Tutorial appendix

Experiment 1

Kinematics of a car-like mobile robot

Mobile robot basic – differential drive

Institute of Informatics VII
Julius Maximilian Universität Würzburg
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The current document is an introduction to the Mobile Robots in the frame of tele-laboratory related to car-like mobile robots. Below text consists in basics about differential drive – the simplest mobile robot, which can be considered as a basic for car-like mobile robots subject.

Differential drive

One of the simplest mobile robot constructions is a chassis with two fixed wheels. Understanding this construction helps you to grasp some basic kinematics of car-like robots. Usually differential drive mobile robots have an additional castor wheel as the third fulcrum. It is usually used for stability. Sometimes roller-balls can be used but from the kinematics point of view, there are no differences in calculations.

As it can rotate freely in all directions, in our calculation we can omit the castor wheel because it only has a very little influence over the robot's kinematics
In case of differential drive, to avoid slippage and have only a pure rolling motion, the robot must rotate around a point that lies on the common axis of the two driving wheels. This point is known as the instantaneous center of curvature (ICC) or the instantaneous center of rotation (ICR). By changing the velocities of the two wheels, the instantaneous center of rotation will move and different trajectories will be followed (Fig. 2).

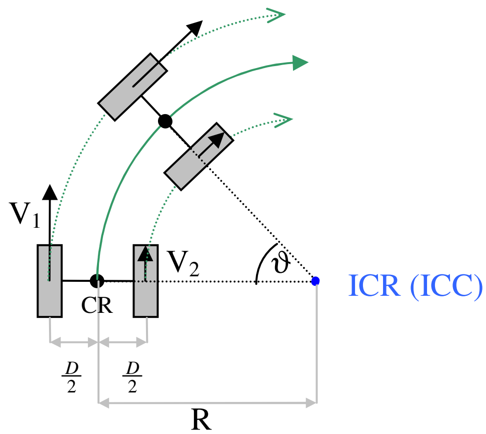

Fig. 1. The differential drive motion example.

At each moment in time the left and right wheels follow a path (Fig. 1) that moves around the ICR with the same angular rate $\omega = \frac{d\vartheta}{dt}$, and thus:

EQ 1

$$\omega \cdot R = v_{CR}$$

$$\omega \cdot (R + \frac{D}{2}) = v_1$$

$$\omega \cdot (R - \frac{D}{2}) = v_2$$

,where v_1 is the left wheel's velocity along the ground, and v_2 is the right wheel's velocity along the ground, and R is the signed distance from the ICC to the midpoint between the two wheels. Note that v_1, v_2, and R are all functions of time. At any moment in time:

EQ 2

$$R = \frac{v_2 + v_1}{v_2 - v_1} \cdot \frac{D}{2}$$

EQ 3

$$\omega = \frac{v_2 - v_1}{D}$$

The velocity of the CR point, which is the midpoint between the two wheels, can be calculated as the average of the velocities v_1 and v_2:

EQ 4

$$V_{CR} = \frac{v_1 + v_2}{2}$$

If $v_1 = v_2$, then the radius R is infinite and the robot moves in a straight line (see Fig. 2a). For different values of v_1 and v_2, the mobile robot does not move in a straight line but rather follows a curved trajectory around a point located at a distance R from CR (Fig. 2b and c – turning around one of the wheels), changing both the robot's position and orientation. If $v_1 = -v_2$, then the radius R is zero and the robot rotates around CR (it rotates in place Fig. 2d).

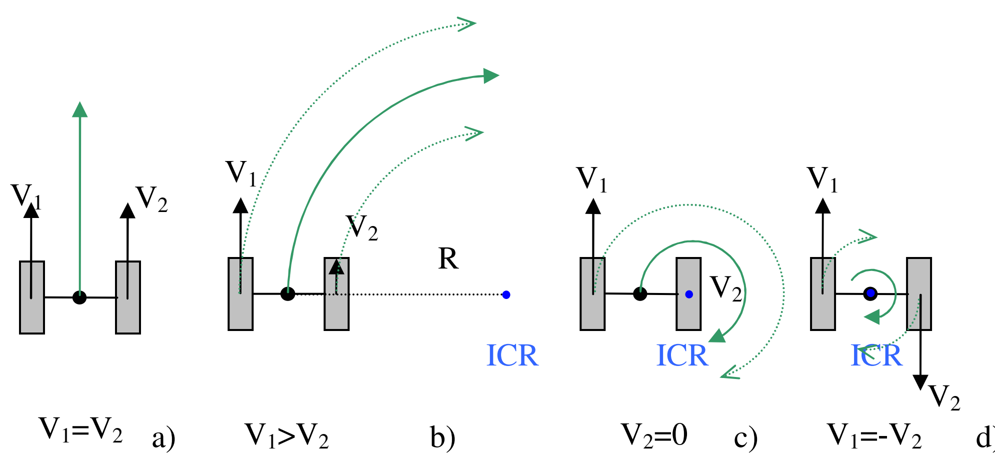

Fig. 2. The different moving possibilities for differential drive a) motion forward or backward, R is infinite, b) turning R>D/2, c) turning R=D/2, d) turning R=0;

A differential drive mobile robot is very sensitive to the relative velocity of the two wheels. Small differences between the velocities provided to each wheel cause different trajectories, not just a slower or faster robot. Differential drive mobile robots typically have to use castor wheels for balance. Thus, differential drive vehicles are sensitive to slight variations in the ground plane

Odometry for differential drive

Let’s try to find formulas by means of which is possible to compute the actual position of the robot.

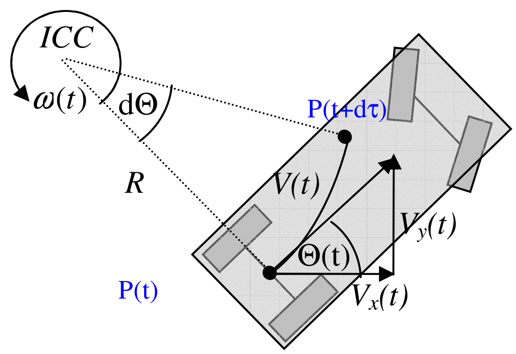

Fig. 3. Odometry for differential drive

Suppose that a differential drive robot is rotating around the point ICC with an angular velocity $\omega(t)$. During the infinite short time $d\tau$ the robot will travel the distance from the point P(t) to P(t+dτ) with a linear velocity V(t) (for more details about the velocity of the CR point see section *Differential Drive*). V(t) has two perpendicular components, one along the X axis - $V_x(t)$, and the other along the Y axis - $V_Y(t)$. For infinite short time we can assume that the robot is moving along a straight line tangent in the point P(t) to the real trajectory of the robot. Based on the two components of the velocity V(t), the traveled distance in each direction can be calculated:

EQ 5

$$dx = V_x(t) \cdot d\tau$$
$$dy = V_y(t) \cdot d\tau$$

, where:

EQ 6

$$V_x(t) = V(t) \cdot \cos\left[\Theta(t)\right]$$
$$V_y(t) = V(t) \cdot \sin\left[\Theta(t)\right]$$

Similarly, the angle of the rotation can be obtained:

EQ 7

$$d\Theta = \omega(t) \cdot d\tau$$

Integrating equation EQ 5 and EQ 7 in the time we obtain:

EQ 8

$$x(t) = \int_0^t V_x(\tau) d\tau + x_0$$

$$y(t) = \int_0^t V_y(\tau) d\tau + y_0$$

$$\Theta(t) = \int_0^t \omega(\tau) d\tau + \Theta_0$$

, where (x_0, y_0, θ_0) – is the initial pose.

Using EQ 7, the equation EQ 8 can be rewritten as:

EQ 9

$$x(t) = \int_0^t V(\tau) \cdot \cos[\Theta(\tau)] d\tau + x_0$$

$$y(t) = \int_0^t V(\tau) \cdot \sin[\Theta(\tau)] d\tau + y_0$$

$$\Theta(t) = \int_0^t \omega(\tau) d\tau + \Theta_0$$

Formulas EQ 9 are valid for all robots capable of moving in a particular direction $\Theta(t)$ at a given velocity $V(t)$. For the special case of differential drive robot, based on EQ 4 and EQ9 we can infer:

EQ 10

$$x(t) = \tfrac{1}{2} \int_0^t [v_1(\tau) + v_2(\tau)] \cdot \cos[\Theta(\tau)] d\tau + x_0$$

$$y(t) = \tfrac{1}{2} \int_0^t [v_1(\tau) + v_2(\tau)] \cdot \sin[\Theta(\tau)] d\tau + y_0$$

$$\Theta(t) = \tfrac{1}{D} \int_0^t [v_2(\tau) - v_1(\tau)] d\tau + \Theta_0$$

, where D is the wheel separation; $v_1(t)$ and $v_2(t)$ are the linear velocities of the left and right wheels respectively.
For a practical realization, the formula EQ 10 can be rewritten for discrete timing:

EQ 11

$$x(k) = \tfrac{1}{2}\sum_{i=1}^{k}\left[v_1(i)+v_2(i)\right]\cdot\cos\left[\Theta(i)\right]\cdot\Delta t + x_0$$
$$y(k) = \tfrac{1}{2}\sum_{i=1}^{k}\left[v_1(i)+v_2(i)\right]\cdot\cos\left[\Theta(i)\right]\cdot\Delta t + y_0$$
$$\Theta(k) = \tfrac{1}{D}\sum_{i=1}^{k}\left[v_2(i)-v_1(i)\right]\cdot\Delta t + \Theta_0$$

, where $x(k)$, $y(k)$, $\theta(k)$ are the components of the pose at the k step of the movement and Δt is the interval (e.g. sampling period) between two sampling times. Depending on the expected accuracy, Δt should be properly selected. $v_1(i)$ and $v_2(i)$ are the values of the velocity for the relevant wheels recorded during the movement. Easier to implement on microcontrollers is the recurrent form:

EQ 12

$$x(k) = \tfrac{1}{2}\left[v_1(k)+v_2(k)\right]\cdot\cos\left[\Theta(k)\right]\cdot\Delta t + x(k-1)$$
$$y(k) = \tfrac{1}{2}\left[v_1(k)+v_2(k)\right]\cdot\sin\left[\Theta(k)\right]\cdot\Delta t + y(k-1)$$
$$\Theta(k) = \tfrac{1}{D}\left[v_2(k)-v_1(k)\right]\cdot\Delta t + \Theta(k-1)$$

The velocities $v_1(i)$ and $v_2(i)$ can be estimated based on encoders, even though it is easier and more precise (without time measurement) to use the distances obtained directly from the sensors, based on the following equation:

EQ 13

$$\left[v_1(k)+v_2(k)\right]\cdot\Delta t = d_1(k)+d_2(k)$$

Where $d_1(k)$ and $d_2(k)$ are the distances traveled in last sampling period. Finally the equation EQ 12 can be rewritten as below:

EQ 14

$$x(k) = \tfrac{1}{2}\left[d_1(k)+d_2(k)\right]\cdot\cos\left[\Theta(k)\right] + x(k-1)$$
$$y(k) = \tfrac{1}{2}\left[d_1(k)+d_2(k)\right]\cdot\sin\left[\Theta(k)\right] + y(k-1)$$
$$\Theta(k) = \tfrac{1}{D}\left[d_2(k)-d_1(k)\right] + \Theta(k-1)$$

Example I:

In this example we will use the equations EQ 10 to determine the position of the robot after movement, assuming that the initial values x_0, y_0, θ_0 are equal to zero. The wheel separation is *D=15cm*. We will draw the characteristic curves for θ(t), x(t), y(t) and the trace of the robot in the X_bY_b coordinate system.
Fig. 4 shows the variation of the velocities for the two wheels.

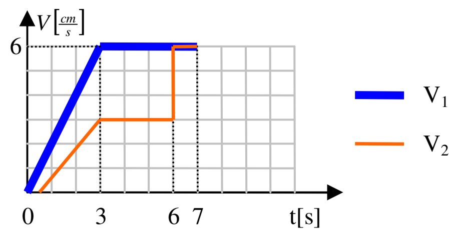

Fig. 4. Velocities of the wheels.

For the analytical calculation of the position, the equation EQ 10 will be used. Because the velocities for each wheel are complex we will use some trick. First, the equation EQ 10 will be used for the interval between 0 and 3 [sec]. The calculated pose for t = 3[sec] will be considered as being the initial pose in the second time interval (time between 3 and 6 [sec]) and then we will go on similarly for the last time interval (6-7[sec]) where pose for t=6 will be considered the initial value. Let's find the pose $\varepsilon(t)$ for the first time interval:

The formulas for the two velocities can be deduced from the graph (Fig. 4):

$$V_1(t) = 2t \quad t \in (0;3\rangle$$
$$V_2(t) = t \quad t \in (0;3\rangle$$

Hence θ(t) (from EQ 10) will be:

EQ 15

$$\Theta(t) = -\tfrac{1}{D}\int_0^t \tau \cdot d\tau + \Theta(0) = \frac{-1}{2D}t^2$$

, where θ(0) = 0 for first case.

The above equation is necessary for calculating *x(t)* and *y(t)* as follows.

EQ 16

$$x(t) = \tfrac{1}{2}\int_0^t 3\tau \cdot \cos\left(\frac{-1}{2D}\tau^2\right) \cdot d\tau + x(0) = \tfrac{3}{2} D \sin\left(\frac{1}{2D}t^2\right)$$

EQ 17

$$y(t) = \tfrac{1}{2}\int_0^t 3\tau \cdot \sin\left(\frac{-1}{2D}\tau^2\right) \cdot d\tau + y(0) = \tfrac{3}{2}D\left[\cos\left(\frac{1}{2D}t^2\right) - 1\right]$$

The equations EQ 15 – EQ 17 stand for the components of the pose $\varepsilon(t)$ for $t \in (0;3\rangle$. Now we can calculate the robot's final position in the first case:

EQ 18

$$x(3) \approx 6{,}65$$
$$y(3) \approx 1$$
$$\Theta(3) = -0{,}3$$

In the second step, as it was mentioned, the values EQ 18 will be used as initial values. The formulas for the two velocities for $t \in (3;6\rangle$, based on Fig. 4 are:

$$V_1(t) = 6 \quad t \in (3;6\rangle$$
$$V_2(t) = 3 \quad t \in (3;6\rangle$$

The formulas EQ 11 have to be changed a little because the integral range is now $t \in (3;6\rangle$:

EQ 19

$$\Theta(t) = -\tfrac{1}{D}\int_3^t 3 \cdot d\tau + \Theta(3) = \frac{-3}{D}(t-3) - 0{,}3$$

EQ 20

$$x(t) = \tfrac{1}{2}\int_3^t 3 \cdot \cos\left[-\tfrac{3}{D}(\tau-3) - 0{,}3\right] \cdot d\tau + x(3) = -\tfrac{1}{2}D\left\{\sin\left[-\tfrac{3}{D}(t-3) - 0{,}3\right] - \sin(0{,}3)\right\} + 6{,}65$$

EQ 21

$$y(t) = \tfrac{1}{2}\int_3^t 3 \cdot \sin\left[-\tfrac{3}{D}(\tau-3) + 0{,}3\right] \cdot d\tau + y(3) = -\tfrac{3}{2}D\left\{\cos\left[\tfrac{3}{D}(t-3) + 0{,}3\right] - \cos(0{,}3)\right\} + 1$$

In the last step, the pose for the time interval $t \in (6;7\rangle$, can be expressed as follows:

EQ 22

$$\Theta(t) = \tfrac{1}{D}\int_{6}^{t} 0 \cdot d\tau + \Theta(6) = -0{,}9$$

EQ 23

$$x(t) = \tfrac{1}{2}\int_{3}^{t} 12 \cdot \cos\left(-0{,}9\right) \cdot d\tau + x(6) = \tfrac{3}{2}\cos(0{,}9)t + 10{,}3$$

EQ 24

$$y(t) = \tfrac{1}{2}\int_{3}^{t} 12 \cdot \sin\left(-0{,}9\right) \cdot d\tau + y(6) = \tfrac{3}{2}\sin(-0{,}9)t - 3{,}5$$

Finally, the robot's pose can be calculated based on the equations (EQ 22 - EQ 24):

EQ 25

$$\begin{aligned} x(7) &\approx 14{,}1 \\ y(7) &\approx -8{,}2 \\ \Theta(7) &= -0{,}9 \end{aligned}$$

The three components of the pose as a function of time have the following form:

EQ 26

$$x(t) = \begin{cases} \tfrac{3}{2} D \sin\left(\dfrac{1}{2D} t^2\right) & t \in (0;3\rangle \\ -\tfrac{1}{2} D\left\{\sin\left[-\tfrac{3}{D}(t-3) - 0{,}3\right] - \sin(0{,}3)\right\} + 6{,}65 & t \in (3;6\rangle \\ \tfrac{3}{2}\cos(0{,}9)t + 10{,}3 & t \in (3;7\rangle \end{cases}$$

EQ 27

$$y(t) = \begin{cases} -\tfrac{3}{2} D\left[\cos\left(\dfrac{1}{2D} t^2\right) - 1\right] & t \in (0;3\rangle \\ -\tfrac{3}{2} D\left\{\cos\left[\tfrac{3}{D}(t-3) + 0{,}3\right] - \cos(0{,}3)\right\} + 1 & t \in (3;6\rangle \\ \tfrac{3}{2}\sin(-0{,}9)t - 3{,}5 & t \in (6;7\rangle \end{cases}$$

EQ 28

$$\Theta(t)=\begin{cases}-\dfrac{1}{2D}t^2 & t\in\langle 0;3\rangle\\ -\dfrac{3}{D}(t-3)-0{,}3 & t\in\langle 3;6\rangle\\ -0{,}9 & t\in\langle 6;7\rangle\end{cases}\qquad [\text{rad}]$$

Using the equations EQ 26, EQ 27, EQ 28, we can draw the required characteristic curves:

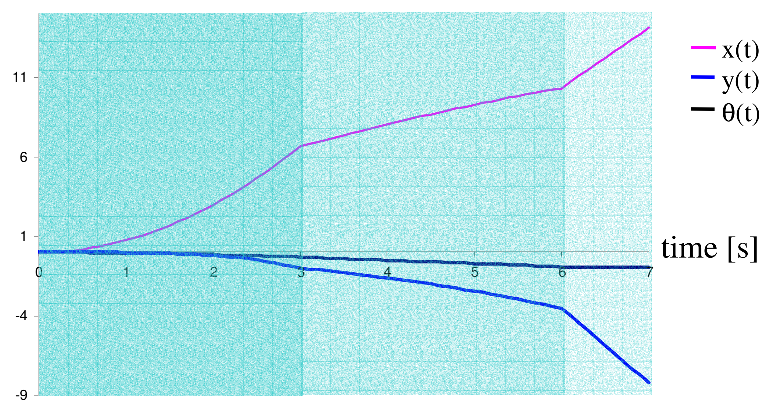

Fig. 5. Pose's components variation.

The graph in base coordinates can be drawn as well:

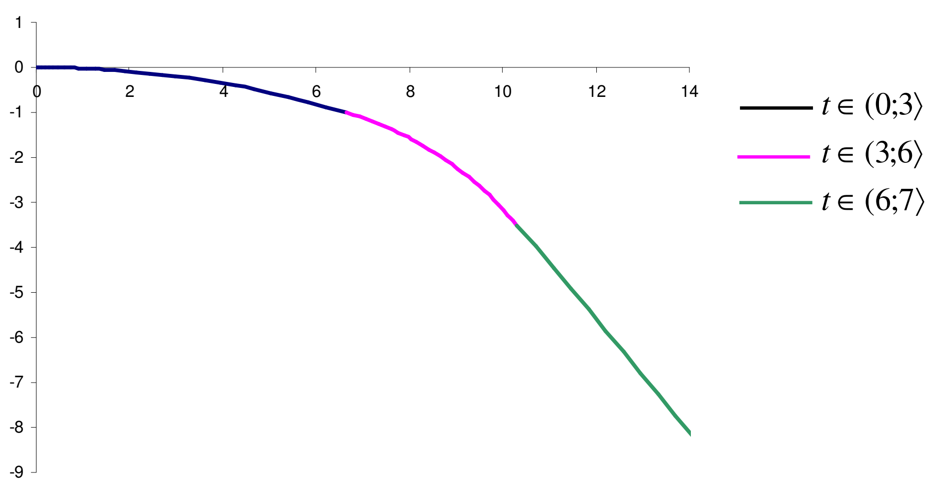

Fig. 6. Robot's trajectory in base coordinates.