the projects is to allow you to perform some in-depth research on an interesting subject, in a much more intensive way than is possible in standard course work. The projects are intentionally open-ended, i.e., after an initial part that will get you started, you determine yourselves in what direction and how far you want to take the project. You should *choose one of the three project options*, not do all of them.

It is strongly recommended that you do not leave the entire project to the last minute but rather commence work on individual parts as soon as you are able to do so.

Group Work

You will be divided into groups of *4–5 students* each.

> Each group member must be familar with and have contributed to each part of the project report. **You may not divide up the work in such a way that only certain members are invovled with certain parts.** In the event of an Honor Code violation (plagiarism or other), all members of the group will be held equally responsible for the violation. Exceptions may only be made, at my discretion, in exceptional situations.

It is therefore all group members' duty to ensure that all collaborators' contributions are plausibly their own and to check on all collaborators' work progress and verify their contributions within reason.

Project Report

The term project will be submitted **electronically only** as a typed report. Handwritten submission will not be accepted! It is recommended that you use a professional type-setting program (such as LaTeX) for your report. Unless you are able to ensure a unified font size and style for formulas and text in Microsoft Word, use of Word is *not recommended*.

Grading Policy

This term project accounts for 10% of the course grade; it will be scored based on

- **Form (2 points):** Does the report contain essential elements, such as a cover page (with title, date, list of authors), a synopsis (abstract giving the main conclusions of the project), table of contents, clear section headings, introduction, clear division into sections and appendices with informative titles and bibliography (if applicable)? Are the pages numbered? Are the text and formulas composed in a unified font? Are all figures (graphs and images) clearly labeled with identifiable source?
- **Language (2 points):** Is the style of english appropriate for a technical report? Do not treat the project as an assignment and simply number your results like part-exercises. Your text should be a single, coherent whole. The text should be a pleasant read for anyone wanting to find out about the subject matter.

  Errors in grammar and orthography (use a spell-checker!) will be penalized. Make sure that the report is interesting to read. Avoid simply repeating sentences by cut-and-paste.
- **Content (6 points):** Are the mathematical and statistical methods and deductions clearly exhibited and easy to follow? Are the conclusions well-supported by the mathematical analysis? It is important to not just copy calculations from elsewhere, but to fully make them your own, adding details and comments where necessary.

All group members will receive the same grade for the term project. (Exceptions are possible in special circumstances.)

On Plagiarism

Study JI's Honor Code carefully. **Any** information from third parties (books, web sites, even conversations) that you use in your project must be accounted for in the bibliography, with a reference in the text. Follow the rules regarding the correct attribution of sources that you have learned in your English course (e.g., Vy100, Vy200). All members of a group are jointly responsible for the correct attribution of all sources in all parts of the project essay, i.e., any plagiarism will be considered a violation of the Honor Code by all group members. Every group member has a duty to confirm the origin of any part of the text.

The following list includes some specific examples of plagiarism:

- Use of any passage of three words or longer from another source without proper attribution. Use of any phrase of three words or more must be enclosed in quotation marks ("example, example, example"). This excludes set phrases (e.g., "and so on", "it follows that") and very precise technical terminology (e.g., "without loss of generality", "reject the null hypothesis") that cannot be paraphrased,
- Use of material from an uncredited source, making very minor changes (like word order or verb tense) to avoid the three-word rule.
- Inclusion of facts, data, ideas or theories originally thought of by someone else, without giving that person (organization, etc.) credit.
- Paraphrasing of ideas or theories without crediting the original thinker.
- Use of images, computer code and other tools and media without appropriate credit to their creator and in accordance with relevant copyright laws.

Project Option 1: A Perfect Pendulum

Pendula[1] have been used for time-keeping and in physical and engineering applicastions for millenia. Their periodic motion and simple construction made them well-suited for the construction of clockworks. However, they poseess a basic drawback when used for these purposes: their periods depend on the initial displacement. The present project investigates this effect in detail.

i) Give a definition of a "mathematical pendulum" of length l and mass m. What is the difference to a "physical pendulum"? Show that the energy of a mathematical pendulum is given by

$$E(\theta,\dot\theta)=\frac{1}{2}ml^2\dot\theta^2+mgl(1-\cos\theta),$$

where $\theta=\theta(t)$ is the angle of displacement and $\dot\theta$ is the time-derivative of θ. Here g is the acceleration due to gravity, assumed constant. Derive the *pendulum equation*

$$\frac{d^2\theta(t)}{dt^2}+\frac{g}{l}\sin(\theta(t))=0.$$

ii) Suppose that a pendulum is held at an angle $\theta(0)=\theta_0$ at time $t=0$ and then let go. Show that

$$|\dot\theta|=\sqrt{\frac{2g}{l}(\cos\theta-\cos\theta_0)}.$$

For a single period, the map $t\mapsto\theta$ is bijective and hence invertible. Use the Inverse Function Theorem of Vv186 to derive the equation

$$T=4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}$$

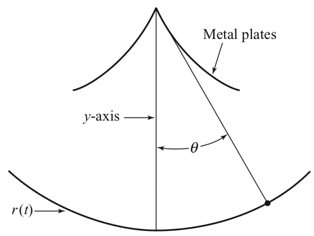
Using an appropriate substitution, show that

$$T=4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\sin^2(\theta_0/2)\sin^2\phi}}$$

Note that this is a complete elliptic integral of the first kind (see Assignment 12 of Vv186). Give a formula relating the period of the pendulum to the arithmetic-geometric mean. Using an appropriate series expansion, show that the period is approximately given by

$$T\approx 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_0^2}{16}\right)\approx 2\pi\sqrt{\frac{l}{g}}.$$

The mass of a simple pendulum moves along a circular path, held there by its string. Suppose the shape of this path could be modified (e.g. by varying the effective length of the string in some way, e.g., using plates). The famous mathematical *tautochrone problem* (sometimes also called the *isochrone problem*) asks whether there exists a path along which the period of such a pendulum would not depend on θ_0. This problem was solved by Christiaan Huygens in 1659 and the main part of this project is concerned with its solution.



Pendulum constrained by plates. Figure taken from [2].

The following questions are based on the section "Huygens Discovers the Isochrone" of [2], which is freely available for download at http://sofia.nmsu.edu/~history/.

iii) Explain why the tautochone must be a path along which

$$\frac{d^2s}{dt^2}=-ks, \tag{1}$$

where t is time and s is the path length. Does a simple pendulum (as discussed in parts i) and ii) above) satisfy this relation?

[1]Either *pendulums* or *pendula* is accepted as the plural of pendulum.

iv) Follow Exercise 3.2 of [2] to deduce that a curve satisfying (1) is a cycloid and give the parametric equations (parametrization) of this cycloid. Note that a proof that a cycloid curve satisfies (1) can be commonly found, e.g., in [3]. However, such a proof depends on already knowing the solution curve. You are asked here to deduce the cycloid from the equation alone.

v) Follow the ideas discussed in Exercises 3.4 and 3.5 of [2] to describe the construction of a tautochronous pendulum.

References

[1] A. Belyaev. Plane and space curves. Curvature. Curvature-based features. www.math.utah.edu/~palais/dnamath/04gm_curves.pdf, 2006. Web. Accessed July 10$^{\text{th}}$, 2016.

[2] R. Knoebel, A. Laubenbacher, R. Lodder, and D. Pengelley. *Mathematical Masterpieces: Further Chronicles by the Explorers*. Undergraduate Texts in Mathematics. Springer, 2007. Several sections can be downloaded here.

[3] E. W. Weisstein. Tautochrone problem. From MathWorld–A Wolfram Web Resource. Web. Accessed April 11$^{\text{th}}$, 2012.

Project 2: The Caustic in the Coffee Cup

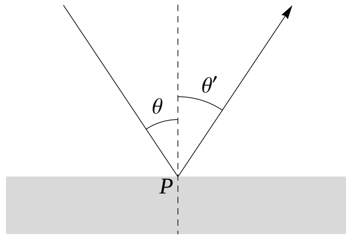
When light shines into a cup of coffee in a certain way, a so-called *coffee cup caustic* appears: a bright circle-like shape of light with a cusp (see figure at right). The goal of this project is to understand this optical phenomenon.

Janot, Gérard. Caustic appearing in a cup of milk coffee. Wikimedia Commons. Wikimedia Foundation. Web. 10 April 2012

In *geometrical optics* one assumes that light "particles" propagate along paths, called *light rays*. This approach does not refklect the wave nature of light (described by the Maxwell equations), but can be justified as an approximation. Since the treatement of light rays in high-school is sufficent for us here, we will not discuss the physical background here.

Consider a single light ray reflected at a point P on a flat surface. The the law of reflection states that

$$\text{angle of incidence } \theta = \text{angle of reflection } \theta'$$



where the angles are with respect to the vertical normal through P. If the surface is curved, the law of reflection remains unchanged, and the reference line becomes the normal vector to the surface curve at P.

i) Let $\{\gamma(p,\cdot)\colon p \in J \subset \mathbb{R}\}$ be a family of parametrized curves, i.e., for every fixed $p \in I$, the points $\gamma(p,t)$, $t \in I \subset \mathbb{R}$, define a cruve in $\mathbb{R}^2$. As an example, plot 20 members of the family of circles $\gamma(p,t) = (p\cos t, p\sin t + 2p)$, $t \in [0, 2\pi)$, $p > 0$.

ii) A curve that is tangent at every point to one of the curves $\gamma(p,\cdot)$ is called an *envelope* of the family of curves. Show that an envelope can be found by solving the equation

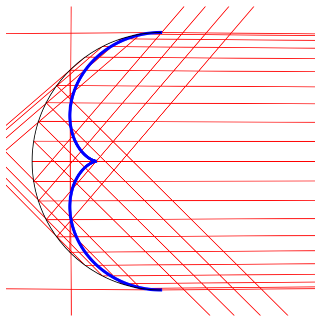
$$\frac{\partial \gamma_1}{\partial p}\frac{\partial \gamma_2}{\partial t} = \frac{\partial \gamma_1}{\partial t}\frac{\partial \gamma_2}{\partial p}$$

where $\gamma(p,t) = (\gamma_1(p,t), \gamma_2(p.t))$. Show that the envelope of the family of circles considered in i) is given by $y = \sqrt{3}|x|$.

iii) The envelope of a reflected family of light rays is called a *catacaustic*. Show that the catacaustic for parallel light rays reflected by a semi-circle is given by the parametric equations

$$x = \frac{1}{4}(3\cos t - \cos(3t)), \qquad y = \frac{1}{4}(3\sin t - \sin(3t))$$

for t in a suitable range. This curve is called a *nephroid*.

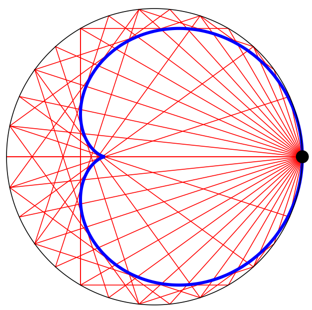


Weisstein, Eric W. Circle Catacaustic. MathWorld–A Wolfram Web Resource. Web. 10 April 2012

iv) In the coffee cup pictured above, the light source is directly above the rim of the cup, so that the light rays emanate from a point on the circle. Show that the catacaustic is given by the *cardioid*

$$x = \frac{2}{3}\cos t(1 + \cos t) - \frac{1}{3}, \qquad y = \frac{2}{3}\sin t(1 + \cos t)$$

for $t \in [0, 2\pi)$.



Weisstein, Eric W. Circle Catacaustic. MathWorld–A Wolfram Web Resource. Web. 10 April 2012

v) Suppose that the family of light rays is parametrized by $\gamma(p,t)$, where $p \in I$ denotes a given ray and $t \in \mathbb{R}$ denotes the point along the ray. Suppose further that light particles (photons) travel along each ray at a constant rate. Explain why the inverse of the determinant of the Jacobian of the map $\gamma\colon I \times \mathbb{R} \to \mathbb{R}^2$ can be used to model the "photon density" at a point $x \in \mathbb{R}^2$. Furthermore, show that this Jacobian vanishes at points of a caustic, leading to an "infinite" photon density.[2] This explains the brightness of the caustic that makes it visible in the picture of the coffee cup above.

[2] Of course, the intensity is not really infinite, but rather the geometrical-optical approximation breaks down.

vi) Create a coffee cup caustic using coffee or another experimental set up. Experiment with different positions of the light source. A beautiful example can be found at [3].

References

[1] A. Belyaev. Plane and space curves. Curvature. Curvature-based features. www.math.utah.edu/~palais/dnamath/04gm_curves.pdf, 2006. Web. Accessed July 10th, 2016.

[2] B. J. Loe and N. Beagley. The coffee cup caustic for calculus students. *The College Mathematics Journal*, 28(4):pp. 277–284, 1997. http://www.jstor.org/stable/2687149.

[3] G. Weir. The coffee cup caustic. http://www.graceweir.com/coffeecupcaustic.html, 2005. Web. Accessed July 10th, 2016.

[4] E. W. Weisstein. Circle catacaustic. From MathWorld–A Wolfram Web Resource. Web. Accessed April 11th, 2012.

Project 3: $\pi = 3.14159265\ldots$

The number π has fascinated mathematicians for millenia. From ancient approximations through Archimedes's circle approximations by polygons through techniques based on calculus, advances in mathematics have been accompanied by new methods for finding ever more decimal digits of π. We have already discovered and proven some fascinating identites, such as

- The Leibniz series $\displaystyle\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$
- The Euler series $\displaystyle\frac{\pi^2}{6} = \sum_{n=0}^{\infty} \frac{1}{n^2}$
- The Wallis formula $\displaystyle\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9}\cdots$

Unfortunately, these formulas are not very practical for actually calculating π to any significant precision because the infinite series/products converge extremely slowly. Both the Leibniz and the Euler series, after summing 10,000 terms, only give $\pi \approx 3.141$. The goal of this project is to investigate efficient algorithms for decimal digits of π. The following exercises are based entirely on [1].

i) Read Section 11.1 of [1]. The Leibniz formula is based on the arctangent series $\arctan x = \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$. *Machin's formula*,
$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right),$$
published in 1706 is a significant improvement on the Leibniz series, because on the one hand $1/239$ is very small and the arctangent series converges rapidly and on the other hand $x = 1/5$ is well-suited for decimal calculations in the arctangent series. Prove Machin's formula as follows: Let $\theta = \arctan(1/5)$. Show that
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{5}{12}$$
and $\tan(4\theta) = 1 + \frac{1}{119}$. Finally establish
$$\tan\left(4\theta - \frac{\pi}{4}\right) = \frac{1}{239}.$$
Many similar formulas based on the arctangent were found and used up to the 1970s for calculations of decimal digits of π. Up to a million digits were obtained in this way.

ii) Very efficient algorithms can be derived from the theory of the arithmetic-geometric mean. We will study Algorithm 2.1 of [1]: Let $x_0 = \sqrt{2}$, $\pi_0 = 2 + \sqrt{2}$, $y_1 = \sqrt[4]{2}$. Define
$$x_n = \frac{1}{2}\left(\sqrt{x_{n-1}} + \frac{1}{\sqrt{x_{n-1}}}\right), \qquad y_{n+1} = \frac{y_n\sqrt{x_n} + 1/\sqrt{x_n}}{y_n + 1}, \qquad \pi_n = \frac{x_n + 1}{y_n + 1}\pi_{n-1}$$
for $n \ge 1$. Then $\pi_n \searrow \pi$,
$$\pi_n - \pi < \frac{1}{10}(\pi_n - \pi)^2 \qquad \text{and} \qquad \pi_n - \pi < 10^{-2^{n+1}}.$$
Thus, in n iterations the algorithm gives at least 2^n decimal digits of π. The theory underlying this algorithm involves various kinds of elliptic integrals. This is the main part of the project: collect the necessary theory from [1] and prove this algorithm. Write an exposition introducing the different types of elliptic integrals and proving the necessary relations. Omit any information that you don't need (for example, theta functions do not play a role here and should not be mentioned). Your derivation should be detailed enough that any other student of Vv285 can follow it without additional references.

iii) Implement the algorithm of ii) in a programming language or a computer algebra system, e.g., Mathematica.

References

[1] J. M. Borwein and P. B. Borwein. *Pi and the AGM.* John Wiley & Sons, New York, 1987. http://as.wiley.com/WileyCDA/WileyTitle/productCd-047131515X.html A copy of this book is available from me upon request.