# Lecture 8: Linearity and Superposition

**OBJECTIVES:**

1. Understand what is meant by a *Linear* Circuit
2. Understand and be able to apply the principle of superposition

**READING**

**Required** :

- Textbook, sections 3.3, pages 100–109

**Optional** : None

## 1 What does it mean for a circuit to be Linear?

Circuits are linear when they can be modeled using only linear elements and independent sources. Further any linear circuit will obey the following 2 principles:

**Homogeneity** : outputs are proportional to the inputs. This is also referred to as **proportionality**.

**Additivity** : output due to multiple inputs can be found by finding the output due to each individual input and then adding them. This is also referred to as **superposition**; we will devote an entire section to this later.

Mathematically these properties are written as:

$$f(Kx) = Kf(x) \qquad \text{proportionality} \tag{1}$$

$$f(x_1 + x_2) = f(x_1) + f(x_2) \qquad \text{superposition} \tag{2}$$

## Lecture 8: Linearity and Superposition

**Example 1:** Solve for $v_{out}$ in the circuit in Figure 1 given $v_{in} = 1V$. Then using the proportionality principle, find the output given $v_{in} = 12V$

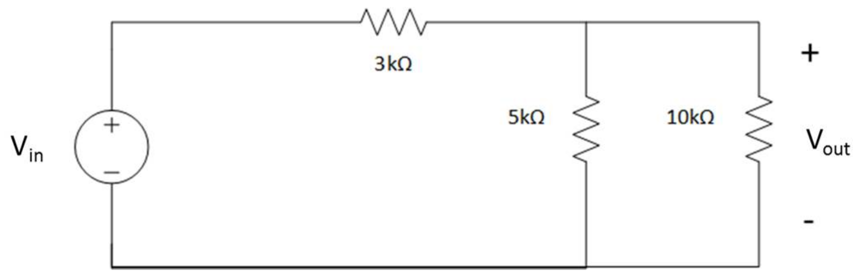
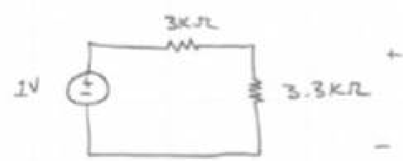



Figure 1: Circuit to accompany example 1

STEP 1: COMBINE 5K & 10K RESISTORS

$$R_{eq} = \frac{(5\times10^3)(10\times10^3)}{15\times10^3} = 3.33\,k\Omega$$




STEP 2: USE VOLTAGE DIVISION TO FIND $V_{out}$

$$V_{out} = (1^V)\left(\frac{3.3K\Omega}{6.3K\Omega}\right) = 0.524\ V$$

STEP 3: USE PROPORTIONALITY TO FIND $V_{out}$ WHEN $V_{in}=12V$

$$V_{out-12} = \left[\frac{12V}{1V}\right](0.524V) = 6.28\,V$$

This also provides a useful tool, the voltage divider, which is just a ratio proportion of voltage drops across resistors.

$$\frac{v_{in}}{R_1 + R_2} = \frac{v_{out}}{R_2} \tag{3}$$

$$v_{out} = \frac{R_2}{R_1 + R_2} v_{in} \tag{4}$$

## 2 Superposition (Additivity)

The goal of superposition is to simplify the analysis of circuits with multiple sources by only having to deal with a single source at a time. Notice, I said it is simpler not shorter.

*The steps for circuit analysis using superposition are:*

1. Turn off *all but one of the independent sources in the circuit*
2. *Solve for output of the circuit based on the one remaining source*
3. *Repeat for every source in the circuit*
4. *Sum the results to get a total output*

**What do we mean by turning off sources?**

**To turn off a voltage source,** we would want the voltage across the two nodes of the source to be zero, so we replace the source with a short circuit

**To turn off a current source,** we want zero current to flow, so we replace it with an open circuit

**Example 2:** Without solving anything else, re-draw the circuit in Figure 2 with each source turned off.

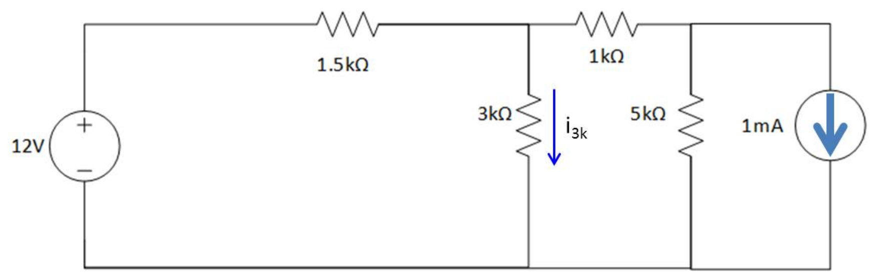
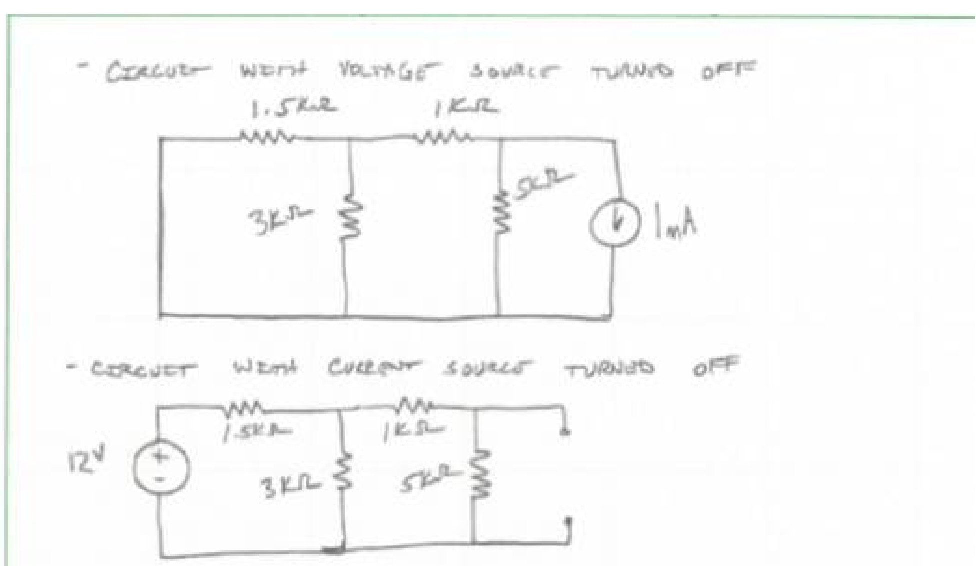



Figure 2: Circuit to accompany example 2



**Example 3:** (Textbook Exercise 3-25) – The circuit in Figure 3 contains two $R$–$2R$ modules. Use superposition to find $v_O$.

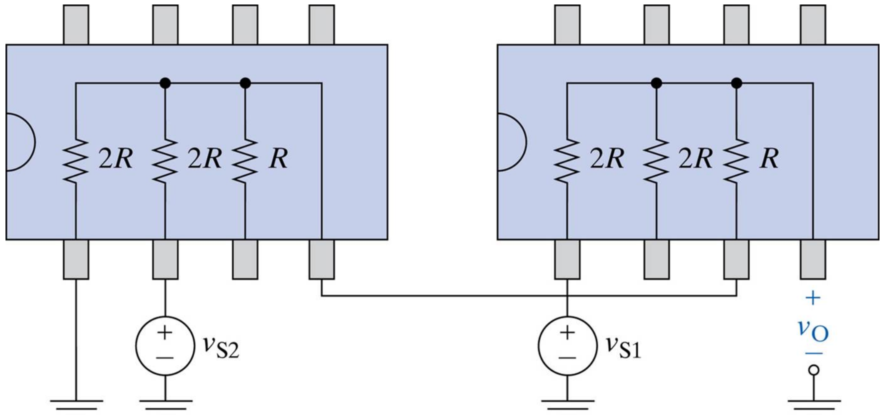
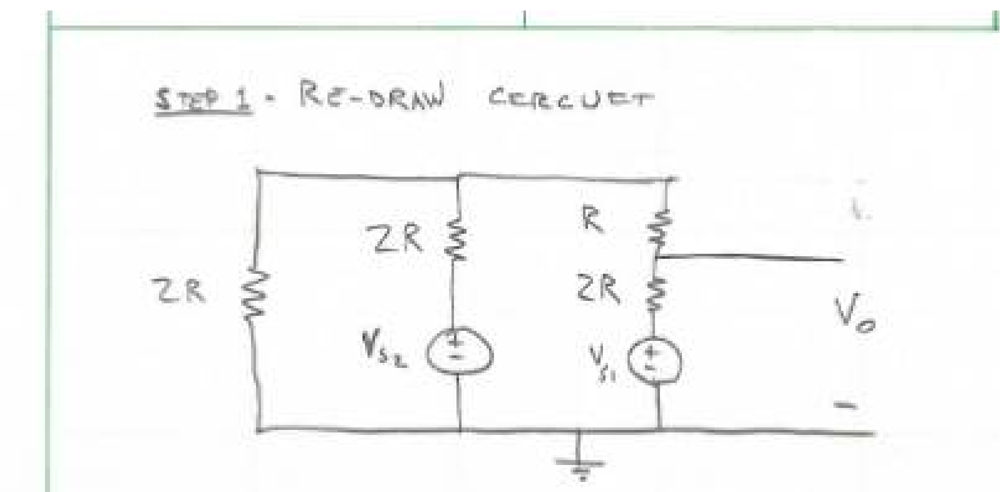
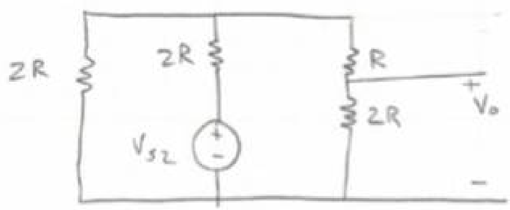
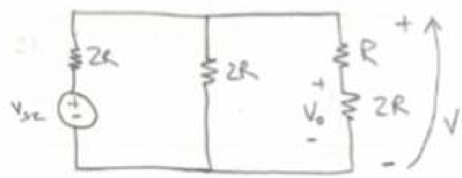



Figure 3: Circuit to accompany example 3. NOTE: In images taken from the textbook, crossed wires are only connected if they are *dotted*. For circuits that I draw on the board or ones I generate for notes, crossed wires are connections. You will need to learn to infer from context...... If there are some *dotted* nodes, crossed wires without dots are not connected; if there are no dots, crossed wires are connected.



STEP 2 - "TURN OFF" $V_{S1}$ & SOLVE FOR $V_o$



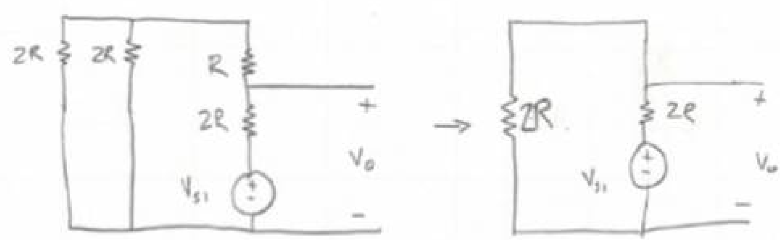




$2R \parallel 3R = \frac{6}{5}R$

$$V' = \left(\frac{\frac{6}{5}R}{\frac{16}{5}R}\right) V_{S2} = \frac{3}{8} V_{S2}$$

$$V_o = \frac{2}{3} V' = \frac{2}{3}\left(\frac{3}{8} V_{S2}\right) = \frac{1}{4} V_{S2}$$

STEP 3 - "TURN OFF" $V_{S2}$ & SOLVE FOR $V_o$




- USE VOLTAGE DIVISION TO FIND $V_{2R}$

$$V_{2R} = \frac{2R}{4R} V_{S1} = \frac{V_{S1}}{2}$$

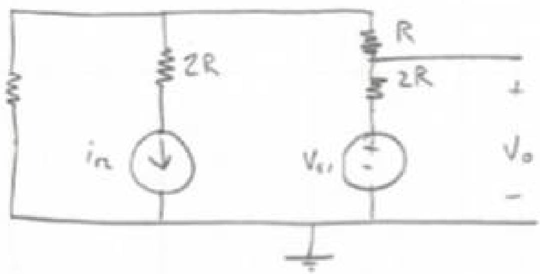
$$V_o = V_{2R}$$

STEP 4 - ADD OUTPUTS FROM STEPS 2 & 3

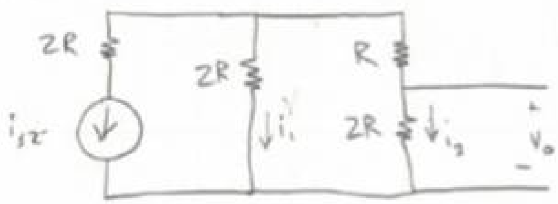
$$V_o = \frac{1}{2} V_{S1} + \frac{1}{4} V_{S2}$$

**Example 4:** (Textbook Exercise 3-26) – Repeat the above example, but replace $v_{s2}$ in Figure 3 with a current source, $i_{s2}$ with the reference arrow pointed toward ground

STEP 1 - DRAW THE CIRCUIT




STEP 2 - "TURN OFF" $V_{s1}$

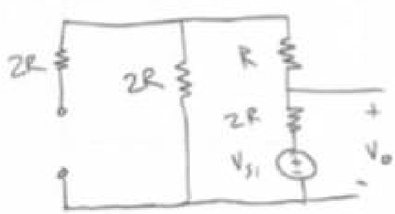



USE CURRENT DIVIDER TO FIND CURRENT IN OUTPUT RESISTOR

$$i_2 = -i_{s2}\left[\frac{2R}{5R}\right] = \frac{2 i_{s2}}{5}$$

$$V_o = -2R \cdot \frac{2}{5} i_{s2} = -\frac{4}{5} i_{s2} R$$

STEP 3 "TURN OFF" $i_{s2}$




$$V_o = \frac{3R}{5R} V_{s1} = \frac{3}{5} V_{s1}$$

STEP 4 - ADD OUTPUTS FROM PREVIOUS STEPS

$$V_o = \frac{3}{5} V_{s1} - \frac{4}{5} i_{s2} R$$

**Example 5:** Use Superposition to find $v_O$ in Figure 4

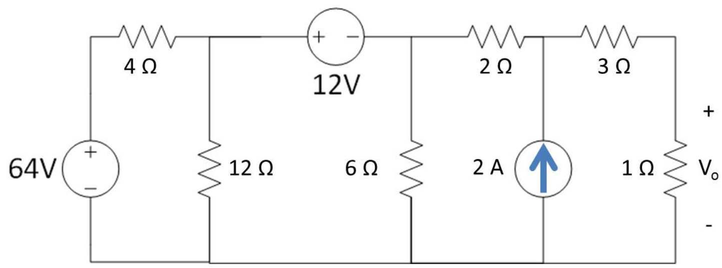
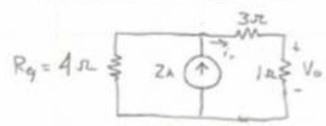



Figure 4: Circuit to accompany example 5

STEP 1 - TURN OFF BOTH VOLTAGE SOURCES


$R_{eq} = (4 \| 12 \| 6) + 2$
$= 4\ \Omega$

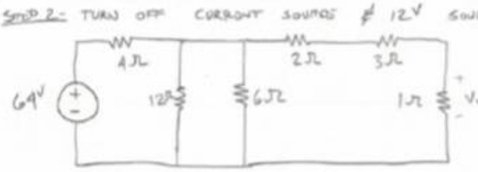
USE CURRENT DIVISION TO FIND RIGHT SIDE CURRENT

$i_r = 2A\left(\frac{4}{8}\right) = 1A$

USE OHM'S LAW TO FIND $V_o$

$V_o = (1\ A)(1\ \Omega) = 1^V$

STEP 2: TURN OFF CURRENT SOURCE & $12^V$ SOURCE





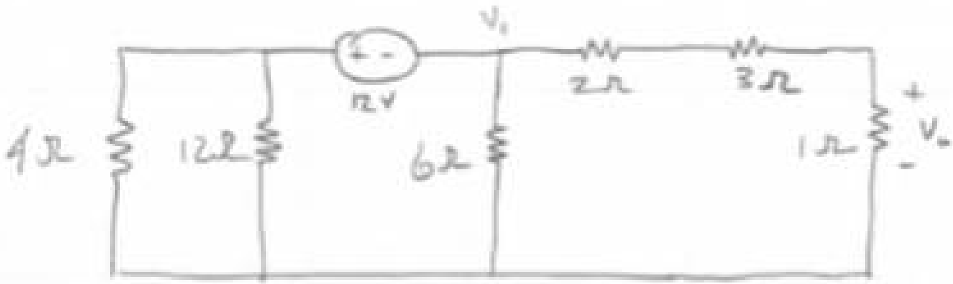
USE VOLTAGE DIVISION TO FIND V'

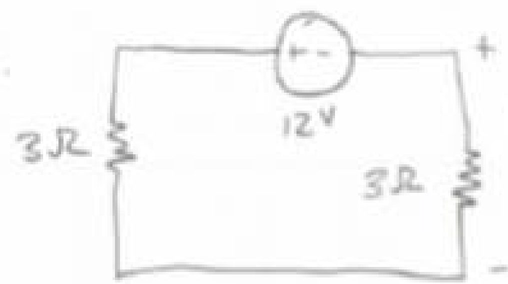
$V' = \left(\frac{2.4}{6.4}\right) 64^V = 24^V$

APPLY VOLTAGE DIVISION AGAIN TO FIND $V_o$

$V_o = \frac{1}{6}(24^V) = 4^V$

STEP 3 - TURN OFF CURRENT SOURCE & 6A SOURCE




$V' = -6V$

USE VOLTAGE DIVISION TO FIND $V_o$

$$V_o = \frac{1}{6}(-6V) = -1V$$

STEP 4 - SUM PREVIOUS STEPS TO GET $V_o$

$$\boxed{V_o = 1V + 4V - 1V = 4V}$$