Note For Algebraic Geometry

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Preface

This note contains solutions to exercises from Hartshorne, as well as selected exercises from Algebraic Geometry taught by Prof. Dr. Daniel Huybrechts at the University of Bonn during the 2024/25 winter and 2025 summer semesters.

I am deeply grateful to Prof. Dr. Huybrechts for his guidance, without which learning the fundamentals of algebraic geometry in one year would not have been possible.

I also thank Wen Ge for kindly sharing the Overleaf document.

Many exercises remain unfinished, and I hope to complete them in the future.

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2.1.1 Preparations

First of all, we introduce some properties of adjoint functors.

Theorem 2.1.1. *Right-adjoint functors commute with limits; Left-adjoint functors commute with colimits.*

Proof. Ch V III Lemma 1.17 of [1]. □

Example 2.1.2 (Example of colimits and limits)**.** 1. Products are a kind of colimits; Coproducts are a kind of colimits.

2. Kernels are a kind of limits; Cokernels are a kind of colimits.

Proof. Ch VIII Example 1.10 and Example 1.11 of [1]. □

Theorem 2.1.3. *Right-adjoint functors are left-exact; Left-exact functors are right-exact.*

Proof. Ch VIII Claim 1.19 of [1]. □

Definition 2.1.4 (Additive Category)**.** An **additive category** is a category $\mathcal{C}$ that satisfies the following properties:

(i) **Zero Object**: $\mathcal{C}$ has a **zero object**, denoted 0, which is both an initial and terminal object. This means for any object X in $\mathcal{C}$, there are unique morphisms $0 \to X$ and $X \to 0$.

(ii) **Binary Products and Coproducts**: For any two objects X, Y in $\mathcal{C}$, the categorical product $X \times Y$ and the categorical coproduct $X \sqcup Y$ exist and are isomorphic. This common object is called the **biproduct** and is denoted $X \oplus Y$.

(iii) **Additive Hom-Sets**: For any two objects X, Y in $\mathcal{C}$, the set of morphisms $\mathrm{Hom}(X, Y)$ is equipped with the structure of an **abelian group**, and composition of morphisms is bilinear. That is, for morphisms $f, f_1, f_2 \in \mathrm{Hom}(X, Y)$ and $g, g_1, g_2 \in \mathrm{Hom}(Y, Z)$, we have:

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2,$$

$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f.$$

(iv) **Additive Identity**: The zero morphism $0 \in \mathrm{Hom}(X, Y)$ acts as the identity element for the abelian group structure on $\mathrm{Hom}(X, Y)$.

Definition 2.1.5 (Abelian Category)**.** An **abelian category** is an additive category $\mathcal{C}$ that satisfies additional properties, making it suitable for homological algebra. Specifically:

(i) **Kernels and Cokernels**: Every morphism $f : X \to Y$ in $\mathcal{C}$ has a **kernel** (denoted $\ker(f)$) and a **cokernel** (denoted $\mathrm{coker}(f)$).

(ii) **Normal Monomorphisms and Epimorphisms**:

- Every monomorphism is the kernel of its cokernel.
- Every epimorphism is the cokernel of its kernel.

(iii) **Finite Limits and Colimits**: $\mathcal{C}$ has all finite limits (e.g., pullbacks) and finite colimits (e.g., pushouts).

(iv) **Exact Sequences**: Short exact sequences behave as expected. A sequence $0 \to A \to B \to C \to 0$ is exact if $A \to B$ is a monomorphism, $B \to C$ is an epimorphism, and the image of $A \to B$ is the kernel of $B \to C$.

Corollary 2.1.6. *Adjoint functors preserve isomorphisms in* ***Abelian Category***

Proof. For the properties of Abelian category, see Ch IX, Section 1 of [1]. $\square$

To solve this problems, we just assume that $Sh_c(X)$ are abelian categories.

Theorem 2.1.7. *Let X be a topology space. For $i : Sh_C(X) \longrightarrow Psh_C(X)$ defined by identity and the sheafification $(-)^+ : Psh_C(X) \longrightarrow Sh_C(X)$, we have*

$$Hom_{Sh}(\mathcal{F}^+, \mathcal{G}) \cong Hom_{Sch}(\mathcal{F}, i(\mathcal{G}))$$

where $\mathcal{F} \in Psh(X)$ and $\mathcal{G} \in Sh(X)$. Thus $(-)^+ \dashv i$.

Proof. Define $\Phi : Hom_{Sh}(\mathcal{F}^+, \mathcal{G}) \longrightarrow Hom_{Psh}(\mathcal{F}, i(\mathcal{G}))$ by $f \mapsto f \circ i_{\mathcal{F}}$ where $i_{\mathcal{F}}$ is the canonical morphism $i_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F}^+$.

Define $\Psi : Hom_{Psh}(\mathcal{F}, i(\mathcal{G})) \longrightarrow Hom_{Sh}(\mathcal{F}^+, \mathcal{G})$ by $g \mapsto \bar{g}$ where $\bar{g}$ is gained by Proposition-Definition 1.2 of [5]:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "i_{\mathcal{F}}"] \arrow[dr, "g"'] & \mathcal{F}^+ \arrow[d, "\exists! \bar{g}"] \\ & i(\mathcal{G}) = \mathcal{G} \end{tikzcd}$$

- : $\Phi \circ \Psi = id$. By the univeral property, $g \mapsto \bar{g} \mapsto \bar{g} \circ i_{\mathcal{F}} = g$.
- : $\Psi \circ \Phi = id$. $f \mapsto f \circ i_{\mathcal{F}} \mapsto \overline{f \circ i_{\mathcal{F}}}$ which is equal to f by the universal property of sheafification.

$\square$

Theorem 2.1.8. *Let X be a topological space. Take $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$. By the universal property of sheaf, we have $\varphi^+ : \mathcal{F}^+ \longrightarrow \mathcal{G}^+$. Then $\varphi_x = \varphi_x^+$ for any $X \in X$.*

Proof. Note that φ^+ is given by

$$\begin{tikzcd} \mathcal{G} \arrow[r, "i_{\mathcal{G}}"] \arrow[d, "\varphi"] & \mathcal{G}^+ \arrow[d, "\exists! \varphi^+"] \\ \mathcal{F} \arrow[r, "i_{\mathcal{F}}"] & \mathcal{F}^+ \end{tikzcd}$$

Taking stalks for this commutative diagram, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{G}_x \arrow[r, "i_{\mathcal{G},x}"] \arrow[d, "\varphi_x"] & \mathcal{G}_x^+ \arrow[d, "\varphi_x^+"] \\ \mathcal{F}_x \arrow[r, "i_{\mathcal{F},x}"] & \mathcal{F}_x^+ \end{tikzcd}$$

Because $i_{\mathcal{G},x}$ is $=$, $\varphi_x = \varphi_x^+$ for any $x \in X$ $\square$

Theorem 2.1.9 (Adjoint Property of f^{-1})**.** . *Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathcal{F}$ on X there is a natural map $f^{-1} f_* \mathcal{F} \to \mathcal{F}$, and for any sheaf $\mathcal{G}$ on Y there is a natural map $\mathcal{G} \to f_* f^{-1} \mathcal{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y,*

$$Hom_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = Hom_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

that is, $f^{-1} \dashv f^*$.

Proof. Given $\varphi : f^{-1}\mathcal{G} \longrightarrow \mathcal{F}$, act it by f_*, we have $f_*(\varphi) : f_*f^{-1}\mathcal{G} \longrightarrow f_*\mathcal{F}$. So we need to conisder $\mathcal{G} \longrightarrow f_*f^{-1}\mathcal{G}$. For the same reason, we need to consider $f^{-1}f_*\mathcal{F} \longrightarrow \mathcal{F}$.

$$\begin{array}{ccccc}\mathcal{G} & & \xrightarrow{\quad\quad\quad i_{\mathcal{G}} \quad\quad\quad} & & f_*f^{-1}\mathcal{G} \\ \mathcal{G}(U) & \longrightarrow & f^{-1}\mathcal{G}(f^{-1}(U)) & \longrightarrow & \varinjlim_{f(f^{-a}(U))=U\subset V} \mathcal{G}(V) = \mathcal{G}(U)\end{array}$$

$$\begin{array}{ccccc}f^{-1}f^* : \mathcal{F} & & \xrightarrow{\quad\quad\quad i_{\mathcal{F}} \quad\quad\quad} & & \mathcal{F} \\ \mathcal{F}(U) & \longrightarrow & \varinjlim_{f(U)\subset V} f^*\mathcal{F}(V) & \longrightarrow & \varinjlim_{f(U)\subset V} \mathcal{F}(f^{-1}V) = \mathcal{F}(U)\end{array}$$

Now define

$$\begin{array}{ccc}\Phi : Hom_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) & \longrightarrow & Hom_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ \varphi & \mapsto & f_*\varphi \circ i_{\mathcal{G}}\end{array}$$

$$\begin{array}{ccc}\Psi : Hom_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}) & \to & Hom_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F})) \\ \psi & \mapsto & i_{\mathcal{F}} \circ f^{-1}\psi\end{array}$$

Because $f_*(i_{\mathcal{F}} \circ f^{-1}\varphi) \circ i_{\mathcal{G}} = \varphi$ and $i_{\mathcal{F}} \circ f^{-1}(f_*\varphi \circ i_{\mathcal{G}}) = \varphi$ (Just write down the morphisms!), thus Φ and Ψ are isomorphisms. Thus

$$\mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

. □

Corollary 2.1.10 (Property of skyscraper). *Let $x \in X$ and $i_x : \{x\} \to X$, we have*

$$Hom_{Sh(x}(i_x^{-1}\mathcal{G}, \mathcal{F}) = Hom_{Sh(X)}(\mathcal{G}, i_{x*}\mathcal{F}).$$

where i_x^{-1} is the taking stalks and i_{x*} is the skyscraper, that is $i_x^{-1} \dashv i_{x*}$.

Proof. Directly use 2.1.9. □

Theorem 2.1.11. *For a sheaf $\mathcal{F}$ on X, the map*

$$\begin{array}{ccc}\Pi i_x : \mathcal{F}(X) & \to & \Pi_{x\in X}\mathcal{F}_x \\ s & \mapsto & s_x\end{array}$$

is an injection.

Proof. If $s_x = 0$, there exists a neighborhood U of x such that $s|_U = 0$. For each $x \in X$, we have such an open set U_x and they form a cover $\mathscr{U} = \{U_x\}_{x\in X}$ of X such that $s|_{U_x} = 0$. Then for s is a section of a sheaf, so s is gained by gluing $s|_{U_x}$. Thus $s = 0$. □

Theorem 2.1.12 (Adjoint Property of $\Gamma(X, -)$). *Let X be a topological space. Then for any $A \in (Ab)$ and $\mathcal{F}$, we have*

$$Hom_{Sh}(A^+, \mathcal{F}) = Hom_{Ab}(A, \Gamma(X, \mathcal{F}))$$

that is, sheafification of a constant presheaf $A \dashv \Gamma(X, -)$.

Proof. Take $f : A^+ \longrightarrow \mathcal{F}$, then $\Gamma(X, f) : \Gamma(X, A^+) \longrightarrow \Gamma(X, \mathcal{F})$. Thus, we consider the morphism $i_A : A \to \Gamma(X, A^+)$. For the same reason, we consider $\rho : \Gamma(X, \mathcal{F})^+ \longrightarrow \mathcal{F}$.

For if $X = U_1 \cup U_2$ with $U_1, U_2 \neq \emptyset$, and $U_1 \cap U_2$. For the sequence in C, the abelian category is exact by the definition of sheaves,

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \longrightarrow \mathcal{F}(U_1 \cap U_2) = \mathcal{F}(\emptyset) = 0$$

we see that $\mathcal{F}(X) = \mathcal{F}(U_1) \oplus \mathcal{F}(U_2)$.

So without lose of generality, we can assume that X is connected. For X is connected, $\Gamma(X, A^+) = A$. So $i_A = id_A$.

Also if $U \subset A$ and $U = \cup U_i$ with each U_i connected. Then $\rho = \Pi_i \rho_{X,U_i} : \Pi_i \Gamma(U_i, \mathcal{F}) \longrightarrow \Pi_i \mathcal{F}_{X,U_i}$.

Then define $\Phi : Hom_{Sh}(A^+, \mathcal{F}) \longrightarrow Hom_{Ab}(A, \Gamma(X, \mathcal{F}))$ by

$$f \mapsto \Gamma(X, f) \circ i_A$$

and $\Psi : Hom_{Ab}(A, \Gamma(X, \mathcal{F})) \longrightarrow Hom_{Sh}(A^+, \mathcal{F})$ by

$$g \mapsto \rho \circ g^+$$

. Then $\Psi \circ \Phi(f) = f$ and $\Psi \circ \Psi(g) = g$. Thus,

$$Hom_{Sh}(A^+, \mathcal{F}) = Hom_{Ab}(A, \Gamma(X, \mathcal{F}))$$

. □

Remark. For A^+, when U is connected, then $A^+(U) = A$: For any cover $U = \cup U_i$, for each U_i there must exist U_j such that $U_i \cap U_j \neq \emptyset$. So, only take the same value in each $\mathcal{F}(U_i)$ can we glue them together. For strict proof, you can use the exact sequence of persheaves above, which is from P25 of [10].

2.1.2 Examples

Example 2.1.13. Let $X = \mathbb{R}$ and $\underline{\mathbb{Z}}$ be the constant presheaf associated to $\mathbb{Z}$. Consider the morphisms of presheaves

$$\underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}^+$$

On $U = (-1, 0) \cup (0, 1)$ we have $\underline{\mathbb{Z}}(U) = \mathbb{Z}$ and $\underline{\mathbb{Z}}^+(U) = \mathbb{Z} \oplus \mathbb{Z}$. So $\underline{\mathbb{Z}}(U) \longrightarrow \underline{\mathbb{Z}}^+(U)$ is not a surjection.

Example 2.1.14. Consider $X = \mathbb{C} - \{0\}$ and $coker^{Pr}(exp)$ with $exp : \mathcal{O}_X \xrightarrow{exp(2\pi i \cdot)} \mathcal{O}_X^*$. By basic complex analysis, $coker^{Pr}(exp) \neq 0$. However, for each $P \in X$, $coker(exp)_P = coker^{Pr}(exp)_P = 0$. By 2.1.11, $coker(exp) = 0$.

Thus, for a presheaf $\mathcal{F}$ over X, as a morphism of presheaves, for $U \subset X$, $\mathcal{F}(U) \longrightarrow \mathcal{F}^+(U)$ may not be injective or surjective.

2.1.3 Exercises

Exercise 2.1.1. *Let A be an abelian group, and define the constant presheaf associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \emptyset$, with restriction maps the identity. Show that the constant sheaf $\mathcal{A}$ defined in the text is the sheaf associated to this presheaf.*

Proof. In order to prove this, we just define

$$\begin{array}{ccc} i : A & \longrightarrow & \mathcal{A} \\ i(U) : A(U) & \longrightarrow & \mathcal{A}(U) \\ a & \mapsto & a \end{array}$$

Note that restrict to $x \in X$, we have an isomorphism $i_x : \mathcal{A}_x = A \longrightarrow A_x = A$.

Now by Proposition-Definition 1.2 of [5], consider

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "i"'] & \mathcal{A}^+ \arrow[d, "\exists!\varphi"] \\ & \mathcal{A} \end{tikzcd}$$

By definition of sheafification, i_{Ax} is an isomorphism. Because $(-)_x$ is a funtor,

$$\varphi_x \circ i_{Ax} = i_x$$

Then φ_x is an isomorphism. By Proposition 1.1 of [5], φ is an isomorphism of sheaves. Thus, $(A)^+ \cong \mathcal{A}$. □

Exercise 2.1.2. *(a) For any morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, show that for each point P,* $(\ker \varphi)_P = \ker(\varphi_P)$ *and* $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$.

(b) Show that φ is injective (respectively, surjective) if and only if the induced map on the stalks φ_P is injective (respectively, surjective) for all P.

(c) Show that a sequence $\ldots \mathcal{F}^{i-1} \xrightarrow{\varphi_i} \mathcal{F}^i \xrightarrow{\varphi_i} \mathcal{F}^{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof.

(a):Take $s_p \in Ker(\varphi_P)$. $\varphi_P(s_P) = 0$. By the definition of taking stalks, there exists a neighborhood U of s such that $\varphi(s)|_U = \varphi(U)(s) = 0$. Then $s_p \in ker(\varphi(U))$. By the definition of colimits, $s_p \in ker(\varphi)_P$.

Take any $s \in ker(\varphi)_P$, there exists a neighborhood U of P, such that $\varphi(U)(s) = 0$. Then $\varphi_P(s) = \varphi(s)_P = 0$. Thus, $ker(\varphi)_P = ker(\varphi_P)$.

Because $i_x^{-1} \dashv i_x^*$, $i_x^{-1} \circ coker(\varphi) = coker \circ i_x^{-1}(\varphi)$. So $coker(\varphi_x) = coker(\varphi)_x$. For $im(\varphi) = ker(coker(\varphi))$, $im(\varphi_x) = im(\varphi)_x$.

(b): If φ is injective then $ker(\varphi) = 0$ and $ker(\varphi_P) = \ker(\varphi)_P = 0$. Thus, φ_P is an injective for each $P \in X$.

If for each $P \in X$, for $s \in \mathcal{G}(X) \longrightarrow \Pi_{P \in X} s_P \Pi_{P \in P} \mathcal{G}_P$ is an injection. So if for each $\varphi(s)_p = \varphi_p(s_p) = 0$, so $\varphi(s) = 0$. Thus, φ is an injection.

For the conditions about surjection, the proof is the same.

(c): If $im(\varphi^{i-1}) = ker(\varphi^i)$, then for each $P \in X$, $im(\varphi^{i-1})_P = ker(\varphi^i)_P$. By (a), $im(\varphi_P^{i-1}) = ker(\varphi_P^i)$. Thus, the sequence of stalks is exact.

If for each $P \in X$, $im(\varphi_P^{i-1}) = ker(\varphi_P^i)$. By the injection $\mathcal{G}(X) \longrightarrow \Pi_{P \in P} \mathcal{G}_P$, $im(\varphi) = ker(\varphi)$. Thus, the sequence is exact. □

Remark. In fact, for $ker(\varphi_P) = ker(\varphi)_P$, we can show this by using the fact that colimitis in abelian group preserve monomorphisms. But I donn't know how to prove this.

Exercise 2.1.3.

(a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathcal{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, and an open set U such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective.

Proof.

(a). $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is surjective if and only if $im\varphi = 0$ if and only if $im(\varphi)_P = im(\varphi_P) = 0$ for each $P \in X$, that is φ_P is surjective for each $P \in X$.

So for $s \in \mathcal{G}(X)$, there exists $(t, V) \in \mathcal{F}_P$ such that $\varphi_P((t, V)) = \varphi(s)_P = s_P$. So there exists an open neighborhood $U_P \subset V$ of P where $\varphi(t)|_{U_P} = s|_{U_P}$. Thus, we can find a cover of X satisfying the condition above.

Conversely, for each $P \in X$,take any $s_P = (s, V) \in \mathcal{G}_P$. Then there exists an open cover $\mathscr{V} = \{V_i\}_{i\in I}$ of V and $P \in V_i$. Note that there exists $t_i \in \mathcal{V}\rangle$ such that $\varphi(t_i) = s|_{V_i}$ and \$$(s, V_i) = (s, V) \in \mathcal{G}_P$. So φ_P is a surjection.

(b). Let $X = \mathbb{C}^\times$ and $\mathbb{Z}^+$ be the constant $\mathbb{Z}$ sheaf on X and $\mathcal{O}$ be the sheaf of holomrophic functions on X and $\mathcal{O}^*$ be the sheaf of non-zero holomorphic functions X. Consider

$$0 \longrightarrow \mathbb{Z}^+ \xrightarrow{i} \mathcal{O} \xrightarrow{exp(2\pi i-)} \mathcal{O}^* \longrightarrow 0$$

$\varphi = exp(2\pi i-)$ is surjective: for each $P \in X$, we can find a open set $U_P \subset X$ such that $0 \notin U_P$. For any $V \subset U$, we can find a cover $\mathscr{V} = \{V_i\}_{i\in I}$ of V then $0 \notin V_i$ for each $i \in I$. Not that for each $f \in \mathcal{O}^*(U)$, if $0 \notin U$, then there exists $g \in \mathcal{O}^*(U)$ such that $exp(2\pi ig) = f$. Thus, each $\varphi(V_i)$ is surjective. By (a), $\varphi|_U$ is surjective. Then for each $P \in U$ φ_P is surjective.

Note that $\mathbb{C}^\times$ can be covered by open sets that don't contain 0. So for each $P \in X$, φ_P is surjective. Thus φ is a surjection of sheaves.

However $\varphi(X)$ is not a surjection. Take $\frac{1}{z} \in \mathcal{O}^*(X)$, there is not elements g in $\mathcal{O}(X)$ such that $exp(2\pi ig) = \frac{1}{z}$. □

Remark. If you want to know details of the example above, I highly recommand you to read a textbook about complex analysis.

Exercise 2.1.4. (a) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves such that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for each U. Show that the induced map $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ of associated sheaves is injective.

(b) Use part (a) to show that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then φ can be naturally identified with a subsheaf of $\mathcal{G}$, as mentioned in the text.

Proof.

(a) Because $ker(\varphi)(U) = 0$, for each $U \subset X$, $ker(\varphi)_x = ker(\varphi^+)_x = 0$ by 2.1.8 for each $x \in X$. Thus $ker(\varphi^+) = 0$. φ^+ is an injection.

(b) For $\mathcal{F}$ and $\mathcal{G}$ are presheaves, we consider the presheaf $im(\varphi)$. Then $i : im(\varphi) \longrightarrow \mathcal{G}$ is an injection of presheavs. Then $i^+ : im(\varphi)^+ \longrightarrow \mathcal{G}^+ = \mathcal{G}$ is an injection of sheaves by (a). □

Exercise 2.1.5. Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

Proof. φ is an isomorphism if and only if φ_P is an isomorphism for each $P \in X$, by Proposition 1.1 [5], if and only if φ_P is an injection and a surjection for each $x \in X$ if and only if φ is injective and surjective for each $x \in X$, by 2.1.2 (b). □

Exercise 2.1.6. (a) Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Show that the natural map of $\mathcal{F}$ to the quotient sheaf $\mathcal{F}/\mathcal{F}'$ is surjective, and has kernel $\mathcal{F}'$. Thus there is an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0.$$

(b) Conversely, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence, show that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$, and that $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.

Proof.
(a) For the exact short sequence of presheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{i} \mathcal{F} \longrightarrow (\mathcal{F}/\mathcal{F}')^{pr} \longrightarrow 0$$

we have the short exact seqence of stalks for colimit is exact in the abelian category,

$$0 \longrightarrow \mathcal{F}'_P \longrightarrow \mathcal{F}_P \longrightarrow (\mathcal{F}/\mathcal{F}')^{pr}_P \longrightarrow 0$$

As $(\mathcal{F}/\mathcal{F}')^{pr}_P = (\mathcal{F}/\mathcal{F}^{'})_P$, we have the exact sequence on each stalks,

$$0 \longrightarrow \mathcal{F}'_P \longrightarrow \mathcal{F}_P \longrightarrow (\mathcal{F}/\mathcal{F}')_P \longrightarrow 0$$

Finally, we get the short exact sequence in $Sh(X)$:

$$0 \longrightarrow \mathcal{F}' \xrightarrow{i} \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow 0$$

(b)By the exactness, we know that $ker(i) = 0$. Thus, $i : \mathcal{F}' \longrightarrow \mathcal{F}$ is an injection. Thus, $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$.

Apply 2.1.2 (c) to

$$0 \longrightarrow \mathcal{F}' \xrightarrow{i} \mathcal{F} \longrightarrow (\mathcal{F}/\mathcal{F}') \longrightarrow 0$$

for ecah $P \in X$, we have a short exact sequence in the abelian category C:

$$0 \longrightarrow \mathcal{F}'_P \xrightarrow{i} \mathcal{F}_P \longrightarrow (\mathcal{F}/\mathcal{F}')_P \longrightarrow 0$$

Thus, $(\mathcal{F}/\mathcal{F}')_P = \mathcal{F}_P/\mathcal{F}'_P$. Apply 2.1.2 (c) to

$$0 \longrightarrow \mathcal{F}' \xrightarrow{i} \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

We get $\mathcal{F}''_P = \mathcal{F}_P/\mathcal{F}'_P = (\mathcal{F}/\mathcal{F}')_P$ for each $P \in X$. Applying 2.1.2 (b), we have

$$\mathcal{F}'' \cong \mathcal{F}/\mathcal{F}'$$

□

Remark. For the proof of (a), if you don't want to use the adjoint functors, you can use the universal property of sheaefification to prove it. But in fact, they are the same by the proof of 2.1.3.

Exercise 2.1.7. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.
(a) Show that $\operatorname{im}\varphi \cong \mathcal{F}/\ker\varphi$.
(b) Show that $\operatorname{coker}\varphi \cong \mathcal{G}/\operatorname{im}\varphi$.

Proof. (a). Note that we have an exact sequence of presheaves

$$0 \longrightarrow \ker^{pr}(\varphi) \xrightarrow{i} \mathcal{F} \xrightarrow{p} im^{pr}(\varphi) \longrightarrow 0$$

and $ker(\varphi) = ker^{pr}(\varphi)$ is a sheaf use the same discussion in 2.1.6 (a), we have an exact seqence of shaves

$$0 \longrightarrow \ker(\varphi) \xrightarrow{i} \mathcal{F} \xrightarrow{p} im(\varphi) \longrightarrow 0$$

Applying 2.1.6 (b), we have

$$im(\varphi) = \mathcal{F}/ker(\varphi)$$

(b). Just note that for ecah $P \in X$, $coker(\varphi)_P = coker^{pr}(\varphi)_P$ and $im(\varphi)_p = im(\varphi)^{pr}(\varphi)_P$ and

$$0 \longrightarrow im(\varphi)^{pr}_P \longrightarrow \mathcal{G}_P \longrightarrow coker^{pr}(\varphi)_P \longrightarrow 0$$

is exact in the alelian category C. So by 2.1.2 (b),

$$0 \longrightarrow im(\varphi) \longrightarrow \mathcal{G} \longrightarrow coker(\varphi) \longrightarrow 0$$

Then use 2.1.6 (b),

$$coker(\varphi) \cong \mathcal{G}/im(\varphi)$$

□

Remark. You can use the method proving (b), to prove (a). But the inverse fails, for $coker^{pr}(\varphi)$ is not a sheaf.

Exercise 2.1.8. For any open subset $U \subseteq X$, show that the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e., if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

is an exact sequence of sheaves, then

$$0 \to \Gamma(U, \mathcal{F}') \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'')$$

is an exact sequence of groups. The functor $\Gamma(U, \cdot)$ need not be exact; see (Ex. 1.21) below.

Proof. Just note that $\Gamma(X, -)$ is a left exact functor by 2.1.12. Then use 2.1.3.

For it is not exact, just see the example of 2.1.3 (b). □

Exercise 2.1.9. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf. It is called the *direct sum* of $\mathcal{F}$ and $\mathcal{G}$, and is denoted by $\mathcal{F} \oplus \mathcal{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X.

Proof. First, if we define $\rho_{\mathcal{F}\oplus\mathcal{G}} = \rho_{\mathcal{F}} \oplus \rho_{\mathcal{G}}$, then for $W \subset V \subset U \subset X$, we have

$$\rho_{\mathcal{F}\oplus\mathcal{G},UW} = \rho_{\mathcal{F},UW} \oplus \rho_{\mathcal{G},UW} = \rho_{\mathcal{F},VW} \circ \rho_{\mathcal{F},UV} \oplus \rho_{\mathcal{G},VW} \circ \rho_{\mathcal{G},UV} = \rho_{\mathcal{F}\oplus\mathcal{G},VW} \circ \rho_{\mathcal{F}\oplus\mathcal{G},UV}$$

so $\mathcal{F} \oplus_{pr} \mathcal{G}$ is a presheaf.

Next we show that $\mathcal{F} \oplus_{pr} \mathcal{G} = (\mathcal{F} \oplus \mathcal{G})^{pr}$. Take any $f : \mathcal{F} \longrightarrow \mathcal{H}$ and $g : \mathcal{H} \longrightarrow \mathcal{G}$ then by the universal property of coproducts in an Abelian categroy, for each $U \subset X$

$$\begin{tikzcd}
 & \mathcal{F}(U) \arrow[dl, "f(U)"'] \arrow[dr, "i_{\mathcal{F}}"] & \\
\mathcal{H}(U) & & \mathcal{F}\oplus\mathcal{G}(U) \arrow[ll, "\exists!\varphi(U)"'] \\
 & \mathcal{G}(U) \arrow[ul, "g(U)"] \arrow[ur, "i_{\mathcal{G}}"'] &
\end{tikzcd}$$

And by the commutativity of $f, g, i_{\mathcal{F}}, i_{\mathcal{G}}$ and the presheaf structure maps $\rho_{\mathcal{H}}, \rho_{\mathcal{F}}, \rho_{\mathcal{G}}, \rho_{\mathcal{F}} \oplus \rho_{\mathcal{G}}$, φ is a morphism between sheaves and it is unique. Thus $\mathcal{F} \oplus_{pr} \mathcal{G} = (\mathcal{F} \oplus \mathcal{G})^{pr}$.

Because in Abeliam group, finite products coincides with finite corproduct. Now, we just view $\mathcal{F} \oplus \mathcal{G}$ as a product.

Because $i(-)$ is a right exact functor 2.1.7, so it commutes with coproudct by 2.1.2 and 2.1.1.

$$\mathcal{F} \oplus_{Sh} \mathcal{G} = i(\mathcal{F} \oplus_{Sh} \mathcal{G}) = i(\mathcal{F}) \oplus_{pr} i(\mathcal{G}) = \mathcal{F} \oplus_{pr} \mathcal{G} = (\mathcal{F} \oplus \mathcal{G})^{pr}$$

Thus, $\mathcal{F} \oplus_{Sh} \mathcal{G} = (\mathcal{F} \oplus \mathcal{G})^{pr}$ as presheaves, so $(\mathcal{F} \oplus \mathcal{G})^{pr}$ has a sheaf structure. Then $\mathcal{F} \oplus_{Sh} \mathcal{G} = (\mathcal{F} \oplus \mathcal{G})^{pr}$ as sheaves. □

Remark. For a presheaf $\mathcal{H}$, if $\mathcal{H} = \mathcal{F}$ for a sheaf $\mathcal{F}$. Then $\mathcal{H}$ is a sheaf. For the sequence in C

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "="] & \mathcal{H}(U) \arrow[r] \arrow[d, "="] & \Pi_i \mathcal{H}(U_i) \arrow[r] \arrow[d, "="] & \Pi_{i,j} \mathcal{H}(U_i \cap U_j) \arrow[d, "="] \\
0 \arrow[r] & \mathcal{F}(U) \arrow[r] & \Pi_i \mathcal{F}(U_i) \arrow[r] & \Pi_{i,j} \mathcal{F}(U_i \cap U_j)
\end{tikzcd}$$

is exact.

Exercise 2.1.10 (Colimit)**.** Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the direct limit of the system $\{\mathcal{F}_i\}$, denoted $\varinjlim \mathcal{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathcal{G}$, and a collection of morphisms $\mathcal{F}_i \to \mathcal{G}$, compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathcal{F}_i \to \mathcal{G}$ such that for each i, the original map $\mathcal{F}_i \to \mathcal{G}$ is obtained by composing the maps $\mathcal{F}_i \to \varinjlim \mathcal{F}_i \to \mathcal{G}$.

Proof. Because $\underset{\rightarrow}{lim}\mathcal{F}_i$ has the presheaf structure given by the universal property: $V \subset U$, then there exists $\rho_{i,UV} : \mathcal{F}_i(U) \to \mathcal{F}_i(V)$

$$\begin{tikzcd}
\mathcal{F}_i \arrow[rr] \arrow[dr, "f_i(U)"] \arrow[ddr, "f_i(V)\circ\rho_{i,UV}"'] & & \mathcal{F}_j \arrow[dl, "f_j(U)"'] \arrow[ddl, "f_j(V)\circ\rho_{j,UV}"] \\
 & \underset{\rightarrow}{lim}\mathcal{F}_i(U) \arrow[d, "\exists! \underset{\rightarrow}{lim}\rho_{UV}"] & \\
 & \underset{\rightarrow}{lim}\mathcal{F}_i(V) &
\end{tikzcd}$$

By the universal property $\underset{\rightarrow}{lim}\rho_{UW} = \underset{\rightarrow}{lim}\rho_{VW} \circ \underset{\rightarrow}{lim}\rho_{UV}$.

AS we claim in 2.1.9, we can see that $\underset{\rightarrow}{lim}\mathcal{F}_i$ is a colimit in PSh. Then for any $\mathcal{F}_i \longrightarrow \mathcal{G}$, it is also a morphism of presheaves. So there exists a unique of persheaves $\varphi : \underset{\rightarrow}{lim}\mathcal{F}_i \longrightarrow \mathcal{G}$. By the universal property of sheafification, there exists a unique map

$$\underset{\rightarrow}{lim}\mathcal{F}_i \longrightarrow (\underset{\rightarrow}{lim}\mathcal{F}_i)^+ \longrightarrow \mathcal{G}$$

where the later morphism is a morphism between sheaves. Thus, $(\underset{\rightarrow}{lim}\mathcal{F}_i)^+$ is a colimit in Sh. □

Exercise 2.1.11. Let $\{\mathcal{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ is already a sheaf. In particular,

$$\Gamma(X, \varinjlim \mathcal{F}_i) = \varinjlim \Gamma(X, \mathcal{F}_i).$$

Proof. Denote $\underset{\rightarrow}{lim}\mathcal{F}_i$ by $\mathcal{F}$. For X is Noetherian, we can take a finite cover $\mathscr{U}$. Take any $s_i \in \mathcal{F}(U_i)$ and $s_j \in \mathcal{F}(U_j)$. By definition of colimit, we can find m such that $\varphi_m : \mathcal{F}_m(U_i) \to \mathcal{F}(U_i)$ such that $s_i^m \in \mathcal{F}_i(U_i) \mapsto s_i$. Similarly, there exists such an n for j. Let $k = max\{m, n\}$ and $s_{ik} \in \mathcal{F}_k(U_i) = \varphi_{ik}(s_i)$, $s_{jk} \in \mathcal{F}(U_j) = \varphi_{ji}(s_j)$. Then $\varphi_k|_{U_i \cap U_j}(s_{ik} - s_{jk}) = 0$ and φ_k is an injection by properties of colimits, so $s_{ik} = s_{ji}$ on $U_i \cap U_j$. So we can glue s_{ik} and s_{jk} to a section $s \in \mathcal{F}_k(U_i \cup U_j) \subset \mathcal{F}(U_i \cup U_j)$, which is unique. Then repeat this process for $U \in \mathscr{U} - \{U_i, U_j\}$. By finite steps, we can glue $(s_i)|_{i \in I}$ into a global section $s \in \mathcal{F}(X)$. $\square$

Remark. Note that colimits are left-exact in Abelian category. Then use the exact sequence in C, the abelian category:

$$0 \longrightarrow \mathcal{F} \longrightarrow \Pi_i \mathcal{F}(U_i) \longrightarrow \Pi_{i,j} F(U_i \cap U_j)$$

we get an exact sequence

$$0 \longrightarrow \underset{\rightarrow}{lim}\mathcal{F}_i \longrightarrow \underset{\rightarrow}{lim}\Pi_\alpha \mathcal{F}_i(U_\alpha) \longrightarrow \underset{\rightarrow}{lim}\Pi_{\alpha,\beta}\mathcal{F}_i(U_\alpha \cap U_\beta)$$

And because **finite** products coincide **finite** coproducts, so $\underset{\rightarrow}{lim}$ commutes with Π_α and $\Pi_{\alpha,\beta}$. So we have an exact sequence

$$0 \longrightarrow \underset{\rightarrow}{lim}\mathcal{F}_i \longrightarrow \Pi_\alpha \underset{\rightarrow}{lim}\mathcal{F}_i(U_\alpha) \longrightarrow \Pi_{\alpha,\beta}\underset{\rightarrow}{lim}\mathcal{F}_i(U_\alpha \cap U_\beta)$$

that is,

$$0 \longrightarrow \mathcal{F} \longrightarrow \Pi_\alpha \mathcal{F}(U_\alpha) \longrightarrow \Pi_{\alpha,\beta}\mathcal{F}(U_\alpha \cap U_\beta)$$

So, $\mathcal{F}$ is a sheaf.

Exercise 2.1.12. (Limit) Let $\{\mathcal{F}_i\}$ be an inverse system of sheaves on X. Show that the presheaf $U \mapsto \varprojlim \mathcal{F}_i(U)$ is a sheaf. It is called the inverse limit of the system $\{\mathcal{F}_i\}$, and is denoted by $\varprojlim \mathcal{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

Proof. Use the univarsal property to identify the presheaf structure of $\underset{\leftarrow}{lim}\mathcal{F}_i$ as a presheaf. Then like in 2.1.10, show that $\underset{\leftarrow}{lim}\mathcal{F}_i$ is a limit in Psh.

Finally, use $i(-)$ is right adjoint 2.1.7 and it commutes with limits 2.1.1. Use the same method in 2.1.9 to show that $\underset{\leftarrow}{lim}\mathcal{F}_i$ is a sheaf.

$\square$

Exercise 2.1.13 (Espace Étale of a Presheaf)**.** (This exercise is included only to establish the connection between our definition of a sheaf and another definition often found in the literature. See for example Godement [1, Ch. II, §1.2].) Given a presheaf $\mathcal{F}$ on X, we define a topological space $\text{Spé}(\mathcal{F})$, called the espace étale of $\mathcal{F}$, as follows. As a set, $\text{Spé}(\mathcal{F}) = \bigcup_{P \in X} \mathcal{F}_P$. We define a projection map $\pi : \text{Spé}(\mathcal{F}) \to X$ by sending $s \in \mathcal{F}_P$ to P. For each open set $U \subseteq X$ and each section $s \in \mathcal{F}(U)$, we obtain a map $\bar{s} : U \to \text{Spé}(\mathcal{F})$ by sending $P \mapsto s_P$, its germ at P. This map has the property that $\pi \circ \bar{s} = \text{id}_U$, in other words, it is a "section" of π over U. We now make $\text{Spé}(\mathcal{F})$ into a topological space by giving it the strongest topology such that all the maps $\bar{s} : U \to \text{Spé}(\mathcal{F})$ for all U, and all $s \in \mathcal{F}(U)$, are continuous. Now show that the sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ can be described as follows: for any open set $U \subseteq X$, $\mathcal{F}^+(U)$ is the set of continuous sections of $\text{Spé}(\mathcal{F})$ over U. In particular, the original presheaf $\mathcal{F}$ was a sheaf if and only if for each U, $\mathcal{F}(U)$ is equal to the set of all continuous sections of $\text{Spé}(\mathcal{F})$ over U.

Proof. I recommand you to jump this exe. For the topology structure of $Sp(X)$, please see P 110 [9]. □

Exercise 2.1.14 (Support)**.** Let $\mathcal{F}$ be a sheaf on X, and let $s \in \mathcal{F}(U)$ be a section over an open set U. The support of s, denoted $\operatorname{Supp} s$, is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathcal{F}_P$. Show that $\operatorname{Supp} s$ is a closed subset of U. We define the support of $\mathcal{F}$, $\operatorname{Supp} \mathcal{F}$, to be $\{P \in X \mid \mathcal{F}_P \neq 0\}$. It need not be a closed subset.

Proof. This is equivalent to proving that

$$V_s = \{P \in X | s_x = 0\}$$

is open. Take any $P \in V_s$, $s_P = 0$. Then by the definition of s_P, there exists a neighborhood U_P of P such $s|_{U_P} = 0$. For any $P \in U_P$, $s_P = s(P) = 0$. Thus $U_P \subset V_s$. So V_s is an open set. □

Exercise 2.1.15. Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf $U \mapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf. It is called the sheaf of local morphisms of $\mathcal{F}$ into $\mathcal{G}$, "sheaf hom" for short, and is denoted $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Proof. For $W \subset V$, define $\rho_{VW} : \mathcal{H}om(\mathcal{F}, \mathcal{G})(V) \to \mathcal{H}om(\mathcal{F}, \mathcal{G})(W)$ by

$$\{f \in Hom(\mathcal{F}|_V, \mathcal{G}|_V)\} \mapsto \{F \in Hom(\mathcal{F}|_W, \mathcal{G}|_W)\}$$

where $F|_{W'} = f|_{W'}$ for any $W^{'} \subset W$.

It will take some time to verify that with ρ_{VW}, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a presheaf.

To verify it is a sheaf, just take U_1, U_2 such that $f_i \in \mathcal{H}om(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i})$ and $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$. Then we verify f_i can be glue to a morphism in $\mathcal{H}om(\mathcal{F}|_{U_1 \cup U_2}, \mathcal{G}|_{U_1 \cup U_2})$.

Note that f_i satisfies $f_i \circ \rho^{\mathcal{F}}_{U_i, U_1 \cap U_2} = \rho^{\mathcal{G}}_{U_i, U_1 \cap U_2} \circ f_i$. Take any $s \in \mathcal{F}(U), U \subset U_1 \cup U_2$. By assumption

$$f_1 \circ \rho^{\mathcal{F}}_{U_1 \cup U_2 \cap, U_1 \cap U_2}(s) = f_2 \circ \rho^{\mathcal{F}}_{U_1 \cup U_2, U_1 \cap U_2}(s)$$

that is, $f_1(s|_{U_1}) = f_2(s|_{U_2})$. So we can glue them together to be an element in $\mathcal{G}(U)$, for $\mathcal{G}$ is a sheaf.

So we can define

$$f_1 \cup f_2 : \mathcal{F}|_{U_1 \cup U_2} \longrightarrow \mathcal{G}|_{U_1 \cup U_2}$$

by $s \mapsto f_1(s) \cup f_2(s)$. As we have shown, it is well-define. Thus, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf. □

Exercise 2.1.16 (Flasque Sheaves)**.** A sheaf $\mathcal{F}$ on a topological space X is flasque if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

(i) Show that a constant sheaf on an irreducible topological space is flasque. See $(I, \S 1)$ for irreducible topological spaces.

(ii) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ is flasque, then for any open set U, the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ of abelian groups is also exact.

(iii) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.

(iv) If $f : X \to Y$ is a continuous map, and if $\mathcal{F}$ is a flasque sheaf on X, then $f_*\mathcal{F}$ is a flasque sheaf on Y.

(v) Let $\mathcal{F}$ be any sheaf on X. We define a new sheaf $\mathcal{G}$, called the sheaf of discontinuous sections of $\mathcal{F}$ as follows. For each open set $U \subseteq X$, $\mathcal{G}(U)$ is the set of all sections of $\mathcal{F}$ over U that are not necessarily continuous maps $s : U \to \bigcup_{P\in U} \mathcal{F}_p$ such that for each $P \in U$, $s(P) \in \mathcal{F}_p$. Show that $\mathcal{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathcal{F}$ to $\mathcal{G}$.

Proof.

(a). For X is irreducible, each open set of X is connected. Thus, for any constant sheaf $\mathcal{F}$, $\mathcal{F}(U) = A$. Thus, $\mathcal{F}(X) = A \longrightarrow \mathcal{F}(U) = A$ is surjective.

(b). For if $\mathcal{F}$ is flasque, $\mathcal{F}|_U$ is also flasque, it is enough to show that

$$0 \longrightarrow \mathcal{F}^{'}(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}^{''}(X) \longrightarrow 0$$

is exact.

First of all, we know that $\Gamma(X, -)$ is left-exact, it remains to show that $\mathcal{F}(X) \longrightarrow \mathcal{F}^{''}(X)$ is surjective.

Take any $c \in \mathcal{F}^{''}(X)$, there exists a cover $\mathscr{U} = \{U_i\}$ of X such that $c = \{c_i\}$ with $c|_{U_i} = c_i$ and there exists $b_i \mapsto c_i$ with $b_i \in \mathcal{F}(U_i)$. Now we need to glue $\{b_i\}$ together. Now that on $U_i \cap U_j$, $b_i|_{U_i\cap U_j} - b_j|_{U_i\cap U_j} \mapsto 0$. So there exists $a_{ij} \in \mathcal{F}^{'}(U_i \cap U_j)$ such that $a_{ij} \mapsto b_i|_{U_i\cap U_j} - b_j|_{U_i\cap U_j}$. By $\mathcal{F}^{'}$ is flasque, we can find $a_i \in \mathcal{F}^{'}(U_i)$ such that $a_i|_{U_i\cap U_j} = a_{ij}$. Consider $b_i^{'} = b_i + a_i$. It satisfies $b_i^{'} \mapsto c_i$ for $a_i \in \mathcal{F}^{'}(U_i)$ and $b_i^{'} - b_j = 0$. Thus, we can glue $b_i^{'}$ and b_j to be a section b^{ij} of $\mathcal{F}(U_i \cap U_j)$ such that $b^{ij} \mapsto c|_{U_i\cap U_j}$. Repeat this method, we can get a global section b of $\mathcal{F}(X)$ such that $b \mapsto c$. Thus, $\mathcal{F}(X) \longrightarrow \mathcal{F}^{''}(X)$ is surjective.

(c). By (b), for any $V \subset U$, we have:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}^{'}(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}^{''}(U) & \longrightarrow & 0 \\
 & & \twoheaddownarrow & & \downarrow & & \twoheaddownarrow & & \\
0 & \longrightarrow & \mathcal{F}^{'}(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{F}^{''}(V) & \longrightarrow & 0
\end{array}$$

For any $s_V \in \mathcal{F}(V)$, there exists $s_U \in \mathcal{F}(U)$ such that $s_U|_V$ and s_V have the same image in $\mathcal{F}^{''}(V)$. Then, there exists $s_U^{'} \in \mathcal{F}(U)$ such that $s_U^{'}|_V = s_V - s_U|_V$. Thus, $s_U^{'} + s_U|_V = s_V$. So $\mathcal{F}$ is flasque.

(d). Just note that if $V \subset U \subset Y$, then $f^{-1}(V) \subset f^{-1}(U) \subset X$. So $f_*\mathcal{F}$ is flasque on Y.

(e). Note that $\mathcal{G}(U) = \Pi_{p\in U}\mathcal{F}_p$. For each $V \subset U$, take any $s_V \in \mathcal{G}(V)$. Just let $s_U \in \mathcal{G}(U)$ be defined by

$$s_U(p) = s_V(p) \text{ if } p \in V; \quad s_U(p) = 0 \text{ Otherwise}$$

then $s_U|_V = s_V$. So $\mathcal{G}$ is flasque.

As for the injection of $\mathcal{F} \longrightarrow \mathcal{G}$, see 2.1.11. □

Exercise 2.1.17 (Skyscraper Sheaves)**.** Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_p(A)$ on X as follows: $i_p(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_p(A)$ is A at every point $Q \in \{P\}^-$, and 0 elsewhere, where $\{P\}^-$ denotes the closure of the set consisting of the point P. Hence the name "skyscraper sheaf." Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\{P\}^-$, and $i : \{P\}^- \to X$ is the inclusion.

Proof. For each $Q \in \{P\}^-$, each $Q \in U$, $P \in U$. Thus for each neighborhood U of Q, $i_P(A)(U) = A$. Taking stalks, $i_P(A)_Q = A$.

Otherwise, $Q \notin \{P\}$, there always exists a neighborhood U of Q such that $P \notin U$, which implies $i_P(A)(U) = 0$ Then by the universal property of colimit, $i_P(A)_Q = 0$.

By definition, $i_*(A)(U) = i_P(A)(i^{-1}(U))$. If $\{P\}^- \cap\ U = \emptyset$, then $i_P(A)(i^{-1})(U) = 0$ for $i^{-1}(U) = \emptyset$ because $P \notin U$. If $\{P\}^- \cap\ U \neq \emptyset$, then then $i_P(A)(i^{-1})(U) = A$ for $i^{-1}(U) = \{P\}$ because $P \in U$.

Taking stalks, $i_*(A)_Q = 0$ if $Q \notin \{P\}^-$ and $i_*(A)_Q = A$ if $Q \in \{P\}^-$. So for each $Q \in X$, $i_*(A)_Q = i_P(A)_Q$. □

Exercise 2.1.18 (Adjoint Property of f^{-1})**.** Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathcal{F}$ on X there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$, and for any sheaf $\mathcal{G}$ on Y there is a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y,

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Hence we say that f^{-1} is a left adjoint of f_*, and that f_* is a right adjoint of f^{-1}.

Proof. 2.1.9. □

Exercise 2.1.19 (Extending a Sheaf by Zero)**.** Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.

(a) Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathcal{F})_p$ of the direct image sheaf on X is $\mathcal{F}_p$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathcal{F}$ instead of $i_*\mathcal{F}$, and say "consider $\mathcal{F}$ as a sheaf on X," when we mean "consider $i_*\mathcal{F}$."

(b) Now let $\mathcal{F}$ be a sheaf on U. Let $j_i(\mathcal{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_i(\mathcal{F}))_p$ is equal to $\mathcal{F}_p$ if $P \in U$, 0 if $P \notin U$, and show that $j_i\mathcal{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathcal{F}$. We call $j_i\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside U.

(c) Now let $\mathcal{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_i(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

Proof.

(a). Note that $(k_*\mathcal{F})_P = \varinjlim_{P \in U} \mathcal{F}(i^{-1}(U)) = \varinjlim_{P \in U} \mathcal{F}(Z \cap U)$

If $P \in Z$, then $U \cap Z$, $P \in U \in Z$ are all open sets in Z that contains P. Thus, $(k_*\mathcal{F})_P = \mathcal{F}_P$.

If $P \notin Z$, then there exists $P \in U$ such that $U \cap Z = \emptyset$. Then $\mathcal{F}(Z \cap U) = 0$. Thus, $(k_*\mathcal{F})_P = 0$.

(b) If $P \in U$, there exists a neighborhood U_P of P such that $U_P \in U$. Then $j_i\mathcal{F}(U_p) = \mathcal{F}(\mathcal{U}_{\checkmark})$. So $j_i\mathcal{F}_P = \varinjlim_{P \in U} \mathcal{F}|_{U_P} = \mathcal{F}_P$.

If $P \notin U$, for any open sets $P \in V$, $V \not\subset U$. So $j_i\mathcal{F}(V) = 0$, which implies $j_i\mathcal{F}_P = 0$.

(c) When $P \in U$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_i(\mathcal{F}|_U)_P & \longrightarrow & \mathcal{F}_P & \longrightarrow & i_*(\mathcal{F}|_Z) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow = & & \downarrow = & & \\ & & \mathcal{F}_P & & \mathcal{F}_P & & 0 & & \end{array}$$

When $P \notin U$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_i(\mathcal{F}|_U)_P & \longrightarrow & \mathcal{F}_P & \longrightarrow & i_*(\mathcal{F}|_Z) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow = & & \downarrow = & & \\ & & 0 & & \mathcal{F}_P & & \mathcal{F}_P & & \end{array}$$

Thus, for all $P \in X$, the sequence is exact at each stalks. So the sequence is exact. $\square$

Remark. $\mathcal{F}|_Z(U) = \underset{\substack{\rightarrow \\ V \supset Z \cup U}}{lim} \mathcal{F}(V)$. So if $p \in \partial Z$, $(\mathcal{F}|_Z)_p = \mathcal{F}_p$.

Exercise 2.1.20 (Subsheaf with Supports)**.** Let Z be a closed subset of X, and let $\mathcal{F}$ be a sheaf on X. We define $\Gamma_z(X, \mathcal{F})$ to be the subgroup of $\Gamma(X, \mathcal{F})$ consisting of all sections whose support (Ex. 1.14) is contained in Z.

(a) Show that the presheaf $V \mapsto \Gamma_{Z\cap V}(V, \mathcal{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathcal{F}$ with supports in Z, and is denoted by $\mathcal{H}^0_Z(\mathcal{F})$.

(b) Let $U = X - Z$, and let $j : U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U).$$

Furthermore, if $\mathcal{F}$ is flasque, the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is surjective.

Proof.

(a) Take $s_i \in \Gamma_{Z\cap U_i}(U_i, \mathcal{F}|_{U_i})$ and $s_1|_{U_1\cap U_2} = s_2|_{U_1\cap U_2}$. For $\mathcal{F}$ is a sheaf, s_1 and s_2 can be glued to be an element in $\Gamma(U_1 \cup U_2, \mathcal{F}|_{U_1\cup U_2})$.

Because $Supp(s_i) \subset Z \cap U_i$. So

$$Supp(s_1 \cup s_2) \subset Supp(s_1) \cup Supp(s_2) \subset (Z \cap U_1) \cup (Z \cap U_2) = Z \cap (U_1 \cup U_2)$$

Thus, $s_1 \cup s_2 \in \Gamma_{Z\cap(U_1\cup U_2)}(U_1 \cup U_2, \mathcal{F}|_{U_1\cup U_2})$.

(b) For $U \cap Z = \emptyset$, $\Gamma_{Z\cap U}(U, \mathcal{F}|_U) = \Gamma(U, \mathcal{F})$. Then, we can see that if $P \in U$,

$$(\mathcal{H}^0_Z)_P = \underset{\rightarrow\ P\in U' \subset U}{lim} \Gamma_{Z\cap U'}(U', \mathcal{F}|_{U'}) = \underset{\rightarrow\ P\in U' \subset U}{lim} \Gamma_{\emptyset}(U', \mathcal{F}|_{U'}) = 0$$

Other case, for $s_p \in \mathcal{F}_p$, if $s_p = 0 \in j_*(\mathcal{F}|_U)$, then there exists $p \in U$ and $s \in \mathcal{F}(U)$ such that $s(p) = s_p$ and $s|_U = 0$. Thus, $supp(s) \subset Z$, that is $s_p \in \mathcal{H}^0_Z(\mathcal{F})_p$. So we have

$$0 \longrightarrow \mathcal{H}^0_Z(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_*(\mathcal{F}|_U)$$

When $\mathcal{F}$ is flasque, for any V, $\mathcal{F}(V) \longrightarrow \mathcal{F}(V \cap U)$ is surjective. For cokernels are a kind of colimits, we see that $\mathcal{F}_p \longrightarrow j_*(\mathcal{F}|_U)_p$ is surjective. $\square$

Remark. $supp(0) = \emptyset$. So for any $Z \subset U$, $supp(0) \in Z$. Thus, $\Gamma_Z(X, \mathcal{F})$ is a subgroup. (More precisely, Abelian group).

Exercise 2.1.21 (Some Examples of Sheaves on Varieties)**.** Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathcal{O}_X$ be the sheaf of regular functions on X (1.0.1).

(i) Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathcal{I}_Y(U)$ is a sheaf. It is called the *sheaf of ideals* $\mathcal{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

(ii) If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

(iii) Now let $X = \mathbf{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Then there is an exact sequence of sheaves on X, where $\mathscr{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$,

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathscr{F} \to 0.$$

Show however that the induced map on global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathscr{F})$ is not surjective. This shows that the global section functor $\Gamma(X, \cdot)$ is not exact (cf. (Ex. 1.8) which shows that it is left exact).

(iv) Again let $X = \mathbf{P}^1$, and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathscr{H}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathscr{H}$. Show that the quotient sheaf $\mathscr{H}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P\in X} i_P(I_P)$, where I_P is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf (Ex. 1.17) given by I_P at the point P.

(v) Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathscr{H}) \to \Gamma(X, \mathscr{H}/\mathcal{O}) \to 0$$

is exact. (This is an analogue of what is called the "first Cousin problem" in several complex variables. See Gunning and Rossi [1, p. 248].)

Proof.

(a). For any $s_i \in \mathcal{I}_Y(U_i)$, if $s_1|_{U_1\cap U_2} = s_2|_{U_1\cap U_2}$, then s_i and s_j can be glued to a section $s_1 \cup s_2$ in $\mathcal{F}(U_i \cup U_j)$. For s_1 and s_2 vanish on Y, so is $s_1 \cup s_2$, that is, $s_1 \cup s_2 \in \mathcal{I}_Y(U_1 \cup U_2)$. Thus, $\mathcal{I}_Y$ is a sheaf.

(b). Just verify at stalks. Consider $p \in X$. If $p \in X - Y$, then $\mathcal{I}_{Y,p} = \mathcal{O}_{X,p}$; If $p \in Y$, $\mathcal{I}_{Y,p} = 0$.
For taking stalks is left adjoint, it commutes with colimits, like cokernels. So $(\mathcal{O}_X/\mathcal{I}_Y)_p = \mathcal{O}_{X,p}/\mathcal{I}_{Y,p}$. Thus, if $p \notin Y$, $(\mathcal{O}_X/\mathcal{I}_Y)_p = \mathcal{O}_{X,p}/\mathcal{I}_{Y,p} = 0 = i_*\mathcal{O}_Y$; if $p \in Y$, $(\mathcal{O}_X/\mathcal{I}_Y)_p = \mathcal{O}_{X,p}/\mathcal{I}_{Y,p} = \mathcal{O}_{X,p} = i_*\mathcal{O}_{Y,p}$. Thus, $\mathcal{O}_X/\mathcal{I}_Y \cong i_*\mathcal{O}_Y$.

(c). Note that $\Gamma(X, \mathcal{O}_X) = k$ and $\Gamma(X, \mathcal{F}) = \mathcal{O}_P \oplus \mathcal{O}_Q$. Note that $k \longrightarrow \mathcal{O}_P \oplus \mathcal{O}_Q$ for they are k-algebras.

(d). Note that $\mathbb{P}^1_k$ is covered by two $\mathbb{A}^1_k$. So $\mathcal{O}_P$ is of the form $k[x]_{(f(x))}$ with f irreducible. Thus, $\mathcal{O}_P \subset K$ for each $P \in \mathbb{P}^1_k$. So there exists a natural injection $\mathcal{O} \longrightarrow \mathcal{K}$.
Because $i_P(I_P)$ is a sheaf on X for each P, by 2.1.9, $\sum_{P\in X} i_P(I_P)$ is a sheaf. We have show that $(\mathcal{K}/\mathcal{O})_p = \mathcal{K}_p/\mathcal{O}_p = K/\mathcal{O}_p$. So $\mathcal{K}/\mathcal{O} \cong \sum_{P\in X} i_P(I_P)$, for the have the same stalks. (Because $\oplus$ is a kind of colimits, so it commutes with taking stalks.)

(e). For $\mathbb{P}^1_k$, each elements in $\sum_{p\in X} i_P(I_p)(X)$ can be written as a $f_1 + f_2 + ... + f_n$ with $f_i \in I_{P_i}$. Note that I_{P_i} are of the form $K/k[x]_{(f(x))}$. So $f_i \in K$, then $f_1 + ... + f_n \in K$. So $\mathcal{K}(X) \longrightarrow \mathcal{K}/\mathcal{O}(X)$ is surjective. $\square$

Remark. By the representation of $\mathcal{K}/\mathcal{O}_X$ in (d), $\mathcal{K}/\mathcal{O}_X$ is flasque.

Exercise 2.1.22 (Glueing Sheaves)**.** Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathscr{F}_j|_{U_i \cap U_j}$ such that

(i) for each i, $\varphi_{ii} = \mathrm{id}$, and

(ii) for each i, j, k, $\varphi_{ik} = \varphi_{jk} \cdot \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$ such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathscr{F}$ is obtained by *glueing* the sheaves $\mathscr{F}_i$ via the isomorphisms φ_{ij}.

Proof. Define $\mathcal{F}(V) = \{(s_i)_i | s_i \in \mathcal{F}^i(U_i)|_V$ and $\psi_{ij} \circ \rho^{\mathcal{F}_i}_{V \cap U_i, V \cap U_{ij}}(s_i) \overset{*}{=} \rho^{\mathcal{F}_j}_{V \cap U_j, V \cap U_{ij}}(s_j)\}$.

(i). Define $\rho_{VW} : \mathcal{F}(V) \to \mathcal{F}(U)$ by $(s_i)_i \mapsto (\rho^{\mathcal{F}_i}_{V \cap U_i, W \cap U_i}(s_i))_i$.

For $\psi_{ij} : \mathcal{F}^i|_{U_{ij}} \to \mathcal{F}^j|_{U_{ij}}$ is an isomorphism,

$$\begin{aligned}\psi_{ij} \circ \rho^{\mathcal{F}_i}_{W \cap U_i, W \cap U_{ij}}(\rho^{\mathcal{F}_i}_{V \cap U_i, W \cap U_i}(s_i)) &= \psi_{ij}(\rho^{\mathcal{F}_i}_{V \cap U_i, W \cap U_{ij}}(s_i)) \\ &= \rho^{\mathcal{F}_j}_{V \cap U_j, W \cap U_{ij}}(s_j) \\ &= \rho^{\mathcal{F}_j}_{W \cap U_j, W \cap U_{ij}}(\rho^{\mathcal{F}_i}_{V \cap U_i, W \cap U_i}(s_j))\end{aligned}$$

So ρ_{VW} is exactly a map between $\mathcal{F}(V)$ and $\mathcal{F}(W)$

Then $\rho_{W,E} \circ \rho_{V,W} = \rho_{V,E}$ for $E \subset W \subset V$:

$$\begin{aligned}\rho_{W,E} \circ \rho_{V,W}(s_i)_i &= \rho^{\mathcal{F}_i}_{W \cap U_i, E \cap U_i}(\rho^{\mathcal{F}_i}_{V \cap U_i, W \cap U_i}(s_i))_i \\ &= (\rho^{\mathcal{F}_i}_{V \cap U_i, E \cap U_i}(s_i))_i \\ &= \rho_{V,E}(s_i)_i\end{aligned}$$

(ii). Let $V = V_1 \cap V_2$. Take $(s_i^1)_i \in \mathcal{F}(V_1)$ and $(s_i^2)_i \in \mathcal{F}(V_2)$ and assume that

$$\rho_{V_1, V_1 \cap V_2}((s_i^1)_i) = \rho_{V_2, V_1 \cap V_2}((s_i^2)_i)$$

Then for s_i^1 and s_i^2,

$$\rho^{\mathcal{F}_i}_{V_1 \cap U_i, V \cap U_{ij}}(s_i^1) = \rho^{\mathcal{F}_i}_{V_2 \cap U_i, V \cap U_{ij}}(s_i^2)$$

For $\mathcal{F}^i$ is a sheaf, there exists a unique $s_i \in \mathcal{F}^i(V \cap U_i)$ such that $\rho^{\mathcal{F}_i}_{V \cap U_i, V_k \cap U_i}(s_i) = s_i^k$ with $k = 1, 2$.

Also, because $(s_i^1)_i$ and $(s_i^2)_i$ satisfy the relations about ψ_{ij} so $(s_i)_i$ also satisfies that. Thus, $(s_i)_i \in \mathcal{F}()V$ and it's unique by the uniqueness of each $\mathcal{F}^i$.

So $\mathcal{F}$ is a sheaf. Define $\psi_i : \mathcal{F}(V) \to \mathcal{F}^i(V)$ by $(s_i)_i \mapsto s_i$, which is obviously a homomorphism in (Ab), by the definition of it.

(iii). ψ_i is an isomrophism.

ψ_i is an injection: if $\psi_i(s_i)_i = \psi_i(s_i^{'})_i$, then for each U_j such that $U_i \cap U_j \neq \emptyset$,

$$\psi_{ij} \circ \rho_{V \cap U_i, V \cap U_{ij}}(s_i) = \psi_{ij} \circ \rho_{V \cap U_i, V \cap U_{ij}(s_i')}$$

which means

$$\rho_{V \cap U_j, V \cap U_{ij}}(s_j) = \rho_{V \cap U_j, V \cap U_{ij}}(s_j^{'})$$

for ψ_{ij} is an isomorphism $s_j = s_j^{'}$.

ψ_i is an surjection. For ψ_{ij} is an isomorphism, for each s_i there exist unique s_j satisfying (*). For each U_k such that $U_{ijk} \neq \emptyset$, we can get a s_k from s_i and s_j, but they are the same: $s_{ik} = s_{jk}$ on U_{ijk} for $\psi_{ik} = \psi_{jk} \circ \psi_{ij}$. Also by ψ_{ij} is an isomorphism (More precisely, for $\psi_{ij}(U_{ijk})$), $s_{ik} = s_{ij}$. Thus, we can get s_k from s_j such that s_k and s_i satisfy (*) on U_{ik}. So we can find $(s_i)_i$ such that $\psi_i(s_i)_i = s_i$.

Thus, ψ_i is an isomorphism. We conclude that $(\mathcal{F}, \psi_i)$ is unique up to isomorphism. $\square$

2.1.4 Addition Exercises

Exercise 2.1.23. Let X be a topological space and $i : Z \longrightarrow X$ the inclusion of a closed subset. Prove the following assertions:

(i) The direct image functor $i_* : Sh(Z) \longrightarrow Sh(X)$ is exact.

(ii) The direct image $i_*\mathcal{I}$ of a flasque sheaf is flasque again.

Proof.

(i). Consider a sheaf $\mathcal{F}$ over Z. For each $P \in X$, if $P \in Z$, then $i_*\mathcal{F}_P = \mathcal{F}_P$ and if $P \notin Z$, then $i_*\mathcal{F}_P = 0$ for there always exists $P \in U_P$ such that $U_P \cap Z = \emptyset$.

Now consider an exact sequence of sheaves over Z:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

and

$$0 \longrightarrow i_*\mathcal{F} \longrightarrow i_*\mathcal{G} \longrightarrow i_*\mathcal{H} \longrightarrow 0$$

As we have see, if $P \in Z$, then their stalks on X are just the same as their stalks on Z. If $P \notin Z$, their stalks are all 0. Hence, on each $P \in X$, the sequence of stalks are exact. Hence the sequence is exact. Thus, i_* is exact.

(ii). We have shown that for any f, f_* maps flasque sheaves to flasque sheaves. $\square$

Remark. If $i : U \hookrightarrow X$ is an inclusion of an open set, (i) may fall.

Consider $X = \mathbb{C}$ and $U = X - \{0\}$. Consider

$$0 \longrightarrow \underline{\mathbb{Z}}^+ \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{O}_U^* \longrightarrow 0$$

is exact on U.

$$0 \longrightarrow i_*\underline{\mathbb{Z}}^+ \longrightarrow i_*\mathcal{O}_U \longrightarrow i_*\mathcal{O}_U^* \longrightarrow 0$$

is not exact on X. More precisely, $i_*\mathcal{O}_{U,0} \longrightarrow i_*\mathcal{O}_{U,0}^*$ is not a surjection by basic complex analysis.

Exercise 2.1.24. Let x be an arbitrary point of a topological space X. Then $i_{x,*}$ is exact.

Proof. Consider the topology on $\{x\}$. For any $\mathcal{F}$ over $\{x\}$, $\mathcal{F}|_x = \mathcal{F}(\{x\})$.

Consider an exact sequence of sheaves over $\{x\}$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

that is,

$$0 \longrightarrow \mathcal{F}(\{x\}) \longrightarrow \mathcal{G}(\{x\}) \longrightarrow \mathcal{H}(\{x\}) \longrightarrow 0$$

is exact.

Consider $\{x\} \subset X$. If $y \notin \{x\}^-$, $i_{x,*}\mathcal{F}_y = 0$. If $y \in \{x\}^-$, any open set U containing y contains x. Hence $i_{x,*}\mathcal{F}_y = \mathcal{F}(\{x\})$. Thus, verifying on stalks, we see that

$$0 \longrightarrow i_{x,*}\mathcal{F} \longrightarrow i_{x,*}\mathcal{G} \longrightarrow i_{x,*}\mathcal{H} \longrightarrow 0$$

is exact. Hence, $i_{x,*}$ is exact. $\square$

2.2 Scheme

2.2.1 Examples

2.2.2 Exercises

Exercise 2.2.1. Let A be a ring, let $X = \text{Spec } A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_{X|D(f)})$ is isomorphic to $\text{Spec } A_f$.

Proof. By definition

$$D(f) = \{\mathfrak{p} \in Spec(A) | f \notin \mathfrak{p}\}$$

By the property of localization,

$$Spec(A_f) = \{\mathfrak{p}_f \in Spec(A) | \mathfrak{p} \cap \{f^n | n \in \mathbb{N}\} = \emptyset\}$$

Then the topology structures of $D(f)$ and $Spec(A_f)$ are homeomorphism by $D_A(g) \cap D_A(f) \longleftrightarrow D_{A_f}(\frac{g}{f})$.

For the sheaf structures, we just need to verify it on stalks.

Take $\mathfrak{p} \in D(f)$, $\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$. For the corresponding $\mathfrak{p}_f \in Spec(A_f)$, $\mathcal{O}_{A_f,\mathfrak{p}_f} = (A_f)_{\mathfrak{p}_\mathfrak{f}}$. By the property of localization,

$$\begin{array}{ccc} (A_f)_{\mathfrak{p}_\mathfrak{f}} & \xrightarrow{\cong} & A_{\mathfrak{p}} \\ \frac{g/f^n}{h/f^m} & \mapsto & \frac{g}{h} f^{n-m} \end{array}$$

with $f, h \notin \mathfrak{p}$, which ensures the morphism defined above is well-defined. It is easy to verify the morphism defined above is an locally-ringed isomorphism of local rings. Thus $\mathcal{O}_{X,\mathfrak{p}} \cong \mathcal{O}_{A_f,\mathfrak{p}_f}$. So $(D(f), \mathcal{O}_X|_{D(f)}) \cong Spec(A_f)$. □

Exercise 2.2.2. Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. We call this the induced scheme structure on the open set U, and we refer to $(U, \mathcal{O}_X|_U)$ as an open subscheme of X.

Proof. Suppose that $(X, \mathcal{O}_X) = \cup_{i \in I}(SpecA_i, \mathcal{O}_{A_i})$ where $\{(SpecA_i, \mathcal{O}_{A_i})\}_{i \in I}$ satisfies the cocyclic condition 2.1.22. Because $\{D(a) | \exists i \in I, a \in A_i\}$ forms a topology basis of X, so in topology $U = \cup_{j \in J} D(a_j)$. Again for X is a scheme, $\{(D(a_j), \mathcal{O}|_{D(a_j)})\}$ satisfies the cocyclic condition. Thus $(U, \mathcal{O}_X|_U) = \cup_{j \in J}(D(a_j), \mathcal{O}|_{D(a_j)})$ is a scheme. □

Exercise 2.2.3. A scheme $(X, \mathcal{O}_X)$ is reduced if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

(a) Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.

(b) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{\text{red}}$, where for any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme. We call it the reduced scheme associated to X, and denote it by X_{red}. Show that there is a morphism of schemes $X_{\text{red}} \to X$, which is a homeomorphism on the underlying topological spaces.

(c) Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{\text{red}}$ such that f is obtained by composing g with the natural map $Y_{\text{red}} \to Y$.

Proof. (Proposition 4.2 of [10])

(a) (i)$\Longrightarrow$: If there exists P such that $\mathcal{O}_{X,P}$ is not nilpotent, then there exists $s_P \in \mathcal{O}_{X,P}$, $s_P \neq 0$ and $n \in \mathbb{N}$ such that $(s_P)^n = (s)_P^n = 0$. Because (X, O_X) is a ringed space, so all morphisms are ring homomorphism, which implies $(s_P)^n = (s)_P^n = 0$. Then there exists $P \in U$ such that $s^n|_U = 0$. Note that for $s_P \neq 0$, $s|_U \neq 0$, which contradicts to the definition of reduced scheme. So each stalk is nilpotent.

$\Longleftarrow$: Just assume that $(X, \mathcal{O}_X)$ is not reduced. Then there exists $U \in X$ such that $\mathcal{O}_X(U)$ is not nilpotent. Take $s \neq 0 \in \mathcal{O}_X(U)$. There must exist $P \in U$ such that $s_P \neq 0$. O.W. by 2.1.11. Then $\mathcal{O}_{X,P}$ is not nilpotent for $(s_P)^n = (s)_P^n = 0$, which leads to a contradiction.

(b). Because any ring homomorphism maps nilpotent elements to nilpotent elements, the presheaf structure of $\mathcal{O}^{pr}_{X^{red}}$ is defined by

$$\rho^{red}(U) : s \mapsto \rho(s)$$

and let $\mathcal{O}_X^{red} = \mathcal{O}^{pr}_{X^{red}}$. Note that they have the same stalks.

Suppose that $P = \mathfrak{p} \in Spec(A)$. Note that for $f \notin \mathcal{N}(A)$, $\mathcal{O}(D(f))^{red} = (A_f)^{red} = A_f^{red}$, while for $f \in (N)(A)$, $V(f) = V(f^n) = X$ for some n. Thus, $\{D(f)|f \notin \mathcal{N}(A)\} \cup \{D(0)\}$ forms a topology basis for $Spec(A)$. Then we have:

$$\mathcal{O}_X^{red}|_{Spec(A),\mathfrak{p}} = \{\frac{a}{b}|a \in A^{red}, b \notin \mathfrak{p}\}$$

which is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}^{red}$ and reduced.

Because $Spec(A) = Spec(A^{red})$by the fact that $A^{red} = \cap_{\mathfrak{p},prime}\mathfrak{p}$ and their stalks are isomorphic, so $(Spec(A), \mathcal{O}_A) \cong (SpecA^{red}, \mathcal{O}_{A^{red}})$. For isomorphisms preserves nilpotemt elements, the gluing functions of $\{(Spec(A), \mathcal{O}_A)\}$ are still the glue functions of $\{(Spec(A^{red}), \mathcal{O}_{A^{red}}\}$. Thus $(X^{red}, \mathcal{O} = \cup(Spec(A^{red}, \mathcal{O}_{red}))$ is a scheme and it is reduced.

For X^{red} is homeomorphism to X, this is because $Spec(A) = Spec(A^{red})$.

(c). For any A and a reduecd ring B with $\varphi : A \longrightarrow B$, we have

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "\varphi"] & A^{red} \arrow[d, "\exists!\overline{\varphi}"] \\ & B \end{tikzcd}$$

by the universal property of nilpotent elements.

Take *Spec* for the diagram above:

$$\begin{tikzcd} Spec(A) & Spec(A^{red}) \arrow[l, "Spec(i)"'] \\ & Spec(B) \arrow[ul, "Spec(\varphi)"] \arrow[u, "\exists! Spec(\overline{\varphi})"'] \end{tikzcd}$$

where the uniqueness comes from $\Gamma(Spec(A), \mathcal{O}_A) = A$.

Suppose that $Y = \cup Spec(A_i)$ and $\varphi^{-1}(Spec(A_i)) = \cup_j Spec(B_{ij})$. Then for $\varphi|_{Spec(B_{ij})}$: $Spec(B_{ij}) \longrightarrow Spec(A_i)$, applying the above discussion, there is unique map $\overline{\varphi|_{Spec(B_{ij})}} : Spec(B_{ij}) \longrightarrow$

$Spec(A_i)_{red}$. Use the gluing functions of $\{Spec(Bij)\}$ to get a morphism $\overline{\varphi}_i : \cup_j Spec(B_{ij}) \longrightarrow Spec(A_i)_{red}$. By the relation between the gluing functions of $\{Spec(A_i)\} = \{Spec(A_i)_{red}\}$ and $\{\cup_j Spec(B_{ij})\}$ we can get a scheme morphism $\overline{\varphi} : X \longrightarrow Y_{red}$ such that $\varphi = \overline{\varphi} \circ i$ where i is the natural map $Y_{red} \longrightarrow Y$. □

Exercise 2.2.4. Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f : X \to \text{Spec } A$, we have an associated map on sheaves $f^* : \mathcal{O}_{\text{Spec } A} \to f_*\mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map

$$\alpha : \text{Hom}_{\text{Sch}}(X, \text{Spec } A) \to \text{Hom}_{\text{Ring}}(A, \Gamma(X, \mathcal{O}_X)).$$

Show that α is bijective (cf. (I, 3.5) for an analogous statement about varieties).

Proof. Use the method in proving 2.1.9. We need to consider:

$$A \longrightarrow \Gamma \circ SpecA; \qquad X \longrightarrow Spec \circ \Gamma(X, \mathcal{O}_X)$$

.

For the first morphism: $i_A : A \longrightarrow \Gamma \circ SpecA = \Gamma(SpecA, \mathcal{O}_A) = A$ so it is just the identity of A i.e. id_A.

For the second one: We first assume that $X = Spec(B)$, then then the morphism is just id_B. Suppose that $X = \cup_i B_i$, then we have $\rho_{X,B_i} : \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(SpecB_i, \mathcal{O}_{B_i} = B_i)$. Then we have $Spec(\rho_{X,B_i}) : Spec(B_i) \longrightarrow Spec(\Gamma(X, \mathcal{O}_X))$. Then glue $Spec(\rho_{X,B_i})$ with the gluing functions of $\{Spec(B_i, \mathcal{O}_{B_i})\}$ to get

$$\cup Spec(\rho_{X,B_i}) : X \longrightarrow Spec(\Gamma(X, \mathcal{O})_X)$$

.

Now assume that $X = \cup Spec(B_i)$, we define

$$\begin{aligned} \alpha : Hom_{Sch}(X, SpecA) &\longrightarrow Hom_{Rings}(A, \Gamma(X, \mathcal{O}_X)) \\ f &\longmapsto \Gamma(f) \circ id_A \\ \beta : Hom_{Rings}(A, \Gamma(X, \mathcal{O}_X) &\longrightarrow Hom_{Sch}(X, SpecA) \\ g &\longmapsto Spec(g) \circ \cup Spec(\rho_{X,B_i}) \end{aligned}$$

Then

$$\alpha \circ \beta(g) = \Gamma(Spec(g) \circ \cup Spec(\rho_{X,B_i}) \circ id_A = g$$

$$\beta \circ \alpha(f) = Spec(\Gamma(f) \circ id_A) \circ \cup Spec(\rho_{x,B_i})$$

□

Exercise 2.2.5. Describe Spec $\mathbb{Z}$, and show that it is a final object for the category of schemes, i.e., each scheme X admits a unique morphism to Spec $\mathbb{Z}$.

Proof. $Spec(\mathbb{Z}) = \{(p) | \text{p is prime in } \mathbb{Z}\}$. Then $Spec(\mathbb{Z})_{(p)} = \{\frac{a}{b} | a \in \mathbb{Z}, (b) \notin (p)\}$. This is a local ring with maxmial ideal $(p)\mathbb{Z}_{(p)}$ so $Spec\mathbb{Z}$ is a shceme.

Suppose that $X = \cup Spec(A_i)$. Then for $\mathbb{Z}$ is the initial object in *Rings*, there exists a morphism $i : \mathbb{Z} \longrightarrow A$ by $1 \mapsto e_A$. For any $\mathfrak{p} \in A$ that is prime, assume that $(p) = i^{-1}(\mathfrak{p})$. Then $\mathbb{Z}_{(p)} \longrightarrow A_{\mathfrak{p}}$ maps $\frac{a}{b}$ to $\frac{i(a)}{i(c)}$ and maps $(p)\mathbb{Z}_{((p)}$ to $\mathfrak{p}A\mathfrak{p}$. So $Spec(i) : Spec(A) \mapsto Spec(Z)$ is a morphism between schemes and it is unique.

Gluing $Spec(A_i)$ with the gluing functions of $\{Spec(A_i), \mathcal{O}_{A_i}\}$, there is a unique morphism between schemes $X \longrightarrow Spec(\mathbb{Z})$. Thus $Spec(\mathbb{Z})$ is a final object in *Sch*. □

Exercise 2.2.6. Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1. Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R.)

Proof. Just note that (0) is the finial object in *Rings*. Repeat the proof 2.2.5. $\square$

Exercise 2.2.7. Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the residue field of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of Spec K to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Proof. Given any $Spec(K) \longrightarrow X$, suppose that $(0) \mapsto x \in X$. Consider $i : x \hookrightarrow X$. We have a morphism between stalks $\varphi : \mathcal{O}_{X,x} \longrightarrow K_{(0)} = K$. For φ is a locally ringed homomorphism, it maps $\mathfrak{m}_x$ to (0) Thus, we have $i : \mathcal{O}_{X,x}/\mathfrak{m}_x = k(x) \longrightarrow K/(0) = K$.

Conversely, given any $k(x) \hookrightarrow K$. For $i : \{x\} \hookrightarrow X$, we can define a scheme structure on $\{x\}$, that is, $(\{x\}, \mathcal{O}_K)$. Consider

$$(i, i^\#) : (\{x\}, \mathcal{O}_K) \longrightarrow (X, \mathcal{O}_X)$$

such that $i^\# : \mathcal{O}_X \longrightarrow i^{-1}\mathcal{O}_K$. Taking stalks, we define $i_x : \mathcal{O}_{X,x} \longrightarrow K$ by $a \in \mathcal{O}_{x,x} \xrightarrow{p} \bar{a} \in k(x) \hookrightarrow K$. On other stalks $\mathcal{O}_{X,p} \longrightarrow 0$. Thus, $i^\#$ is a locally ringed space. So $(i, i^\#)$ is a morphism between schemes. $\square$

Exercise 2.2.8. Let X be a scheme. For any point $x \in X$, we define the Zariski tangent space T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\epsilon]/\epsilon^2$ be the ring of dual numbers over k. Show that to give a k-morphism of Spec $k[\epsilon]/\epsilon^2$ to X is equivalent to giving a point $x \in X$, rational over k (i.e., such that $k(x) = k$), and an element of T_x.

Proof. Note that $Spec(k[\epsilon]/(\epsilon^2)) = \{(\epsilon)\}$

Given $\varphi : Spec(k[\epsilon]/(\epsilon^2)) \longrightarrow X$. Suppose that (ϵ) is mapped to x. Then we have

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "\varphi^\#_{(\epsilon)}"] & \mathcal{O}_{(\epsilon)} \\ & k \arrow[ul, "i^\#_X"] \arrow[u, "i^\#"'] \end{tikzcd}$$

For all these are locally ringed homomorphism, they induce:

$$\begin{tikzcd} k(x) \arrow[r, "\varphi^\#_{(\epsilon)}"] & k \\ & k \arrow[ul, "i^\#_X"] \arrow[u, "i^\#"'] \end{tikzcd}$$

Thus, $k(x) = k$.

For $\varphi^\#((\epsilon)) \subset m_x$, we map ϵ to $\mathfrak{m}_{\mathfrak{x}}$ and $\varphi^\#(\epsilon^2) \in \mathfrak{m}_x^2$ $\epsilon^2 \in (\epsilon^2) = 0 \in k[\epsilon]/(\epsilon^2)$. So ϵ is an element in T_x. (In fact, $\epsilon = 0$ or $\epsilon \in \mathfrak{m}_x - \mathfrak{m}_x^2$).

Given any point x and $y \in \mathfrak{m}_x$. Then define a scheme $(x, Spec(k[\epsilon]/(\epsilon^2))$ and $i : (x, Spec(k[\epsilon]/(\epsilon^2))) \longrightarrow (X, \mathcal{O}_\S)$ given by

$$i_x : \mathcal{O}_{X,x} \longrightarrow k[\epsilon]/(\epsilon^2)$$

with $y \mapsto \epsilon$. Then this is a locally ringed homomorphism. So i is a morphism between schemes. $\square$

Remark. Note that $k[\epsilon]/(\epsilon^2)$ is not integral so $(0) \notin Spec(k[\epsilon]/(\epsilon^2))$

Exercise 2.2.9. If X is a topological space, and Z an irreducible closed subset of X, a generic point for Z is a point ζ such that $Z = \{\zeta\}^-$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

Proof. Suppose that $X = \cup Spec(A_i)$ and the generic point P of the irreducible subset Z lies in $Spec(A_i) \cap Z$. By Ch1 Example 1.1.3. of [5], $Spec(A_i) \cap Z$ is irreducible. By Ch1 Corollary 1.5 of [5]. There exists some prime ideal $\mathfrak{p}$ such that $V(\mathfrak{p}) = \{P\}^-$.

Now, suppose that P' is another generic point of Z. Then $P' \in Z \cap Spec(A_i)$. Because $Z - Z \cup Spec(A_i)$ is closed, if $P' \notin Z \cap Spec(A_i)$, then the closure of P' doesn't contain $Spec(A_i)$. Thus $P' \in V(\mathfrak{p})$. Because $V(\mathfrak{p})$ has only one generic point, that is, $\mathfrak{p}$. So $P = P' = \mathfrak{p}$. □

Remark. Later, we will see that $V_{\mathfrak{p}} = Spec(A/\mathfrak{p})$ is integral. It has a unique minimal prime ideal (0), corresponding to the generic point of $V_{\mathfrak{p}}$.

Exercise 2.2.10. Describe Spec $\mathbb{R}[x]$. How does its topological space compare to the set $\mathbb{R}$? To $\mathbb{C}$?

Proof. Note that $\mathbb{R}[x]$ is a PID. So

$$Spec(\mathbb{R}[x]) = \{(f) | f \in \mathbb{R}[x] \text{ and f is irreducible}\} \cup \{(0)\}$$

. Fot the sheaf sturcture, just note that $\mathcal{O}_{X,(f)} = \mathbb{R}[x]_{(f)}$ and $\mathcal{O}_{X,(0)} = \mathbb{R}(x)$

For $\mathbb{C}[x]$, By Hilbert's Nullstellensatz,

$$Spec(\mathbb{C}[x]) = \{(x-c) | c \in \mathbb{C}\} \cup \{(0)\}$$

with $\mathcal{O}_{X,(x-c)} = \mathbb{C}[x]_{x-c}$ and $\mathcal{O}_{X,(0)} = \mathbb{C}(x)$

□

Exercise 2.2.11. Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe Spec $k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

Proof. For any field k, $k[x]$ is a PID. Then

$$Spec(\mathbb{F}_p[x]) = \{(f) \subset \mathbb{F}_p[x] | f \text{ is irreducible}\} \cup \{(0)\}$$

By the property of a field, we just assume p is a prime number.

Now we focus on the irreducible polynomials in $\mathbb{F}_p[x]$. By Ch VII Corollary 5.6 of [1],any irreducible polynomial of deg d is a factor of $x^{p^d} - x$ with d.

For such a $f(x)$ with $deg(f) = d$

$$\mathbb{F}_p[x]_{(f)}/(f)\mathbb{F}_p[x]_{(f)} = \mathbb{F}_{p^d}$$

because $\mathbb{F}_p/(f(x)) = \mathbb{F}_{p^d}$. So the residue field of $k((f(x))) = \mathbb{F}_{p^d}$.

Then we just need to know the number of elements in $\mathbb{F}_p[x]$ with degree d: Let M_n be the number of irreducible polynomials of deg n. Then

$$p^d = \sum_{n|d} nM_n$$

Use the Möbius inversion

$$M_d = \sum_{n|d} \mu(n) q^{\frac{n}{d}}$$

where μ is the Möbius function. For the details of this function, see P20 Theorem [7]. □

Exercise 2.2.12. Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possibly infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$, and let it have the induced scheme structure (Ex. 2.2). Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ such that (1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$, and (2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$. Then show that there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}. We say that X is obtained by glueing the schemes X_i along the isomorphisms φ_{ij}. An interesting special case is when the family X_i is arbitrary, but the U_{ij} and φ_{ij} are all empty. Then the scheme X is called the disjoint union of the X_i, and is denoted $\coprod X_i$

Proof. For the sheaf structure, by 2.1.22, the sheaf structure of these schemes can be glue together.

Then, for X_i are glued together on open sets, $X = \cup X_i$ is still locally isomorphic to affine schemes. So X is a scheme. □

Exercise 2.2.13. A topological space is quasi-compact if every open cover has a finite subcover.

(a) Show that a topological space is noetherian (I, §1) if and only if every open subset is quasi-compact.

(b) If X is an affine scheme, show that $\mathrm{sp}(X)$ is quasi-compact, but not in general noetherian. We say a scheme X is quasi-compact if $\mathrm{sp}(X)$ is.

(c) If A is a noetherian ring, show that $\mathrm{sp}(\mathrm{Spec}\, A)$ is a noetherian topological space.

(d) Give an example to show that $\mathrm{sp}(\mathrm{Spec}\, A)$ can be noetherian even when A is not.

Proof.

(a). ($\Longrightarrow$): For any open set U, if U is covered by $\{U_i\}$, then $\{U_i^C\}$ forms a set of closed sets. Consider

$$U_1^C \supseteq U_1^C \cap U_2^C \supseteq ... \supseteq \cap_{i=1}^n U_i^C \supseteq ...$$

For X is Noetherian, there exists N such that $\cap_{i=1}^N U_i^C = \cap_{i=1}^\infty U_i^C = U^C$. So U is covered by $\{U_1, U_2, ..., U_n\}$.

($\Longleftarrow$): For any open set sequence:

$$V_1 \supseteq V_2 \supseteq ... \supseteq V_n \supseteq ...$$

Consider $V = \cap_i^\infty V_i$, which is a closed set and V^C is covered by $\{V_i^C\}_{i=1}^\infty$. For by assumption, we assume that V^C is covered by $\{V_i^C\}_{i=1}^N$, which implies $\cup_i^N V_i^C = V^C$. Then $V = \cap_{i=1}^N V_i$. So X is Noetherian.

(b). A set is quasi-compact if and only if for any closed set V and $\{V_i\}$ such that $\cap_i^\infty V_i = \emptyset$, there exists N such that $\cap_{i=1}^N V_i = \emptyset$.

For any $V(a)$ that is closed in $X = Spec(A)$, suppose that $\cap_{i=1}^\infty V(\mathfrak{a}_i) = V(1) = \emptyset$. Then $\sum_i^\infty \mathfrak{a}_i = 1$ i.e. $1 = \sum_{i=1}^N b_i a_i$ with $a_i \in \mathfrak{a}_i$ and $b_i \in A$. Then $\cap_{i=1}^N V(\mathfrak{a}_i) = V(1) = \emptyset$. Thus, X is quasi-compact.

Just note that $Spec(k[x_1, x_2, ..., x_n, ...])$ is quasi-compact but $k[x_1, x_2, ..., x_n, ..]$ it is not Noetherian:

$$(x_1) \subseteq (x_1, x_2) \subseteq ... \subseteq (x_1, x_2, ..., x_n) \subseteq ...$$

Then

$$V(x_1) \supseteq V(x_1, x_2) \supseteq ... \supseteq V(x_1, x_2, ..., x_n) \supseteq ...$$

(c) For any

$$V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \supseteq ... \supseteq V(\mathfrak{a}_n) \supseteq ...$$

we have

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq ... \subseteq \mathfrak{a}_n \subseteq ...$$

For A is Noetherian, there exists N such that $\mathfrak{a}_N = \mathfrak{a}_{N+i}$ for ant $i \geq 0$, that is, $V(\mathfrak{a}_N) = V(\mathfrak{a}_{i+N})$ for any $i \geq 0$. So $Spec(A)$ is Noetherian.

(d). Consider $Spec(k[x_1, x_2, ..., x_n]/(x_1^2, ..., x_n^2, ..) = \{(x_1, ..., x_n, ...)\})$. As a topology space, it is Noetherian. However, $k[x_1, ..., x_n, ...]/(x_1^2, ..., x_n^2, ...)$ is not a Noether Ring. □

Exercise 2.2.14. (a) Let S be a graded ring. Show that $\operatorname{Proj} S = \emptyset$ if and only if every element of S_+ is nilpotent.

(b) Let $\varphi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{p \in \operatorname{Proj} T | p \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of $\operatorname{Proj} T$, and show that φ determines a natural morphism $f : U \to \operatorname{Proj} S$.

(c) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

(d) Let V be a projective variety with homogeneous coordinate ring $S(1, 2)$. Show that $t(V) \cong \operatorname{Proj} S$.

Proof.

(a).($\Longrightarrow$): $Proj(S) = \emptyset$ means that there is no homogeneous prime ideal $\mathfrak{p} \subset S$ such that $S^+ \not\subseteq \mathfrak{p}$, i.e. All homogeneous prime ideals contains S^+.

For any prime ideal $\mathfrak{p} \subseteq S$, $\mathfrak{p}^h = \oplus_{i=0}^{\infty} \mathfrak{p} \cap S_i$ is a homogeneous prime ideal. Then $\mathfrak{p}^h \subset S^+$. Note that $\mathfrak{p} \subset \mathfrak{p}^+$ and

$$\mathcal{N}(S) = \cap_{prime} \mathfrak{p} \supset \cap \mathfrak{p}^h \supset S^+$$

($\Longleftarrow$): Note that $\cap_{prime} \mathfrak{p} = \mathcal{N}(S) \supseteq S^+$. So each homogeneous prime ideal contains S^+, which implies $Proj(S) = \emptyset$.

(b). For any $\mathfrak{p} \in Proj(T)$, $\mathfrak{p}$ is homogeneous and $T^+ \not\subseteq \mathfrak{p}$. Suppose that $\mathfrak{p} \in U$ and take $t \notin \mathfrak{p}$ and $t \in \varphi(S_+)$ homogeneous. Then for any $\mathfrak{q} \in D_+(t)$, $\mathfrak{q} \notin \varphi(S_+)$. Hence U is open.

On U, we define $f : U \longrightarrow Proj(S)$ and $f_{\mathfrak{p}} : S_{(\varphi^{-1}(\mathfrak{p}))} \longrightarrow T_{(\mathfrak{p})}$ by $\frac{a}{b} \mapsto \frac{\varphi(a)}{\varphi(b)}$ where a, b are homogeneous of the same degree and $b \notin \varphi^{-1}(\mathfrak{p})$. Next we show that this is a locally ringed morphism: The maximal ideal of $S_{\varphi^{-1}(\mathfrak{p})}$ is $\varphi^{-1}(\mathfrak{p})_{\varphi^{-1}(\mathfrak{p})} \cap S_{\varphi^{-1}(\mathfrak{p})}$, that is,

$$\{\frac{a}{b} | a \in \varphi^{-1}(\mathfrak{p}), b \text{ homogeneous of the same degree}\}$$

and for φ preserves the degree, $f_{\mathfrak{p}}(\varphi^{-1}(\mathfrak{p})_{\varphi^{-1}(\mathfrak{p})} \cap S_{\varphi^{-1}(\mathfrak{p})})$ is

$$\{\frac{a}{b} | a \in \mathfrak{p}, b \text{ homogeneous of the same degree}\}$$

which is the maximal ideal of $T_{(\mathfrak{p})}$. So f is a morphism of schemes.

(c). Consider $S := \mathbb{C}[x^2, x^3, y] \hookrightarrow T := \mathbb{C}[x, y]$, which satisfies $T_0 = S_0 = \mathbb{C}$. Then for a homogeneous prime ideal $\mathfrak{p} \subseteq \mathbb{C}[x, y]$, $i^{-1}(\mathfrak{p}) = \mathfrak{p} \cap \mathbb{C}[x^2, x^3, y]$ is also a homogeneous prime ideal of $\mathbb{C}[x^2, x^3, y]$. Finally, we see that $\mathbb{C}[x^2, x^3, y]_t = \mathbb{C}[x, y]_t$, for any $t \in T$ homogeneous, which implies $f : Proj(T) \longrightarrow Proj(S)$ is an isomorphism between schemes.

(d). *** □

Exercise 2.2.15. (a) Let V be a variety over the algebraically closed field k. Show that a point $P \in t(V)$ is a closed point if and only if its residue field is k.

(b) If $f : X \to Y$ is a morphism of schemes over k, and if $P \in X$ is a point with residue field k, then $f(P) \in Y$ also has residue field k.

(c) Now show that if V, W are any two varieties over k, then the natural map

$$\mathrm{Hom}_{a_k}(V, W) \to \mathrm{Hom}_{a_k}(t(V), t(W))$$

is bijective. (Injectivity is easy. The hard part is to show it is surjective.)

Proof.
(a).***

(b). For

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i_X"'] & Y \arrow[d, "i_Y"] \\ & Spec(K) \end{tikzcd}$$

Taking stalks, we have For

$$\begin{tikzcd} \mathcal{O}_{X,P} & \mathcal{O}_{Y,f(P)} \arrow[l, "f_P^\#"'] \\ & \mathcal{O}_{Spec(K),i_X(P)} \arrow[ul, "{i_X}^\#_P"] \arrow[u, "{i_Y}^\#_{f(P)}"'] \end{tikzcd}$$

Because these are locally ringed morphisms, we have

$$\begin{tikzcd} k(P) = k & k(f(P)) \arrow[l, "f_P^\#"'] \\ & k \arrow[ul, "{i_X}^\#_P"] \arrow[u, "{i_Y}^\#_{f(P)}"'] \end{tikzcd}$$

So $k(Y) = k$.

(c).*** □

Exercise 2.2.16. Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.

(a) If $U = \operatorname{Spec} B$ is an open affine subscheme of X, and if $\bar{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

(b) Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$. [Hint: Use an open affine cover of X]

(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. (This hypothesis is satisfied, for example, if $\mathrm{sp}(X)$ is noetherian.) Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

(d) With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.

Proof.

(a). For $f_x = \overline{f}_x$ when $x = \mathfrak{p} \in U$, $\overline{f}_x \notin \mathfrak{m}_x = \mathfrak{p}A_{\mathfrak{p}} \iff \frac{\overline{f}}{1} \notin \mathfrak{p}A_{\mathfrak{p}} \iff \overline{f} \notin \mathfrak{p}$. So $U \cap X_f = D(\overline{f})$. Because $U \cap X_f$ and $D(\overline{f})$ have the same stalks, they are isomorphic as schemes.

(b). Suppose that $X = \cup_i^n Spec(A_i)$. Then $X_f \cap Spec(A_i) = D(\overline{f}_i)$ where $\overline{f}_i$ is the restriction of f on $Spec(A_i)$ and the restriction of a on $Spec(A_i)$ is $\overline{a}_i$. Then $\overline{a}_i = 0$ implies that $\overline{a}_i = 0 \in \Gamma(D(\overline{f}), \mathcal{O}_X|_{Spec(A_i)}) = A_{i\overline{f}_i}$ that is, there exists n_i such that

$$\overline{f}_i^{n_i}\overline{a}_i = 0$$

Let $n = \Pi_{i=1}^n n_i$. Because for any $x \in Spec(A_i)$, $(f^n a)_x = f_x^n a_x = \overline{f}_{i,x}^n \overline{a}_{i,x} = 0$. So at each point $x \in X$, we have $(f^n a)_x = 0$ and then $f^n a = 0$ by 2.1.11.

(c). Again suppose that $X = \cup_{i=1}^n Spec(A_i)$ with $U_i = Spec(A_i)$. Then for each $b \in \ U \cap X_f = D(f_i) = A_{i\,f_i}$, there exists n_i and $a_i \in A_i$ such that

$$b_i = \frac{a}{f_i{}^{n_i}} \iff b_i f_i{}^{n_i} = \frac{a_i}{1}$$

where $b_i = b|_{U_i}$ and $f_i = f|_{U_i}$

Note we need to glue bf^{n_i} together. Suppose that $U_i \cap U_j = \cup U_{ijk}$ with $U_{ijk} = Spec(A_{ijk})$. Then $b_{ijk} - b_{jik} = 0$ where $b_{ij} = b|_{U_i}|_{U_{ij}}$. That is,

$$b_i f_i^{n_i} - b_j f_j^{n_j} = 0 \text{ on } Spec(A_{ijk\,f_{ijk}})$$

So we can find n_{ijk} such that $(b_i f_i^{n_i} - b_j f_j^{n_j}) f_{ijk}^{n_{ijk}} = 0$ on the whole U_{ijk}. Let $n_{ij} = \Pi_k n_{ijk}$.

$$(b_i f_i^{n_i} - b_j f_j^{n_j}) f_{ijk}^{n_{ij}} = 0 \text{ on } U_i \cap U_j = Spec(A_{ij})$$

Thus, just take $n = \Pi_{ij} n_{ij}$. Then all $(\frac{a_i}{1})(f_i)^{\frac{n}{n_i}}$ can be glued together, denoted $a \in \Gamma(X, \mathcal{O}_X) = A$. And restriction at $x \in X_f$, we have $(bf^n)_x = a_x$. So bf^n is the restriction of a on X_f.

(d). For elements in A_f are of the form $\frac{a}{f^n}$ for some n and $a \in A$. By (c), we have $\Gamma(X_f, \mathcal{O}_{X_f}) \hookrightarrow A_f$.

For $f_x \notin \mathfrak{m}_x$, f_x is unit at each stalk. So f^{-1} exists in $\Gamma(X_f, \mathcal{O}_{\mathcal{X}_{\{}})$. Thus, for any element $\frac{a}{f^n}$ in A_f, $a|_{X_f} f^{-n} \in \Gamma(X_f, \mathcal{O}_{X_f})$. Thus, $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.

□

Exercise 2.2.17. (a) Let $f : X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

(b) A scheme X is affine if and only if there is a finite set of elements $f_1, \ldots, f_r \in A = \Gamma(X, \mathcal{O}_X)$, such that the open subsets X_f are affine, and $f_1, \ldots, f_r$ generate the unit ideal in A. [Hint: Use (Ex. 2.4) and (Ex. 2.16d) above.]

Proof.

(a). Note that $f^{-1}(Y) = X$. Thus, for $Y = \cup U_i$, $X = \cup f^{-1}(U_i) \cong \cup U_i$. Thus, f is an isomorphism.

(b). ($\Longrightarrow$): $X = Spec(A)$, then just take $1 \in A$, we have the properties we need.

($\Longleftarrow$): For any $\mathfrak{p} \in X$. There must exist X_{f_i} such that $\mathfrak{p} \in X_{f_i}$. Otherwise, all $f_i \in \mathfrak{p}$, which is contradict to the assumption that $f_1, f_2, ..., f_n$ generate the unit ideal in A. Because $\sum a_i f_i = u$ for $a_i \in A$. For u is a unit, so u_x is also unit. However,$\forall i$, $f_i \in \mathfrak{p}$, $u \in \mathfrak{p}$, which is not a unit. So we have

$$X = \cup X_{f_i}$$

For $X_{f_i} \cap X_{f_j}$ is just the subset of X_{f_i} such that the stalk at x of restriction of f_j on it, $\overline{f}_j$, isn't contained in the $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$. For X_{f_i} is affine, $X_{f_j} \cap X_{f_i}$ is also affine by 2.2.16 (a). Hence $X_{f_j} \cap X_{f_i}$ is quasi-compact.

Now, we can use 2.2.16 (b): $\Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}}) = A_{f_i}$. For X_{f_i} is affine, $(X_{f_i}, \mathcal{O}_{X_{f_i}}) \cong (A_{f_i}, \mathcal{O}_{A_{f_i}})$.

Finally, by

$$A = \sum(f_i)$$

so $Spec(A) = \cup Spec(A_f) = X$. □

Exercise 2.2.18. In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.

(a) Let A be a ring, $X = \operatorname{Spec} A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

(b) Let $\varphi : A \to B$ be a homomorphism of rings, and let $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is dominant, i.e., $f(Y)$ is dense in X.

(c) With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

(d) Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then φ is surjective. [Hint: Consider $X' = \operatorname{Spec}(A/\ker\varphi)$ and use (b) and (c).]

Proof.

(a). ($\Longrightarrow$): $D(f) = D(f^n)$ for any n. f is nilpotent, so there exists N such that $D(f) = D(f^N) = D(0) = \emptyset$.

($\Longleftarrow$): $D(f) = \emptyset$ so $V(f) = X$, which implies that $f \in \mathfrak{p}$ for any prime ideal $\mathfrak{p}$. $f \in \cap_{prime}\mathfrak{p} = \mathcal{N}(A)$.

(b). φ is injective if and only if $f^\# : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is injective.

For $f^\# : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is injective if and only if for each $x \in X$, $f^\#_x$ is an injcetion, we can proof this as following:

($\Longrightarrow$): **Use something in Sheaves of Mod**: Consider B as an A-module. Then $f^\#$ is a morphism between $\mathcal{O}_A$-module. Then $f^\#(Spec(A)) = \varphi(Spec(A))$ and $ker(f^\#)(SpecA) = 0$. By Chapter 2 Corollary 5.5 of [5],

$$Mod_A(M) \xrightarrow{1:1} Mod_{\mathcal{O}_A}(\mathcal{F})$$

we see that $ker(f^\#) = 0$. Thus, $f^\#$ is an injection.

($\Longleftarrow$) Now for $f^\#$ is an injection. For the sequence of presheaves is exact and $ker(f^\#)$ is 0, then for each $U \subset X$, $f^\#(U)$ is an injection. Take $U = X$.

(c). Suppose that $B = A/ker(\varphi)$. Then

$$Spec(B) = \{\text{prime ideals of } A \text{ containing } I = ker(\varphi)\} = V(I)$$

So $Spec(B)$ is homeomorphic to $V(I)$, which is a close subset of $Spec(A)$.

Use something in Sheaves of Mod: Consider B as an A-module. Then $f^\#$ is a morphism between $\mathcal{O}_A$-module. Then $f^\#(Spec(A)) = \varphi(Spec(A))$ and $im(f^\#)(SpecA) = 0$. By

$$Mod_A(M) \xrightarrow{1:1} Mod_{\mathcal{O}_A}(\mathcal{F})$$

we see that $im(f^{\#}) = 0$. Thus, $f^{\#}$ is a suejection.

(d).

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "\varphi"'] & A/ker(\varphi) \arrow[d, "i"] \\ & B \end{tikzcd}$$

where i is an injection.Then, we have

$$\begin{tikzcd} Spec(A) & Spec(A/ker(\varphi)) \arrow[l, "Spec(\pi)"'] \\ & Spec(B) \arrow[u, "Spec(i)"'] \arrow[ul, "Spec(\varphi)"] \end{tikzcd}$$

where $Spec(i)$ is injective by (b). Then for any $x \in Spec(B)$, we have

$$\begin{tikzcd} \mathcal{O}_{A,x} \arrow[r, "p_x^{\#}"] \arrow[dr, "\varphi_x^{\#}"'] & \mathcal{O}_{A/ker(\varphi),x} \arrow[d, "i_x^{\#}"] \\ & \mathcal{O}_x \end{tikzcd}$$

where $i_x^{\#}$ is still an injection. Because φ_x is an surjection, so $i_x^{\#}$ is a surjection. Hence it is a bijection, then isomorphism.

Then suppose that B is homeomophic to $V(I) \subset Spec(A/ker(\varphi))$. As we have proved above, they have the same scheme structure. So $(Spec(B), \mathcal{O}) \cong (V(I), \mathcal{O}_{V(I)}) = (Spec(A/I), \mathcal{O}_{A/I})$. Thus, taking global section, we have

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "\varphi"'] & A/I \arrow[d, "i"] \\ & B \end{tikzcd}$$

where i is an isomorphism. So φ is an injection. □

Remark. For $\varphi : A \longrightarrow B$, consider $(f, f^{\#}) : Spec(B) \longrightarrow Spec(A)$ then for any $\mathfrak{p} \in Spec(A)$, $f_{\mathfrak{p}}^{\#} : A_{\mathfrak{p}} \longrightarrow f_*\mathcal{O}_{B,\mathfrak{p}}$ is given by $A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}}$ where B is viewed as a A-module and has a local ring structure with maximal ideal $\mathfrak{q}B_{\mathfrak{p}}$ if $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$.

Exercise 2.2.19. Let A be a ring. Show that the following conditions are equivalent:

(a) Spec A is disconnected;

(b) there exist nonzero elements $e_1, e_2 \in A$ such that $e_1e_2 = 0, e_1^2 = e_1, e_2^2 = e_2, e_1 + e_2 = 1$ (these elements are called orthogonal idempotents);

(c) A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.

Proof.

(i)$\Longrightarrow$(ii): Suppose that $X = U_1 \cup U_2$ and $U_1 \cap U_2 =$. For X is quasi-compact, we can just assume that $U_i = D(f_i)$. Taking $1 \in \Gamma(U_1, \mathcal{O}_1)$ and $0 \in \Gamma(U_2, \mathcal{O}_2)$, we can glue them to a global section of X, denoted by e_1, such that $e_1|_{U_1} = 1$ and $e_1|_{U_2} = 0$. Similarly, we can have $s_2|_{U_1} = 0$ and $s_2|_{U_2} = 1$. It is easy to verify that e_1, e_2 satisfy the properties above.

(ii)$\Longrightarrow$(iii). Just define $A_1 = e_1A$, then for $e_1^2 = e_1$, A_1 has a ring structure. Similarly, e_2A has a ring structure.

Define $A \longrightarrow A_1 \times A_2$ by $e_1 \mapsto (e_1, 0)$ and $e_2 \mapsto (0, e_2)$. By $e_1 + e_2 = 1$, for any $a \in A$, $a = e_1 a + e_2 a$. Thus, we can see that the ring homomorphism defined above is an isomorphism.

(iii)$\Longrightarrow$(i). Just note that $D((0,1)) \cup D((1,0)) = D((0,0)) = Spec(A)$ and $D((0,1))^c = D((1,0))$. $Spec(A)$ is disconnected.

□

2.2.3 Additional Exercises

Exercise 2.2.20. Give an example of a scheme X such that the sets X_a do not form a basis of the topology.

Proof. For $(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k})$, we have seen that $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = k$. Take any $c \in k$ with $c \neq 0$ and any $x = [a_0 : a_1] \in \mathbb{P}^1_k$. $\mathcal{O}_x = k[x]_{(x - \frac{a_0}{a_1})}$, supposing that $a_1 \neq 0$. Then we see that

$$c \notin \mathfrak{m}_x \mathcal{O}_{\mathbb{P}^1_k, x}$$

Hence $X_c = X$. Thus, $\{X_c\}$ doesn't form a topology basis for $\mathbb{P}^1_k$. □

Exercise 2.2.21. Describe the points of the following schemes for $k = \overline{k}$.

(i). $Proj(\mathbb{Z}[x])$;

(ii). $\mathbb{P}^1_k$;

(iii). $\mathbb{P}^2_k - D_+(x^2 + y^2 - z^2)$;

(iv). $Proj(k[x,y]/(x^2, y^2))$.

Proof.

(i). Let $S = \mathbb{Z}[x]$. Note that the prime ideals in $\mathbb{Z}[x]$ are of the form (p), $(f(x))$, $(p, f(x))$ with $f(x)$ irreducible and p prime.

Note that (p) is a homogeneous primed ideal. If $(f(x))$ is homogeneous, $f(x) = x$ hence $S^+ = (x)$. If $(p, f(x))$ is homogeneous, $f(x) = x$. But still $S^+ \subset (p, x)$. So $Proj(\mathbb{Z}[x]) = \{(0), (p)\}$.

(ii). $\mathbb{P}^n_k = \{(0), (\lambda x_0 - \mu x_1)\}$ with $\lambda, \mu \in k$.

(iii). $V_+(x^2 + y^2 - z^2)$

(iv). Let $S = k[x,y]/(x^2, y^2)$. For all elements in S^+ is nilpotent, $Proj(S) = \emptyset$. □

2.3 First Properties of Schemes

2.3.1 Preparations

Theorem 2.3.1. *Let X be a scheme. Then X is integral if and only if X is reduced and irreducible.*

Proof. ($\Longrightarrow$): For X is integral, at each $P \in X$, $\mathcal{O}_{X,P}$ is integral, hence reduced.

If X is not irreducible, then there exists two open sets U, V such that $U \cap V = \emptyset$. Then $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \times \mathcal{O}(V)$ which is not integral, which leads to a contradiction.

($\Longleftarrow$): Take $f, h \in \mathcal{O}_X(U)$ and suppose that $fg = 0$. Let $X = \{x \in U | f_x \notin \mathfrak{m}_x\}$ and $Y = \{x \in U | g_x \notin \mathfrak{m}_x\}$. Then X, Y are closed in U and $X \cup Y = U$. For X is irreducible, so is U. Hence, w.l.o.g., $X = U$. Then on each affine subset of U, say $Spec(A)$, $f \in \mathfrak{p}$ for each $\mathfrak{p} \in A$. For A is reduced, $f \in \cap \mathfrak{p} = \mathcal{N}(A) = \{0\}$. For $\mathcal{O}_X$ is a sheaf, $f = 0$. Thus, $\mathcal{O}_X(U)$ is integral for each U. Hence X is integral. ☐

2.3.2 Exercises

Exercise 2.3.1. Show that a morphism $f : X \to Y$ is locally of finite type if and only if for every open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \text{Spec } A_j$, where each A_j is a finitely generated B-algebra.

Lemma 2.3.2. *For $\varphi : A \longrightarrow B$,*

$$Spec(\varphi)^{-1}(D(a)) = D(\varphi(a))$$

Proof. For $Spec(\varphi) : Spec(B) \longrightarrow Spec(A)$ and $\mathfrak{p} \subset B$, $D(a) = \{\mathfrak{p} \in Spec(A) | a \notin \mathfrak{p}\}$, then $Spec(\varphi)(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$.

Take $\mathfrak{p} \in Spec(\varphi)^{-1}(D(a))$, then $\varphi^{-1}(\mathfrak{p})$ doesn't contain $a \notin \mathfrak{p}$. If $\varphi(a) \in \mathfrak{p}$, then $a \notin \varphi^{-1}(\mathfrak{p})$. Thus, $\varphi(a) \notin \mathfrak{p}$. So $Spec^{-1}(\varphi)(D(a)) \subset D(\varphi(a))$.

If take any $\mathfrak{p} \in D(\varphi(a))$ such that $\varphi(a) \notin \mathfrak{p}$, then a is not contained in $a \in \varphi^{-1}(\mathfrak{p})$. So $Spec(\varphi)(D(\varphi(a))) \subset D(a)$, that is, $D(\varphi(a)) \subset Spec(\varphi)^{-1}(D(a))$. ☐

Proof. ($\Longrightarrow$): We first assume that $f : X \longrightarrow Spec(A)$ and $f^{-1}(Spec(A)) = \cup Spec(B_i)$ such that B_i is finitely generated A-algebra with $\varphi_i = \Gamma(f_i) : A \longrightarrow B_i$ deduced by $f|_{U_i} = f_i : Spec(B_i) \longrightarrow Spec(A)$. Then for any open set $Spec(A_a)$, $f^{-1}(Spec(A_a)) = \cup Spec(B_{i,\varphi_i(b)})$. Because we have $\varphi_{i,a} : A_a \longrightarrow B_{i,\varphi_i(b)}$, each $B_{i,\varphi_i(b)}$ is a finitely generated A_a-algebra. **You may need exactness of localization to prove this point.**

Suppose that $f : X \longrightarrow Y$ with $Y = \cup Spec(A_i)$. For any $U = Spec(C)$, $U \cap Spec(A_i) = \cup Spec(A_{a_{ij}})$, and by the discussion before, we see that $\varphi^{-1}(Spec(A_{a_{ij}})) = \cup Spec(B_{ijk})$ and B_{ijk} is a finitely generated $A_{a_{ij}}$-algebra.

Note suppose that $\cup D(c_{ijl}) = Spec(A_{a_{ij}})$. For $Spec(A_{a_{ij}})$ is quasi-compact, $Spec(A_{a_{ij}})$ is covered by finite $D(c_{ijl}) = D(\Pi_{l=1}^n c_{ijl})$. Thus, $Spec(A_{a_{ij}}) = Spec(C_{c_{ij}})$, implying $A_{a_{ij}} = C_{c_{ij}}$ for some $c_{ij} \in C$. By the property of localization, $A_{a_{ij}} = C_{c_{ij}}$ is a finitely generated C-algebra. Thus, for $f^{-1}(Spec(C)) = \cup_{i,j,k} Spec(A_{ijk})$, B_{ijk} is finitely generated C-algebra.

($\Longleftarrow$): Trivial. ☐

Exercise 2.3.2. A morphism $f : X \to Y$ of schemes is quasi-compact if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasi-compact if and only if for every open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

Proof. ($\Longrightarrow$): First, assume that $f : X \longrightarrow Spec(B)$ and $f^{-1}(Spec(B))$ is quasi-compact. Then $f^{-1}(Spec(B)) = \cup_{i=1}^n Spec(A_i)$. For any $Spec(B_b) \subset Spec(B)$, as we have discussed before,

$f^{-1}(B_b) = \cup_{i=1}^n Spec(A_{i,\varphi_i(b)})$ with the symbols defined in 2.3.1. For each $Spec(A_{i,\varphi_i(b)})$ is quasi-compact, so $f^{-1}(B_b)$ is quasi-compact.

Suppose that Y is covered by $Spec(A_i)$ and $f^{-1}(Spec(A_i))$ is quasi-compact. Taking $U = Spec(C) \subset Y$. Then U is cover by $Spec(A_{i,a_j})$. For U is quasi-compact, it can be covered by finitely many $Spec(A_{i,a_j})$, that is, $U = \cup_{i=1,j=1}^{n,m} Spec(A_{i,a_j})$. And $f^{-1}(U) = \cup_{i=1,j=1}^{n,m} f^{-1}(Spec(A_{i,a_j}))$, with each $f^{-1}(Spec(A_{i,a_j}))$ is quasi-compact as we have shown. So $f^{-1}(U)$ is quasi-compact. □

Exercise 2.3.3. (a) Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

(b) Conclude from this that f is of finite type if and only if for every open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \text{Spec } A_j$, where each A_j is a finitely generated B-algebra.

(c) Show also if f is of finite type, then for every open affine subset $V = \text{Spec } B \subseteq Y$, and for every open affine subset $U = \text{Spec } A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.

Proof.

(a). ($\Longrightarrow$): If Y is covered by $Spec(B_i)$ and $f^{-1}(Spec(B_i)) = \cup_{j=1}^n Spec(A_{ij})$ with A_{ij} finitely generated B_i-algebra, then it is obviously locally of finite type. Because all $Spec(B_i)$ and $Spec(A_{ij})$ are quasi-compact.

($\Longleftarrow$): Suppose that Y is covered by $Spec(B_i)$ and $f^{-1}(Spec(B_i)) = \cup_{i=1}^n Spec(A_{ij})$, for f is quasi-compact. and A_{ij} is finitely generated B_j-algebra. Thus, f is of finite type.

(b). By 2.3.1 and 2.3.2, f is locally of finite type if and only if for each $Spec(C) \subset Y$, $f^{-1}(Spec(C)) = \cup Spec(A_i)$ with A_i finitely generated A_i-algebra and f is quasi-compact if and only if $f^{-1}(Spec(C))$ is quasi-compact. So f is of finite type if and only if for each $Spec(C) \subset Y$, $f^{-1}(Spec(C)) = \cup_{i=1}^n Spec(A_i)$ with A_i finitely generated A_i-algebra.

(c). Directly form (b). □

Exercise 2.3.4. Show that a morphism $f : X \to Y$ is finite if and only if for every open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ is affine, equal to $\text{Spec } A$, where A is a finite B-module.

Proof. ($\Longrightarrow$): First, consider $f : X \to Spec(B)$ such that $f^{-1}(Spec(B)) = Spec(A)$, with A is a finite generated B-module. Then for any $f^{-1}(Spec(B_b)) = Spec(A_{\varphi(b)})$, by the fact that $\varphi_a : B_b \longrightarrow A_{\varphi(b)}$ is given by $\frac{x}{b} \mapsto \frac{\varphi(x)}{\varphi(b)}$, A_a is a finitely generated B_b-module.

If Y is covered by $Spec(B_i)$ associated to f, take any $Spec(C) \subset Y$. For $Spec(C)$ is quasi-compact, assume that $Spec(C) = \cup_{i=1,j=1}^{n,n_i} Spec(B_{i,b_j})$. Let $b_i = \Pi_{j=1}^{n_i} b_{ij}$, then $Spec(C) = \cup Spec(B_{i,b_i})$. and $f^{-1}(Spec(C)) = \cup Spec(A_{i,\varphi_i(b_i)})$. Note that each $Spec(B_{i,b_i}) \cong Spec(C_{c_i})$ for some $c_i \in C$.

W.L.O.G., we just assume that $Spec(B) = \cup Spec(B_{b_i})$ and $X = \cup Spec(A_i)$ with finitely many i and A_i is finitely generated B_i-module. Let $A = \Gamma(X, \mathcal{O}_X)$ and then we have $\varphi : C \longrightarrow A$ deduced by $f : X \longrightarrow Spec(C)$. Then $A_i = X_{\varphi(b_i)}$ for a morphism between schemes is a locally ringed morphism.(**Verify!**) For $Spec(B) = \cup Spec(B_{b_i})$ So $\{b_i\}$ generates the unit of C. So $\{\varphi(b_i)\}$ generates unit of A. By 2.1.17, $X = Spec(A)$.

For each A_i, it is finitely generated by $\frac{\varphi(b_{i1})}{\varphi(b_i)^{k_i}}, ..., \frac{\varphi(b_{ij_i})}{\varphi(b_i)^{k_i}}$ with $b_{ij} \in B^{\oplus n_{ij}}$. Taking $a \in A$, then on $Spec(A_i) = D(\varphi(b_i))$,

$$\frac{a}{1} = \sum_{j=1}^{i_j} \varphi(b'_{ij}) \frac{\varphi(b_{ij})}{\varphi(b_i)^k}$$

Then there exists f_i^l such that

$$\varphi(b_i)^{l_i}(\varphi(b_i)^{k_i}a - \sum_{j=1}^{i_j}\varphi(b'_{ij})\varphi(b_{ij})) = 0$$

Note for $D(b_i) = D(b_i^{l_i+k_i})$ and $\cup D(\varphi(b_i)) = Spec(B)$, we have $\sum b' b^{l_i+k_i} = 1$, then

$$a = \sum \varphi(b'_i)\varphi(b_i)^{l_i}\varphi(b_i)^{k_i}a = \sum \varphi(b'_i)\varphi(b'_{ij})\varphi(b_i)^{l_i}\varphi(b_{ij})$$

. Here we view $b_{ij} \in B^{\oplus n}$. □

Exercise 2.3.5. A morphism $f : X \to Y$ is quasi-finite if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.

(a) Show that a finite morphism is quasi-finite.

(b) Show that a finite morphism is closed, i.e., the image of any closed subset is closed.

(c) Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

Proof. (a). Suppose that $y \in Spec(A_i) = f^{-1}(Spec(B_i))$ with A_i finitely generated B_i-module. Then $f^{-1}(y) = Spec(k(y) \otimes_{B_i} A_i)$. Because $k(y)$ is a field and A_i is a finitely generated B_i-module, $k(y) \otimes_{B_i} A_i$ is a k-vector space, by Chapter 2, Section 2, Exercise 2 of [2]. Then $f^{-1}(y) = Spec(k^n)$, which implying that $f^{-1}(y)$ is a finite set.

(b). First, we show that for $f : Spec(A) \longrightarrow Spec(B)$, deduced by $\varphi : A \to B$, with A is a finite generated B-mod, f is closed.

For any $Z = V(I) \subset Spec(A)$, we have need to show that $f(Z) = V(\varphi^{-1}(Z))$. Take any $I \subset \mathfrak{p}$ that $\mathfrak{p}$ is prime. $\varphi^{-1}\mathfrak{p}$ must contain $\varphi^{-1}(I)$. Thus $f(Z) \subset V(\varphi^{-1}(Z)$. Because A is a finite generated B-mod, A is intergal over B. For any $\varphi^{-1}(I) \subset \mathfrak{q}$, by going up theorem, Chapter 5, Section 5, Theorem 5.11. of [2] , there exists $\mathfrak{p} \subset A$ such that $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. So $f(V(I)) = V(f^{-1}(I))$.

Now consider the general case, suppose that $Y = \cup Spec(B_i)$ and $Spec(A_i) = f^{-1}(Spec(B_i))$. For U is closed on X, $U \cap Spec(A_i)$ is a closed set of $Spec(A_i)$, then for $f|_{Spec(A_i)} : Spec(A_i) \longrightarrow Spec(B_i)$ is closed as we have shown, $f(U \cap Spec(A_i))$ is closed. Thus, on the whole X, f is closed.

(c). See additional exercises. □

Exercise 2.3.6. Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. It is called the function field of X, and is denoted by $K(X)$. Show also that if $U = \text{Spec } A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.

Proof. If $X = Spec(A)$ with A integral, then the generic point of X is $(0) \subset A$ and $\mathcal{O}_{X,(0)} = A_{(0)} = K(A)$.

Suppose that $\xi \in Spec(A_i)$ for $X = \cup Spec(A_i)$ with A_i integral. Then ξ is also a generic point of $Spec(A_i)$, thus $\xi = (0) \in Spec(A_i)$ and $\mathcal{O}_{X,\xi} = \mathcal{O}_{A_i,(0)} = A_{i(0)} = K(A_i)$. □

Exercise 2.3.7. A morphism $f : X \to Y$, with Y irreducible, is generically finite if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y. A morphism $f : X \to Y$ is dominant if $f(X)$ is dense in Y. Now let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite. [Hint: First show that the function field of X is a finite field extension of the function field of Y.]

Proof. For X, Y are integral, denote their generic points by η_X, η_Y. For $\overline{f(\eta_X)} \supseteq f(\overline{\eta_X}) = f(X)$ and $\overline{f(X)} = Y$, $\overline{f(\eta_X)} = Y$, that is, $f(\eta_X) = \eta_Y$. So f induces a morphism between $\mathcal{O}_{X,\eta_X} \longrightarrow \mathcal{O}_{Y,\eta_Y}$, that is $K(X) \longrightarrow K(Y)$.

Assume that $f : Spec(A) \longrightarrow Spec(B)$ with $Spec(A) \subset X$ and $Spec(B) \subset Y$. For the generic point of $Spec(A)$ is mapped to the generic point of B, $\varphi : B \longrightarrow A$ is an injection for $ker(\varphi) = \varphi^{-1}((0)) = (0)$. For $f^{-1}(\eta_Y)$ is finite, there are only finitely many prime ideal $\mathfrak{p}$ of A such that $\varphi^{-1}(\mathfrak{p}) = \mathfrak{p} \cap B = (0)$, that is, the extension $K(X) \subset K(Y)$ is finite by going up theorem, and hence if A is finitely generated by $\{a_i\}_{i=1}^n$ as B-algebra, then there exists $f_i \in B[x]$ such that $f_i(a_i) = 0$.

Suppose that $f_i = b_i x^{n_i} + \sum_{j<n_i} b_{ij} x^j$. Consider $b = \Pi_{i=1}^n b_i$, there exists monic polynomial $f_i^{'} \in B_b[x]$ such that $f_i^{'}(\frac{a_i}{1}) = 0$ for $\frac{a_i}{1} \in A_b$ and generates A_b as a finitely generated B_b-algebra. Thus, A_b is a finitely generated B_b-module. So $f^{-1}(D(b)) \longrightarrow D(b)$ is finite. For Y is irreducible, $D(b)$ is dense in Y. □

Remark. Giving a dominant morphism between integral schemes is equivalent to giving a morphism between them such that maps the generic point to the generic point.

Exercise 2.3.8. A scheme is normal if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \text{Spec } A$ of X, let $\bar{A}$ be the integral closure of A in its quotient field, and let $\bar{U} = \text{Spec } \bar{A}$. Show that one can glue the schemes $\bar{U}$ to obtain a normal integral scheme $\bar{X}$, called the normalization of X. Show also that there is a morphism $\bar{X} \to X$, having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\bar{X}$. If X is of finite type over a field k, then the morphism $\bar{X} \to X$ is a finite morphism. This generalizes (I, Ex. 3.17).

Proof. □

Exercise 2.3.9. Recall that in the category of varieties, the Zariski topology on the product of two varieties is not equal to the product topology (I, Ex. 1.4). Now we see that in the category of schemes, the underlying point set of a product of schemes is not even the product set.

(a) Let k be a field, and let $A_k^1 = \text{Spec } k[x]$ be the affine line over k. Show that $A_k^1 \times_{\text{Spec } k} A_k^1 \cong A_k^2$, and show that the underlying point set of the product is not the product of the underlying point sets of the factors (even if k is algebraically closed).

(b) Let k be a field, let s and t be indeterminates over k. Then $\text{Spec } k(s)$, $\text{Spec } k(t)$, and $\text{Spec } k$ are all one-point spaces. Describe the product scheme $\text{Spec } k(s) \times_{\text{Spec } k} \text{Spec } k(t)$.

Proof.

(a). For $k[x_1] \otimes_k k[x_2] = k[x_1, x_2]$, so $Spec(k[x_1]) \times_k Spec(k[x_2]) = Spec(k[x_1, x_2])$, that is, $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1 = \mathbb{A}_k^2$. Note that $(x - y) \in sp(\mathbb{A}_k^2)$ is not contained in $sp(\mathbb{A}_k^1) \times sp(\mathbb{A}_k^1)$.

(b). Suppose that $k(s) = k[x]/(f(x))$, where f is a irreducible polynomial such that $f(s) = 0$ and $k(t) = k[y]/(g(y))$ where g is a irreducible polynomial such that $f(t) = 0$. Then $k(s) \otimes_k k(t) = k[x, y]/(f(x), g(y))$. □

Exercise 2.3.10. (a) If $f : X \to Y$ is a morphism, and $y \in Y$ a point, show that $\text{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

(b) Let $X = \text{Spec } k[s, t]/(s - t^2)$, let $Y = \text{Spec } k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \to s$. If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fibre X_y consists of two points, with residue field k. If $y \in Y$ corresponds to $0 \in k$, show that the fibre X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η. (Assume k algebraically closed.)

Proof.

(a). If $y \in Spec(B) \subset Y$, then $f^{-1}(y) \in \cup Spec(A_i)$.

Now consider $f : Spec(A) \longrightarrow Spec(B)$ and the corresponding ring homomorphism $\varphi : B \longrightarrow A$. Suppose that $y = \mathfrak{q} \in Spec(B)$, then $sp(f^{-1}(y)) = \{\mathfrak{p} \in Spec(A) | \mathfrak{q} = \varphi^{-1}(\mathfrak{p})\}$.

Note that $k(y) = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ and $k(y) \otimes_B A = B_{\mathfrak{q}} \otimes_B A/\mathfrak{q}B_{\mathfrak{q}} \otimes_B A$ by Chapter 2 Exercise 2 of [2]. Note that $B_{\mathfrak{q}} \otimes_B A = A_{\mathfrak{q}}$ with its ideals correspond to $\mathfrak{p} \in Spec(A)$ such that $\mathfrak{p} \subset \varphi(\mathfrak{q})$,that is, $\varphi^{-1}(\mathfrak{p}) \subset \mathfrak{q}$. For $A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$,

$$Spec(A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}) = \{\mathfrak{p}_{\mathfrak{q}} \in Spec(A_q) | \mathfrak{p}_{\mathfrak{q}} \supset \varphi(\mathfrak{q})A_q\} \xrightarrow{1:1} \{\mathfrak{p} \in Spec(A) | \mathfrak{q} \subset \varphi^{-1}(\mathfrak{q}) \text{ and } \mathfrak{p} \supset \varphi^{-1}(q)\}$$

Thus, with the induced topology, $X_y = f^{-1}(y)$.

(b). For k is algebraically closed and $y = (s-a)$ with $a \neq 0$, then $k(y) = k[s]/(s-a)$. Note that

$$k[s]/(s-a) \otimes_{k[s]} k[s,t]/(s-t^2) = k[s,t]/(s-a, s-t^2) = k[s,t]/(s-a, t^2-a)$$

So, $X_y = Spec(k[s,t]/(s-a,t^2-a)) = \{(s-a, t-a^{\frac{1}{2}}), (s-a, t-a^{\frac{1}{2}})\}$ by Hilbert's Nullstellensatz.

For k is algebraically closed and $y = (s-a)$ with $a \neq 0$, then $k(y) = k[s]/(s-a)$. Note that

$$k[s]/(s) \otimes_{k[s]} k[s,t]/(s-t^2) = k[s,t]/(s, s-t^2) = k[s,t]/(s,t^2)$$

So, $X_y = Spec(k[s,t]/(s,t^2)) = \{(s,t)\}$ by Hilbert's Nullstellensatz. Note that $k[t,s]/(t,s^2)$ is not reducible.

For $a = 0$, $k(y) = K(k[s]) = k(s)$ and

$$k(s) \otimes_{k[s]} k[s,t]/(t^2-s) = k[s] \otimes_{k[s]} k[s,t]/k(s) \otimes (t^2-s) = k(s)[t]/(t^2-s)$$

For $t^2 - s$ is irreducible in $k(s)[t]$, so $k(s)[t]/(s-t^2)$ is a field and for $deg(s-t^2)$=2. $[k(s)[t] : k(s)] = 2$. □

Exercise 2.3.11. (a) Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.

(b) If Y is a closed subscheme of an affine scheme $X = \text{Spec } A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $a \subseteq A$ as the image of the closed immersion Spec $A/a \to$ Spec A. [Hints: First show that Y can be covered by a finite number of open affine subsets of the form $D(f_i) \cap Y$, with $f_i \in A$. By adding some more f_i with $D(f_i) \cap Y = \emptyset$, if necessary, show that we may assume that the $D(f_i)$ cover X. Next show that $f_1, \ldots, f_r$ generate the unit ideal of A. Then use (Ex. 2.17b) to show that Y is affine, and (Ex. 2.18d) to show that Y comes from an ideal $a \subseteq A$.] Note: We will give another proof of this result using sheaves of ideals later (5.10).

(c) Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'. We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.

(d) Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. We call Y the scheme-theoretic image of f. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

Proof. We first prove (b): Suppose that $Y = \cup Spec(B_i)$. Then for Y is a closed immersion, for $Spec(B_i)$ there exists open set $U \subset X$ such that $U \cap Y = Spec(B_i)$. For $Spec(B_i)$ is quasi-compact, we can find finite $D(f_{ij})$ such that $D(f_{ij}) = D(f_i) \cap Y = Spec(B_i)$ where $f_i = \Pi_{i=1}^{j_i} f_{ij}$.

For $X - Y$ is open, it can be covered by some $D(f_k)$. Now $\{D(f_i)\} \cup \{D(f_k)\}$ forms a cover of $X = Spec(A)$ and by the fact that $Spec(A)$ is quasi-compact, we can find finitely many $D(f_i)$ and $D(f_k)$ cover $Spec(A)$ and $D(f_i) \cap Y = Spec(B_i)$ cover Y, that is, $Y = \cup(X_{f_i} \cap Y) = X_{\overline{f_i}}$ where $\overline{f_i} = f_i|_Y$. More precisely because for each $y \in Y$, $i_y^{\#} : \mathcal{O}_{X,y} \to \mathcal{O}_{Y,y}$ is surjective, mapping unit to unit. Thus, if $f_y \notin \mathfrak{m}_{X,y}$ then $\overline{f}_y \notin \mathfrak{m}_{Y,y}$.

Now for $\cup_{i=1}^n D(f_i) \cup \cup_{k=1}^m D(f_k) = X$, so $\{f_i\}_{i=1}^n$ and $\{f_k\}_{k=1}^m$ generates A. Note that $O_X(X) \longrightarrow O_Y(Y)$ is a ring homomorphism, so $\overline{f_i}$ generates a unit of $\mathcal{O}_Y(Y)$. So Y is affine.

Suppose that $Y = Spec(B)$, by 2.1.22, $\varphi : A \to B$ deduced by $i_Y : Y \to X$ is a surjection. So $B = A/ker(\varphi)$.

(a). For affine case: Because $\otimes$ is a right-exact functor. If $A \to B$ is surjective, so is $A \otimes_A A' \longrightarrow B \otimes_A A' = A'$. Thus $X' \times_X Y \to X'$ is a closed immersion.

Note that $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_Y$ is surjective if and only if for each $x \in X$, $f_x^{\#}$ is surjective. Thus, the gluing of closed immersions is also a closed immersions. Thus, closed immersions are stable under base change.

(c). We just show that $i_{Y'} : Y' \to X$ is a closed immerison then $i_Y : Y \to X$ is a closed immersion.

Note that we have $r : Y \to Y'$ with $r_y^{\#} : \mathcal{O}_{Y',y} \longrightarrow \mathcal{O}_{Y,y} = \mathcal{O}_{Y',y}/\mathcal{N}(\mathcal{O}_{Y',y})$ surjective. Then $r : Y' \to Y$ is a closed immersion.

For $i_Y = i_Y^{Y'} \circ i_{Y'}$ and the composition of closed immersions is also a closed immersion. $i_Y : Y \to X$ is a closed immersion.

(d). **Wait**

□

Exercise 2.3.12. (a) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Ex. 2.14) is equal to Proj T, and the morphism f : Proj $T \to$ Proj S is a closed immersion.

(b) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X =$ Proj S defined as image of the closed immersion Proj $S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geq d_0} I_d$. Show that I and I' determine the same closed subscheme.

We will see later (5.16) that every closed subscheme of X comes from a homogeneous ideal I of S (at least in the case where S is a polynomial ring over S_0).

Proof.

(a). For φ is surjective and preserves degree, so $\varphi(S_+) = T_+$. Thus, $U = \{\mathfrak{p} \in Proj(T) | T_+ \not\subset \mathfrak{p}\}$. Hence $U = Proj(T)$ and we have the morphism $f : Proj(T) \longrightarrow Proj(S)$ deduced by φ.

On any principal open subset $D_+(f)$ of $Proj(S)$,

$$S_{(f)} \longrightarrow S_{(\varphi(f))}$$

given by $\frac{g}{f^n} \longrightarrow \frac{\varphi(g)}{\varphi(f)^n}$ is a surjection. Hence $Spec(T_{\varphi(f)}) \longrightarrow Spec(S_{(f)})$ is a closed immersion for any homogeneous element f.

(b). Let $\varphi_1 : S \longrightarrow S_1 := S/I$ and $\varphi_2 : S \longrightarrow S_2 := S/I^{'}$ with $I^{'} = \oplus_{d \geq d_0}(I_d)$. Then $S_{1(\varphi_1(f))} = S_{2(\varphi_2(f))}$:

Because when $s \in S$ and when $deg(s) > d_0$, $\varphi_1(s) = \varphi_2(s)$. For any $\frac{\varphi_1(g)}{\varphi_1(f)^n} \in S_{1(\varphi_1(f))}$ there exists N with $N + ndeg(f) > d_0$ such that $\frac{\varphi_1(g)\varphi_1(f^N)}{\varphi_1(f)^{n+N}} = \frac{\varphi_1(gf^N)}{\varphi_1(f^{n+N})} = \frac{\varphi_2(gf^N)}{\varphi_2(f^{n+N})} \in S_{2\varphi_2(f)}$. Hence

$S_{1\varphi_1(f)} \subset S_{2\varphi_2(f)}$. Similary, $S_{2\varphi_2(f)} \subset S_{1\varphi_1(f)}$. Hence $S_{2\varphi_2(f)} = S_{1\varphi_1(f)}$ for any homogeneous element $f \in S$. So $Proj(S_1) = Proj(S_2)$. □

Exercise 2.3.13. (a) A closed immersion is a morphism of finite type.

(b) A quasi-compact open immersion (Ex. 3.2) is of finite type.

(c) A composition of two morphisms of finite type is of finite type.

(d) Morphisms of finite type are stable under base extension.

(e) If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.

(f) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and if f is quasi-compact, and $g \circ f$ is of finite type, then f is of finite type.

(g) If $f : X \to Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.

Proof. By 2.3.3 (b), we just need to verify, for any $Spec(B) \subset Y$, $f^{-1}(Spec(B))$ can be covered by a finite number of open affine $U_i = Spec(A_i)$, with A_i finitely generated B-algebra.

(a). Suppose that $f : X \longrightarrow Y$ is a closed immersion. If $Spec(B) \subset Y$, then $f^{-1}(Spec(B)) = Spec(B) \cap X \longrightarrow Spec(B)$ is a closed immersion. Thus, $f^{-1}(Spec(B)) = Spec(B/I)$, where B/I is a finitely generated B-algebra.

(b). Suppose that $f : X \to Y$ is such a morphism. Consider $Spec(B) \subset Y$. Then $f^{-1}(Spec(B)) = Spec(B) \cap X$, which is isomorphic to an open subscheme of $Spec(B)$. Suppose that $Spec(B) \cap X = \cup Spec(B_{f_i})$. Then for f is quasi-compact, $Spec(B) \cap X = \cup_{i=1}^n Spec(B_{f_i})$, where each B_{f_i} is a finitely generated B-algebra.

(c). Trivial.

(d). For if $X = \cup Spec(A_i)$, $Y = \cup Spec(B_i)$ and $S = \cup Spec(C_i)$, $X \times_S Y = \cup_{i,j,k} Spec(A_i \otimes_{C_k} B_i)$, we just consider the following case:

Consider $f : X \to S$ is of finite type and $g : Y \longrightarrow S$. Then for any $Spec(B_j) \subset Y$, w.l.o.g. we can assume that $g(Spec(B_j)) \subset Spec(C_k)$ and $f^{-1}(Spec(C_k)) = \cup_{i=1}^n Spec(A_i)$. Then for $p_Y : X \times_S Y \longrightarrow Y$, $p_Y^{-1}(Spec(B_j)) = \cup_{i=1}^n Spec(B_j \otimes_{C_k} A_i)$ and $B_j \otimes_{C_k} A_i$ are finitely generated B_j-algebra for A_i are finitely generated C_k-algebra.

(e). For if $X = \cup Spec(A_i)$, $Y = \cup Spec(B_i)$ and $S = \cup Spec(C_i)$, $X \times_S Y = \cup_{i,j,k} Spec(A_i \otimes_{C_k} B_i)$, we just consider the following case:

Consider $f : X \to S$ and $g : Y \to S$. Then $f^{-1}(Spec(C_k)) = \cup_{i=1}^n Spec(A_i)$, $g^{-1}(Spec(C_k)) = \cup_{j=1}^m Spec(A_i)$ and $(f \times_S g)^{-1}(Spec(C_k)) = \cup_{i,j=1}^{n,m} Spec(A_i \otimes_{C_k}) B_j$. If B_j and A_i are finitely generated C_k-algebras, so is $A_i \otimes_{C_k} B_i$.

Thus, $f \times_k g : X \times_S Y \longrightarrow S$ is of finite type.

(f). Consider for any $Spec(C) \subset Z$, $g^{-1}(Spec(C)) = \cup_{i=1}^n Spec(B_i)$ and for f is quasi-compact, we can assume that $f^{-1}(Spec(B_i)) = \cup_{j=1}^{m_i} Spec(A_{ij})$.

Then $(g \circ f)^{-1}(Spec(C)) = \cup_{i,j} Spec(A_{ij})$ and A_{ij} are finitely generated C-algebras. Note that B_i are also finitely generated C-algebras, so A_{ij} are finitely generated B_i-algebras.

Now that $f^{-1}(Z) = Y$. So Y can be cover by $Spec(B_i)$ as above. Thus, f is of finite type.

(g). Note that any finitely generated B-algebra A with B Noetherian, A is also Noetherian. This claim is trivial.

□

Exercise 2.3.14. If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.

Proof. Suppose that $X = \cup_{i=1}^{n} Spec(A_i)$ with A_i finite generated k-algebras. Consider $MaxSpec(A_i)$, the set of closed points are the set of all maximal ideals. If there exists $D(f)$ such that there is not maximal ideal of A_i contained in $D(f)$. Then $f \in \mathfrak{m}_i$ for each maximal ideal $\mathfrak{m}_i$. Note that k is Jacobson ($\mathfrak{N}(k) = \bigcap_{\mathfrak{m} \text{ maximal ideal}} \mathfrak{m}$), then A_i is also Jacobson by Chapter 5 Exercise of [2]. So f is nilpotent, and $D(f) = \emptyset$.

For counter example, just consider $Spec(k[[t]]) = \{(t), (0)\}$. $(t) \notin D(t)$. □

Exercise 2.3.15. et X be a scheme of finite type over a field k (not necessarily algebraically closed).

(a) Show that the following three conditions are equivalent (in which case we say that X is geometrically irreducible).

(i) $X \times_k K$ is irreducible, where K denotes the algebraic closure of k. (By abuse of notation, we write $X \times_k K$ to denote $X \times_{\text{Spec } k} \text{Spec } K$).

(ii) $X \times_k k_s$ is irreducible, where k_s denotes the separable closure of k.

(iii) $X \times_k K$ is irreducible for every extension field K of k.

(b) Show that the following three conditions are equivalent (in which case we say X is geometrically reduced).

(i) $X \times_k K$ is reduced.

(ii) $X \times_k k_p$ is reduced, where k_p denotes the perfect closure of k.

(iii) $X \times_k K$ is reduced for all extension fields K of k.

(c) We say that X is geometrically integral if $X \times_k K$ is integral. Give examples of integral schemes which are neither geometrically irreducible nor geometrically reduced.

Proof. (a). `https://stacks.math.columbia.edu/tag/0364`Stack Project.

(b). `https://stacks.math.columbia.edu/tag/035U`Stack Project.

(c). Note that $Spec(\mathbb{R}/(x^2+1))$ is an integral scheme. $Spec(\mathbb{R}/(x^2+1)) \times_{\mathbb{R}} \mathbb{C} = Spec(\mathbb{C}/(x^2+1)) = \{(x-i), (x+i)\}$, which is not geometrically irreducible.

Not geometrically reduced. Any example???

□

Remark. For any $k \subset K$, there is a surjection: $A \to A \otimes_k K$, thus $Spec(A) \times_k Spec(K) \to Spec(A)$ is a surjection. ($\varphi^{-1}(\mathfrak{p} \otimes_k K) = \mathfrak{p}$.)

Exercise 2.3.16. Let X be a noetherian topological space, and let $\mathcal{P}$ be a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathcal{P}$ holds for every proper closed subset of Y, then $\mathcal{P}$ holds for Y. (In particular, $\mathcal{P}$ must hold for the empty set.) Then $\mathcal{P}$ holds for X.

Proof. Let $S = \{V_\alpha\}_{\alpha \in \Lambda}$ be the set of closed subsets that don't have $\mathscr{P}$. Note that if a subset Y doesn't have $\mathscr{P}$, then there exists a proper closed subset of Y, denoted by Y_1, such that doesn't have $\mathscr{P}$. For X is Noetherian, we have a sequence

$$Y = Y_0 \supset Y_1 \supset Y_2 \supset ... \supset Y_n \supset ...$$

such that $\exists N$ such that $V_N = V_{N+1} = V_{N+2} = ...$. For V_N doesn't have $\mathscr{P}$, but it doesn't have proper subset of V_N satisfying $\mathscr{P}$, which leads to a contradiction. Thus, each proper closed subset of X has $\mathscr{P}$, so is X. □

Exercise 2.3.17. A topological space X is a Zariski space if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point (Ex. 2.9). For example, let R be a discrete valuation ring, and let $T = \text{sp}(\text{Spec } R)$. Then T consists of two points $t_0 =$ the maximal ideal, $t_1 =$ the zero ideal. The open subsets are $\emptyset$, $\{t_1\}$, and T. This is an irreducible Zariski space with generic point t_1.

(a) Show that if X is a noetherian scheme, then $\text{sp}(X)$ is a Zariski space.

(b) Show that any minimal nonempty closed subset of a Zariski space consists of one point. We call these closed points.

(c) Show that a Zariski space X satisfies the axiom T_0: given any two distinct points of X, there is an open set containing one but not the other.

(d) If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X.

(e) If x_0, x_1 are points of a topological space X, and if $x_0 \in \{x_1\}^-$, then we say that x_1 specializes to x_0, written $x_1 \to x_0$. We also say x_0 is a specialization of x_1, or that x_1 is a generization of x_0. Now let X be a Zariski space. Show that the minimal points, for the partial ordering determined by $x_1 > x_0$ if $x_1 \to x_0$, are the closed points, and the maximal points are the generic points of the irreducible components of X. Show also that a closed subset contains every specialization of any of its points. (We say closed subsets are stable under specialization.) Similarly, open subsets are stable under generization.

(f) Let t be the functor on topological spaces introduced in the proof of (2.6). If X is a noetherian topological space, show that $t(X)$ is a Zariski space. Furthermore X itself is a Zariski space if and only if the map $\alpha : X \to t(X)$ is a homeomorphism.

Proof.

(a). Note that X is Noetherian, which implies that it can be covered by finitely many $Spec(A_i)$ with A_i Noetherian, and each $Spec(A_i)$ with A_i Noetherian is a Noetherian space. Thus, X is a Noetherian space. So X is a Noetherian space.

As we have shown in 2.2.9, each irreducible closed subset of a scheme has only one generic point. So $sp(X)$ is a Zariski space.

(b). Let Z be a minimal nonempty closed subset of a Zariski space. Then Z is obviously irreducible so it have a unique generic point. Suppose that Z has another point x. Then $\{x\}^- \subset Z$, which implies that $\{x\}^- = Z$ by the irreduciblity of Z. Thus x is also generic point, which leads to a contradiction.

(c). Given any x, y, if $x \notin \{y\}^-$ or $y \notin \{x\}^-$, we are done. If $x \in \{y\}^-$ and $y \in \{x\}^-$, then $\{x\} \subset \{y\}$ and $\{x\}^- \supset \{y\}^-$. At this case, $\{x\}^- = \{y\}^-$. Note that $\{x\}^-$ is an irreducible closed set and x is the generic point and so is $\{y\}^-$. By the uniqueness of the generic point, $x = y$.

(b). For the closure of generic point is X, this is trivial.

(d). Suppose that x_0 is a minimal point. If $x_0 > x_1$, that is, $x_1 \in \{x_0\}^-$, then $x_1 = x_0$. Thus, $\{x_0\}^- = \{x_0\}$, implying x_0 is a closed point.

Suppose that y_0 is a maximal point. Suppose that $y_0 \in Z$, where Z is an irreducible component of X. Let y_1 be the generic point. Then $y_1 > y_0$. For y_0 is a maximal point, $y_0 = y_1$, that is, y_0 is a generic point.

Let Z be a closed set. Then for any point $x \in Z$, $\{x\}^- \subset Z$. If x is a specialization of x, $y \in \{x\}^- \subset Z$.

Let U be an open set. Then for any point $x \in U$. If y is a generization of x, that is, $x \in \{y\}^-$. Then any open neighborhood of x contains y, implying $y \in U$.

(e). ***

(f). *** □

Exercise 2.3.18. Let X be a Zariski topological space. A constructible subset of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that (1) every open subset is in $\mathfrak{F}$, (2) a finite intersection of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the complement of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.

(a) A subset of X is locally closed if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible if and only if it can be written as a finite disjoint union of locally closed subsets.

(b) Show that a constructible subset of an irreducible Zariski space X is dense if and only if it contains the generic point. Furthermore, in that case it contains a nonempty open subset.

(c) A subset S of X is closed if and only if it is constructible and stable under specialization. Similarly, a subset T of X is open if and only if it is constructible and stable under generization.

(d) If $f : X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.

Exercise 2.3.19. The real importance of the notion of constructible subsets derives from the following theorem of Chevalley—see Cartan and Chevalley [1, exposé 7] and see also Matsumura [2, Ch. 2, §6]: let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then the image of any constructible subset of X is a constructible subset of Y. In particular, $f(X)$, which need not be either open or closed, is a constructible subset of Y. Prove this theorem in the following steps.

(a) Reduce to showing that $f(X)$ itself is constructible, in the case where X and Y are affine, integral noetherian schemes, and f is a dominant morphism.

(b) In that case, show that $f(X)$ contains a nonempty open subset of Y by using the following result from commutative algebra: let $A \subseteq B$ be an inclusion of noetherian integral domains, such that B is a finitely generated A-algebra. Then given a nonzero element $b \in B$, there is a nonzero element $a \in A$ with the following property: if $\varphi : A \to K$ is any homomorphism of A to an algebraically closed field K, such that $\varphi(a) \neq 0$, then φ extends to a homomorphism φ' of B into K, such that $\varphi'(b) \neq 0$. [Hint: Prove this algebraic result by induction on the number of generators of B over A. For the case of one generator, prove the result directly. In the application, take $b = 1$.]

(c) Now use noetherian induction on Y to complete the proof.

(d) Give some examples of morphisms $f : X \to Y$ of varieties over an algebraically closed field k, to show that $f(X)$ need not be either open or closed.

Exercise 2.3.20. Let X be an integral scheme of finite type over a field k (not necessarily algebraically closed). Use appropriate results from (I, §1) to prove the following.

(a) For any closed point $P \in X$, $\dim X = \dim \mathcal{O}_P$, where for rings, we always mean the Krull dimension.

(b) Let $K(X)$ be the function field of X (Ex. 3.6). Then $\dim X = \text{tr.d.}\, K(X)/k$.

(c) If Y is a closed subset of X, then $\text{codim}(Y, X) = \inf\{\dim \mathcal{O}_{P,X} | P \in Y\}$.

(d) If Y is a closed subset of X, then $\dim Y + \text{codim}(Y, X) = \dim X$.

(e) If U is a nonempty open subset of X, then $\dim U = \dim X$.

(f) If $k \subseteq k'$ is a field extension, then every irreducible component of $X' = X \times_k k'$ has dimension $= \dim X$.

Proof.

(a). If $X = Spec(k[x_1, ..., x_n]/\mathfrak{p})$ with $\mathfrak{p}$ a prime ideal, then if $P \in X$ is a closed point then P corresponds to a maximal ideal $\mathfrak{m}$ of $k[x_1, .., x_n]/\mathfrak{p}$. $dim(X) = dim(k[x_1, ..., x_n]) - ht(\mathfrak{p})$ and $ht_X(\mathfrak{m}) = ht_{k[x_1,...,x_n]}(\mathfrak{m}) - ht(\mathfrak{p})$. For $\mathfrak{m}$ is also maximal in $k[x_1, ..., x_n]$, $ht(\mathfrak{m}) = dim(k[x_1, ..., x_n])$. Hence $dim(X) = ht(\mathfrak{m}) = dim(\mathcal{O}_{X,P})$.

(b). Note that $Frac(\mathcal{O}_P) = K(X)$. By Chapter 1 Theorem 1.8 A of [5], $dim(\mathcal{O})_P = tr.d.K(\mathcal{O}_P)/k = tr.d.K(X)/k$.

(c). Suppose that Y is defined by an ideal $I \subset \mathfrak{p}_Y$. Then

$$cod(Y, X) = \sup_i \text{length}\{\mathfrak{p} \subseteq \mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq ... \subseteq \mathfrak{p}_i \subset I\}$$

For each $P \in Y$, P is define by a prime ideal $\mathfrak{p}_P$ such that $I \subset \mathfrak{p}_P$ and

$$dim(O_{X,P}) = \sup_i \text{length}\{\mathfrak{p} \subseteq \mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq ... \subseteq \mathfrak{p}_i = \mathfrak{p}_P\}$$

Thus, $codim(Y, X) = inf\{\dim \mathcal{O}_{X,P} | P \in Y\}$.

(d). If Y is irreducible, then Y is defined by a prime ideal $\mathfrak{p}$ of $A = k[x_1, ..., x_n]/\mathfrak{p}$. By (2), we know that $codim(X) = dim\mathcal{O}_{\mathfrak{p},X} = dimA_{\mathfrak{p}} = dimA/\mathfrak{p}$. By Chapter 1, Theorem 1.8 A of [5], $dim(\mathfrak{p}) + dim(A/\mathfrak{p}) = dim(A)$, that is, $dimY + codim(Y, X) = dim(X)$.

For Y is not irreducible, take the maximal chain of irreducible closed subsets:

$$Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq ... \supseteq Y_n$$

Then $dim(Y_0) = dim(Y)$ and $codim(Y_0, X) = codim(Y, X)$. Just like the discussion above, we still have $dim(Y) + dim(Y, X) = dim(X)$.

(e). Suppose $U = V(I)^c$. Then there exists closed point in U: If not, $I \subset \bigcap_{\mathfrak{m} \text{ maximal ideal}} \mathfrak{m} = \mathcal{N}(k[x_1, ..., x_k]/\mathfrak{p})$. Thus, $U = X^C = \emptyset$, which leads to a contradiction.

Note that for any closed point $P \in Y$, $\mathcal{O}_{Y,P} = \mathcal{O}_{X,P}$. They have the same dimension. Thus, $dim(U) = dim(X)$.

(f). ***

For generally cases, just know that $dim(X) = dim(Spec(A))$ with $Spec(A) \subset X$:

First of all, note that X is Noetherian for each finitely generated k-alg is Noetherian. So assume that

$$Y_0 \supseteq Y_1 \supseteq Y_2 ... \supseteq Y_n$$

is the longest chain of irreducible closed subsets. Suppose $Y_n \cap Spec(A) \neq \emptyset$, then for all $i = 0, ..., n-1$, $Y_i \cap Spec(A)$ is not empty. Note that

$$Y_0 \cap A \supseteq Y_1 \cap A \supseteq Y_2 \cap A ... \supseteq Y_n \cap A$$

is a chain of irreducible closed subsets of $Spec(A)$. So $dim(Spec(A)) \geq dim(X)$. Note that a chain of irreducible closed subsets of $Spec(A)$ is also a chain of irreducible closed subsets of X. $dim(Spec(A)) = dim(X)$. By (a) and (e), each $Spec(A_i) \cap Spec(A_j)$ contains closed points, so $dim(Spec(A_i)) = dim(Spec(A_j)) = dim \mathcal{O}_{X,P}$, implying $dim(Spec(A)) = dim(Spce(A_i))$ for each $Spec(A_i)$. Finally, we can see that for any $P \in X$ closed,

$$dim(O_{X,P}) = dim(\mathcal{O}_{A_i,P}) = dim(Spec(A_i)) = dim(Spec(A)) = dim(X)$$

.

For the (b),(c) and (d), this will be trivial using the discussion above. □

Exercise 2.3.21. Let R be a discrete valuation ring containing its residue field k. Let $X = \text{Spec } R[t]$ be the affine line over Spec R. Show that statements (a), (d), (e) of (Ex. 3.20) are false for X.

Proof. Suppose that the only maximal ideal of R is $\mathfrak{m}_R$. By Chapter 11 Exercise 7 of [2].

$$dim(X) = 2$$

(a). Take $u \notin \mathfrak{m}$, which implies u is a unit in R. So $(tu - 1)$ is a maximal prime ideal of $R[t]$ and then $(tu - 1)$ is a closed point of X. $dim(P) = dim(R_{(ut-1)}) = 1 \neq dim(X)$

(d). Consider $\mathfrak{p} = (tu - 1)$ with u given above. Then $V(\mathfrak{p})$ is a field hence has dimension 0. For $\mathfrak{p}$ has hight 1, $codim(V(\mathfrak{p})) = 1$. Hence $dim(V(\mathfrak{p})) + codim(V(\mathfrak{p})) = 1 \neq dim(X)$.

(c). Consider $\mathfrak{q} = \mathfrak{m}_R[t] \subset R[t]$. Consider $D(\mathfrak{q})$: $R[t]_{\mathfrak{q}} \cong K(R)[t]$ with $K(R)$ the fraction field of R. So $dim(U) = dim(R[t]_m) = 1 \neq dim(X)$. □

Remark. The ideals of the form

$$(mt - 1)$$

with $m \in \mathfrak{m}$ are maximal, for $R[x]/(ut-1) = R$ and $R[x]/(mt-1) = R[m^{-1}]$ where R is an integral domain and $R[m^{-1}] \cong k$.

Exercise 2.3.22.

Proof. □

2.3.3 Additional Exercises

Exercise 2.3.23. Recall that a topology space is Noetherian if it satisfies the descending chain condition for closed subsets, i.e. for any sequence $Z_1 \supseteq Z_2 ...$ of closed subsets there exists an index n such that $Z_n = Z_{n+1} = ...$

(i) Show that a topological space is Noetherian is Noetherian if and only if every open subset is quasi-compact.

(ii) Construct an example of a non-Noetherian scheme $(X, \mathcal{O}_X)$ with X Noetherian.

Proof.

(i). ($\Longrightarrow$): Let $U \subset X$ be an open subset with $\mathscr{U}$ a cover of U. Consider $U_1 \subseteq U_1 \cup U_2 \subseteq ... \subseteq \cup_{i=1}^{n} U_i \subseteq ....$ For X is Noetherian, there exists N, such that, $\cup_{i=1}^{N} U_i = \cup_{i=1}^{N+1} U_i = ...$ Hence U is covered by $\{U_1, ..., U_N\}$.

($\Longleftarrow$): For any sequence $Z_1 \supseteq Z_2 ...$ of closed subsets, consider $Z = \cap_{i=1}^{\infty} Z_i$, which is a closed subset. By definition $\{Z_i^C\}$ forms a cover of Z^C. Because Z^C is quasi-compact, it can be covered by finitely many $\{Z_i^C\}$, that is, there exists N such that $Z_N = Z_{N+1} = ...$

(ii). Consider $A = k[x_0, ..., x_n, ...]/(x_0^2, ..., x_n^2, ...)$. Note that $\mathcal{N}(A) = k[x_0, ..., x_n, ..]/(x_0, ..., x_n, ...) \cong k$. Hence as topological spaces $X = X_{red} = Spec(\mathcal{N}(A))$, which is a unique point. So X is a Noetherian topological space.

Note that A is not Noetherian:

$$(x_0) \subseteq (x_0, x_1) \subseteq ... \subseteq (x_0, x_1, ..., x_n, ...) \subseteq ...$$

Thus, $X = Spec(A)$ is not a Noetherian scheme. □

Exercise 2.3.24. Consider the subscheme $X \subset \mathbb{A}^2_{\mathbb{Z}}$ given by $x_1x_2^2 - m$ for some $m \in \mathbb{Z}$. Study the fibers of $X \longrightarrow Spec(\mathbb{Z})$. Which ones are irreducible?

Proof. Points in $Spec(\mathbb{Z})$ are {(0),(2),...,(p),...} with p prime. Because $\mathcal{O}_{\mathbb{Z},(p)} = \mathbb{Z}_{(p)}$, $k((p)) = \mathbb{F}_p$ and $k((0)) = \mathbb{Q}$. Then, for any (p), consider the fiber product

$$k(p) \times_{\mathbb{Z}} X = Spec(\mathbb{Z}[x_1, x_2]/(x_1x_2^2 - m) \otimes_{\mathbb{Z}} \mathbb{F}_p) = Spec(\mathbb{F}_p[x_1, x_2]/(x_1x_2^2 - m))$$

Hence, when $p|m$, $k(p) \times_{\mathbb{Z}} X = Spec(\mathbb{F}_p[x_1, x_2]/(x_1x_2^2 - m))$, which is not irreducible ($\mathbb{Z}[x_1, x_2]/(x_1x_2)$ has two minimal prime ideals (x_1), (x_2)). When $p \nmid m$, $x_1x_2^2 - m$ is not irreducible by directly computing.

Consider

$$k((0)) \times_{\mathbb{Z}} X = Spec(\mathbb{Z}[x_1, x_2]/(x_1x_2^2 - m) \otimes_{\mathbb{Z}} \mathbb{Q}) = Spec(\mathbb{Q}[x_1, x_2]/(x_1x_2^2 - m))$$

which is obviously irreducible.

SUMMIT: Then fiber of $X \longrightarrow Spec(\mathbb{Z}$ over (p) is irreducible if and only if $p \nmid m$ or $p = 0$. □

Exercise 2.3.25. Find examples for the following phenomena:

(i) Show that there exists a morphism $X \longrightarrow Y$ with Y integral and such that X_y are irreducible without X being irreducible.

(ii) Show that there exists a morphism $X \longrightarrow Spec(k[x])$ with X integral, the generic fiber X_η non-empty and integral but no closed fiber integral. Here, k is any field.

(iii) Show that there exists a morphism $X \longrightarrow Spec(\mathbb{Q}[x])$ with X integral and infinitely many irreducible and infinitely many reducible closed fibers.

Proof.

(i). Assume that k is algebraically closed. Consider $X = Spec(k[x, y]/(xy))$. Let $\varphi : X \longrightarrow \mathbb{A}^1_k$ be the morphism given by $i : k[x] \hookrightarrow k[x, y]/(xy)$. For $k[x]$ is a PID, each closed point of $\mathbb{A}^1_k$ is of the form $(f(x))$ for an irreducible polynomial $f(x) = (x - \lambda)$ with $\lambda \in k$. Then

$$k(P) \times_{\mathbb{A}^1_k} X = Spec(k[x, y]/(xy) \otimes_{k[x]} k(p)) = Spec(k[x, y]/(x - \lambda, xy)$$

When $\lambda \neq 0$, $k(P) \times_{\mathbb{A}^1_k} X = Spec(k[x, y]/(x - \lambda, y)) = Spec(k)$. When $\lambda = 0$, $k(P) \times_{\mathbb{A}^1_k} X = Spec(k[x, y]/(x)) = Spec(k[y]) = \mathbb{A}^1_k$ Hence $k(P) \times_{\mathbb{A}^1_k}$ is always irreducible at closed point of $\mathbb{A}^1_k$.

At generic point, $k(P) = k$,

$$k(P) \times_{\mathbb{A}^1_k} X = Spec(k[x, y]/(xy) \otimes_{k[x]} k)) = Spec(k[y])$$

Hence at generic point, $k(P) \times_{\mathbb{A}^1_k} X$ is irreducible.

Note that (xy) is not a prime ideal. Hence X is not integral. So we get the corresponding examples.

(ii). Let k be an algebraically closed field. Consider $A = Spec(k[x,y]/(x-y^2))$. Let $X = Spec(A)$ and $X \longrightarrow Spec(k[x])$ be the morphism defined by $k[x] \hookrightarrow k[x,y]/(x-y^2)$.

Closed points P of $Spec(k[x]$ are of the form $(x-\lambda)$ with $\lambda \in k$. Then

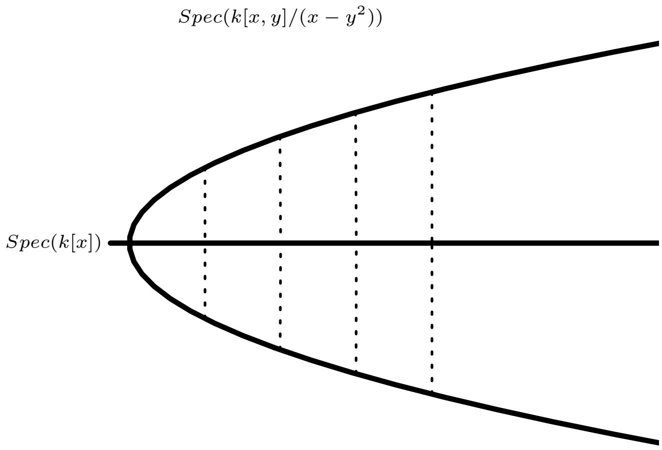
$$k(P) \times_{\mathbb{A}^1_k} Spec(A) = Spec(k[x,y]/(x-y^2) \otimes_{k[x]} k[x]/(x-\lambda)) = Spec(k[x,y]/(x-\lambda, y^2-\lambda))$$

Hence $k(P) \times_{\mathbb{A}^1_k} Spec(A)$ is not integral for $(y^2-\lambda)$ is not prime.

Consider $P = (0)$, then

$$k(P) \times_{\mathbb{A}^1_k} Spec(A) = Spec(k[x,y]/(x-y^2) \otimes_{k[x]} K(k[x])) = Spec(K(k[x])[y]/(x-y^2))$$

which is integral for $K(k[x])[y]/(x-y^2)$ is integral.



(iii). Consider $A = \mathbb{Q}[x,y]/(x-y^2)$. Then for closed points $(x-r)$, we have

$$k(P) \times_{\mathbb{A}^1_{\mathbb{Q}}} Spec(A) = Spec(\mathbb{Q}[x,y]/(x-y^2) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x-r)) = Spec(\mathbb{Q}[x,y]/(x-r, y^2-r))$$

If $r \in \mathbb{Q}^2$, then $y^2 - r$ is reducible. Hence $k(P) \times_{\mathbb{A}^1_{\mathbb{Q}}} Spec(A)$ is reducible. If $r \in \mathbb{Q} - \mathbb{Q}^2$, then $(x-r, y^2-r)$ is a prime ideal. Hence $k(P) \times_{\mathbb{A}^1_{\mathbb{Q}}} Spec(A)$ is integral and thus irreducible. $\square$

Exercise 2.3.26. Consider the natural morphism $\mathbb{A}^2_{\bar{\mathbb{Q}}} \longrightarrow \mathbb{A}^2_{\mathbb{Q}}$ and dethermine the images of the following points.

(i) $(x-\sqrt{2}, y-\sqrt{2})$; (ii) $(x-\sqrt{2}, y-\sqrt{3})$; (iii) $(\sqrt{2}x-\sqrt{3}y)$

Proof. Note that this morphism is given by

$$i : \mathbb{Q}[x, y] \hookrightarrow \bar{\mathbb{Q}}[x, y]$$

(i). $i^{-1}(x - \sqrt{2}, y - \sqrt{2}) = (x - \sqrt{2}, y - \sqrt{2}) \cap \mathbb{Q}[x, y] = (x^2 - 2, x - y)$.

(ii). $i^{-1}(x - \sqrt{2}, y - \sqrt{3}) = (x - \sqrt{2}, y - \sqrt{3}) \cap \mathbb{Q}[x, y] = (x^2 - 2, y^2 - 3)$.

(iii). $i^{-1}(\sqrt{2}x - \sqrt{3}y) = (\sqrt{2}x - \sqrt{3}y) \cap \mathbb{Q}[x, y] = (2x^2 - 3y^2)$. □

Exercise 2.3.27. Verify the following assertions

(i) A morphism $f : X \longrightarrow Y$ of schemes which is surjective, of finite type and qausi-finite, need not be finite.

(ii) 'quasi-finite' and 'injective' are not preserved under base change.

(iii) 'being an open/closed immersion' are preserved under base-change.

(iv) 'having reduced/integral/connected fibers' is not preserved under base change

Proof.

(i). Suppose k is algebraically closed. Let $X = Spec(k[x, y]/(xy-1)) \sqcup Spec(k[x, y]/((x-1)y-1))$. Consider $X \longrightarrow \mathbb{A}^1_k$ given by $k[x] \hookrightarrow k[x, y]/(xy - 1)$ and $k[x] \hookrightarrow k[x, y]/((x - 1)y - 1)$.

Then $X \longrightarrow \mathbb{A}^1_k$ is surjective. Because $k[x, y]/(xy-1)$ and $k[x, y]/((x-1)y-1)$ are both finitely generated $k[x]$-algebra, $X \longrightarrow \mathbb{A}^1_k$ is of finite type.

Consider the closed point of $\mathbb{A}^1_k$. Then for $(x - \lambda)$, $\lambda \neq 0, 1$,

$$k(P) \times_{\mathbb{A}^1_k} Spec(k[x, y]/(xy - 1)) = Spec(k[x, y]/(x - \lambda, \lambda y - 1))$$

Similarly, $k(P) \times_{\mathbb{A}^1_k} Spec(k[x, y]/((x - 1)y - 1))$ is a point.

When $\lambda = 0$, $k(P) \times_{\mathbb{A}^1_k} Spec(k[x, y]/(xy - 1)) = \emptyset$ but $k(P) \times_{\mathbb{A}^1_k} Spec(k[x, y]/((x - 1)y - 1))$ is a point..

When $\lambda = 1$, $k(P) \times_{\mathbb{A}^1_k} Spec(k[x, y]/((x - 1)y - 1)) = \emptyset$, but $k(p) \times_{\mathbb{A}^1_k} Spec(k[x, y]/(xy - 1)$ is a point.

When P is the generic point, $k(P) \times_{\mathbb{A}^1_k} Spec(k[x, y]/(xy - 1)) = Spec(K(k[x])[y]/(xy - 1)$ which is a point. Similarly for another part. Hence, $X \longrightarrow \mathbb{A}^1_k$ is quasi-finite.

Because $k[x, y]/(xy - 1) \cong k[x, x^{-1}]$ is not a finitely generated $k[x]$-module. The morphism is not finite.

(ii). 'Injective' is not preserved under base change: $Spec(\mathbb{C}) \longrightarrow Spec(\mathbb{R})$. $Spec(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow Spec(\mathbb{R})$ where $Spec(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ contains two points.

'Quasi-finite' is not preserved under base change: Consider $Spec(\bar{\mathbb{Q}}) \longrightarrow Spec(\mathbb{Q})$ Then $Spec(\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}) \longrightarrow Spec(\bar{\mathbb{Q}})$ is not quasi-finite. For $Spec(\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}) = \sqcup_{Gal(\bar{\mathbb{Q}}/\mathbb{Q})} Spec(\bar{\mathbb{Q}})$.

(iii). Trivial.

(iv). $Spec(\mathbb{C}) \longrightarrow Spec(\mathbb{R})$. $Spec(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow Spec(\mathbb{R})$ where $Spec(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ contains two points. And note that $\mathbb{C} \oplus \mathbb{C}$ is not integral. Thus, 'having integral/connected fibers' is not preserved under base change.

Conisder $Spec(\mathbb{R}[x]/(x^2+1)) \longrightarrow Spec(\mathbb{R})$.

$$Spec(\mathbb{R}[x]/(x^2+1) \otimes_{\mathbb{R}} \mathbb{F}_2) = Spec(\mathbb{F}_2[x]/(x^2+1)) = Spec(\mathbb{F}_2[x]/(x-1)^2)$$

Hence 'having reduced fibers' is not preserved under base change. $\square$

Exercise 2.3.28. Let $f : X \longrightarrow Y$ be a morphism of schemes.

(i). Assume that f is a surjective immersion of schemes. Show that for every field K and every morphism $g : Spec(K) \longrightarrow Y$, there exists a unique morphism $h : Spec(K) \longrightarrow X$ such that $f \circ h = g$.

(ii). Assume that the diagonal $\Delta : X \longrightarrow X \times_Y X$ is surjective. Show that for all fields K the map

$$\pi : Hom(Spec(K), X) \longrightarrow Hom(Spec(K), Y)$$

given by $h \mapsto f \circ h$ is injective.

Proof.

(i). Assume that $h((0)) = \mathfrak{p} \in Spec(A) \subset X$. Then we have a locally ringed morphism $g^\# : A_\mathfrak{p} \longrightarrow K$ such that $\mathfrak{p}A_\mathfrak{p} = {g^\#}^{-1}(0)$. Hence, we have $g^\# : A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \longrightarrow K$.

For f is a surjective immerison, there exists a unique point $\mathfrak{q} \in Spec(B)$ such that $f(\mathfrak{q}) = \mathfrak{p}$ and $f(Spec(B)) \subset Spec(A)$. Also because f is an immersion, $f^\# : A_\mathfrak{p} \longrightarrow B_\mathfrak{q}$ is a locally ringed surjection. Hence, we have a surjective, $f^\# : A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \longrightarrow B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$. Hence, we see that $k(\mathfrak{q}) = k(\mathfrak{p}) \subset K$. There exists a unique morphism $h^\# : k(\mathfrak{q}) \longrightarrow K$ such that $h^\# \circ f^\# = g^\#$. Moreover, there is a unique morphism $h : Spec(K) \longrightarrow X$ such that $g = f \circ h$.

(ii). At this case, Δ is a surjective immersion of schemes. By (i), we see that

$$Hom(Spec(K), X) \longrightarrow Hom(Spec(K), X \times_Y X)$$

given by $g \mapsto \Delta \circ g$ is an isomorphism. Hence, we can rewrite π as

$$Hom(Spec(K), X \times_Y X) \longrightarrow Hom(Spec(K), Y)$$

given by $g \mapsto f \circ p \circ g$, where $p \circ \Delta = id_X$. For $X \times_Y X$ is the pullback of $f : X \longrightarrow Y$. π is an injection by the universal property of pullback. $\square$

2.4 Separated and Proper Morphisms

2.4.1 Preparations

Theorem 2.4.1. *A projective morphism of Noetherian schemes is proper.*

Proof. Let $f : X \longrightarrow Y$ be a projective morphism. Then

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_Y \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where i is a closed immersion. So we just need to show that $p : \mathbb{P}^n_Y \longrightarrow Y$ is proper. Because

$$\begin{tikzcd} \mathbb{P}^n_Y \arrow[r, "p"] \arrow[d, "p'"] & Y \arrow[d] \\ \mathbb{P}^n_{\mathbb{Z}} \arrow[r, "p_{\mathbb{Z}}"] & Spec(\mathbb{Z} \end{tikzcd}$$

we only need to show that $p_{\mathbb{Z}} : \mathbb{P}^n_{\mathbb{Z}} \longrightarrow Spec(\mathbb{Z})$ is proper.

For $D_+(x_i)$ forms a cover of $\mathbb{P}^n_{\mathbb{Z}}$ and each $D_+(x_i) \cong Spec(\mathbb{Z}[x_0, x_1, ..., x_n])$, $p_{\mathbb{Z}}$ is of finite morphism.

Also, use the cover above. Fixing any $D_+(x_i)$, we see that $D_+(x_i) = Spec(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}) \longrightarrow D_+(x_i) \times_{\mathbb{Z}} D_+(x_j) = Spec(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}, \frac{y_0}{y_j}, ..., \frac{y_n}{y_j})$ is a closed immersion, for each j. Thus, $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$ is a closed immersion. Hence $p_{\mathbb{Z}}$ is separated.

Finally, we use valuation criteria to show that $p_{\mathbb{Z}}$ is universally closed: Consider

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ Spec(R) \arrow[r] & Spec(\mathbb{Z}) \end{tikzcd}$$

Suppose that $Spec(K)$ is mapped to $D_+(x_0)$. Then we have

$$\begin{tikzcd} K & \mathbb{Z}[\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}] \arrow[l] \\ R \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u] \end{tikzcd}$$

Denote the valuation on R by v_R, w.l.o.g. we can assume that $v_R(\frac{x_1}{x_0}) \leq ... \leq v_R(\frac{x_n}{x_0})$. If $v_R(\frac{x_1}{x_0}) \geq 0$, then see that $v_d(\frac{x_i}{x_0}) \geq 0$ for each i. Hence, we have $\mathbb{Z}[\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}] \longrightarrow R$ makes the diagram commutes. If $v_R(\frac{x_1}{x_0}) < 0$, then $(\frac{x_1}{x_0})$ is not mapped to $(0) \subset K$. Hence, at this case, $Spec(K)$ is also contained in $D_+(x_1)$ and $v_R(\frac{x_0}{x_1}) > 0$ with

$$v_P(\frac{x_i}{x_1}) = v_P(\frac{x_i}{x_0})v_R(\frac{x_0}{x_1}) \leq v_P(\frac{x_{i+1}}{x_0})v_R(\frac{x_0}{x_1}) = v_P(\frac{x_{i+1}}{x_1})$$

So there is a morphism to $D_+(x_1)$ from $Spec(R)$. To there is always at least one morphism form $Spec(R)$ to $\mathbb{P}^n_{\mathbb{Z}}$ makes the diagram commutes. Hence $p_{\mathbb{Z}}$ is universally closed.

Hence $p_{\mathbb{Z}}$ is of finite type, separated and universally closed. It is proper. □

Lemma 2.4.2. *The base change of a surjective morphism is surjective.*

Proof. See Stalks Project. □

Theorem 2.4.3. *Consider*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f_X"'] & & Y \arrow[dl, "f_Y"] \\ & S & \end{tikzcd}$$

if f_Z is universally closed and f is surjective, then f_Y is universally closed.

Proof. For any $S^{'} \longrightarrow S$, consider

$$\begin{tikzcd} X \times_S S^{'} \arrow[rr, "f^{'}"] \arrow[dr, "f^{'}_X"'] & & Y \times_S S^{'} \arrow[dl, "f^{'}_Y"] \\ & S^{'} & \end{tikzcd}$$

Then $f^{'}_X$ is closed for f_X is universally closed and $f^{'}$ is surjective for surjection is stable under base change. Hence for any closed subset $V \subset Y \times_S S^{'}$, ${f^{'}}^{-1}(V)$ is closed and because $f^{'}$ is surjective, $f^{'}_Y(V) = f^{'}_X(f^{-1}(V))$ which is closed. So f_Y is universally closed. □

2.4.2 Exercise

Exercise 2.4.1. Show that a finite morphism is proper.

Proof. Let $f : X \longrightarrow Y$ be a finite morphism.

We just need to verfiy that f is of finite type, separated and universally closed.

(i). By 2.3.5 (a), f is of finite type.

(ii). Let $U = Spec(B)$ be an affine open set of Y. Then $V = f^{-1}(U) = Spec(A) \subset X$. $f|_V : Spec(A) \longrightarrow Spec(B)$ is affine, implying that $f|_V$ is separated. Now Y is covered by $\{U_i\}$ such that $f|_{f^{-1}(U_i)}$ is separated, by Chapter 2 Corollary 4.6, f is separated.

(iii). First, we show that being a finite morphism is preserved under base change:

Given any $g : Y^{'} \longrightarrow Y$. Suppose that Y is covered by $\{Spec(B_i)\}$, while X is covered by $\{Spec(A_i)\}$ with $f^{-1}(Spec(B_i)) = Spec(A_i)$ and $Y^{'}$ is covered by $\{Spec(C_{ij})\}$ where $g^{-1}(Spec(B_i))$ is covered by $Spec(B_{ij})$.

For any $Spec(B_{ij})$, $p_{Y'}^{-1}(Spec(C_{ij})) = Spec(A_i \otimes_{B_i} C_{ij})$. For A_i is a finitely generated B_i-module, $A_i \otimes_{B_i} C_{ij}$ is a finitely generated C_{ij}-module. So $p_{Y'}$ is finite.

By 2.3.5, every finite morphism is closed. So every finite morphism is universally closed. Thus, f is universally closed.

□

Remark. Some results:

- Finite $\Longrightarrow$ Affine
- Affine $\Longrightarrow$ Separated
- Finite $\Longrightarrow$ Universally closed.

Remark. We need to know more about $p_Y^{-1}(Spec(C_{ij}))$. I mean the strict proof.

My Gauss: For generally case, if $f^{-1}(Spec(B_i)) = \cup Spec(A_{ik})$, then $f^{-1}(Spec(C_{ij})) = \cup Spec(C_{ij} \otimes_{B_i} A_{ik})$. Just by the definition of fiber product.

Exercise 2.4.2. Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Show that $f = g$. Give examples to show that this result fails if either

(a) X is nonreduced, or

(b) Y is nonseparated.

[Hint: Consider the map $h : X \to Y \times_S Y$ obtained from f and g.]

Proof. Consider the pullback

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "l"] & Y \arrow[d, "\Delta_Y"] \\ X \arrow[r, "f_1 \times_S f_2"] & Y \times_S Y \end{tikzcd}$$

Then l is a closed immersion for being a closed immersion is preserved under base change.

Because we have $p_{1,Y} \circ \Delta_Y = p_{2,Y} \circ \Delta_Y = id_Y$ and $p_{1,Y} \circ f_1 \times_S f_2 = f_1$, $p_{2,Y} \circ f_1 \times_S f_2 = f_2$, $f_1 \circ l = f_2 \circ l = f$.

Use the universal property of the pullback, we have

$$\begin{tikzcd} U \arrow[dr, dashed, "\exists! j"] \arrow[drr, bend left, "f_1|_U = f_2|_U"] \arrow[ddr, bend right, "i"'] & & \\ & Z \arrow[r, "f"] \arrow[d, "l"] & Y \arrow[d, "\Delta_Y"] \\ & X \arrow[r, "f_1 \times_S f_2"] & Y \times_S Y \end{tikzcd}$$

Note that j is an injection and for Z is closed, $Z = \overline{U} = X$ in topology. Thus, $Z = X$ as schemes with X reduced. Then $l = id_X$, so $f_1 = f_2 = f$. $\square$

Remark. For the first pullback, $Z = \{x \in X | f_1(x) = f_2(x)\}$ and it is directly given by the topology structure of X and Y. We know that closed maps are stable under base change. Hence l is closed and then $Z \subset X$ is closed when Y is separated. This is what 'separated' really means in topology.

Remark. Existence of $f_1 \times_S f_2$: For f_1, f_2 are S-scheme morphisms, $f_Y \circ f_i = f_X$ for $i = 1, 2$.

Remark. Suppose X is reduced. Just consider the affine case, that is, $X = Spec(A)$. If $Z \subset X$ is a closed immersion, then $Z = Spec(A/I)$ for some I. In topology, $Z = X$ if and only if $I \subset \mathcal{N}(A)$. However, $\mathcal{N}(A) = 0$. Thus, $Z = X$ as schemes. For general cases, take $X = \cup Spec(A_i)$ with A_i reduced.

Exercise 2.4.3. Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. Then $U \cap V$ is also affine. Give an example to show that this fails if X is not separated.

Proof. Consider the fiber product

$$\begin{tikzcd} X \times_S X \arrow[r, "p_{X,1}"] \arrow[d, "p_{X,2}"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

and $\Delta : X \longrightarrow X \times_S X$ which is a closed immersion hence affine.

For $p_{X,2} \circ \Delta = id_X$, $U = \Delta_X^{-1} \circ p_2^{-1}(U)$, $V = \Delta_X^{-1} \circ p_2^{-1}(V)$ and $U \cap V = \Delta_X^{-1} \circ p_2^{-1}(U \cap V) = \Delta_X^{-1}(p_2^{-1}(U) \cap p_2^{-1}(V)$.

Suppose that $S = Spec(R)$ and $X = \cup Spec(A_i)$. Then $p_2^{-1}(U) = \cup U \times_R Spec(A_i)$, $p_2^{-1}(V) = \cup V \times_R Spec(A_i)$ and $p_2^{-1}(U) \cap p_2^{-1}(V) = U \cap V$ which is affine. So $U \cap V = \Delta_X^{-1}(p_2^{-1}(U) \cap p_2^{-1}(V)$ is affine.

For counter example, suppose that X is the affine plane with two origins and $V, U \cong \mathbb{A}_k^1$ with different origins. Then $U \cap V \cong \mathbb{A}_k^2 - \{0\}$, which is not affine. $\square$

Exercise 2.4.4. Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure (Ex. 3.11d) is proper over S. We refer to this result by saying that "the image of a proper scheme is proper." [Hint: Factor f into the graph morphism $\Gamma_f : X \to X \times_S Y$ followed by the second projection p_2, and show that Γ_f is a closed immersion.]

Proof. Consider the diagram:

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "f_Z"'] & X \arrow[r, "f"] \arrow[d, "f_X"] & Y \arrow[dl, "f_Y"] \\ & S & \end{tikzcd}$$

with f_Z proper, f_X, f_Y of finite type and separated, i closed immersion. For $f_Z = f_Y \circ f \circ i$ and f_Z is proper and f_Y is separated, $f \circ i$ is proper hence universally closed. So $f \circ i$ is closed then by the closeness of i, f is closed. Thus, $f(Z)$ is closed in Y.

Because $i^f : f(Z) \longrightarrow Y$ is a closed immersion, it is of finite type and separated.

Now consider

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "f_Z"'] & f(Z) \arrow[r, "i^f"] \arrow[d, "f_{f(Z)}"] & Y \arrow[dl, "f_Y"] \\ & S & \end{tikzcd}$$

For $f : Z \longrightarrow f(Z)$ is surjective and f_Z is universally closed, $f_{f(Z)}$ is universally closed by 2.4.3. Hence $f_{f(Z)}$ is proper. $\square$

Remark. If $g \circ f$ and f are universally closed, then g is also universally closed:

Consider any base change, $g' \circ f'$ and f' is closed. So is g'.

Exercise 2.4.5. Let X be an integral scheme of finite type over a field k, having function field K. We say that a valuation of K/k (see I, §6) has center x on X if its valuation ring R dominates the local ring $\mathcal{O}_{x,x}$.

(a) If X is separated over k, then the center of any valuation of K/k on X (if it exists) is unique.

(b) If X is proper over k, then every valuation of K/k has a unique center on X.

(c) Prove the converse of (a) and (b). [Hint: While parts (a) and (b) follow quite easily from (4.3) and (4.7), their converses will require some comparison of valuations in different fields.]

(d) If X is proper over k, and if k is algebraically closed, show that $\Gamma(X, \mathcal{O}_X) = k$. This result generalizes (I, 3.4a). [Hint: Let $a \in \Gamma(X, \mathcal{O}_X)$, with $a \notin k$. Show that there is a valuation ring R of K/k with $a^{-1} \in m_R$. Then use (b) to get a contradiction.]

Note. If X is a variety over k, the criterion of (b) is sometimes taken as the definition of a complete variety.

Proof.

(a). Suppose we have two point x, y with $\mathcal{O}_{X,x}, \mathcal{O}_{X,y}$ dominated by R. Then consider

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(R) \arrow[r] & Spec(k) \end{tikzcd}$$

Because $K = K(X)$ and X is integral, just let $Spec(K)$ be mapped to the generic point of X. (By 2.2.7 such, a morphism exists)

Suppose $x = \mathfrak{p} \in Spec(A)$ with A an integral domain and a finitely generated k-algebra. Using $A \hookrightarrow A_{\mathfrak{p}} \hookrightarrow R$, we can define

$$Spec(R) \longrightarrow Spec(A)$$

such that the generic point of $Spec(R)$, (0), is mapped to the generic point of $Spec(A)$, which is the generic point of X and $\mathfrak{m}_R$ is mapped to x. This is a morphism between schemes, for on $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Spec(R),\mathfrak{m}_R}$, $i^{-1}(\mathfrak{m}_R) = \mathfrak{m}_{A_{\mathfrak{p}}}$. Denote the morphism by, φ_x, such that the following diagram is commutative.

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(R) \arrow[r] \arrow[ur, dashed, "\varphi_x"] & Spec(k) \end{tikzcd}$$

Note that for y, we have φ_y. However by valuation criteria, X is not separated, which leads to a contradiction.

(b). For X is proper, there must exist a center x for any valuation on K/k:

For $f_x^{\#} : \mathcal{O}_{X,x} \longrightarrow R_{\mathfrak{m}} = R$. We can see that R dominates $\mathcal{O}_{X,x}$.

By (a), the center is unique.

(c).***

(d). Suppose that $a \in \Gamma(X, \mathcal{O}_X)$ but $a \notin k$. Then we can find a valuation on K/k and define a valuation ring R, with $v(a^{-1}) > 0$.

(In fact, for a is transcendental over k, we can find $K \supset k(a) \supset k$. The precise structure of R is $k[[a^{-1}]]$ with the fraction field $k((a^{-1}))$.)

Then we have a unique morphism such that the following diagram commutative:

$$\begin{tikzcd} Spec(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(R) \arrow[r] \arrow[ur, dashed, "\exists! \varphi"] & Spec(k) \end{tikzcd}$$

We have a map $f : \Gamma(X, \mathcal{O}_X) \longrightarrow R$, which implies on K, $v(a) \geq 0$, for $i_X = i_R \circ f$.

Thus, $v(1) = v(a^{-1}) + v(a) > 1$, which leads to a contradiction. □

Exercise 2.4.6. Let $f : X \to Y$ be a proper morphism of affine varieties over k. Then f is a finite morphism. [Hint: Use (4.11A).]

Proof. We need following lemmas:

Lemma 2.4.4. *For any subring $A \subset K$, there exists a valuation ring R such that $A \subset B \subset K$.*

Lemma 2.4.5. *Let $\varphi : A \longrightarrow B$ be a ring homomorphism of finite type. If B is integral over A, then B is a finite A-algebra.*

Proof. $B = A[x_1, ..., x_n]/I$. B is integral over A, which implies for each x_i, there exists a monic polynomial $f_i \in A[t]$ such that $f_i(x_i) = 0$. Then

$$B \subset A[x_1, ..., x_n]/(f_1(x_1), ..., f_n(x_n))$$

which is a finite A-algebra. □

Since f is locally given by $f : Spec(B) \longrightarrow Spec(A)$ with $Spec(B) = f^{-1}(Spec(A))$, deduced by $\varphi : A \longrightarrow B$, it is enough to show that B is integral over A. For $\varphi(A)$ is a subring of $K(B)$, by [Theorem A.4.11 [5]], the integral closure of $\varphi(A)$ is the intersection of all valuation rings in $K(B)$ containing $\varphi(A)$. Take any valuation ring R containing A. Since $\varphi : A \longrightarrow B$ is proper, there exists

$$\begin{tikzcd} K(B) & B \arrow[l, hook'] \arrow[dl, dashed] \\ R \arrow[u, hook] & A \arrow[l, hook'] \arrow[u, "\varphi"'] \end{tikzcd}$$

which implies $B \subset R$. Hence, B lies in the integral closure of A and hence B is finite over A. $\square$

Remark. This exercise also implies:

$$\text{Proper} + \text{Affine} \implies \text{Finite}$$

Exercise 2.4.7. Schemes Over $\mathbb{R}$. For any scheme X_0 over $\mathbb{R}$, let $X = X_0 \times_{\mathbb{R}} \mathbb{C}$. Let $\alpha : \mathbb{C} \to \mathbb{C}$ be complex conjugation, and let $\sigma : X \to X$ be the automorphism obtained by keeping X_0 fixed and applying α to $\mathbb{C}$. Then X is a scheme over $\mathbb{C}$, and σ is a semi-linear automorphism, in the sense that we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{C} & \xrightarrow{\alpha} & \operatorname{Spec} \mathbb{C}. \end{array}$$

Since $\sigma^2 = id$, we call σ an involution.

(a) Now let X be a separated scheme of finite type over $\mathbb{C}$, let σ be a semilinear involution on X, and assume that for any two points $x_1, x_2 \in X$, there is an open affine subset containing both of them. (This last condition is satisfied for example if X is quasi-projective.) Show that there is a unique separated scheme X_0 of finite type over $\mathbb{R}$, such that $X_0 \times_{\mathbb{R}} \mathbb{C} \cong X$, and such that this isomorphism identifies the given involution of X with the one on $X_0 \times_{\mathbb{R}} \mathbb{C}$ described above.

(b) Show that X_0 is affine if and only if X is.

(c) If X_0, Y_0 are two such schemes over $\mathbb{R}$, then to give a morphism $f_0 : X_0 \to Y_0$ is equivalent to giving a morphism $f : X \to Y$ which commutes with the involutions, i.e., $f \circ \sigma_X = \sigma_Y \circ f$.

(d) If $X \cong \mathbb{A}_{\mathbb{C}}$, then $X_0 \cong \mathbb{A}_{\mathbb{R}}$.

(e) If $X \cong \mathbb{P}^1_{\mathbb{C}}$, then either $X_0 \cong \mathbb{P}^1_{\mathbb{R}}$, or X_0 is isomorphic to the conic in $\mathbb{P}^2_{\mathbb{R}}$ given by the homogeneous equation $x_0^2 + x_1^2 + x_2^2 = 0$.

Proof.

(a). Consider that X is affine, that is, $X = Spec(A)$ with A an integral domain. Because we have

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d] & X \arrow[d] \\ Spec(\mathbb{C}) \arrow[r, "\alpha"] & Spec(\mathbb{C}) \end{tikzcd}$$

then we have a morphism $A \xrightarrow{\sigma} A$ such that

$$\begin{tikzcd} A \arrow[r, "\sigma"] & A \\ \mathbb{C} \arrow[u] \arrow[r, "\alpha"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

commutes and consider $A^\sigma = \{a \in A | \sigma(a) = a\}$. We have a morphism $\mathbb{C}^\alpha \longrightarrow A^\sigma$. So A^σ is an $\mathbb{R}$-algebra.

For $A^\sigma \otimes_{\mathbb{R}} \mathbb{C} \cong A$, we have $Spec(A^\sigma) \times_{\mathbb{R}} \mathbb{C} = Spec(A)$. By the construction above, we can see that such X_0 is unique.

If $X = \cup Spec(A_i)$, for each $Spec(A_i)$, we have $Spec(B_i) \times_{\mathbb{R}} \mathbb{C} \cong Spec(A_i)$. Now, we only need to glue $Spec(B_i)$ together.

For $Spec(A_i)$ and $Spec(A_j)$ are affine, $Spec(A_i) \cap Spec(A_j)$ is still affine by 2.4.3. Suppose $Spec(A_i) \cap Spec(A_j) = Spec(A)$. Then we have $Spec(B)$ such that $Spec(B) \times_{\mathbb{R}} \mathbb{C} \cong Spec(A)$. Then we can glue $Spec(B_i)$ and $Spec(B_j)$ along $Spec(B)$ by the uniqueness of X_0. I don't know how to give the precise gluing functions.

(b). As we have shown.

(c).***

(d). For $\mathbb{A}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \cong \mathbb{A}^1_{\mathbb{C}}$. Then, use the uniqueness of X_0. □

Exercise 2.4.8. Let $\mathcal{P}$ be a property of morphisms of schemes such that:

(a) a closed immersion has $\mathcal{P}$;

(b) a composition of two morphisms having $\mathcal{P}$ has $\mathcal{P}$;

(c) $\mathcal{P}$ is stable under base extension.

Then show that:

(d) a product of morphisms having $\mathcal{P}$ has $\mathcal{P}$;

(e) if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathcal{P}$ and g is separated, then f has $\mathcal{P}$;

(f) If $f : X \to Y$ has $\mathcal{P}$, then $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ has $\mathcal{P}$.

[Hint: For (e), consider the graph morphism $\Gamma_f : X \to X \times_Z Y$ and note that it is obtained by base extension from the diagonal morphism $\Delta : Y \to Y \times_Z Y$.]

Proof.

(d). Just by (b),(c).

(e). Use magic square.

(f). By the definition of X_{red}, we see that $X_{red} \longrightarrow X$ is a closed immersion.

$$\begin{array}{ccc} X_{red} & \xrightarrow{f_{red}} & Y_{red} \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

$i_Y \circ f_{red} = f \circ i_X$ has $\mathscr{P}$. i_Y has $\mathscr{P}$. By (e), f_{red} has $\mathscr{P}$. □

Exercise 2.4.9. Show that a composition of projective morphisms is projective. [Hint: Use the Segre embedding defined in (I, Ex. 2.14) and show that it gives a closed immersion $\mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{rs+r+s}$.] Conclude that projective morphisms have properties (a)-(f) of (Ex. 4.8) above.

Proof. First of all, we show that

(a). A closed immersion is projective

Suppose that $f : X \longrightarrow Y$ is a closed immersion. Then consider the diagram

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[dr, "f"'] & \mathbb{P}_Y^0 = Y \arrow[d, "i_Y"] \\ & Y \end{tikzcd}$$

Thus, f is projective.

(b). A composition of two projective morphisms is projective:

Suppose that $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two projective morphisms. We have

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}_{\mathbb{Z}}^n \times_{\mathbb{Z}} Y \arrow[d, "p_Y"] & \\ & Y \arrow[r, hook] \arrow[dr, "g"'] & \mathbb{P}_{\mathbb{Z}}^m \times_{\mathbb{Z}} Z \arrow[d, "p_Z"] \\ & & Z \end{tikzcd}$$

For the serge embedding $\mathbb{P}_{\mathbb{Z}}^n \hookrightarrow \mathbb{P}_{\mathbb{Z}}^{(n+1)(m+1)-1}$, Apply the magic square, we have

$$\begin{tikzcd} \mathbb{P}_{\mathbb{Z}}^n \times_{\mathbb{Z}} Y \arrow[r] \arrow[d] & \mathbb{P}_{\mathbb{Z}}^{(n+1)(m+1)-1} \times_{\mathbb{Z}} Z \arrow[d] \\ \mathbb{P}_{\mathbb{Z}}^n \times_{\mathbb{P}_{\mathbb{Z}}^n} Y \arrow[r] & \mathbb{P}_{\mathbb{Z}}^{(n+1)(m+1)-1} \times_{\mathbb{P}_{\mathbb{Z}}^n} Z \end{tikzcd}$$

Because $\mathbb{P}_{\mathbb{Z}}^n \times_{\mathbb{Z}} \mathbb{P}_{\mathbb{Z}}^m = \mathbb{P}_{\mathbb{Z}}^{(n+1)(m+1)-1}$, $\mathbb{P}_{\mathbb{Z}}^{(n+1)(m+1)-1} \times_{\mathbb{P}_{\mathbb{Z}}^n} Z = \mathbb{P}_{\mathbb{Z}}^n \times_{\mathbb{Z}} \mathbb{P}_{\mathbb{Z}}^m \times_{\mathbb{P}_{\mathbb{Z}}^n} Z = \mathbb{P}_{\mathbb{Z}}^m \times_{\mathbb{Z}} Z$. We have the diagram:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}_{\mathbb{Z}}^n \times_{\mathbb{Z}} Y \arrow[r] \arrow[d, "p_Y"] & \mathbb{P}_{\mathbb{Z}}^{(n+1)(m+1)-1} \times_{\mathbb{Z}} Z \arrow[d] \\ & Y \arrow[r, hook] \arrow[dr, "g"'] & \mathbb{P}_{\mathbb{Z}}^m \times_{\mathbb{Z}} Z \arrow[d, "p_Z"] \\ & & Z \end{tikzcd}$$

For being a closed immersion is preserved under base change, so $\mathbb{P}_Y^n \longrightarrow \mathbb{P}_Z^{(m+1)(n+1)-1}$ is a closed immersion. Thus, $X \longrightarrow \mathbb{P}_{\mathbb{Z}}^{(\gtrdot+\nVdash)(\ltimes+\nVdash)-\nVdash}$ is a closed immersion. Thus, $g \circ f$ is projective.

(c). For any $f : Y^{'} \longrightarrow Y$, consider the magic square:

$$\begin{tikzcd} X \times_Y Y^{'} \arrow[r] \arrow[d] & \mathbb{P}_Y^n \times_Y Y^{'} \arrow[d] \\ X \arrow[r] & \mathbb{P}_Y^n \end{tikzcd}$$

For $X \longrightarrow \mathbb{P}_Y^n$ is a closed immersion, which is preserved under base change and $\mathbb{P}_Y^n \times_Y Y^{'} = \mathbb{P}_{Y'}^n$, $X \times_Y Y^{'} \longrightarrow \mathbb{P}_{Y'}^n$ is a closed immersion. Thus, being projective is preserved under base change. $\square$

Exercise 2.4.10. Chow's Lemma. This result says that proper morphisms are fairly close to projective morphisms. Let X be proper over a noetherian scheme S. Then there is a scheme X' and a morphism $g : X' \to X$ such that X' is projective over S, and there is an open dense subset $U \subseteq X$ such that g induces an isomorphism of $g^{-1}(U)$ to U. Prove this result in the following steps.

(a) Reduce to the case X irreducible.

(b) Show that X can be covered by a finite number of open subsets $U_i, i = 1, \ldots, n$, each of which is quasi-projective over S. Let $U_i \to P_i$ be an open immersion of U_i into a scheme P_i which is projective over S.

(c) Let $U = \bigcap U_i$, and consider the map

$$f : U \to X \times_S P_1 \times_S \cdots \times_S P_n$$

deduced from the given maps $U \to X$ and $U \to P_i$. Let X' be the closed image subscheme structure (Ex. 3.11d) $f(U)$. Let $g : X' \to X$ be the projection onto the first factor, and let $h : X' \to P = P_1 \times_S \cdots \times_S P_n$ be the projection onto the product of the remaining factors. Show that h is a closed immersion, hence X' is projective over S.

(d) Show that $g^{-1}(U) \to U$ is an isomorphism, thus completing the proof.

Exercise 2.4.11. If you are willing to do some harder commutative algebra, and stick to noetherian schemes, then we can express the valuative criteria of separatedness and properness using only discrete valuation rings.

(a) If $\mathcal{O}, m$ is a noetherian local domain with quotient field K, and if L is a finitely generated field extension of K, then there exists a discrete valuation ring R of L dominating $\mathcal{O}$. Prove this in the following steps. By taking a polynomial ring over $\mathcal{O}$, reduce to the case where L is a *finite* extension field of K. Then show that for a suitable choice of generators $x_1, \ldots, x_n$ of m, the ideal $a = (x_1)$ in $\mathcal{O}' = \ell\,[x_2/x_1, \ldots, x_n/x_1]$ is not equal to the unit ideal. Then let p be a minimal prime ideal of a, and let $\mathcal{O}'_p$ be the localization of $\mathcal{O}'$ at p. This is a noetherian local domain of dimension 1 dominating $\mathcal{O}$. Let $\bar{\mathcal{O}}'_p$ be the integral closure of $\mathcal{O}'_p$ in L. Use the theorem of Krull-Akizuki (see Nagata [7, p. 115]) to show that $\bar{\mathcal{O}}'_p$ is noetherian of dimension 1. Finally, take R to be a localization of $\bar{\mathcal{O}}'_p$ at one of its maximal ideals.

(b) Let $f \colon X \to Y$ be a morphism of finite type of noetherian schemes. Show that f is separated (respectively, proper) if and only if the criterion of (4.3) (respectively, (4.7)) holds for all *discrete* valuation rings.

Proof.

(a). **Wait**

(b). We replace the symbol $\mathcal{O}_{\mathfrak{m}}$ by S.
Consider

$$\begin{array}{ccccc}
Spec(L) & \longrightarrow & Spec(K) & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
Spec(R) & \longrightarrow & Spec(S) & \longrightarrow & Spec(k)
\end{array}$$

with L, R given by (a).

We show that giving a morphism $Spec(S) \longrightarrow X$ is equivalent to giving a morphism $Spec(R) \longrightarrow X$.

Let $\varphi : Spec(S) \longrightarrow X$. Then suppose that $\mathfrak{m}_S \mapsto x \in X$. Then $\mathcal{O}_{X,x} \subset S$ is dominated by S. For S is also dominated by R, $\mathbb{O}_{X,x}$ is dominated by $\mathcal{O}_{X,x}$. Thus, we have $Spec(R) \longrightarrow X$.

Let $\psi : Spec(R) \longrightarrow X$ with $\mathfrak{m}_R \mapsto x$. Then $\mathcal{O}_{X,x}$ is dominated by R. For $K(\mathcal{O}_{X,x}) = K$, $\mathcal{O}_{X,x} \subset S$. For $i^{Spec(R)}_{\mathcal{O}_{X,x}} = i^{Spec(R)}_{Spec(S)} \circ i^{Spec(S)}_{\mathcal{O}_{X,x}}$ and S is dominated by R, $\mathcal{O}_{X,x}$ is dominated by S. So there exists a morphism $Spec(R) \longrightarrow X$. $\square$

Exercise 2.4.12. Examples of Valuation Rings. Let k be an algebraically closed field.

(a) If K is a function field of dimension 1 over k (I, §6), then every valuation ring of K/k (except for K itself) is discrete. Thus the set of all of them is just the abstract nonsingular curve C_K of (I, §6).

(b) If K/k is a function field of dimension two, there are several different kinds of valuations. Suppose that X is a complete nonsingular surface with function field K.

(1) If Y is an irreducible curve on X, with generic point x_1, then the local ring $R = \mathcal{O}_{x_1,X}$ is a discrete valuation ring of K/k with center at the (nonclosed) point x_1 on X.

(2) If $f\colon X' \to X$ is a birational morphism, and if Y' is an irreducible curve in X' whose image in X is a single closed point x_0, then the local ring R of the generic point of Y' on X' is a discrete valuation ring of K/k with center at the closed point x_0 on X.

(3) Let $x_0 \in X$ be a closed point. Let $f\colon X_1 \to X$ be the blowing-up of x_0 (I, §4) and let $E_1 = f^{-1}(x_0)$ be the exceptional curve. Choose a closed point $x_1 \in E_1$, let $f_2\colon X_2 \to X_1$ be the blowing-up of x_1, and let $E_2 = f_2^{-1}(x_1)$ be the exceptional curve. Repeat. In this manner we obtain a sequence of varieties X_i with closed points x_i chosen on them, and for each i, the local ring $\mathcal{O}_{x_{i+1},X_{i+1}}$ dominates $\mathcal{O}_{x_i,X_i}$. Let $R_0 = \bigcup_{i=0}^{\infty} \mathcal{O}_{x_i,X_i}$. Then R_0 is a local ring, so it is dominated by some valuation ring R of K/k by (I, 6.1A). Show that R is a valuation ring of K/k, and that it has center x_0 on X. When is R a discrete valuation ring?

Note. We will see later (V, Ex. 5.6) that in fact the R_0 of (3) is already a valuation ring itself, so $R_0 = R$. Furthermore, every valuation ring of K/k (except for K itself) is one of the three kinds just described.

Proof.
(a). For every valuation ring R, $R \subset K$. Thus, $dim(R) \leq 1$. Thus, the only non-trivial prime ideal of R is $\mathfrak{m}_R$.

(b1). Suppose $x_1 = \mathfrak{p} \in Spec(A)$. Then $K(A_{\mathfrak{p}}) = K$. Because Y is irreducible, then $Y \cap Spec(A) = Spec(A/I)$ has the minimal polynomial $\mathfrak{p}$. Thus, $codim(Y, X) = \inf\{dim\mathcal{O}_{X,P} | P \in Y\} = dim\mathcal{O}_{X,\mathfrak{p}} = dimA_{\mathfrak{p}}$. For X is integral, so is $A_{\mathfrak{p}}$, which implies $(0) \in Spec(A_{\mathfrak{p}})$. For $A_{\mathfrak{p}}$ is a local ring and $dim(A_{\mathfrak{p}}) = 1$, $Spec(A_{\mathfrak{p}}) = \{\mathfrak{p}A_{\mathfrak{p}}, (0)\}$.

□

2.4.3 Addition Exercises

Exercise 2.4.13. Decide which ones of the following morphisms are universally closed:

(i) $\mathbb{A}^1_k \longrightarrow Spec(k)$;

(ii) $\mathbb{P}^1_k \longrightarrow Spec(k)$;

(iii) $V(x) \cong \mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k$;

(iv) $Spec(\overline{\mathbb{Q}}) \longrightarrow Spec(\mathbb{Q})$.

Proof.
(i)$\mathbb{A}^1_k \longrightarrow Spec(k)$ is not universally closed: Consider $\pi : \mathbb{A}^2_k \longrightarrow \mathbb{A}^1_k$ given by the base change over $\mathbb{A}^1_k$. For $V(xy-1)$ is mapped to $\mathbb{A}^1_k - \{0\}$, so π is not closed.

(ii). $\mathbb{P}^1_k \longrightarrow Spec(k)$ is universally closed. For $\mathbb{P}^1_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ is universally closed and 'being universally closed' is preserved under base change. So $\mathbb{P}^1_k \longrightarrow Spec(k)$ is universally closed.

(iii). This map is a closed immersion. Hence it is universally closed.

□

Exercise 2.4.14. Affine line with two origins is not separated.

Proof. Let X be the affine line with two origins. Suppose that X is covered by $U_1 := \mathbb{A}^1_k$ and $U_2 := \mathbb{A}^1_k$ such that they are glued by $id : U_1 - \{0\} \longrightarrow U_2 - \{0\}$. Then $X \times_k X$ is covered by $\{U_i \times_k U_j\}$ with $i, j = 1, 2$.

Consider $\Delta_X : X \longrightarrow X \times_k X$.

$$\Delta_X|_{U_i} : U_i \longrightarrow U_i \times_k U_i$$

We see that $\Delta_X(U_i) \cap U_1 \times_k U_2 \cong \mathbb{A}^1_k - \{0\}$ which is open in $U_1 \times_k U_2$. Hence $\Delta_X(X) \cap U_1 \times_k U_2 = \mathbb{A}^1_k - \{0\}$ is not closed. Thus, Δ_X is not a closed immersion. □

Remark. To compute $\Delta_X(U)$

$$\begin{tikzcd}
U \arrow[dr, dashed, "\Delta_X"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, "id_U"'] & & \\
& X \times_k X \arrow[r] \arrow[d] & X \arrow[d] \\
& U \arrow[r] & Spec(k)
\end{tikzcd}$$

For $i : U \longrightarrow X$, it is given by $i_1 : k[x] \longrightarrow k[x]$ and $i_2 : k[x] \longrightarrow k[x, x^{-1}]$. Then $im(i_1 \otimes_k id_U) = k[x, x]$, which is given by $f(x)g(y) \in k[x] \otimes_k k[y] \mapsto f(x)g(x)$ and $im(i_2 \otimes_k id_U) = k[x, x, x^{-1}] = k[x, x^{-1}]$, which is given by $f(x)g(y) \in k[x] \otimes_k k[y] \mapsto f(x)g(x) \in k[x, x, x^{-1}]$. Hence, we see that $\Delta_X(U)|_{U_1 \times_k U_2} \cong \mathbb{A}^1_k - \{0\}$.

Exercise 2.4.15. Let X and Y be two affine schemes over a scheme S.

(i). Assume that S is separated over $Spec(\mathbb{Z})$. Show that $X \times_S Y$ is affine.

(ii). Given an example where S is not separated and $X \times_S Y$ is not affine.

Proof.

(i). Note that we have

$$\begin{tikzcd}
X \arrow[dr] & & \\
& S \arrow[r] & \mathbb{Z} \\
Y \arrow[ur] & &
\end{tikzcd}$$

By Magic square, we have the pullback

$$\begin{tikzcd}
X \times_S Y \arrow[r] \arrow[d] & X \times_{\mathbb{Z}} Y \arrow[d] \\
S \times_S S \arrow[r] & S \times_{\mathbb{Z}} S
\end{tikzcd}$$

For $S \longrightarrow S \times_Z S$ is a closed immersion, so is $X \times_S Y \longrightarrow X \times_{\mathbb{Z}} Y$ a closed immersion. Because X and Y are affine, $X \times_{\mathbb{Z}} Y$ is also affine. Thus, $X \times_S Y$ is affine.

(ii). Let S be the affine plane with two origins, which is not separated. Consider $X = \mathbb{A}^2_k \hookrightarrow S$ and $Y = \mathbb{A}^2_k \hookrightarrow S$ where O_X and O_Y are mapped to the two different origins.

Then $X \times_S Y = (X \cap Y, \mathcal{O}_{X\cap Y}) = (\mathbb{A}^2_k - \{0\}, \mathcal{O}_{\mathbb{A}^2_k - \{0\}})$, which is not affine. $\square$

Exercise 2.4.16. Given an example of

(i). A local ring A such that $Spec(A)$ is neither reduced nor irreducible.

(ii). A morphism of integral schemes $f : X \longrightarrow Y$ which is not injective and such that there exists an open cover $X = \cup_{i\in I} U_i$ such that every $f|_{U_i}$ is an isomorphism onto its image.

Proof.

(i). Let $A = k[x, y]/(x^2y)$. Then A is not reduced and has two minimal prime ideals (x) and (y). Thus, $Spec(A)$ is not reduced and irreducible.

(ii). Let X be the affine line with two origins and $\varphi : X \longrightarrow \mathbb{A}^1_k$ be two morphism glued two origins together.

Note that X is covered by two affine lines U_i and $f|_{U_i} : \mathbb{A}^1_k \xrightarrow{\cong} \mathbb{A}^1_k$. $\square$

Exercise 2.4.17. Consider the morphism $\varphi : \mathbb{A}^1_k \longrightarrow \mathbb{A}^2_k$ given by $t \mapsto (t^2, t^3)$. Decide whether φ is quasi-finite, finite proper, affine, projective, a closed immersion.

Proof. This morphism is defined by

$$f : k[x, y] \longrightarrow k[t]; \quad x \mapsto t^2, y \mapsto t^3$$

So φ is finite, for $k[t]$ forms a finitely generated $k[x, y]$ module under f.

For a finite morphism is proper, so φ is proper.

For a finite morphism is quasi-finite, φ is qausi-finite.

Note that $\varphi^{-1}(\mathbb{A}^2_k) = \mathbb{A}^1_k$, φ is affine.

If φ is a closed immersion, then f is a surjection. So φ is not a closed immersion.

For φ is finite, it is obviously projective. $\square$

Remark. Using the same method, we can show that both Veronese embedding and Serge embedding are finite.

Exercise 2.4.18. Let $f, g : X \longrightarrow S$ be a morphism of schemes, where X is integral and S is separated and S suppose that f and g agree when restricted to an open non-empty subset $U \subset X$. Show that then $f = g$.

Proof. Let $Z = \{x \in X | f(x) = g(x)\}$. Consider the pullback

$$\begin{array}{ccc} Z & \longrightarrow & S \\ \Big\downarrow i & & \Big\downarrow \Delta_S \\ X & \longrightarrow & S \times_{\mathbb{Z}} S \end{array}$$

For Δ_S is closed, so is i. Hence Z is a closed subset of X. For X is irreducible, U is dense over X. Because $U \subset Z \subset X$ and $\overline{U} = X$, $Z = X$. Hence $f = g$. $\square$

Remark. When S is not separated, this assertion will be false. Just take S to be the affine line with two origins and consider f and g such that f,g agree on $\mathbb{A}^1_k - \{0\}$ and map origins to two different origins of S.

2.5 Sheaves of Modules

2.5.1 Preparations

Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. Then for any $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{O}_Y$-module $\mathcal{G}$, we have

$$Hom_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong Hom_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

Proof. First of all given any $g \in Hom_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$, $f_*\mathcal{G}$ gives a morphism form $f_*f^*\mathcal{G}$ to $f_*\mathcal{F}$. Thus, it is enough to give an canonical morphism form $\mathcal{G}$ to $f_*f^*\mathcal{G}$.

Given any U, $f_*f^*\mathcal{G}(U) = f^*\mathcal{G}(f^{-1}(U))$. As a presheaf, $f^*\mathcal{G}(f^{-1}(U)) = \mathcal{G}(f(f^{-1}(U)))\otimes_{\mathcal{O}_Y(f(f^{-1}(U)))} \mathcal{O}_X(U) = \mathcal{G}(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{O}_X(f^{-1}(U))$. Hence $f_*f^*\mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X$ as sheaves. By the morphism between presheaves

$$\mathcal{G}(U) \hookrightarrow \mathcal{G}(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{O}_X(f^{-1}(U))$$

, using the universal property of sheafification, we define natural morphism $\mathcal{G} \longrightarrow \mathcal{G} \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X$.

Because $\mathcal{G} \otimes^{pr} \mathcal{F}$ is given by the sheafification of presheaves $\mathcal{G} \otimes \mathcal{F} := \mathcal{G}(U) \otimes \mathcal{F}(U)$, we define $f^*f_*\mathcal{F}$ to $\mathcal{F}$ by the morphism of presheaves

$$f^{-1}f_*\mathcal{F} \otimes^{pr}_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow \mathcal{F}$$

Finally, we verify that as presheaveswe have the adjoint relation above. Use the universal property of sheafification, we have the adjoint relation above. □

Remark. Giving the whole proof will be very complex, but not difficult.

Remark. We claim that

$$f^*Hom_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \cong Hom_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$$

.

We just need to show that $f^*Hom_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \cong Hom_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$.

$(\Longleftarrow)$: Trivial.

$(\Longrightarrow)$:Given any $g : f^*\mathcal{G} \longrightarrow f^*f_*\mathcal{G}$, composing with the canonical morphism, we have $g' \in Hom_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong Hom_{\mathcal{O}_Y}(\mathcal{G}, f^*\mathcal{F})$. For f^*g with $g \in Hom_{\mathcal{O}_Y}(\mathcal{G}, f^*\mathcal{F})$, this map is just the identity.

Hence, $f^*Hom_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \cong Hom_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$.

Theorem 2.5.1. *Let A be a ring and $X = Spec(A)$. Also let $A \longrightarrow B$ be a ring homomorphism and let $f : Spec(B) \longrightarrow Spec(A)$ be the corresponding morphism, then:*

(a) The map $M \longrightarrow \widetilde{M}$ gives an exact, fully faithful functor from the category of A-module to the category of $\mathcal{O}_X$-module.

(b) $(M \otimes_A N)^\sim \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$

(c) $(\oplus M_i)^\sim = \oplus \widetilde{M}_i$

(d) $f_*(\widetilde{N}) = ({}_AN)^\sim$

(f) $f^*(\widetilde{M}) = (M \otimes_A B)^\sim$

Proof.

(a). Exactness comes from the exactness of localization of A-modules.

(b). By $(M \otimes_A N)_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$.

(c). By $(\oplus M_i)_{\mathfrak{p}} = \oplus M_{i,\mathfrak{p}}$.

(d). If we denote $A \longrightarrow B$ by φ, then $f^{-1}(D(a)) = D(\varphi(a))$ as we have shown before. Then:

$$f_*(\widetilde{N})(D(a)) = \widetilde{N}(D(\varphi)(a)) = N_{\varphi(a)} = ({}_AN)^{\sim}(D(a))$$

(f). For any $\mathfrak{p} \subset B$,

$$f^*(\widetilde{M})_{\mathfrak{p}} = M_{f(\mathfrak{p})} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} = (M \otimes_A B)^{\sim}_{\mathfrak{p}}$$

□

Theorem 2.5.2. *For $X = Spec(A)$,*

$$QCoh(A) \xrightarrow{1:1} Mod(A)$$

and if A is Noetherian, we have

$$Coh(A) \xrightarrow{1:1} FinMod(A)$$

Theorem 2.5.3. *For any scheme X, $QCoh(X)$, the category of quasi-coherent sheaves is an abelian category, that is, the kernel, cokernel, image of any morphism between quasi-coherent sheaves are also quasi-coherent. If X is Noetherian, so is $Coh(X)$, the category of coherent shaves.*

Proof. These propositions are local. So we just assume that $X = Spec(A)$ affine. By, 2.5.2, it comes directly form the fact that A-modules are abelian categories. Moreover, when A is Noetherian, Fin-A-modules are abelian categories. □

Theorem 2.5.4. *Let $f : X \longrightarrow Y$ be a morphism of schemes.*

(a) If $\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$ sheaf of $\mathcal{O}_Y$, then $f^*\mathcal{G}$ is also a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

(b) If X and Y are both Noetherian, and if $\mathcal{G}$ is coherent, $f^*(\mathcal{G})$ is still coherent.

(c) Assume that either X is Noetherian, or f is quasi-compact and separated. Then if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, $f_*\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules.

Proof.

(a). This proposition is local. So just assume that $X = Spec(B)$ and $Y = Spec(A)$ are affine. If $\mathcal{G}$ is qausi-coherent, $\mathcal{G} = \widetilde{M}$ for some A-module M 2.5.2. By 2.5.1, $f^*\mathcal{F} = (M \otimes_A B)$, with $M \otimes_A B$ a B-module.

(b). Similar to (b), we assume A and B are Noetherian. Then at this case if M is a finitely generated A-module, $M \otimes_A B$ is a finitely generated B-module.

(c).

□

Remark. For (a), (b), note that $f^* \dashv f_*$, so f^* is a right-exact functor. By 2.5.7, locally if $\mathcal{G}$ is locally quasi-coherent, we have the exact sequence

$$\oplus \mathcal{O}_Y^m|_U \longrightarrow \oplus \mathcal{O}_Y^n|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

Then on $f^{-1}(U)$, we have

$$\oplus f^*\mathcal{O}_Y^m|_{f^{-1}(U)} \longrightarrow \oplus f^*\mathcal{O}_Y^n|_{f^{-1}(U)} \longrightarrow f^*\mathcal{F}|_{f^{-1}(U)} \longrightarrow 0$$

Note that $f^*\mathcal{O}_Y = \mathcal{O}_X$. So $\mathcal{F}$ is quasi-coherent.

Theorem 2.5.5. *Let X be an affine scheme and $0 \longrightarrow \mathcal{F}^{'} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{''} \longrightarrow 0$ be an exact sequence of $\mathcal{O}_X$-modules, and assume that $\mathcal{F}^{'}$ is quasi-coherent. Then*

$$0 \longrightarrow \Gamma(X, \mathcal{F}^{'}) \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}^{''}) \longrightarrow 0$$

is exact.

Theorem 2.5.6. *Let X be a scheme. For any closed subscheme Y, the corresponding ideal sheaf $\mathcal{I}_Y$ is a quasi-coherent sheaf of ideals on X. Conversely, any quasi-coherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closed subscheme of X.*

Proof. By 2.1.21 (b), we have the exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow i_Y^*(\mathcal{O}_Y) \longrightarrow 0$$

For i_Y is a closed immersion, i_Y is separated. Thus, $i_*\mathcal{O}_Y$ is quasi-coherent over X. Thus, by 2.5.3, $\mathcal{I}_Y$ is quasi-coherent.

Conversely, given any $\mathcal{I}$ that is a quasi-coherent of ideals of $\mathcal{O}_X$. Then just let $Y = supp \mathcal{O}_X/\mathcal{I}$. Then by 2.1.14, Y is a subspace of X and $(Y, \mathcal{O}_X/\mathcal{I})$ is a ring space with $\mathcal{I}_Y = \mathcal{I}$. The uniqueness is trivial. To show that Y is a closed immersion, just see that locally on $U = Spec(A)$, $\mathcal{I}|_U = \widetilde{\mathfrak{a}}$ for some ideal $\mathfrak{a}$ of A since $\mathcal{I}$ is quasi-coherent. Again, for $\mathcal{I}$ is quasi-coherent, so is $\mathcal{O}_X/\mathcal{I}$. Using the fact that $\Gamma(U, -)$ is exact for quasi-coherent sheaf, we see that $\Gamma(U, \mathcal{O}_X/\mathcal{I}) = A/\mathfrak{a}$. Hence

$$(Y \cap U, \mathcal{O}_X/\mathcal{I}_Y) = (Y \cap U, (A/\mathfrak{a})^{\sim}) \cong Spec(A/\mathfrak{a})$$

, which is a closed subscheme of U. Thus, $(Y, \mathcal{O}_X/\mathcal{I})$ is a closed subseheme of X. □

Sheaves on Projective scheme

Theorem 2.5.7. *Let S be a graded ring and let $X = Proj(S)$. Assume that S is generated by S_1 as an S_0-algebra.*

(a) The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf on X.

(b) For any graded S-module M, $\widetilde{M}(n) \cong (M(n))^{\sim}$. In particular, $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(m+n)$.

Proposition 2.5.8. *Let A be a ring and $S = A[x_0, ..., x_n]$ with $n \geq 0$, and $X = Proj(S)$. Then*

$$\Gamma^*(\mathcal{O}_X) \cong S$$

2.5.2 Examples

Example 2.5.9 (Quasi-Coherent Sheaves)**.**

(i).Let X be an integral scheme. K_X is quasi-coherent. Consider the generic point of X, we have $\{\eta_X\} = Spec(K_X)$ which is Noetherian. Hence consider $i : \eta_X \hookrightarrow X$. For $\{\eta_X\}$ is Noetherian, $i_*\mathcal{O}_{\eta_X}$ is quasi-coherent over X, which is just $\mathcal{K}_X$.

(ii). For $X = Spec(A)$ is an integral scheme, let P be a closed point of X. Then $i_{P,*}(k(P))$ is a quasi-coherent sheaf: $i_{P,*}(k(P))_Q = 0$ if $Q \neq P$ and $i_{P,*}(k(P))_Q = 0$ if $Q = P$. Consider $k(P)^{\sim}$ with $k(P) = A_P/\mathfrak{m}_P$. Suppose that $P = \mathfrak{p}$ and $Q = \mathfrak{q}$. Then

$$k(P)^{\sim}_{\mathfrak{q}} = k(P)_{\mathfrak{q}} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A A_{\mathfrak{q}} = A/\mathfrak{p} \otimes_A A_{\mathfrak{q}} = 0$$

if $\mathfrak{p} \neq \mathfrak{q}$ and $= k(P)$ when $\mathfrak{p} = \mathfrak{q}$. Then, we see that $k(P)^{\sim} = i_{P,*}(k(P))$.

Another method to prove (ii), because $\{P\} = Spec(k(P))$ is Noetherian, i_* maps quasi-coherent sheaves to quasi-coherent sheaves. Hence $i_*k(P)$ is quasi-coherent over X.

(iii). Let $\mathcal{I}_{P,Q}$ be the sheaves of regular functions vanishing at P, Q. Then $\mathcal{I}_{P,Q}$ is quasi-coherent. Because we have

$$0 \longrightarrow \mathcal{I}_{P,Q} \longrightarrow \mathcal{O}_X \longrightarrow i_*(k(P)) \oplus i_*(k(Q)) \longrightarrow 0$$

and for $i_*(k(P))$ and $i_*(k(Q))$ are quasi-coherent, so are $i_*(k(P)) \oplus i_*(k(Q))$ and $\mathcal{I}_{P,Q} = ker(\mathcal{O}_X \longrightarrow i_*(k(P)) \oplus i_*(k(Q)))$.

Remark. $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A A_{\mathfrak{q}} = 0$ when $\mathfrak{p} \not\subset \mathfrak{q}$:

Consider $1 \in A_{\mathfrak{q}}$, we can find an element a in $\mathfrak{p} - \mathfrak{q}$, then $\frac{a}{1}$ is a unit in $A_{\mathfrak{q}}$ but $0 \in A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. For any $\frac{x}{y} \otimes \frac{c}{d} \in A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A A_{\mathfrak{q}}$, then $\frac{x}{y} \otimes \frac{c}{d} = \frac{ax}{y} \otimes \frac{c}{ad} = 0$. Thus, $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \otimes_A A_{\mathfrak{q}} = 0$.

Moreover, we can use this to generalize the result above.

Example 2.5.10 (Non-quasi-Coherent Sheaves)**.**

(i). For any integral scheme X, $\mathcal{O}_X^*$ is not quasi-coherent for $\mathcal{O}_X^*$ doesn't carry any $\mathcal{O}_X$-module. ($\mathcal{O}_{X,P}^*$ is not an abelian group at each $P \in X$.)

(ii). For any integral scheme, if P is not the generic point, then $i_{P,*}(\mathcal{O}_{X,P})$ is not quasi-coherent. Suppose $P = \mathfrak{p} \subset A$ and $i_{P,*}(\mathcal{O}_{X,P})$ is quasi-coherent. Then on $U = Spec(A)$, $i_{P,*}(\mathcal{O}_{X,P})|_U = \widetilde{M}$ with M an A-module. $i_{P,*}(\mathcal{O}_{X,P})(U) = \mathcal{O}_{X,P} = A_{\mathfrak{p}}$, because the presheaf $i_{P,*}\mathcal{O}_{X,P}$ is also a sheaf. Then $A_{\mathfrak{p}} = M$. However at the generic point of U, $\widetilde{M}_{(0)} = M_{(0)} = A_{(0)}$ but $i_{P,*}\mathcal{O}_{X,P}{}_{(0)} = A_{\mathfrak{p}}$, which leads to a contradiction.

Example 2.5.11 (Coherent Sheaves)**.**

(i). For any $X = V_+(F) \subset \mathbb{P}_k^n$ with $deg(F) = d$, then $i_{X,*}\mathcal{O}_X$ is coherent. Note that the exact sequence

$$0 \longrightarrow S(-d) \xrightarrow{\cdot F} S \longrightarrow S/(F) \longrightarrow 0$$

gives the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(-d) \longrightarrow \mathbb{O}_{\mathbb{P}_k^n} \longrightarrow i_{X,*}\mathcal{O}_X \longrightarrow 0$$

Then $i_{X,*}\mathcal{O}_X = coker(\mathcal{O}_{\mathbb{P}_k^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}_k^n})$ is coherent for $\mathbb{P}_k^n$ is Noetherian and both $\mathcal{O}_{\mathbb{P}_k^n}(-d)$ and $\mathcal{O}_{\mathbb{P}_k^n}$ are coherent.

Remark. Using this, we see that $i_{x,*}(k(x))$ is coherent in $\mathbb{P}_k^n$ for any $x \in \mathbb{P}_k^n$ comparing with the fact that $i_{x,*}(\mathcal{O}_{X,x})$ is not coherent over $\mathbb{P}_k^n$.

Example 2.5.12 (Non-Coherent Sheaves).

(i). Consider $i : D_+(x_0) \hookrightarrow \mathbb{P}^1_k$. $i_*\mathcal{O}_{D_+(x_0)}$ is not coherent. Consider $i_*\mathcal{O}_{D_+(x_0)}|_{D_+(x_1)}$. If $i_*\mathcal{O}_{D_+(x_0)}$ is coherent, so is $i_*\mathcal{O}_{D_+(x_0)}|_{D_+(x_1)}$ over $D_+(x_1)$, which implies $\Gamma(D_+(x_1), i_*\mathcal{O}_{D_+(x_0)}|_{D_+(x_1)})$ is a finitely generated $k[\frac{x_0}{x_1}]$-module. However, $\mathcal{O}_{D_+(x_0)}(D(x_1)) = k[\frac{x_1}{x_0}]_{\frac{x_1}{x_0}} = k[\frac{x_0}{x_1}, \frac{x_1}{x_0}]$ which is not a finitely generated $k[\frac{x_0}{x_1}]$-module.

2.5.3 Exercises

Exercise 2.5.1. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the dual of $\mathcal{E}$, denoted $\tilde{\mathcal{E}}$, to be the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

(a) Show that $(\tilde{\mathcal{E}})^{\sim} \cong \mathcal{E}$.

(b) For any $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \tilde{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{F}$.

(c) For any $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}))$.

(d) (Projection Formula). If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, and if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}$.

Proof. W.L.O.G., we just assume that $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})(U) = \mathscr{H}om_{\mathcal{O}_{X}(U)}(\mathscr{E}(U), \mathscr{F}(U))$

(a). Suppose that $\mathscr{E}|_U \cong \mathcal{O}_X^n|_U$, then for

$$\mathscr{H}om(\mathscr{H}om(\mathcal{O}_X^n, \mathcal{O}_X), \mathcal{O}_X) = \mathcal{O}_X^n$$

Thus, $\check{\check{\mathscr{E}}} = \mathscr{E}$.

(b). Use the fact that $Hom_A(M, N) = \check{M} \otimes_A N$ when N is a free A-mod.
(c). Use the fact that $Hom_A(M \otimes_A N, L) \cong Hom_A(N, Hom_A(M, L))$ when M is a free A-mod.
(d). For any $g \in Hom(\mathcal{F}, f^*\mathcal{G})$, we have $f_*(g) : f_*\mathcal{F} \longrightarrow f_*f^*\mathcal{G} = \mathcal{G}$. Thus,

$$f_*(Hom(\mathcal{F}, f^*\mathcal{G})) = Hom(f^*\mathcal{F}, \mathcal{G})$$

Thus, we have

$$f_*\mathscr{H}om_{\mathcal{O}_X}(\check{\mathscr{F}}, f^*\mathscr{E}) = \mathscr{H}om_{\mathcal{O}_Y}(f_*\check{\mathscr{F}}, \mathscr{E}) = f_*\mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{E}$$

□

Remark. No details. All are form Stack Project. ****

Remark. $\mathscr{H}om(\mathscr{F}, \mathscr{G})_p = Hom(\mathscr{F}_p, \mathscr{G}_p)$ is not a trivial result. But when X is a Noetherian scheme and $\mathscr{F}$ is coherent and $\mathscr{G}$ is an $\mathcal{O}_X$-module, we have $\mathscr{H}om(\mathscr{F}, \mathscr{G})_p = Hom(\mathscr{F}_p, \mathscr{G}_p)$. Then just verfiy all above on stalks, which will be very easy. See

Remark. At beginning, I just want to prove (d) by

$$\begin{aligned} f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{E})(V) &= \mathscr{F}(f^{-1}(V)) \otimes_{\mathcal{O}_X(f^{-1}(V))} f^*\mathscr{E}(f^{-1}(V)) \\ &= \mathscr{F}(f^{-1}(V)) \otimes_{\mathcal{O}_X(f^{-1}(V))} \mathscr{E}(V) \otimes_{\mathscr{O}_Y(V)} \mathscr{O}_X(f^{-1}(V)) \end{aligned}$$

But note that, this is just the presheaf structure assosicated $\otimes_{\mathcal{O}_X}$ but not sheaf structure!

Consider

$$k = \mathcal{O}_{\mathbb{P}^n_k}(\mathbb{P}^n_k) = \mathcal{O}(-1) \otimes_{\mathcal{O}_{\mathbb{P}_k}} \mathcal{O}(1)(\mathbb{P}^1_k) \neq \mathcal{O}(-1)(\mathbb{P}^n_k) \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{O}(1)(\mathbb{P}^n_k) = 0$$

for $\mathcal{O}(-1)(\mathbb{P}^n_k) = 0$.

Exercise 2.5.2. Let R be a discrete valuation ring with quotient field K, and let $X = \text{Spec } R$.

(a) To give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R K \to L$.

(b) That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.

Proof.

(a). Note that the the open set of X are $\{X, \emptyset, D(x)\}$ with $\mathfrak{m}_R = (x)$. Note that $\mathcal{O}_X(X) = R, \mathcal{O}_X(D(x)) = K(R) = K$.

Then, given any $\mathcal{F} \in Mod(\mathcal{O}_X)$, we have $\mathcal{F}(X) = M$ is a R-module and $\mathcal{F}(D(x)) = L$ is a K-module with $\rho_{X,D(x)} : M \longrightarrow L$ compatible with $R \hookrightarrow K$, that is, we have a K-module morphism $\rho : M \otimes_R K \longrightarrow L$.

For any such M and L, we define $\mathcal{F} : X \mapsto M$ and $D(x) \mapsto K$. And $\rho_{X,D(x)}$ is defined to be ρ then $\rho_{X,D(x)}$ is compatible with $R \hookrightarrow K$. Thus, $\mathcal{M}$ is a $\mathcal{O}_X$-module.

(b). For any cover of X, it must contain $Spec(R)$. Thus, if $\mathcal{F}$ is a quasi-coherent sheaf, then $\mathcal{F} \cong \widetilde{M}$, with M a R-module. $\mathcal{F}(D(x)) \cong \widetilde{M}(D(x)) = M_x$ with $x \in \mathfrak{m}_R$ and $\mathcal{F}(X) = \widetilde{M}(X) = M$. For M is a R-module $M \otimes_R K \cong M \otimes_R R_x = M_x$. So ρ is an isomorphism.

Conversely, if ρ is an isomorphism: $M_x = M \otimes_R R_x = M \otimes_R K = \mathcal{F}(D(x))$, thus $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$, with $\mathcal{F}(X)$ a R-module. □

Exercise 2.5.3. Let $X = \text{Spec } A$ be an affine scheme. Show that the functors * and Γ are adjoint, in the following sense: for any A-module M, and for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, there is a natural isomorphism

$$\text{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \text{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}).$$

Proof. As before, we need to consider the following two mrophisms:

$$\widetilde{\Gamma(X, \mathcal{F})} \longrightarrow \mathcal{F}$$

$$M \longrightarrow \Gamma(X, \widetilde{M})$$

where by definition, the second morphism is $id_M : M = \Gamma(X, \widetilde{M})$.

We just focus on the first morphism: Just let $M = \Gamma(X, \mathcal{F})$. $\widetilde{M}(D(f)) = M_f = M \otimes_A A_f$. Recall that $\rho^{\mathcal{F}}_{D(f),X} : \mathcal{F}(X) \otimes_A A_f \longrightarrow \mathcal{F}(D(f))$. So $\widetilde{\Gamma(X, \mathcal{F})} \longrightarrow \mathcal{F}$ is just given by $\rho^{\mathcal{F}}_{D(f),X}$. We just define $\rho^{\mathcal{F}}(D(f)) = \rho^{\mathcal{F}}_{D(f),X}$ as a morphism between $\widetilde{\Gamma(X, \mathcal{F})}$ and $\mathcal{F}$.

Now we define

$$Hom_A(M, \Gamma(X, \mathcal{F})) \longrightarrow Hom_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F})$$

by $\varphi : f \mapsto \rho^{\mathcal{F}} \circ \widetilde{f}$ and define

$$Hom_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) \longrightarrow Hom_A(M, \Gamma(X, \mathcal{F}))$$

by $\psi : g \mapsto \Gamma(g)$.

Then

$$\varphi \circ \psi(g) = \varphi(\Gamma(g)) = g$$

and

$$\psi \circ \varphi(f) = f$$

Thus,

$$Hom_A(M, \Gamma(X, \mathcal{F})) \cong Hom_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F})$$

□

Remark. Now we have $\widetilde{-} \dashv \Gamma(X, -)$. So $\widetilde{-}$ is a left-adjoint functor, then right-exact and commutes with $\oplus$ for which is a kind of colimits.

Exercise 2.5.4. Show that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on a scheme X is quasi-coherent if and only if every point of X has a neighborhood U, such that $\mathcal{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves on U. If X is noetherian, then $\mathcal{F}$ is coherent if and only if it is locally a cokernel of a morphism of free sheaves of finite rank. (These properties were originally the definition of quasi-coherent and coherent sheaves.)

Proof. Given any A-module M, let $\{e_i\}_{i\in I}$ be a set of generators of M and $\{k_j\}_{j\in J}$ be the corresponding relation of these generators (some A-linear combinations of $\{e_i\}_{i\in I}$).

Then consider $\varphi : A^{\oplus J} \longrightarrow A^{\oplus I}$ given by $k_j \mapsto k_j$. Then $coker(\varphi) =< e_i >_{i\in I} /im\varphi =< e_i >_{i\in I} /\{k_j\}_{j\in J} = M$.

For a quasi-coherent sheaf $\mathcal{F}$, $\mathcal{F}|_U = \widetilde{M}$. As we have discussed above, we have a exact sequence:

$$\mathcal{O}(U)^{\oplus J} \longrightarrow \mathcal{O}(U)^{\oplus I} \longrightarrow M \longrightarrow 0$$

By right-exactness of $\widetilde{-}$ and it commutes with $\oplus$, we have a exact sequence of sheaves of module:

$$\widetilde{\mathcal{O}(U)}^{\oplus J} \longrightarrow \widetilde{\mathcal{O}(U)}^{\oplus I} \longrightarrow \widetilde{M} \longrightarrow 0$$

that is,

$$\mathcal{O}^{\oplus J}|_U \longrightarrow \mathcal{O}^I|_U \longrightarrow \widetilde{M} \cong \mathcal{F}_U \longrightarrow 0$$

Conversely is trivial.

For a finitely generated A-module, we can let a set of generators of it and the set of relations between generators to be finite.

Using the same way above, we have

$$\mathcal{O}^{\oplus J}|_U \longrightarrow \mathcal{O}^I|_U \longrightarrow \widetilde{M} \cong \mathcal{F}|_U \longrightarrow 0$$

with I, J finite. □

Exercise 2.5.5. Let $f : X \to Y$ be a morphism of schemes.

(a) Show by example that if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.

(b) Show that a closed immersion is a finite morphism (§3).

(c) If f is a finite morphism of noetherian schemes, and if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ is coherent on Y.

Proof.

(a). Consider $i : k \longrightarrow k[x]$ and the morphism deduced by i, $f : \mathbb{A}^1_k \longrightarrow Spec(k)$. $\mathcal{O}_{\mathbb{A}^1_k}$ is coherent over $\mathbb{A}^1_k$ and suppose that $f_*\mathcal{O}_{\mathbb{A}^1_k}$ is coherent over $Spec(k)$. Then the global section of $f_*(\mathcal{O}_{\mathbb{A}^1_k})$ is a finitely generated k-module. Note that $f_*(\mathcal{O}_{\mathbb{A}^1_k})(Spec(k)) = k[x]$, which is not a finitely generated k-module. Thus, $f_*(\mathcal{O}_{\mathbb{A}^1_k})$ is not coherent over $Spec(k)$.

(b). Just note that if $Y = \cup Spec(B_i)$, then $f^{-1}(Spec(B_i)) = Spec(B_i) \cap A \longrightarrow Spec(B_i)$ is a closed immersion. Thus, $Spec(B_i) \cap A = Spec(B_i/I)$ for some ideal of B_i, which is obviously finitely generated B_i-module. So f is finite.

(c). This question is local. We just assume $f : X = Spec(A) \longrightarrow Y = Spec(B)$ with A a finitely generated B-module and B Noetherian.

Let $\mathcal{F}$ be a coherent sheaf on X. Then $\Gamma(X, \mathcal{F}) = M$ is a finitely generated A-module thus a finitely generated B-module.

By Chapter 2 Proposition 5.2 (d) of [5], $f_*(\mathcal{F}) = \widetilde{_BM}$. Thus, $f_*(\mathcal{F})$ is coherent $\square$

Exercise 2.5.6. Support. Recall the notions of support of a section of a sheaf, support of a sheaf, and subsheaf with supports from (Ex. 1.14) and (Ex. 1.20).

(a) Let A be a ring, let M be an A-module, let $X = \operatorname{Spec} A$, and let $\mathcal{F} = \tilde{M}$. For any $m \in M = \Gamma(X, \mathcal{F})$, show that $\operatorname{Supp} m = V(\operatorname{Ann} m)$, where $\operatorname{Ann} m$ is the annihilator of $m = \{a \in A | am = 0\}$.

(b) Now suppose that A is noetherian, and M finitely generated. Show that $\operatorname{Supp} \mathcal{F} = V(\operatorname{Ann} M)$.

(c) The support of a coherent sheaf on a noetherian scheme is closed.

(d) For any ideal $a \subseteq A$, we define a submodule $\Gamma_a(M)$ of M by $\Gamma_a(M) = \{m \in M | a^m m = 0 \text{ for some } n > 0\}$. Assume that A is noetherian, and M any A-module. Show that $\Gamma_a(M)^* \cong \mathcal{H}^0_Z(\mathcal{F})$, where $Z = V(a)$ and $\mathcal{F} = \tilde{M}$. [Hint: Use (Ex. 1.20) and (5.8) to show a priori that $\mathcal{H}^0_Z(\mathcal{F})$ is quasi-coherent. Then show that $\Gamma_a(M) \cong \Gamma_Z(\mathcal{F})$].

(e) Let X be a noetherian scheme, and let Z be a closed subset. If $\mathcal{F}$ is a quasi-coherent (respectively, coherent) $\mathcal{O}_X$-module, then $\mathcal{H}^0_Z(\mathcal{F})$ is also quasi-coherent (respectively, coherent).

Proof.

(a). $Supp(m) = \{x \in Spec(A) | m_x \neq 0\} = \{\mathfrak{p} \subset A | m_{\mathfrak{p}} \neq 0\}$.
If $Ann(m) \not\subset \mathfrak{p}$, then take $a \in Ann(m) \subset A - \mathfrak{p}$. $am = 0$, implying $m_{\mathfrak{p}} = 0$.
If $\mathfrak{p} \notin Supp(m)$, then there exists $a \in A - \mathfrak{p}$ such that $am = 0$, that is, $Ann(m) \not\subset \mathfrak{p}$.
Thus, $D(Ann(m)) = \{x \in Spec(A) | m_x = 0\}$ and $Supp(m) = V(Ann(m))$.

(b). $Supp(\mathcal{F}) = \{x \in Spec(A) | \mathcal{F}_x \neq 0\} = \{\mathfrak{p} \in Spec(A) | M_{\mathfrak{p}} \neq 0\}$. Suppose that M is generated by $\{m_1, m_2, ..., m_n\}$. Then $\{\mathfrak{p} \in Spec(A) | M_{\mathfrak{p}} \neq 0\} = \cup Supp(m_i) = \cup V(Ann(m_i)) = V(\cap Ann(m_i)) = V(Ann(M))$. Thus, $Supp(\mathcal{F}) = V(Ann(M))$.

(c). Still this question is local, we just consider $supp(\mathcal{F}) \cap U$, with $U = Spec(A)$, A Noetherian and $\mathcal{F}|_U = \widetilde{M}$ with M a finitely generated A-module. Then $supp(\mathcal{F} \cap U) = V(Ann(M))$ is closed.

For X is Noetherian, X can be covered by finitely many $Spec(A_i)$. Then $Supp(\mathcal{F})$ is a finite union of closed sets, implying $Supp(\mathcal{F})$ is closed.

(d). For $Z = Spec(A/\mathfrak{a})$, $U = X - Z$ is also Noetherian. So $i_{U*}(\mathcal{F}|_U)$ is quasi-coherent over $X = Spec(A)$ for $\mathcal{F}|_U$ is quasi-coherent over U. For $\mathscr{H}^0_Z(\mathcal{F}) = ker(\mathcal{F} \longrightarrow i_{U*}\mathcal{F}|_U)$, so $\mathscr{H}^0_Z(\mathcal{F})$ is also quasi-coherent under the Abelian category $QCoh(X)$.

If $\mathfrak{a} =< a_1, a_2, ..., a_n >$, then $U = \cup Spec(A_{a_i})$ and them $\Gamma(U, \mathcal{F}_U) = \cap(M_{a_i}) = M_{\mathfrak{a}} = \{\frac{m}{a^n} | a \in \mathfrak{a}\}$. Note that we have an exact sequence of quasi-coherent sheaves:

$$0 \longrightarrow \mathscr{H}^0_Z(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow i_{U*}(\mathcal{F}|_U) \longrightarrow 0$$

then the global sections of them are exact:

$$\mathscr{H}^0_Z(\mathcal{F})(X) = ker(M \longrightarrow M_{\mathfrak{a}})$$

So $\mathscr{H}^0_Z(\mathcal{F}) = \Gamma_{\mathfrak{a}}(\mathcal{F})$.

(e). Just as we have discussed above, this question is local. So by (d), we prove this. $\square$

Remark. I guess that $\Gamma(U, \mathcal{F}_U) = M_{\mathfrak{a}}$. But I don't know how to show this strictly.

Exercise 2.5.7. Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf.

(a) If the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathcal{F}|_U$ is free.

(b) $\mathcal{F}$ is locally free if and only if its stalks $\mathcal{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$.

(c) $\mathcal{F}$ is invertible (i.e., locally free of rank 1) if and only if there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. (This justifies the terminology invertible: it means that $\mathcal{F}$ is an invertible element of the monoid of coherent sheaves under the operation $\otimes$.)

Proof. (a). Suppose that $x = \mathfrak{p} \in Spec(A)$ with A Noetherian. Then $\mathcal{F}_x \cong A_{\mathfrak{p}}^{\oplus n}$ for some n. Suppose that $\mathcal{F}_x$ is generated by $\{\frac{a_1}{s_1}, ..., \frac{a_n}{s_n}\} \subset A_{\mathfrak{p}}$. Let $s = s_1 \cdots s_n$. Then $\frac{a_i}{s_i} \in A_s$, implying $\mathcal{F}|_{D(s)}$ is generated by $\{\frac{a_1}{s_1}, ..., \frac{a_n}{s_n}\}$ as A_s-module.

For $\mathcal{F}$ is coherent, $\mathcal{F}|_{D(s)}$ is finitely presented. So the exists m such that $\mathcal{F}|_U = coker(\varphi)$ with $\varphi : A_s^m \longrightarrow A_s^n$. If φ is represented by $\alpha = (\frac{a_{ij}}{s^{n_{ij}}})_{i,j}$, for $im(\varphi) = 0$ in $A_{\mathfrak{p}}$ i.e. $\alpha = 0$ in $A_{\mathfrak{p}}$, for each a_{ij}, there exists $t_{ij} \in A - \mathfrak{p}$ such that $t_{ij}a_{ij} = 0$. Let $t = (\Pi_{i,j}t_{ij})s$. Then $D(t) \subset D(s)$ and $\alpha|_{D(t)} = 0$. So $\mathcal{F}|_{D(t)} \cong A_t^n$, that is, $\mathcal{F}$ is free on $D(t)$.

(b). ($\Longrightarrow$): By definition of coherent sheaves.
($\Longleftarrow$): By (a).

(c). ($\Longrightarrow$): As we have said, $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)_x = Hom_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{O}_{X,x})$. Because $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} Hom_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{O}_{X,x}) = \mathcal{O}_{X,x}$, so $\mathcal{F} \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) = \mathcal{O}_X$.

($\Longleftarrow$): On $x \in X$, we have $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x = \mathcal{O}_{X,x}$ with $\mathcal{O}_{X,x}$ a local ring and $\mathcal{F}_x$ finitely generated $\mathcal{O}_{X,x}$-module. For $\mathcal{O}_x$ has rank 1, so is $\mathcal{F}_x$. So by (a), $\mathcal{F}$ is locally free of rank 1. □

Remark. By (b), to verify a coherent sheaf is invertible, it is enough to show that it is of rank 1 at each stalk.

Exercise 2.5.8. Again let X be a noetherian scheme, and $\mathcal{F}$ a coherent sheaf on X. We will consider the function

$$\varphi(x) = \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_x} k(x),$$

where $k(x) = \mathcal{O}_x/m_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following results.

(a) The function φ is upper semi-continuous, i.e., for any $n \in \mathbb{Z}$, the set $\{x \in X | \varphi(x) \geq n\}$ is closed.

(b) If $\mathcal{F}$ is locally free, and X is connected, then φ is a constant function.

(c) Conversely, if X is reduced, and φ is constant, then $\mathcal{F}$ is locally free.

Proof.

(a). We prove that $\{x \in X | \varphi(x) < n\}$ is open: Suppose $x = \mathfrak{p} \in U = Spec(A)$ such that $\varphi(x) < 0$ and A is Noetherian.

For $\mathcal{F}$ is coherent, we assume that $\mathcal{F}|_U = \widetilde{M}$ with M a finitely generated A-module and M is generated by $\{m_1, ..., m_k\}$ Then, by assumption, $dim_{k(x)} M_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}M_{\mathfrak{p}} = r < n$. If $\{u_1, ..., u_r\}$ generates $M_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}M$ as a $k(x)$-vector space. By Nakayama lemma (See Chapter VI, Exercise 3.10 of [1]), $\{u_1, ..., u_r\}$ generates $M_{\mathfrak{p}}$ as $A_{\mathfrak{p}}$-module. Suppose $m_i = \sum \frac{a_{ij}}{s_{ij}} u_j$ with $s_{ij} \in A - \mathfrak{p}$.

Let $s = \Pi s_{ij}$. Consider any $\frac{m}{s^n} \in M_s$. It can be written as a A_s linear combination of $\{m_1, ..., m_k\}$. Because $\frac{a_{ij}}{s_{ij}} \in A_s$ for any i, j, M_s is generated by $\{u_i\}$ as A_s-module. Thus, for any $x \in A_s$, $dim_{k(x)} M_x \otimes_{\mathcal{O}_{X,x}} k(x) \leq r$.

(b). If $\mathcal{F}$ is locally free, then $\{x \in X | rank(\mathcal{F}_\S) = n\}$ is an open subset of X by 2.5.7. By (a), it is also closed. For X is connected, $X = \{x \in X | rank(\mathcal{F}_\S) = n\}$. Thus, φ is constant.

(c). Take $x = \mathfrak{p} \in U = Spec(A)$ with A-Noetherian. Suppose that $\mathcal{F}|_U = \widetilde{M}$ with M generated by $\{m_1, ..., m_k\}$ as A-module and $M_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}M_\mathfrak{p}$ is generated by $\{u_1, ..., u_n\}$ as $k(x)$-vector space. By Nakayama lemma, $u_1, ..., u_n$ generates $M_\mathfrak{p}$ as a $A_\mathfrak{p}$-mod.

Suppose that $\sum a_i u_i = 0$ with $a_i \in A_\mathfrak{p}$. Then for any $\mathfrak{q} \subset \mathfrak{p}$, $\{u_i\}$ generates $M_\mathfrak{q}$ as $A_\mathfrak{q}$-module, as we have discussed in (a). Also for $\varphi(x)$ is constant, $\{u_i\}$ are $k(\mathfrak{q})$-linear independent for each $\mathfrak{q} \subset \mathfrak{p}$, which implies $a_i = 0$ in each $A_\mathfrak{q}/\mathfrak{q}A_\mathfrak{q}$. That is, $a_i \in \mathfrak{q}A_\mathfrak{q}$ in each $A_\mathfrak{q} \subset A_\mathfrak{p}$ with $\mathfrak{q} \subset \mathfrak{p}$. Thus, $a_i \in \mathfrak{q}A_\mathfrak{p}$ for each a_i. For A is reduced and so is $A_\mathfrak{p}$, $a_i = 0$ in $A_\mathfrak{p}$. So $\mathcal{F}_x$ is locally free at each point. □

Exercise 2.5.9. Let S be a graded ring, generated by S_1 as an S_0-algebra, let M be a graded S-module, and let $X = \text{Proj } S$.

(a) Show that there is a natural homomorphism $\alpha : M \to \Gamma_*(\tilde{M})$.

(b) Assume now that $S_0 = A$ is a finitely generated k-algebra for some field k, that S_1 is a finitely generated A-module, and that M is a finitely generated S-module. Show that the map α is an isomorphism in all large enough degrees, i.e., there is a $d_0 \in \mathbb{Z}$ such that for all $d \geq d_0$, $\alpha_d : M_d \to \Gamma(X, \tilde{M}(d))$ is an isomorphism. [Hint: Use the methods of the proof of (5.19).]

(c) With the same hypotheses, we define an equivalence relation $\approx$ on graded S-modules by saying $M \approx M'$ if there is an integer d such that $M_{\geq d} \cong M'_{\geq d}$. Here $M_{\geq d} = \bigoplus_{n \geq d} M_n$. We will say that a graded S-module M is quasi-finitely generated if it is equivalent to a finitely generated module. Now show that the functors τ and Γ_* induce an equivalence of categories between the category of quasi-finitely generated graded S-modules modulo the equivalence relation $\approx$, and the category of coherent $\mathcal{O}_X$-modules.

Proof.

(a).

(b).

(c). By Chapter 2, Prop 5.15 [5], at this case, given any quasi-coherent sheaf $\mathcal{F}$, we have

$$\Gamma^*(X, \mathcal{F})^\sim \cong \mathcal{F}$$

By (b), any finitely generated S-module M is equivalent to $\Gamma^*(X, \widetilde{M})$ which satisfies $\Gamma^*(X, \widetilde{M})^\sim = \widetilde{M}$, which is a coherent sheaf.

Now, we need to show that:

(i). For any M that is quasi-finitely generated S-module, $\Gamma^*(X, \widetilde{M}) \approx M$ and $\widetilde{M}'$ is coherent: Suppose $M \approx M'$ and $M_d = M'_d$. By assumption, we can take $D(f)$ with $deg(f) \geq d$ to cover X and then $M_{(f)} = M'_{(f)}$. So $\widetilde{M} = \widetilde{M'}$, which implies $\widetilde{M'}$ is coherent. Then $\Gamma^*(X, \widetilde{M}) = \Gamma^*(X, \widetilde{M'})$. So if $M \approx M'$ with M finitely generated, then

$$M' \approx M \approx \Gamma^*(X, \widetilde{M}) \approx \Gamma(X, \widetilde{M'})$$

(ii). For any coherent sheaf $\mathcal{F}$, $\Gamma^*(X, \mathcal{F})^\sim = \mathcal{F}$, which is just by Chapter 2, Prop 5.15 [5] and $\Gamma^*(X, \mathcal{F}) \approx M$ for some M finitely generated S-module: By Chapter 2, Proposition 5.17 [5], there exists n_0 such that when $n \geq n_0$, $\mathcal{F}(n)$ can be generated by finitely many global sections. $\Gamma(X, \mathcal{F}(n))$ is finitely generated S-module. Thus $\Gamma^*(X, \mathcal{F}) \approx M$ for some finitely generated S-module. $\square$

Remark. We will see that $\mathcal{O}_{X,x}$ is very ample later.

Exercise 2.5.10. Let A be a ring, let $S = A[x_0, \dots, x_r]$ and let $X = \operatorname{Proj} S$. We have seen that a homogeneous ideal I in S defines a closed subscheme of X (Ex. 3.12), and that conversely every closed subscheme of X arises in this way (5.16).

(a) For any homogeneous ideal $I \subseteq S$, we define the saturation $\bar{I}$ of I to be $\{s \in S | \text{for each } i = 0, \dots, r, \text{there is an } n \text{such that } x_i^s \in I\}$. We say that I is saturated if $I = \bar{I}$. Show that T is a homogeneous ideal of S.

(b) Two homogeneous ideals I_1 and I_2 of S define the same closed subscheme of X if and only if they have the same saturation.

(c) If Y is any closed subscheme of X, then the ideal $\Gamma_*(\mathcal{I}_Y)$ is saturated. Hence it is the largest homogeneous ideal defining the subscheme Y.

(d) There is a 1-1 correspondence between saturated ideals of S and closed subschemes of X.

Proof.

(a). Obviously $\bar{I}$ is an ideal. For any $s = s_0 + \dots + s_n$, if $x_i^n(s) \in I$, then $x_i^n s_j \in I_{i+j}$ for I is homogeneous. So each $s_j \in \bar{I}$. Thus, $\bar{I}$ is homogeneous.

(b). Note that $I_{(x_i)} = \bar{I}_{x_i}$: $f : I_{(x_i)} \longrightarrow \bar{I}_{(x_i)}$ is given by $\frac{s}{x_i^n}, s_i \in I_n \mapsto \frac{s}{x_i^n}$ and $g : \frac{s}{x_i^n}, s_i \in \bar{I}_n \mapsto \frac{x_i^m s}{x_i^{m+n}}$ with $x_i^m s \in I$.

($\Longleftarrow$): $\widetilde{I_j}(D_+(x_i)) = Spec(I_{j,(x_i)}) = Spec(I_{(x_i)})$.

($\Longrightarrow$): Suppose that $I = \bar{I}_1$ and $J = \bar{I}_2$, then I and J are saturated and $I_{(x_i)} = J_{(x_i)}$ for each x_i. Take $s \in I$ with $deg(s) = n$, then for an x_i, there exists $s_i \in J$ and m_i such that

$$\frac{s}{x_i^n} = \frac{s_i}{m_i}$$

Take $N = nm_1 \cdots m_n$. Then $x_i^N s \in J$. Thus $I \subset \bar{J} = J$. By the same way, $J \subset I$. Thus, $I = J$.

(c). $\Gamma^*(\mathscr{I}_Y) = \oplus_{n \geq 0} \Gamma(\mathscr{I}_Y(n))$. Suppose $s \in \overline{\Gamma^*(\mathscr{I}_Y)}$, then there exists $x_i \in \Gamma^*(\mathcal{O}_S)$ such that $x_i^n s \in \Gamma^*(\mathscr{I}_Y)$ for all x_i.

For $i : Y \longrightarrow X$ is a closed immersion, $i_*(\mathcal{O}_Y) = \mathscr{I}_Y$ is quasi-coherent. Thus, $\mathscr{I}_Y = \Gamma^*(\mathscr{I}_Y)^\sim$. So, let $M = \Gamma^*(\mathscr{I}_Y)$ then $\mathscr{I}_Y(n) = \widetilde{M}(n)$. Assume that $deg(s) = m$, then $x_i^n s \in \Gamma(\mathscr{I}_Y(m+n))$. If $x_i^n s \in M(n+m)_{(x_i)}$, then $s \in M(m)_{(x_i)}$ by definition, that is, $s = s \in \Gamma(D_+(x_i), \mathscr{I}_Y(m))$. Then s is glued to a global section of $\Gamma(\mathscr{I}_Y(m))$.

So $\Gamma^*(\mathscr{I}_Y)$ is saturation.

(d). By all above. $\square$

Exercise 2.5.11. Let S and T be two graded rings with $S_0 = T_0 = A$. We define the Cartesian product $S \times_A T$ to be the graded ring $\bigoplus_{d \geq 0} S_d \otimes_A T_d$. If $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} T$, show that $\operatorname{Proj}(S \times_A T) \cong X \times_A Y$, and show that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}(S \times_A T)$ is isomorphic to the sheaf $p^*(\mathcal{O}_X(1)) \otimes p^*(\mathcal{O}_Y(1))$ on $X \times Y$. The Cartesian product of rings is related to the Segre embedding of projective spaces (I, Ex. 2.14) in the following way. If $x_0, \dots, x_r$ is a set of generators for S_1 over A, corresponding to a projective embedding $X \subset \mathbb{P}_A^r$, and if $y_0, \dots, y_s$ is a set of generators for T_1, corresponding to a projective embedding $Y \subset \mathbb{P}_A^s$, then $\{x_i \otimes y_j\}$ is a set of generators for $(S \times_A T)_1$, and hence defines a projective embedding $\operatorname{Proj}(S \times_A T) \subseteq \mathbb{P}_A^N$, with $N = rs + r + s$. This is just the image of $X \times Y \subseteq \mathbb{P}^r \times \mathbb{P}^s$ in its Segre embedding.

Proof.

$Proj(S \times_A T) = X \times_A Y$: Take $f \in S_d$ and $g \in T_d$. $(S \times_A T)_{fg} = S_f \otimes_A T_g$: $f : (S \times_A T)_{fg} \longrightarrow S_f \otimes_A T_g$ is given by $\frac{st}{(fg)^n}, s \in S_{d+n}, t \in T_{d+n} \mapsto \frac{s}{f^n}\frac{t}{g^n}$ and $g : S_f \otimes_A T_g \longrightarrow (S \times_A T)_{fg}$ is given by $\frac{s}{f^n}\frac{t}{g^m}, s \in S_{d+n}, t \in T_{d+m} \mapsto \frac{sf^m tg^n}{(fg)^{n+m}}$.

Thus, we have $D+(fg) = D_+(f) \times_A D_+(g)$. By definition of fiber ptodcut, we have $Proj(S \times_A T) = Proj(S) \times_A Proj(T) = X \times_A Y$.

As we have shown, we can see that

$$(S \times_A T)(1)_{fg} = S(1)_f \otimes_A T(1)_g$$

Also by the f, g given above, so $\mathcal{O}(1)$ on $Proj(S \times_A T)$ is isomorphic to $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$. $\square$

Exercise 2.5.12. *Proof.*

(a) Let X be a scheme over a scheme Y, and let $\mathcal{L}$, M be two very ample invertible sheaves on X. Show that $\mathcal{L} \otimes M$ is also very ample. [Hint: Use a Segre embedding.]

(b) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes. Let $\mathcal{L}$ be a very ample invertible sheaf on X relative to Y, and let M be a very ample invertible sheaf on Y relative to Z. Show that $\mathcal{L} \otimes f^*M$ is a very ample invertible sheaf on X relative to Z.

(a). Consider the pullback

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists! i"] \arrow[drr, hook, bend left, "i_1"] \arrow[ddr, hook, bend right, "i_2"'] & & \\
& \mathbb{P}_Y^{mn+n+m} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathbb{P}_Y^m \arrow[d] \\
& \mathbb{P}_Y^n \arrow[r] & Y
\end{tikzcd}$$

Then $\mathcal{O}_{\mathbb{P}_Y^{mn+m+n}}(1) = p_1^*\mathcal{O}_{\mathbb{P}_Y^m}(1) \otimes_Y p_2^*\mathcal{O}_{\mathbb{P}_Y^n}(1)$. So

$$\begin{aligned}
i^*\mathcal{O}_{\mathbb{P}_Y^{mn+m+n}}(1) &= (i_m \circ i_1 \otimes i_n \circ i_2)^* p_1^*\mathcal{O}_{\mathbb{P}_Y^m}(1) \otimes_Y p_2^*\mathcal{O}_{\mathbb{P}_Y^n}(1) \\
&= i_1^*\mathcal{O}_{\mathbb{P}_Y^m}(1) \otimes i_2^*\mathcal{O}_{\mathbb{P}_Y^n}(1) \\
&= \mathcal{L} \otimes \mathcal{M}
\end{aligned}$$

Here i_n and i_m are Serge embeddings.

(b). It is sufficient to show that $f^*\mathcal{M}$ is very ample. Because

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}_{\mathbb{Z}}^n \times_{\mathbb{Z}} Y \arrow[r] \arrow[d, "p_Y"] & \mathbb{P}_{\mathbb{Z}}^{(n+1)(m+1)-1} \times_{\mathbb{Z}} Z \arrow[d] \\
& Y \arrow[r, hook] \arrow[dr, "g"'] & \mathbb{P}_{\mathbb{Z}}^m \times_{\mathbb{Z}} Z \arrow[d, "p_Z"] \\
& & Z
\end{tikzcd}$$

Then $f^*\mathcal{M} = i^*\mathcal{O}_{\mathbb{P}_Z^{mn+m+n}}(1)$ with $i : X \hookrightarrow Z$. Thus, $f^*\mathcal{M}$ is very ample. $\square$

Remark. Please fill the gaps.

Exercise 2.5.13. Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d > 0$, let $S^{(d)}$ be the graded ring $\bigoplus_{n\geq 0} S_n^{(d)}$ where $S_n^{(d)} = S_{nd}$. Let $X = \operatorname{Proj} S$. Show that $\operatorname{Proj} S^{(d)} \cong X$, and that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj} S^{(d)}$ corresponds via this isomorphism to $\mathcal{O}_X(d)$. This construction is related to the d-uple embedding (I, Ex. 2.12) in the following way. If $x_0, \dots, x_r$ is a set of generators for S_1, corresponding to an embedding $X \subseteq \mathbb{P}_A^r$, then the set of monomials of degree d in the x_i is a set of generators for $S_1^{(d)} = S_d$. These define a projective embedding of $\operatorname{Proj} S^{(d)}$ which is none other than the image of X under the d-uple embedding of $\mathbb{P}_A^r$.

Proof.

Note that at this case X can be covered by $D_+(s_i)$ with s_i and $S^{(d)}$ is generated by S_d as S_0-module. Because $D_+(s) = D_+(s^n)$ for any n. X can also be covered by $D(f)$ with $deg(f) = d$.

For any $f \in S_d$, $S_f = S_f^{(d)}$ by definition. Thus, $X = Proj(S^{(d)})$.

Now consider $S^d(1)_f$, which is isomorphic to $S(d)_f$. Thus, $\mathcal{O}(1)$ on $Proj(S^{(d)})$ is isomorphic to $\mathcal{O}(d)$ on X. □

Exercise 2.5.14.

(5.14) Let A be a ring, and let X be a closed subscheme of $\mathbb{P}_A^r$. We define the homogeneous coordinate ring $S(X)$ of X for the given embedding to be $A[x_0, \dots, x_r]/I$, where I is the ideal $\Gamma_*(\mathcal{I}_X)$ constructed in the proof of (5.16). (Of course if A is a field and X a variety, this coincides with the definition given in (I, §2)) Recall that a scheme X is normal if its local rings are integrally closed domains. A closed subscheme $X \subseteq \mathbb{P}_A^r$ is projectively normal for the given embedding, if its homogeneous coordinate ring $S(X)$ is an integrally closed domain (cf. (I, Ex. 3.18)). Now assume that k is an algebraically closed field, and that X is a connected, normal closed subscheme of $\mathbb{P}_A^r$. Show that for some $d > 0$, the d-uple embedding of X is projectively normal, as follows.

(a) Let S be the homogeneous coordinate ring of X, and let $S' = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(n))$. Show that S is a domain, and that S' is its integral closure. [Hint: First show that X is integral. Then regard S' as the global sections of the sheaf of rings $\mathcal{L} = \bigoplus_{n\geq 0} \mathcal{O}_X(n)$ on X, and show that $\mathcal{L}$ is a sheaf of integrally closed domains.]

(b) Use (Ex. 5.9) to show that $S_d = S'_d$ for all sufficiently large d.

(c) Show that $S^{(d)}$ is integrally closed for sufficiently large d, and hence conclude that the d-uple embedding of X is projectively normal. is projectively normal.

(d) As a corollary of (a), show that a closed subscheme $X \subseteq \mathbb{P}_A^r$ is projectively normal if and only if it is normal, and for every $n \geq 0$, the natural map $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ is surjective.

Proof.

(a). For X is connected and normal, X is integral. So S is an integral domain.

Because taking stalks is left-adjoint, it commutes with $\oplus$ which is a kind of colimits. So

$$\mathscr{I}_{\mathfrak{p}} = \oplus_{n\geq 0}\mathcal{O}_X(n)_{\mathfrak{p}} = \{\tfrac{f}{g} | deg(f) \geq deg(g)\}$$

which is normal. Thus, $\mathscr{I}$ is normal. So is S'.

(b). For $S = k[x_0, ..., x_n]/I$ is finitely generated k-module, by 2.1.22. Thus, $S_d = S'_d$ for d sufficiently large.

(c). We chose d sufficiently large, such that $S^{(d)}_{>0} = \oplus_{n>0} S^{'}_{dn}$. For $S^{(d)}_0 = S^{'}_0 = k$, ($S = k[x_1, ..., x_n]/I$, so $S_0 = k$ and $\Gamma(X, \widetilde{S}) = 0$ by direct computation), $S^{(d)} = {S^{'}}^{(d)}$. Then for any $x \in {S^{'}}^{(d)}_m$, we can find $s_i \in S^{'}$ such that

$$\sum s_i x^i = 0$$

If $s_{ij} x^{n_j} \in S_k$, then $\sum s_{ij} x_i^{n_j} \in S^{'}_k$, so we can take each $s_{ij} \in {S^{'}}^{(d)}$. Hence, ${S^{'}}^{(d)}$ is integrally closed. So is $S^{(d)}$.

(d). ($\Longrightarrow$): By definition, X is normal. For X is a closed immersion, we have the surjection $\Gamma^*(\mathbb{P}^n_k) \longrightarrow \Gamma^*(X)$.

($\Longleftarrow$): At this case, $S^{'}$ is integrally closed. For $S^{(d)} = {S^{'}}^(d)$ for sufficiently large d. $X = Proj(S^{'})$. By the surjection, we have a surjection $k[x_1, ..., x_n] \longrightarrow S^{'}$ which keeps the degree. So $Proj(S^{'})$ is given a homogeneous ideal $I^{'}$ of $k[x_1, ..., x_n]$. For I and $I^{'}$ defines the same closed immersion and they are saturated, (for I by definition for $I^{'}$ by 2.5.9 (c)), so $I = I^{'}$ by 2.5.9 (d). Hence $S = S^{'}$. □

Remark. Note that by 2.5.9, every closed subscheme of $\mathbb{P}^n_k$ is of the form

$$Proj(k[x_1, ..., x_n]/I)$$

For more details, see Chapter 2, lemma 3.41 of [10].

Exercise 2.5.15. Extension of Coherent Sheaves. We will prove the following theorem in several steps: Let X be a noetherian scheme, let U be an open subset, and let $\mathcal{F}$ be a coherent sheaf on U. Then there is a coherent sheaf $\mathcal{F}'$ on X such that $\mathcal{F}'|_U \cong \mathcal{F}$.

(a) On a noetherian affine scheme, every quasi-coherent sheaf is the union of its coherent subsheaves. We say a sheaf $\mathcal{F}$ is the union of its subsheaves $\mathcal{F}$ if for every open set U, the group $\mathcal{F}(U)$ is the union of the subgroups $\mathcal{F}(U)$.

(b) Let X be an affine noetherian scheme, U an open subset, and $\mathcal{F}$ coherent on U. Then there exists a coherent sheaf $\mathcal{F}'$ on X with $\mathcal{F}'|_U \cong \mathcal{F}$. [Hint: Let $i : U \to X$ be the inclusion map. Show that $i_*\mathcal{F}$ is quasi-coherent, then use (a).]

(c) With $X, U, \mathcal{F}$ as in (b), suppose furthermore we are given a quasi-coherent sheaf $\mathcal{G}$ on X such that $\mathcal{F} \subseteq \mathcal{G}|_U$. Show that we can find $\mathcal{F}'$ a coherent subsheaf of $\mathcal{G}$, with $\mathcal{F}'|_U \cong \mathcal{F}$. [Hint: Use the same method, but replace $i_*\mathcal{F}$ by $\rho^{-1}(i_*\mathcal{F})$, where ρ is the natural map $\mathcal{G} \to i_*(\mathcal{G}|_U)$.]

(d) Now let X be any noetherian scheme, U an open subset, $\mathcal{F}$ a coherent sheaf on U, and $\mathcal{G}$ a quasi-coherent sheaf on X such that $\mathcal{F} \subseteq \mathcal{G}|_U$. Show that there is a coherent subsheaf $\mathcal{F}' \subseteq \mathcal{G}$ on X with $\mathcal{F}'|_U \cong \mathcal{F}$. Taking $\mathcal{G} = i_*\mathcal{F}$ proves the result announced at the beginning. [Hint: Cover X with open affines, and extend over one of them at a time.]

(e) As an extra corollary, show that on a noetherian scheme, any quasi-coherent sheaf $\mathcal{F}$ is the union of its coherent subsheaves. [Hint: If s is a section of $\mathcal{F}$ over an open set U, apply (d) to the subsheaf of $\mathcal{F}|_U$ generated by s.]

Proof.

(a). Suppose that $\mathcal{F} = \widetilde{M}$ with M a A-module. Then $M = \cup M_\alpha$ with M_α finitely generated for A is Noetherian. Note that $\cup$ is also a kind of colimits, we can see that

$$M_{(\mathfrak{p})} = \cup M_{\alpha,(\mathfrak{p})}$$

Thus, $\mathcal{F} = \cup \mathcal{F}_\alpha$ with $\mathcal{F}_\alpha = \widetilde{M_\alpha}$ with M_α finitely generated A-module.

(b). At this case $U = D(f)$ for some f. For $i_*\mathcal{F}$ is quasi-coherent in X. $i_*\mathcal{F} = \mathcal{F}_\alpha$ with M_α finitely generated A-module. For A is Noetherian, there are only finitely many M_α such that $\cup M_{\alpha,f} = M_{\mathcal{F}}$ as finitely generated A_f-module. Thus, just take $\mathcal{F}' = \cup_{\alpha=1}^n \mathcal{F}_\alpha$, which is a coherent sheaf for $\cup_{\alpha=1}^n M_\alpha$ is also finitely generated. Then $\mathcal{F}'|_U = i_*\mathcal{F}|_U = \mathcal{F}$.

(c). Note that $\rho^{-1}(i_*\mathcal{F}) \subset \mathcal{G}$ is still quasi-coherent. By the construction in (b), we can fine a coherent subsheaf $\mathcal{F}' \subset \rho^{-1}(i_*\mathcal{F}) \subset \mathcal{G}$ such that $\mathcal{F}'|_U = \mathcal{F}$.

(d). For X is Noetherian scheme, X can be covered by $Spec(A_i), i = 1, 2, ..., n$ with A_i Noetherian. If there exists
$$\mathcal{F}'_{n-1} \subset \mathcal{G}|_{(\cup_{i=1}^n Spec(A_i))}$$
on $\cup_{i=1}^{n-1} Spec(A_i)$ coherent such that $\mathcal{F}'_{n-1}|_{U\cap(\cup_{i=1}^{n-1} Spec(A_i))} = \mathcal{F}|_{U\cap(\cup_{i=1}^{n-1} Spec(A_i))}$.

Then on $(\cup_{i=1}^n Spec(A_i)) \cup U)$, $\mathcal{F}'_{n-1}$ and $\mathcal{F}$ can be glued together to be a coherent sheaf, denoted by $\mathcal{F}_{n-1}$. Then on $Spec(A_n)$, for $\mathcal{F}_{\backslash-\infty}|_{(\cup_{i=1}^n Spec(A_i))\cap Spec(A_i)}$ which is coherent, we can find a coherent morphism $\mathcal{F}_\backslash \subset \mathcal{G}_{Spec(A_n)}$ on $Spec(A_n)$ such that $\mathcal{F}_n|_{(\cup_{i=1}^n Spec(A_i))\cap Spec(A_i)} = \mathcal{F}_{n-1}|_{(\cup_{i=1}^n Spec(A_i))\cap Spec(A_i)}$. Then they can be glued to a coherent sheaf, $\mathcal{F}'$ on X and $\mathcal{F}' \subset \mathcal{G}$.

Thus, we just need induct on the number of covers of X. Then by (b), (c), we prove this.

(e). Obviously, $\cup\mathcal{F}_\alpha \subset \mathcal{F}$. Conversely, consider any section of $\mathcal{F}$. By (d), there also exists a subsheaf $\mathcal{F}_\alpha \subset \mathcal{F}$ such that $\mathcal{F}_\alpha|_U = (s)_U$ for an open set U. That is, for any U, $\mathcal{F}(U) \subset \cup\mathcal{F}_\alpha(U)$. Thus, $\mathcal{F} = \cup\mathcal{F}_\alpha$. $\square$

Exercise 2.5.16. Tensor Operations on Sheaves. First, we recall the definitions of various tensor operations on a module. Let A be a ring, and let M be an A-module. Let $T^n(M)$ be the tensor product $M \otimes \ldots \otimes M$ of M with itself n times, for $n \geq 1$. For $n = 0$, we put $T^0(M) = A$. Then $T(M) = \bigoplus_{n\geq 0} T^n(M)$ is a (noncommutative) A-algebra, which we call the tensor algebra of M. We define the symmetric algebra $S(M) = \bigoplus_{n\geq 0} S^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes y - y \otimes x$, for all $x, y \in M$. Then $S(M)$ is a commutative A-algebra. Its component $S^n(M)$ in degree n is called the n-th symmetric product of M. We denote the image of $x \otimes y$ in $S(M)$ by xy, for any $x, y \in M$. As an example, note that if M is a free A-module of rank r, then $S(M) \cong A[x_1, \ldots, x_r]$.

We define the exterior algebra $\bigwedge(M) = \bigoplus_{n\geq 0} \bigwedge^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes x$ for $x \in M$. Note that this ideal contains all expressions of the form $x \otimes y + y \otimes x$, so that $\bigwedge(M)$ is a skew-commutative graded A-algebra. This means that if $u \in \bigwedge^n(M)$ and $v \in \bigwedge^n(M)$, then $u \wedge v = (-1)^n v \wedge u$ (here we denote by $\wedge$ the multiplication in this algebra; so the image of $x \otimes y$ in $\bigwedge^n(M)$ is denoted by $x \wedge y$). The n-th component $\bigwedge^n(M)$ is called the n-th exterior power of M.

Now let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the tensor algebra, symmetric algebra, and exterior algebra of $\mathcal{F}$ by taking the sheaves associated to the presheaf, which to each open set U assigns the corresponding tensor operation applied to $\mathcal{F}(U)$ as an $\mathcal{O}_X(U)$-module. The results are $\mathcal{O}_X$-algebras, and their components in each degree are $\mathcal{O}_X$-modules.

(a) Suppose that $\mathcal{F}$ is locally free of rank n. Then $T^n(\mathcal{F})$, $S^n(\mathcal{F})$, and $\bigwedge^n(\mathcal{F})$ are also locally free, of ranks n^r, $\binom{n+r-1}{r-1}$, and $\binom{n}{r}$ respectively.

(b) Again, let $\mathcal{F}$ be locally free of rank n. Then the multiplication map $\bigwedge^n \mathcal{F} \otimes \bigwedge^{n-r} \mathcal{F} \to \bigwedge^n \mathcal{F}$ is a perfect pairing for any r, i.e., it induces an isomorphism of $\bigwedge^n \mathcal{F}$ with $(\bigwedge^{n-r} \mathcal{F})^r \otimes \bigwedge^n \mathcal{F}$.

As a special case, note that if $\mathcal{F}$ has rank 2, then $\mathcal{F} \cong \mathcal{F}^r \otimes \bigwedge^n \mathcal{F}$.

(c) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of locally free sheaves. Then for any r, there is a finite filtration of $S^n(\mathcal{F})$,

$$S^r(\mathcal{F}) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p/F^{p+1} \cong S^p(\mathcal{F}') \otimes S^{r-p}(\mathcal{F}'')$$

for each p.

(d) Same statement as (c), with exterior powers instead of symmetric powers. In particular, if $\mathcal{F}, \mathcal{F}'$ have ranks n, n' respectively, there is an isomorphism

$$\bigwedge^n \mathcal{F} \cong \bigwedge^n \mathcal{F}' \otimes \bigwedge^n \mathcal{F}''.$$

(e) Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. Then f^* commutes with all the tensor operations on $\mathcal{F}$, i.e., $f^*(S^n(\mathcal{F})) = S^n(f^*\mathcal{F})$, etc.

Proof.

(a). For localization is exact, $(\mathcal{F}/\mathcal{G})_\mathfrak{p} = \mathcal{F}_\mathfrak{p}/\mathcal{G}_\mathfrak{p}$.

For $\mathcal{F} \otimes \mathcal{G}$ is defined by the presheaf. Thus, $(\mathcal{F} \otimes \mathcal{G})_\mathfrak{p} = \mathcal{F}_\mathfrak{p} \otimes \mathcal{G}_\mathfrak{p}$.

Thus, $T^r(\mathcal{F})_\mathfrak{p} = T^r(\mathcal{F}_\mathfrak{p})$. For $\mathcal{F}$ is locally free of rank n, $\mathcal{F}_\mathfrak{p}$ is free of rank n. By theory of multilinear algebra $T^r(\mathcal{F})_\mathfrak{p} = T^r(\mathcal{F}_\mathfrak{p})$ is free of rank n^r. So $T^r(\mathcal{F})$ is locally free of rank n^r, by 2.5.7.

Similarly, we show that $S^r(\mathcal{F}$ is locally free of ranke $\binom{n+r-1}{n-1}$ and $\bigwedge^r \mathcal{F}$ is locally free of rank $\binom{n}{r}$.

(b).

Claim 2.5.13. *Let M be a free A-module of rank n. Then there exists $\bigwedge^n M \otimes \bigwedge^{n-r} M \longrightarrow \bigwedge^n M$ is a perfect paring for any r.*

This claim ensures the morphism $\bigwedge^n \mathcal{F} \otimes \bigwedge^{n-r} \mathcal{F} \longrightarrow \bigwedge^n \mathcal{F}$ exists locally.

Claim 2.5.14. *Let M_1 and M_2 be two free A-modules of rank n. If there exists an isomorphism $\varphi : M_1 \longrightarrow M_2$, then there exists a diagram*

$$\begin{tikzcd}
\bigwedge^n M_1 \otimes \bigwedge^{n-r} M_1 \arrow[r] \arrow[d, "\exists! \bigwedge^n \otimes \bigwedge^{n-r} \varphi"] & \bigwedge^n M_1 \arrow[d, "\exists! \bigwedge \varphi"] \\
\bigwedge^n M_2 \otimes \bigwedge^{n-r} M_2 \arrow[r] & \bigwedge^n M_2
\end{tikzcd}$$

The existence of this diagram ensures there exists a gluing function between affine open sets and the uniqueness implies that the gluing functions satisfy the cocyclic condition. Hence, the map can be glued to be a global morphism

$$\bigwedge^n \mathcal{F} \otimes \bigwedge^{n-r} \mathcal{F} \longrightarrow \bigwedge^n \mathcal{F}$$

Just as we stated at the remark of 2.5.1, $(\check{\mathcal{F}})_\mathfrak{p} = (\mathcal{F}_\mathfrak{p})\check{}$. Then by multilinear algebra, $(\bigwedge^r \mathcal{F})_\mathfrak{p} = ((\bigwedge^{n-r} \mathcal{F})\check{} \otimes \bigwedge^n \mathcal{F})_\mathfrak{p}$.

(c).

Claim 2.5.15. *Let M, N, L be a free A-module and we have a s.e.s.*

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

Then for any r, there exists a finite filtration of $S^n(N)$,

$$S^n(N) = N^0 \supseteq N^1 \supseteq ... \supseteq N^r \supseteq 0$$

such that

$$N^p/N^{p+1} \cong S^p(M) \otimes S^{r-p}(L)$$

Proof. **Wait!** □

This claim ensures that the filtration exists locally.

Claim 2.5.16. *Let (M_1, N_1, L_1) and (M_2, N_2, L_2) be two free A-modules short exact sequence. If there exists an isomorphism $\varphi : (M_1, N_1, L_1) \longrightarrow (M_2, N_2, L_2)$, it induces an unique isomorphism between two filtration.*

Proof. **Wait!** □

The existence ensures there exists a gluing function between affine open sets and the uniqueness implies that the gluing functions satisfy the cocyclic condition. Hence, the filtration can be glued to a global filtration.

(d).Because $f^{-1}(\mathcal{O})_x = \mathcal{O}_{f(x)}$, we can see that

$$(f^*(\mathcal{L}))_{\mathfrak{p}} = \mathcal{L}_{f(\mathfrak{p})} \otimes_{\mathcal{O}_{Y,f(\mathfrak{p})}} \mathcal{O}_{X,\mathfrak{p}}$$

So, if $\mathcal{L}$ is locally free over Y, $f^*(\mathcal{L})$ is locally free over X.

By 2.5.1 (c), for any $\mathcal{P}$ over X and $\mathcal{F}$, $\mathcal{G}$ locally free, we have:

$$\begin{aligned} Hom(f^*\mathcal{F} \otimes f^*\mathcal{G}, \mathcal{P}) &= Hom(f^*\mathcal{F}, Hom(f^*\mathcal{G}, \mathcal{P})) \\ &= Hom(\mathcal{F}, f_*Hom(f^*\mathcal{G}, \mathcal{P})) \\ &= Hom(\mathcal{F}, Hom(\mathcal{G}, f_*\mathcal{P})) \\ &= Hom(\mathcal{F} \otimes \mathcal{G}, f_*\mathcal{P}) \\ &= Hom(f^*(\mathcal{F} \otimes \mathcal{G}), \mathcal{P})) \end{aligned}$$

Thus, $f^*(\mathcal{F} \otimes \mathcal{G}) = f^*\mathcal{F} \otimes f^*\mathcal{G}$.

□

Remark. Just use the fact that taking stalks is exact and it commutes with finite coproduct. We can show that

$$\mathcal{F}_{\mathfrak{p}} = \mathcal{F}^{'}_{\mathfrak{p}} \oplus \mathcal{F}^{''}_{\mathfrak{p}}$$

for all of them are locally free and

$$\otimes^r(\mathcal{F}^{'}_{\mathfrak{p}} \oplus \mathcal{F}^{''}{}_{\mathfrak{p}}) = \oplus_p(\otimes^{r-p}\mathcal{F}^{'}_{\mathfrak{p}} \otimes \otimes^p\mathcal{F}^{''}_{\mathfrak{p}})$$

By basic multilinear algebra, we have

$$S^r(\mathcal{F}^{'}_{\mathfrak{p}} \oplus \mathcal{F}^{''}{}_{\mathfrak{p}}) = \oplus_p(S^{r-p}\mathcal{F}^{'}_{\mathfrak{p}} \otimes S^p\mathcal{F}^{''}_{\mathfrak{p}})$$

and

$$\bigwedge^r(\mathcal{F}^{'}_{\mathfrak{p}} \oplus \mathcal{F}^{''}{}_{\mathfrak{p}}) = \oplus_p(\bigwedge^{r-p}\mathcal{F}^{'}_{\mathfrak{p}} \otimes \bigwedge^p\mathcal{F}^{''}_{\mathfrak{p}})$$

Exercise 2.5.17. Affine Morphisms. A morphism $f : X \to Y$ of schemes is affine if there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is affine for each i.

(a) Show that $f : X \to Y$ is an affine morphism if and only if for every open affine $V \subseteq Y$, $f^{-1}(V)$ is affine. [Hint: Reduce to the case Y affine, and use (Ex. 2.17).]

(b) An affine morphism is quasi-compact and separated. Any finite morphism is affine.

(c) Let Y be a scheme, and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras (i.e., a sheaf of rings which is at the same time a quasi-coherent sheaf of $\mathcal{O}_Y$-modules). Show that there is a unique scheme X, and a morphism $f : X \to Y$, such that for every open affine $V \subseteq Y$, $f^{-1}(V) \cong \text{Spec } \mathcal{A}(V)$, and for every inclusion $U \subset V$ of open affines of Y, the morphism $f^{-1}(U) \subset f^{-1}(V)$ corresponds to the restriction homomorphism $\mathcal{A}(V) \to \mathcal{A}(U)$. The scheme X is called **Spec** $\mathcal{A}$. [Hint: Construct X by gluing together the schemes Spec $\mathcal{A}(V)$, for V open affine in Y.]

(d) If $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_Y$-algebra, then $f : X = \text{Spec } \mathcal{A} \to Y$ is an affine morphism, and $\mathcal{A} \cong f_*\mathcal{O}_X$. Conversely, if $f : X \to Y$ is an affine morphism, then $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, and $X \cong \text{Spec } \mathcal{A}$.

(e) Let $f : X \to Y$ be an affine morphism, and let $\mathcal{A} = f_*\mathcal{O}_X$. Show that f_* induces an equivalence of categories from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of quasi-coherent $\mathcal{A}$-modules (i.e., quasi-coherent $\mathcal{O}_Y$-modules having a structure of $\mathcal{A}$-module). [Hint: For any quasi-coherent $\mathcal{A}$-module $\mathcal{M}$, construct a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$, and show that the functors f_* and $\sim$ are inverse to each other.]

Proof.
(a).

(b). Recall that
1. $Spec(A)$ is quasi-compact for any A.
2. $Spec(A) \longrightarrow Spec(B)$ is separated for any A, B.

And both quasi-comapctness and separeteness are local properties, we see that affine morphisms are quasi-compact and separated.

(c). Let $\{V_i = Spec(A_i)\}$ be an affine cover of Y. Then for any $W \subset V_i \cap V_j$ with W affine, we have

$$\begin{tikzcd} \mathcal{A}(V_i) \arrow[r, "\rho^{\mathcal{A}}_{V_iW}"] & \mathcal{A}(W) \arrow[r, "\varphi^{\mathcal{A}}_{ij}"] & \mathcal{A}(W) & \mathcal{A}(V_j) \arrow[l, "\rho^{\mathcal{A}}_{V_jW}"'] \\ \mathcal{O}_Y(V_i) \arrow[u] \arrow[r, "\rho_{V_iW}"] & \mathcal{O}_Y(W) \arrow[u] \arrow[r, "\varphi_{ij}"] & \mathcal{O}_Y(W) \arrow[u] & \mathcal{O}_Y(V_j) \arrow[u] \arrow[l, "\rho_{V_jW}"'] \end{tikzcd}$$

where $\rho^{\mathcal{A}}$ and ρ are the restriction maps and φ are gluing functions.

Note that for W is affine and $Spec(\mathcal{A}(W)) \subset Spec(\mathcal{A}(V_i)) \cap Spec(\mathcal{A}(V_j))$, we can define $\psi_{ij}|_W : Spec(\mathcal{A}(V_j))|_W \xrightarrow{\cong} Spec(\mathcal{A}(V_i))|_W$ deduced by $\varphi^{\mathcal{A}}_{ij}$.

Now we glue $\psi_{ij}|_W$ to be a gluing functions for between $Spec(\mathcal{A}(V_i))$ and $Spec(\mathcal{A}(V_j))$ on $Spec(\mathcal{A}(V_i)) \cap Spec(\mathcal{A}(V_j))$: For V_i and V_j are affine, they are separated. Hence if we choose $\{W_k\}$ a series of affine open sets to cover $V_i \cap V_j$ then $W_{k_1} \cap W_{k_2}$ is still affine and then $\{Spec(\mathcal{A}(W_k))\}$ forms an affine cover of $Spec(\mathcal{A}(V_i)) \cap Spec(\mathcal{A}(V_j))$ and $Spec(\mathcal{A}(W_{k_1})) \cap Spec(\mathcal{A}(W_{k_2})) = Spec(\mathcal{A}(W_{k_1} \cap W_{k_2}))$. Use the diagram above, replacing V_i by W_{k_i} and W by $W_{k_1} \cap W_{k_2}$ and note that $\{\varphi^{\mathcal{A}}_{k_1k_2}\}$ satisfies the gluing criteria for any k_1, k_2 ,hence $\psi|_W$ can be glued to a morphism between $Spec(\mathcal{A}(V_i))$ and $Spec(\mathcal{A}(V_j))$ on $Spec(\mathcal{A}(V_i)) \cap Spec(\mathcal{A}(V_j))$ denoted by ψ_{ij}.

Then it is easy to check that ψ_{ij} satisfies the gluing criteria. Hence $\{Spec(\mathcal{A}(V_i)), \psi_{ij}\}$ can be glued to a scheme X, denoted by $Spec(\mathcal{A})$.

For the uniqueness, just using 2.2.12 to construct the isomorphisms.

(d). Take any affine open set $V \subset Y$. By definition of $Spec(\mathcal{A})$, $f^{-1}(V) = Spec(\mathcal{A}(V))$, that is, $\mathcal{A}(V) = \mathcal{O}_X(f^{-1}(V))$. Hence, $\mathcal{A} = f_*\mathcal{O}_X$.

Conversely, assume $f : X \longrightarrow Y$ is affine and $\mathcal{A} = f_*\mathcal{O}_X$. Then for any affine open set $V \subset Y$, $f^{-1}(V) = Spec(A_V)$ for some ring A_V. Because $\mathcal{A}(V) = \mathcal{O}_X(f^{-1}(V))$, $\mathcal{A}(V) = A_V$. Hence, we see that $f^{-1}(V) = Spec(\mathcal{A}(V))$. Then using the uniqueness in (c), $X \cong Spec(\mathcal{A})$.

(e). By (d), we just let $X = Spec(\mathcal{A})$.

Now we need to show that we have

$$\{\text{quasi-coherent } \mathcal{O}_X\text{-module}\} \cong \{\text{quasi-coherent } \mathcal{A}\text{-module}\}$$

with $F : \mathcal{N} \mapsto f_*\mathcal{N}$ given $\mathcal{N} \in QCoh(X)$.

First of all, we define $G : \mathcal{M} \mapsto \widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}}$ is given as following: Choose an affine open set V, then $\mathcal{M}|_V = M$ with M an $\mathcal{A}(V)$-module. for $\mathcal{M}$ is an $\mathcal{A}$-module. Then, on $f^{-1}(V) = Spec\mathcal{A}(V)$, we define $\widetilde{\mathcal{M}}|_{f^{-1}(V)}$ to be $\widetilde{M}$. Then use the same arguments in (c), we see that $\mathcal{M}|_{f^{-1}(V)}$ can be glued to be a sheaf on X. For locally, $\widetilde{\mathcal{M}}|_{f^{-1}(V)} = \widetilde{M}$ with M an $\mathcal{O}_X(f^{-1}(V)) = \mathcal{A}(V)$-module. $\widetilde{M} \in QCoh(X)$.

By the definition of F and G, taking any affine cover on Y, check that $F \circ G = id$ and $G \circ F = id$. Then F, G gives the equivalence between these two categories.

$\square$

Exercise 2.5.18. Vector Bundles

Let Y be a scheme. A (geometric) vector bundle of rank n over Y is a scheme X and a morphism $f : X \to Y$, together with additional data consisting of an open covering $\{U_i\}$ of Y, and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbb{A}^n_{U_i}$, such that for any i, j, and for any open affine subset $V = \text{Spec } A \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{A}^n_V = \text{Spec } A[x_1, \ldots, x_n]$ is given by a linear automorphism θ of $A[x_1, \ldots, x_n]$, i.e., $\theta(a) = a$ for any $a \in A$, and $\theta(x_i) = \sum a_{ij}x_j$ for suitable $a_{ij} \in A$.

An isomorphism $g : (X, f, \{U_i\}, \{\psi_i\}) \to (X', f', \{U'_i\}, \{\psi'_i\})$ of one vector bundle of rank n to another one is an isomorphism $g : X \to X'$ of the underlying schemes, such that $f = f' \circ g$, and such that X, f, together with the covering of Y consisting of all the U_i and U'_i, and the isomorphisms ψ_i and $\psi'_i \circ g$, is also a vector bundle structure on X.

(a). Let $\mathcal{E}$ be a locally free sheaf of rank n on a scheme Y. Let $S(\mathcal{E})$ be the symmetric algebra on $\mathcal{E}$, and let $X = \text{Spec } S(\mathcal{E})$, with projection morphism $f : X \to Y$. For each open affine subset $U \subseteq Y$ for which $\mathcal{E}|_U$ is free, choose a basis of $\mathcal{E}$, and let $\psi : f^{-1}(U) \to \mathbb{A}^n_U$ be the isomorphism resulting from the identification of $S(\mathcal{E}(U))$ with $\mathcal{O}(U)[x_1, \ldots, x_n]$. Then $(X, f, \{U\}, \{\psi\})$ is a vector bundle of rank n over Y, which (up to isomorphism) does not depend on the bases of $\mathcal{E}_U$ chosen. We call it the geometric vector bundle associated to $\mathcal{E}$, and denote it by $\mathbb{V}(\mathcal{E})$.

(b). For any morphism $f : X \to Y$, a section of f over an open set $U \subseteq Y$ is a morphism $s : U \to X$ such that $f \circ s = \text{id}_U$. It is clear how to restrict sections to smaller open sets, or how to glue them together, so we see that the presheaf $U \mapsto \{\text{set of sections of } f \text{ over } U\}$ is a sheaf of sets on Y, which we denote by $\mathcal{S}(X/Y)$. Show that if $f : X \to Y$ is a vector bundle of rank n, then the sheaf of sections $\mathcal{S}(X/Y)$ has a natural structure of $\mathcal{O}_Y$-module, which makes it a locally free $\mathcal{O}_Y$-module of rank n. [Hint: It is enough to define the module structure locally, so we can assume $Y = \text{Spec } A$ is affine, and $X = \mathbb{A}^n_Y$. Then a section $s : Y \to X$ comes from an A-algebra homomorphism $\theta : A[x_1, \ldots, x_n] \to A$, which in turn determines an ordered n-tuple $\langle \theta(x_1), \ldots, \theta(x_n) \rangle$ of elements of A. Use this correspondence between sections s and ordered n-tuples of elements of A to define the module structure.]

(c). Again let $\mathscr{E}$ be a locally free sheaf of rank n on Y, let $X = \mathbb{V}(\mathscr{E})$, and let $\mathcal{L} = \mathcal{S}(X/Y)$ be the sheaf of sections of X over Y. Show that $\mathcal{L} \cong \mathscr{E}^*$, as follows. Given a section $s \in \Gamma(V, \mathscr{E}^*)$ over any open set V, we think of s as an element of $\text{Hom}(\mathscr{E}|_V, \mathcal{O}_V)$. So s determines an $\mathcal{O}_V$-algebra homomorphism $S(\mathscr{E}|_V) \to \mathcal{O}_V$. This determines a morphism of spectra $V = \text{Spec } \mathcal{O}_V \to \text{Spec } S(\mathscr{E}|_V) = f^{-1}(V)$, which is a section of X/Y. Show that this construction gives an isomorphism of $\mathscr{E}^*$ to $\mathcal{L}$.

(d). Summing up, show that we have established a one-to-one correspondence between isomorphism classes of locally free sheaves of rank n on Y, and isomorphism classes of vector bundles of rank n over Y. Because of this, we sometimes use the words "locally free sheaf" and "vector bundle" interchangeably, if no confusion seems likely to result.

Proof.

(b). It is enough to define the module section $s : Y \longrightarrow X$ locally. Without loss of generality, let $Y = Spec(A)$ then $X = Spec(A[x_1, ..., x_n])$.

Then a section $s : Y \longrightarrow X$ is uniquely determined by ring homomorphism: $A[x_1, ..., x_n] \longrightarrow A$, that is, we have the following one to one corresponding

$$\mathscr{S}(X/Y)(V) = \{s : Y \longrightarrow X \text{ sections}\} \cong Hom_{Ring}(A[x_1, ..., x_n], A)$$

Let $e_i : A[x_1, ..., x_n] \longrightarrow A$ given by $x_i \mapsto 1$ and $x_j \mapsto 0$ when $j \neq i$. Then $\{e_i\}$ forms an A-basis for $Hom_{Ring}(A[x_1, ..., x_n], A)$, which is a free A-module of rank n. Hence locally, $\mathscr{S}(X/Y)$ is a free $\mathcal{O}_Y$-module of rank n.

(c). Note that $\check{\mathscr{E}} = \mathcal{H}om(\mathscr{E}, \mathcal{O}_X)$. Hence $\Gamma(V, \check{\mathscr{E}}) = Hom(\mathscr{E}|_V, \mathcal{O}|_V)$.

Now let $V = Spec(A)$ be an affine open set. Then $V \subset Y$. Then for V is affine, $s \in \check{\mathscr{E}}(V) = Hom(\mathscr{E}|_V, \mathcal{O}_V) = Hom_A(A, \Gamma(V, \mathscr{E}|_V)) = Hom_A(A, \mathscr{E}(V))$ by 2.5.3.

Then given any $s \in \check{\mathscr{E}}(V)$, we have an ring homomorphism $S(\mathscr{E}(V)) \longrightarrow A$ given by $s \in Hom_{ring}(A, \mathscr{E}(V))$ and this morphism uniquely gives $V = Spec(A) \longrightarrow Spec(\mathscr{E}(V)) \in \mathscr{S}(V)$.

Conversely, given any $s : Spec(A) \longrightarrow Spec(S(\mathscr{E}(V))) \in \mathscr{S}(V)$, it uniquely given a ring homomorphism $S(\mathscr{E}(V)) \longrightarrow A$. Restrict the morphism on $\mathscr{E}(V)$, we get a $A - mod$ homomorphism $\mathscr{E}(V) \longrightarrow A$.

Then, we can check that this corresponding gives an isomorphism between $\check{\mathscr{E}}(V)$ and $\mathscr{S}(V)$ which is compatible with the **gluing functions** between this two sheaves. Hence $\mathscr{L} \cong \check{\mathscr{E}}$.

(d). We can see that $X = V(\mathscr{E}) \mapsto \mathscr{S}(X/Y) = \check{\mathscr{E}}$ and $\mathscr{E} \longrightarrow V(\check{\mathscr{E}})$ gives the equivalence.

□

2.5.4 Additional Exercises

Exercise 2.5.19. Let X be the affine scheme $\text{Spec}(A)$ and $\mathcal{F}$ a quasi-coherent sheaf on X. Prove the following assertions:

(i) Assume $0 = s|_{D(a)} \in \Gamma(D(a), \mathcal{F})$ for some $s \in \Gamma(X, \mathcal{F})$ and $a \in A$. Then there exists $n > 0$ such that $a^n \cdot s = 0$ in $\Gamma(X, \mathcal{F})$.

(ii) Let $t \in \Gamma(D(a), \mathcal{F})$. Then there exists $n > 0$ and $s \in \Gamma(X, \mathcal{F})$ such that $s|_{D(a)} = a^n \cdot t$ in $\Gamma(D(a), \mathcal{F})$.

Proof. This is just to show that $\mathcal{F} = \widetilde{M}$ with M a A-module and $\mathcal{F}|_{D(a)} = \widetilde{M}_a$.

For $\mathcal{F}$ is quasi-finite and X is quasi-compact, we can cover X with finitely many $\{U_i = Spec(A_{a_i})\}$ with $\mathcal{F}|_{U_i} = \widetilde{M_i}$ with M_i A_{a_i}-module.

(i). $D(a)$ is covered by $D(aa_i)$. $s|_{D(a)} = 0$ is equivalent to $s|_{D(aa_i)} = 0$ for each i. There exists n_i such that $(aa_i)^{n_i}s = 0$. For $D(a_i)$ cover $Spec(A)$, there exists b_i such that $\sum b_i a_i^{n_i} = 1$ and $s = \sum b_i a_i^{n_i} s$. Take $N = max\{n_i\}$. Then $a^N s = 0$.

(ii). For $t \in \Gamma(D(a), \mathcal{F})$, then $\frac{t}{1} \in \Gamma(D(aa_i), \mathcal{F}|_{D(aa_i)})$, that is, $(aa_i)^{n_i}t = m_i$ for some $m_i \in M_i$ such that m_i and m_j agree on $D(aa_ia_j)$. Now we need to glue m_i together: For m_i and m_j agree on $D(aa_ia_j)$, there exists, n_{ij} such that $(aa_ia_j)^{n_{ij}}(m_i - m_j) = 0$. We can see that $a^{n_{ij}}m_i$ and $a^{n_{ij}}m_j$ agree on $D(a_ia_j)$. Thus, we can take $N_1 = max\{n_{ij}\}$, $\{a^{N_1}m_i\}$ forms a global section of X, denoted by s. Note that $s = a^{N_1+n_i}a_i^{n_i}t$. Thus, we can find $s|_{D(a)} = a^N t$ for some N. (Use $\cup D((aa_i)^{n_i}) = D(a)$ to show this.)

□

Exercise 2.5.20. Let (X, O_X) be a ringed space and consider $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(X, O_X)$. Show that for all $x \in X$ there exists a natural isomorphism

$$(\mathcal{F} \otimes_{O_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{O_{X,x}} \mathcal{G}_x.$$

Prove that $\mathcal{F} \otimes_{O_X} (\cdot) : \mathrm{Mod}(X, O_X) \to \mathrm{Mod}(X, O_X)$ and $f^* : \mathrm{Mod}(Y, O_Y) \to \mathrm{Mod}(X, O_X)$ for a morphism of ringed spaces $f : (X, O_X) \to (Y, O_Y)$ are both right exact functors. Describe examples showing that in general they are not left exact.

Proof. Because $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Hence $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_P = (\mathcal{F} \otimes^{pr}_{\mathcal{O}_X} \mathcal{G}) = \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{G}_P$.

Because $\mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} (-)$ is left-exact in $\mathcal{O}_{X,P}$-module, $\mathcal{F} \otimes_{\mathcal{O}_X} (-)$ is right exact.

For $f^* = f^{-1} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, because f^{-1} is exact and $(-) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ is right-exact, f^* is right-exact.

To show that $\mathcal{F} \otimes_{\mathcal{O}_X} (-)$ is not left-exact, just consider the morphism of $\underline{\mathbb{Z}}$-module.

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2} \underline{\mathbb{Z}} \xrightarrow{p} \underline{\mathbb{Z}/2\mathbb{Z}} \longrightarrow 0$$

Tensoring with $\underline{\mathbb{Z}/2\mathbb{Z}}$, we get:

$$0 \longrightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\cdot 2} \underline{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{p} \underline{\mathbb{Z}/2\mathbb{Z}} \longrightarrow 0$$

$\cdot 2$ is not injective, hence this sequence is not exact.

Similarly, we can show that f^* is also not left-exact. □

Exercise 2.5.21. Is f_* or f^* compatible with tensor products?

Proof. f^* is compatible with tensor products:

$$f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G})_P = \mathcal{F}_P \otimes_{\mathcal{O}_{Y,P}} \mathcal{G}_{f(P)} \otimes_{\mathcal{O}_{Y,P}} \mathcal{O}_{X,P}$$

$$(f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_P = \mathcal{F}_{f(P)} \otimes_{\mathcal{O}_{Y,f(P)}} \mathcal{O}_{X,P} \otimes_{\mathcal{O}_{XP}} \mathcal{G}_{f(P)} \otimes_{\mathcal{O}_{Y,f(P)}} \mathcal{O}_{X,P}$$

Hence $f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G})_P = (f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})_P$. So $f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$.

f_* is not compatible wit tensor products in general: Note that $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Z} \neq \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}$. Take $X = (\mathbb{R}, \underline{\mathbb{Z}})$ and $Y = (\mathbb{R}, \underline{\mathbb{Q}})$ with $(f, f^\#)$ is defined to be $f = id$ and $f^\#(U) : \underline{\mathbb{Z}}(U) \hookrightarrow \underline{\mathbb{Q}}(U)$. □

Exercise 2.5.22. Consider the standard open set $Spec(k[x]) = D_+(x_0) \subset \mathbb{P}^1_k = Proj(k[x_0, x_1])$, where $x = \frac{x_1}{x_0}$. Let $\mathcal{F} = \widetilde{M}$ be the coherent sheaf on $D_+(x_0)$ corresponding to the $k[x]$-module $M = k[x]/(x^2 - 1)$. Describe a coherent extension of $\mathcal{F}$ to $\mathbb{P}^1_k$, that is, a coherent sheaf $\mathcal{G}$ on $\mathbb{P})k^1$ with $\mathcal{G}|_{D_+(x_0)} = \mathcal{F}$, in terms of a $k[x_0, x_1]$-module. Is the extension unique?

Proof.

Note that $M = k[x]/(x^2 - 1) = k[\frac{x_1}{x_0}]/(\frac{x_1^2}{x_0^2} - 1)$. Hence we can view M as $\widetilde{N}|_{D_+(x_0)}$ with $N = k[x_0, x_1]/(x_1^2 - x_0^2)$.

Note that we have

$$0 \longrightarrow S(-2) \xrightarrow{\cdot(x_1^2 - x_0^2)} S \longrightarrow N \longrightarrow 0$$

then we have:

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\cdot(x_1^2 - x_0^2)} \mathcal{O} \longrightarrow \widetilde{N} \longrightarrow 0$$

Hence $coker(\cdot(x_1^2 - x_0^2)) = \widetilde{N}$, which implies $\widetilde{N}$ is coherent.

Let $x = [0 : 1]$. Consider $i_{x,*}(k(x))$ which is coherent over $\mathbb{P}^1_k$ and $i_{x,*}(k(x))|_{D_+(x_0)} = 0$. Hence $\widetilde{N} \oplus i_{x,*}(k(x))$ is coherent on $\mathbb{P}^1_k$ and $\widetilde{N} \oplus i_{x,*}(k(x))|_{D_+(x_0)} = \mathcal{F}$. □

Exercise 2.5.23. Consider the standard standard open set $D(x) \subset \mathbb{A}^1_k$ and the coherent sheaf $\mathcal{F} = \widetilde{M}$, where M is the $k[x, x^{-1}]$-module $k[x]/(x - 1) \oplus k[x, x^{-1}]$. Describe a coherent extension of $\mathcal{F}$ to $\mathbb{A}^1_k$, i.e. a coherent sheaf $\mathcal{G}$ on $\mathbb{A}^1_k$ with $\mathcal{G}|_{D(x)} \cong \mathcal{F}$. Is the extension unique?

Proof. Note that every coherent sheaf $\mathcal{G}$ on $\mathbb{A}^1_k$ can be expressed as $\widetilde{N}$ with N a finitely generated $k[x]$-module and then $\widetilde{N}|_{D(x)} = \widetilde{N_x}$.

Note that $(k[x]/(x-1))_x = k[x]/(x-1)$ for x is unit in $k[x]/(x-1)$ and $k[x]_x = k[x, x^{-1}]$. So let $N = k[x]/(x-1) \oplus k[x]$, which is a finitely generated $k[x]$-module and $\widetilde{N}|_{D(x)} = \mathcal{F}$ with $\widetilde{N}$ coherent.

Note that $i_{(x),*}(k((x)))$ is a coherent sheaf over $\mathbb{A}^1_k$ and $i_{(x),*}k((x))|_{D(x)} = 0$. So $N \oplus i_{(x),*}(k(x))$ is coherent and $N \oplus i_{(x),*}(k(x))^{\sim}|_{D(x)} = \mathcal{F}$. Hence, the extension is not unique. □

Exercise 2.5.24. Let $f : X \longrightarrow Y$ be a morphism of schemes. Show that f is separated if and only if for every Y-scheme Z, every right-inverse of the base change $f_Z : X \times_Y Z \longrightarrow Z$ is a closed immersion.

Proof. ($\Longrightarrow$): For any $g : Z \longrightarrow X \times_Y Z$ such that $f_Z \circ g = id_Z$, g is a closed immersion.

By Magic square, we have the pullback

$$\begin{array}{ccc} X \times_Y Y & \longrightarrow & X \times_Y X \\ \downarrow & & \downarrow \\ Z \times_Y Y & \longrightarrow & Z \times_Y X \end{array}$$

$X = X \times_Y Y \longrightarrow X \times_Y X$ is a closed immersion.

($\Longleftarrow$): Just take $Z = X$. □

Exercise 2.5.25. Let X ne a scheme and let $\{\mathcal{F}_i\}$ be a family of quasi-coherent subsheaves $\mathcal{F}_i$ of a quasi-coherent sheaf $\mathcal{F}$ on X.

(i). Let $\sum_{i \in I} \mathcal{F}_i$ be the sheafification of the presheaf

$$U \mapsto \sum_{i \in I} \mathcal{F}_i(U) \subset \mathcal{F}(U)$$

Show that $\sum_{i \in I} \mathcal{F}_i$ is quasi-coherent.

(ii). Let $\cap_{i\in I}\mathcal{F}_i$ be the presheaf

$$U \mapsto \cap_{i\in I}\mathcal{F}_i(U) \subset \mathcal{F}(U)$$

Show that $\cap_{i\in I}\mathcal{F}_i$ is a sheaf.

(iii). Show that if I is finite, then $\cap_{i\in I}\mathcal{F}$ is a quasi-coherent sheaf.

(iv). Let $X = Spec(k[x])$. Show that the sheaf $\cap_{n=0}^{\infty}\widetilde{(x^n)}$ is not quasi-coherent.

Proof.

(i). As presheaf $\sum_{i\in I}^{pr}\mathcal{F}_i$ is the image of the natural map between presheaves $\oplus_{i\in I}\mathcal{F}_i \longrightarrow \mathcal{F}$. Thus, $\sum_{i\in I}^{pr}\mathcal{F}_i = im(\oplus\mathcal{F}_i \longrightarrow \mathcal{F}$ is quasi-coherent for $QCoh(X)$ is an abelian category (which implies $\oplus\mathcal{F}_i$ is quasi-coherent and $im = ker(coker)$ is also quasi-coherent.)

(ii). Suppose that $U = U_1 \cup U_2$. Take $s_j \in \cap_{i\in I}\mathcal{F}_i(U_j)$ and assume that $s_1|_{U_1\cap U_2} = s_2|_{U_1\cap U_2}$. Then for each $\mathcal{F}_i$ is a sheaf, s_1 and s_2 can be glued to be a section in $\mathcal{F}_i(U_1 \cup U_2)$ for each i. Hence s_1 and s_2 can be glued to be a section in $\cap\mathcal{F}_i$. Hence $\cap\mathcal{F}_i$ is a sheaf.

(iii). Consider $\mathcal{F} \longrightarrow \Pi\mathcal{F}/\mathcal{F}_i$ which has kernel $\cap_{i\in I}\mathcal{F}_i$. Because $\mathcal{F}/\mathcal{F}_\rangle = im(\mathcal{F}_i \longrightarrow \mathcal{F})$ and both $\mathcal{F}_i$ and $\mathcal{F}$ are quasi-coherent, $\mathcal{F}/\mathcal{F}_\rangle$ is also quasi-coherent for each $i \in I$.

For $QCoh(X)$ is an abelian category, finite product coincides with finite sum. Hence $\Pi\mathcal{F}/\mathcal{F}_\rangle$ is quasi-coherent for I is finite. So as the kernel of a morphism between two quasi-coherent sheaves, $\cap\mathcal{F}_i$ is quasi-coherent.

(iv). Denote $\cap\widetilde{(x^n)}$ by $\mathcal{G}$. For $\mathcal{O}(X) \longrightarrow \Pi\mathcal{O}/\widetilde{(x^n)}(X)$ is just $k[x] \longrightarrow \Pi k[x]/(x^n)$, $\mathcal{G}(X) = 0$. Hence, if $\mathcal{G}$ is quasi-coherent, then $\mathcal{G} = 0$. Now, consider $\mathcal{O}(D(x)) = k[x, x^{-1}]$. However, $\mathcal{O}/(\widetilde{(x^n)})(D(x)) = k[x]/(x^n)^{\sim}(D(x)) = 0$. That's $\cap\widetilde{(x^n)}(D(x)) = k[x, x^{-1}]$. Hence $\cap\widetilde{(x^n)}$ is not quasi-coherent. $\square$

Remark. Here, we see that $\Pi_{i\in I}\mathcal{F}_i$ may not be quasi-coherent.

Exercise 2.5.26. Let X be a Noetherian scheme and let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of coherent $\mathcal{O}_x$-modules. Show that

$$U := \{x \in X | \varphi_x \text{ is injective }\}$$

is open in X, where $\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ denotes the induced morphism on stalks for $x \in X$.

Proof. For X is Noetherian, $Coh(X)$ is an abelian category. Hence, $ker(\varphi)$ is a coherent sheaf. Note that

$$ker(\varphi)_0 := \{x \in X | ker(\varphi)_x = 0\} = \{x \in X | \varphi_x \text{ is injective }\}$$

So it is enough to show that $ker(\varphi)_0$ is open. Consider $\mathfrak{p} \in Spec(A) \subset ker(\varphi)_0$. Then there exists a finitely generated A-module M such that $ker(\varphi)|_{Spec(A)} = \widetilde{M}$ and $M_\mathfrak{p} = 0$. Suppose that M is generated by $s_0, ..., s_n$. Then for each s_i, there exists $a_i \in A - \mathfrak{p}$ such that $a_i s_i = 0$. Take $D_+(a_1...a_n) \subset Spec(A)$. Then $\widetilde{M}|_{D(a_1...a_n)} = M_{a_1...a_n}{}^{\sim} = 0$ for $a_1...a_n s_i = 0$. Note that $\mathfrak{p} \in D(a_1...a_n)$ and $D(a_1...a_n)$ is open. We see that $ker(\varphi)_0$ is open. $\square$

Exercise 2.5.27. Let $U = \mathbb{A}_k^2 - \{0\}$ and write $j : U \longrightarrow \mathbb{A}_k^2 = X$ for the canonical embedding and $p : U \longrightarrow \mathbb{P}_k^1$ for the canonical projection onto the projective line.

(i). Show that $p^*\mathcal{O}(1) \cong \mathcal{O}_U$.

(ii). Show that $j_*\mathcal{O}_U = \mathcal{O}_X$.

(iii). Let $\varphi : \mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1} \longrightarrow \mathcal{O}(1)$ be the canonical surjection onto the twisting sheaf $\mathcal{O}(1)$. Show that the induced map

$$j_*p^*\varphi : \mathcal{O}_X \cong j_*p^*\mathcal{O}_{\mathbb{P}_k^1}^2 \longrightarrow j_*p^*\mathcal{O}(1) \cong \mathcal{O}_X$$

has a non-trivial cokernel.

Proof.

(i). For the pullback of a line bundle is also a line bundle, $p^*\mathcal{O}(1)$ is still a line bundle. As we have shown, U is intergal and hence $Pic(U) = Cl(U) = 0$. Thus, $p^*\mathcal{O}(1) \cong \mathcal{O}_U$.

(ii). For U can be cover by $Spec(k[x, y, y^{-1}])$ and $Spec(k[x, x^{-1}, y])$, U is Noetherian. Hence, $j_*\mathcal{O}_U$ is quasi-coherent over X. So it is enough to show that $\Gamma(X, j_*\mathcal{O}_U) = \Gamma(U, \mathcal{O}_U) = k[x, y]$, for $QCoh(Spec(A)) = Mod(A)$.

Again, note that U can be cover by $Spec(k[x, y, y^{-1}])$ and $Spec(k[x, x^{-1}, y])$, so $\Gamma(U, \mathcal{O}_U) = k[x, y]$. So $j_*\mathcal{O}_U \cong \mathcal{O}_X$.

(iii). Note that the φ is given by

$$\Phi : e_1 k[y_1, y_2] \oplus e_2 k[z_1, z_2] \longrightarrow (x_1, x_2); \quad e_1 \mapsto x_1, e_2 \mapsto x_2$$

which is an morphism preserving the degree between two graded ring.

Consider $\varphi|_{D_+((x_1,x_2))}$ which givens

$$\Phi' : e_1 k[y_1, y_2]_{(y_1,y_2)} \oplus e_2 k[z_1, z_2]_{(z_1,z_2)} \longrightarrow S(1)_{(x_1,x_2)}$$

is not a surjection: there is no element mapped to $x_1, x_2 \in S(1)_{(x_1,x_2)}$. For $coker(j_*p^*\varphi)$ is quasi-coherent hence it is $\widetilde{M}$ for some M. If $coker(j_*p^*\varphi) = 0$, then $M = 0$ and finally, $\widetilde{M}|_U = coker(\Phi') = 0$, which leads to a contradiction. $\square$

2.6 Divisors

($*$) X is a Noetherian integral separated scheme which is regular of codimension 1, that is, every local ring $\mathcal{O}_{X,P}$ of dimension 1 is regular.

Example 2.6.1. $\mathbb{A}_k^n$ satisfies ($*$) for any $n \in \mathbb{N}$.

Proof. For $\mathbb{A}_k^n = Spec(k[x_1, x_2, ..., x_n])$, $\mathbb{A}_k^n$ is a Noetherian integral separated scheme. Note that

$$\{\text{prime ideals in } A_{\mathfrak{p}}\} \xrightarrow{1:1} \{\text{prime ideals } \mathfrak{q} \subset \mathfrak{p}\}$$

For $k[x_1, x_2, ..., x_n]$ is factorial, if $ht(\mathfrak{p}) = 1$, $\mathfrak{p} = (f)$ with f irreducible. So if $dim(\mathcal{O}_{X,P}) = 1$, $\mathcal{O}_{X,P} = A_{(f)}$ with f irreducible. $\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2 = (f)A_{(f)}/(fA_{(f)})^2 = kf$. Hence $dim_k \mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2 = 1$, which implies $\mathbb{A}_k^n$ is regular of codimension 1. □

Example 2.6.2. For 'being regular of codimension 1' is a local property and $\mathbb{P}_k^n$ is covered by some $\mathbb{A}_k^n$, $\mathbb{P}_k^n$ is regular of codimension 1. Because $\mathbb{P}_k^n$ is a Noetherian integral separated scheme, $\mathbb{P}_k^n$ also satisfies ($*$).

We first need to introduce the following propery:

Proposition 2.6.3. *Let X be an integral scheme and U be an open subset of X, then there is a corresponding between the set of closed irreducible subspaces of U and those of X via*

$$Z \mapsto \bar{Z}$$

Proof. If Z is a closed irreducible subset of X, then if $Z \cap U = (Z_1 \cap U) \cup (Z_2 \cap U)$ with Z_1 and Z_2 are closed. Then $Y = Y_1 \cup Y_2 \cup U^c$, which leads to a contradiction.

If $\bar{Z} = Z_1 \cup Z_2$ with Z_1 and Z_2 proper closed subset of Z , then $Z = (Z_1 \cap Z) \cup (Z_2 \cup Z)$. Note that if $Z_1 \cap Z = \emptyset$, then $Z \subset Z_2$ and then $\bar{Z} \subset Z_2$, which leads to a contradiction. □

Corollary 2.6.4. *For any $Y \subset X$ closed,*

$$codim(Y, X) = codim(Y \cap U, U)$$

By **??** (b)-1, let y be the generic point of Y and suppose $y \in Spec(A)$. Then $Y \cap A = Spec(A/I)$ for some I. Because Y is integral, $Y \cap A = Spec(A/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Note that at this case, the generic point of Y, (0) corresponds to $y = \mathfrak{p} \in A$. For Y is of codimension 1, there is on non-trivial prime ideal $\mathfrak{q} \subset A$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Thus, $\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$ has only one non-trivial prime ideal $\mathfrak{p}A_{\mathfrak{p}}$.

2.6.1 Preparation

A Weil Divisor

Theorem 2.6.5. *Let X satisfy ($*$), and let $f \in K^*$ be a nonzero function on X. Then $v_Y(f) = 0$ for all except finitely many prime divisors.*

Proof. Suppose that $f = g/h \in K(X) = Q(\mathcal{O}(U))$ for some non-empty affine open subset $U \cong Spec(A) \subset X$ and $g, h \in \mathcal{O}(U)$. By passing to $U \cap X_h$, we can assume that $f \in \mathcal{O}(U) \cong A$. Now, $Z = X - U$ is closed subset of X. Since X is Noetherian, Z contains at most finitely irreducible components, and hence finitely many prime divisors of X.

Now, it suffices to show that, there are only finitely many prime divisors $Y \cap U \neq \emptyset$ such that $v_Y(f) \neq 0$. Since $f \in \mathcal{O}(U) \cong A$, $v_Y(f) \geq 0$ for each Y. For each $Y \cap U = Spec(A/\mathfrak{p})$, v_Y is the valuation associated to $A_{\mathfrak{p}}$ with $ht(\mathfrak{p}) = 1$. So for a prime divisor, $v_Y(f) > 0$ if and only if $f \in \mathfrak{p}A_{\mathfrak{p}}$ if and only if $f \in \mathfrak{p}$ if and only if $Y \subset V(f)$. Since $V(f)$ is closed, it contains at most finitely irreducible components, that is, only finitely many Y such that $v_Y(f) > 0$. □

Example 2.6.6. Consider $X = \mathbb{A}^2_k$ and $Y_i = V(x_i) = Spec(k[x_1, x_2]/(x_i))$. For $f = \frac{x_1}{x_2}$. $f \in (x_1, x_2)k[x_1, x_2]_{(x_1)}$ so $v_{Y_1}(f) = 1$ and $\frac{1}{f} \in (x_1, x_2)k[x_1, x_2]_{(x_2)}$ so $v_{Y_2}(f) = -1$.

Thus, $(f) = Y_1 - Y_2$.

Theorem 2.6.7. *Let X satisfying (*), let Z be a proper closed subset of X, and let $U = X - Z$. Then:*

(a) there is a surjective homomorphism $Cl(X) \longrightarrow Cl(U)$ defined by $D = \sum n_i Y_i \mapsto \sum n_i(Y_i \cap U)$;

(b) if $codim(Z, X) \geq 2$, then $Cl(X) \longrightarrow Cl(U)$ is an isomorphism;

*(c) if Z is an **irreducible** subset of codim 1, then there exists an exact sequence*

$$\mathbb{Z} \longrightarrow Cl(X) \longrightarrow Cl(U) \longrightarrow 0$$

where the first map is given by $1 \longrightarrow Z$.

Proof.

(a). For U is an open subset of X, U also satisfies X. Hence, we only need to show that if $Y \cap U \neq \emptyset$, $Y \cap U$ is a prime divisor of U. By computing the stalks of $\mathcal{O}_{Y,p}$ with $p \in U$. We see that $Y \cap U$ is integral. For $codim(Y \cap U, U) = codim(Y, X) = 1$, $Y \cap U$ is a prime divisor of U.

For v_Y and $v_{Y \cap U}$, if $y \in X$ is a generic point of Y. If $y \notin U$, then $Y = \{\bar{y}\} \subset U^C$, that is, $Y \cap U = \emptyset$. So if $Y \cap U \neq \emptyset$, $y \in U$ and then $\mathcal{O}_{U,y} = \mathcal{O}_{X,y}$. Thus, $v_Y = v_{Y \cap U}$ if $Y \cap U \neq \emptyset$.

(b). Any rational function f on X is of the form $\frac{g}{h}$ with g, h homogeneous of the same degree. Thus, $(f) = (g) - (h) = 0$.

(c). Directly from (a) and (c). □

Theorem 2.6.8. *Let X satisfy $(*)$. Then $X \times_k \mathbb{A}^1_k$ also satisfies $(*)$ and $Cl(X) \cong Cl(X \times_k \mathbb{A}^1_k)$*

Let's compute $Cl(X)$ for some X.

Example 2.6.9.

(i). For $\mathbb{A}^1_k$ with k algebraically closed, $Cl(\mathbb{A}^1_k) = 0$. Just note that prime divisors of $\mathbb{A}^1_k$ are all closed points $(\lambda) \in \mathbb{A}^1_k$.

Note that for (λ), its corresponding valuation ring is $k[x]_{(x-\lambda)}$ with $v_\lambda(x - \lambda) = 1$. Thus, $(\lambda) = (x - \lambda)$ so it is principal.

For general k, consider $\mathbb{A}^1_k = Spec(k[x])$ any closed subscheme Y with $codim(Y) = 1$ of $\mathbb{A}^n_k$ has the form $V_+((f))$ for some irreducible (f), for $k[x]$ is factorial. Thus, $Cl(\mathbb{A}^1_k) = 0$.

Note for $\mathbb{A}^n_k$ satisfies $(*)$, $Cl(A^n_k) = Cl(\mathbb{A}^{n-1}_k \times_k \mathbb{A}^1_k) = Cl(\mathbb{A}^{n-1}_k)$. So by induction $\mathbb{A}^n_k = 0$ (Also, $Cl(\mathbb{A}^n_k)$ can be directly computed for $k[x_1, ..., x_n]$ is also factorial.)

(ii). For $\mathbb{P}^1_k$ with k algebraically closed, $Cl(\mathbb{P}^1_k) = \mathbb{Z}$. Note that the prime divisors of $\mathbb{P}^1_k$ are all closed points (λ_0, λ_1).

Suppose that $\lambda_0, \lambda_1 \neq 0$. For (λ_0, λ_1)'s corresponding valuation ring is $(\frac{x_0}{x_1} - \frac{\lambda_0}{\lambda_1})k[\frac{x_0}{x_1}]_{(\frac{x_0}{x_1} - \frac{\lambda_0}{\lambda_1})}$. For another $(\lambda_1^{'}, \lambda_2^{'})$ we do the same assumption, then for

$$f = \frac{\frac{x_0}{x_1} - \frac{\lambda_0}{\lambda_1}}{\frac{x_0}{x_1} - \frac{\lambda_0^{'}}{\lambda_1^{'}}} = \frac{\lambda_1 x_0 - \lambda_0 x_1}{x_0 \lambda_1^{'} - \lambda_0^{'} x_1} \frac{\lambda_1^{'}}{\lambda_1} \in K(\mathbb{P}^1_k)$$

and $(f) = (\lambda_0, \lambda_1) - (\lambda_0^{'}, \lambda_1^{'})$. If $\lambda_1^{'} = 0$, $\lambda_0^{'} \neq 0$. We can use the same method to show that $(\lambda_0^{'}, \lambda_1^{'}) \sim (\lambda_0, \lambda_1)$.

So any $(\lambda_1^{'}, \lambda_2^{'}) \sim (\lambda_1, \lambda_2)$. Thus, $Cl(\mathbb{P}_k^1) = \mathbb{Z}$.

(iii). For any $Z \subset X$ with Z is a finite union closed points. For Z doesn't contain any subscheme of codimension 1, $Cl(Z) = 0$. For example, $Z = V_+(x_0^2 + x_1^2) \subset \mathbb{P}_k^1$ with k algebraically closed.

(iv). For any open subsecheme $U \subset \mathbb{A}_k^n$, $Cl(U) = 0$. This is just because $Cl(\mathbb{A}_k^n) \longrightarrow Cl(U)$ is a surjection and $Cl(\mathbb{A}_k^n) = 0$. For example, $Cl(\mathbb{A}_k^n - \{(0)\}) = 0$.

B Cartier Dvisior

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{O}_X^*/\mathcal{K}_X^* \longrightarrow 0$$

we have the exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow \Gamma(X, \mathcal{K}_X^*) \xrightarrow{p} \Gamma(X, \mathcal{O}_X^*/\mathcal{K}_X^*)$$

Define the Cartier divisor to be

$$CoCl(X) = \Gamma(X, \mathcal{O}_X^*/\mathcal{K}_X^*)/Im(p)$$

Thus, for any $s \in CoCl(X)$, we can represented it by

$$\{U_i, f_i \in \mathcal{K}_X^*(U_i), \frac{f_i}{f_j} \in \mathcal{O}_X^*(U_{ij})\}$$

by the left-exactness of $\Gamma(X, -)$. (To satisfy the gluing condition, $p(\frac{f_i}{f_j}) = 1$ on U_{ij}. Then $\frac{f_i}{f_j} \in \mathcal{O}_X(U_{ij})$)

C Picard Group

By **??**, we see that

$$Pic(X) = \check{H}^1(X, \mathcal{O}_X^*) = H^1(X, \mathcal{O}_X^*)$$

D Relation between Cartier Divisor and Picard Group

Given any $D := f = \{U_i, f_i \in \mathcal{K}_X^*(U_i), \frac{f_i}{f_j} \in \mathcal{O}_X^*(U_{ij})\} \in CaCl(X)$, define $f\mathcal{O}_X$ by

$$\mathcal{L}(\mathcal{D})|_{U_i} := f_i^{-1}\mathcal{O}_X|_{U_i}$$

and gluing functions $\varphi_{ij} = f_i/f_j \in \mathcal{O}_X^*(U_{ij})$. This is a sheaf 2.1.22. And it is isomorphic to $\mathcal{O}_X$ as $\mathcal{O}_X$-module for $f_i \in \mathcal{K}_X^*(U_i)$.

Theorem 2.6.10. *Let X be a scheme. Then:*

(a) for any Cartier divisor D, $\mathcal{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathcal{L}(D)$ gives a $1-1$ corresponding between Cartier divisors on X and invertible subsheaves of $\mathcal{K}$;

(b) $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$;

(c) $D_1 \sim D_2$ if and only if $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$.

Proof.
(a). As we stated before, $\mathcal{L}(D)$ is a line bundle.

(d). Use the construction of $\mathcal{L}(D)$.

(c). ($\Longrightarrow$): Trivial.
($\Longleftarrow$):Note that if $f^{'}$ defines the same line bundle, then $f_i^{'}/f_i \in \mathcal{O}^*(U_i)$ and $f_i^{'}/f_i \times f_j/f_j^{'} = 1$. Take $g_i = f_i^{'}/f_i$. Then $g_i/g_j = f_i^{'}/f_i \times f_j/f_j^{'} = 1$. Thus, $g := \cup g_i \in \Gamma(X, \mathcal{O}^*)$. So $f - f^{'} = (g)$. Thus, $f \sim f^{'}$.
□

So we have a well-define morphsim

$$CaCl(X) \longrightarrow Pic(X)$$

Theorem 2.6.11. *When X is integral, then*

$$CaCl(X) = Pic(X)$$

Proof. For $\mathcal{K}_X^*$ is a constant sheaf on X and X is irreducible, $\mathcal{K}_X^*$ is flasque. Hence $H^1(X, \mathcal{K}_X^*) = 0$ Using the long exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow \Gamma(X, \mathcal{K}_X^*) \xrightarrow{p} \Gamma(X, \mathcal{O}_X^*/\mathcal{K}_X^*) \longrightarrow Pic(X) \longrightarrow H^1(X, \mathcal{K}_X^*)$$

We see that $CaCl(X) = Pic(X)$. □

Definition 2.6.12. A Cartier divisor on a scheme is effective if it can be represented by $\{U_i, f_i\}$ with $f_i \in \Gamma(U_i, \mathcal{O}_X|_{U_i})$.

In this case, we can define the associated subscheme of *codim* 1, Y, to be the closed subscheme defined by the sheaf of ideals which is locally generated by f_i, that is, $\mathcal{I}|_{U_i} = f_i\mathcal{O}|_{U_i}$, which is a quasi-coherent ideal of $\mathcal{O}_X$ and then $Y = supp(\mathcal{O}_X/\mathcal{I})$.
Intuitively, locally $Y|_{U_i} = \{f_i = 0\}$.

Theorem 2.6.13. *Let D be an effective Cartier divisor on a scheme X, and let Y be the associated locally principal closed subscheme. Then $\mathcal{I}_Y \cong \mathcal{L}(-D)$.*

Proof. By definition $\mathcal{L}(-D)|_{U_i} = f_i\mathcal{O}_X|_{U_i}$ which coincide the ideal defined by $\mathcal{I}_Y$. □

Finally we have an exact sequence

$$0 \longrightarrow \mathcal{L}(-D) \longrightarrow \mathcal{O}_X \longrightarrow i_{Y,*}\mathcal{O}_Y \longrightarrow 0$$

Remark (Geometric Meaning of $\mathcal{L}(-D)$)**.** As we have shown in 2.1.21 (a), $\mathcal{I}_Y(U)$ represents the regular functions vanishing at all points in $Y \cap U$. So $\mathcal{L}(-D)$ represents the regular functions vanishing at all points at all in the closed subseheme defined by D. For example, later, we will see that over a curve X, $\mathcal{L}(D - P)$ represents the regular functions vanishing at D with a pole at P.

E Relation between Cartier Divisors and Weil Divisors

Where does the valuation of prime divisor Y comes from?
See **??** (a).
Given any $f \in CaCl(X)$, for any prime divisor Y, $v_Y(\frac{f_i}{f_j}) = 0$ because $\frac{f_i}{f_j} \in \mathcal{O}_X^*(U_{ij})$. So we can define $v_Y(f) := v_{Y \cap U_i}(f_i)$ for some U_i, which is well-defined. And note that if $f \sim f^{'}$, $f_i/f_i^{'} \in \mathcal{O}_X^*(U_i)$. So $v_Y : CaCl(X) \longrightarrow \mathbb{Z}$ is well-defined.

Lemma 2.6.14. *If A is a factorial integral domain, then $Cl(A) := Cl(Spec(A)) = 0$.*

Proof. Just note that for any closed subscheme of codimension 1 Y of $Spec(A)$. $Y = V(\mathfrak{p})$. For A is a factorial integral domain, $\mathfrak{p} = (f)$. Then $Y = (f)$. $\square$

Remark. For A is factorial, if $ht(\mathfrak{p}) = 1$, then $\mathfrak{p} = (f)$:

Supppse $\mathfrak{p}$ is of height 1, and let $h \in \mathfrak{p}$ and $f|h$ with f irreducible. Then $0 \subsetneq (f) \subset \mathfrak{p}$. Thus, $\mathfrak{p} = (f)$.

Theorem 2.6.15. *Then map $CaCl(X) \longrightarrow Cl(X)$ given by $f \longrightarrow \sum v_Y(f)Y$ is injective and is an isomorphism if X is factorial.*

Proof. For X is factorial, w.l.o.g. we assume that X is covered by $\{U_i = Spec(A_i)\}$ with A_i factorial.

Given any $D = \sum n_Y Y$, consider $D_{U_i} = \sum n_Y Y \cap U_i$. For A_i is factorial, $Cl(Spec(A_i)) = 0$. Thus, $D_{U_i} = (f_i)$ for some $f_i \in K(A_i) = \mathcal{K}_X(U_i)$. Now, we show that $\{U_i, f_i\}$ forms a Cartier divisor:

(a) For A_i is an integral domain, $f_i \in \mathcal{K}_X^*(U_i)$.

(b) For $D_{U_i}|_{U_i \cap U_j} - D_{U_i}|_{U_i \cap U_j} = 0$, $(f_i)|_{U_{ij}} - (f_j)|_{U_{ij}} = 0$, that is, $(f_i/f_j)|_{U_{ij}} = 0$. So for any $Y \cap U_{ij} \neq \emptyset$, $v_{Y \cap U_{ij}}(f_i/f_j) = 0$, which implies $f_i/f_j \in \mathcal{O}_X^*(U_{ij})$ for U_{ij} is still affine and factorial and $A_{ij}^* = A_{ij} - \cap_{ht(\mathfrak{p})=1}\mathfrak{p}$.

So $f \mapsto \sum v_Y(f)Y$ is a surjection.

For injectivity, if $v_{Y \cap U_i}(f_i^{'}) = v_{Y \cap U_i}(f_i)$ for each prime divisor Y. Then $v_{Y \cap U_i}(f_i^{'}/f_i) = 0$ for each Y_i. Using the same argument, $A_{ij}^* = A_{ij} - \cap_{ht(\mathfrak{p})=1}\mathfrak{p}$, $f_i^{'}/f_i \in \mathcal{O}^*(U_i)$. Thus, $f^{'} \sim f$. $\square$

Proposition 2.6.16. *Let A be a Noetherian domain. Then A is a unique factorization domain if and only if $X = Spec(A)$ is normal and $Cl(X) = 0$.*

F Examples

Proposition 2.6.17. *Let X be $\mathbb{P}_k^n$ over a field k. For any divisor $D = \sum n_i Y$, define the degree of $D = \sum n_i deg(Y_i)$, where $deg(Y_i)$ is the degree of hypersurface Y_i. Let H be the hyperplane $x_0 = 0$. Then:*

(a) If D is a divisor of degree d, then $D \sim dH$;

(b) for any $f \in K^*$, $deg(f) = 0$;

(c) the degree function gives an isomorphism $deg : Cl(X) \longrightarrow \mathbb{Z}$.

Proof.

As $\mathbb{P}_k^n$ is covered by $\mathbb{A}_k^n$ with $\mathbb{A}_k^n = Spec(k[\frac{x_0}{x_i}, ... \frac{x_n}{x_i}])$ which are factorial, every prime divisor of $\mathbb{P}_k^n$ is of the form $V_+(g)$ with g homogeneous and irreducible.

(a). Suppose $D = \sum n_Y Y$ with $Y = V_+(f)$ and $(f) = Y$ with $deg(Y) = deg((f))$. So $D = (\Pi_i f_i^{n_i})$ with f_i homogeneous and irreducible. If $degD = d$, then $\Pi f_i^{n_i} x_0^{-d} = \frac{g}{h} \in K^*(\mathbb{P}_k^n)$ with g, h homomogeneous with the same degree. Thus, $D - dH = (f)$ for $f \in K^*(\mathbb{P}_k^n)$.

(b). If $f \in K^*(\mathbb{P}_k^n)$, then $f = \frac{g}{h}$ with g, h homogeneous of the same degree. Suppose $g = \Pi_i g_i^{n_i}$ and $h = \Pi_j h_j^{m_j}$ with g_i, h_j homogeneous and irreducible. Then $(f) = \sum n_i(g_i) - \sum m_i(h_i)$ and $deg(f) = \sum n_i deg(g_i) - m_j deg(h_j) = 0$.

(c). Directly from (a) and (b). $\square$

Corollary 2.6.18. $Pic(\mathbb{P}^n_k) = \mathbb{Z}$

Proof. Note that $\mathbb{P}^n_k$ is integral and factorial. □

More precisely, $Pic(\mathbb{P}^n_k) = <\mathcal{O}(1)>_{\mathbb{Z}}$. Because $Pic(X) = Cl(X)$ is generated by $H = \{x_0 = 0\}$ and the line bundle define by H is:

On $U_i = Spec(k[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}])$, $H \cap U_i = (x_i)$ for $(x_0) - (x_i) = (\frac{x_0}{x_i})$ with $\frac{x_0}{x_i} \in K^*(\mathbb{P}^n_k)$. So the line bundle $\mathcal{H}$ associated to H is locally represented as $x_i \mathcal{O}_X|_{U_i}$. Note that

$$k[x_0, ..., x_n](-1)_{(x_i)} \overset{x_i}{\cong} x_i k[x_0, ..., x_n]_{(x_i)}$$

and $x_i \in \mathcal{O}^*(U_i)$. So $\mathcal{H} \cong \mathcal{O}(1)$ as line bundles.

2.6.2 Exercises

Exercise 2.6.1. Let X be a scheme satisfying $(*)$. Then $X \times \mathbb{P}^n$ also satisfies $(*)$, and $\mathrm{Cl}(X \times \mathbb{P}^n) \cong (\mathrm{Cl}\ X) \times \mathbb{Z}$

Proof.

For $\mathbb{P}^n$ is a Noetherian integral separated scheme which is regular of codimension 1.

Consider $Z = p^{-1}([1, 0, ..., 0, 0])$ where $p : X \times_k \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k$. Then consider $U = p^{-1}([0, 0, ..., 0, 1])^C \cong X \times_k \mathbb{A}^n_k$ and the exact sequence:

$$\mathbb{Z} \longrightarrow Cl(X \times_k \mathbb{P}^n_k) \longrightarrow Cl(X \times_k \mathbb{A}^n_k) \longrightarrow 0$$

where $i : \mathbb{Z} \longrightarrow Cl(X \times_k \mathbb{P}^n_k)$ is given by $1 \mapsto Z$.

We need to verify i is an injection. If $nZ \sim 0$, then there exists a function $f \in K(X \times_k \mathbb{P}^n_k)$ such that $D(f) = nZ$, i.e. $v_Z(f) = n$ and $v_Y(f) = 0$ for any other $Y \neq Z$. Note that $K(X \times_k \mathbb{P}^n_k) = K(X) \otimes_k k(t_1, ..., t_n) =: K(t_1, ..., t_n)$ with $t_i = \frac{x_1}{x_0}$, $K(X)$ the rational functions on X and $k(t_1, ..., t_n)$ the rational functions on $\mathbb{P}^n_k$. Suppose $f = t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n} \frac{f(t_1, ..., t_n)}{g(t_1, ..., t_n)}$ with $f, g \in K(X)[t_1, ..., t_n]$ having no factors of the form t_i. Note that if $m_i \neq 0$, on $Y = p^{-1}([0, ..., 0, 1, ..., 0])$, $v_Y(f) = m_i \neq 0$, which leads to a contradiction.

So $f = \frac{f(t_1, ..., t_n)}{g(t_1, ..., t_n)}$ with $f, g \in K(X)[t_1, ..., t_n]$ prime. So $f \in K(t_1, ..., t_n)^*$, implying $v(f) = 0$. Thus, $nZ \sim 0$ if and only if $n = 0$, that is, i is injective.

Now we have the exact sequence in the category of Abelian groups:

$$0 \longrightarrow \mathbb{Z} \longrightarrow Cl(X \times_k \mathbb{P}^n_k) \longrightarrow Cl(X \times_k \mathbb{A}^n_k) \longrightarrow 0$$

Note that we have an isomorphism which is composition of the following morphisms

$$Cl(X) \longrightarrow Cl(X \times_k \mathbb{P}^n_k) \longrightarrow Cl(X \times_k \mathbb{A}^n_k) = Cl(X)$$

So the exact sequence splits, that is,

$$Cl(X \times_k \mathbb{P}^n_k) = Cl(X) \oplus \mathbb{Z}$$

□

Exercise 2.6.2. *Proof.*

(a).

(b). Let $D = (f)$ with $f \in K(\mathbb{P}^{\ltimes}_{\urcorner})$ and $\bar{f} \in K(X)$ be the image of f under the map $K(\mathbb{P}^n_k) \longrightarrow K(X)$ deduced by $i : X \to \mathbb{P}^n_k$. Then as we defined in (a), $D.X = (\bar{f})$ which is principal.

(c).

(d). Suppose that $D = (g)$ for some $g \in K(X)$. Take $f \in i^{-1}(g)$ where $i : K(\mathbb{P}^n_k) \longrightarrow K(X)$ is an endomorphism 2.5.14. And by (b), we see that $D = (f).X$. Hence, by (c), $deg(D) = 0$.

□

Exercise 2.6.3.

Exercise 2.6.4. Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \ldots, x_n]$ be a square-free nonconstant polynomial, i.e., in the unique factorization of f into irreducible polynomials, there are no repeated factors. Let $A = k[x_1, \ldots, x_n, z]/(z^2 - f)$. Show that A is an integrally closed ring. [Hint: The quotient field K of A is just $k(x_1, \ldots, x_n)[z]/(z^2 - f)$. It is a Galois extension of $k(x_1, \ldots, x_n)$ with Galois group $\mathbb{Z}/2\mathbb{Z}$ generated by $z \mapsto -z$. If $\alpha = g + hz \in K$, where $g, h \in k(x_1, \ldots, x_n)$, then the minimal polynomial of α is $X^2 - 2gX + (g^2 - h^2 f)$. Now show that α is integral over $k[x_1, \ldots, x_n]$ if and only if $g, h \in k[x_1, \ldots, x_n]$. Conclude that A is the integral closure of $k[x_1, \ldots, x_n]$ in K.]

Proof.
Consider $K = Frac(A)$. For f is square-free, $z^2 - f$ is irreducible. Taking any $\frac{g+zh}{g^{'}+zh^{'}} \in K$,

$$\frac{g + zh}{g^{'} + zh^{'}} = \frac{g + zh}{g^{'} + zh^{'}} \frac{g^{'} - zh^{'}}{g^{'} - zh^{'}} = \frac{gg^{'} + zg^{'}h - zgh^{'} - z^2hh^{'}}{g^{'2} - z^2h^{'2}} = \frac{gg^{'} - fhh^{'} + (g^{'}h - gh^{'})z}{g^{'2} - fh^{'2}}$$

Note that $\frac{gg^{'} - fhh^{'}}{g^{'2} - fh^{'2}}, \frac{g^{'}h - gh^{'}}{g^{'2} - fh^{'2}} \in k(x_1, ..., x_n)$. So any $\alpha \in K$ can be written as $h + zg$ with $h, g \in k(x_1, ..., x_n)$.

Note that for such α

$$\alpha^2 - 2h\alpha - h^2 - fg^2 + 2h^2 = \alpha^2 - 2h\alpha + (h^2 - fg^2) = 0$$

So if $g, h \in k[x_1, ..., x_n]$, α is integral over $k[x_1, ..., x_n]$. Thus, A is integral over $k[x_1, ..., x_n]$, which implies A is integrally closed. □

Remark. Why do we need f to be square free.

Exercise 2.6.5.

Proof.
(a). When $r \geq 2$, let $f = i(x_1^2 + ... + x^r)$. Then f is square-free. Using 2.6.4, $k[x_0, ..., x_r]/(x_0^2 - f^2) = k[x_0, ..., x_n]/(x_0^2 + x_1^2 + ... + x_r^2)$ is norm. Hence, X is normal when $r \geq 2$.

□

Exercise 2.6.6.

Exercise 2.6.7.

Exercise 2.6.8. (a) Let $f : X \to Y$ be a morphism of schemes. Show that $\mathcal{L} \mapsto f^*\mathcal{L}$ induces a homomorphism of Picard groups, $f^* : \text{Pic } Y \to \text{Pic } X$.

(b) If f is a finite morphism of nonsingular curves, show that this homomorphism corresponds to the homomorphism $f^* : \text{Cl } Y \to \text{Cl } X$ defined in the text, via the isomorphisms of (6.16).

(c) If X is a locally factorial integral closed subscheme of $\mathbb{P}^n_k$, and if $f : X \to \mathbb{P}^n$ is the inclusion map, then f^* on Pic agrees with the homomorphism on divisor class groups defined in (Ex. 6.2) via the isomorphisms of (6.16).

Proof.

(a). Because $f^{-1}(\mathcal{O})_x = \mathcal{O}_{f(x)}$, we can see that

$$(f^*(\mathcal{L}))_{\mathfrak{p}} = \mathcal{L}_{f(\mathfrak{p})} \otimes_{\mathcal{O}_{Y,f(\mathfrak{p})}} \mathcal{O}_{X,\mathfrak{p}}$$

So, if $\mathcal{L}$ is locally free of rank 1 over Y, $f^*(\mathcal{L})$ is locally free of rank 1 over X.

So it is enough to show that f^* commutes with $\otimes$, which we have show in 2.5.16 (e). □

Exercise 2.6.9.

Exercise 2.6.10. The Grothendieck Group $K(X)$. Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on X, by the subgroup generated by all expressions $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$, whenever there is an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of coherent sheaves on X. If $\mathcal{F}$ is a coherent sheaf, we denote by $\gamma(\mathcal{F})$ its image in $K(X)$.

(a) If $X = \mathbb{A}^1_k$, then $K(X) \cong \mathbb{Z}$.

(b) If X is any integral scheme, and $\mathcal{F}$ a coherent sheaf, we define the *rank* of $\mathcal{F}$ to be $\dim_K \mathcal{F}_\xi$, where ξ is the generic point of X, and $K = \mathcal{O}_\xi$ is the function field of X. Show that the rank function defines a surjective homomorphism rank : $K(X) \to \mathbb{Z}$.

(c) If Y is a closed subscheme of X, there is an exact sequence

$$K(Y) \to K(X) \to K(X - Y) \to 0,$$

where the first map is extension by zero, and the second map is restriction. [Hint: For exactness in the middle, show that if $\mathcal{F}$ is a coherent sheaf on X, whose support is contained in Y, then there is a finite filtration $\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots \supseteq \mathcal{F}_n = 0$, such that each $\mathcal{F}_i/\mathcal{F}_{i+1}$ is an $\mathcal{O}_Y$-module. To show surjectivity on the right, use (Ex. 5.15).]

For further information about $K(X)$, and its applications to the generalized Riemann-Roch theorem, see Borel-Serre [1], Manin [1], and Appendix A.

Proof.

(a). First, we see that

$$\{\text{coherent sheaves on } \mathbb{A}^1_k\} \xrightarrow{1:1} \{\text{finitely generated } k[x] \text{ modules}\}$$

By the classification of finitely generated PID modules, each finitely generated $k[x]$-mod M can be written as $k[x]^{\oplus n} \oplus Tor(M)$. Thus, we define $rk(M) = n$.

We will show that for any exact sequence in the category of finitely generated $k[x]$-modules,

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

$rk(M) = rk(N) + rk(L)$.

Suppose $N = k[x]^{rkN} \oplus Tor(N)$. For $N \subset M$, free elements of N are free in M and torsion elements of N are torsion in M . Thus, $L = M/N = k[x]^{rkM}/k[x]^{rkN} \oplus TorM/TorN$. So $Free(L) = Free(k[x]^{rkM}/k[x]^{rkN})$. By general theory of linear algebra, we have $rk(L) = rk(M) - rk(N)$. Thus, $K(X) \longrightarrow \mathbb{Z}$, $\gamma(\mathcal{F}) = \gamma(\widetilde{M}) \longrightarrow rkM$ is well-defined for $rk(M) = rk(N) + rk(L)$.

If $rkL = 0$, L is torsion, then there exists an exact sequence

$$0 \longrightarrow k[x]^n \longrightarrow k[x]^n \longrightarrow L \longrightarrow 0$$

Then, $-L = k[x]^n - k[x]^n - L$. Thus, $\gamma(L) = 0$.

For $rk(\widetilde{k[x]^n}) = n$, $K(X) \cong \mathbb{Z}$.

(b). For localization is exact, we have an exact sequence of k-vector spaces

$$0 \longrightarrow \mathcal{F}'_\xi \longrightarrow \mathcal{F}_\xi \longrightarrow \mathcal{F}''_\xi \longrightarrow 0$$

Then $dim\mathcal{F}_\xi = dim\mathcal{F}'_\xi + dim\mathcal{F}''_\xi$. So $rk\mathcal{F} = rk\mathcal{F}' + rk\mathcal{F}''$. Thus, we have a well-defined morphism: $rk : K(X) \longrightarrow \mathbb{Z}$.

Note that $rk\mathcal{O}_X^n = n$. So rk is surjective.

(c). For any coherent sheaf on $X - Y$, which is an open subset of X, it can be extended to a coherent sheaf on X, we see that

$$K(X) \longrightarrow K(X - Y)$$

is surjective.

For any coherent sheaf $\mathcal{F} \in ker(K(X) \longrightarrow K(X - Y))$, we need to show that $\mathcal{F} \in K(Y)$.

For any $U = Spec(A) \subset X$ with A Noetherian, $U \cap Y = Spec(A/I)$ for some I. For any coherent sheaf $\mathcal{F}$ on X, $\mathcal{F}|_U = \widetilde{M}$ with M finitely generated A-module and then $i_*i^*\mathcal{F} = M \otimes_A A/I^\sim = \widetilde{M/IM}$. Let $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_i = \ker(\mathcal{F}_{i-1} \longrightarrow i_*i^*\mathcal{F}_{i-1})$, which is coherent for $Coh(X)$ is Abelian. Then locally, $\mathcal{F}_{i-1}|_U = I^{i-1}M$. For M is finitely generated and A is Noetherian, there exists a finite filtration:

$$M \supseteq IM ... \supseteq I^n M$$

For X is Noetherian, we then have a finite coherent sheaves filtration:

$$\mathcal{F}_0 \supseteq \mathcal{F}_1 ... \supseteq \mathcal{F}_n$$

Note that $\mathcal{F}_{i-1}/\mathcal{F}_{i-1} = i_*i^*\mathcal{F}_{i-1}$, which are extensions of coherent sheaves on Y. Thus, $\gamma(\mathcal{F}) = (\sum \gamma(\mathcal{F}_{i-1}/\mathcal{F}_i)) \in K(Y)$, for $\gamma(\mathcal{F})$ is zero in $K(X - Y)$. (Note that $\gamma(\mathcal{F}_i) \in K(X - Y)$.)

□

Exercise 2.6.11. (*) The Grothendieck Group of a Nonsingular Curve. Let X be a nonsingular curve over an algebraically closed field k. We will show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$, in several steps.

(a) For any divisor $D = \sum n_i P_i$ on X, let $\psi(D) = \sum n_i \gamma(k(P_i)) \in K(X)$, where $k(P_i)$ is the skyscraper sheaf k at P_i and 0 elsewhere. If D is an effective divisor, let $\mathcal{O}_D$ be the structure sheaf of the associated subscheme of codimension 1, and show that $\psi(D) = \gamma(\mathcal{O}_D)$. Then use (6.18) to show that for any D, $\psi(D)$ depends only on the linear equivalence class of D, so ψ defines a homomorphism $\psi : \operatorname{Cl} X \to K(X)$.

(b) For any coherent sheaf $\mathcal{F}$ on X, show that there exist locally free sheaves $\mathcal{E}_0$ and $\mathcal{E}_1$ and an exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$. Let $r_0 = \operatorname{rank} \mathcal{E}_0$, $r_1 = \operatorname{rank} \mathcal{E}_1$, and define

$$\det \mathcal{F} = \left(\bigwedge^{r_0} \mathcal{E}_0 \right) \otimes \left(\bigwedge^{r_1} \mathcal{E}_1 \right)^{-1} \in \operatorname{Pic} X.$$

Here $\bigwedge$ denotes the exterior power (Ex. 5.16). Show that $\det \mathcal{F}$ is independent of the resolution chosen, and that it gives a homomorphism $\det : K(X) \to \operatorname{Pic} X$. Finally show that if D is a divisor, then $\det(\psi(D)) = \mathcal{L}(D)$.

(c) If $\mathcal{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence $0 \to \mathcal{L}(D)^{\oplus r} \to \mathcal{F} \to \mathcal{T} \to 0$, where $\mathcal{T}$ is a torsion sheaf. Conclude that if $\mathcal{F}$ is a sheaf of rank r, then $\gamma(\mathcal{F}) - r\gamma(\mathcal{O}_X) \in \operatorname{Im}\psi$.

(d) Using the maps ψ, det, rank, and $1 \mapsto \gamma(\mathcal{O}_X)$ from $\mathbb{Z} \to K(X)$, show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

Without loss of generality, we assume that C is integral. (Since C is non-singular, it is reduced. Then we just take a irreducible component of C.)

(a). We need to show that if $D = \sum_i n_i P_i$, then $\gamma(\mathcal{O}_D) = \sum_i n_i \gamma(k(P_i))$.

First, we show that when $D = nP$, then $\gamma(\mathcal{O}_D) = n\gamma(k(P))$. Since P is a closed point over a non-singular curve X, $\mathcal{O}_{X,P}$ is a DVR. Suppose that $P \in U := Spec(A)$ and $P := \mathfrak{p} \in U$. Let $f \in A$ be the function defining D, that is, under the morphism $A \longrightarrow A_{\mathfrak{p}}$, $\frac{f}{1} = \pi^n$ with π the uniformiser of A_{p}. Then we get a filtration

$$A_{\mathrm{p}}/f \supset \pi A_{\mathrm{p}}/f \supset ... \supset \pi^i A_{\mathrm{p}}/f \supset ... \supset \pi^n A_{\mathfrak{p}}/f = 0$$

Now that $\mathcal{O}_D = Spec(A/f) = Spec(A_{\mathrm{p}}/f) = i_{X,*}(A_{\mathfrak{p}}/f)$ and we get a filtration over X given by

$$\mathcal{O}_D \supset i_{P,*}\pi A_{\mathrm{p}}/f \supset ... \supset i_{P,*}\pi^i A_{\mathrm{p}}/f \supset ... \supset i_{P,*}\pi^n A_{\mathfrak{p}}/f = 0$$

Since P is a closed point, $i_{P,*}$ is exact and hence $i_{P,*}(\pi^i A_{\mathfrak{p}}/f)/i_{P,*}(\pi^{i+1}A_{\mathrm{p}}/f) \cong i_{P,*}k(P)$, so we have

$$\gamma(\mathcal{O}_D) = \sum \gamma(i_{P,*}(\pi^i A_{\mathfrak{p}}/f)/i_{P,*}(\pi^{i+1}A_{\mathrm{p}}/f)) = \sum_{i=1}^{n-1} \gamma(k(P)) = n\gamma(k(P))$$

Next, we prove the general case. If $D = \sum_i n_i P_i$, take $D_i = n_i P_i$. $\mathcal{O}_D = \oplus_i D_i$ and $\gamma(\mathcal{O}_D) = \sum_i \gamma(\mathcal{O}_{D_i}) = \sum n_i \gamma(k(P_i))$.

For any D, we have

$$0 \longrightarrow \mathscr{L}(-D) \longrightarrow \mathcal{O}_X \longrightarrow i_{D,*}\mathcal{O}_D \longrightarrow 0$$

and $D \sim D^{'}$ if and only if $\mathscr{L}(D) \cong \mathscr{L}(D^{'})$, $\gamma(\mathcal{O}_D) = \gamma(\mathcal{O}_{D'})$.

(b). By [Proposition II.6.7. [5]], X is a projective scheme. Then since $\mathcal{F}$ is coherent sheaf, there exists an very ample line bundle $\mathcal{L}$. (For curves, it always exits. See [Corollary IV.3.2. [5]]) Then by Serre's Theorem [Proposition II. Theorem 5.17 [5]], $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated, that is,

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^n$$

By [Theorem II.5.19 [5]], $H^0(X, \mathcal{F} \otimes \mathcal{L}^n)$ is a k-vector space. Hence, tensoring the surjection by $\mathcal{L}^{-n}$ on both side, we have a locally free sheaf $\mathcal{E}_0$ such that $\varphi : \mathcal{E}_0 \twoheadrightarrow \mathcal{F}$.

Then we only need to show that $\ker(\varphi)$ is a locally free sheaf. Since φ_x is a submodule of $\mathcal{E}_x$, it is torsion-free. Because $\mathcal{O}_{X,x}$ is a DVR, by the classification theorem of PID, $\ker(\varphi)_x$ is locally free.

Now, if we have two free resolution of $\mathcal{F}$, that is, $0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$ and $0 \longrightarrow \mathcal{E}_1^{'} \longrightarrow \mathcal{E}_0^{'} \longrightarrow \mathcal{F} \longrightarrow 0$. Then consider the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[d] & \mathcal{E}_0 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{E}_0^{'} \oplus \mathcal{E}_0 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}_0^{'} \arrow[r, "="] \arrow[d] & \mathcal{E}_0^{'} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

with $\mathcal{K} = \ker(\mathcal{E}_0^{'} \oplus \mathcal{E}_0 \twoheadrightarrow \mathcal{F})$, which is also locally free. By nine lemma, $0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}_0^{'} \longrightarrow 0$ is exact. Similary, we have $0 \longrightarrow \mathcal{E}_1^{'} \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}_0 \longrightarrow 0$ is exact. Then, we have

$$
\begin{aligned}
\det(\mathcal{E}_0) \otimes \det(\mathcal{E}_1)^{-1} &= \det(\mathcal{E}_0) \otimes \det(\mathcal{E}_0^{'}) \otimes \det(\mathcal{E}_0^{'})^{-1} \otimes \det(\mathcal{E}_1)^{-1} \\
&= \det(\mathcal{E}_0) \otimes \det(\mathcal{E}_0^{'}) \otimes \det(\mathcal{K})^{-1} \\
&= \det(\mathcal{E}_0^{'}) \otimes \det(\mathcal{E}_1)^{-1}
\end{aligned}
$$

Hence, $\det(\mathcal{F})$ is well-defined.

Consider $0 \longrightarrow \mathcal{F}^{'} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{''} \longrightarrow 0$. Take free resolutions $0 \longrightarrow \mathcal{E}_1^{'} \longrightarrow \mathcal{E}_0^{'} \longrightarrow \mathcal{F}^{'} \longrightarrow 0$ and $0 \longrightarrow \mathcal{E}_1^{''} \longrightarrow \mathcal{E}_0^{''} \longrightarrow \mathcal{F}^{''} \longrightarrow 0$. Then we consider

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{E}_1^{'} \arrow[r] \arrow[d] & \mathcal{E}_0^{'} \arrow[r] \arrow[d] & \mathcal{F}^{'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{E}_0^{''} \oplus \mathcal{E}_0^{'} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}_1^{''} \arrow[r] \arrow[d] & \mathcal{E}_0^{''} \arrow[r] \arrow[d] & \mathcal{F}^{''} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Then $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}_0^{''} \otimes \mathcal{E}_0^{'} \longrightarrow \mathcal{F} \longrightarrow 0$ is a free resolution of $\mathcal{F}$. Then by nine lemma, we can easily show that $\det(\mathcal{F}) = \det(\mathcal{F}^{'}) \otimes \det(\mathcal{F}^{''})$. Hence, $\det : K(X) \longrightarrow \mathrm{Pic}(X)$ given by $\gamma(\mathcal{F}) \longmapsto \det(\mathcal{F})$ is well-defined and a homomorphism.

Remark. To construct the last big exact sequence, we need to construct $\mathcal{E}_0^{''} \oplus \mathcal{E}_0^{'} \twoheadrightarrow \mathcal{F}$. To do this, note that $\mathcal{E}_0^{''}$ is locally free hence projective. For more details, see [P 237 [5]].

Finally, if D is a divisor, then we have

$$0 \longrightarrow \mathscr{L}(-D) \longrightarrow \mathcal{O}_X \longrightarrow i_{D,*}\mathcal{O}_D \longrightarrow 0$$

Then, $\det(\psi(\mathcal{O}_D)) = \det(\gamma(\mathcal{O}_D)) = \det(\mathcal{O}_D) = \mathscr{L}(D)$.

(c). Since $\mathcal{F}_\eta$ is locally free of rank r, there exists an open neighborhood U of η such that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$ by 2.5.7. Since C is integral (**??** implies $\dim D = 0$), Let $D = X - U$ which is a closed subset containing only finitely many $P_1, ..., P_n$. Now, we have

$$\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$$

It is enough to prove that this exact sequence can be lifted to X.

Let D be effective the Cartier divisor defined by $\{(U_i, f_i), (U, 1)\}$ with U_i affine. Consider $s|_{U_i \cap U} \in \mathcal{F}(D(f_i))$. By [Lemma II.5.3. [5]], there exists m_i such that $sf_i^{m_i} \in \mathcal{F}(U_i) \cong \mathcal{F} \otimes \mathcal{L}(-eD)(U_i)$. Take $m = \max_i m_i$. Then for each f_i, $sf_i^m \in \mathcal{F} \otimes \mathcal{L}(-mD)$. Since the gluing function of $\mathcal{F} \otimes \mathcal{L}(-mD)$ is given by $\{1 \cdot (f_j/f_i)^m\}$, $\{sf_i^m\}$ glues to a global section of $\mathcal{F} \otimes \mathcal{L}(-mD)$. Then we see that $\mathcal{O}|_U \longrightarrow \mathcal{F}|_U$ lifts to a morphism $\mathcal{O}_X \longrightarrow \mathcal{F} \otimes \mathcal{L}(-mD)$. (Recall that $Hom(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}(X)$).

Moreover, for $(\mathcal{O}|_U)^{\oplus r} \longrightarrow \mathcal{F}|_U$ given by $(s_1, ..., s_n)$, for each s_j there exists m_j lifts $\mathcal{O}|_U \longrightarrow \mathcal{F}|_U$ to $\mathcal{O}_X \longrightarrow \mathcal{F}$. Let $e = max_j\{m_j\}$. Then we have a morphism

$$\varphi : \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{F} \otimes \mathcal{L}(-eD)$$

Since $\mathcal{L}(-eD)|_U = \mathcal{O}_U$, we see that its restriction on U is just as above.

Now, consider $\ker(\varphi)_x$ for each $x \in X$. Since it is a sub module of $\mathcal{O}_{X,x}^{\oplus r}$, it is torsion free. However, since $\mathcal{O}_{X,x}$ is a DVR, by the classification theorem of PID, $\mathcal{O}_{X,x}$ is free for each x. By 2.5.7, $\ker(\varphi)$ is a locally free sheaf. By 2.5.8, $\dim \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is constant. Since $\ker(\varphi)_\eta = 0$, we see that $\ker(\varphi) = 0$. Hence φ is an injection.

Let $\mathcal{T}' = \operatorname{coker}(\varphi)$. Since $\mathcal{T}'_\eta = 0$, we see that $\mathcal{T}'$ is torsion. Moreover, $\mathcal{T}'$ is supported over $X - U$ which contains only finitely many points.

Now we get the short exact sequence

$$0 \longrightarrow \mathcal{L}(eD)^{\oplus r} \longrightarrow \mathcal{F} \longrightarrow \mathcal{T}' \otimes \mathcal{L}(eD) \longrightarrow 0$$

Let $\mathcal{T} = \mathcal{T}' \otimes \mathcal{L}(eD)$ which is also a torsion sheaf and supported on finitely many points.

Consider $0 \longrightarrow \mathcal{L}(-eD) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{eD} \longrightarrow 0$. We see that

$$\gamma(\mathcal{L}(eD)) = \gamma(\mathcal{O}_X) + \gamma(\mathcal{L}(eD) \otimes \mathcal{O}_{eD})$$

Since $\mathcal{L}(eD) \otimes i_{D,*}\mathcal{O}_D = i_{D,*}(i_D^*\mathcal{L}(D) \otimes \mathcal{O}_D) = i_{D,*}\mathcal{O}_D$ since $\dim D = 1$. (By Grothendieck vanishing theorem, $\operatorname{Pic}(D) = 0$ since $H^1(D, \mathcal{O}_D^*) = 0$. But for this, you should see 3.4.5). Hence,

$$\gamma(\mathcal{L}(eD)) = \gamma(O_X) + \gamma(\mathcal{O}_X)$$

where $\gamma(\mathcal{O}_X) \in Im(\psi)$.

Since $\mathcal{T}$ is supported on finitely many points, $\mathcal{T} := \sum_j i_{P_j,*}(M_j)$. Consider the Jordan-Hölder filtration of M_j

$$M_j \supset M_{j_1} \supset ... \supset M_{j_n} \supset M_{j_{n+1}} 0$$

such that $M_{j_k}/M_{j_{k+1}}$ is a simple module over $\mathcal{O}_{X,P}$. For $\mathcal{O}_{X,P}$ is a local ring with residue field $k(P)$, $M_{j_k}/M_{j_{k+1}} = k(P)$. (Suppose that M is a simple module. Take $m \in M$. For $A \cdot m$ is a sub-module of M, $A/Ann(m) \cong M$. Again for M is simple, $A/Ann(M) \cong k(P)$). Thus, by the filtration, $\gamma(\mathcal{T}) = \sum n_i \gamma(P_i) \in Im(\psi)$.

(d). First of all, there exists an short exact sequence

$$0 \longrightarrow \operatorname{Pic}(X) \xrightarrow{\psi} K(X) \xrightarrow{\text{rank}} \mathbb{Z} \longrightarrow 0$$

For each $P \in X$, $k(P)$ has rank 0. Hence, rank $\circ\, \psi = 0$. Take any $\sum_i n_i \mathcal{F}_i \in K(X)$, $\sum_i n_i \gamma(\mathcal{F}_i) - \sum_i n_i r_i \mathcal{O}_X \in Im(\psi)$ with $r_i = \text{rank}(\mathcal{F}_i)$. Hence, $\text{rank}(\sum_i n_i \gamma(\mathcal{F}_i)) = \text{rank}(\sum n_i r_i \mathcal{O}_X)$. If

$$\text{rank}(\sum_i n_i \gamma(\mathcal{F}_i)) = \text{rank}(\sum n_i r_i \mathcal{O}_X) = 0$$

, then $\sum n_i r_i = 0$. That is, $\sum_i n_i \gamma(\mathcal{F}_i) \in Im(\psi)$. Thus, the short sequence is exact.

Note that $\mathbb{Z} \longrightarrow K(X) \longrightarrow \mathbb{Z}$ given by the composition of $1 \longmapsto \gamma(\mathcal{O}_X)$ and rank is identity. The short exact sequence splits, that is,

$$K(X) = Pic(X) \oplus \mathbb{Z}$$

Moreover, the inverse of ψ is given by det. Note that

$$K(X) \xrightarrow{\cong} \text{Pic}(X) \oplus \mathbb{Z}$$
$$\gamma(\mathcal{F}_i) \longmapsto (\det(\mathcal{F}_i), r_i)$$

with $r_i = \text{rank}(\mathcal{F})$ and the group structure on $\text{Pic}(X) \oplus \mathbb{Z}$ is given by

$$(\mathcal{L}_1, r_1) \cdot (\mathcal{L}_\in, r_2) = (\mathcal{L}_1 \otimes \mathcal{L}_2, r_1 + r_2)$$

Exercise 2.6.12. Let X be a complete nonsingular curve. Show that there is a unique way to define the *degree* of any coherent sheaf on X, $\deg \mathcal{F} \in \mathbb{Z}$, such that:

1. If D is a divisor, $\deg \mathcal{L}(D) = \deg D$;
2. If $\mathcal{T}$ is a *torsion sheaf* (meaning a sheaf whose stalk at the generic point is zero), then $\deg \mathcal{T} = \sum_{P \in X} \text{length}(\mathcal{T}_P)$; and
3. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence, then $\deg \mathcal{F} = \deg \mathcal{F}' + \deg \mathcal{F}''$.

By 2.6.11, we consider the composition of det and $\deg_X : \text{Pic}(X) \longrightarrow \mathbb{Z}$. Then it is easy to see that deg satisfies (1),(3). For (2), since $\mathcal{F}$ is torsion free, there exists an open neighborhood U of η such that $\mathcal{F}$ is zero by 2.5.7. Hence, $\mathcal{F}$ only supported at finitely many points. Using Jordan-Hölder theorem, (2) is obvious.

For uniqueness, let $\deg^{'}$ be another map satisfying these properties and $\varphi = \deg^{'} - \deg$. Then $\varphi|_{\text{Pic(X)}\times 1} = 0$ by (1). For $\text{Pic}(X) \times n$ with $n \geq 1$, use (3) and induction to show that $\varphi|_{\text{Pic}(X)\times n} = 0$ for any $n \geq 1$: For any coherent sheaf $\mathcal{F}$, there exists $\mathcal{L} \in \text{Pic}(X)$ such that

$$0 \longrightarrow \mathcal{F}^{'} \longrightarrow \mathcal{F} \longrightarrow \mathcal{L} \longrightarrow 0$$

See [Remark 26.132. [4]]. For on $\text{Pic}(X) \times 0$, by (2), this is obvious. Thus, φ is just 0.

2.6.3 Additional Exercises

Exercise 2.6.13. Let X be $\mathbb{P}^1_k$ over a field k. Then $\mathcal{K}^*_X / \mathcal{O}^*_X$ is flasque.

Proof. For any U, $\Gamma(U, \mathcal{K}^*_X/\mathcal{O}^*_X)$ represents the Cartier divisors over U. Note that U is integral and factorial, we see that $\Gamma(U, \mathcal{K}^*_X/\mathcal{O}^*_X)$ is the Weil divisors over U.

When $codim(U) = 1$, we have the exact seqence

$$\mathbb{Z} \longrightarrow Pic(X) \longrightarrow Pic(U) \longrightarrow 0$$

while when $codim(U) \leq 2$, $Pic(X) = Pic(U)$. Hence, $\Gamma(X, \mathcal{K}^*_X/\mathcal{O}^*_X) \longrightarrow \Gamma(U, \mathcal{K}^*_X/\mathcal{O}^*_X)$ is surjective for any U. Hence, $\mathcal{K}^*_X/\mathcal{O}^*_X$ is flasque. □

2.7 Projective Morphisms

2.7.1 Preparations

A Morphisms to $\mathbb{P}_k^n$

Theorem 2.7.1. *Let A be a ring and let X be a scheme over A.*

(a) If $\varphi : X \longrightarrow \mathbb{P}_k^n$ is an A-morphism, then $\varphi^*(\mathcal{O}_{\mathbb{P}_k^1}(1))$ is an invertible sheaf on X, which is generated by the global section $s_i = \varphi^*(x_i), i = 0, 1, 2, ..., x_n$;

(b) Conversely, if $\mathcal{L}$ is an invertible sheaf on X, and if $s_0, s_1, ..., s_n \in \Gamma(X, \mathcal{L})$ are global setions which generates $\mathcal{L}$, then there exists a unique A-morphism $\varphi : X \longrightarrow \mathbb{P}_k^n$ such that $\mathcal{L} \cong \varphi^*(\mathcal{O}_{\mathbb{P}_k^n}(1))$ and $s_i = \varphi^*(x_i)$ under the isomorphism.

Proof.
(a). First, we show that: **Pullback of a globally generated line bundle is also globally generated:**

Suppose that $\mathcal{L}$ is globally generated by s_i. Then $\mathcal{L}_P =< s_{1,P}, ..., s_{n,P} >_{\mathcal{O}_{Y,P}}$. $f^*\mathcal{L}_Q = \mathcal{L}_{f(Q)} \otimes_{\mathcal{O}_{Y,f(Q)}} \mathcal{O}_{X,Q}$. Because $\mathcal{L}_{f(Q)} =< s_{1,f(Q)}, ..., s_{n,f(Q)} >_{\mathcal{O}_{Y,f(Q)}}$, $f^*\mathcal{L}_Q =< s_1(f(Q)), ..., s_n(f(Q)) >_{\mathcal{O}_{X,Q}}$. Thus, $f^*\mathcal{L}$ is globally generated.

For $\mathcal{O}_{\mathbb{P}_A^n}(1)$ is globally generated so is $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbb{P}_A^n}(1)$. As we show above, $\mathcal{L}$ is generated by $\varphi^* x_i$ where x_i is the global generators of $\mathcal{O}_{\mathbb{P}_A^n}(1)$.

(b). Suppose that $\mathcal{L}$ is globally generated by $s_1, ..., s_n \in \Gamma(X, \mathcal{L})$. Consider $X_{s_i} := \{P \in X | s_i \notin \mathfrak{m}_P \mathcal{L}_P\}$. $\{X_{s_i}\}$ forms an open cover of X, for $\{s_i\}$ generates $\mathcal{L}$ globally. On X_{s_i}, we see that $\frac{s_j}{s_i} \in \Gamma(X_{s_i}, \mathcal{O}_{X_{s_i}})$. Hence we define

$$A[x_1, ..., x_n] \longrightarrow \Gamma(X_{s_i}, \mathcal{O}_{X_{s_i}})$$

by $x_i \mapsto \frac{s_j}{s_i}$. Using 2.2.4, there is a unique morphism $\varphi_i : X_{s_i} \longrightarrow Spec(A[x_1, ..., x_n])$. Because φ_i satisfies the gluing criteria compatible with X and $\mathbb{P}_k^n$. Hence, we get a morphism $\varphi : X \longrightarrow \mathbb{P}_k^n$ with $\varphi^*(x_i) = s_i$. □

B Criterias for a morphism to be a closed immersion

Theorem 2.7.2. *Let $\varphi : X \longrightarrow \mathbb{P}_A^n$ be a morphism of schemes over A, corresponding to an invertible sheaf $\mathcal{L}$ on X and sections $s_0, ..., s_n \in \Gamma(X, \mathcal{L})$ as above. Then φ is a closed immersion if and only if*

(1) each open set $X_i = X_{s_i}$ is affine, and

(2) for each i, the map of rings $A[y_1, ..., y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i})$ defined by $y_i \mapsto \frac{s_j}{s_i}$ is surjective.

Proof.
($\Longrightarrow$): For $\varphi_{X_i} : X_i \longrightarrow D_+(x_i)$ with $D_+(x_i)$ is affine, if φ is a closed immersion, so is f_{X_i}. Hence $X_i = Spec(A[y_1, ..., y_n]/I)$ with $y_j = \frac{x_j}{x_i}$, which implies (1) and (2).

($\Longleftarrow$): (1) and (2) imply locally φ is a closed immersion. So φ is a closed immersion. □

Theorem 2.7.3. *Let k be* ***an algebraically closed field****, X be a projective scheme over k and let $\varphi : X \longrightarrow \mathbb{P}_k^n$ be a morphism corresponding to $\mathcal{L}$ and $s_0, ..., s_n \in \Gamma(X, \mathcal{L})$ as above. Let $V \subset \Gamma(X, \mathcal{L})$ be the subspace spanned by the s_i. Then φ is a closed immersion if and only if*

(1) elements of V separate points, i.e., for ***any two distinct closed points*** *$P, Q \in X$, there is an $s \in V$ such that $s \in \mathfrak{m}_P \mathcal{L}_P$ but $s \notin \mathfrak{m}_Q \mathcal{L}_Q$ or more precisely $s(P) = 0$ and $s(Q) \neq 0$;*

(2) *elements of V separate tangent vectors, i.e., for* **each closed point** $P \in X$*, the set* $\{s \in V, s_P \in \mathfrak{m}_P\mathcal{L}_P\}$ *spans the k-vector space* $\mathfrak{m}_P\mathcal{L}_P/\mathfrak{m}_P^2\mathcal{L}_P$.

C Ample Invertible shaeves

Theorem 2.7.4. *Let $\mathcal{L}$ be an invertible sheaf on a Noetherian scheme X. Then the following conditions are equivalent:*

(i) *$\mathcal{L}$ is ample;*

(ii) *$\mathcal{L}^m$ is ample for each $m > 0$;*

(iii) *$\mathcal{L}^m$ is ample for some m.*

Proof. (i)$\Longrightarrow$(ii) and $(ii) \Longrightarrow (iii)$ are trivial. Just consider $(iii) \Longrightarrow (i)$:
Consider

$$\mathcal{F}^{'} = \mathcal{F} \oplus (\mathcal{F} \otimes \mathcal{L}) \oplus ... \oplus (\mathcal{F} \otimes \mathcal{L}^{m-1})$$

For X is Noeterian, $Coh(X)$ is an abelian category, hence $\mathcal{F}^{'}$ is also coherent.

Then there exists n_0 and $n \geq n_0$ such that $\mathcal{F}^{'} \otimes \mathcal{L}^{mn}$ is globally generated. Hence, $\mathcal{F}\otimes\mathcal{L}^{mn}, ..., \mathcal{F}\otimes \mathcal{L}^{m(n+m-1)}$ are globally generated when $n \geq n_0$, that is, when $\mathcal{F} \otimes \mathcal{L}^k$ is globally generated when $k \geq mn_0$. Hence $\mathcal{L}$ is ample. □

Theorem 2.7.5. *Let X be a scheme of finite type over a Noetherian ring A, and let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample over $Spec(A)$ for some $m > 0$.*

Remark. This theorem implies that if X is a scheme of finite type over k, then

$$\mathcal{L} \text{ very ample} \Longrightarrow \mathcal{L} \text{ ample}$$

D Linear system

Definition 2.7.6. A Cartier divisor $D \in CoCl(X)$ is called effective if it can be represented by $\{U_i, f_i \in \mathcal{K}(U_i)\}$ such that $f_i \in \mathcal{O}(U_i)$.

As we have stated in 2.6, an effective Cartier divisor can define a locally principal closed subscheme Y, which can also defines an effective Weil Divisor. More precisely, when X is factorial, there exists an $1-1$ corresponding between effective Cartier divisors and effective Weil divisor.

Definition 2.7.7. Let D be a Cartier divisor represented by $\{U_i, f_i\}$. For any $s \in \Gamma(X, \mathcal{L}(D))$, we define $(s)_0$ to be the Catrier divisor represented by $\{U_i, g_i\}$ with $g_i := \varphi|_{U_i}(s_i)$, where $\varphi : \mathcal{L}(D) \xrightarrow{\cong} \mathcal{O}_X$.

In fact, locally $\varphi|_{U_i} : \mathcal{L}(D)|_{U_i} = f_i^{-1}\mathcal{O}_X|_{U_i} \xrightarrow{\cdot f_i} \mathcal{O}_X|_{U_i}$.

Theorem 2.7.8. *Let X be a nonsingular projective variety over the algebraically closed field k. Let D_0 be a divisor on X and let $\mathcal{L} \cong \mathcal{L}(D_0)$ be the corresponding linearly equivalent to D_0. Then:*

(a) *for each nonzero $s \in \Gamma(X, \mathcal{L})$, the divisor of zeros $(s)_0$ is an effective divisor linear equivalent to D_0;*

(b) *every effective divisor linearly equivalent to D_0 is $(s)_0$ for some $s \in \Gamma(X, \mathcal{L})$;*

(c) *two sections $s, s^{'} \in \Gamma(X, \mathcal{L})$ have the same divisor of zeros if and only if there is a $\lambda \in k^*$ such that $s^{'} = \lambda s$.*

Proof. Let D be a Cartier divisor represented by $\{U_i, f_i\}$.

(a). Take $s \in \Gamma(X, \mathcal{L}(D)) \subset \Gamma(X, \mathcal{K}^*) = K^*(X)$. Suppose s is represented by $f \in K(X)$. Then $(s)_0$ is given by $\{U_i, f_i f\}$. For $f_i f \in \Gamma(U_i, \mathcal{L}(D))$, $(s)_0$ is an effective divisor and $D = (s)_0 + (f)$ with $f \in K^*(X)$. Thus, $D \sim (s)_0$.

(b). Suppose that $D = D_0 + (f)$ with $f \in K^*(X)$. D is given by $\{U_i, g_i\}$ with $g_i^{-1} = f_i^{-1} f^{-1}$. For D is effective, $g_i \in \Gamma(U_i, \mathcal{O}_X|_{U_i})$. Hence $f|_{U_i} = g_i f_i^{-1} \in \Gamma(U_i, \mathcal{L}(D))$, which implies $f \in \Gamma(X, \mathcal{L}(D))$. By definition, we see that $(f)_0 = D$.

(c). ($\Longrightarrow$): Suppose that $(s)_0 = (s^{'})_0$, that is, they define the same Cartier divisor. On U_i, $f_i s|_{U_i} = g_i f_i s^{'}|_{U_i}$ with $\{g_i \in O_X^*(U_i)\} \in \Gamma(X, \mathcal{O}_X^*)$. Hence $s/s^{'} \in \Gamma(X, \mathcal{O}_X^*) = k^*$ by 2.4.5 (d).

($\Longleftarrow$): Trivial. □

Definition 2.7.9. A complete linear system on a non-singular projective variety is defined as the set of all effective linear equivalent to some given divisor D_0, denoted by $|D_0|$.

By theorems above, we see that

$$|D_0| = \frac{\Gamma(X, \mathcal{L}(D_0)) - 0}{k^*}$$

which allows us to compute its dimension by Riemann-Roch formula. It will tell us, when there is a map form X to $\mathbb{P}_k^n$ for some n.

Definition 2.7.10. A linear system $\mathfrak{d}$ is a subset of $|D_0|$, which is a linear subspace for the projective space structure of $|D_0|$.

Proposition 2.7.11. *If* $\varphi : \mathbb{P}_n^k \longrightarrow \mathbb{P}_m^k$ *is given by*

$$\varphi^* x_i \mapsto s_i(x_1, x_2, ..., x_n)$$

for $i = 1, 2, ..., m$, *then* φ *is given by*

$$(x_1 : x_2 : ... : x_n) \mapsto (s_1(x_1, ..., x_n) : s_2(x_1, ..., x_n) : ... : s_m(x_1, ..., x_n))$$

Proof. By the uniqueness stated on Theorem 7.1 [5], we just need to verify that $\varphi : \mathbb{P}_{\ltimes}^{\daleth} \longrightarrow \mathbb{P}_m^k$ given by

$$(x_1 : x_2 : ... : x_n) \mapsto (s_1(x_1, ..., x_n) : s_2(x_1, ..., x_n) : ... : s_m(x_1, ..., x_n))$$

satisfying

$$\varphi^* x_i \mapsto s_i(x_1, x_2, ..., x_n)$$

for $i = 1, 2, ..., m$.

Now, recall the definition of $\varphi^*(\mathbb{P}^{\gtrdot}_{\daleth})$, $\varphi^*(f) = f \circ \varphi$ for $f \in H^0(\mathbb{P}_k^m, \mathcal{O}_{\mathbb{P}_k^m}(1))$

Note that $x_i \in H^0(\mathbb{P}_k^m, \mathcal{O}_{\mathbb{P}_k^m}(1)) = Span_k\{x_0, x_1, ..., x_n\}$. Then

$$\varphi^* x_i = x_i \circ (s_0(x_1, x_2, .., x_n), ..., s_m(x_1, x_2, ..., x_n)) = s_i(x_1, ..., x_n)$$

with $i = 1, 2, ..., m$. □

E Proj, $\mathbb{P}(\mathscr{E})$ and Blowing Up

Recall the definition of $\mathbb{P}(\mathscr{E})$ on X with $\mathscr{E}$ a quasi-coherent sheaf.

Proposition 2.7.12. *Let $X, \mathscr{E}, \mathbb{P}(\mathscr{E})$ be in the definition. Then:*

(a). If rank $\mathscr{E} \geq 2$, there is a canonical isomorphism of graded $\mathcal{O}_X$-algebras $\mathscr{I} = \bigoplus_{l\in\mathbb{Z}} \pi_*(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(l))$, with the grading on the right hand side given by l. In particular, for $l < 0$, $\pi_*(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(l)) = 0$ and $l \geq 0$, $\pi_*(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(l)) = S^l(\mathscr{E})$.

(b). There is a natural surjective morphism $\pi_*\mathscr{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$.

Proposition 2.7.13. *Let $X, \mathscr{E}, \mathbb{P}(\mathscr{E})$ be in the definition. Let $g : X \longrightarrow Y$ be any morphism. Then there exists an 1-1 corresponding*

$$\left\{f : Y \longrightarrow \mathbb{P}(\mathscr{E}) \middle| \begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "g"'] & \mathbb{P}(\mathscr{E}) \arrow[d, "\pi"] \\ & X \end{tikzcd} \right\} \leftrightarrow \{(\mathcal{L}, \varphi) | \mathcal{L} \in Pic(Y), g^*\mathscr{E} \twoheadrightarrow \mathcal{L}\}$$

Definition 2.7.14. Let $\beta : \widetilde{X} \longrightarrow X$ be a blow up at $Z \subset X$. For $Y \subset X$ such that $Y \not\subseteq Z$. Then

(1). The total transformation of Y is $\beta^{-1}(Y)$;

(2). The restrict transformation of Y is the closure of $\beta^{-1}(Y - Z) \subset \beta^{-1}(Y)$.

Proposition 2.7.15. *If $U \subset X$ is an open subsecheme and we have $\beta : \mathrm{Bl}_Z X \longrightarrow X$, then $\mathrm{Bl}_{U\cap Z} U \cong \beta^{-1}(U)$.*

Proposition 2.7.16. *If $Y \subset X$ is closed and we have $\beta : \mathrm{Bl}_Z X \longrightarrow X$, then $\mathrm{Bl}_{Z\cap Y} Y$ is the strict transformation.*

2.7.2 Examples

Example 2.7.17 (Blowing-up can not kill singularity)**.** The blow-up at $V(x_1, x_2^2) \subset \mathbb{A}_k^2$ is singular. Now, let's compute it:

Let $I = (x_1, x_2^2)$ and $\widetilde{X} = \mathrm{Ptoj}(S)$ with $S = \oplus_{i\geq 0} I^i$. We can define

$$\varphi : A[y_1, y_2] \longrightarrow S \tag{2.7.1}$$
$$y_1 \longmapsto x_1 \tag{2.7.2}$$
$$y^2 \longmapsto x_2^2 \tag{2.7.3}$$

which induces a closed immersion $\widetilde{X} \hookrightarrow \mathbb{P}_A^1$ with $A = k[x_1, x_2]$. On $V_+(y_1)$, we see that $\widetilde{X}|_{V_+(y_1)} = V(x^2 - y_2 x_1)$, which is singular at (0,0,0), and over $D_+(y_2)$, $\widetilde{X}|_{V_+(y_2)} = V(y_1 x_2^2 - x_1)$.

However, denoting $\beta : \widetilde{X} \longrightarrow \mathbb{A}_k^2$, note that the exceptional divisor $\beta^{-1}(0,0)$ is not a projective line. Just note that $\beta^{-1}(V(x_1, x_2^2))|_{V_+(y_1)} \cong V(x_1, x_2^2) \subset \mathbb{A}_k^2$ which is also non-reduced.

Example 2.7.18. Let $Q \subset \mathbb{A}_k^3$ be defined by $x_1^2 + x_2^2 + x_3^2 = 0$, which is singular at O. Then $\mathrm{Bl}_O Q$ is smooth.

Let compute the first blow up. We have already known $\beta : \mathrm{Bl}_O \mathbb{A}_k^3 \longrightarrow \mathbb{A}_k^3$ and hence $\mathrm{Bl}_O Q = \beta^{-1}(Q - O)^-$. Then we see that

$$\begin{aligned} \mathrm{Bl}_O Q|_{V_+(y_1)} &= (V(x_1y_2 - x_2, x_1y_3 - x_3, x_2y_3 - x_3y_2, x_1^2 + x_2^2 + x_3^2) \cap D(x_1, x_2, x_3))^- \\ &= (V(x_1y_2 - x_2, x_1y_3 - x_3, 1 + y_1^2 + y_2^2) \cap D(x_1, x_2, x_3))^- \\ &= V(x_1y_2 - x_2, x_1y_3 - x_3, 1 + y_1^2 + y_2^2) \end{aligned}$$

Note that the Jacobi is

$$J = \begin{pmatrix} x_1 & 0 & y_2 & -1 & 0 \\ 0 & x_1 & y_3 & 0 & -1 \\ 2y_2 & 2y_3 & 0 & 0 & 0 \end{pmatrix}$$

which is of rank 3 over $V(x_1y_2 - x_2, x_1y_3 - x_3, 1 + y_1^2 + y_2^2)$. Thus. $\mathrm{Bl}_O Q|_{V_+(y_1)}$ is smooth and similarly, we can see that $\mathrm{Bl}_O Q|_{V_+(y_2)}$ and $\mathrm{Bl}_O Q|_{V_+(y_3)}$ are smooth.

2.7.3 Exercises

Exercise 2.7.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $f : \mathcal{L} \to \mathcal{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism. [Hint: Reduce to a question of modules over a local ring by looking at the stalks.]

Exercise 2.7.2. Let X be a scheme over a field k. Let $\mathcal{L}$ be an invertible sheaf on X, and let $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ be two sets of sections of $\mathcal{L}$, which generate the same subspace $V \subseteq \Gamma(X, \mathcal{L})$, and which generate the sheaf $\mathcal{L}$ at every point. Suppose $n \leq m$. Show that the corresponding morphisms $\varphi : X \to \mathbb{P}_k^n$ and $\psi : X \to \mathbb{P}_k^m$ differ by a suitable linear projection $\mathbb{P}^m - L \to \mathbb{P}^n$ and an automorphism of $\mathbb{P}^n$, where L is a linear subspace of $\mathbb{P}^m$ of dimension $m - n - 1$.

Proof. By the define we have:

$$\begin{array}{ccccccc} X & \xrightarrow{\varphi} & \mathbb{P}_k^n & \qquad & X & \xrightarrow{\psi} & \mathbb{P}_k^m \\ \mathcal{O}_{\mathbb{P}_k^n(1)} & \xrightarrow{\varphi^*} & \mathcal{L} & & \mathcal{O}_{\mathbb{P}_k^m(1)} & \xrightarrow{\psi^*} & \mathcal{L} \\ x_i & \longmapsto & s_i & & y_j & \longmapsto & t_j \end{array}$$

We want to construct a morphism $\rho : \mathbb{P}_k^m \longrightarrow \mathbb{P}_k^n$.

Because $\{s_i\}$ and $\{t_j\}$ generated $\mathcal{L}$, suppose that $s_i = \sum a_{ij} t_j$. Then

$$\psi^*(s_i) = \sum a_{ij} \psi^*(t_j) = \sum a_{ij} y_j = u_i$$

which inspires us to define $\rho^* x_i \mapsto u_i$ such that the diagram

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\varphi^*"] \arrow[dr, "\psi^*"'] & \mathcal{O}_{\mathbb{P}_k^n}(1) \arrow[d, "\rho^*"] \\ & \mathcal{O}_{\mathbb{P}_k^m}(1) \end{tikzcd}$$

is commutative.

Now using ρ^* by Theorem 7.1 [5], $\rho : \mathbb{P}_k^m \longrightarrow \mathbb{P}_k^n$ is defined to be

$$\cup X_{u_i} \longrightarrow \mathbb{P}_k^n$$

Consider $L = Z(u_1, ..., u_n)$ which is the common zeros of $u_1, ..., u_n$ in $\mathbb{P}_k^m$. Then ρ is given by

$$\mathbb{P}_k^m - L \longrightarrow \mathbb{P}_k^n$$

Then $dim L = m - n - 1$. (We can need $\{u_i\}$ to be linearly independent.) □

Remark. We can see that $\mathbb{P}_k^m \longrightarrow \mathbb{P}_k^n$ is given by

$$(y_0 : y_1 : ...; y_m) \mapsto (\sum a_{1j} y_j : \sum a_{2j} y_j : ... : \sum a_{nj} y_j)$$

Exercise 2.7.3. Let $\varphi : \mathbb{P}^n_k \to \mathbb{P}^m_k$ be a morphism. Then:

(a) either $\varphi(\mathbb{P}^n) = \mathrm{pt}$ or $m \geq n$ and $\dim \varphi(\mathbb{P}^n) = n$;

(b) in the second case, φ can be obtained as the composition of (1) a d-uple embedding $\mathbb{P}^n \to \mathbb{P}^N$ for a uniquely determined $d \geq 1$, (2) a linear projection $\mathbb{P}^N - L \to \mathbb{P}^m$, and (3) an automorphism of $\mathbb{P}^m$. Also, φ has finite fibres.

Proof.

(a) Consider $\varphi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k$.

Now that we can use $\mathcal{L} = \varphi^*(\mathcal{O}_{\mathbb{P}^1_k}(1))$ to describe φ by Theorem 7.1 [5]. Note that the set of line bundles on $\mathbb{P}^n_k$, i.e. $Pic(\mathbb{P}^n_k)$ is of the form $\mathcal{O}_{\mathbb{P}^n_k}(d)$, with $d \in \mathbb{Z}$ by Corollary 6.16[5]. Then $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \in \mathbb{Z}$.

CASE 1: $d < 0$, at this case $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = 0$.

CASE 2: $d = 0$, $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = k$. Suppose that x

$$\varphi^* x_i = s_i$$

for $i = 1, ..., m$. Then φ is given by

$$(x_1 : x_2 : ... : x_n) \mapsto (s_1 : ... : s_m)$$

Thus, $\varphi(\mathbb{P}^n_k) = \{pt\}$.

CASE 3: When $m < n$, $d \geq 1$.

Suppose that $s_i \in k[y_1, .., y_m]_d$ for $i = 1, 2, ..., m$. Then the morphism is defined to be

$$\cup X_{s_i} \longrightarrow \mathbb{P}^n_k$$

However, for $m < n$, $Z(s_1, ..., s_m)$ can not be empty. So there exist $P \in \mathbb{P}^n_k$ can not be mapped to $\mathbb{P}^m_k$. Thus φ is not a morphism form $\mathbb{P}^n_k$ to $\mathbb{P}^m_k$.

SUMMIT: When $m < n$, $im(\varphi)$ can not be a point.

CASE 4: When $m \geq n$, $d \geq 1$.

We will prove this by induction. When m=n, if $\varphi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k$ is a surjetive, then $dim(im\varphi) = dim(\mathbb{P}^m_k) = n$. If not, for $\varphi(\mathbb{P}^n_k)$ is closed in $\mathbb{P}^m_k$ by 2.4.3, take $P \notin im(\varphi)$.Then we have

$$\varphi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k - P \cong \mathbb{P}^{m-1}_k$$

By the SUMMIT above, $im(\varphi)$ is just a point.

Now consider the general cases. We fix n and induct on m. We have seen that when $m = n$, $dim(im(\varphi)) = n$ or $\varphi(\mathbb{P}^n_k) = \{pt\}$.

If $\varphi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k$ is surjetive, then $dim(im\varphi) = dim(\mathbb{P}^m_k) = m \geq n$ and $dim(im(\varphi)) \leq n$ by the property of morphisms. Then $dim(im(\varphi)) = n$. If not (by the same argument above), then

$$\varphi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k - P \cong \mathbb{P}^{m-1}_k$$

By indcutive hypothesis, $dim(im(\varphi)) = n$ or $im(\varphi) = \{pt\}$.

(b) Consider the Veronese embedding,

$$\mathbb{P}^n_k \longrightarrow \mathbb{P}^N_k$$

with $N = \binom{N+d}{d} - 1$, given by

$$x_i \mapsto x_1^{j_i} x_2^{j_2} ... x_n^{j_n}$$

with $j_1 + j_2 + ... + j_n = n + d$. Then by Exe **??**, we have

$$\mathbb{P}_k^N - L \longrightarrow \mathbb{P}_k^m$$

Thus φ can be decomposed to

$$\mathbb{P}_k^n \longrightarrow \mathbb{P}_k^N, \mathbb{P}_k^N - L \longrightarrow \mathbb{P}_k^m, \mathbb{P}_k^m \xrightarrow{\cong} \mathbb{P}_k^m$$

Let $y \in \mathbb{P}_k^m$ be a closed point. For φ is dominant, $dim(\varphi^{-1}(y)) = dim(X) - dim(\operatorname{im}\varphi) = 0$. Thus, $\varphi^{-1}(y)$ is finite. Thus φ is quasi-finite.

Because,

$$\begin{tikzcd} \mathbb{P}_k^n \arrow[r, "\varphi"] \arrow[dr, "p_n"'] & \mathbb{P}_k^m \arrow[d, "p_m"] \\ & Spec(k) \end{tikzcd}$$

is commutative and p_n. p_m are proper. Then φ is finite. □

Exercise 2.7.4.

(a) Use (7.6) to show that if X is a scheme of finite type over a noetherian ring A, and if X admits an ample invertible sheaf, then X is separated.

(b) Let X be the affine line over a field k with the origin doubled (4.0.1). Calculate Pic X, determine which invertible sheaves are generated by global sections, and then show directly (without using (a)) that there is no ample invertible sheaf on X.

Proof.

(a). By the assumption, we have $f : X \longrightarrow Spec(A)$ such that f is of finite type. We know that there exists some n, such that $i : X \longrightarrow P_A^n$ is a clsoed immersion, for X admits an ample sheaf $\mathcal{L}$ and then X admits a very ample sheaf $\mathcal{L}^{\otimes n}$ for some n. Thus, i is proper. Because, $\pi : \mathbb{P}_k^A \longrightarrow Spec(A)$ is also proper, then $f = \pi \circ i$ is also proper. Thus, f is separated i.e. X is separated over A.

(b). **IF WE USE (a)**: Because the line of double points is quasi-finite over k[x], which is a Noetherian ring for k is a field. Thus, if it admits any ample line bundle, then it is separated. But we know that the line of double point is not separated. It has no ample line bundle. □

Exercise 2.7.5. Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathcal{L}, \mathcal{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a noetherian ring A.

(a) If $\mathcal{L}$ is ample and $\mathcal{M}$ is generated by global sections, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

(b) If $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then $\mathcal{M} \otimes \mathcal{L}^n$ is ample for sufficiently large n.

(c) If $\mathcal{L}, \mathcal{M}$ are both ample, so is $\mathcal{L} \otimes \mathcal{M}$.

(d) If $\mathcal{L}$ is very ample and $\mathcal{M}$ is generated by global sections, then $\mathcal{L} \otimes \mathcal{M}$ is very ample.

(e) If $\mathcal{L}$ is ample, then there is an $n_0 > 0$ such that $\mathcal{L}^n$ is very ample for all $n \geq n_0$.

Proof.

(a). Because $\mathcal{M}$ is generated by global sections. So for any $m \in \mathbb{N}$, $\mathcal{M}^{\otimes m}$ is generated by global sections.

Now take any $\mathcal{F} \in Coh(X)$. Because $\mathbb{L}$ is ample, then there exists $n \in \mathbb{N}$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. So

$$\mathcal{F} \otimes (\mathcal{L} \otimes \mathcal{M})^{\otimes n} = \mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{M}^{\otimes n}$$

is globally generated. Thus, $\mathcal{L} \otimes \mathcal{M}$ is globally generated.

(b). For $\mathcal{M}$ is coherent, there exists $n-1 \in \mathbb{N}$ such that

$$\mathcal{M} \otimes \mathcal{L}^{\otimes (n-1)}$$

is generated by global sections. By (a),

$$\mathcal{M} \otimes \mathcal{L}^{n} = (\mathcal{M} \otimes \mathcal{L}^{n-1}) \otimes \mathcal{L}$$

is ample.

(c). First of also, we see that there exists $n \in \mathbb{N}$ such that

$$M \otimes \mathcal{L}^{\otimes n}$$

is generated by global sections.

Because $\mathcal{M}$ is ample, then $\mathcal{M}^{\otimes (n-1)}$ is also ample by Proposition 7.5 **??**. Again by (a)

$$(\mathcal{M} \otimes \mathcal{L})^{\otimes n} = \mathcal{M}^{\otimes (n-1)} \otimes M \otimes \mathcal{L}^{\otimes n}$$

is also ample. Again by Proposition 7.5 **??**,

$$\mathcal{L} \otimes \mathcal{M}$$

is ample.

(d)Because $\mathcal{L}$ is very ample and $\mathcal{M}$ is globally genertated, then there exist a closed immersion:

$$i : X \hookrightarrow \mathbb{P}^n_l$$

and a morphism

$$\varphi : X \longrightarrow \mathbb{P}^k_m$$

with $i^*\mathcal{O}_{\mathbb{P}^n_k} = \mathcal{L}$ and $\varphi^*\mathcal{O}_{\mathbb{P}^m_k} = \mathcal{M}$.

Consider $\varphi \otimes id_X : X \longrightarrow X \times_k \mathbb{P}^m_k$ given by the fiber product and it is a closed immersion, for $p_X \circ \varphi \otimes id_X = id_X$ with $p_X : X \times_k \mathbb{P}^m_k \longrightarrow X$. Finally note that $\varphi \otimes {id_X}^* \mathcal{O}_{\mathbb{P}^n_k}(1) \otimes \mathcal{L} = \mathcal{M} \otimes \mathcal{L}$.

Then consider $id_m \otimes i : \mathbb{P}^m_k \times_k X \longrightarrow \mathbb{P}^n_k \times_k \mathbb{P}^m_k$ deduced by the fiber product. Using Magic Square in P164 [12],

$$\begin{array}{ccc} \mathbb{P}^m_k \times_k X & \longrightarrow & \mathbb{P}^n_k \times_k \mathbb{P}^m_k \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{P}^n_k \end{array}$$

and the fact that closed immersions are stable under base change, we see that $id_m \otimes i$ is a clsoed immerison and $id_m \otimes i^*\mathcal{O}_{\mathbb{P}^m_k}(1) \otimes \mathcal{O}_{\mathbb{P}^n_k}(1) = \mathcal{O}_{\mathbb{P}^n_k}(1) \otimes \mathcal{L}$

Finally, note that the Serge embedding

$$i_s : \mathbb{P}^n_k \times_k \mathbb{P}^m_k \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}_k$$

$$(x_1 : x_2 : ... : x_m : y_1 : ... : y_m) \mapsto (x_1 y_1 : ... : x_1 y_m : .... : x_n y_1 : ... : x_n y_m)$$

is given by

$$i_S^* \mathcal{O}_{\mathbb{P}^{(n+1)(m+1)-1}_k}(1) = \mathcal{O}_{\mathbb{P}^m_k}(1) \otimes \mathcal{O}_{\mathbb{P}^n_k}(1)$$

$$z_{ij} \mapsto x_i y_j$$

.

Let $f = i_s \circ id_m \otimes i \circ \varphi \otimes id_X$. f is a closed immersion and $f^* O_{\mathbb{P}^{(m+1)(n+1)-1}}(1) = \mathcal{L} \otimes \mathcal{M}$. So $\mathcal{L} \otimes \mathcal{M}$ is very ample.

You'd better write a diagram to understand the relations of these maps

(e). For $\mathcal{L}$ is coherent, there exists $m_0 \in \mathbb{N}$ such that $\forall n \geq m_0$

$$\mathcal{L}^{\otimes n}$$

is globally generated.

Because $\mathcal{L}$ is ample, there exist $d \in \mathbb{N}$ such that

$$\mathcal{L}^{\otimes d}$$

is very ample.

By (d) and take $n_0 = m_0 + d$, then for any $n \geq n_0$, $\mathcal{L}^{\otimes n}$ is very ample. □

Exercise 2.7.6 (The Riemann-Roch Problem)**.** Let X be a nonsingular projective variety over an algebraically closed field, and let D be a divisor on X. For any $n > 0$ we consider the complete linear system $|nD|$. Then the Riemann-Roch problem is to determine $\dim |nD|$ as a function of n, and, in particular, its behavior for large n. If $\mathcal{L}$ is the corresponding invertible sheaf, then $\dim |nD| = \dim \Gamma(X, \mathcal{L}^n) - 1$, so an equivalent problem is to determine $\dim \Gamma(X, \mathcal{L}^n)$ as a function of n.

(a) Show that if D is very ample, and if $X \subset \mathbb{P}^n_k$ is the corresponding embedding in projective space, then for all n sufficiently large, $\dim |nD| = P_X(n) - 1$, where P_X is the Hilbert polynomial of X (I, §7). Thus in this case $\dim |nD|$ is a polynomial function of n, for n large.

(b) If D corresponds to a torsion element of $\operatorname{Pic} X$, of order r, then $\dim |nD| = 0$ if $r \mid n$, and -1 otherwise. In this case the function is periodic of period r.

It follows from the general Riemann-Roch theorem that $\dim |nD|$ is a polynomial function for n large, whenever D is an ample divisor. See (IV, 1.3.2), (V, 1.6), and Appendix A. In the case of algebraic surfaces, Zariski [7] has shown for any effective divisor D, that there is a finite set of polynomials

Exercise 2.7.7. Let $X = \mathbb{P}^2_k$, and let $|D|$ be the complete linear system of all divisors of degree 2 on X (conics). D corresponds to the invertible sheaf $\mathcal{O}(2)$, whose space of global sections has a basis $x^2, y^2, z^2, xy, xz, yz$, where x, y, z are the homogeneous coordinates of X.

(a) The complete linear system $|D|$ gives an embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$, whose image is the Veronese surface (I, Ex. 2.13).

(b) Show that the subsystem defined by $x^2, y^2, z^2, y(x-z), (x-y)z$ gives a closed immersion of X into $\mathbb{P}^4$. The image is called the Veronese surface in $\mathbb{P}^4$. Cf. (IV, Ex. 3.11).

(c) Let $b \subseteq |D|$ be the linear system of all conics passing through a fixed point P. Then b gives an immersion of $U = X - P$ into $\mathbb{P}^4$. Furthermore, if we blow up P, to get a surface $\tilde{X}$, then this map extends to give a closed immersion of $\tilde{X}$ in $\mathbb{P}^4$. Show that $\tilde{X}$ is a surface of degree 3 in $\mathbb{P}^4$, and that the lines in X through P are transformed into straight lines in $\tilde{X}$ which do not meet. $\tilde{X}$ is the union of all these lines, so we say $\tilde{X}$ is a ruled surface (V, 2.19.1).

Proof.
(a). As we discuss above the Veronese embedding gives

$$i_S^*\mathcal{O}_{\mathbb{P}^5_k}(1) \longrightarrow \mathbb{O}_{\mathbb{P}^2_k}(2)$$

by $x_1 \mapsto x^2, x_2 \mapsto y^2, x_3 \mapsto z^2 \ldots$. Then use the uniqueness of Theorem 7.1 [5]. The embedding given by $|D|$ is just the Veronese embedding.

(b). Now we see that

$$\begin{aligned} \varphi : \mathbb{P}^2_k &\longrightarrow \mathbb{P}^3_k \\ (x:y:z) &\longmapsto (x^2, y^2, z^2, y(x-z), (x-y)z)) \end{aligned}$$

$X_{x^2} = D_+(x)$ which is affine and $k[y_1, y_2, y_3, y_4] \longrightarrow k[\frac{y}{x}, \frac{z}{x}]$ given by

$$y_1 \mapsto \frac{y^2}{x^2}, y_2 \mapsto \frac{z^2}{x^2}, y_3 \mapsto \frac{y(x-z)}{x^2}, y_4 \mapsto \frac{(x-y)z}{x^2}$$

is surjective. Similarly for X_{y^2} and X_{z^2}.

$X_{y(x-z)}$ is $D(\frac{x}{y} - \frac{z}{y}) \subset D+(y)$.Thus, $X_{y(x-z)} = Spec(k[\frac{x}{y}, \frac{z}{y}]_{\frac{x}{y}-\frac{z}{y}})$ with $k[y_1, y_2]$ and $k[y_1, y_2, y_3, y_4] \longrightarrow k[\frac{x}{y}, \frac{z}{y}]_{\frac{x}{y}-\frac{z}{y}}$ given by

$$y_1 \mapsto \frac{x^2}{y(x-z)}, y_2 \mapsto \frac{y^2}{y(x-z)}, y_3 \mapsto \frac{z^2}{y(x-z)}, y_4 \mapsto \frac{(x-y)z}{y(x-z)}$$

is surjctive. (Let $s = \frac{x}{y} - \frac{z}{y}, t = \frac{x}{y} + \frac{z}{y}$. Write $k[\frac{x}{y}, \frac{z}{y}]_{\frac{x}{y}-\frac{z}{y}} = k[s,t]_s = k[s, s^{-1}, t]$. Then it is easy to see that the morphism is surjective.) Similary for $X_{(x-y)z}$.

Hence $\mathbb{P}^2_k \longrightarrow \mathbb{P}^3_k$ defined by the linear system $x^2, y^2, z^2, y(x-z), (x-y)z$ is a closed immersion. □

Exercise 2.7.8. Let X be a noetherian scheme, let $\mathcal{E}$ be a coherent locally free sheaf on X, and let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the corresponding projective space bundle. Show that there is a natural 1-1 correspondence between sections of π (i.e., morphisms $\sigma : X \to \mathbb{P}(\mathcal{E})$ such that $\pi \circ \sigma = \mathrm{id}_X$) and quotient invertible sheaves $\mathcal{E} \to \mathcal{L} \to 0$ of $\mathcal{E}$.

Proof. By 2.7.13, we have the following corresponding:

$$\left\{ f : X \longrightarrow \mathbb{P}(\mathscr{E}) \;\middle|\; \begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "id"'] & \mathbb{P}(\mathscr{E}) \arrow[d, "\pi"] \\ & X \end{tikzcd} \right\} \leftrightarrow \{(\mathcal{L}, \varphi) | \mathcal{L} \in Pic(x), id^*\mathscr{E} \twoheadrightarrow \mathcal{L}\}$$

Note that $f : X \longrightarrow \mathbb{P}(\mathscr{E})$ is just a section of π. □

Exercise 2.7.9. Let X be a regular noetherian scheme, and $\mathcal{E}$ a locally free coherent sheaf of rank ≥ 2 on X.

(a) Show that $\operatorname{Pic}\mathbb{P}(\mathcal{E}) \cong \operatorname{Pic}X \times \mathbb{Z}$.

(b) If $\mathcal{E}'$ is another locally free coherent sheaf on X, show that $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}')$ (over X) if and only if there is an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.

Proof. (a).First, now that we should assume that X is irreducible. Consider $X = Spec(k \oplus k) = Spec(k) \amalg Spec(k)$. Since $Pic(Spec(k)) = 0$, $Pic(X) = Pic(Spec(k)) \oplus Pic(Spec(k)) = 0$. However, for $\mathscr{E} = \mathcal{O}_X^{\oplus 2}$, $\mathbb{P}(\mathscr{E}) = \mathbb{P}_k^1 \amalg \mathbb{P}_k^1$. Hence, $Pic(\mathbb{P}(\mathscr{E})) = \mathbb{Z} \times \mathbb{Z} \neq Pic(X) \times \mathbb{Z}$.

From now on, we just assume that X is integral: Since X is regular and each regular ring is a domain [P184 Corollary 13.6 [8]], X is reduced and hence integral.

We can define the map

$$\begin{aligned} Pic(X) \times \mathbb{Z} &\longrightarrow Pic(\mathbb{P}(\mathscr{E})) \\ (\mathcal{L}, n) &\longmapsto \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \end{aligned}$$

First, we verify the the map is injective: If $\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) = \mathcal{O}_{\mathbb{P}(\mathscr{E})}$, by 2.7.12 and projection formula 2.5.1, $\pi_*(\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)) = \mathcal{L} \otimes \pi_*\mathcal{O}_{\mathbb{P}(\mathscr{E})}(n) = \mathcal{L} \otimes S^n(\mathscr{E}) = \mathcal{O}_X$. By comparing the rank of these two bundle, $n = 0$ and hence $\mathcal{L} = \mathcal{O}_X$.

Next, we verify that the map is surjective: We can choose a cover $\{U_i = Spec(A_i)\}$ of X such that $U_i \cap U_j$ is also affine.

Since U_i is affine, it is separated. So each U_i is a Noetherian integral separated scheme and it is regular of codimension 1 (Note that at any point $x \in X$, $\mathcal{O}_{X,x}$ is regular.) For each regular local ring is factorial [P184 [8]] and U_i is integral, $Cl(X) = Pic(X)$. Since if A_i is regular, $A_i[x_1, ..., x_n]$ is also regular [P193 Exe13.1 [8]], we see that $\mathbb{P} \times U$ is also regular and hence $Cl(\mathbb{P}^n \times U) = Pic(\mathbb{P}^n \times U)$. Now by 2.6.1,

$$Pic(\mathbb{P}^n \times U_i) = Pic(U_i) \times \mathbb{Z}$$

where the isomorphism is just defined as above.

Now, consider $\mathcal{L} \in Pic(\mathbb{P}(\mathscr{E}))$. Then $\{\mathbb{P}(S^{\cdot}(\mathscr{E}(U_i))) \times U_i\} \cong \{\mathbb{P}_{A_i}^n \times U_i\}$ forms a cover of $\mathbb{P}(\mathscr{E})$. To simplify, we denote $\mathbb{P}_{A_i}^n \times U_i$ by V_i. Then $\mathcal{L}|_{V_i}$ can be written as

$$\mathcal{L}|_{V_i} = \pi_{U_i}^*\mathcal{L}_i \otimes \mathcal{O}_{\mathbb{P}_{A_i}^n}(n_i)$$

Restricting $\mathcal{L}$ on $V_i \cap V_j$, then

$$\mathcal{L}|_{V_i \cap V_j} = \pi_{U_i \cap U_j}^*(\mathcal{L}_i|_{U_i \cap U_j}) \otimes \mathcal{O}_{\mathbb{P}_{A_i}^n}(n_i)|_{\mathbb{P}_{A_{ij}}^n} = \pi_{U_i \cap U_j}^*(\mathcal{L}_j|_{U_i \cap U_j}) \otimes \mathcal{O}_{\mathbb{P}_{A_j}^n}(n_j)|_{\mathbb{P}_{A_{ij}}^n}$$

By the injectivity we have proved, $n_i = n_j$ and $\mathcal{L}_i|_{U_i \cap U_j} = \mathcal{L}_j|_{U_i \cap U_j}$. Hence, we $\mathcal{L}_i$ can be glued to $\mathcal{L}_X \in Pic(X)$. Since any two sets in $\{U_i\}$ have non-empty intersection, let $n = n_i$. Then locally, $\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathscr{E})}(-n)|_{V_i} = \pi_{U_i}^*\mathcal{L}_i$. Hence, globally, $\mathcal{L} = \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathscr{E})}(n)$.

Hence, by 2.6.1 and the fact that each

(b). ($\Longrightarrow$): Let $f : \mathbb{P}(\mathscr{E}) \longrightarrow \mathbb{P}(\mathscr{E}^{'})$ be the isomorphism with g the inverse and suppose $f^*\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) = \mathcal{O}_{\mathbb{P}(\mathscr{E}')}(n^{'}) \otimes \pi^{'*}\mathcal{L}^{'}$ with $\mathcal{L}^{'} \in Pic(X)$. Then

$$\begin{aligned} \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) &= g^*f^*\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) \\ &= g^*\mathcal{O}_{\mathbb{P}(\mathscr{E}')}(n^{'}) \otimes g^*\pi^{'*}\mathcal{L}^{'} \\ &= g^*\mathcal{O}_{\mathbb{P}(\mathscr{E}')}(n^{'}) \otimes \pi^*\mathcal{L}^{'} \end{aligned}$$

$$= g^*(\mathcal{O}_{\mathbb{P}(\mathscr{E}')}(1))^{\otimes n'} \otimes \pi^*\mathcal{L}'$$

So we suppose that $g^*(\mathcal{O}_{\mathbb{P}(\mathscr{E}')}(1)) = \mathcal{O}_{\mathbb{P}(\mathscr{E})}(n) \otimes \pi_*\mathcal{L}$. By the injectivity of (a), we have

$$nn' = 1, \mathcal{L}^{\otimes n} \otimes \mathcal{L}' = \mathcal{O}_X$$

Case I: if $n = -1$ and $n' = -1$. Then $f^*\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) = \mathcal{O}_{\mathbb{P}(\mathscr{E}')}(-1) \otimes {\pi'}^*\mathcal{L}$. Then, locally on an affine open set $U = Spec(A)$, $f : \mathbb{P}_A \longrightarrow \mathbb{P}_A$ is given by $f^*\mathcal{O}(1) = \mathcal{O}(-1)$ which cannot be an isomorphism, which leads to a contradiction.

Case II: If $n = 1$ and $n' = -1$. Then $f^*\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) = \mathcal{O}_{\mathbb{P}(\mathscr{E}')}(1) \otimes {\pi'}^*\mathcal{L}$. Use projective formula and $\pi'_*f^* = \pi_*$(Because π is flat, then use the property of flatness.), we have $\mathscr{E} = \mathscr{E}' \otimes \mathcal{L}$.

($\Longleftarrow$): By 2.7.12, we have

$$\pi^*\mathscr{E} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$$

and then

$$\pi^*\mathcal{E}' \twoheadrightarrow \pi^*\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*\mathcal{L}$$

which induces a morphism $\mathbb{P}(\mathscr{E}) \longrightarrow \mathbb{P}(\mathscr{E}')$. Similarly, we have $\mathbb{P}(\mathscr{E}') \longrightarrow \mathbb{P}(\mathscr{E})$. And the composition is given by $\pi^*\mathscr{E} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$, which implies the composition morphism is just identity. Hence $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$.

□

Exercise 2.7.10. $\mathbb{P}^n$-Bundles Over a Scheme. Let X be a noetherian scheme.

(a) By analogy with the definition of a vector bundle (Ex. 5.18), define the notion of a projective n-space bundle over X, as a scheme P with a morphism $\pi : P \to X$ such that P is locally isomorphic to $U \times \mathbb{P}^n$, $U \subseteq X$ open, and the transition automorphisms on $\operatorname{Spec} A \times \mathbb{P}^n$ are given by A-linear automorphisms of the homogeneous coordinate ring $A[x_0, \ldots, x_n]$ (e.g., $x'_i = \sum a_{ij}x_j, a_{ij} \in A$).

(b) If $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on X, then $\mathbb{P}(\mathcal{E})$ is a $\mathbb{P}^n$-bundle over X.

(c) Assume that X is regular, and show that every $\mathbb{P}^n$-bundle P over X is isomorphic to $\mathbb{P}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ on X.
Hint: Let $U \subseteq X$ be an open set such that $\pi^{-1}(U) \cong U \times \mathbb{P}^n$, and let $\mathcal{L}_0$ be the invertible sheaf $\mathcal{O}(1)$ on $U \times \mathbb{P}^n$. Show that $\mathcal{L}_0$ extends to an invertible sheaf $\mathcal{L}$ on P. Then show that $\pi_*\mathcal{L} = \mathcal{E}$ is a locally free sheaf on X and that $P \cong \mathbb{P}(\mathcal{E})$. Can you weaken the hypothesis "X regular"?

(d) Conclude (in the case X regular) that we have a 1-1 correspondence between $\mathbb{P}^n$-bundles over X, and equivalence classes of locally free sheaves $\mathcal{E}$ of rank $n+1$ under the equivalence relation $\mathcal{E}' \sim \mathcal{E}$ if and only if $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on X.

Proof. (a). This is just the definition of projecticve bundles.

(b). Consider a affine cover $\{U_i = Spec(A_i)\}$ of X. Then locally, on U_i, $\mathscr{E}(U_i) \cong A_i^{\oplus(n+1)}$ and $S^{\cdot}(\mathscr{E})(U_i) \cong A_i[x_0, x_1, ..., x_n]$. By the definition of $\mathbb{P}(\mathscr{E}) \longrightarrow X$, on $\pi^{-1}(U_i)$, $\pi|_{\pi^{-1}(U_i)}$ is just $Proj(A_i[x_0, ..., x_n]) \cong \mathbb{P}_{A_i}^n = U \times \mathbb{P}^n \longrightarrow U_i = Spec(A_i)$.

On each affine open subset $V = Spec(A) \subset U_i \cap U_j$, the transition automorphism is given by A-linear automorphism

$$\mathscr{E}|_{U_i}(V) = A^{\oplus n} \longrightarrow \mathscr{E}|_{U_j}(V) = A^{\oplus n}$$

which induces an A-linear automorphism φ_{ij}

$$S^{\cdot}(\mathscr{E}|_{U_i})(V) = A[x_0, ..., x_n] \longrightarrow S^{\cdot}(\mathscr{E}|_{U_j})(V) = A[x_0, ..., x_n]$$

Note that φ_{ij} is a morphism preserving the degree of two graded ring. By **??**, we have a morphism

$$Proj(S^{\cdot}(\mathscr{E}|_{U_i})(V)) \longrightarrow Proj(S^{\cdot}(\mathscr{E}|_{U_j})(V))$$

which is just the gluing function from $\pi_{U_i}^{-1}(V)$ to $\pi_{U_j}^{-1}(V)$. By the definition of projective bundle, $\pi : \mathbb{P}(\mathscr{E}) \longrightarrow X$ is just a projective bundle.

(c). Find the solution!!! Let $\pi : P \longrightarrow X$ be any projective bundle and $\{U_i = Spec(A_i)\}$ be a cover of X such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^n$ and $U_i \cap U_k$ is also affine.

By the hint, we let $\mathcal{L}_i$ be the invertible sheaf on $U \times \mathbb{P}^n$. Then for $V = U_i \cap U_j$, the transitive automorphism

$$\varphi_{ij}|_{V\times\mathbb{P}^n} : U_i \times \mathbb{P}^n|_{V\times\mathbb{P}^n} \longrightarrow U_j \times \mathbb{P}^n|_{V\times\mathbb{P}^n}$$

induces a map

$$\pi_i|_V^*\mathcal{N} \otimes \mathcal{O}_{\mathbb{P}^n_{A_i}}(l) = \varphi_{ij}^*\mathcal{L}_j|_{V\times\mathbb{P}^n}$$

with $\mathcal{N} \in Pic(V)$. By the same argument in 2.7.9, we see that $l = 1$. Hence

$$\pi_i|_V^*\mathcal{N} \otimes \mathcal{L}_i|_{V\times\mathbb{P}^n} = \varphi_{ij}^*\mathcal{L}_j|_{V\times\mathbb{P}^n}$$

By flatness, we have

$$\pi_i^* i_{VU_i*}\mathcal{N} \otimes \mathcal{L}_i|_{V\times\mathbb{P}^n} = \varphi_{ij}^*\mathcal{L}_j|_{V\times\mathbb{P}^n}$$

Now fix U_0 and $\mathcal{L}_0$ in $U_0 \times \mathbb{P}^n$. Since X is Noetherian, it can be covered by finitely many affine open sets. Then we can use

$$\pi_i|_V^*\mathcal{N} \otimes \mathcal{L}_i|_{V\times\mathbb{P}^n} = \varphi_{ij}^*\mathcal{L}_j|_{V\times\mathbb{P}^n}$$

to extend $\mathcal{L}_{\prime}$ to $\mathcal{L}$ such that $\mathcal{L}|_{U_0} = \mathcal{L}$.???

On each $\pi_i : U_i \times \mathbb{P}^n \longrightarrow U_i$, consider $\pi_{i_*}\mathcal{L}_i$. Since U_i is affine and π_i is proper, $\pi_*\mathcal{L}_i$ is coherent on U_i [[5] P115 Caution 5.8.1]. For $\pi_{i_*}\mathcal{L}_i(U_i) = H^0(U_i \times \mathbb{P}^n, \mathcal{O}(1)) = Span_{A_i}[x_0, ..., x_n]$, we see that $\pi_{i*}\mathcal{L}_i$ is locally free of rank $n+1$.

Using the same method we can see that $\{\pi_{i*}\mathcal{L}_i\}$ can be glued to a morphism on X, denoted by $\mathscr{E}$ and $\mathscr{E} = \pi_*\mathcal{L}$. Then since the gluing function of $\mathscr{E}$ is given by the transitive automorphism on P, the gluing function of $\mathbb{P}(\mathscr{E})$ is the same as the transitive automorphism on P. Hence, $P \cong \mathbb{P}(\mathscr{E})$.

Remark: X should be regular, see [**https://math.stackexchange.com/questions/4013148/what-is-the-pro**

(d).Directly from (c) and 2.7.9-(b).

□

Exercise 2.7.11. On a noetherian scheme X, different sheaves of ideals can give rise to isomorphic blown up schemes.

(a) If $\mathcal{I}$ is any coherent sheaf of ideals on X, show that blowing up $\mathcal{I}^d$ for any $d \geq 1$ gives a scheme isomorphic to the blowing up of $\mathcal{I}$ (cf. Ex. 5.13).

(b) If $\mathcal{I}$ is any coherent sheaf of ideals, and if $\mathcal{J}$ is an invertible sheaf of ideals, then $\mathcal{I}$ and $\mathcal{I} \cdot \mathcal{J}$ give isomorphic blowings-up.

(c) If X is regular, show that (7.17) can be strengthened as follows. Let $U \subseteq X$ be the largest open set such that $f : f^{-1}(U) \to U$ is an isomorphism. Then $\mathcal{I}$ can be chosen such that the corresponding closed subscheme Y has support equal to $X - U$.

Proof. To begin with, we just assume that X admits an affine cover $\{U_i = Spec(A_i)\}$ with $\{U_i \cap U_j = Spec(A_{ij})\}$ affine.

(a). Locally, on U_i, we have a grades ring homomorphism $\varphi_i : \oplus_k \mathscr{I}^k(U_i) \longrightarrow \oplus_k \mathscr{I}^{dk}(U_i)$ deduced by $a \mapsto a^d$. This homomorphism induces $Proj(\varphi_i) : Proj(\oplus_k \mathscr{I}^{dk}(U_i)) \longrightarrow Proj(\oplus_k \mathscr{I}^k(U_i))$. By 2.5.13, $Proj(\varphi_i)$ induces an isomorphism.

Moreover, note that the gluing functions of $\mathscr{I}$ induces the same gluing functions for $\mathbb{P}(\mathscr{I})$ and $\mathbb{P}(\mathscr{I}^d)$, that is, we have the commutative diagram:

$$\begin{array}{ccccccc}
\oplus_k \mathscr{I}^k(U_i) & \xrightarrow{\oplus\rho_{I,U_iU_{ij}}} & \oplus_k \mathscr{I}^k(U_{ij}) & \xrightarrow{\oplus g_{I,U_iU_j}} & \oplus_k \mathscr{I}^k(U_{ij}) & \xleftarrow{\oplus\rho_{I,U_jU_{ij}}} & \oplus_k \mathscr{I}^k(U_j) \\
\downarrow{\varphi_i} & & \downarrow{\varphi_{ij}} & & \downarrow{\varphi_{ij}} & & \downarrow{\varphi_j} \\
\oplus_k \mathscr{I}^{dk}(U_i) & \xrightarrow{\oplus\rho_{I,U_iU_{ij}}} & \oplus_k \mathscr{I}^{dk}(U_{ij}) & \xrightarrow{\oplus g_{I,U_iU_j}} & \oplus_k \mathscr{I}^{dk}(U_{ij}) & \xleftarrow{\oplus\rho_{I,U_jU_{ij}}} & \oplus_k \mathscr{I}^{dk}(U_j)
\end{array}$$

Hence, we see that $\mathbb{P}(\mathscr{I}) \cong \mathbb{P}(\mathscr{I}^d)$ as blowings-up, meaning that the isomorphism $Proj(\varphi)$: $\mathbb{P}(\mathscr{I}) \longrightarrow \mathbb{P}(\mathscr{I}^d)$ commutes with π and π^d.

(b). We have now that if $\mathscr{I}' = \mathscr{I} * \mathscr{L}$, then $\mathbb{P}(\mathscr{I}') \cong \mathbb{P}(\mathscr{I})$ as blowings-up. It is enough to show that $\mathscr{I} * \mathscr{L} \cong \oplus_k(\mathscr{I} \cdot \mathscr{L})^d$. Note that locally, we have

$$\mathscr{I}^n(U_i) \otimes \mathscr{L}^n(U_i) \cong \mathscr{I}(U_i)^n \cdot \mathscr{L}(U_i)^n$$

since $\mathscr{I}(U_i)$ is a finitely generated A_i-module.

Then just verifying the gluing criteria as above, we see that $\mathscr{I} * \mathscr{L} \cong_k \oplus(\mathscr{I} \cdot \mathscr{L})^k$ with the same graded structure. Hence, $\mathbb{P}(\mathscr{I}) \cong \mathbb{P}(\mathscr{I} \cdot \mathscr{L})$ as blowings-up.

(c). Without loss of generality, we assume that X is irreducible. Hence, X is integral.

Let $Z = V(\mathscr{I})$ and we will verify that $U = X - Z$ is just the maximal open subset. If U' is an affine open set of X and $U' \cap Z \neq \emptyset$,, then suppose $Z \cap U' \subset U' = Spec(A')$ is given by $I' \subset A'$. Then $\mathscr{I}|_{U'} = I^\sim$ and $\pi^{-1}(U') = Proj(\oplus_k I^k) \times U'$ which is not an isomrophsim.

Conversely, if $U' = Spce(A') \cap Z = \emptyset$, then $\mathscr{I}|_{U'} \cong \mathcal{O}_{U'}$. At this case, $Proj(\oplus_k \mathcal{O}^k_{U'}) = \emptyset$ since $\mathcal{O}^k_{U'} = \mathcal{O}_{U'}$ and so $\pi^{-1}(U') = U'$.

Hence, $U = X - Z$ is just the maxiaml open set such that $\mathscr{I}$ supports on it. Now choose the ideal sheaf defining Z, denote by $\mathscr{L}$. Then there exists $\mathscr{N} \in Pic(X)$ such that $\mathscr{L} = \mathscr{I} \otimes \mathscr{N}$ (Just choose $f_i \in A_i^\times$ to construct the local isomorphism, then show that $\{f_i\}$ forms a Cartier divisor. Since X is regular and integral, this Cartier divisor corresponds to an invertible sheaf). By (b), $\mathbb{P}(\mathscr{I}) \cong \mathbb{P}(\mathscr{L})$. $\square$

Remark. If $I \subset A$ is an ideal and M is a finitely-generated A-module. Then

$$M \otimes_A I = IM$$

Exercise 2.7.12. Let X be a noetherian scheme, and let Y, Z be two closed subschemes, neither one containing the other. Let $\tilde{X}$ be obtained by blowing up $Y \cap Z$ (defined by the ideal sheaf $\mathcal{I}_Y + \mathcal{I}_Z$). Show that the strict transforms $\tilde{Y}$ and $\tilde{Z}$ of Y and Z in $\tilde{X}$ do not meet.

Proof. We just consider this question locally. Consider $U = Spec(A)$ with Y, Z defined by I_Y and I_Z, with $i_Y : Spec(A/I_Y) \hookrightarrow U = Spec(A)$ and $i_Z : Spec(A/I_Z) \longrightarrow U = Spec(A)$. Then for Y, $i_Y^{-1}(\mathscr{I}_X + \mathscr{I}_Y)\mathcal{O}_Y(Y) = (I_Z + I_Y)A/I_Y = (I_Y + I_Z)/I_Y$ and $i_Y^{-1}(\mathscr{I}_X + \mathscr{I}_Y)\mathcal{O}_Y = ((I_Y + I_Z)/I_Y)^\sim$. Then the restriction transformation is defined to be $\pi_Y : \widetilde{Y} = Proj(\oplus_k((I_Z + I_Y)/I_Y))^k) \longrightarrow Y$ and similarly, the restriction transformation $\pi_Z : \widetilde{Z} := Proj(\oplus_k((I_Z + I_Y)/I_Z)^k) \longrightarrow Z$.

Now we can see that $\widetilde{Y} \subset \widetilde{X}$ is given by the projection

$$p_{\widetilde{Y}} : \oplus_k(I_Z + I_Y)^k \longrightarrow \oplus_k((I_Z + I_Y)/I_Y)^k$$

with $ker(p_Y) = \oplus_k I_Y^k$. Now, if $\mathfrak{p} \subset \oplus_k((I_Z + I_Y)/I_Y)^k$ is a homogeneous prime ideal, then $p_Y^{-1}(\mathfrak{p})$ must contain $\oplus_k I_Y^k$, that is, if $\mathfrak{p} \in \widetilde{Y}$, then $\mathfrak{q} = i_{\widetilde{Y}}(\mathfrak{p})$ must contain $\oplus_k I_Y^k$. Similarly, if $\mathfrak{q}$ lies in

$\widetilde{Z}$, then $\mathfrak{q}$ must contains $\oplus_k I_Z^k$. Hence, we see that $I_Y, I_Z \subset \mathfrak{q}_1$ and so $\mathfrak{q} = I_Y + I_Z$. Since $\mathfrak{q}$ is a homogeneous prime ideal, $\oplus_k (I_Z + I_Y)^k \subset \mathfrak{q}$. Such $\mathfrak{q}$ doesn't exist in $\widetilde{X}$. Hence, $\widetilde{Y} \cap \widetilde{Z} = \emptyset$. $\square$

Exercise 2.7.13. A Complete Nonprojective Variety. Let k be an algebraically closed field of char $\neq 2$. Let $C \subseteq \mathbb{P}_k^2$ be the nodal cubic curve $y^2z = x^3 + x^2z$. If $P_0 = (0, 0, 1)$ is the singular point, then $C - P_0$ is isomorphic to the multiplicative group $G_m = \operatorname{Spec} k[t, t^{-1}]$ (Ex. 6.7). For each $a \in k$, $a \neq 0$, consider the translation of G_m given by $t \mapsto at$. This induces an automorphism of C which we denote by φ_a. Now consider $C \times (\mathbb{P}^1 - \{0\})$ and $C \times (\mathbb{P}^1 - \{\infty\})$. We glue their open subsets $C \times (\mathbb{P}^1 - \{0, \infty\})$ by the isomorphism $\varphi : \langle P, u\rangle \mapsto \langle \varphi_u(P), u\rangle$ for $P \in C$, $u \in G_m = \mathbb{P}^1 - \{0, \infty\}$. Thus we obtain a scheme X, which is our example. The projections to the second factor are compatible with φ, so there is a natural morphism $\pi : X \to \mathbb{P}^1$.

(a) Show that π is a proper morphism, and hence that X is a complete variety over k.

(b) Use the method of (Ex. 6.9) to show that $\operatorname{Pic}(C \times \mathbb{A}^1) \cong G_m \times \mathbb{Z}$ and $\operatorname{Pic}(C \times (\mathbb{A}^1 - \{0\})) \cong G_m \times \mathbb{Z} \times \mathbb{Z}$.
Hint: If A is a domain and if $*$ denotes the group of units, then $(A[u])^* \cong A^*$ and $(A[u, u^{-1}])^* \cong A^* \times \mathbb{Z}$.

(c) Now show that the restriction map $\operatorname{Pic}(C \times \mathbb{A}^1) \to \operatorname{Pic}(C \times (\mathbb{A}^1 - \{0\}))$ is of the form $\langle t, n\rangle \mapsto \langle t, 0, n\rangle$, and that the automorphism φ of $C \times (\mathbb{A}^1 - \{0\})$ induces a map of the form $\langle t, d, n\rangle \mapsto \langle t, d + n, n\rangle$ on its Picard group.

(d) Conclude that the image of the restriction map $\operatorname{Pic} X \to \operatorname{Pic}(C \times \{0\})$ consists entirely of divisors of degree 0 on C. Hence X is not projective over k and π is not a projective morphism.

Proof. $\square$

Exercise 2.7.14. (a) Give an example of a noetherian scheme X and a locally free coherent sheaf $\mathcal{E}$, such that the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$ is not very ample relative to X.

(b) Let $f : X \to Y$ be a morphism of finite type, let $\mathcal{L}$ be an ample invertible sheaf on X, and let $\mathcal{S}$ be a sheaf of graded $\mathcal{O}_X$-algebras satisfying (†). Let

$$P = \operatorname{Proj} \mathcal{S},$$

let $\pi : P \to X$ be the projection, and let $\mathcal{O}_P(1)$ be the associated invertible sheaf. Show that for all $n > 0$, the sheaf $\mathcal{O}_P(1) \otimes \pi^* \mathcal{L}^n$ is very ample on P relative to Y.
Hint: Use (7.10) and (Ex. 5.12).

Proof. (a). Consider $X = \mathbb{P}_{\mathbb{C}}^1$ and $\mathscr{E} = \mathcal{O}_X(-1)$. Consider $\pi : \mathbb{P} := \mathbf{Proj}(\mathscr{E}) \longrightarrow X$. Note that $\pi_* \mathcal{O}_{\mathbb{P}}(1) = \mathscr{E}$. Now assume that $O_{\mathbb{P}}(1)$ is very ample, which implies it is globally generated. Since $\pi_* \mathcal{O}_{\mathbb{P}}(1) = \mathscr{E}$, $\mathcal{O}_{\mathbb{P}}(1)(\mathbb{P}) = \mathscr{E}(X) = 0$. By

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \otimes \mathcal{O}_{\mathbb{P}} \stackrel{\epsilon}{\twoheadrightarrow} \mathcal{O}_{\mathbb{P}}(1)$$

$\mathcal{O}_{\mathbb{P}}(1) = 0$, which is even not an invertible sheaf. Hence, $\mathcal{O}_{\mathbb{P}}(1)$ is not very ample.

More generally, $\mathcal{O}_{\mathbb{P}}(1)$ is not globally generated.

(2). By [[5] P161 Prop 7.10 (b)], we can choose n large enough such that $\mathcal{O}_P(1) \otimes \pi^* \mathscr{L}^{n-1}$ is very ample and by 2.5.12,

$$\mathcal{O}_P(1) \otimes \pi^* \mathscr{L}^{n-1} \otimes \pi^* \mathscr{L} = \mathcal{O}_P(1) \otimes \pi^* \mathscr{L}^n$$

is very ample relative to Y. $\square$

2.7.4 Addition Exercises

Exercise 2.7.15 (Serge Embedding)**.** Let $B = A[x_0, ..., x_n]$ and $C = A[y_0, ..., y_m]$, $\mathbb{P}^n_A = Proj(B)$, $\mathbb{P}^m_A = Proj(C)$. Show that the sections $x_0y_0, ..., x_ny_m$ of $\mathcal{L} := p_1^*\mathcal{O}_{\mathbb{P}^n_A}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^m_A}$ induce a morphism $s : \mathbb{P}^n_A \times_A \mathbb{P}^m_A \hookrightarrow \mathbb{P}^N_A$ with $N = mn + n + m$ and s is a closed immersion.

Proof.

For $Proj(B)$ is covered by $\{D_+(x_i)\}_{i=0}^n$ and $Proj(C)$ is covered by $\{D_+(y_j)\}_{j=0}^m$, $Proj(B) \times_A Proj(C)$ is cover by $\{D_+(x_i) \times_A D_+(y_j)\}_{i,j=0}^{n,m}$. Because $Proj(B) \times_A Proj(C) \cong Proj(B \times_A C)$, $\Gamma(X, \mathcal{O}_X(1)) =< x_iy_j | i = 0, ..., n; j = 0, ..., m >_A$. Hence $\mathcal{O}_X$ is globally generated by $\{x_iy_j | i = 0, ..., n; j = 0, ..., m\}$ and we have the morphism

$$X \longrightarrow \mathbb{P}^N_A$$

with $N = nm + m + n$. Note that $X_{x_iy_j} = D_+(x_iy_j) \cong D_+(x_i) \times_A D_+(y_j)$ by . So $X_{x_iy_j}$ is affine and then the morphism

$$A[z_1, ..., z_{mn}] \longrightarrow \Gamma(X_{x_iy_j}, \mathcal{O}_{X_{x_iy_j}})$$

given by $z_{rs} \mapsto \frac{x_ry_s}{x_iy_j}$ is surjective. Thus, s is a closed embedding. □

Exercise 2.7.16 (Veronese Embedding)**.** Conisder $S = k[x_0, ..., x_n]$ show that the surjection

$$k[y_0, ..., y_N] \longrightarrow S^{(d)}$$

mapping y_i to the $i - th$ monomial of degree d in the variables x_i defines a closed embedding

$$\mathbb{P}^n_k \hookrightarrow \mathbb{P}^N_k$$

with $N = \binom{n+d}{d}$

Proof. Note that $\mathcal{O}(d)$ is globally generated by $\{x_0^{l_0}x_1^{l_1}...x_n^{l_n} | l_0 + l_1 + ... + l_n = d\}$ (See Chapter 3, Section 3.5 of [5]) . Hence we have a projective morphism

$$\mathbb{P}^n_k \longrightarrow \mathbb{P}^N_k$$

Consider $X_{x_0^{l_0}x_1^{l_1}...x_n^{l_n}} = D_+(x_0^{l_0}x_1^{l_1}...x_n^{l_n}) \cong Spec(k[x_0, x_1, ..., x_n]_{x_0^{l_0}x_1^{l_1}...x_n^{l_n}})$. So $X_{x_0^{l_0}x_1^{l_1}...x_n^{l_n}}$ is affine and the map

$$k[y_1, ..., y_N] \longrightarrow k[x_0, x_1, ..., x_n]_{(x_0^{l_0}x_1^{l_1}...x_n^{l_n})}$$

given by $y_I \mapsto \frac{x^I}{x_0^{l_0}x_1^{l_1}...x_n^{l_n}}$ with $I = (i_0, .., i_n)_{\sum i_j = d} \neq (l_0, .., l_n)$ is obviously a surjective by the definition of localization at homogeneous ideals. Hence the projective map is a closed embedding. □

Remark. When k is algebraically closed, the morphism is given by

$$[\lambda_0 : ... : \lambda_n] \mapsto [\lambda_0^d : ... : \lambda^I : ... : \lambda_n^d]$$

on closed points.

For $[\lambda_0 : ... : \lambda_n]$, its corresponding homogeneous prime ideal is $(x_j\lambda_i - x_i\lambda_j)_{i,j} \subset S$ and under the morphism $k[y_0, ..., y_N] \longrightarrow S^{(d)}$, the preimage of $(x_j\lambda_i - x_i\lambda_j)_{i,j}$ is just $(\lambda^Iy^J - \lambda^Jy^I)$.

Question: How to show

$$(x_j\lambda_i - x_i\lambda_j)_{i,j} \cap S^{(d)} = (x_0^{i_0}...x_n^{i_n}\lambda_0^{j_0}...\lambda_n^{j_n} - x_0^{j_0}...x_n^{j_n}\lambda_0^{i_0}...\lambda_n^{i_n})$$

Exercise 2.7.17. Let V be a linear system, then $Bs(V) \subset X$ is a closed subset.

Proof. Note that $s_P \in \mathfrak{m}_P\mathcal{L}_P$ if and only if $s(P) = 0 \in k = \mathcal{L}_P/\mathfrak{m}_P\mathcal{L}_P$. $\square$

Exercise 2.7.18. Consider $\mathbb{P}^1_k \times \mathbb{P}^1_k \cong Q = V_+(x_0x_3 - x_1x_2)$ and use the fact that $Pic(Q) = \mathbb{Z} \oplus \mathbb{Z}$, that is, every invertible sheaf on Q is isomorphic to a unique $\mathcal{O}(a,b) := p_1^*\mathcal{O}(a) \otimes p_2^*\mathcal{O}(b)$. When is $Q(a,b)$ very ample?

Proof. When $a \le 0$, $\mathcal{O}(a,b)|_{\mathbb{P}^1_k \times_k k(P)}$ is trivial. Hence $Q(a,b)$ can not be ample thus not very ample.

Hence when $a \le 0$ or $b \le 0$, $\mathcal{Q}(a,b)$ is not very ample or ample.

When $a > 0$, $\mathcal{O}(a)$ is very ample and ample in $\mathbb{P}^1_k$. Consider

$$\begin{tikzcd} \mathbb{P}^1_k \times_k \mathbb{P}^1_k \arrow[r, hook, "v"] \arrow[dr, hook] & \mathbb{P}^a_k \times_k \mathbb{P}^b_k \arrow[d, hook, "s"] \\ & \mathbb{P}^{ab+a+b}_k \end{tikzcd}$$

where v is deduced by the Veronese embedding $\mathbb{P}^1_k \longrightarrow \mathbb{P}^a_k$ and s is the Serge embedding.

Now that $s^*\mathcal{O}(1) = \mathcal{O}(1) \otimes \mathcal{O}(1)$ and $v^*(\mathcal{O}(1) \otimes \mathcal{O}(1)) = Q(a,b)$. Hence $Q(a,b)$ is very ample and then $Q(a,b)$ is ample. $\square$

Exercise 2.7.19. Let k be an algebraically closed field. Describe the scheme $X \subset \mathbb{P}^1_k \times \mathbb{P}^1_k \times \mathbb{P}^1_k$ as the zero locus of a setion of a line bundle for which the fiber of the first projection $X \longrightarrow \mathbb{P}^1_k$ over a closed point $[t_0 : t_1]$ is the curve $X_{[t_0:t_1]} \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ described by $x_0^2y_1t_1 = (x_0^2 + x_1^2)y_0t_0$. Find closed points for which the fiber is irreducible, non-reduced, and reducible, respectively.

Proof.

(i). X is defined by the section $x_0^2y_1t_1 - (x_0^2 + x_1^2)y_0t_0 \in H^0(\mathbb{P}^1_k \times_k \mathbb{P}^1_k \times_k \mathbb{P}^1_k, p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(2) \otimes p_3^*\mathcal{O}(1))$.

(ii). At $[t_0 : t_1] = [0 : 1]$, $X_{[0:1]}$ is defined by $x_0^2y_1$. Without loss of generality, we consider $(x_0 : x_1) \times (y_0 : y_1)$ with $x_1 \ne 0$ and $y_0 \ne 0$. Then $X \cap (D_+(x_1) \times_k D_+(y_0)) \subset D_+(x_1) \times_k D_+(y_0) \cong \mathbb{A}^2_k$ is given by x^2y. Hence $X \cap (D_+(x_1) \times_k D_+(y_0))$ is reducible and non-reduced. Hence, $X_{[0:1]}$ is reducible and non-reduced.

(iii). At $[t_0 : t_1] = [1 : 1]$, X is defined by $x_0^2y_1 - (x_0^2 + x_1^2)y_0 = 0$. Consider the cover of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$: $\{D_+(x_i) \times D_+(y_j)\}_{i,j}$. On the cover $D_+(x_0) \times D_+(y_0)$, X is defined by $y - (1 + x^2)$, which is a prime ideal of $k[x,y]$. Hence $X \cap D_+(x_0) \times D_+(y_0)$ is integral. Moreover, each $X \cap D_+(x_i) \times D_+(y_j)$ is integral. Hence $X_{[1:1]}$ is irreducible and reduced.

(iv). At $[t_0 : t_1] = [1 : 0]$, X is defined by $(x_0^2 + x_1^2)y_0$. Also, on $D_+(x_0) \times_k D_+(y_1)$, X is defined by $(1 + x^2)y$, which is not irreducible but reduced. Thus, $X_{[1:0]}$ is reducible and reduced. $\square$

Exercise 2.7.20. Determine the base locus of the linear system $\{t_0(x_0^4 + x_1^4 + x_2^4 + x_3^4) + t_1x_0x_1x_2x_4\} \subset |\mathcal{O}(4)|$ on $\mathbb{P}^3_k$ where $char(k) \ne 2$. Prove that the irreducible components of the base locus are curves, find their genus.

Remark. By Künneth formula, we have

$$H^i(X, \mathcal{F} \otimes \mathcal{G}) = \oplus_{p+q=i} H^q(X, \mathcal{F}) \otimes H^p(X, \mathcal{G})$$

Proof. Let X be $Bs(V) = V_+(x_0^4 + x_1^4 + x_2^4 + x_3^4) \cap V_+(x_0x_1x_2x_3) = \cup_{i=0}^3 (V_+(x_0^4 + x_1^4 + x_2^4 + x_3^4) \cap V_+(x_i))$. Note that $V_+(x_i) \cong \mathbb{P}^3_k$ and $X \cap V_+(x_i)$ is isomorphic to the curve in $\mathbb{P}^2_k$ defined $x_0^4 + x_1^4 + x_2^4$, which is integral.

For the $Y = V_+(x_0^4 + x_1^4 + x_2^4) \subset \mathbb{P}_k^2$, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^2}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}_k^2} \longrightarrow i_{Y,*}\mathcal{O}_Y \longrightarrow 0$$

Then $H^1(Y, \mathcal{O}_Y) = H^2(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-4))$. Hence $g(Y) = dim_k H^1(Y, \mathcal{O}_Y) = dim_k H^2(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-4)) = 3$. $\square$

Exercise 2.7.21. Let k be an algebraically closed field, $char(k) = 3$. Consider the scheme $X \subset \mathbb{P}_k^1 \times \mathbb{P}_k^2$ for which the fibers of the first projection $X \longrightarrow \mathbb{P}_k^1$ over closed points $[t_0 : t_1]$ are the curve $X_{[t_0:t_1]} \subset \mathbb{P}_k^2$ given by $t_0(x_0^3 + x_1^3 + x_2^3) + t_1 x_0 x_1 x_2 = 0$. Describe X as the zero locus of a section of a line bundle. Find a closed point for which the fiber is irreducible and a closed point for which the fiber is reducible. Is the generic fiber integral?

Proof.
(i). X is defined by the section $t_0(x_0^3 + x_1^3 + x_2^3) + t_1 x_0 x_1 x_2 \in H^0(\mathbb{P}_k^1 \times_k \mathbb{P}_k^2, p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(3))$.

(ii). At $[t_0 : t_1] = [0 : 1]$, X is given by $x_0 x_1 x_2$. Thus, $X = \cup_{i=0}^2 V_+(x_i)$, which is reducible.

(iii). At $[t_0 : t_1] = [1 : 0]$, X is given by $x_0^3 + x_1^3 + x_2^3$. Note that at $D_+(x_0)$, $X \cap D_+(x_0)$ is defined by $1 + x^3 + y^3$, which is irreducible for $char(k) = 3$. Moreover, $X \cap D_+(x_i)$ is irreducible. Hence X is irreducible.

(iv). At the generic point of $\mathbb{P}_k^1$, $\mathcal{O}_{\mathbb{P}_k^1, P} = k(t)$ with $t = \frac{t_1}{t_0}$. Hence X is defined by $x_0^3 + x_1^3 + x_2^3 + t x_0 x_1 x_2$ over $\mathbb{P}_{k(t)}^2$. For $1 + x^3 + y^3 + txy$ is irreducible in $k(t)[x, y]$ with $char(k) = 3$, X is integral. $\square$

Exercise 2.7.22. Let k be an algebraically closed field with $char(k) \neq 2$. Consider the graded morphism of graded k-algebra

$$\varphi : k[x_0, x_1] \longrightarrow k[x_0, x_1, x_2, x_3]/(\sum_{i=0}^{3} x_i^2)$$

given by $x_i \mapsto x_i$.
(i). Show that φ induces a morphism of schemes $f : V_+(\sum_{i=0}^3 x_i^2)\ V_+(x_0, x_1) \longrightarrow \mathbb{P}_k^1$.
(ii). Calculate the fiber F of f over the point $[\lambda, \mu] \in \mathbb{P}_k^1$ with $\lambda, \mu \in k$.
(iii). Show that F is connected for all $[\lambda, \mu] \in \mathbb{P}_k^1$.

Proof.
(i). For $V_+(\sum_{i=0}^3 x_i^2)$ is a closed subscheme of $\mathbb{P}_k^3$ and $\varphi(x_i) = x_i$, φ defined a morphism $D_+(x_0) \cup D_+(x_1) \subset V_+(\sum_{i=0}^3 x_i^2) \longrightarrow \mathbb{P}_k^1$. Moreover, note that $D_+(x_0) \cup D_+(x_1) \subset V_+(\sum_{i=0}^3 x_i^2) = V_+(\sum_{i=0}^3 x_i^2)\ V_+(x_0, x_1)$.

(ii). Consider $[\lambda, \mu] \in D_+(x_0)$.

$$\begin{aligned} f^{-1}([\lambda, \mu]) &= k[\frac{x_1}{x_0}]/(\frac{x_1}{x_0} - \frac{\mu}{\lambda}) \otimes_{k[\frac{x_1}{x_0}]} k[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}]/(1 + \frac{x_1^2}{x_0^2} + \frac{x_2^2}{x_0^2} + \frac{x_3^2}{x_0^2}) \\ &= k[\frac{x_2}{x_0}, \frac{x_3}{x_0}]/(1 + \frac{\lambda^2}{\mu^2} + \frac{x_2^2}{x_0^2} + \frac{x_3^2}{x_0^2}) \end{aligned}$$

Fibers at $D_+(x_1)$ are the same.
(iii). For $char(k) \neq 2$, $1 + \frac{\lambda^2}{\mu^2} + x^2 + y^2$ is irreducible if $1 + \frac{\lambda^2}{\mu^2} \neq 0$.

If $1 + \frac{\lambda^2}{\mu^2} = 0$, then $Spec(k[x, y]/(x^2 + y^2)) = Spec(k[x, y]/(x + iy)(x - iy))$, which is a union of two line in $\mathbb{A}_k^2$. Hence, the fiber is also connected. $\square$

Exercise 2.7.23. Let D be an effective Cartier divisor on X. Show that the open immersion $l : X - Y_D \hookrightarrow X$ is an affine morphism, where Y_D is the morphism defined by D.

Proof. We have seen that "being an affine morphism " is a local property. We can take $X = Spec(A)$ and D is defined by $a \in A$, that is, $Spec(A/(a))$. Hence $X - Y_D = D(a)$ and then $X - Y_D \longrightarrow X$ is an open immersion. $\square$

Exercise 2.7.24. Let k be a field and let $X = V_+(f_d) \subset \mathbb{P}^n_k$ be a hypersurface given by a homogeneous polynomial f_d of degree d. Consider the Veronese embedding $v_d : \mathbb{P}^n_k \longrightarrow \mathbb{P}^N_k$ with $N = \binom{n+d}{n} - 1$.

(i). Show that X is the intersection of $v_d(\mathbb{P}^n_k)$ with a hyperplane $V_+(g)$, where $g \in \Gamma(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N_k}(1))$.

(ii). $\mathbb{P}^1_k - X$ is affine.

Proof.

(i). By definition of v_d^*, $v_d^*(\mathcal{O}_{\mathbb{P}^N_k}(1)) = \mathcal{O}_{\mathbb{P}^n_k}(d)$ with $f_d \in \mathcal{O}_{\mathbb{P}^n_k}(d)$. Then there exists $g \in \Gamma(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N_k}(1))$ such that $v_d^*(g) = f_d$. Hence

$$v_d(\mathbb{P}^n_k - X) = v_d(D_+(v_d^*(g)) \subset D_+(g)$$

Thus, $V_+(g) \subset v_d(X) \cup v_d(\mathbb{P}^n_k)^c$ and then $V_+(g) \cap v_d(\mathbb{P}^n_k) \subset v_d(X)$. Hence $V_+(g) \cap v_d(\mathbb{P}^n_k) = v_d(X)$ by the definition of Veronses embedding (which implies $v_d(X) \subset V_+(g)$).

(ii). Note that $\mathbb{P}^N_k - V_+(g) \cong \mathbb{A}^N_k$ is affine. Since $\mathbb{P}^n_k \longrightarrow \mathbb{P}^N_k$ is a closed immersion, so is $\mathbb{P}^n_k \cap D_+(g) \longrightarrow D_+(g)$ a closed immersion, that is, $\mathbb{P}^n_k - X \longrightarrow \mathbb{A}^N_k$ is a closed immersion. So $\mathbb{P}^n_k - X$ is affine. $\square$

Remark.

$$v_d(\mathbb{P}^n_k - X) = v_d(D_+(v_d^*(g)) \subset D_+(g)$$

This is because $\varphi : X_{s_i} = D(\varphi^* x_i) \longrightarrow D_+(x_i)$ and $V(\varphi^* x_i) \longrightarrow V_+(x_i)$ if φ is given by $s_i = \varphi^* x_i$.

Exercise 2.7.25. Let k be an algebraically closed field of characteristic $cher(k) = 3$. Consider the two homogeneous polynomials $s_0 = x_0^2 + x_1^2 + x_2^2$ and $s_1 = x_0 x_1 x_2$.

(i). Determine the underlying set of the base locus $V_+(s_0) \cap V_+(s_1)$ of the linear system $< s_0, s_1 > \subset H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(3))$. What's the cardinality of this set?

(ii). Conisder the rational map

$$\mathbb{P}^2_k - (V_+(s_0) \cap V_+(s_1)) \longrightarrow \mathbb{P}^1_k$$

given by $[x_0, x_1, x_2] \mapsto [s_0, s_1]$. Show that the fiber F of f over $[\lambda, \mu]$ is $V_+(\mu s_0 - \lambda s_1)$.

Proof.

(i). For $char(k) = 3$, $s_0 = (x_0 + x_1 + x_2)^3$. Thus, $V_+(s_0) = V_+(x_0 + x_1 + x_2)$. $V_+(s_0) \cap V_+(s_1) = V_+(x_0 + x_1 + x_2) \cap (V_+(x_0) \cup V_+(x_1) \cup V_+(x_2))$. So

$$V_+(s_0) \cap V_+(s_1) = \{[0 : 1 : -1], [1 : 0 : -1], [1 : -1 : 0]\}$$

(ii). Because $f^*(x_i) = s_i$, locally f can be written as

$$f|_{D_+(s_i)} : D_+(s_i) \longrightarrow D_+(x_i)$$

Suppose that $x = [\lambda : \mu]$ with $\mu \neq 0$. Then

$$f^{-1}(x) = \frac{k[\frac{x_0}{x_1}]}{(\frac{x_0}{x_1} - \frac{\lambda}{\mu})} \otimes_{k[\frac{x_0}{x_1}]} k[y_0, y_1, y_2]_{(s_1)}$$

with $x_i \mapsto s_i$. Hence $f^{-1}(x) = k[x_0, x_1, x_2]_{(s_1)}/(\frac{s_0}{s_1} - \frac{\lambda}{\mu}) = V_+(\mu s_0 - \lambda s_1) \cap U$. So, without loss of generality, $f^{-1}([\lambda : \mu]) = V_+(\mu s_0 - \lambda s_1) \cap U$. $\square$

Exercise 2.7.26. Let $\varphi : \mathbb{P}^n_k \longrightarrow X$ be a morphism of projective k-scheme. Show that either the image of φ consists of a single point or that φ is quasi-finite.

Proof.

For X is a projective k-scheme, then there exists m such that $X \hookrightarrow \mathbb{P}^m_k$ is a closed immersion. Without loss of generality, just take $X = \mathbb{P}^m_k$.

Consider $\varphi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k$. φ is completely determined by $\varphi^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ which is a line bundle over $\mathbb{P}^n_k$. Hence, we assume that $\varphi^*\mathcal{O}_{\mathbb{P}^m_k}(1) = \mathcal{O}_{\mathbb{P}^n_k}(d)$ for some d.

When $d < 0$, $\Gamma(\mathbb{P}^n_k, \varphi^*\mathcal{O}_{\mathbb{P}^m_k}(1))$ is empty, which leads to a contradiction.

When $d = 0$, $\Gamma(\mathbb{P}^n_k, \varphi^*\mathcal{O}_{\mathbb{P}^m_k}(1)) = k$. Hence φ is constant.

When $d > 0$, consider for a point P, consider $F := \varphi^{-1}(P) = \mathbb{P}^n_k \times_{\mathbb{P}^m_k} k(P)$

$$\begin{tikzcd}
\mathbb{P}^n_k \times_{\mathbb{P}^m_k} k(P) \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathbb{P}^n_k \arrow[d, "\varphi"] \\
Spec(k(P)) \arrow[r, "i"] & \mathbb{P}^m_k
\end{tikzcd}$$

For $Pic(Spec(k(P)))$ is trivial, $i^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is trivial and then $p_2^* \circ i^*\mathcal{O}_{\mathbb{P}^m_k}(1)$ is trivial over F, i.e. the structure sheaf of F. Then $\mathcal{O}_F = p_2^* \circ i^*\mathcal{O}_{\mathbb{P}^m_k}(1) = p_1^*\mathcal{O}_{\mathbb{P}^n_k}(d)$. For $\mathcal{O}_{\mathbb{P}^n_k}(d)$ is vary ample and p_2 is a closed immersion, $\mathcal{O}_F$ is very ample. Then there is a closed immersion of $F \longrightarrow \mathbb{P}^N_k$ (In fact, $N = \binom{n+d}{n} - 1$), which is constant. Hence, F is finite. So, φ is quasi finite. □

Exercise 2.7.27. Let X be a projective scheme over $k = \overline{k}$ and let $\mathcal{L} \in Pic(X)$. Show that $Bs(\mathcal{L}^n) \subset Bs(\mathcal{L})$ for any $n > 0$. Show a example such that for $n > m$, $Bs(\mathcal{L}^n) \not\subset Bs(\mathcal{L}^m)$.

Proof. Take any $x \in Bs(\mathcal{L}^n)$. Then for any $s \in \Gamma(X, \mathcal{L})$, $s_x \in \mathfrak{m}_x$. By Künneth formula, $H^0(X, \mathcal{L}^n) = \otimes_n H^0(X, \mathcal{L})$. Hence, for any $s \in H^0(X, \mathcal{L})$, $s^n \in H^0(X, \mathcal{L}^n)$ and $s^n_x = (s_x)^n \in \mathfrak{m}_x$ implying $s_x \in \mathfrak{m}_x$. Hence $Bs(\mathcal{L}^n) \subset Bs(\mathcal{L})$.

Consider the elliptic curve C over $\mathbb{C}$. Then $Pic^0(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$. Thus, we can take $\mathcal{L} \in Pic^0(C)$ such that $\mathcal{L}$ is of order 2. Then $\mathcal{L}^2 \cong \mathcal{O}_X$. Hence $Bs(\mathcal{L}^2) = \emptyset$. However, $\mathcal{L}^3 \cong \mathcal{L}$ with $Bs(\mathcal{L}^3) \neq \emptyset$. Hence $Bs(\mathcal{L}^2) \not\subset Bs(\mathcal{L}^3)$. □

2.8 Differentials

2.8.1 Preparations

2.8.2 Examples

Example 2.8.1 (Some Differentials)**.** Let $A = k[x_1, x_2]/\mathfrak{a}$. When $\mathfrak{a} = (x_1x_2 + x_1^3 + x_2^3)$, we have the exact sequence

$$\mathfrak{a}/\mathfrak{a}^2 \xrightarrow{d} \Omega_{k[x_1,x_2]/k} \otimes_{k[x_1,x_2]} A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

We know that $\Omega_{k[x_1,x_2]/k} = k[x_1, x_2]dx_1 \oplus k[x_1, x_2]dx_2$ and $\mathrm{Im}(d) = \langle (x_2 + 3x_1^2)dx_1 + (x_1 + 3x_2^2)dx_2 \rangle$. Hence

$$\Omega_{A/k} = \frac{Adx_1 \oplus Adx_2}{\langle (x_2 + 3x_1^2)dx_1 + (x_1 + 3x_2^2)dx_2 \rangle}$$

Similarly, we can compute this for any $\mathfrak{a} = (f(x_1, x_2))$.

Example 2.8.2 (Non-exactness of the conormal sequence)**.** Consider $A = k[x, y]/\mathfrak{a}$ with $\mathfrak{a} = (x^2, y^2)$. Then

$$\mathfrak{a}/\mathfrak{a}^2 \xrightarrow{d} \Omega_{k[x,xy]/k} \otimes_{k[x_1,x_2]} A \longrightarrow \Omega_{A/k} \longrightarrow 0$$

Note that $d(x^3) = 3x^2dx = 0$ in $Adx \oplus Ady$. For x^3 is not zero in $\mathfrak{a}/\mathfrak{a}^2$, d is not an injection.

2.8.3 Exercises

Exercise 2.8.1. Here we will strengthen the results of the text to include information about the sheaf of differentials at a not necessarily closed point of a scheme X. (a) Generalize (8.7) as follows. Let B be a local ring containing a field k, and assume that the residue field $k(B) = B/\mathfrak{m}$ of B is a separably generated extension of k. Then the exact sequence of (8.4A),

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

is exact on the left also. [Hint: In copying the proof of (8.7), first pass to $B/\mathfrak{m}^2$, which is a complete local ring, and then use (8.25A) to choose a field of representatives for $B/\mathfrak{m}^2$.]

(b) Generalize (8.8) as follows. With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $= \dim B + \mathrm{tr.d.}\, k(B)/k$.

(c) Strengthen (8.15) as follows. Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $x \in X$, not necessarily closed, show that the local ring $\mathcal{O}_{X,x}$ is a regular local ring if and only if the stalk $(\Omega_{X/k})_x$ of the sheaf of differentials at x is free of rank n.

(d) Strengthen (8.16) as follows. If X is a variety over an algebraically closed field k, then $U = \{x \in X \mid \mathcal{O}_{X,x} \text{ is a regular local ring}\}$ is an open dense subset of X.

Proof.

(a). Note that B is a k-algebra and $C = B/\mathfrak{m} = k(B)$. So we have

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes_B k(B) \longrightarrow \Omega_{k(B)/k} \longrightarrow 0$$

To verify δ is an injection, it is equivalent to verify we have a surjection

$$\delta^* : Hom_{k(B)}(\Omega_{B/k} \otimes_B k(B), k(B)) \longrightarrow Hom_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$$

Note that $Hom_{k(B)}(\Omega_{B/k} \otimes_B k(B), k(B)) = Hom_B(\Omega_{B/k}, k(B)) = Der_k(B, k(B))$. Then we consider the map

$$\delta^* : Der_k(B, k(B)) \longrightarrow Hom_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$$

given by $[d : B \longrightarrow k(B)] \mapsto [d|_{\mathfrak{m}} : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow k(B)]$. Since for any $f, g \in \mathfrak{m}$, $d(fg) = fd(g) + d(f)g$ with $d(f), d(g) \in k(B) = B/\mathfrak{m}$, $d(fg) = 0$. That's why we can define the map on $\mathfrak{m}/\mathfrak{m}^2$.

Now, given $h \in Hom_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$, we want to find $d \in Der_k(B, k(B))$ such that $\delta^*(d) = h$. Since $k(B)/k$ is separated, by [[5] P167 Theorem 8.25A], there exist $K \subset B$ such that the composition

$$K \hookrightarrow B \twoheadrightarrow k(B)$$

is an isomorphism. For each $b \in B$, consider

$$\begin{aligned} K \longrightarrow B &\longrightarrow k(B) \\ b &\longmapsto \lambda \\ \lambda \longmapsto \lambda & \end{aligned}$$

Take $c = b - \lambda$ and define $d : b \mapsto h(\bar{c})$ with $\bar{c}$ the image of c under $B \longrightarrow k(B)$. Then we can verify the d is a k-derivation from B to $k(B)$:

$$\begin{aligned} d(b_1 b_2) = d((\lambda_1 + c_1)(\lambda_2 + c_2)) &= d(\lambda_1\lambda_2 + c_1\lambda_2 + c_2\lambda_1 + c_1c_2) \\ &= d(c_1\lambda_2 + c_2\lambda_1 + c_1c_2) \\ &= \lambda_2 d(c_1) + \lambda_1 d(c_2) \\ &= b_1 d(b_2) + b_2 d(b_1) \end{aligned}$$

and by this definition $d|_{\mathfrak{m}} = h$.

(b). ($\Longrightarrow$): Suppose that $\Omega_{B/k}$ is free of rank $\dim B + tr.d.k(B)/k$. By (a), $\dim \Omega_{B/k} \otimes k(B) = \dim \Omega_{k(B)/k} + \dim \mathfrak{m}/\mathfrak{m}^2$. Since $k(B)/k$ is perfect, $k(B)/k$ is separable [P27 Theorem 4.7A [5]] and then $dim(\Omega_{k(B)/k}) = tr.d.k(B)/k$. Hence, we see that $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim B$, which implies B is a local regular ring.

($\Longleftarrow$):Again, by the exact sequence in (a), we have

$$\dim \Omega_{B/k} \otimes k(B) = \dim \mathfrak{m}/\mathfrak{m}^2 + \dim \Omega_{k(B)/k} = \dim \mathfrak{m}/\mathfrak{m}^2 + tr.d.k(B)/k$$

Since $B = A_{\mathfrak{p}}$ is a local regular ring, then $\dim B = \dim \mathfrak{m}/\mathfrak{m}^2$. Now, consider the fraction field K of B, then $\Omega_{K/k} = \Omega_{B/k} \otimes_B K$ by $\Omega_{S^{-1}B/k} = S^{-1}\Omega_{B/k}$ (taking $S = B - 0$), which implies $\dim_K \Omega_{B/k} \otimes_B K = tr.d.K/k$ by [P27 Theorem 4.8A [5]]. By [P6 Theorem 1.8A [5]], $\dim A = tr.d.K/k = ht.\mathfrak{p} + \dim A/\mathfrak{p}$ with $\dim A/\mathfrak{p} = tr.d.Frac(A/\mathfrak{p})/k = tr.d.k(B)/k$ (Note that $B/\mathfrak{m} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = S^{-1}(A/\mathfrak{p}) = Frac(A/\mathfrak{p})$ with $S = A - \mathfrak{p}$). Since $B = A_{\mathfrak{p}}$, $\dim B = ht.\mathfrak{p}$. So, we have

$$dim_K \Omega_{B/k} \otimes_B K = \dim \mathfrak{m}/\mathfrak{m}^2 + tr.d.k(B)/k$$

Then by [P174 Lemma 8.9 [5]], $\Omega_{B/k}$ is free of rank $\dim \mathfrak{m}/\mathfrak{m}^2 + tr.d.k(B)/k$.

(c). Since $S^{-1}\Omega_{B/A} = \Omega_{S^{-1}B/A}$, $\Omega_{O_{X,x}/k} = (\Omega_{X/k})_x$. By (b), $O_{X,x}$ is a local regular ring if and only if $(\Omega_{X/k})_x$ is free of rank $\dim O_{X,x} + tr.d.k(x)/k$.

Now use **??**. Taking the closure of x, by (b), $\dim\{x\} = tr.d.k(x)/k$ and by (c), $codim(\{x\}, X) = \dim \mathcal{O}_{X,x}$ and finally by (c), $n = \dim X = \dim \mathcal{O}_{X,x} + tr.d.k(x)/k$. Hence, $O_{X,x}$ is a local regular ring if and only if $(\Omega_{X/k})_x$ is free of rank n.

(d). By **??**, if $(\Omega_{X/k})_x$ is locally free of rank n, then there is a neighborhood of x such that $\Omega_{X/k}$ is free of rank n. Then by (c), $\{x \in X | x \text{ is a regular point of } X\}$ is open. $\square$

Exercise 2.8.2. Let X be a variety of dimension n over k. Let $\mathcal{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subseteq \Gamma(X, \mathcal{E})$ be a vector space of global sections which generate $\mathcal{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x\mathcal{E}_x$. Conclude that there is a morphism $\mathcal{O}_X \to \mathcal{E}$ giving rise to an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{E}' \to 0$$

Exercise 2.8.3 (Product Schemes)**.** (a) Let X and Y be schemes over another scheme S. Use (8.10) and (8.11) to show that

$$\Omega_{X\times_S Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}.$$

(b) If X and Y are nonsingular varieties over a field k, show that

$$\omega_{X\times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y.$$

(c) Let Y be a nonsingular plane cubic curve, and let X be the surface $Y\times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$ (I, Ex. 7.2). This shows that the arithmetic genus and the geometric genus of a nonsingular projective variety may be different.

Proof.
(a). Consider the pullback

$$\begin{array}{ccc} X\times Y & \xrightarrow{p_Y} & Y \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{f_X} & S \end{array}$$

Then by [P176 Proposition 8.11 [5]], we have

$$p_Y^*\Omega_{Y/S} \xrightarrow{f_1} \Omega_{X\times Y/S} \xrightarrow{g_1} \Omega_{X\times Y/Y} \longrightarrow 0$$

$$p_X^*\Omega_{X/S} \xrightarrow{f_2} \Omega_{X\times Y/S} \xrightarrow{g_2} \Omega_{X\times Y/X} \longrightarrow 0$$

By [P173 Proposition 8.2A [5]], we have $\Omega_{X\times Y/Y} = p_X^*\Omega_{X/S}$ and $\Omega_{X\times Y/X} = p_Y^*\Omega_{Y/S}$. Hence, it is enough to verify that $g_1\circ f_2 = id$ are inverse to each other and $g_2\circ f_1 = id$ are inverse to each other.

To do this, we just verify this locally, take $S = Spec(A)$ and $X = Spec(B_1)$ with $Y = Spec(B_2)$ and let $C = B_1\otimes_A B_2$. Note that

$$\begin{aligned} g_1 : \Omega_{C/A} &\longrightarrow \Omega_{C/B_2} \\ d_{C/A}c &\longmapsto d_{C/B_2}c \end{aligned}$$

and

$$\begin{aligned} f_2 : \Omega_{B_1/A}\otimes_{B_1} C &\longrightarrow \Omega_{C/A} \\ d_{B_1/A}(b_1)\otimes c &\longmapsto b_1 d_{B_1/A}(c) \end{aligned}$$

and

$$\begin{aligned} \Omega_{B_1/A}\otimes_{B_1} C &\xrightarrow{\cong} \Omega_{C/B_2} \\ d_{B_1/C}(b_1)\otimes c &\longmapsto b_1 d_{C/B_2}(c) \end{aligned}$$

Hence $g_1\circ f_2 = id$ and then f_2 is injective, that is, there exists a short exact sequence

$$0 \longrightarrow p_X^*\Omega_{X/S} \xrightarrow{f_2} \Omega_{X\times Y/S} \xrightarrow{g_2} p_Y^*\Omega_{Y/S} \longrightarrow 0$$

For we also have $f_1\circ g_2 = id$, the short exact sequence splits, which implies $\Omega_{X\times Y/S} = p_Y^*\Omega_{Y/S} \oplus p_X^*\Omega_{X/S}$. (b). Since $\Omega_{X\times Y/S} = p_X^*\Omega_{X/S}\oplus p_Y^*\Omega_{Y/S}$. We see that $\omega_{X\times Y} = p_X^*\omega_{X/S}\otimes p_Y^*\omega_{Y/S}$.

(c). Let $Y = V(f) \subset \mathbb{P}_k^2$ with $deg(f) = 3$ be a cubic plane curve.

To compute $g_a(Y\times Y)$, we need **??**: By (b), $p_a(Y) = \frac{1}{2}(3-1)(2-1) = 1$. Then use (e), $p_a(Y\times Y) = p_a(Y)p_a(Y) + 2(-1)^1 p_a(Y) = -1$.

To compute $p_g(Y\times Y)$, we need to use Künneth formula: $H^0(Y\times Y,\omega_{Y/k}\otimes\omega_{Y/k}) = H^0(Y,\omega_{Y/k})\otimes H^0(Y,\omega_{Y/k})$. For Y, we have the exact sequence

$$0 \longrightarrow \mathscr{I}_Y/\mathscr{I}_Y^2 \longrightarrow \Omega_{\mathbb{P}^2_k/k}\otimes\mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0$$

with $\mathscr{I}_Y \cong O_{\mathbb{P}^2_k}(-3)$. So $\omega_{Y/k} = \mathcal{O}_{\mathbb{P}^2_k}(3)\otimes\omega_{\mathbb{P}^2_k/k}\otimes O_Y = \mathcal{O}_Y$ by Euler equation. Hence, $H^0(Y,\omega_{Y/k}) = H^0(Y,\mathcal{O}_Y)$ which implies $p_g(Y\times Y) = h^0(Y\times Y,\omega_{Y\times Y}) = 1$. □

Remark 2.8.3. Here, we define $p_g(Y) = H^0(Y,\omega_Y)$. But to identify this definition with before, you need to first go through [Chapter III.7 [5]].

Exercise 2.8.4 (Complete Intersections in $\mathbb{P}^n$)**.** A closed subscheme Y of $\mathbb{P}^n_k$ is called a (strict, global) complete intersection if the homogeneous ideal I of Y in $S = k[x_0,\dots,x_n]$ can be generated by $r = \mathrm{codim}(Y,\mathbb{P}^n)$ elements (I, Ex. 2.17).

(a) Let Y be a closed subscheme of codimension r in $\mathbb{P}^n$. Then Y is a complete intersection if and only if there are hypersurfaces (i.e., locally principal subschemes of codimension 1) $H_1,\dots,H_r$, such that $Y = H_1\cap\ldots\cap H_r$ as schemes, i.e., $\mathcal{I}_Y = \mathcal{I}_{H_1}+\ldots+\mathcal{I}_{H_r}$. [Hint: Use the fact that the unmixedness theorem holds in S (Matsumura [2, p. 107]).]

(b) If Y is a complete intersection of dimension ≥ 1 in $\mathbb{P}^n$, and if Y is normal, then Y is projectively normal (Ex. 5.14). [Hint: Apply (8.23) to the affine cone over Y.]

(c) With the same hypotheses as (b), conclude that for all $l\geq 0$, the natural map $\Gamma(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(l))\to\Gamma(Y,\mathcal{O}_Y(l))$ is surjective. In particular, taking $l=0$, show that Y is connected.

(d) Now suppose given integers $d_1,\dots,d_r\geq 1$, with $r<n$. Use Bertini's theorem (8.18) to show that there exist nonsingular hypersurfaces $H_1,\dots,H_r$ in $\mathbb{P}^n$, with $\deg H_i = d_i$, such that the scheme $Y = H_1\cap\ldots\cap H_r$ is irreducible and nonsingular of codimension r in $\mathbb{P}^n$.

(e) If Y is a nonsingular complete intersection as in (d), show that $\omega_Y\cong\mathcal{O}_Y(\sum d_i - n - 1)$.

(f) If Y is a nonsingular hypersurface of degree d in $\mathbb{P}^n$, use (c) and (e) above to show that $p_g(Y) = \binom{d-1}{n}$. Thus $p_g(Y) = p_a(Y)$ (I, Ex. 7.2). In particular, if Y is a nonsingular plane curve of degree d, then $p_g(Y) = \frac{1}{2}(d-1)(d-2)$.

(g) If Y is a nonsingular curve in $\mathbb{P}^3$, which is a complete intersection of nonsingular surfaces of degrees d,e, then $p_g(Y) = \frac{1}{2}de(d+e-4)+1$. Again the geometric genus is the same as the arithmetic genus (I, Ex. 7.2).

Proof. (a). To begin with, we have the following claims:

Claim 2.8.4. *If $I\subset S = k[x_1,...,x_n]$, a homogeneous ideal, can be generated by r elements, then it can be generated by r homogeneous elements.*

Proof. Since I is a homogeneous ideal, it can be generated by homogeneous elements. Suppose that $\{f_1,...,f_s\}_{s\in S}$ is the minimal set of homogeneous generators of I. Now, we define

$$\varphi : I \longrightarrow k^{\oplus S}$$
$$\sum g_i f_i \longmapsto (g_i)$$

This map is well-defined: If $\sum g_i f_i = 0$, then $g_i\in(x_1,...,x_n)$. Since if $g_{s_0}(0) = c_{s_0}\neq 0$, consider the sum of homogeneous elements of $deg(f_{s_0})$: $c_{s_0}f_{s_0}+\sum g'_{s\neq s_0}f_s = 0$, which implies f_{s_0} can be generated by $\{f_s\}_{s\neq s_0}$, contradicting to the hypothesis that $\{f_s\}_{s\in S}$ is the minimal set of homogeneous generators of I.

Now, φ is a surjection by definition and for I can be generated by r elements, $k^{\oplus r}$ can also be generated by r elements, which implies $r\geq |S|$. Hence, if $|S| = r$, we are done. Otherwise, we can extend $\{f_i\}_{i\in S}$ to be a set containing r homogeneous elements such that generates I. □

Claim 2.8.5. *If H is a hypersurface in $\mathbb{P}^n_k$, then $H = V_+(f)$ for some homogeneous element f.*

Proof. Suppose that H is given by the saturation ideal I_H and $\mathscr{I}_H$ is the ideal sheaf of H. Then by **??**, $\Gamma_*(\mathscr{I}_H) = I_H$ and $H = Proj(k[x_0, ..., x_n]/I_H)$.

Since $\mathscr{I}_H$ is locally principal, $\mathscr{I}_H$ is an invertible sheaf over $\mathbb{P}^n_k$. We assume that $\mathscr{I}_H \cong \mathcal{O}_{\mathbb{P}^n_k}(-d)$ for some $d \geq 0$, that is, $I_H \cong k[x_0, ..., x_n](-d)$ as graded $k[x_0, ..., x_n]$-mod. Since $k[x_0, ..., x_n](-d)$ is a free $k[x_0, ..., x_n]$-module generated by 1, hence, I_H is an ideal generated by the corresponding element in $f \in I$, such that $f \mapsto 1 \in k[x_0, ..., x_n]$. Hence, $I_H = (f)$ with $deg(f) = d$. □

Now, we start to prove this result.

($\Longrightarrow$): Suppose that Y is given by I such that I can be generated by r elements. By the first claim, I can be generated by r homogeneous elements, that is, $I = (f_1, ..., f_r)$ with f_i homogeneous. Then $I = (f_1) + ... + (f_r)$ and $Y = \cap V_+(f_i)$. For each i, take $H_i = V_+(f_i)$, which is a hypersurface.

($\Longleftarrow$): By the second claim, we assume that $H_i = V_+(f_i)$ for some homogeneous element f_i. Then $Y = V_+(f_1, ..., f_r)$. Suppose that $I_Y = \Gamma^*(\mathscr{I}_Y)$. Then I_Y is the saturation of $I = (f_1, ..., f_r)$. Now we will use unmixedness theorem to show that I is saturated:

Theorem 2.8.6. *(Unmixedness Theorem [3]) Let R be a ring. If $I = (x_1, \ldots, x_n)$ is an ideal generated by n elements such that $\operatorname{codim} I = n$, then all minimal primes of I have codimension n. If R is Cohen–Macaulay, then every associated prime of I is minimal over I.*

Note that $Ass(I) = \{Ass(m) | m \in R/I\}$. We know that $S = k[x_0, ..., x_n]$ is Cohen-Macaulay, hence for $I = (f_1, ..., f_r)$, $Ass(I)$ is just the set of minimal prime ideals of I and $(x_0, ..., x_n) \notin Ass(I)$. So if $x_i y \in I$ for each i, then $y \in I$. Take any $y \in \overline{I}$. For each i, there exists n_i such that $x_i^{n_i} y \in I$. By discussion before, we see that $y \in I$. Hence $I = I_Y$.

(b). Suppose that $I_Y = (f_1, ..., f_r)$ with f_i homogeneous and hence

$$Y = Proj(k[x_0, ..., x_n]/(f_1, ..., f_r)) \subset \mathbb{P}^n_k, \quad C(Y) = Spec(k[x_0, ..., x_n]/(f_1, ..., f_r)) \subset \mathbb{A}^{n+1}_k$$

Since Y is normal, by [P186 Proposition 8.23 [5]], Y is regular of codimension 1. Hence, $C(Y)$ is also regular of codimension 1:

Since Y is closed, $\dim Y + codim(Y, \mathbb{P}^n_k) = dimn$ by 2.3.20. For $\dim C(Y) = \dim Y + 1$ by **??** and $C(Y) \subset \mathbb{A}^{n+1}_k$ is also closed, $codim(C(Y), \mathbb{A}^{n+1}_k) = 1$.

Again, by [P186 Proposition 8.23 [5]], $C(Y)$ is normal, which implies $k[x_0, ..., x_n]/(f_1, ..., f_r)$ is normal. So Y is projectively normal by **??**.

(c). By 2.5.14, $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(l)) \longrightarrow \Gamma(Y, \mathcal{O}_Y(l))$ is surjective.When $l = 0$, we have $k \twoheadrightarrow \Gamma(Y, \mathcal{O}_Y) := A$ with $A = k[x_0, ..., x_n]/(f_1, ..., f_r)$. Then $A = 0$ or $A = k$. For the first case $Y = \emptyset$. For the later case, $C(Y) = Spec(k)$ which is connected. Then $Y = \theta(C(Y))$ with θ is the projection form $\mathbb{A}^{n+1}_k$ to $\mathbb{P}^n_k$.

(d). To use Bertini's theorem, we should assume that k is algebraically closed and $char(k) = 0$.

First of all, given d_1, there exists H_1 with degree d_i such that H_1 is non-singular, like $H_1 = V_+(x_0^{d_1} + ... + x_n^{d_1})$. Now, we use induction to show that we can always find $H_1, ..., H_r$ such that $H_1 \cap ... \cap H_r$ is non-singular in $\mathbb{P}^n_k$.

Suppose that we are given $H_1 \cap ... \cap H_{r-1}$ which is non-singular and $deg(H_i) = d_i$. Then consider the Veronese embedding of degree d_r,

$$H_1 \cap ... \cap H_{r-1} \subset \mathbb{P}^n_k \xrightarrow{i} \mathbb{P}^N_k$$

Then, using Bertini's theorem, we can find $H_r' \in \mathbb{P}^N_k$ with $deg(H_r') = 1$ such that $H_1 \cap ... \cap H_r'$ is non-singular in $\mathbb{P}^N_k$. Then $H_r = i^{-1}(H_r')$ is a hypersurface of degree d_r and $H_1 \cap ... \cap H_{r-1} \cap H_r = H_1 \cap ... \cap H_r'$, which implies $H_1 \cap ... \cap H_{r-1} \cap H_r$ is non-singular.

(e). Suppose that $Y = V_+(f_1, ..., f_r)$ with $deg(f_i) = d_i$ and $X = V_+(f_1, ..., f_{r-1})$ Then consider the short exact sequence:

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_{\mathbb{P}^n_k}|_Y \longrightarrow \mathcal{N}_{Y/\mathbb{P}^n_k} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^n_k}|_X \longrightarrow \mathcal{N}_{X/\mathbb{P}^n_k} \longrightarrow 0$$

By [P182 Proposition 8.20], we have

$$\omega_Y = \omega_X|_Y \otimes \mathcal{O}_Y(-d_r)$$

. Then use induction to get that $\omega_Y = \mathcal{O}_Y(n - \sum_{i=1}^r d_i)$.

(f). Now, we see that $\mathscr{I}_Y = \mathcal{O}_{\mathbb{P}^n_k}(-d)$. Tensoring the exact sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow i_{Y*}\mathcal{O}_Y \longrightarrow 0$$

with $\mathcal{O}_{\mathbb{P}^n_k}(d-n-1)$, we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(d-n-1) \longrightarrow i_{Y*}\omega_Y \longrightarrow 0$$

Taking global section, we have

$$0 \longrightarrow H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-n-1)) \longrightarrow H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(d-n-1)) \longrightarrow H^0(Y, \omega_Y)$$

By (c), the last arrow is surjection, hence

$$0 \longrightarrow H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-n-1)) \longrightarrow H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(d-n-1)) \longrightarrow H^0(Y, \omega_Y) \longrightarrow 0$$

is exact. $p_g(Y) = h^0(\mathbb{P}^1_k, \mathcal{O}(d-n-1)) = \binom{d-1}{n}$ since $h^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-n-1)) = 0$.

(e). We just repeat what we have done in (e). Tensoring

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow i_{Y*}\mathcal{O}_Y \longrightarrow 0$$

with $\mathcal{O}_{\mathbb{P}^n_k}(d+e-n-1)$, we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(e-n-1) \oplus \mathcal{O}_{\mathbb{P}^n_k}(d-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(d+e-n-1) \longrightarrow i_{Y*}\omega_Y \longrightarrow 0$$

Again, by (c), there exists an exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(e-n-1)) \oplus H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d-n-1)) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d+e-n-1)) \longrightarrow H^0(Y, \omega_Y \longrightarrow)0$$

Hence

$$\begin{aligned} p_g(Y) &= h^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d+e-n-1)) - h^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(e-n-1)) - h^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d-n-1)) \\ &= \binom{d+e-1}{n} - \binom{e-1}{n} - \binom{d-1}{n} \end{aligned}$$

Taking $n = 3$, $p_g(Y) = \frac{1}{2}de(d+e-4) + 1$

$\square$

Exercise 2.8.5 (8.5. Blowing up a Nonsingular Subvariety)**.** As in (8.24), let X be a nonsingular variety, let Y be a nonsingular subvariety of codimension $r \geq 2$, let $\pi : \tilde{X} \to X$ be the blowing-up of X along Y, and let $Y' = \pi^{-1}(Y)$.

(a) Show that the maps $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \tilde{X}$, and $\mathbb{Z} \to \operatorname{Pic} \tilde{X}$ defined by $n \mapsto$ class of nY', give rise to an isomorphism $\operatorname{Pic} \tilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

(b) Show that $\omega_{\tilde{X}} \cong \pi^*\omega_X \otimes \mathcal{L}((r-1)Y')$. [Hint: By (a) we can write in any case $\omega_{\tilde{X}} \cong \pi^*\mathcal{M} \otimes \mathcal{L}(qY')$ for some invertible sheaf $\mathcal{M}$ on X, and some integer q. By restricting to $\tilde{X} - Y' \cong X - Y$, show that $\mathcal{M} \cong \omega_X$. To determine q, proceed as follows. First show that $\omega_{Y'} \cong \pi^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)$. Then take a closed point $y \in Y$ and let Z be the fibre of Y' over y. Then show that $\omega_Z \cong \mathcal{O}_Z(-q-1)$. But since $Z \cong \mathbb{P}^{r-1}$, we have $\omega_Z \cong \mathcal{O}_Z(-r)$, so $q = r - 1$.]

Proof. (a). By [P133 Proposition 6.5 [5]], taking $U = \widetilde{X} - Y^{'} = X - Y$, we have

$$\mathbb{Z} \longrightarrow Cl(\widetilde{X}) \longrightarrow Cl(U) \longrightarrow 0$$

with $Cl(U) = Cl(X)$ for $codim(Y, X) \geq 2$. Note that the first map is given by $n \longrightarrow nY^{'}$. We aim to prove that this map is an injection:

If $nY^{'} \sim 0$, there exists a ration function f on $\widetilde{X}$ such that f vanishes on $Y^{'}$. If such rational function f exists, we can restrict f on X vanishing on Y. However, $Z(f)$ is of codimension 1 so such f doesn't exist. Thus, if $nY^{'} \sim 0$, $n = 0$.

For the second map $\varphi : Cl(\widetilde{X}) \longrightarrow Cl(\widetilde{X} - Y^{'})$ is given by $V \mapsto V \cap (\widetilde{X} - Y^{'})$ has a right inverse

$$\phi : Cl(\widetilde{X} - Y^{'}) \longrightarrow Cl(\widetilde{X})$$

given by $V \mapsto V$. Hence, the exact sequence splits and $Cl(\widetilde{X}) = Cl(X) \oplus \mathbb{Z}$.

Since X and $\widetilde{X}$ are both integral varieties, we can identify their zero divisors as their Picard groups. Hence, we have

$$Pic(\widetilde{X}) = Pic(X) \oplus \mathbb{Z}$$

deduced by $\pi^* : Pic(X) \longrightarrow Pic(\widetilde{X})$ and $n \longmapsto nY^{'}$.

(b). Suppose that $\omega_{\widetilde{X}} = \pi^*\mathcal{M} \otimes \mathscr{L}(qY^{'})$. Consider $i_1 : \widetilde{X} - Y^{'} \hookrightarrow \widetilde{X}$. Then we see that

$$\omega_{X-Y} = i^*\omega_{\widetilde{X}} = i^*_{X-Y}\mathcal{M}$$

since $i^*\mathscr{L}(qY^{'}) = 0$ and $\Omega_{X-Y/k} = \Omega_{\widetilde{X}/k}|_{X-Y}$. As we have shown above, $Pic(X - Y) \cong Pic(X)$ and $i^*_{X-Y}\omega_X = \omega_{X-Y}$. Hence $\mathcal{M} = \omega_X$.

Now that the ideal of $Y^{'}$ in $\widetilde{X}$ is just $\mathscr{I}_Y = \mathscr{L}(-Y^{'})$. By [P167 Exe.22.3D [12]], $\mathscr{L}(Y) = \mathcal{O}_{\widetilde{X}}(-1)$. Now, since $\mathscr{I}_Y$ is locally free, $\mathscr{I}_{Y'}/\mathscr{I}^2_{Y'} = \mathscr{L}(-Y^{'}) \otimes i_{Y'*}\mathcal{O}_{Y'}$, which implies $\mathscr{N}_{Y'/\widetilde{X}} = \mathscr{L}(Y^{'})|_{Y'} = \mathcal{O}_{Y'}(-1)$. By [P182 Prop 8.20 [5]], $\omega_{Y'} \cong \omega_{\widetilde{X}} \otimes \mathcal{O}_{Y'} \otimes \mathscr{N}_{Y'/\widetilde{X}} = \pi^*\omega_X \otimes \mathcal{O}_{Y'}(-1-q)$.

Now consider the pullback:

$$\begin{array}{ccccc} Z & \xrightarrow{\quad i_Z \quad} & Y^{'} & & \\ \Big\downarrow{\scriptstyle p_Z} & & \Big\downarrow{\scriptstyle \pi} & & \\ \{y\} = Spec(k) & \xrightarrow{i_y} & Y & \xrightarrow{\ g\ } & Spec(k) \end{array}$$

Then $Z = Y^{'} \times_Y \{y\} = Y^{'} \times_{Spec(k)} \{y\}$. Using 2.8.3 (b), $\omega_Z = p^*_Z\omega_{\{y\}} \otimes i^*_Z\omega_{Y'}$. Since $g \circ i_y = id_{Spec(k)}$, $\Omega_{\{y\}/k} = 0$ and hence $\omega_{\{y\}} = \mathcal{O}_y$. So $\omega_Z = i^*_Z\omega_{Y'} = \mathcal{O}_Z(-q-1)$. Note that $i^*_Z\pi^* = p^*_Z i^*_Y$ implies $i^*_Z\pi^*\omega_X$ is trivial.

Finally, since $Z \cong \mathbb{P}^{r-1}_k$, $\omega_Z = \mathcal{O}_Z(-r)$ by the Euler's equation. Hence, $-r = -q - 1$ and $\omega_{\widetilde{X}} = \pi^*\omega_X \otimes \mathscr{L}((r-1)Y^{'})$ □

Remark. Using Eular sequence to reprove this. See 3.8.4.

Exercise 2.8.6. The following result is very important in studying deformations of nonsingular varieties. Let k be an algebraically closed field, let A be a finitely generated k-algebra such that $\operatorname{Spec} A$ is a nonsingular variety over k. Let $0 \to I \to B' \to B \to 0$ be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f : A \to B$. Then there exists a k-algebra homomorphism $g : A \to B'$ making a commutative diagram

We call this result the infinitesimal lifting property for A. We prove this result in several steps.

(a) First suppose that $g : A \to B'$ is a given homomorphism lifting f. If $g' : A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, which we can consider as an element of $\operatorname{Hom}_A(\Omega_{A/k}, I)$. Note that since $I^2 = 0$, I has a natural structure of B-module and hence also of A-module. Conversely, for any $\theta \in \operatorname{Hom}_A(\Omega_{A/k}, I)$, $g' = g + \theta$ is another homomorphism lifting f. (For this step, you do not need the hypothesis about $\operatorname{Spec} A$ being nonsingular.)

(b) Now let $P = k[x_1, \dots, x_n]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there does exist a homomorphism $h : P \to B'$ making a commutative diagram,

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
J & & I \\
\downarrow & & \downarrow \\
P & \xrightarrow{h} & B' \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

and show that h induces an A-linear map $\overline{h} : J/J^2 \to I$.

(c) Now use the hypothesis $\operatorname{Spec} A$ nonsingular and (8.17) to obtain an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0.$$

Show furthermore that applying the functor $\operatorname{Hom}_A(\cdot, I)$ gives an exact sequence

$$0 \to \operatorname{Hom}_A(\Omega_{A/k}, I) \to \operatorname{Hom}_P(\Omega_{P/k}, I) \to \operatorname{Hom}_A(J/J^2, I) \to 0.$$

Let $\theta \in \operatorname{Hom}_P(\Omega_{P/k}, I)$ be an element whose image gives $\overline{h} \in \operatorname{Hom}_A(J/J^2, I)$. Consider θ as a derivation of P to B'. Then let $h' = h - \theta$, and show that h' is a homomorphism of $P \to B'$ such that $h'(J) = 0$. Thus h' induces the desired homomorphism $g : A \to B'$.

Proof. To begin with, we assume that

$$0 \longrightarrow I \longrightarrow B^{'} \xrightarrow{\varphi} B \longrightarrow 0$$

(a). First, we show that if $g, g^{'}$ both lift f. Then $g, g^{'}$ define the same A-module structure of I: Take any $a \in A$ and $b \in I$. Then since both $g, g^{'}$ lift f, $\theta(a) \in I$. Then $\theta(a)b = 0$ for $I^2 = 0$. Hence, $g^{'}(a)b = g(a)b$.

Since all maps are k-algebra homomorphisms, $\theta(a) = g^{'}(a) - g(a) = 0$. Take any $a_1, a_2 \in A$, we have

$$\theta(a_1 a_2) = g(a_1)g(a_2) - g^{'}(a_1)g^{'}(a_2)$$

$$= g(a_1)\theta(a_2) + \theta(a_1)g^{'}(a_2)$$

Since the image of θ lies in I, $\theta(a_1a_2) = a_1\theta(a_2) + \theta(a_1)a_2$. So θ is of course a k-derivation.

Conversely, if θ is a k-derivation. It is enough to show that $\varphi \circ \theta = 0$: This is trivial since the image of θ lies in I.

(b). Suppose that the image of x_i in A is $\bar{x}_i$. Since $B^{'} \longrightarrow B$ is surjective, For each $\bar{x}_i$, there exist b_i such that $b_i \longmapsto f(\bar{x}_i)$. Then we define $h : P \longrightarrow B$ by $x_i \longrightarrow b_i$.

Now, consider $a \in J$. $h(a) \longmapsto 0$ under $B^{'} \longrightarrow B$ since in A, $\bar{a} = 0$ and hence $f(\bar{a}) = 0$. Thus, $h(a) \in I$. Since $I^2 = 0$, $h : J \longrightarrow I$ is in fact $\bar{h} : J/J^2 \longrightarrow I$.

(c). Since $Ext^1_A(\Omega_{A/k}, I) = 0$ for $\Omega_{A/k}$ is a free A-module, we have the exact sequence.

Now that θ deduce

$$\begin{tikzcd} & J/J^2 \arrow[d, "d"] \arrow[dr, "\bar{h}"] & & \\ \theta^{'} : P \arrow[r, "d"] & \Omega_{P/k} \arrow[r, "\theta"] & I \arrow[r, hook] & B^{'} \end{tikzcd}$$

Then for any $a \in J$, we see that $\theta^{'}(a) = h(a)$ by the commutative diagram above. Thus, we get $h^{'} : P \longrightarrow B^{'}$ such that $h^{'}(J) = 0$, which induces $g : A \longrightarrow B^{'}$.

□

Exercise 2.8.7.

Exercise 2.8.8. *Proof.* Let $f : X \dashrightarrow X^{'}$ be the birational map and $V \subset X$ be the largest open set such that $f : V \longrightarrow X^{'}$ represents f. Then f induces a map $f^*\Omega_{X'/k} \longrightarrow \Omega_{V/k}$. Since $X^{'}$ and V are non-singular, $\Omega_{X'/k}$, $\Omega_{V/k}$ are locally free. Since f^* commutes with $\otimes$, we have $f^* \wedge^q \Omega_{X'/k} \longrightarrow \wedge^q\Omega_{V/k}$. Now, suppose $U \subset V$ is the open subset such that $f(U)$ is open and $f : U \longrightarrow f(U)$ is an isomorphism. Then $f^*\wedge^q\Omega_{X'/k}|_{f(U)} \cong \wedge^q\Omega_{V/k}|_U$, that is, $H^0(f(U), \wedge^q\Omega_{X'/k}|_U) = H^0(U, \wedge^q\Omega_{V/k}|_U)$.

Note that we have a natural map $H^0(X^{'}, \wedge^q\Omega_{X'/k}) \longrightarrow H^0(f(U), \Omega_{X'/k}|_{f(U)})$ given by restriction. Since $f(U)$ is dense over $X^{'}$ and a non-zero global section of a locally free sheaf cannot vanish on a dense open set, the natural map is an injection.

Now, we have a commutative diagram

$$\begin{tikzcd} H^0(X^{'}, \wedge^q\Omega_{X'/k}) \arrow[r, "f^*"] \arrow[d, "p_1"] & H^0(V, \wedge^q\Omega_{V/k}) \arrow[d, "p_2"] \\ H^0(f(U), \wedge^q\Omega_{X'/k}|(f(U)) \arrow[r, "\cong"] & H^0(U, \wedge^q\Omega_{V/k}|_U) \end{tikzcd}$$

Since p_1 is injective and the diagram is commutative f^* is injective.

Now, by 2.5.7, we choose an affine cover $\{U_i\}$ of X such that $\wedge^q\Omega_{X/k}|_{U_i}$ is free. Now, as discussion in [Theorem II.8.19. [5]] (using the same proof to show that), $H^0(U_i, \mathcal{O}_{U_i}) = H^0(U_i \cap V, \mathcal{O}_{U_i\cap V})$, that is, $H^0(U_i, \wedge^q\Omega_{X/k}|_{U_i}) = H^0(U_i \cap V, \wedge^q\Omega_{X/k}|_{V\cap U_i})$ since $\wedge^q\Omega_{X/k}|_{U_i}$ is free on U_i. By the gluing of sheaves, $H^0(X, \wedge^q\Omega_{X/k}) = H^0(V, \wedge^q\Omega_{X/k}|_V) = H^0(V, \wedge^q\Omega_{V/k})$. Hence we have the inclusion $H^0(X^{'}, \wedge^q\Omega_{X'/k}) \subset H^0(X, \wedge^q\Omega_{X/k})$. Since the map is birational, we have the inverse inclusion. Hence, $H^0(X^{'}, \wedge^q\Omega_{X'/k}) = H^0(X, \wedge^q\Omega_{X/k})$.

For ω^n_X and $\omega^n_{X'}$, use the same method. □

2.8.4 Addition Exercises

After you finish Chapter 3:

Exercise 2.8.9 (Complete intersections in $\mathbb{P}^4$)**.** Let $X \subset \mathbb{P}^4_k$ be a two-dimensional smooth complete intersection of degree (d_1, d_2), i.e. $X = V_+(f_1, f_2)$ with $f_i \in H^0(\mathbb{P}^4_k, \mathcal{O}(d_i))$.

(i) Describe the normal bundle of X in $\mathbb{P}^4_k$.

(ii) Determine for which d_1, d_2 the boundary map $H^0(X, \mathcal{N}_{X/\mathbb{P}^4}) \to H^1(X, \mathcal{T}_X)$ of the normal bundle sequence is surjective and discuss its kernel.

Proof. (a). First we need some results in commutative algebra: Let $x, y \in A$ be a regular sequence and then let $I = (x, y)$. Then there exists an isomorphism

$$\begin{aligned} A/I\ \overline{x} \oplus A/I\ \overline{y} &\longrightarrow I/I^2 \\ (\bar{a}, \bar{b}) &\longmapsto \bar{a}\overline{x} + \bar{b}\overline{y} \end{aligned}$$

This map is well-defined and it is a surjection since I is generated by x, y. It is injective since if $\bar{a}\bar{x} + \bar{b}\bar{y} = 0$, $\bar{a}\bar{y} = 0$ in $A/(x)$, which implies $\bar{a} \in (x) \subset I$.

Back to our question, since X is a complete intersection in $\mathbb{P}^4_k$ of degree (d_1, d_2), we see that f_1, f_2 are regular in $A := k[x_0, ..., x_4]$. Since $I/I^2 = A/I\bar{f}_1 \oplus A/I\bar{f}_2$, by comparing the degree of I/I^2 and $A/I\bar{f}_1 \oplus A/I\bar{f}_2$, we see that $\mathrm{Proj}(A/I\bar{f}_1) \cong \mathcal{O}_X(-d_1)$ and so $\mathcal{I}_X/\mathcal{I}_X^2 \cong \mathcal{O}_X(-d_1) \oplus \mathcal{O}_X(-d_2)$, which implies $\mathcal{N}_{X/\mathbb{P}^4_k} = \mathcal{O}_X(d_1) \oplus \mathcal{O}_X(d_2)$.

(b). Since $X \subset \mathbb{P}^4_k$ is a closed subscheme, we consider

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^4_k}|_X \longrightarrow \mathcal{N}_{X/\mathbb{P}^4_k} \longrightarrow 0$$

with $\mathcal{N}_{X/\mathbb{P}^n_k} = \mathcal{O}_X(d_1) \oplus \mathcal{O}_X(d_2)$. Hence, we first try to compute $H^1(X, \mathcal{O}_X(d_1))$ and $H^1(X, \mathcal{O}_X(d_2))$. For X is a closed subscheme of dimension 2, by 3.5.5 (c), $H^1(X, \mathcal{O}_X(d)) = 0$ for all $d \in \mathbb{Z}$. Hence, $H^1(X, \mathcal{N}_{X/\mathbb{P}^4_k}) = 0$. Now, we just need to consider $H^1(X, \mathcal{T}_{\mathbb{P}^4_k}|_X)$ since we have

$$H^0(X, \mathcal{T}_{\mathbb{P}^4_k}|_X) \longrightarrow H^0(X, \mathcal{N}_{X/\mathbb{P}^4_k}) \longrightarrow H^1(X, \mathcal{T}_X) \longrightarrow H^1(X, \mathcal{T}_{\mathbb{P}^4_k}|_X) \longrightarrow H^1(X, \mathcal{N}_{X/\mathbb{P}^4_k})$$

To compute $H^1(X, \mathcal{T}_{\mathbb{P}^4_k}|_X)$, we will use the Euler sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus 5} \longrightarrow \mathcal{T}_{\mathbb{P}^4_k}|_X \longrightarrow 0$$

to get a long exact sequence

$$0 \longrightarrow H^1(X, \mathcal{T}_{\mathbb{P}^4_k}|_X) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X(1))^{\oplus 5}$$

We know that when $H^2(X, \mathcal{O}_X) = 0$, then $H^1(X, \mathcal{T}_{\mathbb{P}^4_k}|_X) = 0$. By Serre duality and 3.5.5 (a), this happens when $d_1 + d_2 \leq 4$. $\square$

Remark. Since the Euler sequence in $\mathbb{P}^n_k$ is an exact sequence over locally free bundle, restricting to X, we get a new exact sequence.

We can generalize the isomorphism to get the following results: If $Y \subset \mathbb{P}^n_k$ is a complete intersection defined by $f_1, ...f_r$ with deg $f_i = d_i$, then

$$\mathcal{N}_{Y/\mathbb{P}^n_k} \cong \bigoplus_{i=1}^{r} \mathcal{O}_Y(d_i)$$

In fact, there exists a long exact sequence to describe complete intersections:

Exercise 2.8.10. Let A be a ring and $a_1, \ldots, a_n \in A$ a sequence of elements. Recall that a sequence is called regular, if for any $1 \leq i \leq n$ the element a_i is not a zero-divisor in $A/(a_1, \ldots, a_{i-1})$. Consider the complex

$$K_i^* := \left(\cdots \to 0 \to A \xrightarrow{a_i} A \to 0 \to \cdots \right)$$

concentrated in degrees -1 and 0, where the only non-zero differential is multiplication by a_i. Let K^* be the tensor product $K^* := K_1^* \otimes_A \cdots \otimes_A K_n^*$, so that K^* is concentrated in degrees $-n, \ldots, 0$.

Assuming that the sequence $a_1, \ldots, a_n$ is regular, prove that $H^i(K^*) = 0$ for $i < 0$ and $H^0(K^*) = A/(a_1, \ldots, a_n)$. The complex K^* is called Koszul complex and it gives a free resolution of the A-module $A/(a_1, \ldots, a_n)$.

Now let $\mathcal{E}$ be a locally free sheaf of finite rank r on a scheme X and $s \in H^0(X, \mathcal{E})$ a section which is locally given by a regular sequence of elements in the corresponding ring (this is true if the zero locus of s has codimension equal to the rank of $\mathcal{E}$).

Let $Y \subset X$ be the zero locus of s. Show that the local Koszul complexes glue into a global locally free resolution of $\mathcal{O}_Y$ (which is also called Koszul complex):

$$0 \to \bigwedge^r \mathcal{E}^* \to \bigwedge^{r-1} \mathcal{E}^* \to \cdots \to \bigwedge^2 \mathcal{E}^* \to \mathcal{E}^* \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

The differentials are given by convolution with s.

Proof. Let $\mathcal{E}$ be a locally free. Then there exists an affine cover $\{U_i\}$ such that $\mathcal{E}|_{U_i} = \mathcal{O}_{U_i}^{\oplus r}$. Suppose that $U_i = \operatorname{Spec}(A_i)$ and $Y|_{U_i}$ is defined by $(a_1, ..., a_r)$. Then there exists

$$0 \longrightarrow \bigwedge^r A_i^{\oplus r} \longrightarrow ... \longrightarrow \bigwedge^2 A_i^{\oplus r} \longrightarrow A_i^{\oplus r} \longrightarrow A_i \longrightarrow A_i/(a_1, ..., a_r) \longrightarrow 0$$

Over $U_i \cap U_j$, note that we can cover it by an affine cover such that the intersection of any two open sets is affine. Hence, we can glue the long exact sequence over $U_i \cap U_j$ and then glue this to the whole scheme X. $\square$

2.9 Formal Schemes

2.9.1 Preparations

Example 2.9.1.

(1). $A = \mathbb{Z}$a and $\mathfrak{m} = (p)$. Then $\widehat{A} \cong \mathbb{Z}_p$;

(2). $A = k[x_1, ..., x_n]$ and $\mathfrak{m} = (x_1, ..., x_n)$. Then $\widehat{A} \cong k[[x_1, ..., x_n]]$, the power series ring.

Proposition 2.9.2. *Let A be a Noetherian ring and $I \subset A$ be an ideal. Then*

(1). The completion induces an exact functor from finitely generated A-mod to finitely generated $\widehat{A}$-mod;

(2). If M is finitely generated over A, then $\widehat{M} = M \otimes_A \widehat{A}$;

(3). $\widehat{A}$ is flat over A;

(4). $\widehat{A}$ is Noetherian;

(5). If $(A, \mathfrak{m})$ is local, then $(\widehat{A}, \widehat{\mathfrak{m}})$ is a local ring and $gr_{\mathfrak{m}}(A) = gr_{\widehat{\mathfrak{m}}}(\widehat{A})$;

(6). If $(A, \mathfrak{m})$ is a local ring, the followings are equivalent: (i). A is regular; (ii). $\widehat{A}$ is regular; (iii). $gr_{\mathfrak{m}}(A) \cong gr_{\widehat{\mathfrak{m}}}(\widehat{A}) \cong K[[x_1, ..., x_d]]$ with $K = A/\mathfrak{m}$ and $d = \dim A$.

Lemma 2.9.3 (Hensel's Lemma)**.** *Let $(\widehat{A}, \widehat{\mathfrak{m}})$ be a complete Noetherian local ring and $f(T) \in \widehat{A}[T]$ be a polynomial. If $\overline{f}(t) \in K[t]$ with $K = \widehat{A}/\widehat{\mathfrak{m}}$ has a simple root, then $f(T) \in \widehat{A}[T]$ has a simple root.*

Example 2.9.4. Let k be an algebraically closed field of characteristic 0. Let $C_1 = V(y^2 - x^3 - x^2)$ and $C_2 = V(y^2 - x^3)$ and $O = (0, 0)$. Compute $\widehat{\mathcal{O}_{C_1,O}}$ and $\widehat{\mathcal{O}_{C_2,O}}$.

Note that $\mathcal{O}_{C_1,O} = k[x, y]_{(x,y)}/(y^2 - x^3 - x^2)$. By 2.9.2 (2),

$$\widehat{\mathcal{O}_{C,O}} = k[x, y]_{(x,y)}/(y^2 - x^3 - x^2) \otimes_{k[x,y]_{(x,y)}} \widehat{k[x, y]_{(x,y)}} = k[[x, y]]/(y^2 - x^3 - x^2)$$

By Hensel's lemma, $T^2 - (x + 1) = 0$ has a solution in $\widehat{A}$. Let $s = \sqrt{x + 1}$. Then we see that $y^2 - x^3 - x^2 = y^2 - x^2 s^2 = (y - xs)(y + xs)$. Let $u = y - sx$ and $v = y + sx$. Since s is invertible (write the expansion of s and use Hensel's lemma), $\widehat{\mathcal{O}_{C,O}} = k[[u, v]]/(uv)$.

Similarly, we have

$$\widehat{\mathcal{O}_{C,O}} = k[x, y]_{(x,y)}/(y^2 - x^3) \otimes_{k[x,y]_{(x,y)}} \widehat{k[x, y]_{(x,y)}} = k[[x, y]]/(y^2 - x^3)$$

which is not isomorphic to $k[[u, v]]/(uv)$, since x^3 is not a square in $k[[x, y]]$ by its degree, that is odd.

More precisely, we can show that $A \hookrightarrow \widehat{A}$ is faithfully flat:

Definition 2.9.5. Let $\varphi : A \longrightarrow B$ be a homomorphism of rings. We say that φ is faithfully flat if

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

of A-mod is exact if and only if

$$0 \longrightarrow M_1 \otimes_A B \longrightarrow M_2 \otimes_A B \longrightarrow M_3 \otimes_A B \longrightarrow 0$$

Lemma 2.9.6. *Let $\varphi : A \longrightarrow B$ be a homomorphism of rings. Then*

(1). If $\varphi : A \longrightarrow B$ is faithfully flat and $M \otimes_A B = 0$, then $M = 0$;

(2). If φ is faithfully flat, then φ is injective;

(3). If φ is faithfully flat, then for any A-mod M, $M \longrightarrow M \otimes_A B$ is an injection;

(4). If φ is faithfully flat and $I \subset A$ is an ideal, then $IB \cap A = I$;

(5). If $\varphi : A \longrightarrow B$ is a homomorphism of local rings and φ is flat, then φ is faithfully flat.

Proof.

(1). Consider $0 \longrightarrow M \longrightarrow 0$. Since tensoring B, the exact sequence is exact, $0 \longrightarrow M \longrightarrow 0$ is exact. Thus, $M = 0$.

(2). Note that if $\ker(\varphi) \otimes_A B = 0$ since $A \otimes_A B \cong B \longrightarrow B \otimes_A B$ is an injective. By (1), $\ker(\varphi) = 0$;

(3). For $M \otimes_A B \longrightarrow M \otimes_A B \otimes_A B$ admits a section given by $M \otimes_A B \otimes_A B \longrightarrow M \otimes_A B$ given by $m \otimes b \otimes b^{'} \longmapsto m \otimes bb^{'}$, $M \otimes_A B \longrightarrow M \otimes_A B \otimes_A B$ is an injection. Hence, $M \longrightarrow M \otimes_A B$ is an injection like (2).

(4). By (3), we have $A/I \hookrightarrow A/I \otimes_A B \cong B/IB$, which is deduced by $\varphi : A \longrightarrow B \longrightarrow B/I$. Thus, $\varphi^{-1}(IB) = IB \cap A \subset I$. Obviously, $I \subset IB \cap A$.

$\square$

Chapter 3

Cohomology

3.1 Derived Functors

Theorem 3.1.1. *For $\mathcal{I}$ injective and F a left exact functor, $R^iF(\mathcal{I}) = 0$ for $i > 0$*

Proof. First, we know that $R^iF(\mathcal{I})$ is independent of the choice of injective resolutions. So we consider

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}_0 = \mathcal{I} \longrightarrow 0$$

which is an injective resolution of $\mathcal{I}$. Thus,

$$0 \longrightarrow F(\mathcal{I}) \longrightarrow F(\mathcal{I}) \longrightarrow 0$$

So $R^iF(\mathcal{I}) = 0$ when $i > 0$. □

Theorem 3.1.2. *With $F : \mathfrak{A} \longrightarrow \mathfrak{B}$ left-exact, suppose there is a sequence*

$$0 \longrightarrow A \longrightarrow J^0 \longrightarrow J^1 \longrightarrow ...$$

where each J^i is acyclic for F, $i \geq 0$. Then for each $i \geq 0$, there is a natural isomorphism $R^iF(A) = h^i(F(J^\cdot))$.

Prove this, maybe we need to use spectral sequence???

Remark. This theorem tells us that to compute the cohomology of a sheaf, it is sufficient to find a flasque resolution of it.

How to induce the long exact sequence?
For $\mathcal{F}$, given the injective resolution of $\mathcal{F}$, then for the short exact sequence

$$0 \longrightarrow \mathcal{F}^{'} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{''} \longrightarrow 0$$

we have

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow f^{'}_{i-1} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(\mathcal{I}^{'}_i) & \longrightarrow & F(\mathcal{I}_i) & \longrightarrow & F(\mathcal{I}^{''}_i) & \longrightarrow & 0 \\
 & & \downarrow f^{'}_i & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(\mathcal{I}^{'}_{i+1}) & \longrightarrow & F(\mathcal{I}_{i+1}) & \longrightarrow & F(\mathcal{I}^{''}_{i+1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Then we have:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(\mathcal{I}^{'}_i)/im(f^{'}_{i-1}) & \longrightarrow & F(\mathcal{I}_i)/im(f_{i-1}) & \longrightarrow & F(\mathcal{I}^{''}_i)/im(f^{''}_{i-1}) & \longrightarrow & 0 \\
 & & \downarrow f^{'}_i & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(\mathcal{I}^{'}_{i+1}) & \longrightarrow & F(\mathcal{I}_{i+1}) & \longrightarrow & F(\mathcal{I}^{''}_{i+1}) & \longrightarrow & 0
\end{array}$$

Using snake lemma, we have the long exact sequence.

3.2 Cohomology of Sheaves

3.2.1 Preparation

Theorem 3.2.1. *If $(X, \mathcal{O}_x)$ is a ringed space, any injective $\mathcal{O}_X$-module is flasque.*

Proof. For $U \subset X$, consider $\mathcal{O}_U := j_!(\mathcal{O}|_U)$ **??**. For any $V \subset U$, we have an injection $\mathcal{O}_V \longrightarrow \mathcal{O}_U$ by verifying on stalks. For $\mathcal{I}$, we have

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_V \arrow[r, hook] \arrow[d] & \mathcal{O}_U \arrow[dl, dashed, "\exists"] \\ & \mathcal{I} & \end{tikzcd}$$

Thus, we have $Hom_{Sh(X)}(\mathcal{O}_U, \mathcal{I}) \longrightarrow Hom_{Sh}(\mathcal{O}_V, \mathcal{I})$. Note that $Hom_{Sh(X)}(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U)$, implying that $\mathcal{I}$ is flasque. □

Remark. Given any $f \in \mathcal{I}(U)$, we define

$$\mathcal{O}_U(V) \longrightarrow \mathcal{I}(V)$$

by $s_V \mapsto \rho_{UV}(f)s_V$. We can easily verify that this is a morphism between sheaves.

For $g \in Hom_{Sh}(\mathcal{O}_U, \mathcal{I})$, take $f = g(U)1_U \in \mathcal{I}(U)$ where 1_U is the unit in $\mathcal{O}_U$.

These two maps are inverse to each other. So $\mathcal{I}(U) = Hom_{Sh}(\mathcal{O}|_U, \mathcal{I})$.

Theorem 3.2.2. *If $\mathcal{F}$ is flasque sheaf on a topological space X, then*

$$H^i(X, \mathcal{F}) = 0$$

for $i \geq 0$.

Proof. Consider an injective $\mathcal{I}$ such that

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{R} \longrightarrow 0$$

with $\mathcal{R} = \mathcal{F}/\mathcal{I}$. Then we have

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{R}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{I})$$

For $\mathcal{I}$ is injective, $H^p(X, \mathcal{I}) = 0$ when $p > 0$. Thus, $H^i(X, \mathcal{F}) = H^{i+1}(X, \mathcal{R})$ when $i > 0$.

By 2.1.16 (b), $H^1(X, \mathcal{F}) = 0$. Because $\mathcal{I}$ is also flasque, $\mathcal{R}$ is also flasque by 2.1.16 (c). Hence by induction, $H^i(X, \mathcal{F}) = 0$ for $i > 0$. □

Theorem 3.2.3. *Let X be a Noetherian topological space. Then colimits preserve $H^i(X, -)$.*

Proof. Sketch: Use the exactness of *colimit* in the category of abelian and use the universal property of δ-functor. □

3.2.2 Exercises

Exercise 3.2.1. (a) Let $X = \mathbf{A}_k^1$ be the affine line over an infinite field k. Let P, Q be distinct closed points of X, and let $U = X - \{P, Q\}$. Show that $H^1(X, \mathcal{I}_U) \neq 0$.

(b) More generally, let $Y \subseteq X = \mathbf{A}_k^n$ be the union of $n+1$ hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X, \mathcal{I}_U) \neq 0$. Thus the result of (2.7) is the best possible.

Proof.

(a). Note that, we have

$$0 \longrightarrow \underline{\mathbb{Z}}_U \longrightarrow \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}_{P,Q} \longrightarrow 0$$

For X is integral, so it is irreducible, then $\underline{\mathbb{Z}}$ is flasque. So we have:

$$0 \longrightarrow \Gamma(X, \underline{\mathbb{Z}}_U) \longrightarrow \Gamma(X, \underline{\mathbb{Z}}) \longrightarrow \Gamma(X, \underline{\mathbb{Z}}_{P,Q}) \longrightarrow H^1(X, \underline{\mathbb{Z}}_U) \longrightarrow 0$$

If $H^1(X, \underline{\mathbb{Z}}_U) = 0$. Then $\Gamma(X, \underline{\mathbb{Z}}) \longrightarrow \Gamma(X, \underline{\mathbb{Z}}_{P,Q})$ is surjective, that is, $\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ is a surjection, which leads to a contradiction. $\square$

Exercise 3.2.2. Let $X = \mathbf{P}^1_k$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{K}/\mathcal{O} \to 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{C}$. Conclude from (II, Ex. 1.21e) that $H^i(X, \mathcal{C}) = 0$ for all $i > 0$.

Proof.

For $\mathbb{P}^1_k$ is integral, it is irreducible. So $\mathcal{K}$ is flasque on X. For $\mathcal{K}/\mathcal{O} \cong \sum i_{P*}(I_P)$, so $\mathcal{K}/\mathcal{O}$ is also flasque. So, $0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{O} \longrightarrow 0$ is a flasque resolution of $\mathcal{O}$.

For $\mathcal{K}$ and $\mathcal{K}/\mathcal{O}$ are flasque, $H^i(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{K}/\mathcal{O}) = 0$ when $i > 0$. Then $H^i(X, \mathcal{O}) = 0$ for $i > 1$.

For $i = 1$, just note that we have the exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{O}) \longrightarrow \Gamma(X, \mathcal{K}) \longrightarrow \Gamma(X, \mathcal{K}/\mathcal{O}) \longrightarrow 0$$

and $H^1(X, \mathcal{F}) = 0$. Then $H^1(X, \mathcal{O}) = 0$. $\square$

Exercise 3.2.3. Let X be a topological space, let Y be a closed subset, and let $\mathcal{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathcal{F})$ denote the group of sections of $\mathcal{F}$ with support in Y (II, Ex. 1.20).

(a) Show that $\Gamma_Y(X, \cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$. We denote the right derived functors of $\Gamma_Y(X, \cdot)$ by $H^i_Y(X, \cdot)$. They are the *cohomology groups* of X *with supports* in Y, and coefficients in a given sheaf.

(b) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, with $\mathcal{F}'$ flasque, show that

$$0 \to \Gamma_Y(X, \mathcal{F}') \to \Gamma_Y(X, \mathcal{F}) \to \Gamma_Y(X, \mathcal{F}'') \to 0$$

is exact.

(c) Show that if $\mathcal{F}$ is flasque, then $H^i_Y(X, \mathcal{F}) = 0$ for all $i > 0$.

(d) If $\mathcal{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X - Y, \mathcal{F}) \to 0$$

is exact.

(e) Let $U = X - Y$. Show that for any $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$\begin{gathered} 0 \to H^0_Y(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}|_U) \to \\ \to H^1_Y(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(U, \mathcal{F}|_U) \to \\ \to H^2_Y(X, \mathcal{F}) \to \dots \end{gathered}$$

(f) **Excision**. Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathcal{F}$,

$$H^i_Y(X, \mathcal{F}) \cong H^i_Y(V, \mathcal{F}|_V).$$

Proof.

(a). $\Gamma_Y(X, \mathcal{F}^{'}) \longrightarrow \Gamma_Y(X, \mathcal{F})$ is injective if $\mathcal{F}^{'} \longrightarrow \mathcal{F}$ is injective:

For any $a \in \Gamma_Z(X, \mathcal{F}^{'}) \subset \Gamma(X, \mathcal{F}^{'}) \subset \Gamma(X, \mathcal{F})$, we see that $supp(a) \subset Y$. So $s \in \Gamma_Y(X, \mathcal{F})$. And it is injective for $\Gamma(X, \mathcal{F}^{'}) \subset \Gamma(X, \mathcal{F})$.

$\Gamma_Y(X, \mathcal{F}^{'}) \xrightarrow{i} \Gamma_Y(X, \mathcal{F}) \xrightarrow{f} \Gamma_Y(X, \mathcal{F}^{''})$ is exact at $\Gamma_Y(X, \mathcal{F})$. For any $b \in ker(f)$, $b \in \Gamma(X, \mathcal{F}^{'})$. Because $supp(b) \subset Y$, $b \in \Gamma_Y(X, \mathcal{F}^{'})$.

Thus, $\Gamma_Y(X, -)$ is left-exact.

(b). We just need to show that $\Gamma_Y(X, \mathcal{F}) \xrightarrow{f} \Gamma_Y(X, \mathcal{F}^{''})$ is a surjection. For $\mathcal{F}^{'}$ is flasque, $\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}^{''})$ is surjective.

For $c \in \Gamma_Y(X, \mathcal{F}^{''})$, there exists $b \in \Gamma(X, \mathcal{F})$ such that $f(b) = c$. Note that $f(b|_{X-Y}) = f(b)|_{X-Y} = 0$. Thus, $b|_{X-Y} \in \Gamma(X, \mathcal{F}^{'})$. For $\mathcal{F}^{'}$ is flasque, there exists $a \in \Gamma(X, \mathcal{F}^{'})$ such that $a|_{X-Y} = b|_{X-Y}$. Then $f(b-a) = c$ and $supp(b-a) \subset Y$. So $\Gamma_Y(X, \mathcal{F}) \longrightarrow \Gamma_Y(X, \mathcal{F}^{''})$ is surjective.

(c). Consider the injectives resolution $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}$. Let $\mathcal{G} = \mathcal{F}/\mathcal{I}$, then we have

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

For $\mathcal{F}$ and $\mathcal{I}$ are flasque, so is $\mathcal{G}$. For $\mathcal{I}$ is injective, $H^1_Y(X, \mathcal{I}) = 0$ for $i > 0$. By the exact sequence in (b), we see that $H^1_Y(X, \mathcal{F}) = 0$. For $\mathcal{G}$ is flasque, $H^1(X, \mathcal{G}) = 0$ and $H^2_Y(X, \mathcal{F}) = H^1(X, \mathcal{G}) = 0$. Repeat this process, we see that $H^i_Y(X, \mathcal{F}) = 0$ for any $i > 0$.

(d). Note that we have:

$$0 \longrightarrow \mathcal{H}^0_Y(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_*(\mathcal{F}) \longrightarrow 0$$

where $j : X - Y \hookrightarrow X$.

Then we have

$$0 \longrightarrow \Gamma_Y(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X - Y, \mathcal{F})$$

For $\mathcal{F}$ is flasque, $\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X - Y, \mathcal{F})$ is surjective. Thus, we have the exact sequence:

$$0 \longrightarrow \Gamma_Y(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X - Y, \mathcal{F}) \longrightarrow 0$$

(e). Consider an injective resolution $\{\mathcal{I}_i\}$ of $\mathcal{F}$. $\mathcal{I}_i|_{X-Y}$ is an injective resolution of $\mathcal{F}|_U$,for $\mathcal{F}(V)|_{X-Y} = \mathcal{F}(V \cap (X - Y))$.

For each $\mathcal{I}_i$ is flasque, we have the exact sequence

$$0 \longrightarrow \Gamma_Y(X, \mathcal{I}_i) \longrightarrow \Gamma(X, \mathcal{I}_i) \longrightarrow \Gamma(X - Y, \mathcal{I}_i) \longrightarrow 0.$$

Thus, we get a diagram:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_Y(X, \mathcal{I}_i) & \longrightarrow & \Gamma(X, \mathcal{I}_i) & \longrightarrow & \Gamma(X - Y, \mathcal{I}_i|_U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Thus, we have the long the exact sequence.

(f). □

Exercise 3.2.4. Mayer-Vietoris Sequence. Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\ldots \to H^i_{Y_1 \cap Y_2}(X, \mathcal{F}) \to H^i_{Y_1}(X, \mathcal{F}) \oplus H^i_{Y_2}(X, \mathcal{F}) \to H^i_{Y_1 \cup Y_2}(X, \mathcal{F}) \to$$
$$\to H^{i+1}_{Y_1 \cap Y_2}(X, \mathcal{F}) \to \ldots$$

Proof. By the definition of the sheaf, we have the exact sequence

$$0 \longrightarrow \Gamma_{Y_1 \cap Y_2}(X, \mathcal{F}) \longrightarrow \Gamma_{Y_1}(X, \mathcal{F}) \oplus \Gamma_{Y_2}(X, \mathcal{F}) \longrightarrow \Gamma_{Y_1 \cup Y_2}(X, \mathcal{F})$$

So it remains to show that for any injective $\mathcal{I}$, we have the exact sequence

$$0 \longrightarrow \Gamma_{Y_1 \cap Y_2}(X, \mathcal{I}) \longrightarrow \Gamma_{Y_1}(X, \mathcal{I}) \oplus \Gamma_{Y_2}(X, \mathcal{I}) \longrightarrow \Gamma_{Y_1 \cup Y_2}(X, \mathcal{I}) \longrightarrow 0$$

Use 3.2.3(d), we have

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_{Y_1 \cap Y_2}(X, \mathcal{I}) \arrow[r] \arrow[d] & \Gamma_{Y_1}(X, \mathcal{I}) \oplus \Gamma_{Y_2}(X, \mathcal{I}) \arrow[r] \arrow[d] & \Gamma_{Y_1 \cup Y_2}(X, \mathcal{I}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X, \mathcal{I}) \arrow[r] \arrow[d] & \Gamma(X, \mathcal{I}) \oplus \Gamma(X, \mathcal{I}) \arrow[r] \arrow[d] & \Gamma(X, \mathcal{I}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X - Y_1 \cap Y_2, \mathcal{I}) \arrow[r] \arrow[d] & \Gamma(X - Y_1, \mathcal{I}) \oplus \Gamma(X - Y_2, \mathcal{I}) \arrow[r] \arrow[d] & \Gamma(X - Y_1 \cup Y_2, \mathcal{I}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the last row is exact by the definition of sheaf and $\mathcal{I}$ is flasque (so that $\Gamma(X - Y_1, \mathcal{I}) \longrightarrow \Gamma(X - Y_1 \cup Y_2, \mathcal{I})$ is surjective), the second row is exact by the definition of the map and each column is exact by 3.2.3 (d) for $\mathcal{I}$ is flasque.

By 9-lemma, the first column is exact. We are done. □

Exercise 3.2.5.

Exercise 3.2.6.

Exercise 3.2.7. We will compute this by *Čech* cohomology.

3.3 Cohomology of a Noetherian Affine Scheme

Lemma 3.3.1. *Let I be an injective module over a Noetherian ring A. Then the sheaf $\widetilde{I}$ on $X = Spec(A)$ is flasque.*

Theorem 3.3.2. *Let $X = Spec(A)$ be the spectrum of a Noetherian ring A. Then for all quasi-coherent sheaves $\mathcal{F}$ on X,*

$$H^i(X, \mathcal{F}) = 0$$

when $i > 0$.

Proof. Suppose $\mathcal{F}$ is a quasi-coherent sheaf over X. Then $\mathcal{F} = \widetilde{M}$ for M a A-module. Consider an injectives resolution of $0 \longrightarrow M \longrightarrow I^{\cdot}$. For localization is exact, we have $0 \longrightarrow \mathcal{F} \longrightarrow \widetilde{I}^{\cdot}$ which is a flasque resolution of $\mathcal{F}$. Taking global setion, we have $0 \longrightarrow M \longrightarrow I^{\cdot}$ which is exact. Thus, $H^i(X, \mathcal{F}) = 0$ when $i > 0$. □

Lemma 3.3.3. *Let X be a Noetherian scheme, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $\mathcal{F}$ can be embedded in a flasque, quasi-coherent sheaf of $\mathcal{G}$.*

Proof. Take a finite affine cover $U_i = Spec(A_i)$ of X where A_i are Noetherian. Consider $\mathcal{F}|_{U_i} = \widetilde{M_i}$ with M_i an A_i-module and $0 \longrightarrow M_i \longrightarrow I_i$ where I_i is an injective in A_i-module. Then $\widetilde{I_i}$ is flasque, so is $\mathcal{I}_i := i_{U,*}\widetilde{I_i}$ and $\mathcal{G} = \oplus \mathcal{I}_i$. Because X and U_i are Noetherian, each $\mathcal{I}_i = i_{U,*}(\widetilde{I_i})$ is quasi-coherent and so $\mathcal{G}$.

By verifying on stalks, we have

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$$

where $\mathcal{G}$ is quasi-coherent and flasque. □

Theorem 3.3.4 (Serre)**.** *Let X be a Noetherian scheme. Then the following conditions are equivalent:*

(i) X is affine;

(ii) $H^i(X, \mathcal{F}) = 0$ for all $\mathcal{F}$ quasi-coherent and $i > 0$;

(iii) $H^i(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$;

3.4 Čech Cohomology

3.4.1 Preparation

Lemma 3.4.1. *For any sheaf of abelian groups $\mathcal{F}$ on X, the complex $\mathcal{C}^{\cdot}(\mathfrak{U}, \mathcal{F})$ is a resultion of $\mathcal{F}$, that is, there exists a natural map $\epsilon : \mathcal{F} \longrightarrow \mathcal{C}^0$ such that the sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \longrightarrow ...$$

is exact.

Theorem 3.4.2. *If $\mathcal{F}$ is a flasque sequence sheaf of abelian groups on X. Then for all $p > 0$,*

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = 0$$

Proof. We need to show that $\mathcal{C}^p(\mathfrak{U}, \mathcal{F})$ is still flasque. For $\mathcal{F}$ is flasque, so is $\mathcal{F}|_{U_{i_0 \cdots i_p}}$ and $i_{U_{i_0 \cdots i_p}, *}\mathcal{F}|_{U_{i_0 \cdots i_p}}$. Note that direct product preserves flasque. So $\mathcal{C}^p(\mathfrak{U}, \mathcal{F})$ is flasque for every $p > 0$.

Now,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \longrightarrow ...$$

is a flasque resolution of $\mathcal{F}$. So

$$H^i(X, \mathcal{F}) = h_i(\Gamma(\mathcal{C}^{\cdot}(\mathfrak{U}, \mathcal{F}))) = \check{H}^i(\mathfrak{U}, \mathcal{F})$$

Thus, $\check{H}^i(\mathfrak{U}, \mathcal{F}) = 0$ for $i > 0$. $\square$

Theorem 3.4.3. *Let X be **a Noetherian separated scheme**, let $\mathfrak{U}$ be **an open affine cover** of X, and let $\mathcal{F}$ be **a quasi-coherent sheaf** on X. Then for all $p \geq 0$, the natural map gives isomorphisms*

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) \xrightarrow{\cong} H^p(X, \mathcal{F})$$

Proof. Consider the affine cover $\mathfrak{U} = \{U_i\}$ and $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$ with $\mathcal{G}$ quasi-coherent and flasque.

Then we have

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{R} \longrightarrow 0$$

For $QCoh(X)$ is an abelian category, $\mathcal{R}$ is also quasi-coherent. Hence on $U_{i_0 \dots i_p}$, we have

$$0 \longrightarrow \mathcal{F}(U_{i_0 \dots i_p}) \longrightarrow \mathcal{G}(U_{i_0 \dots i_p}) \longrightarrow \mathcal{R}(U_{i_0 \dots i_p}) \longrightarrow 0$$

Because direct product preserves exactness we have

$$0 \longrightarrow \mathcal{C}^{\cdot}(\mathfrak{U}, \mathcal{F}) \longrightarrow \mathcal{C}^{\cdot}(\mathfrak{U}, \mathcal{G}) \longrightarrow \mathcal{C}^{\cdot}(\mathfrak{U}, \mathcal{R}) \longrightarrow 0$$

Thus, we have the long exact sequence

$$0 \longrightarrow \check{H}^0(\mathfrak{U}, \mathcal{F}) \longrightarrow \check{H}^0(\mathfrak{U}, \mathcal{G}) \longrightarrow \check{H}^0(\mathfrak{U}, \mathcal{R}) \longrightarrow \check{H}^1(\mathfrak{U}, \mathcal{F}) \longrightarrow \check{H}^1(\mathfrak{U}, \mathcal{G}) \longrightarrow ...$$

For $\mathcal{G}$ is flasque, $\check{H}^p(\mathfrak{U}, \mathcal{G}) = 0$ for $p > 1$. Thus, $\check{H}^{p+1}(\mathfrak{U}, \mathcal{F}) = \check{H}^p(\mathfrak{U}, \mathcal{R})$ and moreover $H^{p+1}(X, \mathcal{F}) = H^p(X, \mathcal{G})$ when $p > 1$.

So it is enough to show that $H^1(X, \mathcal{F}) = \check{H}^1(\mathfrak{U}, \mathcal{F})$: For any sheaf $\mathcal{H}$, $\check{H}^0(\mathfrak{U}, \mathcal{H}) = H^0(X, \mathcal{H})$ by the definition of sheaves. We have a natural map form $\check{H}^p(\mathfrak{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$. And note that $H^1(X, \mathcal{G})$ and $\check{H}^1(\mathfrak{U}, \mathcal{G}) = 0$, by five lemma, we see that $H^1(\mathfrak{U}, \mathcal{F}) = H^1(X, \mathcal{F})$. $\square$

3.4.2 Exercises

Exercise 3.4.1. Let $f : X \to Y$ be an affine morphism of noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathcal{F}$ on X, there are natural isomorphisms for all $i \geq 0$,

Proof. Chose an affine cover $\mathfrak{V} = \{V_i\}_{i\in I}$ of Y. Then $\mathfrak{U} = \{U_i = f^{-1}(V_i)\}_{i\in I}$ is an affine cover of X. For both X and Y are separated, $V_{i_0 i_1 \dots i_p}$ and $U_{i_0 i_1 \dots i_p}$ are both affine.

For any affine open set $f : Spec(B) \longrightarrow U = Spec(A)$ and a quasi-coherent sheaf $\mathcal{F}$, $\mathcal{F} = \widetilde{M}$ with M an B-module and $f_*(\mathcal{F}) = \widetilde{M}$, regarding M as a A-module.

Thus, $\mathcal{F}(U_{i_0 i_1 \dots i_n})_*\mathcal{F}(V_{i_0 i_1 \dots i_n})$. as abelian groups. So

$$\check{H}^i(\mathfrak{U}, \mathcal{F}) \cong \check{H}^i(\mathfrak{V}, f_*\mathcal{F})$$

By Theorem 4.5,

$$H^i(X, \mathcal{F} \cong H^i(Y, f_*\mathcal{F})$$

□

Exercise 3.4.2. Prove Chevalley's theorem: Let $f : X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.

(a) Let $f : X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathcal{M}$ on X, and a morphism of sheaves $\alpha : \mathcal{O}_Y^r \to f_*\mathcal{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

(b) For any coherent sheaf $\mathcal{F}$ on Y, show that there is a coherent sheaf $\mathcal{G}$ on X, and a morphism $\beta : f_*\mathcal{G} \to \mathcal{F}^r$ which is an isomorphism at the generic point of Y.
Hint: Apply $\mathcal{H}om(\cdot, \mathcal{F})$ to α and use (II, Ex. 5.17e).

(c) Now prove Chevalley's theorem. First use (Ex. 3.1) and (Ex. 3.2) to reduce to the case X and Y integral. Then use (3.7), (Ex. 4.1), consider ker β and coker β, and use noetherian induction on Y.

(a).

(b).

(c).

Exercise 3.4.3. Let $X = \mathbb{A}_k^2 = \operatorname{Spec} k[x, y]$, and let $U = X - \{(0, 0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathscr{O}_U)$ is isomorphic to the k-vector space spanned by $\{\frac{1}{x^i y^j} \mid i, j < 0\}$. In particular, it is infinite-dimensional. (Using (3.5), this provides another proof that U is not affine—cf. (I, Ex. 3.6).)

Proof. Note that $U = \mathbb{A}_k^2 - \{(0, 0)\}$ can be covered by $U_1 = (\mathbb{A}_k^1 - \{0\}) \times_k \mathbb{A}_k^1$ and $U_2 = \mathbb{A}_k^1 \times_k (\mathbb{A}_k^1 - \{0\})$. Because $\mathbb{A}_k^1 = Spec(k[x, x^{-1}])$, $U_1 = Spec(k[x, x^{-1}, y])$ and $U_2 = Spec(k[x, y, y^{-1}])$, which are affine. and $U_1 \cap U_2 = (\mathbb{A}_k^1 - \{0\}) \times_k (\mathbb{A}_k^1 - \{0\}) = Spec(k[x, y, x^{-1}, y^{-1}])$.

For $\Gamma(U_1, \mathcal{O}_{U_1}) \oplus \Gamma(U_2, \mathcal{O}_{U_2}) \longrightarrow \Gamma(U_1 \cap U_2, \mathcal{O}_U)$ is given by $(f(x, x^{-1}, y), g(x, y, y^{-1})) \mapsto f(x, x^{-1}, y) - g(x, y, y^{-1})$. So

$$H^0(U, \mathcal{O}_U) = \check{H}^0(\mathfrak{U}, \mathcal{O}_U) = k[x, y]$$

$$H^1(X, \mathcal{O}_U) = \check{H}^1(\mathfrak{U}, \mathcal{O}_U) = k[x^{-1}, y^{-1}]$$

By Serre's theorem, if U is affine, $H^1(U, \mathcal{O}_U) = 0$, which leads to a contradiction. Thus, U is not affine. □

Remark. As we have shown before, $U \cap V$ can also be computed as

$$U \cap V = U \times_{\mathbb{A}^2_k} V = Spec(k[x,y,x^{-1}] \otimes_{k[x,y]} k[x,y,y^{-1}]) = Spec(k[x,y,x^{-1},y^{-1}])$$

Exercise 3.4.4. On an arbitrary topological space X with an arbitrary abelian sheaf $\mathcal{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if one takes the limit over all coverings.

(a) Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{V} = (V_j)_{j \in J}$, together with a map $\lambda : J \to I$ of the index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology, for any abelian sheaf $\mathcal{F}$, and for each i,

$$\lambda^i : \tilde{H}^i(\mathfrak{U}, \mathcal{F}) \to \tilde{H}^i(\mathfrak{V}, \mathcal{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\lim_{\mathfrak{U}} \tilde{H}^i(\mathfrak{U}, \mathcal{F}).$$

(b) For any abelian sheaf $\mathcal{F}$ on X, show that the natural maps (4.4) for each covering

$$\tilde{H}^i(\mathfrak{U}, \mathcal{F}) \to H^i(X, \mathcal{F})$$

are compatible with the refinement maps above.

(c) Now prove the following theorem. Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups. Then the natural map

$$\lim_{\mathfrak{U}} \tilde{H}^1(\mathfrak{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$$

is an isomorphism. [*Hint*: Embed $\mathcal{F}$ in a flasque sheaf $\mathcal{G}$, and let $\mathcal{R} = \mathcal{G}/\mathcal{F}$, so that we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0.$$

Define a complex $D(\mathfrak{U})$ by

$$0 \to C(\mathfrak{U}, \mathcal{F}) \to C(\mathfrak{U}, \mathcal{G}) \to D(\mathfrak{U}) \to 0.$$

Then use the exact cohomology sequence of this sequence of complexes, and the natural map of complexes

$$D(\mathfrak{U}) \to C(\mathfrak{U}, \mathcal{R}),$$

and see what happens under refinement.]

Proof. (a).

(b).

(c). By hint, we assume that $D^{\cdot}(\mathfrak{U})$ to be the image of $C^{\cdot}(\mathfrak{U}, \mathcal{F}) \longrightarrow C^{\cdot}(\mathfrak{U}, \mathcal{G})$, then we have an exact sequence of complexes

$$0 \longrightarrow C^{\cdot}(\mathfrak{U}, \mathcal{F}) \longrightarrow C^{\cdot}(\mathfrak{U}, \mathcal{G}) \longrightarrow D^{\cdot}(\mathfrak{U}) \longrightarrow 0$$

Taking the cohomology, we see that

$$H^i(C^{\cdot}(\mathfrak{U}, \mathcal{F})) = \check{H}^i(\mathfrak{U}, \mathcal{F}); \quad H^i(C^{\cdot}(\mathfrak{U}, \mathcal{G})) = \check{H}^i(\mathfrak{U}, \mathcal{G});$$

For $\mathcal{G}$ is flasque, $\check{H}^i(\mathfrak{U}, G) = 0$ for $i \geq 0$. And by the definition of sheaf, $\check{H}^0(X, \mathcal{F}) = H(X, \mathcal{F})$ and $\check{H}^0(X, \mathcal{G}) = H(X, \mathcal{G})$, hence $\varinjlim \check{H}^0(X, \mathcal{F}) = H(X, \mathcal{F})$ and $\varinjlim \check{H}^0(X, \mathcal{G}) = H(X, \mathcal{G})$.

By the construction of $\varinjlim$, it is exact at the category of abelian groups. So it is enough to show that $\varinjlim H^0(D^{\cdot}(\mathfrak{U})) = \varinjlim \check{H}^0(\mathfrak{U}, \mathcal{R}) = H^0(X, \mathcal{R})$. Then by five lemma,

$$\begin{array}{ccccccccc}
H^0(\mathfrak{U}, \mathcal{G}) & \longrightarrow & \varinjlim H^0(\mathfrak{U}, \mathcal{R}) & \longrightarrow & \varinjlim H^1(\mathfrak{U}, \mathcal{F}) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(X, \mathcal{G}) & \longrightarrow & \varinjlim H^0(X, \mathcal{R}) & \longrightarrow & \varinjlim H^1(X, \mathcal{F}) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

we can show that $\varinjlim H^1(\mathfrak{U}, \mathcal{F}) = H^1(X, \mathcal{F})$

Consider $s \in \varinjlim \check{H}^0(\mathfrak{U}, \mathcal{R})) = \varinjlim ker(C^0(\mathfrak{U}, \mathcal{R}) \longrightarrow C^1(\mathfrak{U}, \mathcal{R})$. Then there exists a covering $\mathfrak{U}$, such that $s \in ker(C^0(\mathfrak{U}, \mathcal{R}) \longrightarrow C^1(\mathfrak{U}, \mathcal{R})$. For $\mathcal{R}$ is the sheafification of the presheaf $(\mathcal{G}/\mathcal{F})^{Pr}$, for each U_i, there exists a cover $\mathfrak{U}_i$ of U_i, such that $s|_{U_{ij}} \in \mathcal{G}(U_{ij})/\mathcal{F}(U_{ij})$. For $s_m|_{U_m \cap U_n} - s_n|_{U_n \cap U_m} = 0$, so is $\{s|_{U_{ij}}\} \in D^0(\mathfrak{V})$, where $\mathfrak{V} = \{U_{ij}\}$ as defined above. Thus, $s \in ker(D^0(\mathfrak{V}) \longrightarrow D^1(\mathfrak{V}))$, so

$$\varinjlim H^0(\mathfrak{U}, \mathcal{R}) \subset \varinjlim H^0(D^{\cdot}(\mathfrak{U}))$$

For $\mathcal{F}$ is the sheafification of $\mathcal{G}/\mathcal{F}$, for each $\mathfrak{U}$, $H^0(D^{\cdot}(\mathfrak{U})) \subset H^0(\mathfrak{U}, \mathcal{R})$. Again, by the exactness of $\varinjlim$, we have

$$\varinjlim H^0(\mathfrak{U}, \mathcal{R}) = \varinjlim H^0(D^{\cdot}(\mathfrak{U}))$$

□

Exercise 3.4.5. For any ringed space $(X, \mathcal{O}_X)$, let $\operatorname{Pic} X$ be the group of isomorphism classes of invertible sheaves (II, §6). Show that $\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^{\times})$, where $\mathcal{O}_X^{\times}$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X)$, with multiplication as the group operation. [*Hint*: For any invertible sheaf $\mathcal{L}$ on X, cover X by open sets U_i on which $\mathcal{L}$ is free, and fix isomorphisms $\varphi_i : \mathcal{O}_{U_i} \to \mathcal{L}|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_i \cap U_j}$ with itself. These isomorphisms give an element of $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^{\times})$. Now use (Ex. 4.4).]

Proof. Consider an isomorphism $f : \mathcal{L} \longrightarrow \mathcal{M}$ between line bundles. Then there exists a cover $\mathfrak{U}$ of X such that $f_{U_i} : \mathcal{L}|_{U_i} \longrightarrow \mathcal{M}|_{U_i}$. More precisely, we can take $\mathfrak{U}$ such that $\mathcal{L}|_{U_i} = \widetilde{L}$ and $\mathcal{M} = \widetilde{M}$ with L, M $\mathcal{O}_X(U_i)$-module of rank 1. Then $f|_{U_i}$ is equivalent to an isomorphism between L and M, that is, an element in $\Gamma(U_i, \mathcal{O}_X^*)$. Thus, we can see that $f \in \check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$. Thus, $f \in \check{H}(X, \mathcal{O}_X)$

Conversely, for any $s \in \check{H}^1(X, \mathcal{O}_U^*)$, $s \in \check{H}(\mathfrak{V}, \mathcal{O}_X^*)$. Then s define a line bundle $s\mathcal{O}_X$, which is equivalent to an isomorphism between $\mathcal{O}_X$ and $s\mathcal{O}_X$.

Thus, by $\check{H}(X, \mathcal{O}_X^*) = Pic(X)$. □

Exercise 3.4.6. Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathcal{I}$ be a sheaf of ideals with $\mathcal{I}^2 = 0$, and let X_0 be the ringed space $(X, \mathcal{O}_X/\mathcal{I})$. Show that there is an exact sequence of sheaves of abelian groups on X,

$$0 \to \mathcal{I} \to \mathcal{O}_X^{\times} \to \mathcal{O}_{X_0}^{\times} \to 0,$$

where $\mathcal{O}_X^{\times}$ (respectively, $\mathcal{O}_{X_0}^{\times}$) denotes the sheaf of (multiplicative) groups of units in the sheaf of rings $\mathcal{O}_X$ (respectively, $\mathcal{O}_{X_0}$); the map $\mathcal{I} \to \mathcal{O}_X^{\times}$ is defined by $a \mapsto 1 + a$, and $\mathcal{I}$ has its usual (additive) group structure. Conclude there is an exact sequence of abelian groups

$$\ldots \to H^1(X, \mathcal{I}) \to \operatorname{Pic} X \to \operatorname{Pic} X_0 \to H^2(X, \mathcal{I}) \to \ldots.$$

Proof. By taking stalks, we have

$$0 \longrightarrow \mathcal{I}_p \longrightarrow \mathcal{O}_{X,p}^* \longrightarrow (\mathcal{O}_{X,p}/\mathcal{I}_p)^* \longrightarrow 0$$

For $\mathcal{I}$ is a sheaf of ideals and $\mathcal{I}^2 = 0$, taking any $a \in \mathcal{I}_p$, $(1+a)(1-a) = 1 - a^2 = 1$ and this is obviously injective. So $0 \longrightarrow \mathcal{I}_p \longrightarrow \mathcal{O}_{X,p}^*$ is exact.

If $b + \mathcal{I}_p = 1 + \mathcal{I}_p \in (\mathcal{O}_{X,p}/\mathcal{I}_p)^*$, then $b = 1 + a$ for some $a \in A$. Thus, $b \in im(\mathcal{I}_p \longrightarrow \mathcal{O}_{X,p}^*)$. So $\mathcal{I}_p \longrightarrow \mathcal{O}_{X,p}^* \longrightarrow (\mathcal{O}_{X,p}/\mathcal{I}_p)^*$.

Note that for any $b + \mathcal{I}_p$ that is unit, there exists $c + \mathcal{I}_p$ such that $bc + \mathcal{I}_p = 1 + \mathcal{I}_p$, then $bc = 1 + a$ for some a and $1 + a$ is a unit. so $b \in \mathcal{O}_{X,p}^*$. So $\mathcal{O}_{X,p}^* \longrightarrow (\mathcal{O}_{X,p}/\mathcal{I}_p)^* \longrightarrow 0$

Thus, we have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{O}_{X_0}^* \longrightarrow 0$$

□

Remark. $\mathcal{O}_X^*$ is gained by the sheafification of $U \longrightarrow \mathcal{O}_X(U)^*$.

Exercise 3.4.7. Let X be a subscheme of $\mathbf{P}_k^2$ defined by a single homogeneous equation $f(x_0, x_1, x_2) = 0$ of degree d. (Do not assume f is irreducible.) Assume that $(1, 0, 0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$$

explicitly, and thus show that

$$\dim H^0(X, \mathcal{O}_X) = 1,$$
$$\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2).$$

Proof. In $\{x_1 \neq 0\} \cong \mathbb{A}_k^2$, X is defined by $f(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1})$. Thus, $U = Spec(k[\frac{x_0}{x_1}, \frac{x_2}{x_1}]/(f(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1})))$ and similarly $V = Spec(k[\frac{x_0}{x_2}, \frac{x_1}{x_2}]/(f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)))$. Note that $\{x_1 \neq 0, x_2 \neq 0\} \cong \mathbb{A}_k^2 - \{0\} \times_k \mathbb{A}_k^1 \cong Spec(k[\frac{x_0}{x_2}, \frac{x_1}{x_2}, \frac{x_1}{x_2}^{-1}])$. So $U \cap V = Spec(k[\frac{x_0}{x_2}, \frac{x_1}{x_2}, \frac{x_1}{x_2}^{-1}]/(f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)))$. Finally ,we see that $\Gamma(U, \mathcal{O}_U) \oplus \Gamma(V, \mathcal{O}_V) \longrightarrow \Gamma(U \cap V, \mathcal{O}_{U \cap V})$ is defined by $(h(\frac{x_0}{x_1}, \frac{x_2}{x_1}), g(\frac{x_0}{x_2}, \frac{x_1}{x_2})) \mapsto h(\frac{x_0}{x_1}, \frac{x_2}{x_1}^{-1}) - g(\frac{x_0}{x_2}, \frac{x_1}{x_2})$. Note that $\frac{x_0}{x_1} = \frac{x_0}{x_2}\frac{x_2}{x_1}$. So to simplify, we see that

$$\Gamma(U, \mathcal{O}_U) \oplus \Gamma(V, \mathcal{O}_V) \longrightarrow \Gamma(U \cap V, \mathcal{O}_{U \cap V})$$

$$(h(x, y), g(z, w)) \longmapsto h(xy^{-1}, y^{-1}) - g(x, y)$$

where $U = Spec(k[x, y]/(f(x, 1, y)))$, $V = Spec(k[z, w]/(f(z, w, 1)))$ and $U \cap V = Spec(k[x, y, y^{-1}]/(f(x, y, 1)))$. It is easy to verify that this map is well-defined.

For $(1, 0, 0) \notin X$, we see that $f(1, 0, 0) \neq 0$, that is $f(x_0, x_1, x_2) = ax_0^d + f_1(x_0, x_1, x_2)$ with $a \neq 0$. Thus, w.l.o.g. we just assume $f(x_0, x_1, x_2) = x_0^d + f_1(x_0, x_1, x_2)$.

Note that $h(xy^{-1}, y^{-1})$ contains $x^i y^j$ with $0 < -j < i$. So suppose that $h(xy^{-1}, y^{-1}) - g(x, y) = f(x, y, 1)e(x, y)$. Now, we see that $f(x, y, 1) = x^d + \sum a_{nm} x^n y^m$ with $m + n \leq d$:

(1) For e, we assume that e_0 is the sum of $x^i y^j$ with $j \leq -i - d$. Then $m + j \leq -i - j$, that is, $e_0 f \in Im(\Gamma(U) \longrightarrow \Gamma(U \cap V))$.

(2) Let e_1 be the sum of $x^i y^j$ with $j > 0$. Obviously, $e_1 f \in Im(\Gamma(V) \longrightarrow \Gamma(U \cap V))$.

(3) Let e_2 be the rest items $x^i y^j$ such that $j < 0$ and $-i - d < j$. Then $x^i y^j x^d = x^{i+d} y^j$. So $e_2 f$ can not be contained in any images. Thus, $e_2 = 0$.

As we have conclude that $h(xy^{-1}, y^{-1}) = e_0(x,y)f(x,y,1) + C$ and $g(x,y) = e_1(x,y)f(x,y,1) - C$ Then $g(x,y) = -C \in \Gamma(V, \mathcal{O}_U)$ and

$$\begin{aligned} h(z,w) &= e_0(\frac{z}{w}, \frac{1}{w})f(\frac{z}{w}, \frac{1}{w}, 1) + C \\ &= \frac{1}{w^d}e_0(\frac{z}{w}, \frac{1}{w})f(z,w,1) + C \\ &= C \in \Gamma(U, \mathcal{O}_U) \end{aligned}$$

Thus, $\check{H}^0(\mathfrak{U}, \mathcal{O}) = k$, with $dim(H^0(X, \mathcal{O})) = 1$.

As we have discussed, $x^i y^j \in im(d_0)$ if and only if $-i < j < 0$. Note that $x^d = -f_1(x,y)$. So $\check{H}^1(\mathfrak{U}, \mathcal{O})$ is generated by $\{x^i y^j | -d < -i < j < 0\}$, which has dimension $\binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$. □

Exercise 3.4.8. Cohomological Dimension (Hartshorne [3]). Let X be a noetherian separated scheme. We define the *cohomological dimension* of X, denoted $\mathrm{cd}(X)$, to be the least integer n such that $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > n$. Thus for example, Serre's theorem (3.7) says that $\mathrm{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\mathrm{cd}(X) \leq \dim X$.

(a) In the definition of $\mathrm{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X. Use (II, Ex. 5.15) and (2.9).

(b) If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X. Use (II, 5.18).

(c) Suppose X has a covering by $r+1$ open affine subsets. Use Čech cohomology to show that $\mathrm{cd}(X) \leq r$.

(d) If X is a quasi-projective scheme of dimension r over a field k, then X can be covered by $r+1$ open affine subsets. Conclude (independently of (2.7)) that $\mathrm{cd}(X) \leq \dim X$.

(e) Let Y be a set-theoretic complete intersection (I, Ex. 2.17) of codimension r in $X = \mathbf{P}^n_k$. Show that $\mathrm{cd}(X - Y) \leq r - 1$.

Proof. (a). Note that if $\mathcal{F}$ is quasi-coherent, then $\mathcal{F} = \cup_\alpha \mathcal{F}_\alpha$ with $\mathcal{F}_\alpha$ coherent.

If $H^i(X, \mathcal{F}) = 0$ for any coherent sheaf, then because $\varinjlim$ commutes with H^i on Noetherian topology spaces and $\cup$ is a kind of colimits,

$$H^i(X, \mathcal{F}) = \varinjlim H^i(X, \mathcal{F}_\alpha)$$

So it is sufficient to verify all coherent sheaves on X.

(b).

(c). For X is a Noetherian separated scheme and $\mathcal{F}$ is quasi-coherent,

$$H^i(X, \mathcal{F}) = \check{H}^i(\mathfrak{U}, \mathcal{F})$$

where $\mathfrak{U}$ is an affine cover.

$C^{r+i}(\mathfrak{U}, \mathcal{F}) = \Pi_{i_0 \ldots i_{r+i}} \mathcal{F}(U_{i_0 \ldots i_{r+i}})$. Note that when $i > 0$, $U_{i_0 \ldots i_{r+i}} = \emptyset$. So $C^{r+i}(\mathfrak{U}, \mathcal{F}) = 0$ when $i > 0$. Thus, $\check{H}^{r+i}(\mathfrak{U}, \mathcal{F}) = 0$ for any $i > 0$. Thus, $cd(X) = r$.

(d). Here, we consider $i : X \hookrightarrow \mathbb{P}^n_k$ which is a closed immersion. Then $i : X \cap D_+(x_i) \longrightarrow D_+(x_i)$ is a closed immersion, hence $X_i \cap D_+(x_i)$ is affine. Note that X can be covered by $X \cap D_+(x_i)$ and

X is a Noetherian separated scheme for $i : X \hookrightarrow \mathbb{P}^n_k$ is separated. Thus, by (c), $cd(X) \leq n$.

(e).

$\square$

3.5 The Cohomology of Projective Space

3.5.1 Preparations

Theorem 3.5.1. *Let A be a Noetherian ring, $S = A[x_0, ..., x_r]$ and let $X = \mathbb{P}^n_A$ is an isomorphism of graded S-modules:*

(a) the natural map $S \longrightarrow \Gamma^*(\mathcal{O}_X) = \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ is an isomorphism of graded S-modules;

(b) $H^i(X, \mathcal{O}_X(n)) = 0$ for all $0 < i < r$ and all $n \in \mathbb{Z}$;

(c) $H^r(X, \mathcal{O}_X(-r-1)) \cong A$;

(d) The natural map

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathcal{O}_X(-r-1)) \cong A$$

is a perfect pairing of finitely generated free A-modules, for each $n \in \mathbb{Z}$.

Proof. For $\mathbb{P}^n_k$ is a Noetherian separated scheme, chose the cover $\mathfrak{U} = \{D_+(x_i)\}_{i=0,1,..,n}$, which is an affine cover. By 3.4.3,

$$H^i(X, \mathcal{O}_X) = \check{H}^i(\mathfrak{U}, \mathcal{O}_X)$$

Note that $D(x_i) \cap D(x_j) = D(x_ix_j)$ and $\widetilde{M}|_{D_+(x_{i_0}...x_{i_k})} = M^{\sim}_{(x_{i_0}...x_{i_k})}$

(a). For $\mathcal{F} = \oplus_{n\in\mathbb{Z}}\mathcal{O}_X(n)$, by 3.2.3,

$$H^i(X, \mathcal{F}) = \oplus_{n\in\mathbb{Z}} H^i(X, \mathcal{O}_X(n))$$

and $H^i(X, \mathcal{O}_X(n))$ has an A-module structure. So $H^i(X, \mathcal{F})$ has a A-module structure and we have the $\check{C}$ech complex

$$0 \longrightarrow \Pi S_{x_i} \xrightarrow{d_0} \Pi S_{x_{i_0}x_{i_1}} \longrightarrow .... \longrightarrow \Pi S_{x_0x_1...\check{x}_i...x_r} \xrightarrow{d_r} S_{x_0x_1...x_r} \longrightarrow 0$$

for $\mathcal{F} = (\oplus_{n\in\mathbb{Z}} S(n))^{\sim}$ and $\mathcal{F}|_{D_+(x_{i_0}...x_{i_k})} = S^{\sim}_{x_{i_0}...x_{i_k}}$. Then $H^0(X, \mathcal{F}) = ker(d_0) = S$, with a graded structure.

(b).

For *coker* is a kind of colimits, $coker(\oplus_{n\in\mathbb{Z}} d(n)) = \oplus_{n\in\mathbb{Z}} coker(d(n))$. Since $S_{x_1...x_r} = \oplus S(n)_{(x_1...x_r)}$, given any monomial f, $deg(\frac{f}{(x_1...x_n)^n}) = deg(x_0^{i_0}...x_r^{i_r}) = n$ if and only if $\frac{f}{(x_1...x_n)^n} \in S(n)_{(x_0...x_n)}$. Thus, $\frac{f}{(x_1...x_n)^n} \in coker(d_r)$ and $deg(\frac{f}{(x_1...x_n)^n}) = n$ if and only if $\frac{f}{(x_1...x_n)^n} \in coker(d_r(n))$. To compute $H^r(X, \mathcal{O}_X(n))$, it is enough to compute $H^r(X, \mathcal{F})$.

$\frac{f}{(x_0...x_r)^n} = x_0^{l_0}...x_r^{l_n} \in im(d)$ if and only if there exists $l_0 \geq 0$. (If so, we can chose $\frac{f/x_i^n}{(x_0...\check{x}_i...x_r)^n} \in S_{x_0...\check{x}_i...x_r}$.) Thus, $coker(d_r)$ is spanned by $\{x_0^{l_0}...x_r^{l_n} | l_0 < 0, ..., l_n < 0\}$ as A-module.

(c). Note that the only monomial in the basis of $coker(d)$ is of deg $-r-1$ is $x_0^{-1}x_1^{-1}...x_r^{-1}$. So $H^r(X, \mathcal{O}_X(-r-1)) = coker(d_r(-r-1))$ is spanned by $\{x_0^{-1}x_1^{-1}...x_r^{-1}\}$ as an A-module, which is of *rank* 1.

(d). When $n < 0$, note that $H^0(X, \mathcal{O}_P(n)) = 0$ by (a) and there is no monomial in basis of $coker(d_r)$ of $deg \geq -n-r-1$. Thus, when $n < 0$, $H^r(X, \mathcal{O}_X(-n-r-1)) = 0$. Thus, $H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow A$ is obviously a perfect pairing.

When $n > 0$, then $H^0(X, \mathcal{O}_P(n))$ is spanned by $\{x_0^{l_0}...x_r^{l_r} | l_0 + ... + l_r = n\}$ as A-module by (a) and $H^r(X, \mathcal{O}_X(-n-r-1))$ is spanned by $\{x_0^{l_0}...x_r^{l_r} | l_0 + ... + l_r = -n-r-1\}$ as A-module. For there is a $1-1$ corresponding between

$$\{x_0^{l_0}...x_r^{l_r} | l_0 + ... + l_r = n\} \xrightarrow{1:1} \{x_0^{l_0}...x_r^{l_r} | l_0 + ... + l_r = -n-r-1\}$$

$$x_0^{l_0}...x_r^{l_r} \longmapsto x_0^{-1-l_0}...x_r^{-1-l_r}$$

Thus, we have the perfect pairing $H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow A$.

□

Remark. We need to prove the graded structre of $coker(d)$ and others. Maybe for d preserves deg, but no strict proof!

3.5.2 Examples

For any $X \subset \mathbb{P}_k^n$ with $n \geq 2$, defined by F with $deg(F) > d$, $H^0(X, \mathcal{O}_X) = k$.

Proof. Note that we have the exact sequence

$$0 \longrightarrow S(-d) \xrightarrow{\cdot F} S \longrightarrow S/(F) \longrightarrow 0$$

then we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}_k^n} \longrightarrow i_{X,*}\mathcal{O}_X \longrightarrow 0$$

Taking H^i, we have

$$0 \longrightarrow H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-d)) \longrightarrow H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^1(\mathbb{P}_k^n, \mathcal{O}(-d))$$

Note that $H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-d)) = 0$ and $H^1(\mathbb{P}_k^n, \mathcal{O}(-d)) = 0$ when $n \geq 2$. So $k = H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}) = H^0(X, \mathcal{O}_X)$. □

3.5.3 Exercises

Exercise 3.5.1. Let X be a projective scheme over a field k, and let $\mathcal{F}$ be a coherent sheaf on X. We define the Euler characteristic of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \sum(-1)^i \dim_k H^i(X, \mathcal{F}).$$

If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, show that

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Proof. Note that we have the long exact sequence

$$0 \longrightarrow H^0(X, \mathcal{F}^{'}) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}^{''}) \longrightarrow ...$$

Takin apart this long exact sequence to short exact sequence, using Grothendieck vanishing theorem we can see that

$$\mathcal{X}(\mathcal{F}) = \mathcal{X}(\mathcal{F}^{'}) + \mathcal{X}(\mathcal{F}^{''})$$

□

Exercise 3.5.2. (a) Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathcal{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbb{Q}[z]$, such that $\chi(\mathcal{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$. We call P the Hilbert polynomial of $\mathcal{F}$ with respect to the sheaf $\mathcal{O}_X(1)$. [Hints: Use induction on dim Supp $\mathcal{F}$, general properties of numerical polynomials (I, 7.3), and suitable exact sequences

$$0 \to \mathcal{R} \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{Q} \to 0.]$$

(b) Now let $X = \mathbb{P}^r_k$, and let $M = \Gamma_*(\mathcal{F})$, considered as a graded $S = k[x_0, \ldots, x_r]$-module. Use (5.2) to show that the Hilbert polynomial of $\mathcal{F}$ just defined is the same as the Hilbert polynomial of M defined in (I, §7).

Proof. (a). When $\dim \operatorname{Supp}(\mathcal{F}) = 0$, we note that $\chi(\mathcal{F}(n))$ is a constant function. When $\dim \operatorname{Supp}(\mathcal{F}) = n+1$, we claim that $\dim \operatorname{Supp}(\mathcal{R}), \dim \operatorname{Supp}(\mathcal{Q}) \le n$. Since $\mathcal{R}, \mathcal{Q}$ are coherent, then by induction hypothesis, $\chi(\mathcal{R}(n))$ and $\chi(\mathcal{Q}(n))$ are polynomials. Then for $\chi(\mathcal{F}(n)) = \chi(\mathcal{F}(n-1)) + \chi(\mathcal{Q}(n)) - \chi(\mathcal{R}(n))$ by 3.5.1, $\chi(\mathcal{F}(n))$ is a polynomial.

Now, we prove our claim, that is, $\dim \operatorname{Supp}(\mathcal{R}), \dim \operatorname{Supp}(\mathcal{Q}) \le n$: Let Z be an irreducible component of $\operatorname{Supp}(\mathcal{F})$ and $s : \mathcal{F}(-1) \longrightarrow \mathcal{F}$ be a section in $H^0(X, \mathcal{O}_X)$ that doesn't vanishing at Z. Take the generic point of Z, η. For $\mathcal{F}$ is coherent, $\mathcal{F}_\eta$ and $\mathcal{F}(-1)_\eta$ are finite dimensional $k(Z)$-vector spaces. Since $s_\eta \neq 0$ on Z, $\mathcal{F}(-1)_\eta \xrightarrow{\times s_\eta} \mathcal{F}_\eta$ is an isomorphism by comparing the dimension of the vector spaces. Thus, $\operatorname{Coker}(\mathcal{F})_\eta = 0$ and $\operatorname{Ker}(\mathcal{F})_\eta = 0$. Hence, $\dim \operatorname{Supp}(\mathcal{R}), \dim \operatorname{Supp}(\mathcal{Q}) \le n$ since they don't have η in their support.

(b). By [[5] Theorem III.5.2.] There exists n_0 depending on $\mathcal{F}$, such that for $i > 0$, $H^i(X, \mathcal{F}(n)) = 0$. At this case, $\chi(\mathcal{F}(n)) = h^0(X, \mathcal{F}(n))$. By definition, $H^0(X, \mathcal{F}(n)) = M_n$. Since both $\chi(\mathcal{F}(n))$ and M_n are polynomials and they are equal at infinitely many points ($n \ge n_0$), they are the same. $\square$

Remark. For $\mathcal{O}_X(1)$ is very ample, we can inject X into some $\mathbb{P}^n$. Then, we can take some hypersurface H such that $H \not\subset Z$. s taken above corresponds to such a hypersurface.

Exercise 3.5.3 (Arithmetic Genus)**.** Let X be a projective scheme of dimension r over a field k. We define the arithmetic genus p_a of X by

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1).$$

Note that it depends only on X, not on any projective embedding.

(a) If X is integral, and k algebraically closed, show that $H^0(X, \mathcal{O}_X) \cong k$, so that

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X).$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X).$$

[Hint: Use (I, 3.4).]

(b) If X is a closed subvariety of $\mathbb{P}^r_k$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

(c) If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a birational invariant. Conclude that a nonsingular plane curve of degree $d \ge 3$ is not rational. (This gives another proof of (II, 8.20.3) where we used the geometric genus.)

Proof. (a). By [[5] Theorem I.3.4.], $H^0(X, \mathcal{O}_X) = k$. Thus, $h^0(X, \mathcal{O}_X) = 1$. Hence,

$$p_a(X) = (-1)^r(\sum_{i=1}^{r}(-1)^i h^i(X, \mathcal{O}_X))$$

(b). Recall that $P(0) = \chi(X, \mathcal{O}_X)$.

(c). In the proof of [[5] Theorem II.8.19.], if we let V to be the largest open set for which there is a morphism $f : V \longrightarrow X$ represents $f : X \dashrightarrow Y$, then codim $X - V \geq 2$. Hence, if $f : X \dashrightarrow Y$ is a birational morphism, then f is an isomorphism. Hence, p_a is a birational invariant in curves.

We can compute that $p_a(X) = \binom{d}{2}$ by **??**. When $d \geq 3$, $p_a(X) \neq 0$. Hence, it can not be rational. □

Exercise 3.5.4. (a) Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Show that there is a (unique) additive homomorphism

$$P : K(X) \to \mathbb{Q}[z]$$

such that for each coherent sheaf $\mathcal{F}$ on X, $P(\gamma(\mathcal{F}))$ is the Hilbert polynomial of $\mathcal{F}$ (Ex. 5.2).

(b) Now let $X = \mathbb{P}_k$. For each $i = 0, 1, \ldots, r$, let L_i be a linear space of dimension i in X. Then show that

(i) $K(X)$ is the free abelian group generated by $\{\gamma(\mathcal{O}_{L_i}) | i = 0, \ldots, r\}$, and

(ii) the map $P : K(X) \to \mathbb{Q}[z]$ is injective.

[Hint: Show that (1) $\Rightarrow$ (2). Then prove (1) and (2) simultaneously, by induction on r, using (II, Ex. 6.10c).]

Proof. (a). By 3.5.1, we can just define $P(\gamma(\mathcal{F}))(n) = \chi(\mathcal{F}(n))$.

(b). We first show that (1) $\Longrightarrow$ (2): □

Exercise 3.5.5. Let k be a field, let $X = \mathbb{P}_k$, and let Y be a closed subscheme of dimension $q \geq 1$, which is a complete intersection (II, Ex. 8.4). Then: (a) for all $n \in \mathbb{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

is surjective. (This gives a generalization and another proof of (II, Ex. 8.4c), where we assumed Y was normal.)

(b) Y is connected;

(c) $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbb{Z}$;

(d) $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$.

[Hint: Use exact sequences and induction on the codimension, starting from the case $Y = X$ which is (5.1).]

Proof. We will first prove (a),(c) by induction on the codim Y. If $Y = \mathbb{P}^n_k$, we're done by [[5] Theorem III.5.1.]. Suppose that codim $Y = n + 1$ and $Y = Z \cap V(f)$ for some f of degree d. Then we have

$$0 \longrightarrow \mathcal{O}_Z(-d) \longrightarrow \mathcal{O}_Z \longrightarrow i_{Y*}\mathcal{O}_Y \longrightarrow 0$$

Since $H^1(Z, \mathcal{O}_Z(n)) = 0$ for all n by induction hypothesis, $H^0(Z, \mathcal{O}_Z) \longrightarrow H^0(Y, \mathcal{O}_Y)$ is surjective. Hence, $H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n))$ is surjective.

If $\dim Z = q$, then $\dim Y = q-1$. By the long exact sequence and the fact that $H^i(Z, \mathcal{O}_Z(n)) = 0$ when $0 < i < q$, $H^i(Y, \mathcal{O}_Y(n)) = 0$ for all $0 < i < q - 1$.

Now, we have get (a),(c). Since Y is a complete intersection, $H^0(Y, \mathcal{O}_Y)$ has dimension ≥ 1. Since $H^0(X, \mathcal{O}_X) \longrightarrow H^0(Y, \mathcal{O}_Y)$ is a surjection, $H^0(Y, \mathcal{O}_Y) = k$. By 2.2.19, Y is connected. And by (c), Grothendieck Vanishing theorem and the fact that $H^0(Y, \mathcal{O}_Y) = k$, $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$ □

Exercise 3.5.6. (a) Use the special cases $(q, 0)$ and $(0, q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbb{P}^1$ in Q, to show:

(i) if $|a-b| \leq 1$, then $H^1(Q, \mathcal{O}_Q(a,b)) = 0$;

(ii) if $a, b < 0$, then $H^1(Q, \mathcal{O}_Q(a,b)) = 0$;

(iii) if $a \leq -2$, then $H^1(Q, \mathcal{O}_Q(a,0)) \neq 0$.

(b) Now use these results to show:

(i) if Y is a locally principal closed subscheme of type (a,b), with $a, b > 0$, then Y is connected;

(ii) now assume k is algebraically closed. Then for any $a, b > 0$, there exists an irreducible nonsingular curve Y of type (a,b). Use (II, 7.6.2) and (II, 8.18).

(iii) an irreducible nonsingular curve Y of type (a,b), $a, b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a-b| \leq 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbb{P}^3$. The simplest is the one of type (1,3), which is just the rational quartic curve (I, Ex. 3.18).

(c) If Y is a locally principal subscheme of type (a,b) in Q, show that $p_a(Y) = ab - a - b + 1$. [Hint: Calculate Hilbert polynomials of suitable sheaves, and again use the special case $(q,0)$ which is a disjoint union of q copies of $\mathbb{P}^1$. See (V, 1.5.2) for another method.]

Proof. (a). Note that

$$0 \longrightarrow S(-2) \xrightarrow{\cdot xy - zw} S \longrightarrow S/(xy - zw) \longrightarrow 0$$

induces

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3_k} \longrightarrow i_*\mathcal{O}_Q \longrightarrow 0$$

Tensoring the exact sequence with $\mathcal{O}_{\mathbb{P}^n_k}$, we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(n-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(n) \longrightarrow i_*\mathcal{O}_Q(n) \longrightarrow 0$$

Thus, we can compute $H^1(Q, \mathcal{O}_Q(n)) = 0$ for $H^i(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(n)) = 0$ when $0 < i < 3$.

Consider the exact sequence

$$0 \longrightarrow Q(-1,0) \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \longrightarrow 0$$

given by

$$0 \longrightarrow k[x](-1) \times_k k[y] \xrightarrow{\cdot x} k[x] \times_k k[y] \longrightarrow (k[x]/(x)) \times_k k[y] \longrightarrow 0$$

Tensoring with $Q(a,b)$, we have

$$0 \longrightarrow Q(a-1,b) \longrightarrow Q(a,b) \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(b) \longrightarrow 0$$

Thus, we just need to conisder

$$H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(b)) \longrightarrow H^1(Q, Q(a-1,b)) \longrightarrow H^1(Q, Q(a,b)) \longrightarrow H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(b))$$

When $b > -2$, $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(b)) = 0$.

Case I $a > -2, b > -2$, w.l.o.g. $a \geq b > -2$, then

$$H^1(Q, Q(b+1,b)) = 0$$

for taking $a = b+1$. Next, take $a = b+2$, using $H^1(Q, Q(a,b)) = H^1(Q, Q(b,a))$, we see that

$$H^1(b+2,b) = 0$$

By induction, we see that $H^1(a,b)=0$ for any $a \geq b > -2$.

Case II $a \leq -2, b \leq -2$. W.l.o.g. $a \leq b \leq -2$. Take $a=b$, we see that

$$H^1(Q, Q(b-1,b)) = 0$$

Next take $a = b-1$, we see that

$$H^1(Q, Q(b-2,b)) = 0$$

By induction, we see that $H^1(a,b)=0$ for any $a \leq b \leq -2$.

Case III $a \leq -2, b > -2$. Later we will show that $H^i(Q, Q(0,b)) = H^i(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(b))$ Hence, $H^0(Q, Q(-1,b) = H^1(Q, Q(-1,b)) = 0$. Take $a=-1$, then

$$H^0(Q, Q(-2,b)) = 0$$

for $H^0(Q, Q(-1,b)) = 0$. Because $b > -2$ implies $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(b)) = 0$,

$$H^1(Q, Q(-2,b)) = H^0(P^1, \mathcal{O}_{\mathbb{P}^1_k}(b))$$

which implies $dimH^1(Q, Q(-2,b)) = b+1$. Next take $a=-2$, we see that

$$H^0(Q, Q(-3,b)) = 0; \quad dim(H^1(Q(-3,b))) = dim(H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(b))) + dimH^1(Q, Q(-2,b))$$

Hence, by induction, we see that $dimH^1(Q, Q(a,b)) = (-a-1)(b+1)$ when $a \leq -2, b > 2$.

Another proof of (a): Use Künneth formula. Now that $Q \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ and $\mathcal{O}_Q(a,b) = p_1^*\mathcal{O}(a) \otimes p_2^*\mathcal{O}(b)$.

Theorem 3.5.2 (Künneth Formula). *L Let X_1 and X_2 be projective over k. Suppose that $\mathcal{F}_1 \in$* $\mathrm{Coh}(X_1)$ *and $\mathcal{F}_2 \in$* $\mathrm{Coh}(X_2)$. *Then*

$$H^n(X_1 \times X_2, p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2) = \bigoplus_{i+j=n} H^i(X_1, \mathcal{F}_1) \otimes H^j(X_2, \mathcal{F}_2)$$

Note that

$$h^1(\mathbb{P}^1_k, \mathcal{O}(a)) = h^0(\mathbb{P}^1_k, \mathcal{O}(-2-a)) = \binom{-1-a}{1}, h^0(\mathbb{P}^1_k, \mathcal{O}(a)) = \binom{a}{1}$$

Thus,

$$h^1(Q, \mathcal{O}_Q(a,b)) = \binom{a}{1}\binom{-1-b}{1} + \binom{b}{1}\binom{-1-a}{1}$$

We can use this to get the result we want.

(b). For the following exercises, we will use Künneth formula directly.

(1). Use Künneth. $H^0(Y, \mathcal{O}_Y) = k$. Hence, Y is connected.

(2). Since $\mathcal{O}^1_{\mathbb{P}}(a)$ is very ample when $a > 0$. Thus, we can construct $\mathbb{P}^1_k \hookrightarrow \mathbb{P}^a_k$. Now consider the Serge embedding

$$s : \mathbb{P}^a_k \times \mathbb{P}^b_k \longrightarrow \mathbb{P}^{ab+a+b}_k$$

and $s^*\mathcal{O}_{\mathbb{P}^{ab+a+b}_k}(1) = p_1^*\mathcal{O}_{\mathbb{P}^a_k}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^a_k}(b)$ we can see that $\mathcal{O}_Q(a,b)$ is very ample when $a, b > 0$. Then we embedding $Q \hookrightarrow \mathbb{P}^N_k$ by $\mathcal{O}_Q(a,b)$. By Bertini theorem [[5] Theorem II.8.18.], there exists a hypersurface H such that $H \cap Q$ is nonsingular, which is just the nonsingular curve we want.

(3). By 2.5.14,$Y \subset \mathbb{P}^3_k$ is projectively normal if and only if $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n))$ is surjective. Since we have

$$\begin{tikzcd}
H^0(\mathbb{P}^3_k, \mathcal{O}(n)) \arrow[rr] \arrow[dr] & & H^0(Y, \mathcal{O}_Y(n)) \\
& H^0(Q, \mathcal{O}_Q(n)) \arrow[ur] &
\end{tikzcd}$$

and $H^0(\mathbb{P}^3_k, \mathcal{O}(n)) \longrightarrow H^0(Q, \mathcal{O}_Q(n))$ is surjective by 3.5.5 (a), $H^0(Q, \mathcal{O}_Q(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n))$ is surjective if and only if $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n))$ is surjective.

Now, we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Q(a,b) \longrightarrow \mathcal{O}_Q \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0$$

If $|a-b| \leq 1$, then so is $|(a+n)-(b+n)| \leq 1$. Then by (a), $H^1(Q, \mathcal{O}_Q(a+n, b+n)) = 0$, which implies $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n))$ is surjective. Conversely, if $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n))$ is surjective, $H^1(Q, \mathcal{O}_Q(a+n, b+n)) = 0$ for any n. Suppose $b \geq a$. Take $n = -b$. Then $H^1(Q, \mathcal{O}_Q(a-b, 0)) = 0$ if $a - b = 1$ by (a). Thus, $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n))$ is surjective implying $|a-b| \leq 1$.

(c). Directly comes from Künneth formula. □

Exercise 3.5.7. Let X (respectively, Y) be proper schemes over a noetherian ring A. We denote by $\mathcal{L}$ an invertible sheaf.

(a) If $\mathcal{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathcal{L}$ is ample on Y, where $i : Y \to X$ is the inclusion.

(b) $\mathcal{L}$ is ample on X if and only if $\mathcal{L}_{\mathrm{red}} = \mathcal{L} \otimes \mathcal{O}_{X_{\mathrm{red}}}$ is ample on X_{red}.

(c) Suppose X is reduced. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

(d) Let $f : X \to Y$ be a finite surjective morphism, and let $\mathcal{L}$ be an invertible sheaf on Y. Then $\mathcal{L}$ is ample on Y if and only if $f^*\mathcal{L}$ is ample on X. [Hints: Use (5.3) and compare (Ex. 3.1, Ex. 3.2, Ex. 4.1, Ex. 4.2). See also Hartshorne [5, Ch. I §4] for more details.]

Proof. (a). Since X is proper over a Noetherian ring A, $\mathcal{L}^n$ is very ample for some n. Then it induces a closed embedding $i_X : X \longrightarrow \mathbb{P}^N_k$ for some N. Note that $i_X \circ i$ is a closed embedding form $Y \hookrightarrow \mathbb{P}^N_k$ and $i^*\mathcal{L}^n = (i_X \circ i)\mathcal{O}_{\mathbb{P}^N_k}(1)$. Hence, $i^*\mathcal{L}^n = (i^*\mathcal{L})^n$ is very ample hence ample. By [[5] Proposition II.7.5.], $i^*\mathcal{L}$ is ample.

(b). ($\Longrightarrow$): Just note that we have a closed embedding $X_{red} \hookrightarrow X$.

($\Longleftarrow$): We will use [[5] Proposition II.5.3.] to prove that $\mathcal{L}$ is very ample. Since X is Noetherian and $\mathcal{N}$ is its nilpotent ideal, there exists a filtration

$$\mathcal{N} \supseteq \mathcal{N}^2 \supseteq ... \supseteq \mathcal{N}^r = 0$$

For any coherent sheaf $\mathcal{F}$ on X, there is a filtration

$$\mathcal{F} \supseteq \mathcal{N}\mathcal{F} \supseteq \mathcal{N}^2\mathcal{F} \supseteq ... \supseteq \mathcal{N}^r\mathcal{F} = 0$$

Then we have

$$0 \longrightarrow (\mathcal{N}^{i+1}\mathcal{F}) \otimes \mathcal{L}^n \longrightarrow (\mathcal{N}^i\mathcal{F}) \otimes \mathcal{L} \longrightarrow (\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes \mathcal{L}^n \longrightarrow 0$$

Since $i : X_{red} \hookrightarrow X$ is a closed embedding, it is affine. By 3.4.1 and the projection formula 2.5.1 (d), we have

$$H^m(X, (\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes \mathcal{L}^n) = H^m(X_{red}, i^*(\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes i^*\mathcal{L}^n) = H^m(X_{red}, i^*(\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes \mathcal{L}^n_{red})$$

Since all sheaves are coherent, $H^m(X_{red}, i^*(\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes \mathcal{L}^n_{red}) = 0$ for $n \geq n_0$ and $m > 0$ by [[5] Proposition III.5.3.]. The by the long exact sequence, we have

$$H^m(X, \mathcal{F} \otimes \mathcal{L}) = ... = H^m(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n) = H^m(X, \mathcal{N}^{i+1}\mathcal{F} \otimes \mathcal{L}^n) = ... = H^m(X, \mathcal{N}^r\mathcal{F} \otimes \mathcal{L}) = 0$$

for $\mathcal{N}^r = 0$. Thus, $\mathcal{L}$ is ample.

(c). By (a), if $\mathcal{L}$ is very ample over X, $i^*_{X_i}\mathcal{L}$ is very ample for any X_i. For the other direction, we prove this by induction:

When X is irreducible, we are done. When $X = X_1 \cup ... \cup X_m$, let $\mathcal{I}$ be the ideal sheaf of X_m and $Y = X_1 \cup ... \cup X_{m-1}$. Then for any coherent sheaf $\mathcal{F}$ over X, we have

$$0 \longrightarrow \mathcal{I}\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0$$

Note that $\mathrm{Supp}(\mathcal{Q}) \subset X$ since each stalk of $\mathcal{I}$ is trivial at $X - X_1$. Hence, we can consider $\mathcal{Q}$ as $i_{X*}\mathcal{Q}$. Then tensoring $\mathcal{L}^n$ and note that $\mathrm{Supp}(\mathcal{I}\mathcal{F}) \subset Y$, we have (use 2.4.1 to identify $H^i(X, \mathcal{I}\mathcal{F})$ with $H^i(Y, \mathcal{I}\mathcal{F})$)

$$... \longrightarrow H^i(Y, \mathcal{I}\mathcal{F} \otimes \mathcal{L}^n_Y) \longrightarrow H^i(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^i(X_m, \mathcal{Q} \otimes \mathcal{L}^n) \longrightarrow ...$$

By induction hypothesis, there exists n_0 depending on $\mathcal{F}$ such that $n \geq n_0$, $H^i(Y, \mathcal{I}\mathcal{F} \otimes \mathcal{L}^n)$ and $H^i(X_m, \mathcal{Q} \otimes \mathcal{L}^n) = 0$ for $i > 0$. Hence, $H^i(X, \mathcal{F} \otimes \mathcal{L}) = 0$ for all $i > 0$, which implies $\mathcal{L}$ is ample.

(d). By (b) and (c), we can assume that X, Y are both integral.

($\Longrightarrow$): Since f is finite, it is affine. Hence, for any coherent sheaf $\mathcal{F}$ over X, we have

$$H^i(X, \mathcal{F} \otimes f^*\mathcal{L}^n) = H^i(Y, f_*\mathcal{F} \otimes \mathcal{L}^n)$$

by Projection formula. Since f is finite, $f_*\mathcal{F}$ is still coherent. Thus, there exists n_0 depending on $\mathcal{F}$ such that $n > n_0$, $H^i(X, \mathcal{F} \otimes f^*\mathcal{L}^n) = 0$ for $i > 0$. Hence, $f^*\mathcal{L}$ is coherent.

($\Longleftarrow$): Just like 3.5.2, we use induction on the $\dim \mathrm{Supp}(\mathcal{F})$. When $\dim \mathcal{F} = 0$, by Grothendieck Vanishing theorem, $H^i(Y, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for any $i > 0$ and n. Now, for $\dim \mathrm{Supp}(\mathcal{F}) = n+1$, by 3.4.2 we can find a coherent sheaf $\mathcal{G}$ on X such that

$$u : f_*\mathcal{G} \longrightarrow \mathcal{F}^{\oplus m}$$

is an isomorphism on the generic point. Hence, $\dim \mathrm{Supp}(\mathrm{Ker(u)}) \leq \dim \mathrm{Supp}(\mathcal{F})$ and $\dim \mathrm{Supp}(\mathrm{Coker(u)}) \leq \dim \mathrm{Supp}(\mathcal{F})$ as the same reason in 3.5.2. Hence, there exists $n > n_0$ such that $H^i(Y, \mathrm{Supp}(\mathrm{Ker(u)}) \otimes \mathcal{L}^n) = 0$ and $H^i(Y, \mathrm{Supp}(\mathrm{Coker(u)}) \otimes \mathcal{L}^n) = 0$ for $i > 0$ and $n > n_0$. At these case,

$$H^i(X, \mathcal{G} \otimes f^*\mathcal{L}^n) = H^i(Y, f_*\mathcal{G} \otimes \mathcal{L}^n) = H^i(Y, \mathcal{F}^{\oplus m} \otimes \mathcal{L}^n)$$

Since $f^*\mathcal{L}$ is very ample, we can take $n_1 \geq n_0$ such that when $n > n_1$, $H^i(Y, \mathcal{F}^{\oplus m} \otimes \mathcal{L}^n) = 0$ for all $i > 0$ that is, $H^i(Y, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $i > 0$. Hence, $\mathcal{L}$ is ample.

□

Remark. Can we use Mayer-Victoris Sequence to prove this directly? See 3.2.4.

Exercise 3.5.8. Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.

(a) If X is irreducible and nonsingular, then X is projective by (II, 6.7).

(b) If X is integral, let $\tilde{X}$ be its normalization (II, Ex. 3.8). Show that $\tilde{X}$ is complete and nonsingular, hence projective by (a). Let $f : \tilde{X} \to X$ be the projection. Let $\mathcal{L}$ be a very ample invertible sheaf on $\tilde{X}$. Show there is an effective divisor $D = \sum P_i$ on $\tilde{X}$ with $\mathcal{L}(D) \cong \mathcal{L}$, and such that $f(P_i)$ is a nonsingular point of X, for each i. Conclude that there is an invertible sheaf $\mathcal{L}_0$ on X with $f^*\mathcal{L}_0 \cong \mathcal{L}$. Then use (Ex. 5.7d), (II, 7.6) and (II, 5.16.1) to show that X is projective.

(c) If X is reduced, but not necessarily irreducible, let $X_1, \ldots, X_r$ be the irreducible components of X. Use (Ex. 4.5) to show Pic $X \to \bigoplus$ Pic X_i is surjective. Then use (Ex. 5.7c) to show X is projective.

(d) Finally, if X is any one-dimensional proper scheme over k, use (2.7) and (Ex. 4.6) to show that Pic $X \to$ Pic X_{red} is surjective. Then use (Ex. 5.7b) to show X is projective.

Proof. (a) has been prove by Hartshoren.

(b). Since $f : \widetilde{X} \longrightarrow X$ is finite, it is proper and hence $\widetilde{X}$ is proper. Thus, $\widetilde{X}$ is complete. By [[5] Proposition 8.23.], $\widetilde{X}$ is regular in codimension 1. Note that dim $\widetilde{X} = 1$. Thus, $\widetilde{X}$ is nonsingular by [[5] Theorem I.5.1.]

Since A and $\widetilde{A}$ have the same fraction field, $f : \widetilde{X} \longrightarrow X$ is isomorphism at generic point. Hence, there are only finitely many points that f_x is not an isomorphism. Since $\widetilde{X}$ is projective, there exists a very ample bundle $\mathcal{L}$ over $\widetilde{X}$ that induces $\widetilde{X} \longrightarrow \mathbb{P}^n_k$. Then we can find a hypersurface $H \subset \mathbb{P}^n_k$ such that H doesn't contain any points x that f_x is not an isomorphism. Let $D = X \cap H$. Then $f^*\mathcal{L}(f(D)) = D = \mathcal{L}$. By 3.5.7 (d), $f^*\mathcal{L}(f(D))$ is ample hence there exists a very ample line bundle over X, which implies X is projective.

(c). Again, we will use induction on the number of X's irreducible components. Let $Y = X_2 \cup ... \cup X_r$. Then verifying on stalks, there exists an injection $\mathcal{O}_X^* \hookrightarrow i_{Y_*}\mathcal{O}_Y^* \oplus i_{X_1}\mathcal{O}_{X_1}^*$. Consider its cokernel $\mathcal{Q}$:

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow i_{Y_*}\mathcal{O}_Y^* \oplus i_{X_1}\mathcal{O}_{X_1}^* \longrightarrow \mathcal{Q} \longrightarrow 0$$

Then Supp$(\mathcal{Q}) \subset Y \cap X_1$, which is of dimension 0 since they are both of dimension 1. By Grothendieck Vanishing theorem, $H^1(X, \mathcal{Q}) = 0$. Then apply the long exact sequence, we get

$$H^1(X, \mathcal{O}_X^*) \twoheadrightarrow H^1(Y, \mathcal{O}_Y^*) \oplus H^1(X_1, \mathcal{O}_{X_1}^*)$$

By induction hypothesis and 3.4.5, we get $\text{Pic}(X) \twoheadrightarrow \bigoplus_{i=1}^r \text{Pic}(X_i)$

Now, since each X_i is projective, there exits $\mathcal{L}_i$ that is very ample over X_i. Since the Picard group is given by $\oplus i_{X_i}^*$, there exists $\mathcal{L} \in \text{Pic}(X)$ such that $i_{X_i}^*\mathcal{L} = \mathcal{L}_i$. By 3.5.7 (c), $\mathcal{L}$ is ample hence there exist $\mathcal{L}^n$ that is very ample.

(b). Now that for any ring A, letting N be its nilpotent radical, we have $A^* \twoheadrightarrow (A/N)^*$ (verify directly). There exists a surjection

$$\mathcal{O}_X^* \twoheadrightarrow i^*\mathcal{O}_{X_{red}}^*$$

Let $\mathcal{Q}$ be its cokernel. We have

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X_{red}, \mathcal{O}_{X_{red}}^*) \longrightarrow H^2(X, \mathcal{Q})$$

By Grothendieck vanishing theorem, $H^2(X, \mathcal{Q}) = 0$. Hence, we have $\text{Pic}(X) \twoheadrightarrow \text{Pic}(X_{red})$.

Just like (c), we see that X is projective. □

Exercise 3.5.9. A Nonprojective Scheme. We show the result of (Ex. 5.8) is false in dimension 2. Let k be an algebraically closed field of characteristic 0, and let $X = \mathbb{P}^2_k$. Let ω be the sheaf of differential 2-forms (II, §8). Define an infinitesimal extension X' of X by ω by giving the element $\xi \in H^1(X, \omega \otimes \mathscr{T})$ defined as follows (Ex. 4.10). Let x_0, x_1, x_2 be the homogeneous coordinates of X, let U_0, U_1, U_2 be the standard open covering, and let $\xi_{ij} = (x_j/x_i)d(x_i/x_j)$. This gives a Čech

1-cocycle with values in Ω^1_X, and since $\dim X = 2$, we have $\omega \otimes \mathscr{T} \cong \Omega^1$ (II, Ex. 5.16b). Now use the exact sequence

$$\cdots \to H^1(X,\omega) \to \operatorname{Pic} X' \to \operatorname{Pic} X \xrightarrow{\delta} H^2(X,\omega) \to \cdots$$

of (Ex. 4.6) and show δ is injective. We have $\omega \cong \mathcal{O}_X(-3)$ by (II, 8.20.1), so $H^2(X,\omega) \cong k$. Since $\operatorname{char} k = 0$, you need only show that $\delta(\mathcal{O}(1)) \neq 0$, which can be done by calculating in Čech cohomology. Since $H^1(X,\omega) = 0$, we see that $\operatorname{Pic} X' = 0$. In particular, X' has no ample invertible sheaves, so it is not projective.

*Note.*In fact, this result can be generalized to show that for any nonsingular projective surface X over an algebraically closed field k of characteristic 0, there is an infinitesimal extension X' of X by ω, such that X' is not projective over k. Indeed, let D be an ample divisor on X. Then D determines an element $c_1(D) \in H^1(X,\Omega^1)$ which we use to define X', as above. Then for any divisor E on X one can show that $\delta(\mathcal{L}(E)) = (D.E)$, where $(D.E)$ is the intersection number (Chapter V), considered as an element of k. Hence if E is ample, $\delta(\mathcal{L}(E)) \neq 0$. Therefore X' has no ample divisors.

On the other hand, over a field of characteristic $p > 0$, a proper scheme X is projective if and only if X_{red} is!

Exercise 3.5.10. Let X be a projective scheme over a noetherian ring A, and let $\mathcal{F}^1 \to \mathcal{F}^2 \to \cdots \to \mathcal{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geq n_0$, the sequence of global sections

$$\Gamma(X,\mathcal{F}^1(n)) \to \Gamma(X,\mathcal{F}^2(n)) \to \cdots \to \Gamma(X,\mathcal{F}^r(n))$$

is exact.

Proof. Just decompose the long exact sequence to short exact sequences. For each short exact sequence, we can take n_i such that when $n > n_i$, the short sequence of global sections is exact. Take $N = \max n_i$. Then when $n > N$, we have the long exact sequence we need. □

3.5.4 Addition Exercises

Exercise 3.5.11. Let $H \subset \mathbb{P}^r_k$ be a hypersurface defined by a homogeneous polynomial F of deg d. Compute $P(n) := \mathcal{X}(X,\mathcal{O}_X(n))$

Proof.

Note that at this case $H = Proj(k[x_0,...,x_n]/(F))$ By the exact sequence of $k[x_0,...,x_n]$-module

$$0 \longrightarrow S(-d) \xrightarrow{F} S \longrightarrow S/(F) \longrightarrow 0$$

we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_H \longrightarrow 0$$

Thus, for tensoring a free module is exact, we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(n-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(n) \longrightarrow \mathcal{O}_H(n) \longrightarrow 0$$

Thus, by 3.5.1, we have $\mathcal{X}(X,\mathcal{O}_X(n)) = \mathcal{X}(\mathbb{P}^n_k,\mathcal{O}(n)) - \mathcal{X}(\mathbb{P}^n_k,\mathcal{O}(n-d))$ □

Exercise 3.5.12 (Huybrechts' Exam)**.** Let $H \subset \mathbb{P}^3_k$ be a hypersurface defined by a homogeneous polynomial F of deg 3. Compute $P(n) := \mathcal{X}(X,\mathcal{O}_X(n))$

Proof. $\mathcal{X}(X, \mathcal{O}_X(n)) = \mathcal{X}(\mathbb{P}^3_k, \mathcal{O}(n)) - \mathcal{X}(\mathbb{P}^3_k, \mathcal{O}(n-3))$

To compute this, we just need to compute $H^3(X, \mathcal{O}_X(n))$:

$$dim(H^3(\mathbb{P}^3_k, \mathcal{O}_X(n))) = dim(H^0(\mathbb{P}^3_k, \mathcal{O}(-n-4)))$$

$$dim(H^3(\mathbb{P}^3_k, \mathcal{O}_X(n-3))) = dim(H^0(\mathbb{P}^3_k, \mathcal{O}(-n-1)))$$

Hence

$$dim(H^3(\mathbb{P}^3_k, \mathcal{O}_X(n))) = \binom{-n-4+3}{3}$$

$$dim(H^3(\mathbb{P}^3_k, \mathcal{O}_X(n-3))) = \binom{-n-1+3}{3}$$

We can compute that

$$\mathcal{X}(X, \mathcal{O}_X(n)) = \binom{-n-1}{3} - \binom{-n+2}{3}$$

□

3.6 Ext Groups and Sheaves

3.6.1 Preparations

3.6.2 Exercises

To begin with, you need to first understand 3.6.7

Exercise 3.6.1. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{F}, \mathscr{F}' \in \mathcal{M}\mathrm{ob}(X)$. An extension of $\mathscr{F}'$ by $\mathscr{F}$ is a short exact sequence

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{F}' \to 0$$

in $\mathcal{M}\mathrm{ob}(X)$. Two extensions are isomorphic if there is an isomorphism of the short exact sequences, inducing the identity maps on $\mathscr{F}$ and $\mathscr{F}'$. Given an extension as above consider the long exact sequence arising from $\mathrm{Hom}(\mathscr{F}',)$, in particular the map

$$\delta : \mathrm{Hom}(\mathscr{F}', \mathscr{F}) \to \mathrm{Ext}^1(\mathscr{F}', \mathscr{F}),$$

and let $\xi \in \mathrm{Ext}^1(\mathscr{F}', \mathscr{F})$ be $\delta(1_{\mathscr{F}})$. Show that this process gives a one-to-one correspondence between isomorphism classes of extensions of $\mathscr{F}'$ by $\mathscr{F}$, and elements of the group $\mathrm{Ext}^1(\mathscr{F}', \mathscr{F})$. For more details, see, e.g., Hilton and Stammbach [1, Ch. III].

Proof. By 3.6.7 and [P234 Prop 6.2 [5]], just let $X = Spec(A)$ with A Noetherian and consider $\mathcal{F}' = \widetilde{F'}$ and $\mathcal{F}'' = \widetilde{F''}$.

Consider

$$0 \longrightarrow F' \longrightarrow I \longrightarrow N \longrightarrow 0$$

with I an injcetive. Then we have

$$0 \longrightarrow Hom(F'', F') \longrightarrow Hom(F'', I) \longrightarrow Hom(F'', N) \longrightarrow Ext^1(F'', F') \longrightarrow Ext^1(F'', I)$$

Since I is an injcetive, $Ext^1(F'', I) = 0$. Then $\varphi : Hom(F'', N) \longrightarrow Ext^1(F'', F')$ is a surjection. Taking any $\lambda \in Ext^1(F'', F')$, chose $g \in Hom(F'', N)$ such that $\varphi(g) = \lambda$. Then define

$$\begin{tikzcd}
0 \arrow[r] & F' \arrow[r] \arrow[d, "="] & X \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & F' \arrow[r] & I \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

where X is the pullback of $I \longrightarrow N$ and g.

Check: The upper sequence is exact.

Check: For any $0 \longrightarrow F' \longrightarrow X \longrightarrow F'' \longrightarrow 0$, $X \cong I \times_N F''$ for some $g' : F'' \longrightarrow N$.

Check: If $\varphi(g_1) = \varphi(g_2)$, then the deduced exact sequences are isomorphic. □

Exercise 3.6.2. Let $X = \mathbb{P}^1_k$, with k an infinite field.

(i) Show that there does not exist a projective object $\mathscr{P} \in \mathcal{M}\mathrm{ob}(X)$, together with a surjective map $\mathscr{P} \to \mathcal{O}_X \to 0$. [Hint: Consider surjections of the form $\mathcal{O}_V \to k(x) \to 0$, where $x \in X$ is a closed point, V is an open neighborhood of x, and $\mathcal{O}_V = j_!(\mathcal{O}_X|_V)$, where $j : V \to X$ is the inclusion.]

(ii) Show that there does not exist a projective object $\mathscr{P}$ in either $\mathfrak{Qco}(X)$ or $\mathfrak{Cob}(X)$ together with a surjection $\mathscr{P} \to \mathcal{O}_X \to 0$. [Hint: Consider surjections of the form $\mathscr{L} \to \mathscr{L} \otimes k(x) \to 0$, where $x \in X$ is a closed point, and $\mathscr{L}$ is an invertible sheaf on X.]

Proof. (a). Suppose that there exists such an surjective map $\mathscr{P} \twoheadrightarrow \mathcal{O}_X$.

Claim 3.6.1. *For any closed point $x \in U \subset \mathbb{P}^n_k$, we can find $x \in V$ such that $V \subset U$.*

Let $Y = X - \{x\}$. $\mathcal{O}_V \oplus \mathcal{O}_Y \twoheadrightarrow O_X$ with $\mathcal{O}_V = j_!(\mathcal{O}_X|_V)$ and $\mathcal{O}_Y = j_!(\mathcal{O}_X|_Y)$. Since $\mathscr{P}$ is a projective, we have

$$\begin{tikzcd} \mathcal{O}_V \oplus \mathcal{O}_Y \arrow[r, two heads] & \mathcal{O}_X \arrow[r] & 0 \\ & \mathscr{P} \arrow[ul] \arrow[u, two heads] & \end{tikzcd}$$

Since $\mathcal{O}_V(U) = 0$ and $\mathcal{O}_Y(U) = 0$ by definition of $j_!$, $\mathscr{P}(U) \longrightarrow \mathcal{O}_X(U)$ is zero. However, note that the choice of U is arbitrary, $\mathscr{P}_x \longrightarrow \mathcal{O}_{X,x}$ is zero, which contradicts to the fact that $\mathscr{P} \longrightarrow \mathcal{O}_X$ is surjective.

(b). Suppose that we have a quasi-coherent sheaf $\mathscr{P} \twoheadrightarrow \mathcal{O}_X$. By 2.5.15 (e), $\mathscr{P} = \cup_\alpha \mathscr{P}_\alpha$ such that $\mathscr{P}_\alpha$ is coherent for any α. Since $\mathbb{P}^1_k$ is Noetherian, we can find finitely many α such that $\mathscr{P}' = \cup_{\alpha_i} \mathscr{P}_{\alpha_i}$ satisfies $\mathscr{P}' \longrightarrow \mathcal{O}_X$ is surjective:

Restrict on $D_+(x_0)$ and $D_+(x_1)$ to get α_i's. Since $\mathscr{P}$ is quasi-coherent, on $D_+(x_0)$, $\mathscr{P}(D_+(x_0)) \longrightarrow O_{\mathbb{P}^1_k}(D_+(x_0)) \cong k[x]$ is surjective. Since $k[x]$ is Notherian, $\mathcal{O}_{\mathbb{P}^1_k}(D_+(x_0))$ can be written as the $\sum_{j=1}^{n_0} im(\mathscr{P}_{\alpha_j}(D_+(x_0)) \longrightarrow k[x])$. Hence, we just choose those α_j. Note that we only have two open sets.

By the construction, $\mathscr{P}'$ is coherent. Thus, without loss of generality, we just assume that $\mathscr{P}$ is coherent.

Let $\mathcal{Q} = Ker(\mathscr{P} \longrightarrow \mathcal{O}_X)$. Since $Coh(X)$ is abelian, $\mathcal{Q}$ is coherent. By Serre's Theorem, $H^1(X, \mathcal{Q}(n)) = 0$ for some n large enough, that means, we have the s.e.s.

$$0 \longrightarrow H^0(X, \mathcal{Q}(n)) \longrightarrow H^0(X, \mathcal{P}(n)) \longrightarrow H^0(X, \mathcal{O}_X(n)) \longrightarrow 0$$

Now, choosing a k-rational point x, we have $\mathcal{O}_X \twoheadrightarrow k(x)$ (In fact, $i_x^* k(x)$) and $H^0(X, \mathcal{O}_X(n)) \longrightarrow k(x)$ is surjective. Note that $O_X(-n-1)$ is invertible, $O_X(-n-1) \longrightarrow k(x)$ is also surjective. Consider the commuatative diagram

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, two heads] & k(x) \\ \mathscr{P} \arrow[u, two heads] \arrow[r, dashed, "f"] & \mathcal{O}(-n-1) \arrow[u, two heads] \end{tikzcd}$$

where f exists for $\mathscr{P}$ is a projective. Tensoring the diagram with $\mathcal{O}(n)$ preserves surjections. Taking global sections, we have

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X(n)) \arrow[r, two heads] & k(x) \\ \Gamma(X, \mathscr{P}) \arrow[u, two heads] \arrow[r, dashed, "f"] & \Gamma(X, \mathcal{O}(-1)) \arrow[u] \end{tikzcd}$$

Since $\Gamma(X, \mathcal{O}(-1)) = 0$, this diagram can not be commutative which leads to a contradiction. □

Exercise 3.6.3. Let X be a noetherian scheme, and let $\mathcal{F}, \mathcal{G} \in \mathcal{M}\mathrm{ob}(X)$.

(i) If $\mathcal{F}, \mathcal{G}$ are both coherent, then $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is coherent, for all $i \geq 0$.

(ii) If $\mathcal{F}$ is coherent and $\mathcal{G}$ is quasi-coherent, then $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ is quasi-coherent, for all $i \geq 0$.

Proof. Since coherence and quasi-coherence are locally properties and $\mathscr{E}xt(\mathcal{F}, \mathcal{G})|_U = \mathscr{E}xt(\mathcal{F}|_U, \mathcal{G}|_U)$ [P234 Prop 6.2 [5]]. So we just consider $X = Spec(A)$ with A Noetherian, $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$. Since M is finitely generated A-module with A Noetherian, there exists a free resolution $L_\bullet \longrightarrow M \longrightarrow to$:

$$\cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

with L_i free A-module.

By 3.6.7, $\mathscr{E}xt^i(\mathcal{F},\mathcal{G}) = Ext^i_A(M,N)^\sim = h^i(Hom(L_\bullet,N))$. For L is free A-module $Hom(L_i,N) = L_i^* \otimes N$.

(i). N is also finitely generated, then $Hom(L_i,N) = L_i^* \otimes N$ is finitely generated and $Hom(M,N)$ is also finitely generated.. Since $Coh(A)$ is abelian, $im(Hom(L_{i+1},N) \longrightarrow Hom(L_i,N))$ is finitely generated and so is $ker(Hom(L_i,N) \longrightarrow Hom(L_{i-1},N))$. Hence,

$$Ext^i(M,N) = h^i(Hom(L_\bullet,N)) = \frac{ker(Hom(L_i,N) \longrightarrow Hom(L_{i-1},N))}{im(Hom(L_{i+1},N) \longrightarrow Hom(L_i,N))}$$

is finitely generated, which implies $\mathscr{E}xt^i$ is coherent.

(ii). N is just an A-mod. Just like before, use $QCoh(X)$ is abelian to get $Ext^i(M,N)$ is an A-module. $\square$

Exercise 3.6.4. Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. In this case we say $\mathfrak{Cob}(X)$ has enough locally frees. Then for any $\mathcal{G} \in \mathcal{M}\text{ob}(X)$, show that the δ-functor $(\mathcal{E}xt^i(\cdot,\mathcal{G}))$, from $\mathfrak{Cob}(X)$ to $\mathcal{M}\text{ob}(X)$, is a contravariant universal δ-functor. [Hint: Show $\mathcal{E}xt^i(\cdot,\mathcal{G})$ is coeffaceable (§1) for $i > 0$.]

Proof. Recall that a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is said to be coeffaceable if for any $A \in \mathcal{C}$, there exists $u : P \twoheadrightarrow A$ scuh that $\mathcal{F}(u) = 0$.

Now, we consider $Coh(X)$. Let $\mathcal{F} \in Coh(X)$, then there exists $\mathcal{H} \twoheadrightarrow \mathcal{F}$ with $\mathcal{H}$ locally free. Act the morphism with $\mathscr{E}xt^i(-,\mathcal{G})$. Then we have

$$\mathscr{E}xt^i(u,\mathcal{G}) : \mathscr{E}xt^i(\mathcal{F},\mathcal{G}) \longrightarrow \mathscr{E}xt^i(\mathcal{H},\mathcal{G})$$

Since $\mathcal{H}$ is locally free, taking stalk at $x \in X$:

$$\mathscr{E}xt^i(\mathcal{H},\mathcal{G})_x = Ext^i_{\mathcal{O}_{X,x}}(\mathcal{H}_x,\mathcal{G}_x)$$

by [P235 Prop 6.8 [5]]. Now, we need to following lemma:

Lemma 3.6.2. *If A is a local ring, then any finitely generated projective A-module is free, and conversely, every free module is projective.*

Proof. [See P556 Exe 6.11[1]] A free A-module is of course projective. Since $Ext^1(M \oplus N,-) = Ext^1(M,-) \oplus Ext^1(N,-)$. Now, we prove the converse holds up:

Suppose that we have M a finitely generated projective A-module. Let $n = \dim M \otimes_A k$ and let $g : A^n \longrightarrow M$ be the map deduce $g \otimes_A k : k^n \cong M \otimes_A k$. Since $Coker(g) \otimes_A k = Coker(g)/\mathfrak{m}Coker(g) = 0$ and $Coker(g)$ is finitely generated A-module, $Coker(g) = 0$ by Nakayama. Now, consider

$$0 \longrightarrow L \longrightarrow A^{\oplus n} \xrightarrow{g} M \longrightarrow 0$$

Since $Ext^1(M,L) = 0$, we have an exact sequence

$$0 \longrightarrow Hom(M,L) \longrightarrow Hom(M,A^{\oplus n}) \longrightarrow Hom(M,M) \longrightarrow 0$$

which means there exists $f : A^{\oplus n} \longrightarrow M$ such that $f \circ g = id_M$. Hence, the sequence splits and $A^{\oplus n} = M \oplus L$. Again, using the isomorphism $W \otimes_A k = k^{\oplus n}$, we have $L \otimes_A k = 0$, which implies $L = 0$ by Nakayama. $\square$

Since $\mathcal{H}_x$ is free, it is projective and hence $Ext^i_{\mathcal{O}_{X,x}}(\mathcal{H}_x,\mathcal{G}_x) = 0$ [P533 Proposition 6.14 [1]]. Hence, when $i > 0$, $\mathscr{E}xt^i(u,\mathcal{G}) = 0$, which implies $\mathscr{E}xt^i(-,\mathcal{G})$ is the universal δ-functor. $\square$

Remark. This is an addition to [P234 Prop 6.5 [5]].

Exercise 3.6.5.

Let X be a noetherian scheme, and assume that $\mathfrak{Cob}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathcal{F}$ we define the homological dimension of $\mathcal{F}$, denoted $(\mathcal{F})$, to be the least length of a locally free resolution of $\mathcal{F}$ (or $+\infty$ if there is no finite one). Show:

(i) $\mathcal{F}$ is locally free $\Leftrightarrow \mathcal{E}xt^i(\mathcal{F},\mathcal{G}) = 0$ for all $\mathcal{G} \in \mathcal{M}\mathrm{ob}(X)$;

(ii) $(\mathcal{F}) \leq n \Leftrightarrow \mathcal{E}xt^i(\mathcal{F},\mathcal{G}) = 0$ for all $i > n$ and all $\mathcal{G} \in \mathcal{M}\mathrm{ob}(X)$;

(iii) $(\mathcal{F}) = \sup_x \mathcal{O}_{X,x}\mathcal{F}_x$.

Proof. (a). $\mathcal{F}$ is locally free if and only if $\mathcal{F}_x$ is free if and only if $\mathcal{F}_x$ is projective if and only if $Ext^1_{O_{X,x}}(\mathcal{F}_x,\mathcal{G}_x) = 0$ for any $x \in X$ and $\mathcal{G} \in Mod(X)$ if and only if $\mathscr{E}xt^1(\mathcal{F},\mathcal{G}) = 0$ for any $\mathcal{G} \in Mod(X)$.

(For any $O_{X,x}$-module N, we can define $\mathcal{G} = i_{x,*}\underline{N}$ such that $\mathcal{G} \in Mod(X)$ and $\mathcal{G}_x = N$.)

(b)($\Longrightarrow$): There exists a free resolution of $\mathcal{F}$:

$$0 \longrightarrow \mathscr{L}_r \longrightarrow \cdots \longrightarrow \mathscr{L}_1 \longrightarrow \mathscr{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

with $r \leq n$. Then $\mathscr{E}xt^i(\mathcal{F},\mathcal{G}) = h^i(\mathscr{H}om(\mathscr{L}_\bullet,\mathcal{G})) = 0$ when $i > n \geq r$.

($\Longleftarrow$): Suppose that $hd(\mathcal{F}) = r > n$. Consider

$$0 \longrightarrow \mathscr{L}_r \xrightarrow{f} \cdots \longrightarrow \mathscr{L}_1 \longrightarrow \mathscr{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

with f is not zero. Acting the exact sequence with $\mathscr{H}om(-,\mathscr{L}_r)$, we have $\mathscr{H}om(f,\mathscr{L}_r)$ is not zero. Hence $\mathscr{E}xt^r(\mathcal{F},\mathcal{G}) = h^r(\mathscr{H}om(\mathscr{L}_\bullet,\mathcal{G}))$ is non trivial, since

$$\mathscr{H}om(f,\mathscr{L}_r) : \mathscr{H}om(\mathscr{L}_{r-1},\mathscr{L}_r) \longrightarrow \mathscr{H}om(\mathscr{L}_i,\mathscr{L}_r)$$

can not be surjective:

Consider

$$0 \longrightarrow \mathscr{L}_i \longrightarrow \mathscr{L}_{r-1} \longrightarrow \mathcal{Q} := Coker(f) \longrightarrow 0$$

If $\mathscr{H}om(f,\mathscr{L}_r)$ is a surjection, then it is surjection on stalks, that is,

$$Hom_{\mathcal{O}_{X,x}}(f_x,\mathscr{L}_{r,x}) : Hom_{\mathcal{O}_{X,x}}(\mathscr{L}_{r-1,x},\mathscr{L}_{r,x}) \longrightarrow Hom_{\mathcal{O}_{X,x}}(\mathscr{L}_{r,x},\mathscr{L}_{r,x})$$

is surjective, which implies

$$0 \longrightarrow \mathscr{L}_{r,x} \longrightarrow \mathscr{L}_{i-1,x} \longrightarrow \mathcal{Q}_x \longrightarrow 0$$

splits. Hence, $\mathscr{L}_{r-1,x} = \mathscr{L}_{r,x} \oplus \mathcal{Q}_x$. Since $\mathscr{L}_{r-1,x}, \mathscr{L}_{r,x}$ are free and then projective, $\mathcal{Q}_x$ is also projective and then free, i.e. $\mathcal{Q}$ is a locally free sheaf. Then we have a locally free resolution

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathscr{L}_{r-2} \longrightarrow \cdots \longrightarrow \mathscr{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

contradicting to $hd(\mathcal{F}) = r$.

Hence, $Hom_{\mathcal{O}_{X,x}}(f_x,\mathscr{L}_{r,x})$ is not surjective and $h^r(\mathscr{H}om(\mathscr{L}_\bullet,\mathscr{L}_r))$ is not zero, which contradicts to our assumption.

(c). Suppose that $r = hd(\mathcal{F})$. Consider

$$0 \longrightarrow \mathscr{L}_r \longrightarrow \cdots \longrightarrow \mathscr{L}_1 \longrightarrow \mathscr{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

Since taking stalks is exact and for each x, $\mathscr{L}_{r,x}$ is free and hence projective. We see that $pd_{\mathcal{O}_{X,x}}\mathcal{F}_x \leq r$ for each x, which implies $sup_x pd_{\mathcal{O}_{X,x}}\mathcal{F}_x \leq nd(\mathcal{F})$.

Let $sup_x pd_{\mathcal{O}_{X,x}}\mathcal{F}_x = n$. Then for each $\mathcal{G} \in Mod(X)$, $\mathscr{E}xt^i(\mathcal{F},\mathcal{G}) = 0$ by [P235 Proposition 6.8 [5]] and [P237 Proposition 6.10A (b) [5]]. By what we have shown in (b), $nd(\mathcal{F}) \leq n$.

Hence, $r = n$. □

Exercise 3.6.6. Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10A) as follows.

(i) M is projective if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > 0$. [Hint: Use (6.11A) and descending induction on i to show that $\mathrm{Ext}^i(M, N) = 0$ for all $i > 0$ and all finitely generated A-modules N. Then show M is a direct summand of a free A-module (Matsumura [2, p. 129]).]

(ii) Use (a) to show that for any n, $M \leq n$ if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > n$.

Proof. (a). ($\Longrightarrow$): Since M is projective if and only if $Ext^i(M, -) = 0$ for any $i > 0$.

($\Longleftarrow$): We first claim that if $Ext^i(M, -) = 0$ for any finitely generated A-module. Then so is for $i - 1 > 0$.

Let N be a finitely generated A-module. Consider $L = coker(A^{\oplus n} \twoheadrightarrow N)$ and the s.e.s.

$$0 \longrightarrow L \longrightarrow A^{\oplus n} \longrightarrow N \longrightarrow 0$$

We have

$$Ext^{i-1}(M, A^{\oplus n}) \longrightarrow Ext^{i-1}(M, N) \longrightarrow Ext^i(M, L) \longrightarrow Ext^i(M, A^{\oplus n})$$

Hence, $Ext^{i-1}(M, N) = 0$.

Now, since A is a local regular ring, $Ext^i(M, N) = 0$ for any i large enough and N finitely generated A-module. Using the claim, we see that $Ext^1(M, N) = 0$ for any finitely generated A-module. Let $K = ker(A^{\oplus n} \twoheadrightarrow M)$. It is finitely generated and there exists a s.e.s.

$$0 \longrightarrow K \longrightarrow A^{\oplus n} \longrightarrow M \longrightarrow 0$$

Since $Ext^1(M, K) = 0$, there exists a s.e.s.

$$0 \longrightarrow Hom(M, K) \longrightarrow Hom(M, A^{\oplus n}) \longrightarrow Hom(M, M) \longrightarrow 0$$

which implies the s.e.s. splits. Hence, $A^{\oplus n} = M \oplus K$. For $A^{\oplus n}$ is projective, so is M.

(b). ($\Longrightarrow$): Trivial.($\Longleftarrow$):

By induction on n. For $n = 1$, we are done. Suppose that when $i > n - 1$, $pd(M) \leq n - 1$ if $Ext^i(M, A) = 0$ when $i > n - 1$.

Consider N is a finitely generated A-module and $dp(N) = n$. Let $L_\bullet$ be the projective resolution of N. Since A is a local ring, each L_i is free. Let $K = coker(L_0 \twoheadrightarrow N)$, which is finitely generated. $dp(K) = n - 1$. For we have

$$0 \longrightarrow K \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

there exists

$$Ext^i(L_0, A) \longleftarrow Ext^i(N, A) \longleftarrow Ext^{i-1}(K, A) \longleftarrow Ext^{i-1}(L_0, A)$$

By our hypothesis, $Ext^{i-1}(K, A) = 0$ and $Ext^i(L_0, A) = 0$ for L_0 is projective. Hence, $Ext^i(N, A) = 0$. □

Remark. Note that we taking L_0 in (b) to be finitely generated. In fact, for finitely generated modules, the minimal resolution can be taken within finitely generated projectives without increasing the length, when A is Noetherian.

Exercise 3.6.7. Let $X = \operatorname{Spec} A$ be an affine noetherian scheme. Let M, N be A-modules, with M finitely generated. Then

$$\mathrm{Ext}^i_X(\tilde{M}, \tilde{N}) \cong \mathrm{Ext}^i_A(M, N)$$

and

$$\mathcal{E}xt^i_X(\tilde{M}, \tilde{N}) \cong \widetilde{\mathrm{Ext}^i_A(M, N)}.$$

Proof. First of all, since M is finitely generated, we consider a free resolution of M:

$$\cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

Then $\widetilde{L}_\bullet$ is a free resolution of $\widetilde{M}$. By [P234 Prop 6.5 [5]],

$$\mathscr{E}xt^i(\widetilde{M}, \widetilde{N}) = h^i(\mathscr{H}om(\widetilde{L}_\bullet, \widetilde{N}))$$

Claim 3.6.3. $\mathscr{H}om(\widetilde{L}_i, \widetilde{N}) = Hom_A(L, N)^\sim$

Proof. Since $\widetilde{L}_i$ is free, by 2.5.1 $\mathscr{H}om(\widetilde{L}_i, \widetilde{N}) = \widetilde{L}_i^* \otimes \widetilde{N}$ is quasi-coherent. Hence, $\mathscr{H}om(\widetilde{L}_i, \widetilde{N}) = Hom(\widetilde{L}_i, \widetilde{N})^\sim$. By 2.5.3, $Hom(\widetilde{L}_i, \widetilde{N}) = Hom_A(L_i, N)$. Hence, $\mathscr{H}om(\widetilde{L}_i, \widetilde{N}) = Hom_A(L_i, N)^\sim$. □

Because taking $(\cdot)^\sim$ in $Mod(A)$ is exact,

$$h^i(\mathscr{H}om(\widetilde{L}_\bullet, \widetilde{N})) = h^i(Hom_A(L_\bullet, N)^\sim) = h^i(Hom_A(L_\bullet, N))^\sim = Ext^i(M, N)^\sim$$

Hence $\mathscr{E}xt^i(\widetilde{M}, \widetilde{N}) = Ext^i(M, N)^\sim$.

Again, using the resolution (which is a projective resolution), and $Hom(-, \widetilde{N})$ is contravariant, we have

$$Ext^i(\widetilde{M}, \widetilde{N}) = h^i(Hom(\widetilde{L}_\bullet, \widetilde{N})) = h^i(Hom(L_\bullet, N)) = Ext(M, N)$$

□

Remark. ([Chapter III.1 [5]]) Since $Hom(-, \mathcal{G})$ is contravariant, to get $R^i Hom(\mathcal{F}, G)$, we just need a projective resolution of $\mathcal{F}$:

$$\cdots \longrightarrow \mathcal{P}_2 \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{F} \longrightarrow 0$$

Then $R^i Hom(\mathcal{F}, \mathcal{G}) = h^i(Hom(\mathcal{F}, \mathcal{G}))$.

Remark. Generally, $\Gamma(X, \mathscr{E}xt^i(\mathcal{F}, \mathcal{G})) \neq Ext^i(\mathcal{F}, \mathcal{G})$. Just see [P234 Proposition 6.3 [5]]. But as we have shown, when X is affine, they are always equal.

Exercise 3.6.8. Prove the following theorem of Kleiman (see Borelli [1]): if X is a noetherian, integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).

(i) First show that open sets of the form X_s, for various $s \in \Gamma(X, \mathcal{L})$, and various invertible sheaves $\mathcal{L}$ on X, form a base for the topology of X. [Hint: Given a closed point $x \in X$ and an open neighborhood U of x, to show there is an $\mathcal{L}$ and $s \in \Gamma(X, \mathcal{L})$ such that $x \in X_s \subseteq U$, first reduce to the case that $Z = X - U$ is irreducible. Then let ζ be the generic point of Z. Let $f \in K(X)$ be a rational function with $f \in \mathcal{O}_{X,x}, f \notin \mathcal{O}_{X,\zeta}$. Let $D = (f)_\infty$, and let $\mathcal{L} = \mathcal{L}(D), s \in \Gamma(X, \mathcal{L}(D))$ correspond to D (II, §6).]

(ii) Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus_i \mathcal{L}_i^{n_i}$ for various invertible sheaves $\mathcal{L}_i$ and various integers n_i.

Exercise 3.6.9. Let X be a noetherian, integral, separated, regular scheme. (We say a scheme is regular if all of its local rings are regular local rings.) Recall the definition of the Grothendieck group $K(X)$ from (II, Ex. 6.10). We define similarly another group $K_1(X)$ using locally free sheaves: it is the quotient of the free abelian group generated by all locally free (coherent) sheaves, by the subgroup generated by all expressions of the form $\mathcal{E} - \mathcal{E}' - \mathcal{E}''$, whenever $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of locally free sheaves. Clearly there is a natural group homomorphism $\varepsilon : K_1(X) \to K(X)$. Show that ε is an isomorphism (Borel and Serre [1, §4]) as follows.

(i) Given a coherent sheaf $\mathcal{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathcal{E}_\bullet \to \mathcal{F} \to 0$. Then use (6.11A) and (Ex. 6.5) to show that it has a finite locally free resolution

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0.$$

(ii) For each $\mathcal{F}$, choose a finite locally free resolution $\mathcal{E}_\bullet \to \mathcal{F} \to 0$, and let $\delta(\mathcal{F}) = \sum (-1)^i \gamma(\mathcal{E}_i)$ in $K_1(X)$. Show that $\delta(\mathcal{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K(X)$ to $K_1(X)$, and finally, that it is an inverse to ε.

Exercise 3.6.10 (Duality for a Finite Flat Morphism)**.**

(i) Let $f : X \to Y$ be a finite morphism of noetherian schemes. For any quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$, $\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})$ is a quasi-coherent $f_*\mathcal{O}_X$-module, hence corresponds to a quasi-coherent $\mathcal{O}_X$-module, which we call $f^!\mathcal{G}$ (II, Ex. 5.17e).

(ii) Show that for any coherent $\mathcal{F}$ on X and any quasi-coherent $\mathcal{G}$ on Y, there is a natural isomorphism

$$f_*\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{G}).$$

(iii) For each $i \geq 0$, there is a natural map

$$\varphi_i : \operatorname{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \operatorname{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G}).$$

[Hint: First construct a map

$$\operatorname{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \operatorname{Ext}^i_Y(f_*\mathcal{F}, f_*f^!\mathcal{G}).$$

Then compose with a suitable map from $f_*f^!\mathcal{G}$ to $\mathcal{G}$.]

(iv) Now assume that X and Y are separated, $\mathfrak{Coh}(X)$ has enough locally frees, and assume that $f_*\mathcal{O}_X$ is locally free on Y (this is equivalent to saying f flat—see §9). Show that φ_i is an isomorphism for all i, all $\mathcal{F}$ coherent on X, and all $\mathcal{G}$ quasi-coherent on Y. [Hints: First do $i = 0$. Then do $\mathcal{F} = \mathcal{O}_X$, using (Ex. 4.1). Then do $\mathcal{F}$ locally free. Do the general case by induction on i, writing $\mathcal{F}$ as a quotient of a locally free sheaf.]

Proof. Let $f : X \longrightarrow Y$ be a finite morphism. Then $X = \operatorname{Spec}(f_*\mathcal{O}_X)$, where $\mathcal{A} := f_*\mathcal{O}_X$ is a commutative sheaf of $\mathcal{O}_Y$-algebra in $\operatorname{Coh}(Y)$. Locally, we have $X = \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ and $\mathcal{A} = \widetilde{B}$. Then $\operatorname{Coh}(X)$ is just the category of finite A-module with a B-structure. More generally, $\operatorname{Coh}(X) \cong \operatorname{Coh}(\mathcal{A})$ is the abelian category of coherent $\mathcal{O}_X$-module together with an $\mathcal{A}$-structure. 2.5.17 (d).

(a). Locally, $\mathcal{G} = M$ an A-module and $f^!\mathcal{G} = \operatorname{Hom}_A(B, M)$ is a B-module.

(b). Locally $\mathcal{F} = \widetilde{N}$ with N a B-module and $\mathcal{G} = \widetilde{M}$ with M an A-module. Then

$$\operatorname{Hom}_X(\mathcal{F}, f^!\mathcal{G}) \cong \operatorname{Hom}_B(N, \operatorname{Hom}_A(B, M)) \cong \operatorname{Hom}_A({}_AN, M) \cong \operatorname{Hom}_Y(\mathcal{F}, \mathcal{G})$$

where in the middle, the map is given by

$$\operatorname{Hom}_A({}_AN, M) \longrightarrow \operatorname{Hom}_B(N, \operatorname{Hom}_A(B, M))$$
$$\varphi : N \to M \longmapsto \psi : N \longrightarrow \operatorname{Hom}(B, M) \text{ deduced by the B-module structure of } N(B \longrightarrow N)$$

with converse given by $\psi \longmapsto \varphi : n \mapsto \psi(n)(1_B)$.

(c). Use the universal δ-functor: we have $\operatorname{Hom}(\mathcal{F}, -) \longrightarrow \operatorname{Hom}(f_*\mathcal{F}, f_*(-))$. Thus, there exists $\operatorname{Ext}^i(\mathcal{F}, -) \longrightarrow \operatorname{Ext}^i(f_*\mathcal{F}, f_*(-))$ by the property of universal δ-functor. Thus, we get

$$\operatorname{Ext}^i(\mathcal{F}, f^!\mathcal{G}) \longrightarrow \operatorname{Ext}^i_Y(f_*\mathcal{F}, f_*f^!G)$$

By (ii), we get $f_*f^!\mathcal{G} \longrightarrow \mathcal{G}$ given by $f_*id_{f^!\mathcal{G}}$, which induces φ_i.

(d). 1st Step: Consider $\mathcal{F} = \mathcal{O}_X$. Then

$$\operatorname{Ext}^i(\mathcal{O}_X, f^!\mathcal{G}) \cong H^i(X, f^!\mathcal{G}) \cong H^i(Y, f_*f^!\mathcal{G})$$

since f is finite. For $f_*f^!\mathcal{G}$, we have

$$f_*f^!\mathcal{G} \cong \mathscr{H}om_Y(\mathcal{O}_Y, f_*f^!\mathcal{G}) \overset{f^*\dashv f_*}{\cong} f_*\mathscr{H}om_X(f_*\mathcal{O}_Y, f^!\mathcal{G}) \overset{f_*\dashv f^!}{\cong} \mathscr{H}om_Y(f_*\mathcal{O}_X, \mathcal{G}) \cong \mathcal{G} \otimes f_*\check{\mathcal{O}}_X$$

since $f_*\mathcal{O}_X$ is locally free of finie rank. Thus,

$$\operatorname{Ext}^i_X(\mathcal{O}_X, f^!\mathcal{G}) \cong H^i(Y, \mathcal{G} \otimes (f_*\mathcal{O}_X)\check{}) \cong \operatorname{Ext}^i(\mathcal{O}_Y, \mathcal{G} \otimes (f_*\mathcal{G})\check{}) \cong \operatorname{Ext}^i(f_*\mathcal{O}_X, \mathcal{G})$$

2nd Step: Prove that: φ^i is an isomorphism for $\mathcal{F}, \mathcal{G}$ if and only if φ^i is an isomorphism for $\mathcal{F} \otimes f^*\mathcal{L}, \mathcal{G} \otimes f^*\mathcal{L}$ for $\mathcal{L} \in \operatorname{Pic}(Y)$;

3rd step: Use resolution of $\mathcal{F}$. $\square$

3.7 The Serre Duality

3.7.1 Exercises

Exercise 3.7.1. Let X be an integral projective scheme of dimension ≥ 1 over a field k, and let $\mathcal{L}$ be an ample invertible sheaf on X. Then $H^0(X, \mathcal{L}^{-1}) = 0$. (This is an easy special case of Kodaira's vanishing theorem.)

Proof. Suppose that $H^0(X, \mathcal{L}^{-1}) \neq 0$. Then we can take $s \in H^0(X, \mathcal{L}^{-1})$ and $s \neq 0$ such that s defines a morphism

$$s : \mathcal{L} \longrightarrow \mathcal{O}_X$$

On stalks x where $s_x \neq 0$, φ_x is given by

$$l_x \longmapsto l_x s_x$$

since locally $\mathcal{L}$ is define by $\mathcal{L}|_{U_i} = s|_{U_i}^{-1}\mathcal{O}_{U_i}$. Moreover, we can tensoring both side with $\mathcal{L}$ such that the composition

$$\mathcal{L}^n \longrightarrow \mathcal{L}^{n-1} \longrightarrow ... \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_X$$

is not zero. (Just verify on stalks) Hence, we get $s^{'} \neq 0 \in H^0(X, \mathcal{L}^n)$. Thus, $H^0(X, \mathcal{L}^{-n}) \neq 0$.

Now, since $\mathcal{L}$ is ample over X and X is projective, there exists n such that $\mathcal{L}^n$ is very ample. Let $\varphi : X \longrightarrow \mathbb{P}^n_k$ be the corresponding closed immersion. Then

$$H^0(X, \mathcal{L}^{-n}) = H^0(\mathbb{P}^N_k, \mathcal{O}(-1)) = 0$$

which leads to a contradiction. ($N = \dim \mathbb{P}^N_k \geq \dim X \geq 1$. Hence $H^0(\mathbb{P}^N_k, \mathcal{O}(-1)) = 0$.) □

Exercise 3.7.2. Let $f : X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k, and let ω_Y^0 be a dualizing sheaf for Y.

(i) Show that $f^!\omega_Y^0$ is a dualizing sheaf for X, where $f^!$ is defined as in (Ex. 6.10).

(ii) If X and Y are both nonsingular, and k algebraically closed, conclude that there is a natural trace map $t : f_*\omega_X \to \omega_Y$.

Proof. (a). If $(\omega_Y^0, \mathrm{tr} : H^n(Y, \mathcal{O}_Y) \longrightarrow k)$ is a dualizing sheaf on Y, we will show that $\omega_X^0 := f^!\omega_Y^0$ is a dualizing sheaf with the trace map

$$\begin{tikzcd}
tr_X : H^n(X, \omega_X^0) \arrow[rr, "\mathrm{tr}_X"] \arrow[d, "\cong"] & & k \\
H^n(X, f^!\omega_Y^0) \arrow[d, "\cong"] & & \\
H^n(Y, f_*f^!\omega_Y^0) \arrow[r, "H^n"] & H^n(Y, \omega_Y^0) \arrow[ur, "\mathrm{tr}_Y"'] &
\end{tikzcd}$$

where the morphism at bottom is induced by 3.6.10 (b). To prove this, we have

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{F}, f^!\omega_Y) \times H^n(X, \mathcal{F}) \arrow[r] \arrow[d, "\cong"] & H^n(X, f^!\omega_X^0) \arrow[dr, "\mathrm{tr}_X"] & \\
& & k \\
\mathrm{Hom}(f_*\mathcal{F}, \omega_Y) \times H^n(Y, f_*\mathcal{F}) \arrow[r] & H^n(T, \omega_Y^0) \arrow[ur, "\mathrm{tr}_Y"'] &
\end{tikzcd}$$

where $\mathrm{Hom}(\mathcal{F}, f^!\omega_Y) \times H^n(X, \mathcal{F}) \cong \mathrm{Hom}(f_*\mathcal{F}, \omega_Y) \times H^n(Y, f_*\mathcal{F})$ by 3.6.10 and 3.4.1.

(b). Now that when X, Y are smooth, ω_X, ω_Y exist and also $\omega_X \cong \bigwedge^n \Omega_{X/k}$ and $\omega_Y \cong \bigwedge^n \Omega_{Y/k}$. Since there is a natural morphism $f^*\Omega_{Y/k} \longrightarrow \Omega_{X/k}$ with $f^*\Omega_{Y/k}$ and $\Omega_{X/k}$ locally free of rank n. There exists

$$f^* \bigwedge^n \Omega_{Y/k} \cong f^*\omega_Y \longrightarrow \bigwedge^n \Omega_{X/k} \cong \omega_X$$

□

Exercise 3.7.3. Let $X = \mathbb{P}^n_k$. Show that $H^q(X, \Omega^p_X) = 0$ for $p \neq q$, and $H^q(X, \Omega^p_X) = k$ for $p = q$, $0 \leq p, q \leq n$.

Proof. For $\Omega_{\mathbb{P}^n_k/k}$, we have the Euler sequence:

$$0 \longrightarrow \Omega_{\mathbb{P}^n_k/k} \longrightarrow \mathcal{O}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O} \longrightarrow 0$$

By 2.5.17, we have a filtration:

$$\bigwedge^q \mathcal{O}(-1)^{n+1} = F^0 \supseteq F^1 \supseteq ... \supseteq F^q \supseteq F^{q+1} = 0$$

and have

$$F^s/F^{s+1} = \bigwedge^s \Omega_{\mathbb{P}^n_k/k} \otimes \bigwedge^{q-s} \mathcal{O}$$

However, $\bigwedge^{q-s} \mathcal{O} = 0$ when $q - s > 1$. Hence, $F^s = F^{s+1}$ when $q - r > 1$. Hence, we have $F^0 = F^1 = ... = F^{q-2} = F^{q-1}$ and $F^{q-1}/F^q = \bigwedge^{q-1} \Omega_{\mathbb{P}/k}$,$F^q = \bigwedge^q \Omega_{\mathbb{P}/k}$,that is, we have

$$0 \longrightarrow \bigwedge^q \Omega_{\mathbb{P}/k} \longrightarrow \bigwedge^q \mathcal{O}(-1)^{\oplus(n+1)} \longrightarrow \bigwedge^{q-1} \Omega_{\mathbb{P}^n_k/k} \longrightarrow 0$$

and we will use this short exact sequence to compute $H^p(\bigwedge^q \Omega_{\mathbb{P}_k/k})$.

Note that this short exact sequence deduce

$$... \longrightarrow H^i(X, \bigwedge^q \mathcal{O}(-1)^{\oplus(n+1)}) \longrightarrow H^i(X, \bigwedge^{q-1} \Omega_{\mathbb{P}^n_k/k}) \longrightarrow H^{i+1}(X, \bigwedge^q \Omega_{\mathbb{P}^n_k/k}) \longrightarrow H^{i+1}(X, \bigwedge^q \mathcal{O}(-1)^{\oplus(n+1)}) \longrightarrow ...$$

Claim 3.7.1.

$$\bigwedge^q \mathcal{O}(-1)^{\oplus(n+1)} = \mathcal{O}(-q)^{\oplus\binom{n+1}{q}}$$

Hence, when $i + 1 \leq n$, $H^{i+1}(X, \bigwedge^q \Omega_{\mathbb{P}^n_k/k}) = H^i(X, \bigwedge^{q-1} \Omega_{\mathbb{P}^n_k/k})$. If we assign $H^p(\bigwedge^q \Omega_{\mathbb{P}^n_k/k})$, we have: on each line $q = p + r$ with $r = 1, ..., n$, $H^p(\bigwedge^q \Omega_{\mathbb{P}^n_k/k})$ are equal when $0 \leq p \leq n$. Hence, we have

$$H^p(X, \bigwedge^q \Omega_{\mathbb{P}^n_k/k}) = 0, q > p$$

since $H^0(\mathbb{P}^n_k, \bigwedge^0 \Omega_{\mathbb{P}^n_k/k}) = H^0(\mathbb{P}^n_k, \mathcal{O}) = 0$. By Serre duality, $H^{n-p}(X, \bigwedge^{n-q} \Omega_{\mathbb{P}^n_k/k}) = H^p(X, \bigwedge^1 \Omega_{\mathbb{P}^n_k/k})$. We have

$$H^p(X, \bigwedge^q \Omega_{\mathbb{P}^n_k/k}) = 0, q < p$$

Then for $H^0(X, \mathcal{O}) = k$, we have

$$H^p(X, \bigwedge^q \Omega_{\mathbb{P}^n_k/k}) = k, q = p$$

□

Exercise 3.7.4 (The Cohomology Class of a Subvariety)**.** Let X be a nonsingular projective variety of dimension n over an algebraically closed field k. Let Y be a nonsingular subvariety of codimension p (hence dimension $n-p$). From the natural map $\Omega_X \otimes \mathcal{O}_Y \to \Omega_Y$ of (II, 8.12) we deduce a map $\Omega_X^{n-p} \to \Omega_Y^{n-p}$. This induces a map on cohomology $H^{n-p}(X, \Omega_X^{n-p}) \to H^{n-p}(Y, \Omega_Y^{n-p})$. Now $\Omega_Y^{n-p} = \omega_Y$ is a dualizing sheaf for Y, so we have the trace map $t_Y : H^{n-p}(Y, \Omega_Y^{n-p}) \to k$. Composing, we obtain a linear map $H^{n-p}(X, \Omega_X^{n-p}) \to k$. By (7.13) this corresponds to an element $\eta(Y) \in H^p(X, \Omega_X^p)$, which we call the *cohomology class* of Y.

(i) If $P \in X$ is a closed point, show that $t_X(\eta(P)) = 1$, where $\eta(P) \in H^n(X, \Omega^n)$ and t_X is the trace map.

(ii) If $X = \mathbb{P}^n$, identify $H^p(X, \Omega^p)$ with k by (Ex. 7.3), and show that $\eta(Y) = (\deg Y) \cdot 1$, where $\deg Y$ is its degree as a projective variety (I, §7). [Hint: Cut with a hyperplane $H \subseteq X$, and use Bertini's theorem (II, 8.18) to reduce to the case Y is a finite set of points.]

(iii) For any scheme X of finite type over k, we define a homomorphism of sheaves of abelian groups $d\log : \mathcal{O}_X^* \to \Omega_X$ by $d\log(f) = f^{-1}df$. Here $\mathcal{O}^*$ is a group under multiplication, and Ω_X is a group under addition. This induces a map on cohomology $\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega_X)$ which we denote by c—see (Ex. 4.5).

(iv) Returning to the hypotheses above, suppose $p = 1$. Show that $\eta(Y) = c(\mathcal{L}(Y))$, where $\mathcal{L}(Y)$ is the invertible sheaf corresponding to the divisor Y.

See Matsumura [1] for further discussion.

3.7.2 Additional exercises

Exercise 3.7.5 (Calabi-Yau Varieties)**.** Assume X is a smooth projective variety over a field k. Assume furthermore that $\omega_X \cong \mathcal{O}_X$. (This is sometimes called a *Calabi-Yau variety.*)

(i) The space $H^1(X, \mathcal{T}_X)$ parametrizes first order deformations of X (see Example 9.13.2 in Hartshorne's book). Show that its dimension is a Hodge number of X (i.e. one of the numbers $h^{p,q}(X) = \dim_k H^p(X, \Omega_X^q)$). Which one?

(ii) Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. For which d is X a Calabi-Yau variety? What is the dimension of $H^1(X, \mathcal{T}_X)$ if $n = 3$ and $d = 4$?

Proof.

(1). By Serre Duality, $H^1(X, \mathcal{T}_X) = H^0(X, \mathcal{T}_X^* \otimes \omega_X)^* = H^0(X, \Omega_{X/Y}^{\dim X - 1})^*$. Thus, $h^1(X, \mathcal{T}_X) = h^{0, \dim X - 1}$.

(2). Note that $\omega_X = \mathcal{O}_X(d-n-1)$ by 2.8.4. Thus, if $d = n+1$, then X is Calabi-Yau. When $n = 3$ and $d = 4$, we see that $\dim X = 2$ and $h^1(X, \mathcal{T}_X) = h^{1,1}(X)$. To compute this, we need the following exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-4) \longrightarrow \Omega_{\mathbb{P}_k^3}|_X \longrightarrow \Omega_X \longrightarrow 0$$

$$0 \longrightarrow \Omega_{\mathbb{P}_k^3} \longrightarrow \mathcal{O}_{\mathbb{P}_k^3}(-1)^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbb{P}_k^3} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}_k^3} \longrightarrow i_{X*}\mathcal{O}_X \longrightarrow 0$$

We see that

$$\chi(X, \Omega_X) = \chi(X, \Omega_{\mathbb{P}_k^3}|_X) - \chi(X, \mathcal{O}_X(-4))$$
$$\chi(X, \Omega_{\mathbb{P}_k^3}|_X) = 4\chi(X, \mathcal{O}_X(-1)) - \chi(X, \mathcal{O}_X)$$
$$\chi(X, \mathcal{O}_X(-1)) = \chi(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}_k^3}(-1)) - \chi(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}_k^3}(-5))$$

$$\chi(X, \mathcal{O}_X(-4)) = \chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-4)) - \chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-8))$$
$$\chi(X, \mathcal{O}_X) = \chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}) - \chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-4))$$

Since $\chi(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(n)) = \binom{n+3}{3} - \binom{-n-1}{3}$, we have $\chi(X, \mathcal{O}_X(-1)) = 4, \chi(X, \mathcal{O}(-4)) = 34, \chi(X, \mathcal{O}_X) = 2$. Thus, $\chi(X, \Omega_X) = -20$.

Note that $h^{0,1} = h^{2,1} = 0$ while $h^{0,1} = 0$ by the exact sequence and $h^{2,1} = h^{0,1}$ by Serre Duality. We see that $h^{1,1} = 20$. □

Exercise 3.7.6. Assume X is a Gorenstein (i.e. ω_X is invertible), projective, and equidimensional scheme over k. Let $D \subset X$ be a Cartier divisor. Prove the adjunction formula

$$\omega_D \cong (\omega_X \otimes \mathcal{O}(D))|_D.$$

Here, ω_D is the dualizing sheaf of the Cohen–Macaulay scheme D. Note that we know this formula already for smooth X and D.

Proof. Since D is an effective Cartier divisor, we see that D is smooth over X. Hence, there exists an exact sequence

$$0 \longrightarrow \mathcal{I}_D/\mathcal{I}_D^2 \longrightarrow \Omega_X|_D \longrightarrow \Omega_D \longrightarrow 0$$

By verifying locally, we see that $\mathcal{I}_D/\mathcal{I}_D^2 \cong \mathcal{O}_D(-D)$ (Just like 2.8.4). Thus,

$$\omega_D = \omega_X|_D \otimes \mathcal{O}_D(-D)) = (\omega_X \otimes \mathcal{O}_X(-D))|_D$$

□

Exercise 3.7.7. Assume X is an integral, smooth and projective scheme over a field k of dimension n and let L be an invertible sheaf with $H^0(X, L^m) \neq 0$ for some $m > 0$. Show that $H^n(X, \omega_X \otimes L) \neq 0$ implies L^m is trivial.

Proof. Let $H^n(X, \omega_X \otimes L) \neq 0$. By Serre duality, $H^0(X, L^{-1}) \neq 0$. Thus, $H^0(C, L^{-m}) \neq 0$ by the cup product. Hence, $L^m \cong \mathcal{O}_X$ since $H^0(C, L^m) \neq 0$. □

Exercise 3.7.8. Let X be a smooth irreducible projective surface over a field k such that $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. (This is what is called a K3 surface.) Compute $\chi(X, \mathcal{O}_X)$ and express $\chi(X, L)$ for an invertible sheaf L that admits a smooth curve C in the associated linear system $|L|$ in terms of the genus of C.

Proof. Let C be a curve defined L. By last exercises

$$\omega_C \cong (\omega_X \otimes \mathcal{O}(C))|_C \cong L|_C$$

Since we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow i_{C*}L \longrightarrow 0$$

Hence, $\chi(X, L) = \chi(X, \mathcal{O}_X) + \chi(X, i_{C*}L) = \chi(X, \mathcal{O}_X) + \chi(C, L|_C)$. Note $H^0(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = k$ and $H^1(X, \mathcal{O}_X) = 0$. $\chi(X, \mathcal{O}_X) = 2$.Thus, $\chi(C, L|_C) = h^0(C, \omega_C) - h^1(C, \omega_C) = g_C - 1$. Thus,

$$\chi(X, L) = g_C + 1$$

□

3.8 Higher Direct Images Of Sheaves

3.8.1 Exercises

Exercise 3.8.1. Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf of abelian groups on X, and assume that $R^i f_*(\mathcal{F}) = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geq 0$,

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F}).$$

(This is a degenerate case of the Leray spectral sequence—see Godement [1, II, 4.17.1].)

Proof. We consider the spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_*\mathcal{F})$$

which converges to $F^{p+q} = H^{p+q}(X, \mathcal{F})$. Since $R^i f_*\mathcal{F} = 0$ for $i > 0$, $E_\infty^{p,0} = H^p(Y, f_*\mathcal{F})$ and other terms vanish.

Now, we have a filtration

$$0 \subseteq F^n E^n \subseteq ... \subseteq F^0 E^n = E^n$$

Since $E_\infty^{p,q} = 0$ when $q > 0$ and $F^{n-i}E^n/F^{n-i+1}E^n = E_\infty^{n,i}$, $F^{n-i}E^n = F^{n-i+1}E^n$ when $i > 0$. Hence, the filtration becomes

$$0 \subseteq F^n E^n \subseteq E^n$$

Since $F^n E^n / F^{n+1} E^n = E_\infty^{n,0} = H^n(Y, f^*\mathcal{F})$, we have

$$H^n(Y, f^*\mathcal{F}) = H^n(X, \mathcal{F})$$

□

Remark (Leray Spectral sequence)**.** A spectral sequence in an abelian category $\mathscr{A}$ is

$$(E_r^{p,q}, E^n), n, p, q \in \mathbb{Z}, r \geq 1$$

with all objects in $\mathscr{A}$ together with

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r, q-r-1}$$

such that

(i). $d_r^{p,q} \circ d_r^{p-r, q+r-1} = 0$;

(ii). $E_{r+1}^{p,q} \cong H(E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q} \longrightarrow E_r^{p+r,q-r-1})$

For all p, q, there exists r_0 such that $d_r^{p,q} = 0$ (just assume $E_2^{p,q} \neq 0$ when $p, q \geq 0$.) Then $E_{r_0}^{p,q} \cong E_{r_0+1}^{p,q} \cong ... := E_\infty^{p,q}$ and there exists a filtration

$$F^{p+1}E^n \subseteq ... \subseteq E^n$$

such that

$$\cap F^p E^n = 0, \cup F^p E^n = E^n, F^p E^{p+q} / F^{p+1} E^{p+q} \cong E_\infty^{p,q}$$

We write $E_r^{p,q} \Longrightarrow E^n$, that is, $E_r^{p,q}$ converges to E^n.

Consider the filtration

$$F^{p+1}E^n \subseteq F^p E^n \subseteq ... \subseteq E^n$$

As we have stated above, when $p > n, p < 0$, $E_\infty^{p,n-q} = 0$, which implies $F^p E^n = F^{p+1} E^n = 0$ when $p > n$ and $p < 0$. For the intersection of all $F^p E^n$ is zero and the union of then is E^n, we have

$$0 \subseteq F^n E^n \subseteq ... \subseteq F^0 E^n = E^n$$

Exercise 3.8.2. Let $f : X \to Y$ be an affine morphism of schemes (II, Ex. 5.17) with X noetherian, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Show that the hypotheses of (Ex. 8.1) are satisfied, and hence that $H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F})$ for each $i \geq 0$. (This gives another proof of (Ex. 4.1).)

Proof. Fix $i > 1$. It is enough to prove that $R^i\pi_*\mathcal{F}_y = 0$ for each $y \in Y$. Let $U \subset Y$ be an affine open set containing $y \in Y$. Then $f^{-1}(U)$ is an affine open set in X.Consider $f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$, and $\mathcal{F}|_{f^{-1}(U)}$.

By [Proposition III.8.5 [5]] and [Corollary III.8.2 [5]], we have $R^i\pi_*(\mathcal{F})|_U = R^i f|_{f^{-1}(U)*}(\mathcal{F}|_{f^{-1}U})$ and

$$R^i f_*(\mathcal{F})|_U = H^i(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})^{\sim} = 0$$

since by [Theorem III.3.7 [5]] $H^i(X, \mathcal{F}) = 0$ when $i > 0$ with $f^{-1}(U)$ affine. Hence $R^i f_*\mathcal{F}_y = 0$ for any $y \in Y$. $\square$

Remark (If you have known what is flatness:)**.** Consider $i_U : U \longrightarrow Y$, which is flat and

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & X \\ \Big\downarrow{\scriptstyle f|_{f^{-1}(U)}} & & \Big\downarrow{\scriptstyle f} \\ U & \xrightarrow{\ i_F\ } & Y \end{array}$$

By [Proposition III.9.3 [5]], we have

$$i_U^* R^i\pi_*(\mathcal{F}) = R^i f|_{f^{-1}(U)*}(i_{f^{-1}(U)^*}\mathcal{F})$$

that is $R^i f_*(\mathcal{F})|_U = R^i f|_{f^{-1}(U)*}(\mathcal{F}|_{f^{-1}U})$

Exercise 3.8.3. Let $f : X \to Y$ be a morphism of ringed spaces, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Prove the projection formula (cf. (II, Ex. 5.1))

$$R^l f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong R^l f_*(\mathcal{F}) \otimes \mathcal{E}.$$

Proof. Note that $R^\bullet f_*(- \otimes f^*\mathscr{E})$ and $R^\bullet f_*(-) \otimes \mathscr{E}$ are two δ-functor and $f_*(\mathcal{F} \otimes f^*\mathscr{E}) = f_*(\mathcal{F}) \otimes \mathscr{E}$. So it is enough to show that they are universal δ-functor.

Now for any $\mathcal{F}$, consider $0 \longrightarrow \mathcal{F} \xrightarrow{u} \mathcal{I}$ with $\mathcal{I}$ injective. (such a sequence exists for we assume $Sh((X)$ has enough injectives) Since $R^i\pi_*$ is a universal *delta*-functor, $R^i f_*(\mathcal{I}) = 0$ and hence $R^i f_*(u) \otimes \mathscr{E} = 0$, which implies $R^\bullet f_*(-) \otimes \mathscr{E}$ is effaceable.

Now choosing an affine open set $U \subset Y$, on which $\mathscr{E}|_U$ is free. Then as we have show above

$$\begin{aligned} R^i f_*(\mathcal{I} \otimes f^*\mathscr{E})|_U &= R^i f|_{f^{-1}(U)*}(\mathcal{I}|_{f^{-1}(U)} \otimes f^*\mathscr{E}|_{f^{-1}(U)}) \\ &= R^i f|_{f^{-1}(U)*}(\mathcal{I}|_{f^{-1}(U)} \otimes f^*_{f^{-1}(U)}\mathscr{E}|_U) \\ &= R^i f_*(\mathcal{I} \otimes f^*\mathscr{E})|_U \\ &= R^i f_*(\mathcal{I}|_{f^{-1}(U)} \otimes \mathcal{O}^{\oplus n}_{f^{-1}(U)}) \\ &= R^i f_*(\mathcal{I}|_{f^{-1}(U)})^{\oplus n} \end{aligned}$$

Again, as we have shown $R^i f_*(\mathcal{I}|_{f^{-1}(U)}) = R^i f_*(\mathcal{I})|_U = 0$ since $\mathcal{I}$ is an injective. Thus, $R^i f_*(\mathcal{I} \otimes f^*\mathscr{E})|_U$, which implies $R^\bullet f_*(- \otimes f^*\mathscr{E})$ is effaceable. Hence, it is a universal δ-functor. $\square$

Exercise 3.8.4. Let Y be a noetherian scheme, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of rank $n + 1$, $n \geq 1$. Let $X = \mathbb{P}(\mathcal{E})$ (II, §7), with the invertible sheaf $\mathcal{O}_X(1)$ and the projection morphism $\pi : X \to Y$.

(i) Then $\pi_*(\mathcal{O}_X(l)) \cong S^l(\mathcal{E})$ for $l \geq 0$, $\pi_*(\mathcal{O}_X(l)) = 0$ for $l < 0$ (II, 7.11); $R^i\pi_*(\mathcal{O}_X(l)) = 0$ for $0 < i < n$ and all $l \in \mathbb{Z}$; and $R^n\pi_*(\mathcal{O}_X(l)) = 0$ for $l > -n - 1$.

(ii) Show there is a natural exact sequence

$$0 \to \Omega_{X/Y} \to (\pi^*\mathcal{E})(-1) \to \mathcal{O}_X \to 0,$$

cf. (II, 8.13), and conclude that the relative canonical sheaf $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ is isomorphic to $(\pi^* \wedge^{n+1} \mathcal{E})(-n-1)$. Show furthermore that there is a natural isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$ (cf. (7.1.1)).

(iii) Now show, for any $l \in \mathbb{Z}$, that

$$R^n\pi_*(\mathcal{O}_X(l)) \cong \pi_*(\mathcal{O}_X(-l-n-1))^\vee \otimes (\wedge^{n+1}\mathcal{E}^\vee).$$

(iv) Show that $p_a(X) = (-1)^n p_a(Y)$ (use (Ex. 8.1)) and $p_g(X) = 0$ (use (II, 8.11)).

(v) In particular, if Y is a nonsingular projective curve of genus g, and $\mathcal{E}$ a locally free sheaf of rank 2, then X is a projective surface with $p_a = -g$, $p_g = 0$, and irregularity g (7.12.3). This kind of surface is called a geometrically ruled surface (V, §2).

3.8.2 Additional Exercises

Here are some applications of relative Euler sequence:

Exercise 3.8.5. Let $X = \mathbb{P}(\mathcal{E})$ with $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$ over $\mathbb{P}^1_k$. Compute the dimension of the tangent space of $H^1(X, \mathcal{T}_X)$.

Proof. Consider two short exact squences over X:

$$0 \longrightarrow \check{\Omega}_{X/Y} \longrightarrow \mathcal{T}_X \longrightarrow \pi^*\mathcal{T}_Y \longrightarrow 0$$

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \pi^*\mathcal{E} \otimes \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

First, taking the duality of the second exact sequence, we get

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \pi^*(\mathcal{O} \oplus \mathcal{O}(-1)) \otimes \mathcal{O}_X(1) \longrightarrow \check{\Omega}_{X/Y} \longrightarrow 0$$

Since $R^1\pi_*\mathcal{O}_X = 0$ by 3.8.4, hence we get

$$0 \longrightarrow \pi_*\mathcal{O}_X \longrightarrow (\mathcal{O} \oplus \mathcal{O}(-1)) \otimes \pi_*\mathcal{O}_X(1) \longrightarrow \pi_*\check{\Omega}_{X/Y} \longrightarrow 0$$

Again, by 3.8.4, $\pi_*\mathcal{O}(1) = \mathcal{E}$ and $\pi_*\mathcal{O}_X = \mathcal{O}$. Thus, $h^0(X, \check{\Omega}_{X/Y}) = h^0(\mathbb{P}^1_k, \mathcal{E} \otimes \check{\mathcal{E}}) - h^0(\mathbb{P}^1_k, \mathcal{O}) = 4 - 1 = 3$.

Next, we use the first exact sequence to get that

$$0 \longrightarrow \pi_*\Omega_{X/Y} \longrightarrow \pi_*\mathcal{T}_X \longrightarrow \mathcal{T}_Y \otimes \pi_*\mathcal{O}_X \longrightarrow 0$$

because $R^1\pi_*\Omega_{X/Y} = 0$ by base change and cohomology. $(H^1(X_y, \check{\Omega}_{X/Y}|_{X_y}) = H^1(\mathbb{P}^1_k, \Omega_{\mathbb{P}^1_k/k})$. Since $\mathbb{P}^1_k$ is a curve, $\Omega_{\mathbb{P}^1_k/k} = \omega_{\mathbb{P}^1_k/k} = \mathcal{O}(-2)$. Hence $H^1(X_y, \check{\Omega}_{X/Y}|_{X_y}) = H^1(\mathbb{P}^1_k, \mathcal{O}(2)) = 0)$ Hence, we have

$$h^0(X, \mathcal{T}_X) = h^0(\mathbb{P}^1_k, \mathcal{T}_Y) + h^0(X, \check{\Omega}_{X/Y}) = 6$$

□

Exercise 3.8.6. In 3.8.4 (1).If rank$(\mathcal{E}) > 1$, can $\omega_{\mathbb{P}(\mathcal{E})}$ ever be ample? What about its dual?

(2).Compute $\omega_{\mathbb{P}(\mathcal{O}^{\oplus k})}$ for $k > 1$ and $\omega_{\mathbb{P}(\mathcal{T}_Y)}$, when $Y = \mathbb{P}^n$.

Proof. (1). At closed fiber y, $X_y \longrightarrow X$ is a closed embedding. Hence if $\omega_{\mathbb{P}(\mathcal{E})}$ is ample, so is $\omega_{\mathbb{P}(\mathcal{E})}|_{X_y}$ is ample by 3.5.7. However, $\omega_{\mathbb{P}(\mathcal{E})}|_{X_y} = \mathcal{O}(-n-1)$ is not ample, hence $\omega_{\mathbb{P}(\mathcal{E})}$ can not be ample.

For $\omega^*_{\mathbb{P}(\mathcal{E})}$, it **may** be ample. For example, $Y = \mathrm{Spec}(k)$.

(2). Note that $\omega_{X/Y} = \pi^*\det(\mathcal{E}) \otimes \mathcal{O}_X(-n-1)$ with $n = \mathrm{rank}(\mathcal{E}) - 1$. Note that $\det(\mathcal{O}^{\oplus k}) = \mathcal{O}$ and $\det(\mathcal{T}_Y) = \check{\omega}_{Y/k} = \mathcal{O}(n+1)$. Then use $\omega_X = \pi^*\omega_Y \otimes \omega_{X/Y}$. We have

$$\omega_{X_1} = \pi^*\mathcal{O}(-n-1) \otimes \mathcal{O}_X(-k);\ \omega_{X_2} = \pi^*\mathcal{O}(-n-1)$$

$\square$

Exercise 3.8.7. Determine the ramification divisor of the blow-up $\tilde{\mathbb{A}}^2 := \mathrm{Bl}_0(\mathbb{A}^2) \to \mathbb{A}^2$ of the affine plane in the origin and describe the canonical bundle of $\tilde{\mathbb{A}}^2$.

Proof. Note that $\overset{\sim}{\mathbb{A}}^2 := \mathrm{Bl}_{(0,0)}\mathbb{A}^2$ is a closed subscheme of $X := \mathbb{P}(\mathcal{O} \oplus \mathcal{O})$. Then, $\omega_X = \mathcal{O}_X(-2)$. Since $\overset{\sim}{\mathbb{A}}^2$ is defined by $\mathcal{O}(-1)$, hence $\mathcal{N}_{\overset{\sim}{\mathbb{A}}^2/X} = \mathcal{O}_{\overset{\sim}{\mathbb{A}}^2}(1)$. Thus, $\omega_{\overset{\sim}{\mathbb{A}}^2} = \mathcal{O}_{\overset{\sim}{\mathbb{A}}^2}(-1)$.

Note that the ramification divisor over X related to $f : X \longrightarrow Y$ is defined to be $f^*\omega^*_Y \otimes \omega_X$. (Just remember!) We see that $R = \mathcal{O}_{\overset{\sim}{\mathbb{A}}_k^2}(-1)$. $\square$

3.9 Flat Morphisms

3.9.1 Preparations

Here is a stronger version of [[5] Proposition III.9.5.]

Proposition 3.9.1. *Assume $f : X \longrightarrow Y$ is a morphism between two local Noetherian scheme X, Y. Let $y = f(x)$. Then*

$$\dim \mathcal{O}_{X_y,x} \geq \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

and if f is flat, the equality holds up.

Example 3.9.2. If $\dim \mathcal{O}_{X_y,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$, we can not say f is flat. For example consider $k[x]/(x^2) \longrightarrow k[x]/(x)$ satisfies this property but $k[x]/x$ is not a flat $k[x]/(x^2)$-module.

And we can rewrite [[5] Corollary III.9.6.] as:

Corollary 3.9.3. *If $f : X \longrightarrow Y$ is flat and of finite type, Y is irreducible, then all fibers are of the same dimension* $\dim X - \dim Y$.

Proposition 3.9.4 (Flatness over smooth curves)**.** *Assume $f : X \longrightarrow Y = \mathrm{Spec}(R)$ with R a DVR. Then f is flat if and only if $\overline{X_\eta} = X$.*

Proof. This is just [[5] Proposition III.9.7.] □

3.9.2 Examples

Example 3.9.5 (Flat Morphisms)**.**

(1). For any scheme over k, $Y \longrightarrow \mathrm{Spec}(k)$ is flat. Since for every y, $\mathcal{O}_{Y,y}$ is a k-algebra and every k-algebra is a flat k-module.

(2). The morphism $F : \mathrm{Spec}(\mathbb{F}_p[x]) \longrightarrow \mathrm{Spec}(\mathbb{F}_p[x])$ given by $a \mapsto a^p$ is flat. It is enough to check that $F : \mathbb{F}[x] \longrightarrow \mathbb{F}[x]$ given by $f(x) \mapsto f^p(x)$ is flat. Note that this map is injective, which identify $\mathbb{F}_p[x]$ as $\mathbb{F}_p[x^p]$-module. For $\mathbb{F}_p[x]$ is a free $\mathbb{F}_p[x]$-module generated by $1, x, ..., x^{n-1}$, it is a flat $\mathbb{F}_p = [x]$-module.

(3). A dominant morphism $f : X \to Y$ of integral k-schemes with Y a smooth curve. Let $x \in X$ and $y = f(x)$. Consider $f_y : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$. Since Y is an integral smooth curve, $\mathcal{O}_{Y,y}$ is a DVR. Since over a DVR, a module is flat if and only if it is torsion-free (Need a little effort)

Here it another way to show this fact: Since f is dominant, $\eta_X \in f^{-1}(\eta_Y)$. By the fact that X is integral, $\overline{X}_{\eta_Y} = X$. Hence, f is flat.

Remark. If f is dominant, $\{\eta_X\}$ is dense over X and $f(\{\eta_X\})$ is dense in $f(X)$. Since $f(X)$ is dense in Y, we see that $\{f(\eta_X)\}^- = Y$. Thus, $\eta_Y \in f(\eta_X)$.

Example 3.9.6 (Morphisms that Are Not Flat)**.**

(1). Consider $X = V(3x^2 + 6y^2) \subset \mathbb{A}^2_{\mathbb{Z}}$. Then $X \longrightarrow \mathbb{Z}$ is not flat. Consider the fiber of $X \longrightarrow \mathrm{Spec}(\mathbb{Z})$. Over (0), the fiber is $V(3x^2 + 6y^2) \subset \mathbb{A}^2_{\mathbb{Q}}$, which is of dimension 1. Over (3), the fiber is $V(3x^2 + 6y^2) = V(0) = \mathbb{A}^2_{\mathbb{F}_3}$ which is of dimension 2.

However, if we take X to be the closure of $V(3x^2 + 6y^2) \subset \mathbb{A}^2_{\mathbb{Q}}$ in $\mathbb{A}^2_{\mathbb{Z}}$, the morphism is flat by 3.9.4.

(2).$\pi : \mathrm{Bl}_{(0,0)}\mathbb{A}^2_k \longrightarrow \mathbb{A}^2_k$ is also not flat. We can directly compute the dimension of the fibers.

(3). Let $X = V(x^3 - y^2) \subset \mathbb{A}^2_k$. Consider $\pi : \widetilde{X} \longrightarrow X$ given by

$$\begin{aligned} k[x,y]/(x^3 - y^2) &\longrightarrow k[t] \\ x &\longmapsto t^2 \end{aligned}$$

$$y \longmapsto t^3$$

Note that $(x, y) \hookrightarrow k[x, y]/(x^3 - y^2) =: A$ is an injctive. However, $(x, y) \otimes_A k[t] \longrightarrow k[t]$ is not an injection since $x \otimes t - y \otimes 1 \longmapsto 0$.

(4). The first projection $f : Z \to \mathbb{P}^1_k$, where $Z \subset \mathbb{P}^1_k \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$ is defined by the equation $x_0^2 y_1 - x_0 x_1 y_0$. Here, x_0, x_1 and y_0, y_1 are the coordinates on the first and second factor, respectively. Then f is not flat. We see that $f : ([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0 : x_1]$. Note that at $[0 : 1]$, $f^{-1}([0 : 1]) = \mathbb{P}^1_k$ and at $[1 : 1]$, $f^{-1}([1 : 1]) = V(y_1 - y_0)$. They have different Hilbert polynomials, which means f can not be flat.

3.9.3 Exercises

Exercise 3.9.1. A flat morphism $f : X \to Y$ of finite type of noetherian schemes is open, i.e, for every open subset $U \subseteq X$, $f(U)$ is open in Y. [Hint: Show that $f(U)$ is constructible and stable under generization (II, Ex. 3.18) and (II, Ex. 3.19).]

Proof. By 2.3.19, Chevalley Theorem, $f(U)$ is constructable. By 2.3.18-(b), it is enough to show that $f(U)$ is stable under generization, that is, given $x_0 \in f(U)$ and $x_0 \in \{x_1\}^-$, then $x_1 \in f(U)$.

Since openness is a local property, it is enough to assume $f : Spec(A) \longrightarrow Spec(B)$ with A a flat B-mod deduced by $\varphi : B \longrightarrow A$. Assume $\mathfrak{q} \in Spec(B)$ lies in $f(Spec(A))$, that is, $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$ for some $\mathfrak{p} \in Spec(A)$. Suppose that $\mathfrak{q} \in \{\mathfrak{q}'\}^-$. $\mathfrak{q} \supseteq \mathfrak{q}'$. Since $\varphi : B \longrightarrow A$ is flat, it satisfies going down theorem. Hence, we can find $\mathfrak{p}'$ such that $\mathfrak{p} \supseteq \mathfrak{p}'$ and $\varphi^{-1}(\mathfrak{p}') = \mathfrak{q}'$. □

Exercise 3.9.2. Do the calculation of (9.8.4) for the curve of (I, Ex. 3.14). Show that you get an embedded point at the cusp of the plane cubic curve.

Exercise 3.9.3. Some examples of flatness and nonflatness.

(a) If $f : X \to Y$ is a finite surjective morphism of nonsingular varieties over an algebraically closed field k, then f is flat.

(b) Let X be a union of two planes meeting at a point, each of which maps isomorphically to a plane Y. Show that f is not flat. For example, let $Y = \operatorname{Spec} k[x, y]$ and $X = \operatorname{Spec} k[x, y, z, w]/(z, w) \cap (x + z, y + w)$.

(c) Again let $Y = \operatorname{Spec} k[x, y]$, but take $X = \operatorname{Spec} k[x, y, z, w]/(z^2, zw, w^2, xz - yw)$. Show that $X \cong Y, X$ has no embedded points, but that f is not flat.

Proof. (a). Since X, Y are non-singular, Y is regular and X is Cohen-Macaulay. Now we're going to use "miracle flatness" to prove that f is flat:

Since f is finite and surjective, $\dim X = \dim Y$ (Check locally) and f is quasi-finite, which implies for any $y \in Y$, $f^{-1}(y)$ is non-empty and $\dim f^{-1}(y) = 0 = \dim X - \dim Y$. By "miracle flatness", f is flat.

(b). Now that $\mathbb{P}^2_Y = \operatorname{Proj}(k[x, y][T_0, T_1, T_2])$. Then we can embedding

$$X \longrightarrow \mathbb{A}^2_Y \hookrightarrow \mathbb{P}^2_Y$$

that is, we have a closed immersion from $X \longrightarrow \mathbb{P}^2_Y$. Hence, we can tell whether or not $f : X \longrightarrow Y$ deduced by $\varphi : k[x, y] \longrightarrow k[x, y, z, w]/(z^2, zw, w^2, zx - yw)$ given by $x \mapsto x$ and $y \mapsto y$.

Consider $\mathfrak{p}_1 = (x, y)$ and $\mathfrak{p}_2 = (x - 1, y)$. Then

$$\begin{aligned} f^{-1}(\mathfrak{p}_1) &= Spec(k[x, y, z, w]/(z, w) \cap (x + z, y + w) \otimes_{k[x,y]} k[x, y]/(x, y)) \\ &= Spec(k[z, w]/(z, w)) \end{aligned}$$

$$\cong Spec(k)$$

Hence, $p_{\mathfrak{p}_1} = 1$. For $\mathfrak{p}_2$, we have

$$\begin{aligned} f^{-1}(\mathfrak{p}_1) &= Spec(k[x,y,z,w]/(z,w) \cap (x+z, y+w) \otimes_{k[x,y]} k[x,y]/(x-1,y)) \\ &= Spec(k[z,w]/(z,w) \cap (z+1,w)) \end{aligned}$$

By CRT, $k[z,w]/(z,w) \cap (z+1,w) = k[z,w]/(z,w) \oplus k[x,y]/(z-1,w) \cong k \oplus k$. Hence $p_{\mathfrak{p}_2} = 2$. Thus, $f : X \longrightarrow Y$ is not flat.

For general case, **Wait!**

(c). Again, we use the same method in (b). By directly computing

$$\mathrm{rad}((z^2, zw, w^2, xz - yw)) = (z,w)$$

Hence $X^{red} = Spec(k[x,y,z,w]/(z,w)) \cong Spec(k[x,y]) = Y$.

To show that $f : X \longrightarrow Y$ is not flat, we consider the fibers of some points. Consider $\mathfrak{p}_1 = (x,y), \mathfrak{p}_2 = (x-1,y)$.

$$\begin{aligned} f^{-1}(\mathfrak{p}_1) &= Spec(k[x,y,z,w]/(z^2, w^2, zw, xz - yw) \otimes_{k[x,y]} k[x,y]/(x,y)) \\ &= Spec(k[z,w]/(z^2, w^2, zw)) \\ &\cong Spec(k \oplus k \oplus k) \end{aligned}$$

Hence, $p_{\mathfrak{p}_1} = 3$

$$\begin{aligned} f^{-1}(\mathfrak{p}_2) &= Spec(k[x,y,z,w]/(z^2, zw, w^2, xz - yw) \otimes_{k[x,y]} k[x,y]/(x-1,y) \\ &= k[z,w]/(z,w^2)) \\ &\cong Spec(k \oplus k) \end{aligned}$$

Hence, $p_{\mathfrak{p}_2} = 2$, which implies the Hilbert polynomials over fibers are depend on $y \in Y$. Thus, f is not flat.

As for embedded points, (**Wait**)! □

Exercise 3.9.4 (Open Nature of Flatness.)**.** Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then $\{x \in X | f \text{ is flat at } x\}$ is an open subset of X (possibly empty)—see Grothendieck [EGA IV$_3$, 11.1.1].

Proof. Also, this is a local problem. Let $X = Spec(A)$ and $Y = Spec(R)$ with A a finitely generated R-mods. Then we have

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] & \mathbb{A}^n_R \arrow[dl] \\ Y & \end{tikzcd}$$

Then we need to show that if M is a finitely generated $R[x_1, ..., x_n]$-mod, then

$$\{\mathfrak{p} \in A^n_{\mathbb{R}} | M_{\mathrm{p}} \text{ is a flat } R_{\mathrm{q}}\text{-mod with } \mathrm{q} = \mathrm{p} \cap R\}$$

is an open set.

Also, we need to show that this set is constructible and stable under generalization. First we show that it is stable under generalization: Consider $\mathfrak{p}_1 \subset \mathrm{p}_2$, that is, $\mathfrak{p}_2 \in \{\mathfrak{p}_1\}^-$. If M_{p_2} is flat over R_{p_2}. Now since $\mathfrak{p}_1 \subset \mathrm{p}_2$, we see that R_{p_2} is a flat $R_{\mathfrak{p}_1}$-mod since localization preserves flatness. Hence, $M_{\mathfrak{p}_2} \otimes_{R_{\mathrm{p}_2}} R_{\mathrm{p}_1} = M_{\mathrm{p}_1}$ a flat R_{p_1}-mod.

The key point is to show that this set is constructible. To begin with, we need the following lemma:

Lemma 3.9.7 (Grothendieck's Generic Freeness Lemma)**.** *Suppose that R is a domain, A is a finitely generated R algebra and M is a finitely generated A-mod. Then there exists $f \neq 0$ and $a \in R$ such that M_f is a free R_f-mod.*

Suppose that $\mathfrak{p}_1 \in U$ and $\mathfrak{p}_2 \in \{\mathfrak{p}_1\}^-$, that is, $\mathfrak{p}_1 \subset \mathfrak{p}_2$. By the following lemma, it is enough to show that there is an open subset of $\{x\}^{-1}$ such that $V \subset \{\mathfrak{p}_1\}^- \cap U$:

Lemma 3.9.8. *Let X be a Noetherian topology space. A subset $E \subset X$ is constructible if and only if for every closed irreducible subset $Y \subset X$, $E \cap Y$ contains a non-empty open subset of Y or is nowhere dense in Y.*

Consider $\{\mathfrak{p}_1\}^- = Spec(R[x_1, ..., x_n]/\mathfrak{p}_1)$. Since M is a finitely generated $R[x_1, ..., x_n]$, $M/\mathrm{p}_1 M^\sim = M^\sim|_{\{\mathfrak{p}_1\}^-}$ is a finitely generated $R[x_1, ..., x_n]/\mathfrak{p}_1$. Since $\mathfrak{q}_1 = \mathfrak{p}_1 \cap R$, using Generic Freeness Lemma, then there exists a $f \neq 0$ which lies in $R/\mathfrak{q}_1$ such that $(M/\mathfrak{p}_1 M)_f$ is a free R_f-mod. Take the preimage of $D(f) \in Y$, showing that U is constructible. □

Remark 3.9.9. This prove loses many details, like $M_{\mathfrak{p}_1}$ is a flat $R_{\mathfrak{q}_1}$-mod with $\mathfrak{q}_1 = \mathfrak{p}_1 \cap R$. But I doesn't give more details of that.

Exercise 3.9.5 (Very Flat Families.)**.** For any closed subscheme $X \subseteq \mathbb{P}^n$, we denote by $C(X) \subseteq \mathbb{P}^{n+1}$ the projective cone over X (I, Ex. 2.10). If $I \subseteq k[x_0, \ldots, x_n]$ is the (largest) homogeneous ideal of X, then $C(X)$ is defined by the ideal generated by I in $k[x_0, \ldots, x_{n+1}]$.

(a) Give an example to show that if $\{X_t\}$ is a flat family of closed subschemes of $\mathbb{P}^n$, then $\{C(X_t)\}$ need not be a flat family in $\mathbb{P}^{n+1}$.

(b) To remedy this situation, we make the following definition. Let $X \subseteq \mathbb{P}^n_T$ be a closed subscheme, where T is a noetherian integral scheme. For each $t \in T$, let $I_t \subseteq S_t = k(t)[x_0, \ldots, x_n]$ be the homogeneous ideal of X_t in $\mathbb{P}^n_{k(t)}$. We say that the family $\{X_t\}$ is *very flat* if for all $d \geq 0$,

$$\dim_{k(t)}(S_t/I_t)_d$$

is independent of t. Here $()_d$ means the homogeneous part of degree d.

(c) If $\{X_t\}$ is a very flat family in $\mathbb{P}^n$, show that it is flat. Show also that $\{C(X_t)\}$ is a very flat family in $\mathbb{P}^{n+1}$, and hence flat.

(d) If $\{X_t\}$ is an algebraic family of projectively normal varieties in $\mathbb{P}^n_k$, parametrized by a nonsingular curve T over an algebraically closed field k, then $\{X_t\}$ is a very flat family of schemes.

Proof. First, $\{X_t\}$ is a flat family means that under the map $\pi_2 : \mathbb{P}^n_k \times_k \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$, $\pi_t|_{X_t}$ is flat. For example $X_t := V(xy - tz^2) \subset \mathbb{P}^2_k$ is a flat family since its Hilbert polynomial is constant along t.

(a). Consider $X_t = \{[1:0:0], [0:1:0], [1:1:t]\} \subset \mathbb{P}^2_k$. Then $P_{X_t}(m) = H^0(X_t, \mathcal{O}_{X_t}(m)) = 3$, which is independent of t. Hence, X_t is a flat family. To show that $C(X_t)$ is not a flat family, we need to use Hilbert functions.

If X is defined by the homogeneous ideal $I_X \subset S = k[x_0, x_1, ...x_n]$ and $I_{X_t} = \oplus_{s+t=m}(I_X)_s x^t_{n+1}$ by definition. Then we have

$$H_{C(X)}(m) = \dim_k(I_{C(X)})_m = \sum_{i=0}^{m} \dim_k(I)_i$$

For X_t, we have when $t \neq 0$

$$H_{X_t}(m) = \begin{cases} 0, & m \leq 0 \\ 3, & m \geq 1 \end{cases}$$

and when $t = 0$,

$$H_{X_t}(m) = \begin{cases} 0, & m \leq 0 \\ 2, & m = 1 \\ 3, & m \geq 2 \end{cases}$$

(I will give the computation of the Hilbert functions in the remark). Hence, $H_{C(X_t)}(m)$ differs when m is very large. However, by 2.5.14, we know that when $m \gg 0$,

$$H(m) = P(m)$$

Hence, the Hilbert polynomials of $\{C(X_t)\}$ are not independent of t, which implies $C(X_t)$ is not a flat family.

(b). This is just a notation for our exercise.

(c). Since the Hilbert functions $H_{X_t}(m)$ is independent of t. As we have stated, $P_{X_t}(m) = H_{X_t}(m)$ when $m \gg 0$ for each t. We see that $P_{X_t}(m)$ is also independent of the choice of t. Hence, $\{X_t\}$ is a flat family.

Next, we know that

$$H_{C(X)}(m) = \dim_k (I_{C(X)})_m = \sum_{i=0}^{m} \dim_k (I)_i$$

Hence, $H_{C(X_t)}(m)$ is also independent of the choice of t. By the same arguments, we see that $\{C(X_t)\}$ is also a flat family.

(d) **Wait!** Use Hilbert regularity.

□

Remark 3.9.10 (Hilbert functions)**.**

Exercise 3.9.6. Let $Y \subseteq \mathbb{P}^n$ be a nonsingular variety of dimension ≥ 2 over an algebraically closed field k. Suppose $\mathbb{P}^{n-1}$ is a hyperplane in $\mathbb{P}^n$ which does not contain Y, and such that the scheme $Y' = Y \cap \mathbb{P}^{n-1}$ is also nonsingular. Prove that Y is a complete intersection in $\mathbb{P}^n$ if and only if Y' is a complete intersection in $\mathbb{P}^{n-1}$. [See (II, Ex. 8.4) and use (9.12) applied to the affine cones over Y and Y'.]

Proof. ($\Longrightarrow$): If $Y = \cap_{i=1}^r H_i$, then $Y \cap \mathbb{P}_k^{n-1} = \cap (H_i \cap \mathbb{P}_k^{n-1})$. Since H_i is a hypersurface over $\mathbb{P}_k^n$, $\dim H_i = n-1$. By [Theorem I.7.2 [5]], we know that $\dim H_i \cap \mathbb{P}_k^{n-1} \geq n-2$. Hence, $H_i \subset \mathbb{P}_k^{n-1}$ or $H_i \cap \mathbb{P}_k^{n-1}$ is a hypersurface over $\mathbb{P}_k^{n-1}$, which implies $Y \cap \mathbb{P}_k^{n-1}$ is still a complete intersection.

($\Longleftarrow$):**Wait!** □

Exercise 3.9.7. Let $Y \subseteq X$ be a closed subscheme, where X is a scheme of finite type over a field k. Let $D = k[t]/t^2$ be the ring of dual numbers, and define an infinitesimal deformation of Y as a closed subscheme of X, to be a closed subscheme $Y' \subseteq X \times_k D$, which is flat over D, and whose closed fibre is Y. Show that these Y' are classified by $H^0(Y, \mathcal{N}_{Y|X})$, where

$$\mathcal{N}_{Y|X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y).$$

Proof. Here to prove this results. I think it is necessary to formalize definitions and theorems related to deformation theory:

Definition 3.9.11 (Deformation)**.** Let $D = Spec(k[t]/(t^2))$. A deformation of Y over D in X is a closed subscheme $Y^{'} \subset X^{'}$, flat over D, with $X^{'} = X \times_k D$ such that $Y^{'}_{(t)} \cong Y$.

Here, the $Y^{'}_{(t)} \cong Y$ means that the fiber of $Y^{'}$ over (t) is just isomorphic to Y.

Definition 3.9.12 (Deformation of Affine Case)**.** Let A a k-algebra and I be an ideal of A. Then deformation of I over D in A is an ideal $I^{'} \subset A^{'} = A[t]/(t^2)$ such that
- $A^{'}/I^{'}$ is flat over $k[t]/(t^2)$ and;
- $(A^{'}/I^{'}) \otimes_D k \cong A/I$.

For affine case, $Spec(A)$ is just X and $Spec(A/I) = Y$. Now, we are going to classify all deformations of Y over D in $Spec(A)$.

Proposition 3.9.13. *Let $A^{'} \twoheadrightarrow A$ be a surjection of Noetherian rings whose kernel J has square zero. Then an $A^{'}$-mod M is flat over $A^{'}$ if and only if*
(i). $M = M^{'} \otimes_{A^{'}} A$ is flat over A;
(ii). The natural map $M \otimes_A J \longrightarrow M^{'}$.

Now consider $0 \longrightarrow (t) \longrightarrow D \longrightarrow k \longrightarrow 0$. By 3.9.13 If we already have $(A^{'}/I^{'}) \otimes_D k = A/I$, then the case $A^{'}/I^{'}$ is flat over $k[t]/(t^2)$ is equivalent to
(1). A/I is flat over k;
(2). $A^{'}/I^{'} \otimes_D (t) \longrightarrow A^{'}/I^{'}$ is injective.
Since A/I is a k-alg, it is naturally flat [P 513 [1]]. Hence, (1) automatically holds up. For $(t) \subset D$ is isomorphic to k as D-alg and the isomorphism is given by $k \xrightarrow{\times t} (t)$, the short exact sequece is given by

$$0 \longrightarrow k \xrightarrow{\times t} D \longrightarrow k \longrightarrow 0$$

and then (2) is just $B/I \xrightarrow{\times t} A^{'}/I^{'}$ is injective. Hence, we have the following lemma

Lemma 3.9.14. *$I^{'}$ is a deformation of I if and only if*
- $(A^{'}/I^{'}) \otimes_D k \cong A/I;$
- $0 \longrightarrow A/I \xrightarrow{\times t} A^{'}/I^{'} \longrightarrow A/I \longrightarrow 0$ *is exact.*

Recall that given an inclusion $Y^{'} \hookrightarrow Spec(A^{'})$ is equivalent to give $p : A^{'} \longrightarrow A^{'}/I^{'}$. Then consider the exact sequence

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r, "\times t"] \arrow[d] & I^{'} \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\times t"] \arrow[d] & A^{'} \arrow[r] \arrow[d, "p"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A/I \arrow[r, "\times t"] \arrow[d] & A^{'}/I^{'} \arrow[r] \arrow[d] & A/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By nine lemma, $0 \longrightarrow I \xrightarrow{\times t} I^{'} \longrightarrow I \longrightarrow 0$ is exact if and only if $0 \longrightarrow A/I \xrightarrow{\times t} A^{'}/I^{'} \longrightarrow A/I \longrightarrow 0$ is exact.

Lemma 3.9.15. *Giving an ideal $I^{'}$ such that*

$$0 \longrightarrow I \xrightarrow{\times t} I^{'} \longrightarrow I \longrightarrow 0$$

is exact is equivalent to giving an element in $Hom_A(I, A/I)$.

Proof. Note that $B^{'} = B \oplus tB$ has a B-mod structure.

Suppose we are given such an $I^{'}$. Let $x \in I$. We can find $x + ty \in I^{'}$ such that $x + ty \mapsto x$. If $x + ty_1$ and $x + ty_2$ maps to the same element, by the exactness of the sequence, $y_1 - y_2$ lies in I. Hence, we get a well-defined map

$$\begin{aligned} \varphi : I &\longrightarrow B/I \\ x &\longmapsto \overline{y} \end{aligned}$$

where $x + ty \mapsto x$.

Conversely, given any $\varphi \in Hom_A(I, A/I)$, we can define

$$I^{'} := \{x + ty | x \in I, y \in B \text{ such that } \overline{y} = \varphi(x)\}$$

Then $I^{'}$ is an ideal and the short sequence $0 \longrightarrow I \xrightarrow{\times t} I^{'} \longrightarrow I \longrightarrow 0$ is exact. □

Note that $Hom_B(I, B/I) \cong Hom_{B/I}(I/I^2, B/I)$. Globally, we see that the deformation of Y over D in X is given by

$$Hom_Y(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathcal{O}_Y) = H^0(Y, \mathscr{H}om(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathcal{O}_Y)) = H^0(Y, \mathscr{N}_{Y/X})$$

□

Exercise 3.9.8. Let A be a finitely generated k-algebra. Write A as a quotient of a polynomial ring P over k, and let J be the kernel:

$$0 \to J \to P \to A \to 0.$$

Consider the exact sequence of (II, 8.4A)

$$J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0.$$

Apply the functor $\mathrm{Hom}_A(\cdot, A)$, and let $T^1(A)$ be the cokernel:

$$\mathrm{Hom}_A(\Omega_{P/k} \otimes A, A) \to \mathrm{Hom}_A(J/J^2, A) \to T^1(A) \to 0.$$

Now use the construction of (II, Ex. 8.6) to show that $T^1(A)$ classifies infinitesimal deformations of A, i.e., algebras A' flat over $D = k[t]/t^2$, with $A' \otimes_D k \cong A$. It follows that $T^1(A)$ is independent of the given representation of A as a quotient of a polynomial ring P.

Proof. First of all, let's define what's the meaning of equivalence of deformations:

Definition 3.9.16 (Equivalence classes of Deformation)**.** Two deformations $Y_1^{'}, Y_2^{'}$ are equivalent if and only if there exists an isomorphism $f : Y_1 \longrightarrow Y_2$ such that $f|_{Y_{1(t)}}$ is an isomorphism.

On affine cases, this is equivalent to giving $f_1 : A^{'}/I_1^{'} \longrightarrow A^{'}/I_2^{'}$ such that $f \otimes_D k$ is an isomorphism, that is two six diagrams:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\times t"] \arrow[d] & A^{'} \arrow[r] \arrow[d, "p_i"] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A/I \arrow[r, "\times t"] & A^{'}/I_i^{'} \arrow[r] & A/I \arrow[r] & 0 \end{tikzcd}$$

Then once we give another equivalent deformation of $I^{'}$, there is an another lift of $A^{'} \longrightarrow A \longrightarrow A/I$. Hence, by 2.8.6 (a), the equivalence deformations of $I^{'}$ are all elements in $Hom_{A'}(\Omega_{A'/k}, A/I)$.

Now back to our assumption, we take $A = P$ and $I = J$ such that $P/J = A$. Then

$$Hom_{A'}(\Omega_{A'/k}, A) = Hom_{A'}(\Omega_{P/k} \otimes D, A) = Hom_A(\Omega_{P/k} \otimes A, A)$$

Hence, we have

$$Hom_A(\Omega_{P/k} \otimes A, A) \longrightarrow Hom_A(J/J^2, A) \longrightarrow T^1(A) \longrightarrow 0$$

where $T^1(A)$ is set of the equivalent class of deformations. □

Remark 3.9.17. See [Chapter 1 [6]] for more details.

Exercise 3.9.9. A k-algebra A is said to be rigid if it has no infinitesimal deformations, or equivalently, by (Ex. 9.8) if $T^1(A) = 0$. Let $A = k[x, y, z, w]/(x, y) \cap (z, w)$, and show that A is rigid. This corresponds to two planes in $\mathbb{A}^4$ which meet at a point.

Proof. By the exact sequence

$$Hom_A(\Omega_{P/k} \otimes A, A) \longrightarrow Hom_A(J/J^2, A) \longrightarrow T^1(A) \longrightarrow 0$$

It is enough to show that every map $J/J^2 \longrightarrow A$ factors through $J/J^2 \longrightarrow \Omega_{P/k} \otimes A \longrightarrow A$.

First of all, $J = (x, y) \cap (z, w) = (xz, xw, yz, yw)$. Any A-mod homomorphism $\varphi : J/J^2 \longrightarrow A$ is completely determined by images of xz, xw, yz, yw and we denote them by $a_{xz}, a_{xw}, a_{yz}, a_{yw}$. Moreover, they satisfy $ya_{xz} = xa_{yz}$, $wa_{xz} = za_{xw}$, $ya_{xw} = xa_{yw}$, $wa_{yz} = za_{yw}$. By the exact sequence

$$0 \longrightarrow P/J \longrightarrow P/(x, y) \times P/(z, w) \longrightarrow P/(x, y) + (z, w) \longrightarrow 0$$

with $P = k[x, y, z, w]$, $A = \{(f, g) \in k[x, y] \times k[z, w] \big| f(0, 0) = g(0, 0)\}$. Suppose that $a_{xz} = (f_{xz}(x, y), g_{xz}(z, w))$ and so on. Then we get $yf_{xz} = xf_{yz}$. Hence, there exists $p_1(x, y)$ such that $f_{xz}(x, y) = xp_1(x, y)$. And similarly, we can write

$$\begin{aligned} a_{xz} &= (xp_1(x, y), zq_1(z, w)); \\ a_{yz} &= (yp_1(x, y), zq_2(z, w)); \\ a_{xw} &= (xp_2(x, y), wq_1(z, w)); \\ a_{yw} &= (yp_2(x, y), wq_2(z, w)). \end{aligned}$$

Since $i : J/J^2 \longrightarrow \Omega_{P/A} \otimes_A$ is given by $xz \longrightarrow (xdz + zdx) \otimes 1$ and so on, if we define $dx \mapsto (0, q_1), dy \mapsto (0, q_2), dz \mapsto (p_1, 0), dw \mapsto (p_2, 0)$, it gives a map $\varphi^{'}$ such that $\varphi^{'} \circ i = \varphi$. Hence, we proved the result. □

Exercise 3.9.10. A scheme X_0 over a field k is rigid if it has no infinitesimal deformations.

(a) Show that $\mathbb{P}^1_k$ is rigid, using (9.13.2).

(b) One might think that if X_0 is rigid over k, then every global deformation of X_0 is locally trivial. Show that this is not so, by constructing a proper, flat morphism $f : X \to \mathbb{A}^2$ over k algebraically closed, such that $X_0 \cong \mathbb{P}^1_k$, but there is no open neighborhood U of 0 in $\mathbb{A}^2$ for which $f^{-1}(U) \cong U \times \mathbb{P}^1$.

(c) Show, however, that one can trivialize a global deformation of $\mathbb{P}^1$ after a flat base extension, in the following sense: let $f : X \to T$ be a flat projective morphism, where T is a nonsingular curve over k algebraically closed. Assume there is a closed point $t \in T$ such that $X_t \cong \mathbb{P}^1_k$. Then there exists a nonsingular curve T', and a flat morphism $g : T' \to T$, whose image contains t, such that if $X' = X \times_T T'$ is the base extension, then the new family $f' : X' \to T'$ is isomorphic to $\mathbb{P}^1_{T'} \to T'$.

(a). It is enough to compute $H^1(\mathbb{P}^1_k, \mathcal{T}_{\mathbb{P}^1_k})$. By the Euler sequence,

$$0 \longrightarrow \Omega_{\mathbb{P}^1_k/k} \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-1) \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \longrightarrow 0$$

Taking duality, we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(1) \oplus \mathcal{O}_{\mathbb{P}^1_k}(1) \longrightarrow \mathcal{T}_{\mathbb{P}^1_k} \longrightarrow 0$$

Since $H^2(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = 0$ and $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(1)) = 0$ (By direct computation or by Serre Duality: $H^1(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_k}(1)) = H^0(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_K}(-1) \otimes \omega_{\mathbb{P}^1_k})^* = 0$.) Hence,

$$H^1(\mathbb{P}^1_k, \mathcal{T}_{\mathbb{P}^1_k}) = 0$$

(b). Consider $Y = V((a-1)x^2 + (b-1)y^2 + z^2) \subset \mathbb{A}^2_{(a,b)} \times \mathbb{P}^2_{[x:y:z]}$. Consider $\pi : Y \longrightarrow \mathbb{A}^2_k$. By computing the Hilbert polys, we see that π is flat. Since $\mathbb{P}^2_k \times \mathbb{A}^2_k \longrightarrow \mathbb{A}^2_k$ is projective, it is proper. For $i : Y \longrightarrow \mathbb{P}^2_k \times \mathbb{A}^2_k$ is a closed immersion, π is proper.

Note that $Y_0 = V(x^2 + y^2 - z^2) \subset \mathbb{P}^2_k$. Hence, $g(Y) = 0$, which implies $Y_0 = \mathbb{P}^1_k$. However, on the generic point, $(a-1)x^2 + (b-1)y^2 + z^2$ doesn't have solution over $k(a,b)$, meaning that $Y_{k(a,b)}$ doesn't have any $k(x,y)$-points. Hence, there is no open neighborhood U of 0 such that $\pi^{-1}(U) \cong U \times \mathbb{P}^1$.

(c).

Exercise 3.9.11. Let Y be a nonsingular curve of degree d in $\mathbb{P}^n_k$, over an algebraically closed field k. Show that

$$0 \leq p_a(Y) \leq \frac{1}{2}(d-1)(d-2).$$

Proof. Come back after Chapter IV. □

3.9.4 Additional Exercises

Exercise 3.9.12. Recall that in 3.9.4, we show that $U = \{x \in X \big| f \text{ is flat at } x\}$ is open. Find an example of a finite type morphism $f : X \longrightarrow Y$ such that U is empty.

Proof. Take $p : \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$ which is not flat and note that $\mathrm{Spec}(\mathbb{Z}/p\mathbb{Z}) = \mathrm{Spec}(\mathbb{F}_p)$ which contains only one point. □

Exercise 3.9.13. Let Y be a smooth curve. A finite morphism $f : X \longrightarrow Y$ is flat if and only if $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module.

Proof. For $f : X \longrightarrow Y$ is finite, $(f_*\mathcal{O}_X)_y = \oplus_{x \in f^{-1}(y)} \mathcal{O}_{X,x}$. (You can prove this by considering $A \longrightarrow B$ with B a finite A-algebra.) Then we have:

$$\begin{aligned} f \text{ is flat } &\Longleftrightarrow \forall y \in Y, x \in f^{-1}(y), \mathcal{O}_{X,x} \text{ is flat over } \mathcal{O}_{Y,y} \\ &\Longleftrightarrow \forall y \in Y, x \in f^{-1}(y), \mathcal{O}_{X,x} \text{ is free over } \mathcal{O}_{Y,y} \\ &\Longleftrightarrow \forall y \in Y, , \oplus_{x \in f^{-1}(y)} \mathcal{O}_{X,x} \text{ is free over } \mathcal{O}_{Y,y} \\ &\Longleftrightarrow f_*\mathcal{O}_X \text{ is free over } Y \end{aligned}$$

for over a DVR R, a finitely generated R-module M is flat if and only if it is free. □

3.10 Smoothness

3.10.1 Preparations

Proposition 3.10.1. *Let $f : X \longrightarrow Y$ be morphisms between two smooth schemes. Then the following conditions are equivalent:*

(i). f is smooth of relative dimension n;

(ii). $\Omega_{X/Y}$ is locally free of rank n on X;

(iii). For every closed point $x \in X$, the induced map on Zariski tangent spaces $T_f : T_x \longrightarrow T_y$ is surjective.

Corollary 3.10.2. *Let $f : X \longrightarrow Y$ be a smooth morphism between two smooth schemes, then we have an short exact sequence:*

$$0 \longrightarrow f^*\Omega_{Y/k} \longrightarrow \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Proposition 3.10.3. *Let $f : X \longrightarrow Y$ be a morphism of finite type, X, Y be integral. Assume that $K(X)/K(Y)$ is separable. Then there exists a non-empty open set $U \subseteq X$ such that*

$$f : U \longrightarrow Y$$

is smooth.

Proposition 3.10.4 (Sard)**.** *Assume X, Y smooth over k and* $\operatorname{char}(k) = 0$*. Then there exists a non-empty open set $V \subset Y$ such that $f|_{f^{-1}(V)}$ is smooth.*

Theorem 3.10.5 (Bertini)**.** *Assume that X is quasi-projective and smooth over k. Then for $\mathcal{L} \in$* $\operatorname{Pic}(X)$ *and $V \subset H^0(X, \mathcal{L})$, a base point free linear subspace. Moreover, assume* char $k = 0$ *and $k = \bar{k}$ ($|k| = \infty$ is enough). Then there exists a non-empty subspace U of V, such that $Z := \varphi^{-1}(V_+(s)) \subset X$ is smooth.*

Remark. In the proof we consider $X \longrightarrow \mathbb{P}^n_k$ induced by $\mathcal{L}$. And we take the preimage of degree 1 hypersurfaces in $\mathbb{P}^n_k$.

Remark. Now, return to general Bertini: Consider $X \subset \mathbb{P}^n_k$ with X non-singular. Then we can choose a hypersurface H of any deg such that $X \cap H$ is smooth. Just use the Versonese $X \subset \mathbb{P}^n_k \hookrightarrow \mathbb{P}^{nd}_k$.

3.10.2 Examples

Example 3.10.6 (Smooth Morphisms)**.**

Example 3.10.7 (Morphisms are not smooth)**.** (1). Consider $k[x] \longrightarrow k[y]$ by $x \longmapsto y^2$. Then $k[y] \cong k[x] \oplus yk[x]$ is a flat $k[x]$-module. Consider the fiber at $a \in \mathbb{A}^1_k$. $f^{-1}(a) = V(x^2 - a) \in \mathbb{A}^1_k$. Note that $V(x^2)$ is nonsingular at $x = 0$. Hence, this morphism is not flat.

(2). The natural projection $f : V_+(y^2z - x^3 + x^2z + xz^2 - 10z^4) \subset \mathbb{P}^2_{\mathbb{Z}}$. Over $V_+(x)$, $X|_{V_+(x)} = V(y^2z - 1 + z + z^2 - 10z^3) \subset \mathbb{A}^2_{\mathbb{Z}}$. Let $f(y, z) = y^2z - 1 + z + z^2 - 10z^3$ Note that

$$\begin{cases} \frac{\partial f}{\partial z} = y^2 + 1 + 2z - 30z \\ \frac{\partial f}{\partial y} = 2yz \\ f(y, z) = 0 \end{cases}$$

Over $\mathbb{F}_2$, this system of equations always has a solution y=1, z=1, which implies the fiber at (2) is singular. Hence, this morphism is not smooth.

(3). Consider $f : X \longrightarrow \mathbb{A}^1_{\mathbb{C}}$ given by $(x, y, t) \mapsto t$, where $X \subset \mathbb{A}^3_{\mathbb{C}}$ is defined by $x^2+y^2 = t$. Over $\mathfrak{p} = (x-a)$, $X_{\mathfrak{p}} = V(x^2+y^2-t) \subseteq \mathbb{A}^2_{\mathbb{C}}$. Let $f(x, y) = x^2+y^2-t$. Then

$$\begin{cases} \frac{\partial f}{\partial x} = 2x \\ \frac{\partial f}{\partial y} = 2y \\ f(y, z) = 0 \end{cases}$$

when $a \neq 0$, this system of equations doesn't have solutions and when over $a = 0$, X is just a point. We can see that f is not flat, hence f is not smooth. In fact, f is smooth at $\mathbb{A}^3_k - \{(0, 0, 0)\}$.

Example 3.10.8 (Kate)**.** Consider $X = V_+(\sum_{i=0}^n x_i y_i^q - x_i^q y_i) \subset \mathbb{P}^{2b+1}_{\mathbb{F}_p}$. For any $s \in H^0(\mathbb{P}^{2n+1}_{\mathbb{F}_p}, \mathcal{O}(1))$, $X \cap V_+(s)$ is singular.

Example 3.10.9 (Bertini fails when $p = \mathrm{char}(k) > 0$)**.** Consider $F : \mathbb{P}^1_k \longrightarrow \mathbb{P}^1_k$ given by $[x_0 : x_1] \longmapsto [x_0^p : x_1^p]$. All fibers are isomorphic to the non-reduced scheme $\mathrm{Spec}(k[x]/x^p)$. F is surjective but no smooth fiber. Now that F is induced by $\mathcal{O}(p)$.

Example 3.10.10 (Flatness doesn't implies smoothness)**.** Let $X \subset \mathbb{P}^2_k \times_k \mathbb{P}^1_k$ be the hypersurface defined by $y_0(x_0^3+x_1^3+x_2^3) - y_1 x_0 x_1 x_2$ with the two projections $p_1 : X \to \mathbb{P}^2_k$ and $p_2 : X \to \mathbb{P}^1_k$. Here, x_0, x_1, x_2 and y_0, y_1 are the coordinates on the first and second factor, respectively. For $y \in \mathbb{P}^1_k$, denote by X_y the fiber of y under the second projection p_2. Note that p_2 is flat but not smooth: Over k-points, we have $p_2 : ([x_0 : x_1 : x_2], [y_0 : y_1]) \mapsto [y_0 : y_1]$. In fact, for any $y \in \mathbb{P}^1_k$, X_y is a curve of degree 3 in $\mathbb{P}^2_{k(y)}$. Thus, the Hilbert polynomials are the same for all $y \in \mathbb{P}^1_k$.

However, p_2 is not smooth, since over $y = [0 : 1]$, $X_y = V(x_0 x_1 x_2) \in \mathbb{P}^2_k$ is not reduced hence not smooth.

Example 3.10.11 (Non-smooth morphism between smooth schemes)**.**

Example 3.10.12 (Smooth Morphism between non-smooth schemes)**.**

3.10.3 Exercises

Exercise 3.10.1. To check every local ring of X at a k-point is regular ring, it suffices to apply the Jacobian criteria, because we have the following result:

Claim 3.10.13. *Let $Y \subset \mathbb{A}^n_k$ be an affine variety and $P \in Y$ be a point, then*

$$\dim \mathfrak{m}_P/\mathfrak{m}_P^2 + \mathrm{rk} J = n$$

.

Proof. See [Theorem I.5.1 [5]] □

Recall that, to check the regularity of a scheme, it is enough to check over its closed points and for any $p \in X_{cl}$, $\dim \mathcal{O}_{X,p} = \dim X = 1$. Thus, to verify that X is regular, it is enough to show that $\mathrm{rk} J = 1$ for each $p \in X_{cl}$.

By the definition of X, we have

$$J_p = (2y, px^{p-1}) = (2y, 0)$$

When $\mathrm{char} k \neq 2$, $J_p = 0$ iff $y = 0$. However, since $k_0(t)$ is not perfect, there exists no p such that $y^2 = x^p - t = 0$. When $\mathrm{char} k = 2$, X is the solution of $(x+y)^2 = t$, which is empty.

To show that X is not smooth, we know that X is smooth if and only if $X_{\overline{k}}$ is smooth if and only if $X_{\overline{k}}$ is regular. However, when $\mathrm{char} k \neq 2$, over $\overline{k_0(t)}$, $x^p - t = 0$ does have solution, that is $X_{\overline{k_0(t)}}$ is not regular hence not smooth. When $\mathrm{char} k = 2$, $X = V((x+y+t)^2)$ which is not reduced, hence not regular and then not smooth.

Exercise 3.10.2. By $\Omega_{f^{-1}(U)/U} = \Omega_{X/Y}|_{f^{-1}(U)}$, by [Proposition III.10.4], we need to find U, such that over $f^{-1}(U)$, $\Omega_{X/Y}$ is locally free of $\dim X - \dim Y$. (Note that for U an open subset of Y, $\dim U = \dim Y$ 2.3.20.)

To begin with, we see that for any $x \in X$ and $y = f(x) \in Y$, we have

$$\Omega_{X/Y,x} \otimes_{\mathcal{O}_{X,x}} k(x) \twoheadrightarrow \Omega_{k(x)/k(y)}$$

deduced by $Spec(k(x)) \longrightarrow X_y$ and $X_y \longrightarrow Spec(k(y))$. Note that $\Omega_{X_y/k(y)} = \Omega_{X/Y}|_{X_y}$ [Proposition II.8.11 [5]].

Thus, we see that

$$\dim \Omega_{X/Y,x} \otimes_{\mathcal{O}_{X,x}} k(x) \geq \dim \Omega_{k(x)/k(y)} \geq tr.deg.(k(x)/k(y))$$

by [Theorem II.8.6A [5]]. Since $k(X_y) \subset k(x)$,

$$tr.deg.(k(x)/k(y)) \geq tr.deg(k(X_y)/k(x)) = \dim X_y = \dim X - \dim Y$$

with $f(x) = y$ by 2.3.20 and [Corollary III.9.6. [5]].

Now, we get an very important result:

$$\dim \Omega_{X/Y,x} \otimes k(x) \geq \dim X - \dim Y$$

Next, we consider the set

$$V = \{x \in X | \Omega_{X/Y,x} \text{ is free of rank } \dim X - \dim Y\}$$

This is an open set since "locally free" is an open condition by 2.5.7 and "$\dim \Omega_{X/Y,x} \otimes_{\mathcal{O}_{X,x}} k(x)$" is an open condition by 2.5.8. (We have shown $\dim \Omega_{X/Y,x} \otimes k(x) \geq \dim X - \dim Y$!)

Take $Z = X - V$ and consider $f(Z)$. Since f is flat, it is open and hence $f(Z)$ is closed over Y. Take $U = Y - Z$. Then

$$f^{-1}(U) \subset V = \{x \in X | \Omega_{X/Y,x} \text{ is free of rank } \dim X - \dim Y\}$$

is just the open set we need.

Exercise 3.10.3. *Proof.* (i) $\Longleftrightarrow$ (ii): "$\Longrightarrow$": Since f is étale, f is smooth of relative dimension 0, which means f is flat and $\Omega_{X/Y}$ is locally free of rank 0, that is, $\Omega_{X_y/k(y)} = 0$. By Cohomology and base change, $\Omega_{X/Y} = 0$.

"$\Longleftarrow$": f is flat and $\Omega_{X_y/k(y)} = \Omega_{X/Y}|_{X_y} = 0$. As we have shown that when f is flat

$$\dim \Omega_{X/Y,x} \otimes k(x) \geq \dim X - \dim Y$$

Hence, $\dim X = \dim Y$.

(ii) $\Longleftrightarrow$ (iii):

Lemma 3.10.14. *Assume that $f : X \longrightarrow Y$ is locally of finite type and flat. f is unramified if and only if $\Omega_{X/Y} = 0$.*

Proof. Note that if $x \in X_y \subset X$ then

$$\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y) = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_y\mathcal{O}_{Y,y} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$$

"$\Longrightarrow$": Since f is unramified $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$, $\mathcal{O}_{X_y,x} = k(x)$. $\Omega_{X/Y}|_{X_y} = \Omega_{X_y/k(y)}$. As we have seen in 3.10.14:

$$\Omega_{X_y/k(y),x} = \Omega_{\mathcal{O}_{X_y,x}/k(y)} = \Omega_{k(x)/k(y)} = 0$$

since $k(y) \subset k(x)$ is separable. Hence, $\Omega_{X_y/k(y)} = 0$. Moreover, $\Omega_{X/Y} = 0$ by Cohomology and Base Change.

"$\Longleftarrow$": Assume $\Omega_{X/Y} = 0$. $y = f(x)$. Without loss of generality, we assume that $X_y = Spec(A)$ with A of finite type over $k(y)$ and $\Omega_{A/k(y)} = \Omega_{X/Y}|_{X_y} = 0$.

Let $x = \mathfrak{p}$. We have morphisms $A_{\mathfrak{p}} \twoheadrightarrow k(\mathfrak{p})$ and $k(y) \rightarrow A_{\mathfrak{p}}$. Recall that we have an exact sequence

$$\mathfrak{p}A_{\mathfrak{p}}/(\mathfrak{p}A_{\mathfrak{p}})^2 \longrightarrow \Omega_{A_{\mathfrak{p}}/k(y)} \otimes_{A_{\mathfrak{p}}} k(x) \longrightarrow \Omega_{k(x)/k(y)} \longrightarrow 0$$

By [Proposition II.8.1.A [5]], $\Omega_{A_{\mathfrak{p}}/k(y)} = (\Omega_{A/k(y)})_{\mathfrak{p}}$ and then $\Omega_{A_p/k(y)} = 0$. $\Omega_{k(x)/k(y)} = 0$ which implies $k(y) \subset k(x)$ is separable. Moreover we have an short exact sequence **??**:

$$0 \longrightarrow \mathfrak{p}A_{\mathfrak{p}}/(\mathfrak{p}A_{\mathfrak{p}})^2 \longrightarrow \Omega_{A_{\mathfrak{p}}/k(y)} \otimes_{A_{\mathfrak{p}}} k(x) \longrightarrow \Omega_{k(x)/k(y)} \longrightarrow 0$$

Thus, $\mathfrak{p}A_{\mathfrak{p}} = (\mathfrak{p}A_{\mathfrak{p}})^2$. By Nakayama lemma, $\mathfrak{p}A_{\mathfrak{p}} = 0$, that is, $\mathcal{O}_{X_y,x}$ is a field. Again, by 3.10.14, $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$ □

Remark 3.10.15. We strengthen the result in [Theorem III.10.2 [5]]: Let $f : X \longrightarrow Y$ be a morphism of schemes of finite type over k, then f is smooth of relative dimension n if and only if

(i). f is flat; and

(ii). X_y is equidimensional of dimension n over $k(y)$.

Exercise 3.10.4.

Exercise 3.10.5. *Proof.* We know that $\mathcal{F}$ is locally free over X if and only if $\mathcal{F}_x$ is free $\mathcal{O}_{X,x}$-module for every $x \in X$. Since $\mathcal{O}_{U,u}$ is a flat $\mathcal{O}_{X,x}$-module and $f^*\mathcal{F}_u = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{U,u} = \mathcal{O}_{U,u}^{\oplus n}$ with $x = f(u)$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module:

Lemma 3.10.16. *Let $f : (A, \mathfrak{a}) \longrightarrow B$ be a ring homomorphism of finite type and B is flat over A, then B is faithfully flat over A.*

Proof. We prove that for $N \neq 0$ a $B \otimes_A N \neq 0$. Since $M \neq 0$, we can take $m \neq 0$ and $m \in M$. Consider $A \cdot m = A/I \subset N$, then it is enough to show that $A/I \otimes_A B = B/IB \neq 0$.

Assume that $B/IB = 0$, then $B = IB \subset \mathfrak{a}B \subset B$. By Nakayama lemma, $B = 0$, which leads to a contradiction. □

Lemma 3.10.17. *Let M be faithfully flat A-module. Then $M \otimes_A N$ is flat, then N is also flat.*

Proof. Let $f : L \hookrightarrow L^{'}$ be an injection and consider $ker(f \otimes N)$. We have $L \otimes N \otimes M \hookrightarrow L^{'} \otimes N \otimes M$, that is, $ker(f \otimes N) \otimes M = 0$. However, since M is faithfully flat, $ker(f \otimes N) = 0$. Hence, N is flat. □

By 3.10.16, $\mathcal{O}_{U,u}$ is a faithfully flat $\mathcal{O}_{X,x}$-mod. So is $\mathcal{O}_{U,u}^{\oplus n}$, which is flat. By 3.10.17, $\mathcal{F}_x$ is flat. Now, since $\mathcal{F}_x$ is flat over $\mathcal{O}_{X,x}$, it is free by [Proposition III.9.1A.] □

Remark 3.10.18. It is enough to assume that $f : X \longrightarrow Y$ is flat? (of finite type.)

Exercise 3.10.6. Let Y be the plane nodal cubic curve $y^2 = x^2(x + 1)$. Show that Y has a finite étale covering X of degree 2, where X is a union of two irreducible components, each one isomorphic to the normalization of Y

Proof. Here is a geometric way to tell whether or not a morphism is unramified:

Lemma 3.10.19. *Let X and Y be two schemes and $f : X \longrightarrow Y$ be of finite type. If at each point x, $df_x : \mathcal{T}_{X,x} \longrightarrow \mathcal{T}_{Y,y}$ is injective, then f is unramified.*

Proof. Recall that $\mathcal{T}_X = \mathscr{H}om_X(\Omega_{X/k}, \mathcal{O}_X)$. Hence,

$$df_{X,x} \text{ is injective if and only if } \Omega_{X,x} \otimes k(x) \longrightarrow \Omega_{Y,f(x)} \otimes k(x) \text{ is surjective}$$

So $\Omega_{X/Y,x} \otimes k(x) = 0$ by the short exact sequence

$$f^*\Omega_{Y/k} \longrightarrow \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Since f is of finite type, $\Omega_{X/Y}$ is a finitely generated map. By Nakayama lemma, $\Omega_{X/Y,x} = 0$ for all $x \in X$. Thus, $\Omega_{X/Y} = 0$ and f is unramified. $\square$

Remark. When X is a scheme over a Jacobian ring, it is enough to verify df_x is injective at each closed point. In fact, for any coherent sheaf $\mathcal{M}$, $\mathrm{Supp}(\mathcal{M})$ is a closed subset. For X is a scheme over Jacobian ring, each closed subset must contains at least one closed points. Hence, if for all closed points x, $\mathcal{M}_x \neq 0$, $\mathrm{Supp}(\mathcal{M}) = \emptyset$.

Let $\widetilde{Y}$ be the normalization of Y. Then at each point $\widetilde{y} \in \widetilde{Y}$, $\pi_{\widetilde{y}}$ is an injection (by the picture). Hence, $\widetilde{\pi}$ is unramified. Thus, $\widetilde{Y} \cup \widetilde{Y} \longrightarrow Y$ is unramified.

Note that the fiber of the map at each closed points contains two points. The Hilbert polynomial is constant, which implies such a morphism is flat. Thus, it is étale.

$\square$

Remark. Also, to check whether $f : X \longrightarrow Y$ is flat, it is enough to check Hilbert polynomials over closed points of Y when Y is Noetherian and Jacobson. (Why?)

Exercise 3.10.7 (A linear system with moving singularities.)**.** Let k be an algebraically closed field of characteristic 2. Let $P_1, \ldots, P_7 \in \mathbb{P}^2_k$ be the seven points of the projective plane over the prime field $\mathbb{F}_2 \subseteq k$. Let $\mathfrak{d}$ be the linear system of all cubic curves in X passing through $P_1, \ldots, P_7$.

(a) $\mathfrak{d}$ is a linear system of dimension 2 with base points $P_1, \ldots, P_7$, which determines an inseparable morphism of degree 2 from $X - \{P_i\}$ to $\mathbb{P}^2$.

(b) Every curve $C \in \mathfrak{d}$ is singular. More precisely, either C consists of 3 lines all passing through one of the P_i, or C is an irreducible cuspidal cubic with cusp $P \neq$ any P_i. Furthermore, the correspondence $C \mapsto$ the singular point of C is a 1-1 correspondence between $\mathfrak{d}$ and $\mathbb{P}^2$. Thus the singular points of elements of $\mathfrak{d}$ move all over.

Proof. First of all, $\{P_i\}$ is just the set $\{[0:0:1],[0:1:1],[1:1:1],[0:1:0],[1:1:0],[1:0:0],[1:0:1]\}$.

(a). Suppose that $C = \{f(x,y,z) = \sum a_{ijk}x^iy^jz^k = 0\} \in \mathfrak{d}$. $\{[0:0:1],[0:1:0],[1:0:0]\} \subset C$ implies

$$a_{001} = a_{010} = a_{100} = 0$$

By $\{[1:0:1],[1:1:0],[1:1:0]\} \subset C$,

$$a_{201} + a_{102} = 0, a_{012} + a_{021} = 0, a_{120} + a_{210} = 0$$

Finally, $[1:1:1] \in C$, $a_{111} = 0$. Hence,

$$f = axy(x-y) + byz(y-z) + cxz(x-z) = axy(x+y) + byz(y+z) + cxz(x+z)$$

where the second equality holds up for $\mathrm{char}(k) = 2$. So $\mathfrak{d} =< xy(x+y), yz(y+z), xz(x+z) >$ and $\dim \mathfrak{d} = 2$. Since the base locus of $\mathfrak{d}$ is $\{P_i\}$, $\mathfrak{d}$ gives a morphism

$$\pi : X - \{P_i\} \longrightarrow \mathbb{P}^2$$

$$[x:y:z] \longmapsto [xy(x+y):yz(y+z):xz(x+z)]$$

Over $\pi^{-1}(D_+(x_0))$, the morphism is given by

$$\pi^{'}:\{x,z\in\mathbb{A}_k^2\big|x\neq 0,1\}\longrightarrow\mathbb{A}^2$$
$$(x,z)\longmapsto(\frac{z(1+z)}{x(x+1)},\frac{z(x+z)}{x+1})$$

By directly computation, $\pi^{'}$ is not separable since $\Delta_{\pi'}$ is not a closed embedding ($\{x,z\in\mathbb{A}_k^2\big|x\neq 0,1\}$ is open).

(b). For any f of the from $axy(x+y)+byz(y+z)+cxz(x+z)$ with $a,b,c\in k$, we have

$$\frac{\partial f}{\partial x}=ay^2+cz^2,\frac{\partial f}{\partial y}=ax^2+bz^2,\frac{\partial f}{\partial z}=by^2+cx^2,$$

Over $\mathbb{P}^2$, this system of equations always has solution, which implies this curve is singular. Note that the singular point of C is $[\sqrt{b}:\sqrt{c}:\sqrt{a}]$, which is a 1-1 correspondence between $\mathfrak{d}$ and $\mathbb{P}^2$. □

Exercise 3.10.8 (A linear system with moving singularities contained in the base locus (any characteristic).)**.** In affine 3-space with coordinates x,y,z, let C be the conic $(x-1)^2+y^2=1$ in the xy-plane, and let P be the point $(0,0,t)$ on the z-axis. Let Y_t be the closure in $\mathbb{P}^3$ of the cone over C with vertex P. Show that as t varies, the surfaces $\{Y_t\}$ form a linear system of dimension 1, with a moving singularity at P. The base locus of this linear system is the conic C plus the z-axis.

Proof. First of all recall the definition of cone over C with P in A^3 is

$$\{(x-1+\frac{z}{t})^2+y^2=(1-\frac{z}{t})^2\}$$

Then $Y_t\subset\mathbb{P}^3$ is defined to be

$$\{(tx-tw+z)^2+t^2y^2-(tw-z)^2=0\}=\{t^2x^2-2tx(tw-z)+t^2y^2=0\}$$

Let $f(x,y,z,w)=tx^2-2txw+2xz+ty^2$. Then

$$\frac{\partial f}{\partial x}=2(tx-tw+z),\frac{\partial f}{\partial y}=2ty,\frac{\partial f}{\partial z}=2tx,\frac{\partial f}{\partial w}=2(tx-tw+z)$$

One of the singular points of Y_y is $[0:0:t:1]$. Thus, Y_t moves singularly over $\mathbb{P}^3$. It is easy to see that the base locus of Y_t is

$$\{xz=0\}\cap\{x^2-2xw+y^2=0\}=\{x=y=0\}\cap\{x^2-2xw+y^2=0\}$$

□

Exercise 3.10.9. Let $f:X\to Y$ be a morphism of varieties over k. Assume that Y is regular, X is Cohen–Macaulay, and that every fibre of f has dimension equal to $\dim X-\dim Y$. Then f is flat. [Hint: Imitate the proof of (10.4), using (II, 8.21A).]

Proof. First of all, let's explain why it is enough to show that over all closed points f is flat: We have show that $U=\{x\in X\big|f \text{ is flat at } x\}$ is open. Since X is Jacobson, if U contains all closed points, then $U=X$.

Also, note that since both X and Y are Jacobson, closed points in X are mapped to closed points in Y. See [Lemma 29.16.9. of Stack Project.] From now on, we denote $y=f(x)$.

Now, we start by induction on the dimension of Y. If $\dim Y=0$, then $\mathcal{O}_{Y,y}=\mathrm{Spec}(k)$. For $\mathcal{O}_{X,x}$ is a finite generated k-algebra. It is flat over k.

Suppose that $\dim Y = n$. Since Y is regular, then $\mathfrak{m}_{Y,y}$ is generated by $\alpha_1, ..., \alpha_n$. Under the base change, we see that

$$\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_y = \mathcal{O}_{X,x}/(\alpha_1, ..., \alpha_n)\mathcal{O}_{X,x}$$

and $\dim \mathcal{O}_{X_y,x} = \dim \mathcal{O}_{X,x} - n$. Hence, $\alpha_1, .., \alpha_n$ forms a regular sequence in $\mathcal{O}_{X,x}$ since X is Cohen-Macaulay and by [[5] Theorem II.8.21A (c)]. Hence α_1 is not a zero divisor of $\mathcal{O}_{X,x}$.

By [[5] Lemma III.10.3A], it is enough to show that $\mathcal{O}_{X,x}/\alpha_1\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}/\alpha_1\mathcal{O}_{Y,y}$ for α_1 is not a unit (lying the maximal ideal) and α_1 is not a zero divisor of $\mathcal{O}_{X,x}$: Note that

(1). $\dim \mathcal{O}_{Y,y}/(\alpha_1) = \dim \mathcal{O}_{Y,y} - 1$ and it is still regular;

(2). $\dim \mathcal{O}_{X,x}/\alpha_1\mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x} - 1$, it is Cohen-Macaulay by [[5] Theorem 8.21A (d)] (α_1 is a regular sequence) and it is the pull of $\mathcal{O}_{Y,y}/(\alpha_1)$ related to f.

By induction hypothesis, $\mathcal{O}_{X,x}/\alpha_1\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}/\alpha_1\mathcal{O}_{Y,y}$. □

3.10.4 Unramified and Étale Morphisms

We have already shown:

Theorem 3.10.20. *Let $f : X \longrightarrow Y$ be a morphism of finite type. Then the following conditions are equivalent:*

(1). f is étale;

(2). f is flat and $\Omega_{X/Y} = 0$;

(3). f is flat and unramified.

Proposition 3.10.21. *Let $f : X \longrightarrow Y$ be a morphism of finite type. Then*

(1). If f is smooth, then $T_{X,x} \twoheadrightarrow T_{Y,f(x)}$ for any $x \in X$;

(2). If f is unramified, then $T_{X,x} \hookrightarrow T_{Y,f(x)}$ for any $x \in X$;

(3). If f is étale, then $T_{X,x} \cong T_{Y,f(x)}$ for any $x \in X$.

Let $Z = \mathrm{Spec}(A)$ and $Z_0 = \mathrm{Spec}(A/I)$ with I nilpotent, then we define the following notations:

Definition 3.10.22. A morphism $f : X \longrightarrow Y$ is formally smooth/unramified/étale) if for any

$$\begin{tikzcd} Z_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ Z \arrow[r] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

there exists at least one/at most one/ exactly one g making the diagram commutative.

We can prove that

Theorem 3.10.23. *If f is locally of finite type, then f is smooth/unramifed/étale if and only if f is formally smooth/smooth/unramifed/étale.*

Corollary 3.10.24. *Let $f : X \longrightarrow Y$ be of finite type. Then*

(1). f is smooth if and only if $T_{X,x} \twoheadrightarrow T_{Y,f(x)}$ for any $x \in X$;

(2). f is unramified if and only if $T_{X,x} \hookrightarrow T_{Y,f(x)}$ for any $x \in X$;

(3).f is étale if and only if $T_{X,x} \cong T_{Y,f(x)}$ for any $x \in X$.

Exercise 3.10.10. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms of finite type such that $g \circ f$ is étale and g is unramified. Then f is étale.

Proof. At this case we identify formally unrmaified/étale with unramified/étale. Consider the diagram.

$$\begin{tikzcd}
Z_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
T \arrow[ur, dashed, "H"] \arrow[r, "h"] \arrow[dr, "g\circ h"'] & Y \arrow[d, "g"] \\
 & Z
\end{tikzcd}$$

Since $g \circ f$ is étale there exists unique H such that $g \circ f \circ H = g \circ h$. Since g is unramified, there exists at morst one $h^{'} : T \longrightarrow Y$ such that $g \circ h^{'} = g \circ h$, that is h. Hence, $f \circ H = h$, which implies f is étale. $\square$

Example 3.10.25. (1). Consider $\mathbb{A}^1_k\backslash\{0\} \longrightarrow \mathbb{A}^1_k\ \{0\}$. At x we can see that

$$\begin{aligned} d\varphi_x : T_x &\longrightarrow T_{f(x)} \\ v &\longmapsto 2xv \end{aligned}$$

At $x \neq 0$, this is an isomorphism. Hence, this map is étale.

(2). Consider $X = V(x^3 - y^2) \longrightarrow \operatorname{Spec}(k[x, y]/(x^3 - y^2))$. Note that $\widetilde{X} = \mathbb{A}^1_k$ and $\pi : \widetilde{X} \longrightarrow X$ is given by $x \longrightarrow t^2, y \longmapsto t^3$. At x we can see that

$$\begin{aligned} d\varphi_a : T_a &\longrightarrow T_{f(a)} \\ v &\longmapsto \begin{pmatrix} 2a \\ 3a^2 \end{pmatrix} v \end{aligned}$$

which is 0 at $x = 0$. Hence, f is not unramified. It can not be étale.

(3). Consider $X = V(y^2 - x^2(x + 1)) \subset \mathbb{A}^2_k$. Note that $\widetilde{X} = \mathbb{A}^1_k$ and $\pi : \widetilde{X} \longrightarrow X$ is given by $x \mapsto t^2 - 1$ and $y \mapsto t(t^2 - 1)$. Then

$$\begin{aligned} d\varphi_a : T_x &\longrightarrow T_{f(a)} \\ v &\longmapsto \begin{pmatrix} 2a \\ 3a^2 - 1 \end{pmatrix} v \end{aligned}$$

which is an injective at every point. Hence, it is unramified. However, at $t = 1$ and $t = -1$, $d\varphi_a$ is not surjective. Hence, it can not be étale.

Example 3.10.26 (Directly compute.)**.** For $X = V(y^2 - x^2(x + 1)) \subset \mathbb{A}^2_k$ and $\pi : \widetilde{X} \longrightarrow X$ given by $x \mapsto t^2 - 1$ and $y \mapsto t(t^2 - 1)$, we direcly prove that it is unramified:

Since the singular point of X is $(0, 0)$ and $\pi^{-1}(0, 0) = \pm 1$, over $a \neq \pm 1$, π is an isomprhism by the property of blow up. Over $x = 1$, note that the local ring map is given by

$$(k[x, y]/(y^2 - x(x^2 + 1)))_{(x-1,y-1)} \longrightarrow k[t]_{(t-1)}$$

Note that $(x, y)k[t]_{(t-1)} = (t^2 - 1, t(t^2 - 1))k[t]_{(t-1)} = (t - 1)k[t]_{(t-1)}$ since $t + 1$ and t are invertible in $k[t]_{(t-1)}$. Similarly at $t = -1$. Hence, we see that π is unramified.

Example 3.10.27. Let $\mathcal{L}$ be a line bundle over X such that $\mathcal{L}^n = \mathcal{O}_X$. Then define $\mathcal{A} = \oplus_{i=1}^{n-1} \mathcal{L}^i$. Then $\operatorname{Spec}(\mathcal{A}) \longrightarrow X$ is a étale cover of degree n. To prove this, directly check at a cover of A, such that $\mathcal{L}$ is trivial at each open set.

If you know some things about Étale Toplogy, here are some general results:

Example 3.10.28. Let $f : \mathcal{C} \longrightarrow |\mathcal{O}(2)|_{sm}$ be the universal family of smooth conics in $\mathbb{P}^2_k$ with k algebraically closed. Then each fiber is isomorphic to $\mathbb{P}^1_k$. Then f is locally trivial in the étale topology but not in the Zariski topology.

Example 3.10.29. Every étale double cover of $\mathbb{P}^n_k$ is trivial, that is, it is isomorphic to the union of two disadjoint copies of $\mathbb{P}^n_k$.

3.11 The Theorem of Formal Functions

3.11.1 Preparation

Theorem 3.11.1 (Theorem of Formal Functions)**.** *Let $f : X \longrightarrow Y$ be a proper morphism of local Noetherian schemes. For any coherent sheaf $\mathcal{F}$ over X and $y \in Y$,*

$$(R^i f_* \mathcal{F})\hat{}_y = \varprojlim H^i(X_n, \mathcal{F}|_{X_n})$$

where X_n is given by the pullback

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(\mathcal{O}_y/\mathfrak{m}_y^n) \arrow[r] & Y \end{tikzcd}$$

Definition 3.11.2. Let $f : X \longrightarrow Y$ be a proper morphism between local Noetherian schemes. f is $\mathcal{O}$-connected if $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Theorem 3.11.3 (Zariski Connectness)**.** *If f is $\mathcal{O}$-connected, the fibers of f are connected.*

Theorem 3.11.4 (Stein Factorization)**.** *Let $f : X \longrightarrow Y$ be a proper morphisms between local Noetherian schemes. Then there exists a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & Y' \arrow[dl, "\pi"] \\ Y & \end{tikzcd}$$

such that

(1). g is $\mathcal{O}$-connected and proper;

(2). π is finite.

Remark. In this theorem, $Y' = \mathcal{S}pec(f_*\mathcal{O}_X)$ and $\pi : Y' \longrightarrow Y$ as 2.5.17.

Definition 3.11.5. Let $f : X \longrightarrow Y$ be a proper morphism between local Noetherian schemes. We say $x \in X$ is isolated in its fiber if for $y = f(x)$, $\{x\} \subset f^{-1}(y)$ is an irreducible component.

Theorem 3.11.6 (Grothendieck's Form at ZMT)**.** *Let $f : X \longrightarrow Y$ be a proper morphism between locally Noetherian schemes. Then there exists a diagram*

$$\begin{tikzcd} & X \arrow[dr, "g"] \arrow[dd, "f"] & \\ X_0 \arrow[ur, "\subseteq"] \arrow[rr, "i"] \arrow[dr, "f|_{X_0}"'] & & Y' \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

such that

(1).$X_0 \subseteq X$ is open and it is the subset of points isolated in its fiber;

(2). $f|_{X_0}$ factors on $\pi \circ i$ where i is an open embedding and π is finite;

(3). $f = \pi \circ g$.

Proof. See [12]. □

3.11.2 Examples

Example 3.11.7. We can show that for $f : X \longrightarrow Y$ such that $f_*\mathcal{O}_Y \cong \mathcal{O}_X$ is stable under flat base change. However, the property "f has connected fibers" is not stable under flat base change. Consider

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{C} \oplus \mathbb{C}) & \longrightarrow & \mathbb{S}_{\mathbb{I}}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{R}) \end{array}$$

since $\mathrm{Spec}(\mathbb{C}) \longrightarrow \mathrm{Spec}(\mathbb{R})$ is not proper.

3.11.3 Exercises

Exercise 3.11.1. Show that the result of (11.2) is false without the projective hypothesis. For example, let $X = \mathbb{A}^n_k$, let $P = (0, \dots, 0)$, let $U = X - P$, and let $f : U \to X$ be the inclusion. Then the fibres of f all have dimension 0, but $R^{n-1}f_*\mathcal{O}_U \neq 0$

Proof. By [[5] Proposition 8.5.],

$$R^{n-1}f_*\mathcal{F} = H^{n-1}(U, \mathcal{O}_U)^{\sim}$$

since X is affine. To compute this, use Céch cohomolog: Note that we have an affine cover for U, $\mathscr{U} = \{D(x_i)\}_{i=1}^n$ and

$$\mathcal{C}^{n-1}(\mathscr{U}, \mathcal{O}_U) = H^0(D(x_1) \cap D(x_2) \cap ... \cap D(x_n), \mathcal{O}) = H^0(D(x_1...x_n), \mathcal{O}) = k[x_1, ..., x_n]_{x_1...x_n}$$

$$\mathcal{C}^{n-2}(\mathscr{U}, \mathcal{O}_U) = \prod_{i=1}^n H^0(D(x_1...\hat{x}_i...x_n), \mathcal{O}_U)$$

Since $d_{n-2} : \mathcal{C}^{n-2}(\mathscr{U}, \mathcal{O}_U) \longrightarrow \mathcal{C}^{n-1}(\mathscr{U}, \mathcal{O}_U)$ is not surjective (no element in $\mathcal{C}^{n-1}(\mathscr{U}, \mathcal{O}_U)$ can be mapped to $1/x_1...x_n$), thus $H^{n-1}(U, \mathcal{O}_U) \neq 0$. And $R^{n-1}f_*\mathcal{F} \neq 0$. □

Exercise 3.11.2. Show that a projective morphism with finite fibres (= quasi-finite (II, Ex. 3.5)) is a finite morphism.

Proof. Let $f : X \longrightarrow Y$ be projective. Then by Stein's Factorization, there exists

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & Y' \arrow[dl, "\pi"] \\ Y & \end{tikzcd}$$

Since f and π are both quasi-finite, g is quasi-finite. Since g is $\mathcal{O}$-connected, by Zariski Main Theorem, g's fibres are connected. Hence each fibres of g have only 1 point. By Grothendieck's form on ZMT, $X_0 = X$. Hence, $f : X \longrightarrow Y$ is finite. □

Remark. To summit, we have the following results:

Proposition 3.11.8. *Let $f : X \longrightarrow Y$ be a morphism between two locally Noetherian schemes. The following are equivalent:*

(1). f is finite;

(2). f is affine and proper;

(3). f is quasi-finite and proper.

Exercise 3.11.3. Let X be a normal, projective variety over an algebraically closed field k. Let $\mathfrak{d}$ be a linear system (of effective Cartier divisors) without base points, and assume that $\mathfrak{d}$ is not composite with a pencil, which means that if $f : X \to \mathbb{P}^n_k$ is the morphism determined by $\mathfrak{d}$, then $\dim f(X) \geq 2$. Then show that every divisor in $\mathfrak{d}$ is connected. This improves Bertini's theorem (10.9.1). [Hints: Use (11.5), (Ex. 5.7) and (7.9).]

Proof. Before stating proving this result, we need X to be integral. So that we can apply [[5] Corollary III.7.9.]

First of all, we assume that $f : X \longrightarrow \mathbb{P}^n_k$ is finite. For any $D = \{s = 0\}$ in $\mathfrak{d}$, it is a pullback of some hyperplane H in $\mathbb{P}^n_k$. Hence, $D = \{s = 0\}$ is of codimension 1. Now, by [[5] Corollary III.7.9.], it is enough the line bundle associated to D is ample, that is, $f^*\mathcal{O}(1)$ is ample. Since X is proper ($f : X \longrightarrow \mathbb{P}^n_k$ is finite hence ample), by 3.5.10 (d), $f^*\mathcal{O}(1)$ is ample.

Now, we consider the general cases. Consider the Stein's factorization:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & Y' \arrow[dl, "\pi"] \\ \mathbb{P}^n_k & \end{tikzcd}$$

Since g is surjective and X is irreducible, Y' is also irreducible. By the universal property of Y'_{red}, g can be factorized as

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "g'"] & Y' \\ Y'_{red} \arrow[ur, hook] & \end{tikzcd}$$

Note that Y' and Y'_{red} have the same topology and $Y'_{red} \hookrightarrow Y'$ is a closed immersion. Hence, we can assume that in the Stein's factorization, Y' is reduced, for which it is integral.

Finally, since X is normal, we can show that Y' is normal (**Wait!**). Then by our discussion before, we see that if $D = f^*H$, π^*H is connected. Since g has connected fiber, $D = g^*(\pi^*H)$ is connected. □

Exercise 3.11.4 (Principle of Connectedness.)**.** Let $\{X_t\}$ be a flat family of closed subschemes of $\mathbb{P}^n_k$ parametrized by an irreducible curve T of finite type over k. Suppose there is a nonempty open set $U \subseteq T$, such that for all closed points $t \in U$, X_t is connected. Then prove that X_t is connected for all $t \in T$.

Proof. **Warning: Need Base Change and cohomology**:

Lemma 3.11.9. *Let $f : X \longrightarrow Y$ be a projective, flat morphism with Y Noetherian. If f has geometrically connected fiber, then f is $\mathcal{O}$-connected.*

Proof. Since f has connected fiber, $t \longmapsto h^0(X_y, \mathcal{O}_{X_t}) = 1$ is constant over T. Hence, $f_*\mathcal{O}_X$ is locally free of rank 1 by 3.12.3. Then note that $H^0(Y, f_*\mathcal{O}_X) = H^0(X, \mathcal{O}_X)$ and

$$H^0(Y, (f_*\mathcal{O}_X)^*) = \mathrm{Hom}_Y(f_*\mathcal{O}_X, \mathcal{O}_Y) = \mathrm{Hom}(\mathcal{O}_X, f^*\mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$$

We see that $f_*\mathcal{O}_X = \mathcal{O}_Y$. □

For this exercise, we assume that T is a nonsingular curve. Since $f : X \longrightarrow T$ is flat, it is open. Hence, we have

$$f_*\mathcal{O}_{T,t} \hookrightarrow \prod_{x \in X_t} \mathcal{O}_{X,x}$$

Thus, $f_*\mathcal{O}_X$ is torsion free over T so it is locally free for $\mathcal{O}_{T,t}$ is a PID for any closed $t \in T$.

Restrict f over $f^{-1}(U)$, we see that $f_*\mathcal{O}_X$ is locally free of rank 1. Again using the medthod in the proof the lemma, we can show that

$$H^0(X, f_*\mathcal{O}_X) \neq 0;\ H^0(X, (f_*\mathcal{O}_X)^*) \neq 0$$

which implies $f_*\mathcal{O}_X = \mathcal{O}_T$. Hence $f^{-1}(t)$ is connected for all $t \in T$.

If T is not a nonsingular curve, things will be more complicated (**Wait!**). □

Remark. Here is a useful lemma to tell whether $\mathcal{L}$ over X is trivial or not over a curve.

Lemma 3.11.10. *Let $\mathcal{L}$ be a line bundle over X. Then $\mathcal{L}$ is trivial if and only if $H^0(X, \mathcal{L}) \neq 0$ and $H^0(X, \mathcal{L}^*) \neq 0$.*

Proof. ($\Longrightarrow$): Trivial.

($\Longleftarrow$): Since $H^0(X, \mathcal{L}) \neq 0$ and $H^0(X, \mathcal{L}^*) \neq 0$, $\deg(\mathcal{L}) = 0$. By [[**"cite −hartshorne1977"**] Lemma IV.1.2.], $\mathcal{L}$ is trivial. □

Remark. When f is flat,

$$(f_*\mathcal{O}_X)_y \hookrightarrow \prod_{x \in X_y} \mathcal{O}_{X,x}$$

is an injection. Since

$$(f_*\mathcal{O}_X)_y = \varinjlim \Gamma(f^{-1}(U), \mathcal{O}_X)$$

with U running through all open subset containing y. Suppose $s \in \Gamma(f^{-1}(U), \mathcal{O}_X)$ satisfies $s_x = 0$ for all $x \in X_y$. Then s vanishing in an open subset of X_y, denoted by U. Since f is open, $f(U)$ is open containing y and $U \subset f^{-1}(f(U))$.

Exercise 3.11.5. Let Y be a hypersurface in $X = \mathbb{P}^N_k$ with $N \geq 4$. Let $\hat{X}$ be the formal completion of X along Y (II, §9). Prove that the natural map $\operatorname{Pic}\hat{X} \to \operatorname{Pic} Y$ is an isomorphism. *Hint: Use (II, Ex. 9.6), and then study the maps* $\operatorname{Pic} X_{n+1} \to \operatorname{Pic} X_n$ *for each n using (Ex. 4.6) and (Ex. 5.5).*

Proof. □

Exercise 3.11.6. Again let Y be a hypersurface in $X = \mathbb{P}^N_k$, this time with $N \geq 2$.

(a) If $\mathcal{F}$ is a locally free sheaf on X, show that the natural map

$$H^0(X, \mathcal{F}) \to H^0(\hat{X}, \hat{\mathcal{F}})$$

is an isomorphism.

(b) Show that the following conditions are equivalent:

(i) for each locally free sheaf $\mathfrak{F}$ on $\hat{X}$, there exists a coherent sheaf $\mathcal{F}$ on X such that $\mathfrak{F} \cong \hat{\mathcal{F}}$ (i.e., $\mathfrak{F}$ is algebraizable);

(ii) for each locally free sheaf $\mathfrak{F}$ on $\hat{X}$, there is an integer n_0 such that $\mathfrak{F}(n)$ is generated by global sections for all $n \geq n_0$.

Hint: For (ii) ⇒ (i), show that one can find sheaves $\mathcal{E}_0, \mathcal{E}_1$ on X, which are direct sums of sheaves of the form $\mathcal{O}(-q_i)$, and an exact sequence $\hat{\mathcal{E}}_1 \to \hat{\mathcal{E}}_0 \to \mathfrak{F} \to 0$ on $\hat{X}$. Then apply (a) to the sheaf $\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_0)$.

(c) Show that the conditions (i) and (ii) of (b) imply that the natural map $\operatorname{Pic} X \to \operatorname{Pic}\hat{X}$ is an isomorphism.

Note. In fact, (i) and (ii) always hold if $N \geq 3$. This fact, coupled with (Ex. 11.5) leads to Grothendieck's proof [SGA 2] of the Lefschetz theorem which says that if Y is a hypersurface in $\mathbb{P}_k^N$ with $N \geq 4$, then $\operatorname{Pic} Y \cong \mathbb{Z}$, and it is generated by $\mathcal{O}_Y(1)$. See Hartshorne [5, Ch. IV] for more details.

Exercise 3.11.7. Now let Y be a curve in $X = \mathbb{P}_k^2$.

(a) Use the method of (Ex. 11.5) to show that $\operatorname{Pic} \hat{X} \to \operatorname{Pic} Y$ is surjective, and its kernel is an infinite-dimensional vector space over k.

(b) Conclude that there is an invertible sheaf $\mathfrak{L}$ on $\hat{X}$ which is not algebraizable.

(c) Conclude also that there is a locally free sheaf $\mathfrak{F}$ on $\hat{X}$ so that no twist $\mathfrak{F}(n)$ is generated by global sections. Cf. (II, 9.9.1).

Exercise 3.11.8. Let $f : X \to Y$ be a projective morphism, let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y, and assume that $H^i(X_y, \mathcal{F}_y) = 0$ for some i and some $y \in Y$. Then show that $R^i f_*(\mathcal{F})$ is 0 in a neighborhood of y.

Proof. Directly use Base Change and Cohomology 3.12.3. □

3.11.4 Additional Exercises

Exercise 3.11.9 (Stein factorization and reduced fibers)**.** Let X be a normal variety over k, and C an integral normal curve over k. Let $f : X \to C$ be a proper surjective morphism, and let $\pi : C' \to C$ be the finite morphism in the Stein factorization

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & C' \arrow[dl, "\pi"] \\ C & \end{tikzcd}$$

of f.

(i) Show that C' is normal, and deduce that π is flat.

(ii) Let $c' \in C'$ be a ramification point of π, and $c = \pi(c')$. Show that the fiber $f^{-1}(c)$ is geometrically non-reduced (i.e., the base change of $f^{-1}(c)$ to the algebraic closure $\kappa(c)$ of the residue field of c is non-reduced). In particular, deduce that π is étale as soon as all the fibers of f are geometrically reduced.

(iii) Show that, if $C = \mathbb{P}_k^1$ and all fibers of f are geometrically reduced, then $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}$. In particular the fibers of f are connected.

Proof.
Wait!
□

Exercise 3.11.10 (Rigid I)**.** Let $f : X \to Y$ and $g : X \to Z$ be projective morphisms of varieties (i.e. integral schemes of finite type over a field) with $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ and such that g contracts each fibre of f (i.e. $g(f^{-1}(y))$ is a point for each $y \in Y$). Show that there exists a morphism $h : Y \to Z$ with $h \circ f = g$, that is,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \\ Z \arrow[ur, dashed, "h"] & \end{tikzcd}$$

Hint: Study the image of the morphism $(f,g) : X \to Y \times Z$.

Proof. Let Γ be the image of $(f,g) \in Y \times_k Z$ and p_Y, p_Z be the corresponding projections.

$$\begin{tikzcd}
X \arrow[dr, "f"] \arrow[drr, "g"] \arrow[ddr] & & \\
& \Gamma \arrow[r, "p_Z"] \arrow[d, "p_Y"] & Z \\
& Y &
\end{tikzcd}$$

We want to show that p_Y is an isomorphism hence $h = p_Z \circ p_Y^{-1}$ is what we want.

First of all, since f is surjective and for any $y \in Y$, $p_Y^{-1}(y) = (f,g)(f^{-1}(y)) = \{(f(y), g(f^{-1}(y))) \in \Gamma\}$, which implies p_Y is quasi-finite. Since X is separated and f, g are proper, (f,g) is proper. Hence, Γ is closed in $Y \times_k Z$ and so $p_Y : \Gamma \longrightarrow Y$ is proper. By 3.11.2, we see that p_Y is finite.

Next, since p_Y is affine, let $U := Spec(A) \subset Y$ and consider $p_Y^{-1}(Spec(A)) = Spec(B) \subset \Gamma$, we have

$$H^0(U, f_*\mathcal{O}_X) = H^0(f^{-1}(U), \mathcal{O}_X);\ H^0(V, (f,g)_*\mathcal{O}_X) = H^0((f,g)^{-1}V, \mathcal{O}_X);$$

Let $\mathcal{F} := (f,g)_*\mathcal{O}_X$. We have $\mathcal{F}(V) = \mathcal{O}_\Gamma(V) = A$. Now, we can verify that $\mathcal{F}$ and $\mathcal{O}_\Gamma$ has the same gluing functions, which implies $\mathcal{F} = \mathcal{O}_\Gamma$. Moreover, by n$p_{Y*}\mathcal{O}_\Gamma = \mathcal{O}_Y$ and finiteness, p_Y is an isomorphism. □

Exercise 3.11.11 (Rigid II)**.** Let X, Y, and Z be varieties over an algebraically closed field k, and let X be proper over k. Let $f : X \times Y \to Z$ be a morphism. Assume that there exists a closed point $y_0 \in Y$, such that $f(X \times \{y_0\})$ is a single point in Z. Prove that there exists $g : Y \to Z$, such that $f = g \circ \pi_Y$ where $\pi_Y : X \times Y \to Y$ is the projection, that is,

$$\begin{tikzcd}
X \times_k Y \arrow[r, "f"] \arrow[d, "\pi_Y"] & Z \\
Y \arrow[ur, dashed, "g"] &
\end{tikzcd}$$

Hint: Fix any $x_0 \in X$ and show that $g : Y \to Z$ such that $g(y) = f(x_0, y)$ works.

Proof. Choose a k-rational point x and consider $f|_{(\{x\} \times_k Y)} : Y \longrightarrow Z$, denoted by g. Then, we will show that $g \circ \pi_Y = f$. By 2.4.2, we just need to show that they agree on an non-empty open set of $X \times_k Y$.

Let $\{z\} = f(X \times \{y_0\})$ and U be an affine open set of Z containing z. Then $V := \pi_Y(f^{-1}(Z \backslash U))$ is closed in Y. We claim that $\pi_Y^{-1}(Y \backslash V)$ is the open set we want: If $y \notin V$, then $X \times \{y\}$ is mapped to U. Since $X \times \{y\}$ is proper and U is affine, $f|_{X \times \{y\}} : X \times \{y\} \longrightarrow U$ is constant by 2.2.4 and 2.4.5 (d). Thus, over $\pi_Y^{-1}(V)$, $g \circ \pi_Y = f$. □

Exercise 3.11.12. Let $f : X \to Y$ be a projective morphism with $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Show that for every invertible $\mathcal{L}$ on Y the natural adjunction morphism

$$\mathcal{L} \to f_*f^*\mathcal{L}$$

is an isomorphism. Suppose $\mathcal{L}$ is an invertible sheaf on X. Is the adjunction morphism $f^*f_*\mathcal{L} \to \mathcal{L}$ injective (resp. surjective) in general? Give a proof or provide a counterexample.

Proof. By the projective formula 2.5.1, it is obvious that $\mathcal{L} \longrightarrow f_*f^*\mathcal{L}$ is an isomorphism. For the second statement, consider $\pi : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathrm{Spec}(\mathbb{C})$ and take $\mathcal{L} = \mathcal{O}(2) \in \mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}})$. □

Exercise 3.11.13. Let $f : X \to Y$ be a projective morphism with X and Y locally Noetherian. Show that the connected components of a fibre X_y are in bijection to the maximal ideals of $(f_*\mathcal{O}_X)_y$.

Proof. By Stein's factorization, we can assume that f is finite. Then consider this locally. □

Exercise 3.11.14. Let $\mathcal{L}$ be a globally generated invertible sheaf on a normal projective scheme X over a field k. Consider the induced morphism $\varphi : X \to \mathbb{P}_k^N$. Show that the morphism φ can be decomposed as $\varphi = \pi \circ \varphi'$, with $\varphi' : X \to Z$ projective with connected fibres and $\pi : Z \to \mathbb{P}_k^N$ finite such that:

(i) $\deg(\mathcal{L}|_C) = 0$ for a complete integral curve $C \subset X$ if and only if $\varphi(C) = \mathrm{pt}$.

(ii) Z is normal.

Assume that for every complete curve $C \subset X$ we have $\deg(\mathcal{L}|_C) \neq 0$. Show that then $\mathcal{L}$ is ample.

Proof. Directly by 3.11.9, Z is normal. To key point is to prove (i).

When $\deg(\mathcal{L}|_C) = 0$, hence $h^0(C, \mathcal{L}|_C) \leq 1$ by [[5] Lemma IV.4.1.]. Thus $\varphi(C)$ is a point. The converse will be proved **Later**!

Finally, consider the image of φ in $\mathbb{P}_k^N$. By (i), for every $y \in Y$, $f^{-1}(y)$ contains finitely many points. Hence f is quasi-finite. Since f is proper, f is finite. By 3.5.7, $\mathcal{L} = \varphi^* \mathcal{O}_Y(1)$ is ample since $\mathcal{O}_Y(1)$ is ample. □

3.12 Base Change and Cohomology

3.12.1 Preparations

I will just list some results form Mumford's book Abelian Varieties [[11]].

Theorem 3.12.1. *Assume $f : X \longrightarrow Y$ is proper and $Y = \mathrm{Spec}(A)$ with A Noetherian. Suppose that $\mathcal{F} \in \mathrm{Coh}(X)$ is flat over Y. Then, there exists a complex $K^\bullet$:*

$$0 \longrightarrow K^0 \longrightarrow K^1 \longrightarrow K^2 \longrightarrow ... \longrightarrow K^n \longrightarrow 0$$

of finite projective A-modules K^n such that for any $A \longrightarrow B$, we have

$$H^i(X \times_A B) \cong H^i(K^\bullet \otimes B)$$

Corollary 3.12.2. *Assume $f : X \longrightarrow Y$ is projective and Y is Noetherian, $\mathcal{F} \in \mathrm{Coh}(X)$ is flat over Y. Then*

(1). The map

$$\begin{aligned} Y &\longrightarrow \mathbb{Z} \\ y &\longmapsto h^i(X_y, \mathcal{F}_y) \end{aligned}$$

is upper semi-continuous;

(2). The function

$$\begin{aligned} Y &\longrightarrow \mathbb{Z} \\ y &\longmapsto \chi(X_y, \mathcal{F}_y) \end{aligned}$$

is locally constant. (It is constant on connected component of Y.)

Corollary 3.12.3. *If $f : X \longrightarrow Y$ is projective and Y is Noetherian, $\mathcal{F} \in \mathrm{Coh}(X)$ is flat over Y. If Y is reduced and connected, fixing i, then $y \longmapsto h^i(X_y, \mathcal{F}_y)$ is constant if and only if $R^i f_*\mathcal{F}$ is locally free and $R^i f_*\mathcal{F} \otimes k(y) \cong H^i(X_y, \mathcal{F}_y)$*

Corollary 3.12.4. *Assume $f : X \longrightarrow Y$ is projective, Y is Noetherian and $\mathcal{F} \in \mathrm{Coh}(X)$ is flat over Y. If $H^i(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$, $R^{i-1} f_*\mathcal{F} \otimes k(y) \cong H^{i-1}(X_y, \mathcal{F}_y)$.*

Corollary 3.12.5. *Let $f : X \longrightarrow Y$ be a projective morphism, Y is Noetherian and $\mathcal{F} \in \mathrm{Coh}(X)$ is flat over Y. Fixing i_0, if $R^i f_*\mathcal{F} = 0$, for all $i \geq i_0$, then*

$$H^i(X_y, \mathcal{F}_y) = 0$$

for all $i \geq i_0$.

Corollary 3.12.6 (Seesaw)**.** *Let X be projective over k, geometrically integral and T is of finite type over k. Given $\mathcal{L} \in \mathrm{Pic}(X \times T)$. Consider $Z := \{t \in T \big| \mathcal{L}|_{X \times k(t)}$ is trivial$\}$. Then Z is a closed subset of T.*

Corollary 3.12.7. *Let $f : X \longrightarrow Y$ be a flat projective surjective morphism. All fiber X_y are integral and Y is reduced. If $\mathcal{L} \in \mathrm{Pic}(X)$ satisfies $\mathcal{L}_y \cong \mathcal{O}_{X_y}$ for all $y \in Y$, then there exists $\mathcal{M} \in \mathrm{Pic}(Y)$ such that $\mathcal{L} \cong f^*\mathcal{M}$.*

Proof. 3.12.4 □

Corollary 3.12.8. *If $f : X \longrightarrow Y$ is projective and Y is Noetherian, $\mathcal{L} \in \mathrm{Pic}(X)$, then $A_{\mathcal{L}} := \{y \in Y \big| \mathcal{L}|_{X_y}$ is ample$\}$ is an open subset.*

3.12.2 Examples

Example 3.12.9. For 3.12.2, it is necessary for $\mathcal{F}$ to be flat. For example, let $f : X \longrightarrow \mathbb{A}^1_k$ with $X := \mathrm{Bl}_{(0,0)}\mathbb{A}^1_k$ and $\mathcal{F} := \mathcal{O}(E) \in \mathrm{Pic}(X)$. Then

$$X_y = \begin{cases} pt. \text{ if } y \neq O; \\ E \cong \mathbb{P}^1_k, \text{ if } y = O \end{cases} \qquad \text{and } \mathcal{F}_y = \begin{cases} k(pt.) \text{ if } y \neq O; \\ \mathcal{O}(E)|_E := \mathcal{O}(-1), \text{ if } y = O \end{cases}$$

Then $h^0(\mathcal{F}_y) = \begin{cases} 1, \text{ if } y \neq O; \\ 0, \text{ if } y = O. \end{cases}$ We see that $\{y \in \mathbb{A}^1_k | h^0(X_y, \mathcal{F}_y) \geq 1\}$ is open.

Example 3.12.10. Let E be an elliptic curve over k and $X = E \times_k E$. Let $\Delta, E \times O$ be divisors in the surface X. Let $\mathcal{L} \cong \mathcal{O}(\Delta - E \times O)$. Since $p : X \longrightarrow E$ is flat, $\mathcal{L}$ is flat over E. When $y \neq O$, $\mathcal{L}|_{y\times E} \cong \mathcal{O}(y - O)$. At this case, $H^0(X_y, \mathcal{O}(y - O)) = 0$ since $\deg(\mathcal{L}_y) = 0$. Thus, $p_*\mathcal{L}|_{E-\{O\}} = 0$. Then we can show that $p_*\mathcal{L} = 0$ over E. However, at $y = O$, $p_*\mathcal{L} \otimes k(y) = 0$ but $H^0(X_y, \mathcal{L}_y) = k \neq 0$.

Hence, for

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow p \\ k(y) & \xrightarrow{i} & E \end{array}$$

we don't have $i^*p_*\mathcal{L} = p'_*i'^*\mathcal{L}$. This, is because $i : k(y) \longrightarrow E$ is not flat.

However, since $H^2(X_y, \mathcal{L}_y) = 0$ by Grothendieck vanishing theorem, $R^1f_*\mathcal{L} \otimes k(y) = H^1(X_y, \mathcal{L}_Y)$ by 3.12.4. We have $i^*R^1p_*\mathcal{L} = R^1p_*i'^*\mathcal{L}$. In fact, by Riemann-Roch, we have $H^1(X_y, \mathcal{L}_y) = 0$ when $y \neq O$ and $H^1(X_y, \mathcal{L}_y) = k$ when $y = O$, we can give a direct description of $R^1p_*\mathcal{L}$: it is not locally free. We see that 3.12.3 will fail at some conditions.

3.12.3 Exercises

Exercise 3.12.1. Let Y be a scheme of finite type over an algebraically closed field k. Show that the function

$$\varphi(y) = \dim_k(\mathfrak{m}_y/\mathfrak{m}_y^2)$$

is upper semicontinuous on the set of closed points of Y

Proof. At closed point, by [[5] Proposition 8.7.] $\mathfrak{m}_y/\mathfrak{m}_y^2 \cong \Omega_{Y/k,y} \otimes k(y)$. Since Y is of finite type, $\Omega_{Y/k}$ is a coherent sheaf over Y. By 2.5.8, the function $y \longmapsto \dim \Omega_{Y/k,y} \otimes k(y)$ is upper semi-continuous. Hence, φ is upper semi-continuous. □

Exercise 3.12.2. Let $\{X_t\}$ be a family of hypersurfaces of the same degree in $\mathbb{P}^n_k$. Show that for each i, the function $h^i(X_t, \mathcal{O}_{X_t})$ is a constant function of t.

Proof. It is enough to prove that for any Y that is a subscheme of degree d in $\mathbb{P}^n_k$, then $h^i(Y, \mathcal{O}_Y)$ is independent of Y.

In fact, we have the exact sequence

$$0 \longrightarrow \mathcal{O}(-d) \longrightarrow \mathcal{O} \longrightarrow i_{Y,*}\mathcal{O}_Y \longrightarrow 0$$

By [[5] Theorem III.5.1.], when $0 < i < n$, $H^i(\mathbb{P}^n_k, \mathcal{O}(d)) = 0$ for all $d \in \mathbb{Z}$. So $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < n-1$. When $i = 0$, $H^1(Y, \mathcal{O}_Y) = k$ since $d \geq 1$ implies $H^0(\mathbb{P}^n_k, \mathcal{O}(-d)) = 0$. When $i = n-1$, $\dim H^{n-1}(Y, \mathcal{O}_Y) = \dim H^n(\mathbb{P}^n_k, \mathcal{O}(d)) - \dim H^n(\mathbb{P}^n_k, \mathcal{O})$. Thus, $H^i(Y, \mathcal{O}_Y)$ is independent of the choice of Y but only depends on the degree of Y.

Thus, if $\{X_t\}$ is a family of hypersurfaces of the same degree, $h^i(X_t, \mathcal{O}_{X_t})$ is independent of the choice of t. □

Exercise 3.12.3. Let $X_1 \subseteq \mathbb{P}^4_k$ be the rational normal quartic curve (which is the 4-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^4$). Let $X_0 \subseteq \mathbb{P}^3_k$ be a nonsingular rational quartic curve, such as the one in (I, Ex. 3.18b). Use (9.8.3) to construct a flat family $\{X_t\}$ of curves in $\mathbb{P}^4$, parametrized by $T = \mathbb{A}^1$, with the given fibres X_1 and X_0 for $t = 1$ and $t = 0$. Let $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^4 \times T}$ be the ideal sheaf of the total family $X \subseteq \mathbb{P}^4 \times T$. Show that $\mathcal{I}$ is flat over T. Then show that

$$h^0(t, \mathcal{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$

and also

$$h^1(t, \mathcal{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}.$$

This gives another example of cohomology groups jumping at a special point.

Proof. First of all, let's figure out what X_0, X_1 and X_t: X_1 is the image of Veronese embedding of $\mathbb{P}^1_k$ in $\mathbb{P}^4_k$, that is,

$$X_1 = \{[u^4 : u^3v : u^2v^2 : uv^3 : v^4] \big| [u : v] \in \mathbb{P}^1_k\}$$

and $X_0 = \{[u^4 : u^3v : uv^3 : v^4] \big| [u : v] \in \mathbb{P}^1_k\} \cong \{[u^4 : u^3v : 0 : uv^3 : v^4] \big| [u : v] \in \mathbb{P}^1_k\}$. By the construction,

$$X_t = \{[u^4 : u^3v : tu^2v^2 : uv^3 : v^4] \big| [u : v] \in \mathbb{P}^1_k\}$$

and $X_t \cong X_1$ when $t \neq 1$.

Warning: Use Riemann-Roch to compute the Hilbert polynomial. By Riemann-Roch, for a curve in its Hilbert polynomial is

$$p_C(n) = n\deg(C) + 1 - p_a$$

Note that when C is a curve p_a and p_a is a birational invariant 3.5.3 (b). Hence, $p_g(X_t) = 0$ since X_t is birational to $\mathbb{P}^1_k$ under

$$s \longmapsto (1, s, ts^2, s^3, s^4)$$

Since $v : \mathbb{P}^1_k \longrightarrow X_1$ and $v^*\mathcal{O}_{X_1} = \mathcal{O}_{\mathbb{P}^1_k}(4)$, by [[5] Proposition II.6.8.], $\deg(\mathcal{O}_{X_1}) = 4$. Similarly , we get $\deg(X_0) = 4$. Hence, the Hilbert polynomial at t is always $4n + 1$. Thus, X_t is flat.

By direct computation, we have

$$\mathcal{I}_t = (t^2x_0x_4 - x_2^2,\ t^2x_1x_3 - x_2^2,\ x_0x_2 - tx_1^2,\ x_2x_4 - tx_3^2,\ tx_0x_3 - x_1x_2,\ tx_1x_4 - x_2x_3)$$

when $t \neq 1$, we see that $h^0(X_t) = 1$. When $t = 0$, $I_0 = (x_0)(x_0, x_1, x_2, x_3, x_4)$. It's stature is (x_2). Hence $h^0(I_0) = 0$. $\square$

Remark. It is right to compute the Hilbert polynomials?It is right to compute $h^0(I_0)$?There is something wrong with Hartshrone?

Exercise 3.12.4. Let Y be an integral scheme of finite type over an algebraically closed field k. Let $f : X \to Y$ be a flat projective morphism whose fibres are all integral schemes. Let $\mathcal{L}, \mathcal{M}$ be invertible sheaves on X, and assume for each $y \in Y$ that $\mathcal{L}_y \cong \mathcal{M}_y$ on the fibre X_y. Then show that there is an invertible sheaf $\mathcal{N}$ on Y such that $\mathcal{L} \cong \mathcal{M} \otimes f^*\mathcal{N}$. [Hint: Use the results of this section to show that $f_*(\mathcal{L} \otimes \mathcal{M}^{-1})$ is locally free of rank 1 on Y.]

Proof. It is enough to show that if $\mathcal{L} \in \mathrm{Pic}(X)$ and $\mathcal{L}$ is trivial at X_y. Then $\mathcal{L} = f^*\mathcal{N}$ for some $\mathcal{N} \in \mathrm{Pic}(Y)$: By 3.12.3, $f_*\mathcal{L}$ is locally free of rank 1. Hence there exists a line bundle $\mathcal{N} \in \mathrm{Pic}(Y)$ such that $f_*\mathcal{L} = \mathcal{N}$.

By adjunction formula, we have a natural map $f^*f_*\mathcal{L} \longrightarrow \mathcal{L}$ deduced by

$$\mathrm{Hom}(f_*\mathcal{L}, f_*\mathcal{L}) \cong \mathrm{Hom}(f^*f_*\mathcal{L}, \mathcal{L})$$

We now show that $f^*f_*\mathcal{L} \longrightarrow \mathcal{L}$ is a surjection: Since both sheaves are line bundles, we can verify it is surjective locally. Choose $U = \mathrm{Spec}(R)$ such that $f_*\mathcal{L}$ is trivial. Since

$$R = \mathrm{Hom}(f_*\mathcal{L}, f_*\mathcal{L}) = \mathrm{Hom}(f^*f_*\mathcal{L}, \mathcal{L}) = \mathcal{L}(f^{-1}(U)) = R$$

as R-modules, id_R is mapped to a unit in $\mathcal{L}(f^{-1}(U))$ which corresponding to $\mathcal{O}_{f^{-1}(U)} \longrightarrow \mathcal{L}$ and this map is surjective. Hence, $f^*f_*\mathcal{L} \longrightarrow \mathcal{L}$ is a surjection and then it is an isomorphism by 2.7.1. □

Exercise 3.12.5. Let Y be an integral scheme of finite type over an algebraically closed field k. Let $\mathcal{E}$ be a locally free sheaf on Y, and let $X = \mathbb{P}(\mathcal{E})$ (see (II, §7)). Then show that

$$\mathrm{Pic}\, X \cong (\mathrm{Pic}\, Y) \times \mathbb{Z}.$$

This strengthens (II, Ex. 7.9).

Proof. Consider $\pi : \mathbb{P}(\mathcal{E}) \longrightarrow Y$. Since each fiber is isomorphic to $\mathbb{P}^n_k$ with $n = \mathrm{rank}(\mathcal{E})$. Hence, the Hilbert polynomials will be constant. Thus, π is flat.

For any $\mathcal{L} \in \mathrm{Pic}(X)$ In $\mathbb{P}^n_k$, we can determine any line bundle by its cohomology. Since Y is irreducible and hence it is connected, $y \longmapsto h^i(X_y, \mathcal{L}_y)$ is constant by 3.12.2. Thus, $\deg(\mathcal{L}_t)$ is constant. Suppose that $\deg(\mathcal{L}_t) \equiv l$. Then $\mathcal{L} \otimes \mathcal{O}_X(-l)$ is trivial at each fiber. By 3.12.4, $\mathcal{L} = \mathcal{O}_X(l) \otimes \pi^*\mathcal{N}$ for some $\mathcal{N} \in \mathrm{Pic}(Y)$.

Conversely, just like what we did in **??**, the map $(\mathcal{N}, n) \longrightarrow \mathcal{O}_X(n) \otimes \pi^*\mathcal{N}$ is injective. Thus, $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y) \times \mathbb{Z}$.

□

Exercise 3.12.6. Let X be an integral projective scheme over an algebraically closed field k, and assume that $H^1(X, \mathcal{O}_X) = 0$. Let T be a connected scheme of finite type over k.

(a) If $\mathcal{L}$ is an invertible sheaf on $X \times T$, show that the invertible sheaves $\mathcal{L}_t$ on $X = X \times \{t\}$ are isomorphic, for all closed points $t \in T$.

(b) Show that $\mathrm{Pic}(X \times T) = \mathrm{Pic}\, X \times \mathrm{Pic}\, T$. (Do not assume that T is reduced!)

Cf. (IV, Ex. 4.10) and (V, Ex. 1.6) for examples where $\mathrm{Pic}(X \times T) \neq \mathrm{Pic}\, X \times \mathrm{Pic}\, T$. [Hint: Apply (12.11) with $i = 0, 1$ for suitable invertible sheaves on $X \times T$.]

3.12.4 Additional Exercises

Exercise 3.12.7 (Hodge Bundle)**.** Let $f : X \to Y$ be a smooth projective morphism with X and Y Noetherian and $\dim X = \dim Y + 1$. Show that the Hodge bundles $R^1 f_*\Omega_{X/Y}$ are locally free sheaves. (In characteristic zero, this holds true without the assumption on the dimension.) (Hint: Use cohomology and base change twice, once in degree 2 and once in degree 1.)

Proof. We see that each X_y is a smooth projective over $k(y)$. Since $\mathcal{O}_X$ is flat over Y, the Hilbert polynomials

$$p_{X_y}(m) := \chi(\mathcal{O}_{X_y}(m)) = \deg(\mathcal{O}_{X_y})m + g_{X_y} - 1$$

is a constant polynomial related to $y \in Y$. Hence, $g(X_y) = g$ for all $y \in Y$.

By Grothendieck vanishing theorem, $h^q(X_y, \Omega_{X/Y,y}) = h^q(X_y, \Omega_{X_y/k(y)}) = 0$ when $q \geq 2$. Hence, $R^q f_*\Omega_{X/Y}$ is locally free when $q \geq 2$.

Since f is smooth, $\Omega_{X/Y}$ is locally free by [[5] Proposition III.10.4]. Thus, $\Omega_{X/Y}$ is flat over Y. For the same reason $\Omega^p_{X/Y}$ is flat over Y for all $p \geq 0$.

When $q = 0, p = 0$, $h^0(X_y, (\Omega^q_{X/Y})_y) = h^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$. By 3.12.3, $f_*\Omega^0_{X/Y}$ is locally free.

When $q = 0, p = 1$, $h^0(X_y, (\Omega^q_{X/Y})_y) = h^0(X_y, \Omega_{X_y/k(y)})$. Since X_y is of dimension 1, $\Omega_{X_y/k(y)} = \omega_{X_y/k(y)}$. Thus, $h^0(X_y, (\Omega^q_{X/Y})_y) = h^0(X_y, \omega_{X_y/k(y})) = g_{X_y}$ which is constant as we have discussed before. Again by 3.12.3, $f_*\Omega_{X/Y}$ is locally free.

When $q = 1, p = 0$, $h^1(X_y, \mathcal{O}_{X_y}) = g_{X_y}$ by Serre Duality which is constant as we have seen. Hence $R^1 f_*\mathcal{O}_X$ is locally free.

When $q = 1, p = 1$, $h^1(X_y, \omega_{X_y}) = h^0(X_y, \mathcal{O}_{X_y}) = 1$, which is constant. Thus, $R^1 f_*\Omega_{X/Y}$ is locally free. $\square$

Chapter 4

Curves

4.1 Riemann-Roch Theorem

In this chapter, a curve is a integral scheme of dimension 1, proper over k, all of whose local rings are regular.

4.1.1 Preparations

A Riemann-Roch

Recall the for any divisor D

$$|D| = \frac{\Gamma(X, \mathcal{L}(D)) - 0}{k^*}$$

Hence $dim|D| = dim_k\Gamma(X, \mathcal{L}(D)) - 1$ and we denote $l(D) = dim_k\Gamma(X, \mathcal{L}(D)) = dim_k H^0(X, \mathcal{L}(D))$.

Theorem 4.1.1. *Let D be a divisor of X. Then if $l(D) \neq 0$, we must have $degD \geq 0$. Moreover, if $l(D) \neq 0$ and $deg(D) = 0$, we have $D \sim 0$, that is, $D \cong \mathcal{O}_X$.*

Proof.
If $l(D) \neq 0$, $\Gamma(X, \mathcal{L}(D))$ is not 0. Hence there exists at least one effective divisor D_0 such that $D \sim D_0$ and then $deg(D) = deg(D_0) \geq 0$.

If $deg(D) = 0$, then the effective divisor D_0 has degree 0, i.e., $D_0 = 0$. Hence $D \sim 0$. □

Remark 4.1.2. This well be used to tell when $l(D) = 0$: If $deg(D) < 0$, then $l(D) = 0$.

Theorem 4.1.3 (Riemann-Roch). *Let K be the divisor corresponding to the canonical line bundle of X, then*

$$l(D) - l(K - D) = deg(D) + 1 - g$$

Proof. For $H^0(X, \omega_X \otimes \mathcal{L}^*) = H^1(X, \mathcal{L})$ and $\mathcal{L}(K - D) = \mathcal{L}(K) \otimes \mathcal{L}(D)^*$, via Grothendick vanishing theorem, it is enough to verify

$$\chi(X, \mathcal{L}) = deg(D) + 1 - g$$

where $g = dim_k H^1(X, \mathcal{O}_X)$. Note that $H^0(X, \mathcal{O}_X) = k$. Hence, the formula can be written as

$$\chi(X, \mathcal{L}(D)) - \chi(X, \mathcal{O}_X) = deg(D)$$

Suppose that $D = \sum n_P P$. We show that when $D = P$, we have the statement above. Then induct for $D - P$ to D:

When $D = P$, we have

$$0 \longrightarrow \mathcal{L}(-P) \longrightarrow \mathcal{O}_X \longrightarrow k(p) \longrightarrow 0$$

where $k(p) = i_{p,*}\mathcal{O}_P$, by 2.6.1.D. Note that $\chi(X, k(p)) = 1 = deg(P)$. So we have

$$\chi(X, \mathcal{L}(D)) - \chi(X, \mathcal{O}_X) = deg(D)$$

Then consider

$$0 \longrightarrow \mathcal{L}(D - P) \longrightarrow L(D) \longrightarrow k(p) \longrightarrow 0$$

which is gained by tensoring $\mathcal{L}(D)$ with the exact sequence above. For $\mathcal{L}(D)$ is locally free, it preserves exactness. Then we have

$$\chi(X, \mathcal{L}(D - P)) + 1 = \chi(X, \mathcal{L}(D))$$

Using induction hypothesis, we gain the Riemann-Roch formula. □

B Some Examples

Example 4.1.4. Take $D = 0$. $l(D) = dim_k\Gamma(X, \mathcal{O}_X) = 1$. By Riemann-Roch $1 - l(K) = 1 - g$. Take $D = K$. By Riemann-Roch $l(K) - 1 = deg(K) + 1 - g = g - 1$. Hence $deg(K) = 2g - 2$.

Example 4.1.5. For any curve X, if $g(X) = 0$, $P \sim Q$ for any two point: Take $D = P - Q$. $deg(K) = 2g - 2 = -2$. Hence $deg(K - D) = -2 < 0$, implying $l(K - D) = 0$. By Riemann-Roch, $l(D) = 0 + 1 - g = 1 \neq 0$.

Note that $l(D) \neq 0$ and $deg(D) = 0$. By 4.1.1, $D \sim 0$ i.e. $P \sim Q$.

Example 4.1.6. We say a curve is elliptic if $g(X) = 1$. At this case, $deg(K) = 0$. Now, $l(K) = dim_k H^0(X, \omega_X) = dim_k H^1(X, \mathcal{O}_X) = g(X) = 1$. Then $K \sim 0$.

Example 4.1.7. When $g(X) = 1$, fixing $P_0 \in X$, there is a $1-1$ corresponding between closed points in X and $Pic(X)^0 := \{$ line bundles of degree 0 $\}$ given by $P \mapsto \mathcal{O}_X(P - P_0)$.

We just need to show that given any D with $deg(D) = 0$, there exist unique P such that $D + P_0 \sim P \iff D \sim P - P_0$: $deg(K - D - P_0) < 0$, hence $l(K - D - P_0) = 0$. By Riemann-Roch,

$$l(D + P_0) = 1 + 1 - 1 = 1$$

Thus, $H^0(X, \mathcal{O}_X(D + P_0)) = ks$ for some $s \in H^0(X, \mathcal{O}_X(D + P_0))$. For $(s)_0$ is unique and $deg(D + P_0) = 1$, there is a unique $P = (s)_0$ such that $D + P_0 \sim P$.

4.1.2 Exercise

Exercise 4.1.1. Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

Proof. Take $Q \neq P$. Let $D = -Q + nP$ with $n - 1 > 2g - 2$. Then $deg(K - D) < 0$, implying $l(K - D) = 0$. By Riemann-Roch, $l(D) = n - 1 + 1 - g$. So, we just take $n > g$. $l(D) > 0$. Thus, $H^0(X, \mathcal{L}(D)) \neq 0$.

Take $s \in H^0(X, \mathcal{L}(D)) \subset K^*(X)$. There exists $f \in K^*(X)$ such that $(f^{-1}) = -D + (s)_0$. For $(s)_0$ is an effective divisor, the only pole of f^{-1} is P. □

Exercise 4.1.2. Again let X be a curve, and let $P_1, \ldots, P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_i, and regular elsewhere.

Proof. Take $Q \neq P_i$ for $i = 1, 2, ..., r$. Let $D = -Q + n(P_1 + ... + P_r)$. Then $deg(D) = nr - 1$. Just like 4.1.1, take n large enough. We have the results. □

Exercise 4.1.3.

Exercise 4.1.4.

Exercise 4.1.5. For an effective divisor D on a curve X of genus g, show that $\dim |D| \leq \deg D$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

Proof. I want to know, why if $D > 0$, then $l(K - D) \leq l(D)$: maybe $H^1(X, \mathcal{L}(D)) \subset H^1(X, \mathcal{O}_X)$? I think this can be prove by taking colimit of Cech cohomology. □

Exercise 4.1.6. Let X be a curve of genus g. Show that there is a finite morphism $f : X \to \mathbf{P}^1$ of degree $\leq g + 1$. (Recall that the *degree* of a finite morphism of curves $f : X \to Y$ is defined as the degree of the field extension $[K(X) : K(Y)]$ (II, §6).)

Proof. Take differect points $x_1, ..., x_{g+1}$. By Riemann-Roch

$$l(X, \mathcal{O}_X(x_1 + ... + x_{g+1})) \geq g + 1 + 1 - g = 2$$

Note that $D = x_i$ induces an short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-x_i) \longrightarrow \mathcal{O}_X \longrightarrow i_* k(x_i) \longrightarrow 0$$

Tensoring with $\mathcal{O}(x_1 + ... + x_i)$, we have

$$0 \longrightarrow \mathcal{O}_X(x_1 + ... + x_{i-1}) \longrightarrow \mathcal{O}_X(x_1 + ... + x_i) \longrightarrow i_* k(x_i) \longrightarrow 0$$

Then $dim_k H^0(X, \mathcal{O}_X(x_1 + ... + x_i)) = dim_k H^0(X, \mathcal{O}(x_1 + ... + x_{i+1})) + 1$ or $dim_k H^0(X, \mathcal{O}(x_1 + ... + x_{i+1}))$. Hence, there must exists N such that $H^0(X, \mathcal{O}_X(x_1 + ... + x_N)) = 2$.

Let $\mathcal{L} = \mathcal{O}_X(x_1 + ... + x_N)$. W.l.o.g., we can assume that $\mathcal{L}$ is globally generated. Hence we have $\varphi : X \longrightarrow \mathbb{P}^1_k$ with $\varphi^* \mathcal{O}_{\mathbb{P}^1_k}(1) = \mathcal{L}$.

For both X and $\mathbb{P}^1_k$ are complete non-singular curve, φ is either constant or surjective. By the definition of φ, φ can not be constant, hence φ is surjective and finite by Chapter 2 Proposition 6.8 of [5]. By Proposition 6.9, we see that $\deg(\varphi) = \frac{deg \varphi^* \mathcal{O}_{\mathbb{P}^1_k}(1)}{deg \mathcal{O}_{\mathbb{P}^1_k}(1)} \leq g + 1$. □

4.1.3 Addition Exercises

Exercise 4.1.7. Let X be a projective regular curve over an algebraically closed field k and let $x_1, x_2 \in X$ be two closed points. Show that $\chi(X, \mathcal{O}_X(x_1 - x_2))$ is independent on the choice of x_1, x_2

Proof. Given any $x_1 \in X$, there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-x_1) \longrightarrow \mathcal{O}_X \longrightarrow i_{x_1,*} k(x_1) \longrightarrow 0$$

Tensor with $\mathcal{O}_X(x_2)$ and $\mathcal{O}_X(x_1)$, we have

$$0 \longrightarrow \mathcal{O}_X(x_2 - x_1) \longrightarrow \mathcal{O}_X(x_2) \longrightarrow i_{x_1,*} k(x_1) \longrightarrow 0$$

We see that

$$\chi(X, \mathcal{O}_X(x_2 - x_1)) + \chi(X, i_{x_1,*} k(x_1)) = \chi(X, \mathcal{O}_X(x_2))$$

Note that $\chi(X, i_{x_1,*} k(x_1)) = 1$, and by Riemann-Roch, $\chi(X, \mathcal{O}_X(x_2)) = 1 + 1 - g$. Hence $\chi(X, \mathcal{O}_X(x_2 - x_1)) = 1 - g$ is independent of the choice of x_1, x_2. □

Exercise 4.1.8. Let X be a projective regular curve of genus one over an algebraically closed field k and let $x_1, x_2 \in X$ be two closed points. Show that $H^1(X, \mathcal{O}(x_1 - x_2)) \neq 0$ if and only if $x_1 = x_2$.

Exercise 4.1.9. By Riemann-Roch, we have

$$\chi(X, \mathcal{O}_X(x_1 - x_2)) = 0$$

with Grothendieck Vanishing theorem, we see that

$$H^0(X, \mathcal{O}(x_1 - x_2)) = H^1(X, \mathcal{O}_X(x_1 - x_2))$$

If $x_1 = x_2$, then $\mathcal{O}_X(x_1 - x_2) = \mathcal{O}_X$, then $H^0(X, \mathcal{O}_X) = k$.

If $H^0(X, \mathcal{O}_X(x_1 - x_2)) \neq 0$, then for $deg(\mathcal{O}_X(x_1 - x_2)) = 0$, $x_1 - x_2 \sim 0$.

By the $1 - 1$ correspondence between points in X and $Pic^0(X)$ of elliptic curves, $x_1 - x_2 \sim 0$ if and only if $x_1 = x_2$.

4.2 Hurwitz's Theorem

Let $f : X \longrightarrow Y$ be a morphisms between two curves. The ramification divisor is defines as the following: Consider $y \in Y$ and $P \in f^{-1}(y)$. Define e_P to be the multiplication of P at X. Then define

$$R(f) = \sum_{P \in C} (e_p - 1)P$$

Theorem 4.2.1 (Hurwitz's Formula). *If $f : X \longrightarrow Y$ is a finite morphism between the curves. Then*

$$2g_X - 2 = \deg(f)(2g_Y - 2) + \deg(R(f))$$

4.2.1 Examples

Example 4.2.2 (Application of Hurwitz's Formula).

4.2.2 Exercises

Exercise 4.2.1. We prove this result by induction on n. First all, when $n = 1$, by Hurwitz's formula, we know that $\mathbb{P}^1_k$ admits only trivial étale cover. Now, suppose that $f : X \longrightarrow \mathbb{P}^n_k$ be a finite étale morphism with X irreducible. Now, consider a hypersurface $H \subset \mathbb{P}^n_k$. Then $f^*H \subset X$ is an effective divisor in X, that is, $H \times_{\mathbb{P}^n_k} X$. By [[5] Corollary 7.9.], $H \times_{\mathbb{P}^n_k} X$ is connected and $H \times_{\mathbb{P}^n_k} X \longrightarrow H$ is finite and étale since finite morphisms and étale morphisms are stable under base change. For $H \cong \mathbb{P}^{n-1}_k$, by our induction hypothesis, $H \times_{\mathbb{P}^n_k} X \longrightarrow H$ is just $id : \mathbb{P}^{n-1}_k \longrightarrow \mathbb{P}^{n-1}_k$. Thus, f is just the identity.

Exercise 4.2.2. (a). By Hurwitz's formula, $2g_X - 2 = \deg f(2g_Y - 2) + \deg R(f)$. Hence, $\deg R(f) = 6$. For $e_P \leq \deg f$ for all $P \in X$, $e_P = 2$ at six different points. Thus, $R(f) = \sum_i P_i$ with $P_i \neq P_j$ when $i \neq j$.

Exercise 4.2.3.

Exercise 4.2.4.

Exercise 4.2.5.

Exercise 4.2.6. To prove that $f_*\mathcal{M}$ is locally free, just see 4.2.10.

(a). First of all, we let D be an effective divisor over X. Consider $0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_D \longrightarrow 0$. Since f is affine, locally $R^1 f_*\mathcal{O}_X(-D)|_U = H^1(f^{-1}(U), \mathcal{O}_X(-D))^\sim$. By Serre duality, $H^1(f^{-1}(U), \mathcal{O}_X(-D)) = 0$. Thus, $R^1 f_*\mathcal{O}_X = 0$. Thus, we get

$$0 \longrightarrow f_*\mathcal{O}_X(-D) \longrightarrow f_*\mathcal{O}_X \longrightarrow f_*i_*\mathcal{O}_D \longrightarrow 0$$

By the definition, we see that $f_*i_*\mathcal{O}_D = i_{f(D)*}\mathcal{O}_{f(D)}$ and since $f(D) = \sum n_i f(P_i)$, we see that $f_*i_*\mathcal{O}_D \cong \mathcal{O}_Y(\sum n_i P_i) = \mathcal{O}_Y(f_*D)$. Hence, we have

$$\det(f_*\mathcal{O}_X) = \det f_*\mathcal{O}_X(-D) \otimes \det \mathcal{O}_Y(f_*D)$$

which implies our result.

Now for any divisor D, assume that there exist two effective divisors D_1, D_2 such that $D = D_1 - D_2$. We can see that $f_*D = f_*D_1 - f_*D_2$ by the definition of f_* and consider

$$0 \longrightarrow f_*\mathcal{O}_X(-D_2) \longrightarrow f_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y(f_*D_2) \longrightarrow 0$$

$$0 \longrightarrow f_*\mathcal{O}_X(D_2 - D_1) \longrightarrow f_*\mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_Y(f_*D_1) \longrightarrow 0$$

we get $\det f_*\mathcal{O}_X(D)\otimes\mathcal{O}_Y(f_*D_1)=\det f_*\mathcal{O}_X\otimes\mathcal{O}_Y(f_*D_2)$.

(b). Since if $D_1\sim D_2$, $f_*D_1\sim f_*D_2$ by the definition. Thus, $\mathcal{O}_Y(f_*D_1)\cong\mathcal{O}_Y(f_*D_2)$. Hence,

$$\begin{aligned} f_*:\operatorname{Pic}(X)&\longrightarrow\operatorname{Pic}(Y)\\ D&\longmapsto\det f_*\mathcal{O}_X(D)\end{aligned}$$

is well-defined.

Note that for any point $x\in X$, $f_*f^{-1}(x)=(\deg f)x$. Thus, $f_*f^*D=n\cdot D$ with $n=\deg f$.

(c). Since X and Y are curves, $\Omega_X=\omega_X$ and $\Omega_Y=\omega_Y$, which implies

$$\begin{aligned} f_*\Omega_X&=f_*f^!\Omega_Y\\ &=f_*\mathscr{H}om_X(\mathcal{O}_X,f^!\Omega_Y)\\ &=\mathscr{H}om_Y(f_*\mathcal{O}_X,\Omega_Y)\\ &=(f_*\mathcal{O}_X)\check{}\otimes\Omega_Y\end{aligned}$$

Because for a vector bundle $\mathcal{E}$ of rank r and a line bundle $\mathcal{L}$, there exists $\det(\mathcal{E}\otimes\mathcal{L})=\det\mathcal{E}\otimes\mathcal{L}^{\otimes r}$. We see that

$$\det f_*\Omega_X=\det(f_*\mathcal{O}_X)^{-1}\otimes\Omega_Y^{\otimes n}$$

with $n=\deg f$.

(d). Because $\Omega_X=f^*\Omega_Y\otimes\mathcal{O}_X(R(f))$ and $f_*\Omega_X=\Omega_Y\otimes f_*\mathcal{O}_X(R(f))$, then $\det(f_*\Omega_X)=\Omega_Y^n\otimes\det(f_*\mathcal{O}_X(R(f)))$ where $n=\deg f$. Note that

$$\det(f_*\mathcal{O}_X(R(f)))=(\det f_*\mathcal{O}_X)\otimes\mathcal{O}_Y(f_*R(f))$$

We see that $\mathcal{O}_Y(f_*R(f))=(\det f_*\mathcal{O}_X)^{-2}$. Taking the inverse, we get what we want.

Remark. For any locally free sheaf $\mathcal{E}$ of rank r and invertible sheaf $\mathcal{L}$, we have

$$\det(\mathcal{E}\otimes\mathcal{L})=\det(\mathcal{E})\otimes\mathcal{L}^{\otimes r}$$

Exercise 4.2.7. (a) Since $f:X\longrightarrow Y$ is finite and étale of degree 2. Then there exists an injection $0\longrightarrow\mathcal{O}_Y\longrightarrow f_*\mathcal{O}_X$ since finite étale morphisms are faithfully flat. Now, consider the exact sequence

$$0\longrightarrow\mathcal{O}_Y\longrightarrow f_*\mathcal{O}_X\longrightarrow\mathcal{L}\longrightarrow 0$$

Locally, we have

$$0\longrightarrow\mathcal{O}_{Y,y}\longrightarrow f_*\mathcal{O}_{X,y}\longrightarrow\mathcal{L}_y\longrightarrow 0$$

which implies $\mathcal{L}_y\cong(f_*\mathcal{O}_X)_y/\mathcal{O}_{Y,y}$ is torsion-free $\mathcal{O}_{Y,y}$-module. Since Y is a smooth curve, $\mathcal{O}_{Y,y}$ is a PID. Thus, $\mathcal{L}_y$ is free and $\mathcal{L}$ is a locally free sheaf over Y of rank 1.

By , we have $\mathcal{L}=\det f_*\mathcal{O}_X$ and $\mathcal{L}^2=\mathcal{O}_Y$ since f is unramified.

(b). By **??**, $f:X\longrightarrow Y$ is a finite morphism of degree 2. Since locally $\mathcal{L}\otimes\mathcal{O}_Y$ is isomorphic to A^2, f is flat over Y. Now, it is enough to show that $\Omega_{X/Y}=0$: Consider

$$0\longrightarrow f^*\Omega_Y\longrightarrow\Omega_X\longrightarrow\Omega_{X/Y}\longrightarrow 0$$

We can check locally that $\Omega_{X/Y}=0$. Hence f is étale.

(c).**Wait!**

Remark (Finite Étale Morphisms are Faithfully Flat)**.**

4.2.3 Additional Exercises

Exercise 4.2.8. Let E be a smooth plane cubic over a field k defined by an equation of the form

$$y^2z = x^3 + cx^2z + dxz^2,$$

with $c, d \in k$. Set $e = [0 : 1 : 0]$ to be the origin of the group law on E, and $x = [0 : 0 : 1]$. Show that $x + x = e$ on E, and deduce that $\mathcal{O}_E(x - e)$ is a two-torsion line bundle on E. By Exercise 36, $\mathcal{O}_E(x - e)$ induces an étale double cover $C \to E$, where C is a smooth curve over k. Compute the genus of C.

Proof. Note that $l = \{x = 0\}$ is a line passing through x and e. Since it is the tangent space of E at x, thus $x + x = e$. Recall that the group law over E is defined as $\mathcal{O}(x - e) \otimes \mathcal{O}(y - e) = \mathcal{O}(z - e)$ if $x + y \sim z + e$. Thus, we see that $\mathcal{O}_E(x - e) \otimes \mathcal{O}_E(x - e) = \mathcal{O}_E(e - e) = \mathcal{O}_E$.

Since $\pi : C \longrightarrow E$ is étale, $2g_C - 2 = \deg(\pi)(2g_E - 2)$. Hence ,$g_C = 1$ by Hurwitz. □

Exercise 4.2.9. Let $k = \mathbb{C}$, and consider the curve $X \subset \mathbb{P}^1_k \times_k \mathbb{P}^1_k$ defined by the equation

$$f = x_0^2(y_0^3 + y_1^3) + x_1^2(y_0y_1^2 + y_0^3).$$

Here, x_0, x_1 and y_0, y_1 are the coordinates on the first and second factor, respectively. Show that X is a smooth curve, determine the ramification divisor $R(\pi)$ of the second projection $\pi : X \to \mathbb{P}^1$, and compute the genus of X.

Proof. To prove X is smooth, it is enough to compute the definition function on $D_+(x_i) \times_k D_+(y_j)$ with $i, j = 1, 2$. It is easy.

To compute the genus of X, we need to use Hurwitz formula. Note that $f : ([x_0 : x_1], [y_0 : y_1]) \mapsto [y_0 : y_1]$. For $[y_0 : y_1]$, f has multiple roots when $y_0^3 + y_1^3 = 0$ or $y_0y_1^2 + y_0^3$. Thus, the ramification divisor in X are $X_0 \times [-1 : 1] + X_0 \times [-\zeta_3 : 1] + X_0 \times [-\zeta_3^2 : 1] + X_1 \times [0 : 1] + X_1 \times [-i : 1] + X_1 \times [i : 1]$. Thus $\deg R(\pi) = 6$. Note that $\deg(\pi) = 2$. By Hurwitz, we have

$$2g_X - 2 = 2(2g_Y - 2) + 6$$

Hence, $g_X = 2$. □

Exercise 4.2.10. Let $f : C \to D$ be a morphism between smooth projective curves over a field k and let $\mathcal{L}$ be an invertible sheaf on C.

(i) Show that for f finite the direct image sheaf $f_*\mathcal{L}$ is a locally free sheaf of rank $\deg(f)$.

(ii) What can be said about the higher direct images $R^if_*\mathcal{L}$?

(iii) Describe an example that shows that the smoothness of D is essential in (i). What about the smoothness of C?

Proof. (i). Take $t \in D$. Then $Z = f^*(y) = f^{-1}(y)$ is a divisor in C. Since C and D are smooth and f is finite, $\deg(Z) = \deg(f)$ by [[5] Proposition II.6.8.] Using the exact sequence

$$0 \longrightarrow \mathcal{O}(-Z) \longrightarrow \mathcal{O}_X \longrightarrow i_{Z*}\mathcal{O}_Z 0$$

Tensoring $\mathcal{L}$, we get

$$0 \longrightarrow \mathcal{L}(-Z) \longrightarrow \mathcal{L} \longrightarrow i_{Z*}\mathcal{O}_Z 0$$

since $\mathcal{L}|_Z = \mathcal{O}_Z$ for $\dim Z = 0$. Then we have $h^0(Z, \mathcal{O}_Z) = \chi(X, \mathcal{L}) - \chi(X, \mathcal{L}(-\mathcal{Z})) = \deg Z = \deg f$. Hence, by 3.12.3, $f_*\mathcal{L}$ is locally free of rank $\deg f$.

(ii). Since $h^i(C_y, \mathcal{L}_y) = 0$ when $i \geq 2$ by Grothendieck Vanishing theorem, $R^if_*\mathcal{L}_y \otimes k(y) \cong h^i(C_y, \mathcal{L}_y)$ when $i \geq 1$.

□

4.3 Embeddings in Projective Spaces

4.3.1 Preparations

Definition 4.3.1. A point P is a base point of linear system $\mathfrak{b}$ if $P \in Supp(D)$ for all $D \in \mathfrak{b}$.

For any $D \in \mathfrak{d} \subset |D_0|$, it corresponds to a unique $s \in H^0(X, \mathcal{L}(D))$ up to k^*. And $s(P) = \pi_p(s|_P)$ where $\pi : \mathcal{L}(D_0)_P \longrightarrow \mathcal{L}(D_0)_P/\mathfrak{m}_P\mathcal{L}(D_0)_P$ which is a 1-dimensional k-vector space. Thus, let D be represented by s. Then $supp(D) = \{P \in X | s(P) \neq 0\} = \{P \in X | s_P \notin \mathfrak{m}_P\mathcal{L}(D_0)\}$.

Lemma 4.3.2. *A complete linear system $|D|$ is base-point free if and only if $\mathcal{L}(D)$ is generated generated.*

Proof. Choose a basis $s_0, ..., s_n$ for $H^0(X, \mathcal{L}(D))$, which corresponds to the basis for $|D|$ up to k^*.

($\Longrightarrow$): If $|D|$ is base-point free, then $\{X_{s_i}\}$ with $X_{s_i} := \{P \in X | s_i(P) \neq 0\}$ forms a cover of X. Hence $\mathcal{L}$ is globally generated.

($\Longleftarrow$): If $\mathcal{L}$ is globally generated, X_{s_i} forms a cover of X. Hence $|D|$ is base-point free. $\square$

Now, we can rewrite the criteria for closed immersions.

Theorem 4.3.3. *Let $\varphi : X \longrightarrow \mathbb{P}^n_k$ be a morphism corresponding to the linear system $\mathfrak{d}$ (without base point). Then φ is a closed immersion if and only if*

(1) $\mathfrak{b}$ separates points

(2) $\mathfrak{b}$ generates tangent vectors, i.e. given a closed point $P \in X$ and a tangent vector $t \in T_P(X)$, there is a $D \in \mathfrak{d}$ such that $P \in Supp(D)$ but $t \in T_P(D)$.

By Riemann-Roch, we have the following principal to tell when an embedding of a curve is a closed immersion.

Theorem 4.3.4. *Let D be a divisor on a curve X. Then:*

(a) the complete linear system $|D|$ has no base points if and only if for every point $P \in X$,

$$dim|D - P| = dim|D| - 1$$

(b) D is very ample if and only if for every two points $P, Q \in X$ (including the case $P = Q$),

$$dim|D - P - Q| = dim|D| - 2$$

Proof.

(a). For P, we have the exact sequence:

$$0 \longrightarrow \mathcal{L}(-P) \longrightarrow \mathcal{O}_X \longrightarrow k(P) \longrightarrow 0$$

tensoring $\mathcal{L}(D)$, we have

$$0 \longrightarrow \mathcal{L}(D - P) \longrightarrow \mathcal{L}(D) \longrightarrow k(P) \longrightarrow 0$$

Then,

$$0 \longrightarrow H^0(X, \mathcal{L}(D - P)) \longrightarrow H^0(X, \mathcal{L}(D)) \longrightarrow k$$

So $dim_k H^0(X, \mathcal{L}(D - P)) - dim_k H^0(X, \mathcal{L}(D))$ equals to 0 or 1, that is, $dim|D - P| - dim|D| = 0$ or 1. Consider

$$|D - P| \longrightarrow |D|$$

given by $E \mapsto E + P$. This is obviously an injection. This map is surjective if and only if P is a base-point of D. Hence, $dim|D - P| = dim|D| - 1$ if and only D is base-point free.

(b) First of all, D need to be globally generated, that is, for any P

$$dim|D - P| = dim|D| - 1$$

Then we have a morphism $\varphi : X \longrightarrow \mathbb{P}^n_k$, with $\varphi^*(\mathcal{O}_{\mathbb{P}^n_k}(1)) = \mathcal{L}(D)$. The only left thing is to tell when φ is a closed immersion. Use 4.3.3.

The criteria $\mathfrak{d}$ separates points is equivlent to that $D - P$ is base-point free. Hence, fix P, for any $Q \neq P$, we have

$$dim|D - P - Q| = dim|D - P| - 1 = dim|D| - 2$$

The criteria $\mathfrak{d}$ separates tangent spaces is equivalent to

$$dim|D - 2P| = dim|D| - 2$$

□

Corollary 4.3.5. *Let D be a divisor on a curve X of genus g.*

(a) If $deg(D) \geq 2g$, then $|D|$ has no base-point, that is, $\mathcal{L}(D)$ is globally generated.

(b) If $deg(D) \geq 2g + 1$, then D is very ample.

Proof.

(a). If $deg(D) \geq 2g$, then $deg(K - D) \leq -2$ and $deg(K - D + P) \leq -1$. Hence $l(K - D) = 0$ and $l(K - D + P) = 0$. By Riemann-Roch,

$$l(D) = deg(D) + 1 - g$$

and

$$l(D - P) = deg(D - P) + 1 - g$$

Thus, $l(D - P) = l(D) - 1$, that is, $dim|D - P| = dim|D| - 1$. Hence $|D|$ has on base points.

(b). If $deg(D) \geq 2g + 1$, then $deg(K - D - P - G) \leq -1$ for any $P, Q \in X$. By Riemann-Roch

$$l(D - P - Q) = deg(D - P - Q) + 1 - g$$

$$l(D) = deg(D) + 1 - g$$

Hence, $l(D) = l(D - P - Q) + 2$, that is, $dim|D| = dim|D - P - Q| + 2$. So $|D|$ is very ample. □

Corollary 4.3.6. *A divisor D on a curve X is ample if and only if $deg(D) > 0$.*

Proof.

Suppose that D is ample if and only if $\exists n \in \mathbb{N}$ such that nD is very ample.

If nD is very ample, there exists a closed immersion $\varphi : X \longrightarrow \mathbb{P}^n_k$, such that $\mathcal{L}(nD) = \varphi^*(\mathcal{O}_{\mathbb{P}^n_k}(1))$. Hence, $ndeg(D) > 0$.

If $deg(D) > 0$, then there exists n such that $deg(nD) \geq 2g + 1$, that is, nD is very ample. Hence D is ample. □

4.3.2 Examples

Example 4.3.7. Consider $g(X) = 0$, then D is ample $\Longleftrightarrow$ $deg(D) > 0 \Longleftrightarrow D$ is very ample. $(2g + 1 = 1)$. For a precise description, $X = \mathbb{P}^1_k$.

Example 4.3.8. Consider $g(X) = 1$, then D is very ample if and only if $deg(D) \geq 3$:

If $deg(D) \geq 3 = 2g + 1$, then $deg(D)$ is of course very ample.

If $deg(D) > 0$, $deg(K-D) < 0$. Hence $l(K-D) = 0$. By Riemann-Roch, $l(D) = deg(D)+1-1 = deg(D)$.

If D is very ample and $deg(D) = 2$, $l(D) = 2$ then $dim|D| = 1$, which implies there is an map $\varphi : X \longrightarrow \mathbb{P}^1_k$. By the uniqueness of the existence of projective morphism, φ is a closed immersion corresponding to $\varphi^*(\mathcal{O}_{\mathbb{P}^1_k}(1)) = \mathcal{L}(D)$. However, $\mathbb{P}^1_k$ doesn't contain any subvariety of dim 1 except $\mathbb{P}^1_k$. Hence $X = \mathbb{P}^1_k$. But $g(\mathbb{P}^1_k) = 0$.

If D is very ample and $deg(D) = 1$, then $l(D) = 1$ and $dim|D| = 1$. By 2.5.12, $2D$ is very ample and $deg(2D) = 2$, which is obviously impossible.

If $deg(D) \leq 0$, then D can not be ample. Thus, D is not very ample, for "very ample" implies "ample".

4.3.3 Additional Exercises

Let C be a smooth projective curve of genus one over a field k with a k-rational point $z \in C(k)$. Prove that $\mathcal{O}_C(3z)$ induces an embedding $C \hookrightarrow \mathbb{P}^2$, whose image is a plane cubic.

Find an example of a smooth plane cubic over some field k without any k-rational point.

Proof. Since $\deg(\mathcal{O}_C(3z)) = 3 \geq 2g_c + 1$, $\mathcal{O}_C(3z)$ is very ample. By Serre Duality, $h^1(C, \mathcal{O}_C(3z)) = h^0(C, \omega_C \otimes \mathcal{O}_C(-3z)) = 0$ since $\deg(\omega_C \otimes \mathcal{O}_C(-3z)) = -3 < 0$. By Riemann-Roch, $h^0(C, \mathcal{O}_C(3z)) =$, which means it induces a closed embedding $i : C \hookrightarrow \mathbb{P}^2_k$. Let $Y = \mathrm{Im}(i)$. Then $\deg(Y) = \deg(\mathcal{O}_C(3z)) = 3$.

Consider $E : x^3 + py^3 + p^2z^3 = 0$ in $\mathbb{P}^2_{\mathbb{Q}}$. We can prove that $E(\mathbb{Q})$ doesn't have rational solutions in $\mathbb{Q}$: Suppose that $[x : y : z]$ is a solution with $x, y, z \in \mathbb{Q}$. We can times some integer such that $x, y, z \in \mathbb{Z}$ such that $\gcd(x, y, z) = 1$. If so, we see that $x^3 \equiv 0 \mod p$, that is, $x \equiv 0 \mod p$. Hence $x = px_1$. Then, similarly, we can see that $y \equiv 0 \mod p$ and $z \equiv 0 \mod p$, which contradicts to the fact that $\gcd(x, y, z) = 1$. $\square$

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