Lecture 33:Filters I – Transfer Functions

OBJECTIVES:

1. Understand how to calculate and interpret magnitude and phase responses
2. Understand the concept of a transfer function and its frequency dependence

READING

Required : Filters Handout (Available on Sharepoint), pgs 1-13

Optional :

1 Introduction

Throughout the last 4 lessons, we have been working with complex numbers to solve *single frequency sinusoidal*, steady state circuits. During this lesson, we are going to start allowing frequency, ω, to vary. In order to look a the behavior of a circuit as ω varies, we will learn to build magnitude and phase plots. Since we are no longer fixing ω, we will not calculate impedances and solve circutis in the same manner.

2 Magnitude Plots

Since we have been looking at circuits with a single fixed frequency, our complex numbers (whether voltage, current or impedance) have had a fixed real part and a fixed imaginary part. As stated above, we are now going to allow ω to vary, so we will write our complex numbers in one of the forms below:

$$K[R + j\omega]$$

$$K\left[R + \frac{1}{j\omega}\right]$$

The reason for this will hopefully be obvious later.

Let's look at a couple of impedances to convice ourselves that write any complex number in this form; this will help us later when we want to write transfer functions in standard form; **our standard form will always have j and ω together and the coefficient on highest order of $j\omega$ will always be 1**. Let $Z_L = R_L + j\omega L$ and $Z_C = R_C - \frac{j}{\omega C}$; write both of these impedances in the form $K[R + j\omega]$:

For Z_L, let $K = L$ therfore:

$$Z_L = L[\frac{R}{L} + j\omega]$$

Z_C is a little harder.... We first want o rewrite Z_C to put the j and the ω back together

$$Z_C = R_C + \frac{1}{j\omega C}$$

Next, factor out a $\frac{1}{C}$

$$Z_C = \frac{1}{C}\left[R_C C + \frac{1}{j\omega}\right]$$

Now that we know how to right complex numbers in this *standard form* (we really over use that term), lets solve for the magnitude of a couple of examples and plot that magnitude vs fequency:

First lets find the magnitude of $20 + j\omega$:

$$|100 + j\omega| = \sqrt{100^2 + \omega^2} = \sqrt{10,000 + \omega^2}$$

We can use MATLAB to plot this:

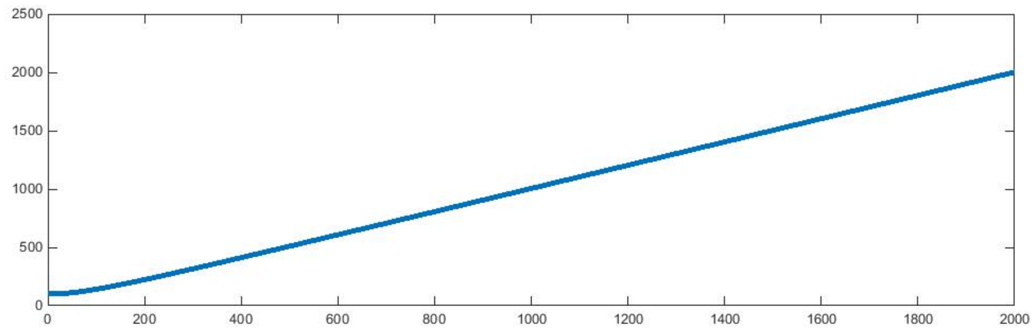


Figure 1: A simple magnitude respsonse

Let's add some complexity by going onto an example from the reading. Lets do a magnitude plot for: $\frac{200}{j\omega+50}$

Start by calculating the magnitude as a function of ω:

$$\left|\frac{200}{j\omega+50}\right| = \frac{|200|}{|j\omega+50|} = \frac{200}{\sqrt{50^2+\omega^2}} = \frac{200}{\sqrt{2500+\omega^2}}$$

We can plot this in MATLAB as well:

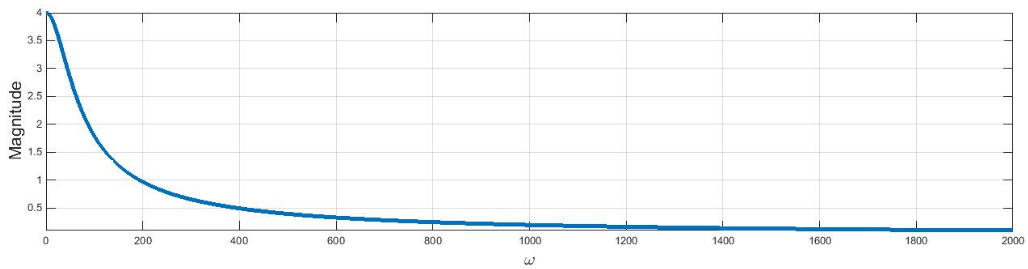


Figure 2: A slightly more complex magnitude respsonse

For the next several lessons we need to get in the habit of using semi-log plots. This will make some of the frequency dependent behavior easier to interpret. Tha magnitude response above replotted on a semilog scale would look like:

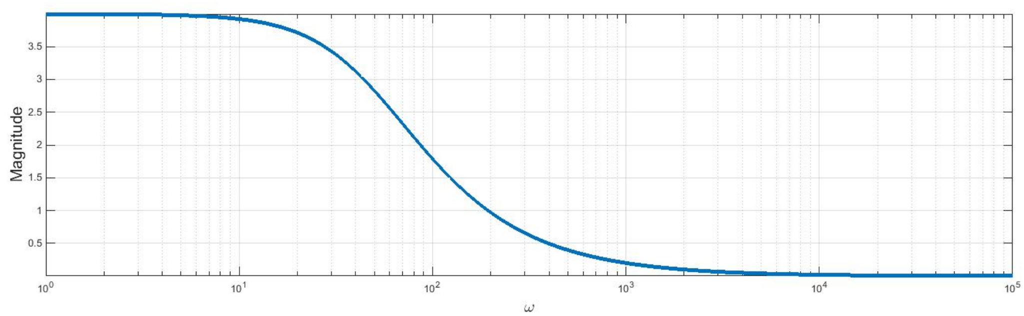


Figure 3: Magnitude response plotted on a semilog (x) scale

If you look closely you will see these plots contain the exact some information.

3 Phase Response

Remember complex numbers have both magnitude (discussed above) and phase. If we look at the complex number $K + j\omega$, the phase is:

$$\theta = \tan^{-1}\left(\frac{\omega}{K}\right)$$

Using our examples above lets do a couple of phase plots. First we will plot the phase of $100 + j\omega$:

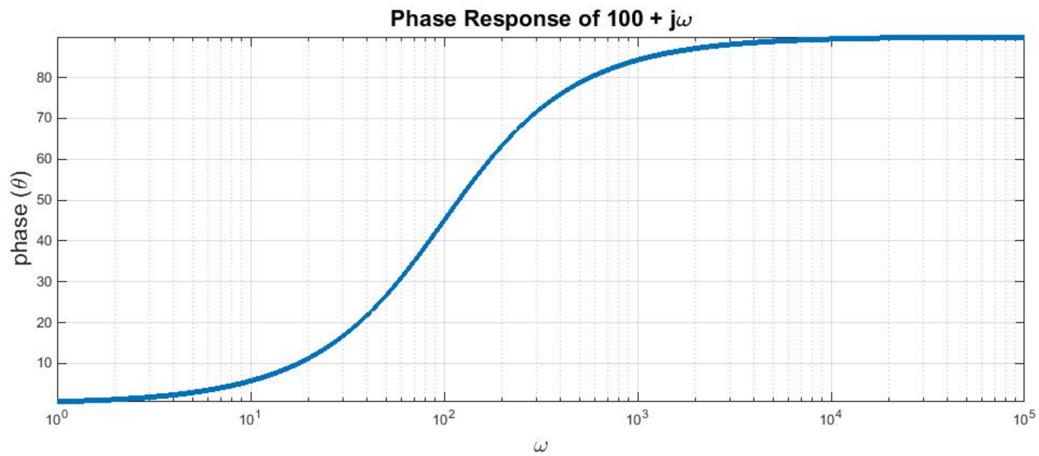


Figure 4: A simple phase response plotted on a semilog (x) scale

Next let's plot the phase response of $\frac{200}{j\omega+50}$

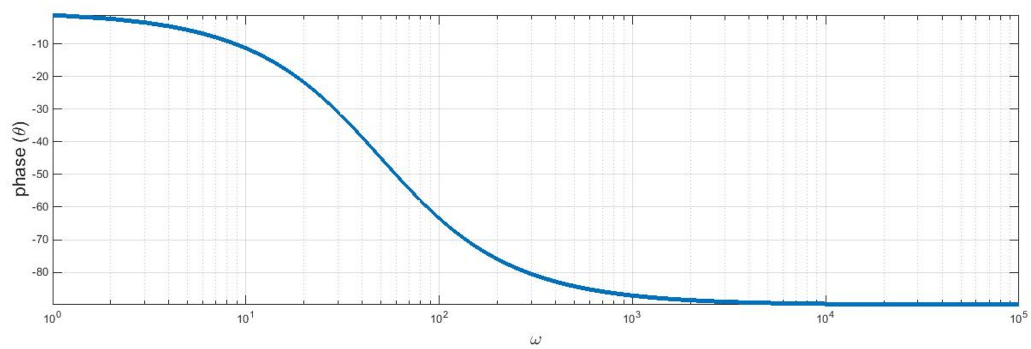


Figure 5: A simple phase response plotted on a semilog (x) scale

4 Transfer Functions

Now that we have developed some knowledge of magnitude and phase responses and their plots, we are going to look at some circuits and develop tranfer functions. The transfer function can be thought of as a complex gain:

$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)}$$

Example 1–To demonstrate this idea lets find the transfer fucntion of the circuit in Figure 6. Assume $C = 1\mu F$ and $R = 1k\Omega$

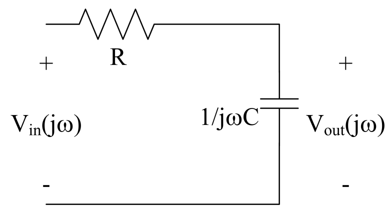


Figure 6: Circuit to accompany Example 1

We can find V_{out} by doing the following voltage division

$$V_{out}(j\omega) = V_{in}(j\omega)\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}$$

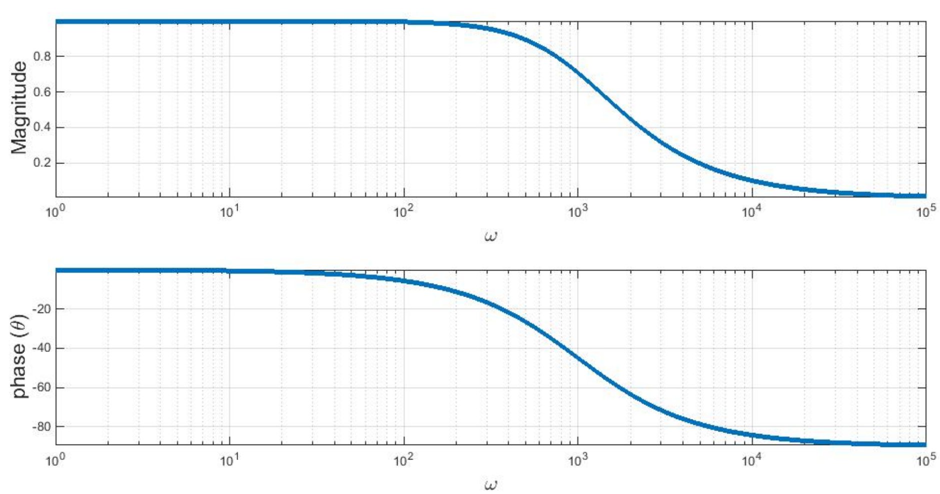
but we need to manipulate this to get a standard form

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$$

Now we plug infor R and C to get our final transfer function:

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{1000}{j\omega + 1000}$$

Now we plot magnitude and phase:



Example 2 – Write a transfer function for the circuit in Figure 7 and then plot the magnitude and phase responses given $C_{in} = 500nF$, $R_{in} = 1k\Omega$, $C_f = 1\mu F$, and $R_f = 1k\Omega$.

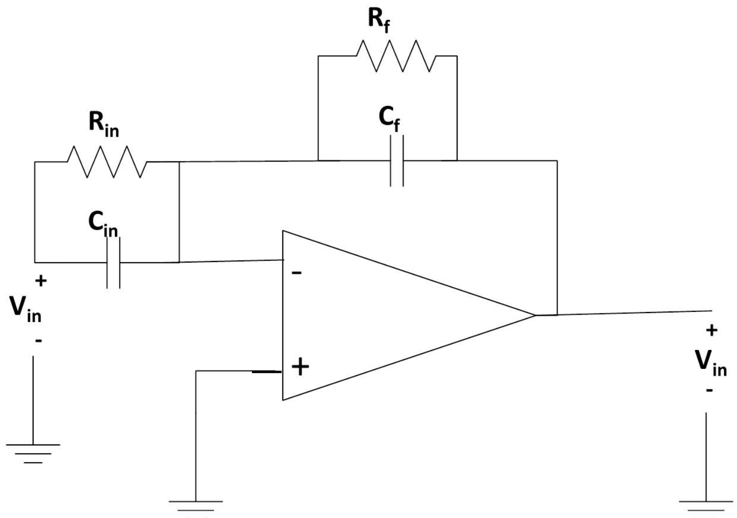


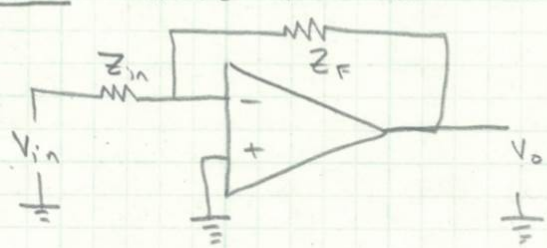
Figure 7: Circuit to accompany Example 2

STEP 1 - SIMPLIFY INPUT AND FEEDBACK IMPEDANCES

$$Z_{in} = \left[\frac{1}{R_{in}} + j\omega C_{in}\right]^{-1} = \frac{R_{in}}{1 + j\omega C_{in} R_{in}}$$

$$Z_F = \frac{R_F}{1 + j\omega C_F R_F}$$ (SAME LOGIC AS ABOVE)

STEP 2 - WRITE KCL AT THE INVERTING NODE



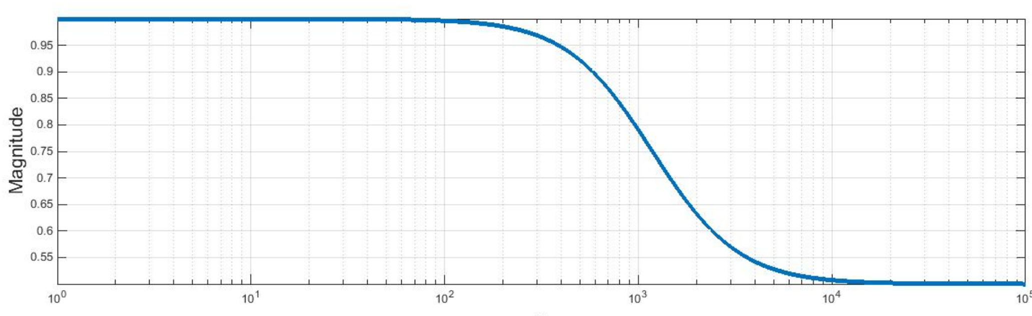
$$\frac{V_{in}}{Z_{in}} = \frac{-V_o}{Z_F} \longrightarrow \frac{V_o}{V_{in}} = \frac{-Z_F}{Z_{in}}$$

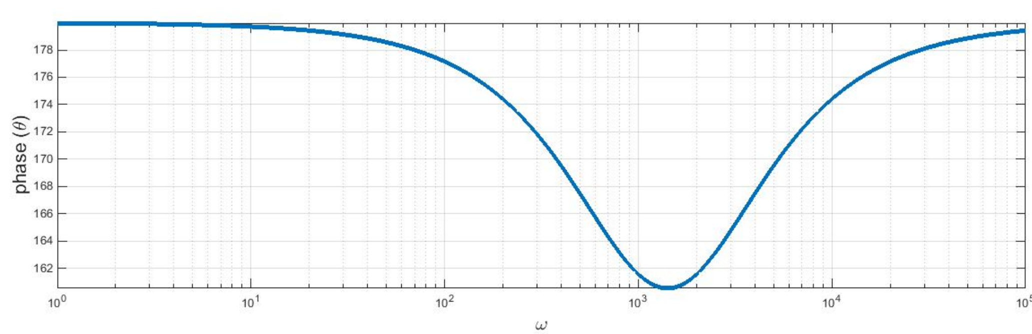
STEP 3 - PLUG IN FOR Z_{in} & Z_F, THEN SIMPLIFY

$$\frac{V_o}{V_{in}} = H(j\omega) = \left[\frac{-R_F}{1 + j\omega C_F R_F}\right]\left[\frac{1 + j\omega C_{in} R_{in}}{R_{in}}\right]$$

$$= \frac{-R_F C_{in} R_{in}}{R_{in} C_F R_F}\left[\frac{\frac{1}{C_{in} R_{in}} + j\omega}{\frac{1}{C_F R_F} + j\omega}\right]$$

$$H(\omega) = \frac{-C_{in}}{C_F}\left[\frac{j\omega + \frac{1}{C_{in} R_{in}}}{j\omega + \frac{1}{C_F R_F}}\right]$$





5 Cascading Transfer Functions

Recall all of our lessons on Op Amps where we showed we could cascade gains. Since transfer functions are just complex gains we can cascade them as well. The primary difference is with Op Amps your first stage could suffer from loading problems, but in general, successive stages were immune to loading; this is not the case for transfer functions in general. In general, any stage can have loading issues when transfer functions are cascaded, but this is easily fixed with our old friend the buffer.

Example 3– For the circuits in Figures 8, 9, and 10, find the transfer functions if $C_1 = C_2$ and $R_1 = R_2$. Does the cascade (or chain) rule work as you would expect? Plot Magnitude and Phase response of each transfer function for $C_1 = C_2 = 1\mu F$and $R_1 = R_2 = 1k\Omega$.

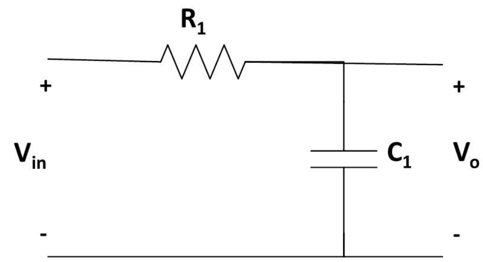


Figure 8: Circuit 1 to accompany Example 3

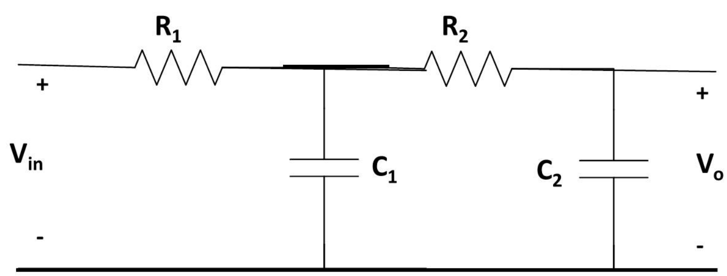


Figure 9: Circuit 2 to accompany Example 3

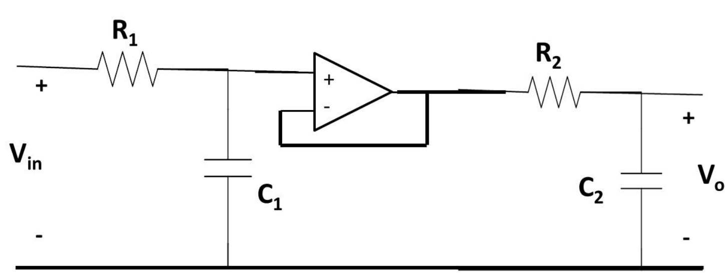
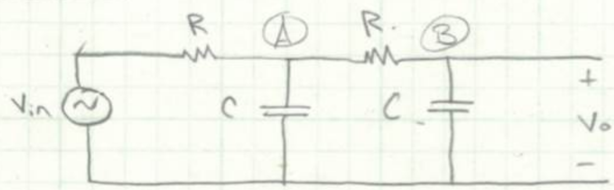


Figure 10: Circuit 3 to accompany Example 3

CIRCUIT 1

$$V_{OUT} = V_{in}\left[\frac{\frac{1}{j\omega C_1}}{\frac{1}{j\omega C_1}+R_1}\right] \rightarrow \frac{V_{OUT}}{V_{in}} = \frac{1}{1+j\omega C_1 R_1} = \boxed{\frac{\frac{1}{R_1 C_1}}{j\omega + \frac{1}{R_1 C_1}}}$$

CIRCUIT 2



$R_1 = R_2 = R$

$C_1 = C_2 = C$

WRITE NODE EQUATIONS FOR A & B $\boxed{V_B = V_o}$

A $$\frac{V_A - V_{in}}{R} + j\omega C V_A + \frac{V_A - V_B}{R} = 0$$

B $$\frac{V_o - V_A}{R} + j\omega C V_o = 0$$

$$V_A = R\left[\frac{V_o}{R} + j\omega C V_o\right] = V_o\left[1 + j\omega RC\right]$$

$$V_o\left[\frac{1+j\omega RC}{R}\right] - \frac{V_{in}}{R} + j\omega\frac{RCV_o + (j\omega)^2 R^2 C^2 V_o}{R}$$

$$+ V_o\left[\frac{1+j\omega RC}{R}\right] - \frac{V_o}{R} = 0$$

$$V_o\left[\frac{1+3j\omega RC + (j\omega)^2 (RC)^2}{R}\right] = \frac{V_{in}}{R}$$

$$\frac{V_o}{V_{in}} = \frac{1}{(j\omega)^2 (RC)^2 + 3j\omega RC + 1} = \frac{\left(\frac{1}{RC}\right)^2}{(j\omega)^2 + j3\omega\left[\frac{1}{RC}\right] + \left[\frac{1}{RC}\right]^2}$$

CIRCUIT 3

BECAUSE THE STAGES ARE SEPERATED BY BUFFERS, $H(j\omega)$ IS EASY

$$H(j\omega) = \left[\frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}\right]\left[\frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}\right]$$

$$H(j\omega) = \frac{\left(\frac{1}{RC}\right)^2}{j\omega^2 + j2\omega\left(\frac{1}{RC}\right) + \left(\frac{1}{RC}\right)^2}$$

NOTICE THIS IS NOT THE SAME AS CIRCUIT 2!