



Lecture 19: Time Varying Signals

OBJECTIVES:

1. Understand and be able to use the unit step function
2. Understand and be able to use the unit ramp function
3. Understand and be able to use the delta function
4. Understand and be able to use exponential functions
5. Understand and be able to use sinusoidal functions

READING

Required :

- Textbook, section 5.1–5.5, pages 240–261

Optional : None

1 Introduction to time varying signals

Until now, we have been looking only at signals (voltage, current) that are constant for all time. They were at a give value prior to our circuit analysis and will remain at the value throughout our analysis. Moving forward, we will need to look at some time varying signals. Specifically we will look at the following signals:

1. *Unit Step function*
2. *Delta function (or impulse function)*
3. *Ramp function*
4. *Exponential function*
5. *Sinusoids*

We will also show how these waveforms can be combined to synthesize more complex waveforms.

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2 The Unit Step Function

We start with the simplest of our new signals. The unit step function is simply zero before time $t < 0$ and equal to 1 for time $t \geq 0$. Mathematically,

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases} \quad (1)$$

The unit step function is shown in Figure 1.

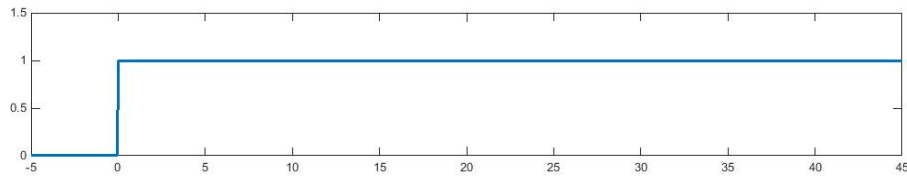
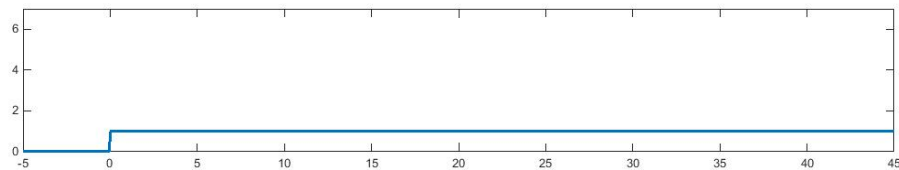


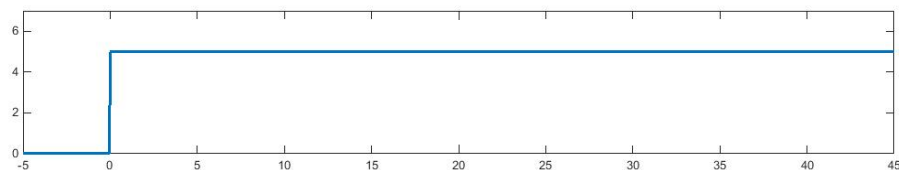
Figure 1: Unit Step Function

Example 1 - Scaling of the unit step – Draw a graph of $5u(t)$

First graph $u(t)$



Multiply entire graph by 5



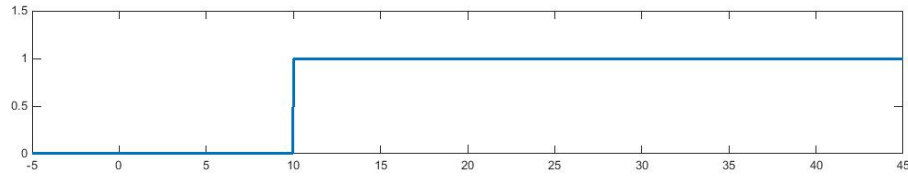
Shifting the unit step– What if I want the step function to turn on at some time other than $t = 0$? The unit step function always turns on when the argument is equal to zero. What if I replace the argument t with $t - t_0$? It should be obvious that $t - t_0 = 0$ only when $t = t_0$.

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Example 2 - Shifting the unit step – Graph $u(t - 10)$

The argument of this step function equals zero when $t = 10$



Adding and subtracting unit step functions—Next we will look at adding and subtracting shifted and scaled step functions to create composite functions.

Example 3 - Creating a pulse—Can we write an equation for a single pulse with amplitude 2. The pulse should be on for $8s \leq t \leq 15s$.

To start lets write an equation for an amplitude 2 step function that turns on at 8 seconds.

$$f(t) = 2u(t - 8) \quad (2)$$

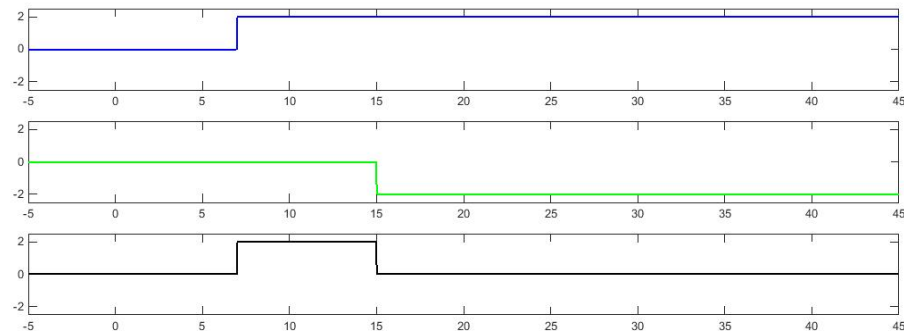
Now we can write an equation for an amplitude -2 step function that starts at 15 seconds

$$f(t) = -2u(t - 15) \quad (3)$$

What if we add these?

$$f(t) = 2u(t - 8) - 2u(t - 15) \quad (4)$$

The result is shown in the graph below:



SOLUTION

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Example 4—Using step functions, write an equation for the plot shown in Figure 2.

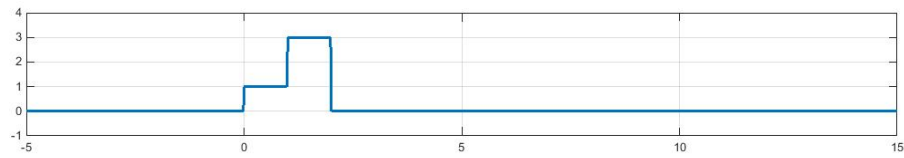


Figure 2: Plot for Example 4

The function turns on with a value of 1 at $t = 0$

$$f(t) = u(t) \tag{5}$$

At time $t = 1$ the value of the function increases to 3 (an increase of 2)

$$f(t) = u(t) + 2u(t - 1) \tag{6}$$

Finally the function turns off at time $t = 2$, (a decrease of 3)

$$f(t) = u(t) + 2u(t - 1) - 3u(t - 2) \tag{7}$$

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3 The Unit Ramp Function

The unit ramp function is defined as

$$r(t) = tu(t) \quad (8)$$

which can be expanded (for clarity)

$$r(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases} \quad (9)$$

It should be obvious that this function (for positive t) is just a ramp with $\text{slope} = 1$. The unit ramp is shown in Figure 3

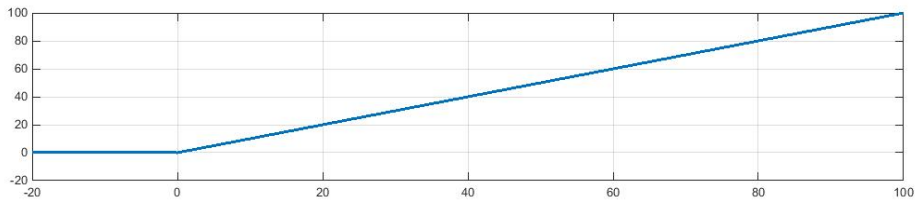
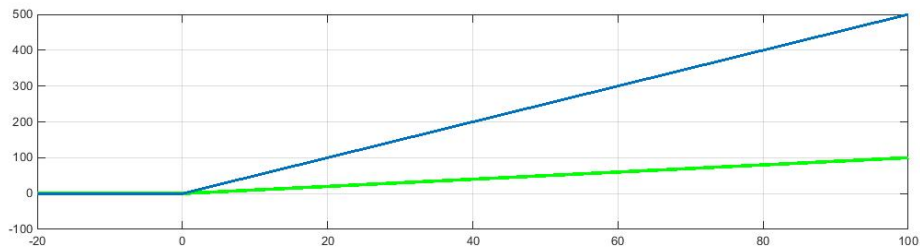


Figure 3: The Unit Ramp

What happens if we multiply the unit ramp by a scalar?

$$A \times r(t) = \begin{cases} A \times 0 = 0 & \text{for } t < 0 \\ A \times t = At & \text{for } t \geq 0 \end{cases} \quad (10)$$

We change the slope for $t > 0$. See figure below to see $r(t)$ and $5r(t)$ on the same graph



Can we shift a ramp function?

Yes!, just like we shift a unit step

$$r(t - t_0) = \begin{cases} 0 & \text{for } t < t_0 \\ t - t_0 & \text{for } t \geq t_0 \end{cases} \quad (11)$$

It should also be clear that you can combine the two properties above to get a shifted and scaled ramp

$$Ar(t - t_0) = \begin{cases} 0 & \text{for } t < t_0 \\ A(t - t_0) & \text{for } t \geq t_0 \end{cases} \quad (12)$$

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4 Delta (or impulse) Function

To understand an impulse function, let's start by writing an equation of a unit area pulse centered on $t = 0$

$$\delta(t) = \frac{1}{T} \left[u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right) \right] \quad (13)$$

What happens if we take the limit of this function as $T \rightarrow 0$?

The pulse gets narrower and the amplitude gets higher, but the area always stays 1. In the limit you have a zero width pulse with infinite amplitude and area 1. Mathematicians will have to forgive me..... We call this a delta function.

Mathematically

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases} \quad (14)$$

It should also be noted that the delta function is related to the unit step by

$$\delta(t) = \frac{\partial u(t)}{\partial t} \quad (15)$$

Since this function is related to the unit step and the unit step can be shifted, so can the delta function.

$$\delta(t - t_0) = \begin{cases} 0 & \text{for } t \neq t_0 \\ \infty & \text{for } t = t_0 \end{cases} \quad (16)$$

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5 Exponential Functions

Pay attention to this section... Exponential functions are important in circuit analysis...

An exponential function has the form

$$f(t) = e^{-(t-t_0)}u(t-t_0) \quad (17)$$

Note: Because of the step function it is always zero prior to $t = t_0$

For $t_0 = 0$ this simplifies to

$$f(t) = e^{-t}u(t) \quad (18)$$

The plot of this function is shown in Figure 4

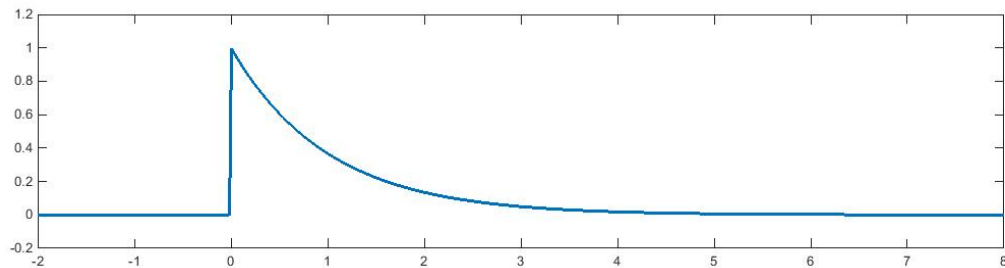


Figure 4: Exponential Function

Does this function ever reach zero? *No, but it will get very close.... As a matter of fact when it reaches 0.67% we will say it is effectively zero.*

How long does it take to decay to this value? – To really talk about the decay time we have to modify our equation a little and introduce the concept of a time constant, T_c . If we include the time constant our equation now looks like

$$f(t) = e^{-\frac{t}{T_c}}u(t) \quad (19)$$

Note: It should be obvious in our simple form of the equation from before $T_c = 1$

To find out how fast this function decays, let's calculate the value of $f(t)$ for different values of t

The table below shows the value of $f(t)$ after integer numbers of time constants have passed.

t	$f(t)$
0	1.000
$1T_c$	0.367
$2T_c$	0.135
$3T_c$	0.049
$4T_c$	0.0183
$5T_c$	0.0067

This shows that an exponential function is effectively zero after 5 time constants (some texts will use 4 time constants).

Much like the unit step function, we can also scale the amplitude of the exponential function. This gives us:

$$f(t) = Ae^{-\frac{t}{T_c}}u(t) \quad (20)$$

Where A is the amplitude and T_c is the time constant (as before).

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Example 5 – Plot $f(t) = 4e^{-1000t}u(t)$ and determine with the function is effectively equal to zero.

First calculate the time constant:

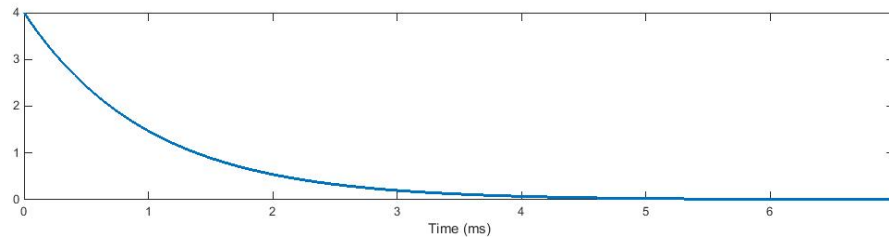
$$T_c = \frac{1}{1000} = 1 \text{ ms} \quad (21)$$

This immediately tells us that the function is effectively zero after 5 ms.

To plot the function we need to calculate a few values.... We will use our table from before. We know our initial value is 4 and after 1 T_c the function has decayed to 36.7% of its initial value, and after 2 T_c it has decayed to 13.5%, etc

t	$f(t)$
0	1.000
1 ms	1.468
2 ms	0.540
3 ms	0.196
4 ms	0.073
5 ms	≈ 0

The plot is shown below:



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6 Sinusoids

Understanding sinusoids is critical to the study of many areas of science and engineering; this is very true in our study of electrical circuits.

Note: All the functions we have defined above are zero for $t < 0$; for our sinusoids we will assume they exist for all time, $-\infty < t < \infty$.

6.1 Sinusoid Basics

A simple sinusoid can be expressed as either a sine or a cosine function

$$\begin{aligned} f_1(t) &= \sin\left\{\frac{2\pi t}{T_0}\right\} \\ f_2(t) &= \cos\left\{\frac{2\pi t}{T_0}\right\} \end{aligned}$$

where T_0 is the period of the sinusoid

Sine and Cosine are shown as functions on time in Figure 5

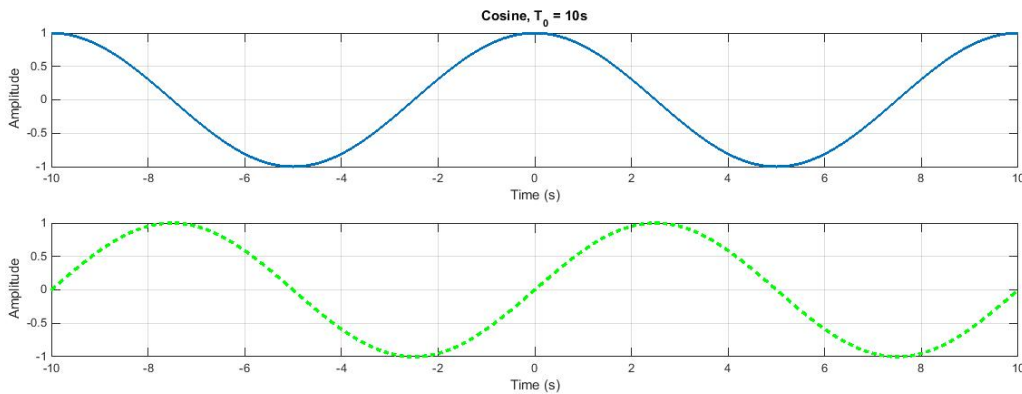


Figure 5: Sin and Cos shown as functions of time

Another way to view *sin* and *cos* is to look at them on the same graph as a function of angle. That is what you see in Figure 6:

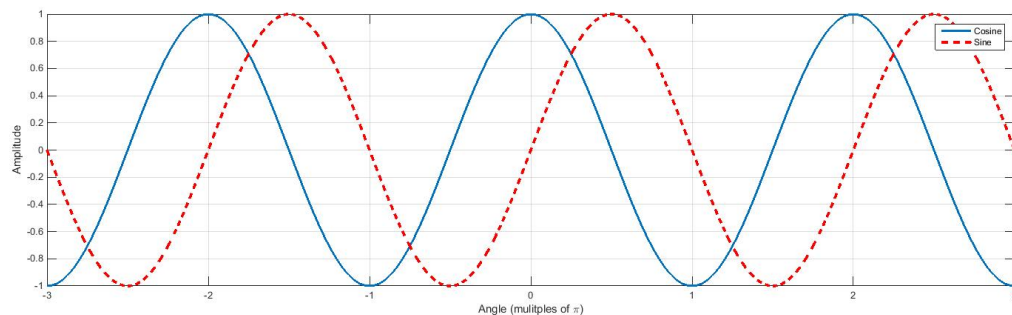


Figure 6: Sin and Cos shown as functions of angle (in multiples of π)

What you should notice here is that the cos wave *leads* the sin wave by 90° or $\frac{\pi}{2}$ rad. They are otherwise identical. This can be written mathematically as

$$\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right) \quad (22)$$

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6.2 Time or Phase Shifted Sinusoids

This section will discuss shifted sinusoids. Mathematically a phase shifted or time shifted sinusoid are the identical (which we will show). The text book refers to this as a general sinusoid and it has the form:

$$f(t) = A \cos \left\{ \frac{2\pi(t - T_s)}{T_0} \right\} \quad (23)$$

where A is the amplitude, T_0 is the period (as before) and T_s is the time shift.

We can expand this equation to show that the time shift can also be represented as a angular phase shift:

$$f(t) = A \cos \left\{ \frac{2\pi t}{T_0} - \frac{2\pi T_s}{T_0} \right\} = A \cos \left\{ \frac{2\pi t}{T_0} + \phi \right\} \quad (24)$$

is should be evident that

$$\phi = -\frac{2\pi T_s}{T_0} \quad (25)$$

ϕ is in radians.

The phase shift is often displayed in degrees instead of radians:

$$\phi = -\frac{360^\circ T_s}{T_0} \quad (26)$$

Care must be taken when doing calculations to not mix degrees and radians!

Example 6 – Plot $f(t) = 4 \cos(\frac{\pi t}{5} - 30^\circ)$ as a function of time:

First find the period T_0 :

$$\frac{2\pi t}{T_0} = \frac{\pi t}{5} \quad (27)$$

Therefore $T_0 = 10$ s

Next, convert 30° to rad:

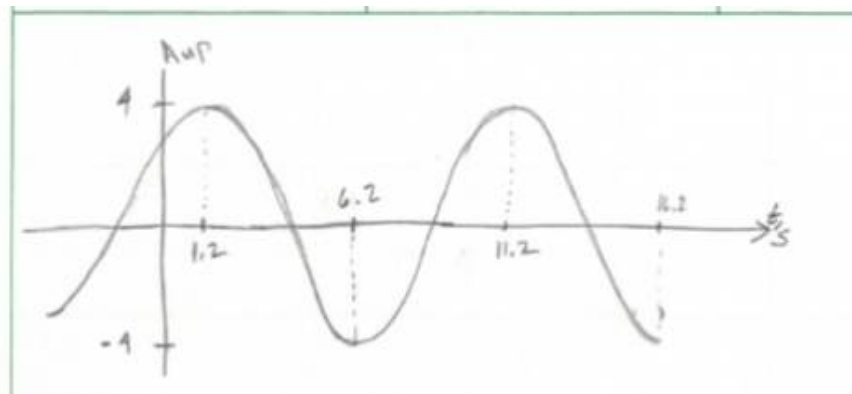
$$-30^\circ * \frac{\pi}{180^\circ} = -\frac{\pi}{6} \quad (28)$$

Next we can convert ϕ to a time shift T_s (remember $T_0 = 10$):

$$-\frac{2\pi T_s}{T_0} = -\frac{2\pi T_s}{10} = -\frac{\pi}{6} \quad (29)$$

Therefore $T_s = 1.2$ s

So we have a cosine with $A = 4$, $T_0 = 10$ s and $T_s = 1.2$ s:



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An alternative version of a general sinusoid can be derived by starting with:

$$f(t) = A \cos \left\{ \frac{2\pi t}{T_0} + \phi \right\} \quad (30)$$

and expanding it using the following trigonometric identity:

$$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y) \quad (31)$$

I will leave the algebra to you, but using the last two equations we can arrive at

$$f(t) = A \cos \phi \cos \left(\frac{2\pi t}{T_0} \right) - A \sin \phi \sin \left(\frac{2\pi t}{T_0} \right) \quad (32)$$

$A \cos \phi$ and $A \sin \phi$ are constants and are referred to as Fourier coefficients.

6.3 Damped Sinusoids

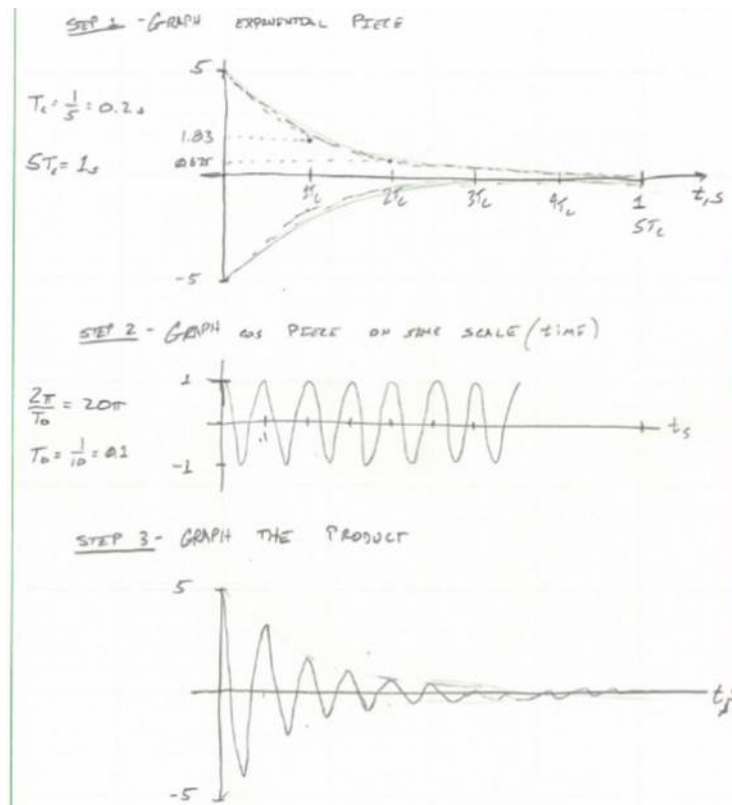
The last functions we will discuss are damped sinusoids; these functions are the product of a decaying exponential and a sinusoid and have the form

$$f(t) = Ae^{-\alpha t} \cos \left\{ \frac{\beta t}{T_0} \right\} \quad (33)$$

where we have neglected any time shifts and α is an attenuation constant (or $\frac{1}{T_c}$) and $\beta = \frac{2\pi}{T_0}$

To graph a damped sinusoid graph the decaying exponential, then mirror it across the t axis, then fill it in with the appropriate sinusoid.

Example 7 – Graph $f(t) = 5e^{5t} \cos(20\pi t)$



OR USE MATLAB!