Solution Manual

System Dynamics

Modeling, Simulation and Control of Mechatronic Systems

5th Edition

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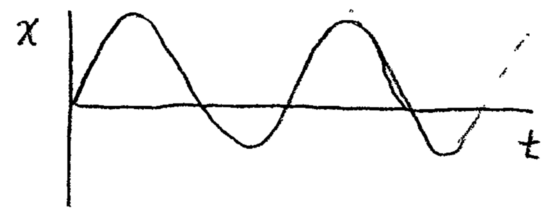
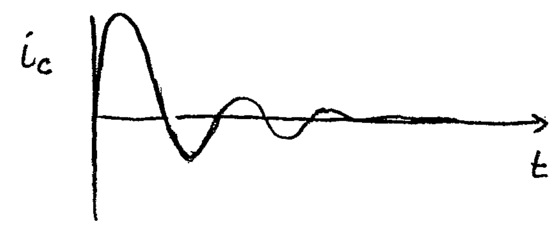
Problems 1.1 to 1.5 are mainly discussion questions.

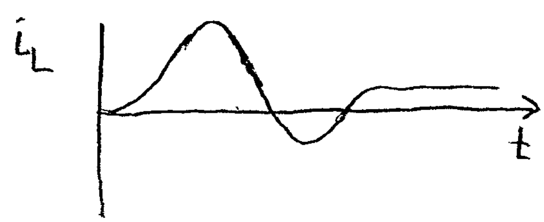
1.6



steady state deflection, $x_0 = \frac{(m+M)g}{k}$

1.7



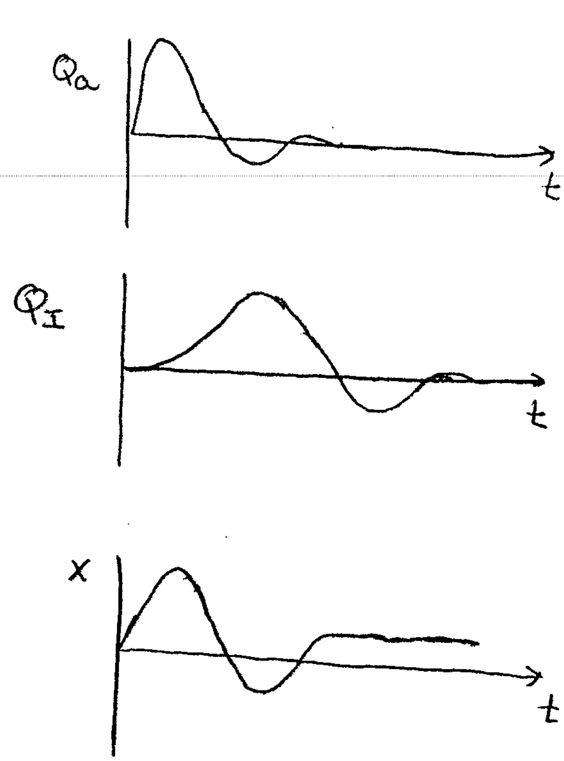


$i_{L_{ss}} = \frac{e_0}{R}$

1.8 Q_a similar to i_c } in prob. 1.7
Q_I similar to i_L

$Q_{I_{ss}} = P_s / R_f$

1.9



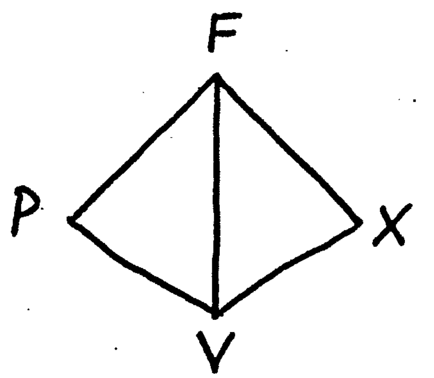
Accumulator

supply Pressure — pipe Tee — fluid line — cylinder — mass

spring

damper

2-1

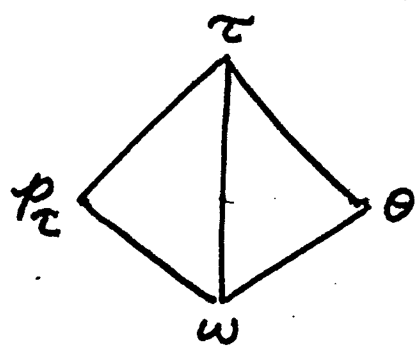


$[F]$ - N

$[V]$ - m/s

$[P]$ - $N \cdot s$

$[X]$ - m

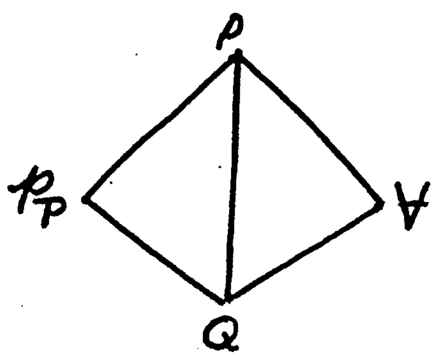


$[\tau]$ - $N \cdot m$

$[\omega]$ - rad/s

$[p_\tau]$ - $N \cdot m \cdot s$

$[\theta]$ - rad

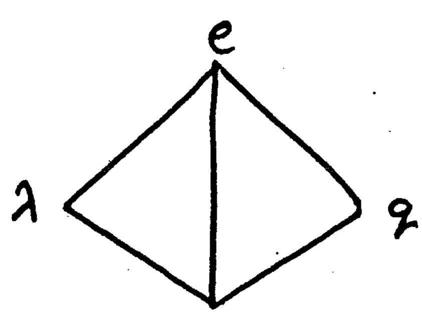


$[P]$ - N/m^2

$[Q]$ - m^3/s

$[p_P]$ - $N \cdot s/m^2$

$[\forall]$ - m^3



$[e]$ - V

$[i]$ - A

$[\lambda]$ - $V \cdot s$

$[q]$ - $A \cdot s = C$

2-2 $\xrightarrow[\omega]{\tau}$ Electric Motor $\xrightarrow[i]{e}$

(a)

$\xrightarrow[\omega]{\tau}$ Hydraulic Pump $\xrightarrow[Q]{P}$

(b)

$\xleftarrow[\omega_1]{\tau}$ Shaft $\xleftarrow[\omega_2]{\tau}$

(c)

$\xrightarrow[V_1]{F}$ Shock Absorber $\xrightarrow[V_2]{F}$

(d)

$\xrightarrow[i_1]{e_1}$ Transistor $\xrightarrow[i_2]{e_2}$

(e)

$\xrightarrow[i]{e}$ Speaker

(f)

$\xrightarrow[\omega]{\tau}$ Crank $\xrightarrow[V]{F}$

(g)

$\xrightarrow[V]{F}$ Wheel $\xleftarrow[\omega]{\tau}$

(h)

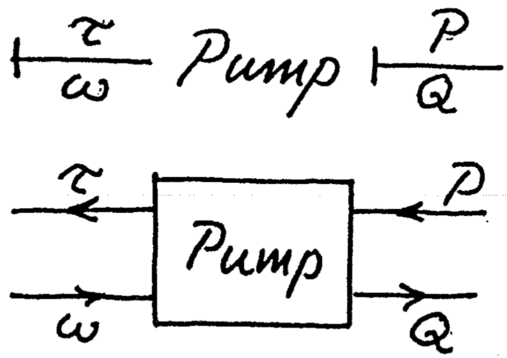
$e_f \downarrow i_f$

$\xrightarrow[\omega]{\tau}$ Motor $\xleftarrow[i_a]{e_a}$

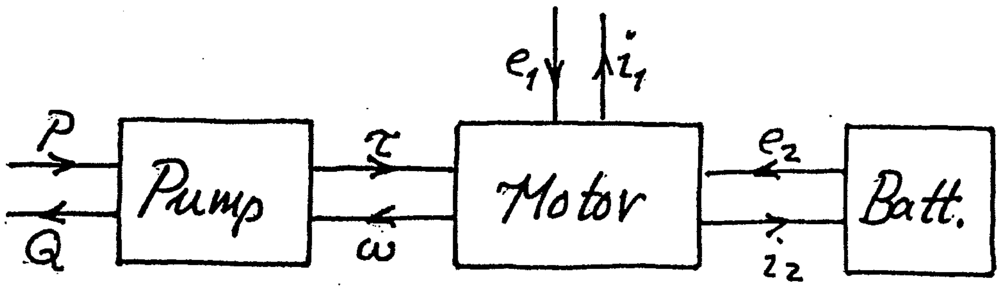
(i)

2-3

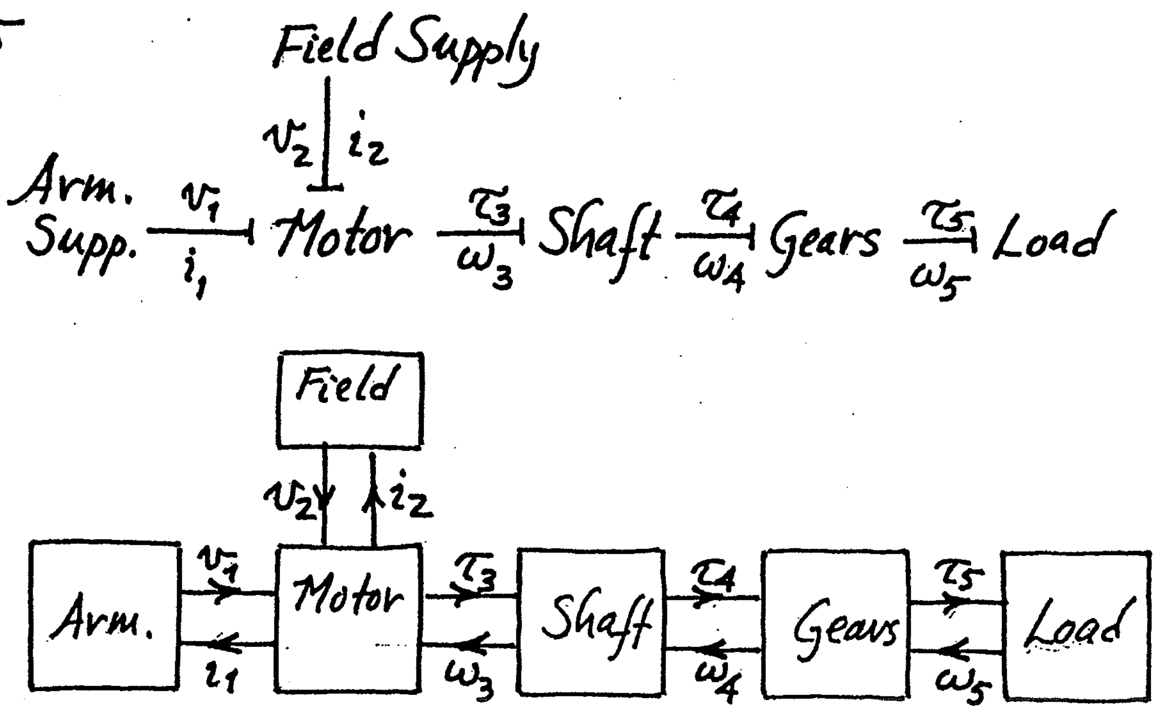
2-3


τ
ω
Pump
P
Q
τ
P
Pump
ω
Q

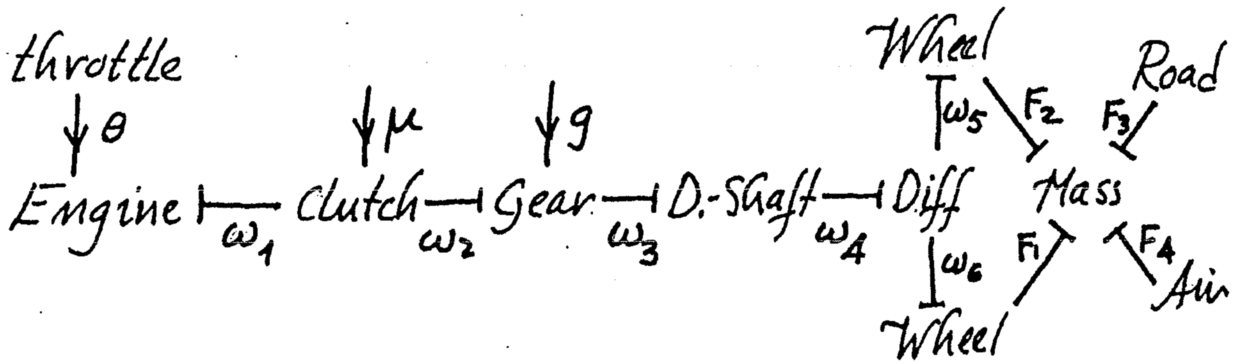
2-4

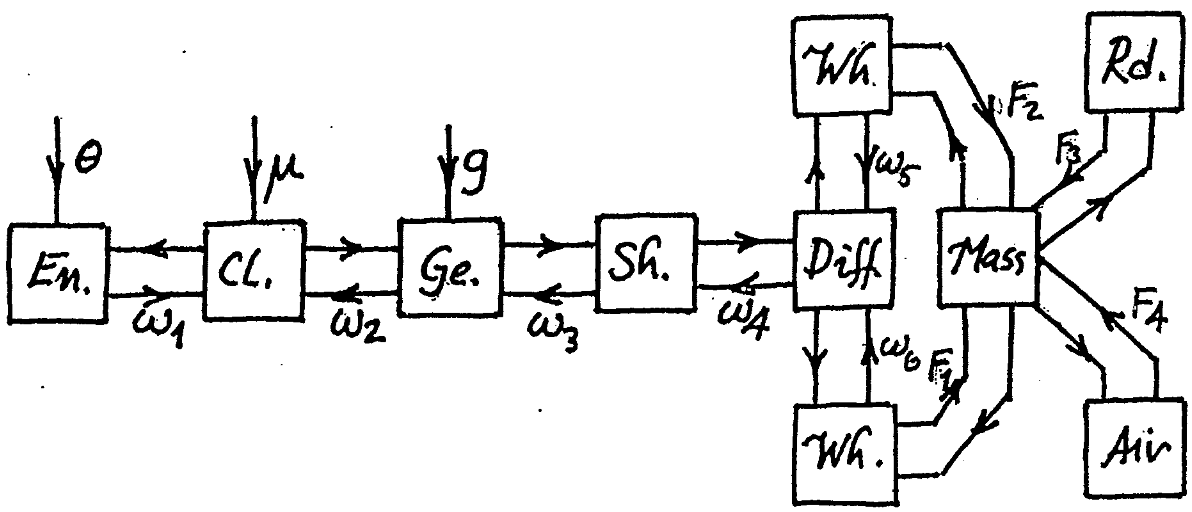

e1
i1
P
Pump
τ
Motor
e2
Batt.
Q
ω
i2

2-5


Field Supply
v2
i2
Arm.
Supp.
v1
i1
Motor
τ3
ω3
Shaft
τ4
ω4
Gears
τ5
ω5
Load
Field
v2
i2
Arm.
v1
i1
Motor
τ3
ω3
Shaft
τ4
ω4
Gears
τ5
ω5
Load

2-6





2-7 Inputs: P, e_1, e_2

Outputs: Q, i_1, i_2

2-8

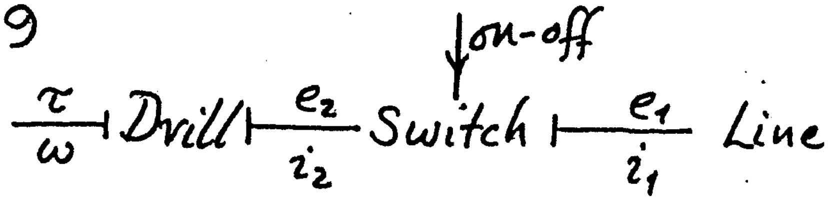
Power × Time = Energy

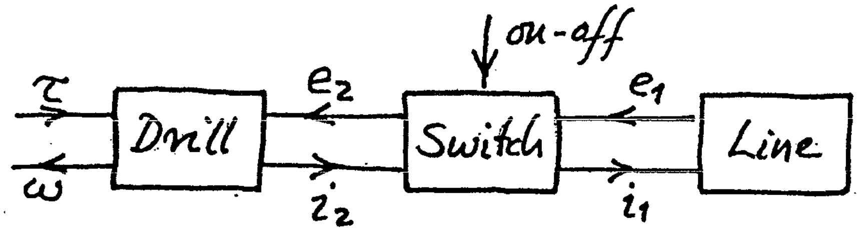
$$P \cdot t = mgh$$

$$100 \cdot t = 10 \cdot (9.81) \cdot 30$$

$$t = 29.43 \text{ s}$$

2-9





2-10 $\quad \tau\omega = PQ$

$$\omega = \frac{P}{\tau} \cdot Q = \frac{7.0\times10^6}{5} Q$$

$$\omega = 1.4\times10^6 Q$$

$$\left[\frac{rad}{s}\right] = \left[\frac{1}{m^3}\right]\cdot\left[\frac{m^3}{s}\right]$$

2.11 using kinematics:

$$x = R\cos\theta + l\cos\alpha$$

$$l\sin\alpha = R\sin\theta$$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{R^2}{l^2}\sin^2\theta}$$

$$\therefore \quad x = R\cos\theta + l\sqrt{1-\frac{R^2}{l^2}\sin^2\theta}$$

Then $\dot{x} = -v = -R\sin\theta\,\dot{\theta} + l\,\frac{1}{2}\left[1-\frac{R^2}{l^2}\sin^2\theta\right]^{-1/2}\left(-\frac{R^2}{l^2}2\sin\theta\cos\theta\right)\dot{\theta}$

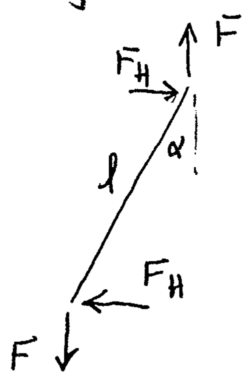
or

$$v = \underbrace{\left[R\sin\theta + \frac{l\left(\frac{R}{l}\right)^2\sin\theta\cos\theta}{\sqrt{1-\left(\frac{R}{l}\right)^2\sin^2\theta}}\right]}_{m(\theta)}\dot{\theta}$$

$$\therefore \quad v = m(\theta)\,\omega$$

$$\tau = m(\theta)\,F$$

using Forces and moments:

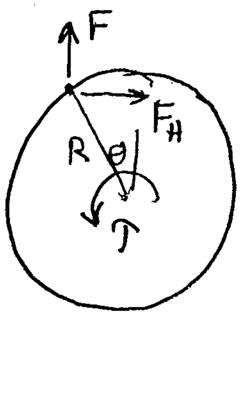


moment equilibrium for rod:

$$F\,l\sin\alpha = F_H\,l\cos\alpha$$

or

$$F_H = F\sin\alpha/\cos\alpha$$



moment equilibrium for crank:

$$\tau = F_H R\cos\theta + F R\sin\theta$$

$$= F\frac{\sin\alpha}{\cos\alpha}R\cos\theta + F R\sin\theta$$

use $\sin\alpha = \frac{R}{l}\sin\theta$

$$\cos\alpha = \sqrt{1-\left(\frac{R}{l}\right)^2\sin\theta}$$

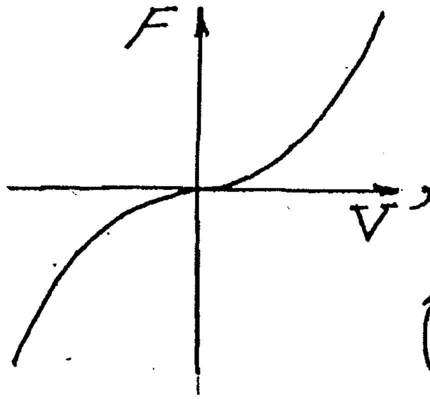
substitute and you will end up with:

$$\tau = m(\theta)\,F$$

2.12

element	inputs	outputs
supply pressure	Q_s	P_s
Tube	P_s, P_a, Q_ℓ	Q_s, Q_a, P_c
Accumulator	Q_a	P_a
Cylinder	P_c, v_m	Q_ℓ, F_m
Mass	F_m, F_s, F_d	v_m, v_s, v_d
Spring	v_s	F_s
Damper	v_d	F_d

3-1



$$\vdash\!\xrightarrow[V]{F} R,$$

$$V = \sqrt{\frac{|F|}{A}} \cdot \operatorname{sgn} F$$

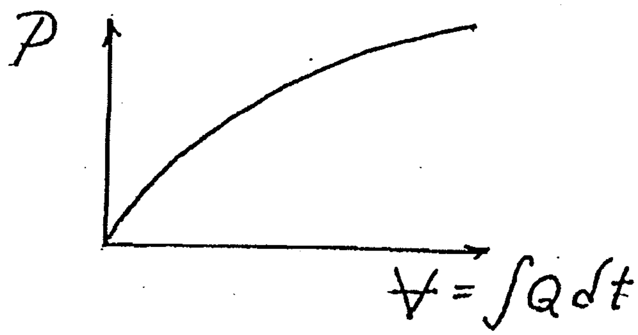
$$\xrightarrow[V]{F}\!\dashv R$$

3-2 $hA = \forall = \int Q\,dt,$

$$P = \rho g h = \frac{\rho g}{A} \int Q\,dt$$

$$\therefore\ C = A/\rho g$$

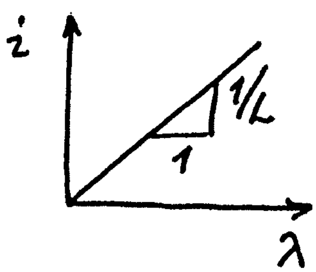
3-3

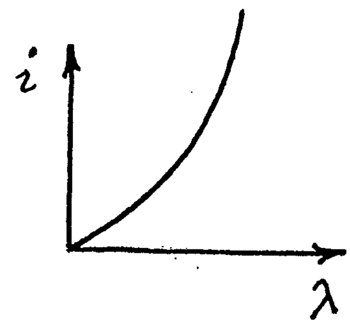


3-4

$$\xrightarrow[\dot{x}]{F} C\,, \quad F = \left(\frac{3EI}{L^3}\right) x\,.$$

3-5 $Li = \lambda = \int e\,dt$



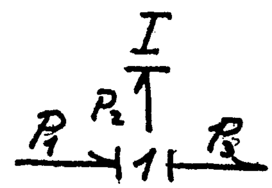


$i = i(\lambda)$ or $\lambda = \lambda(i)$

$$\frac{di}{dt} = \frac{d\,i(\lambda)}{d\lambda} \cdot \frac{d\lambda}{dt} \qquad \frac{d\lambda}{dt} = e = \frac{d\,\lambda(i)}{di} \cdot \frac{di}{dt}$$

$$= \frac{d\,i(\lambda)}{d\lambda} \cdot e$$

3-6 $\bar{F} = m\bar{a} = m\dot{\bar{v}}$

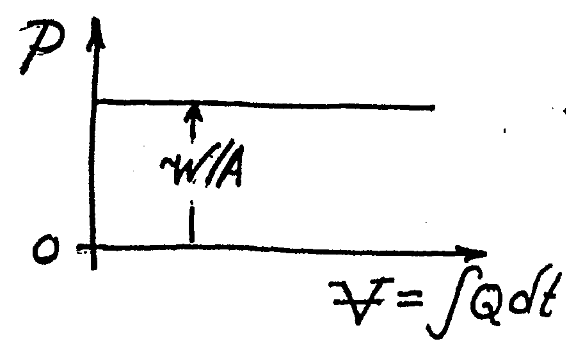


$$P_1 A - P_3 A = \rho A L \cdot \frac{dQ_2/A}{dt}$$

or $$P_1 - P_3 = \left(\frac{\rho L}{A}\right) \cdot \frac{dQ_2}{dt} = P_2$$

$$p_{P_2} = \int (P_1 - P_3)\,dt = \rho \frac{L}{A} \cdot Q_2$$

3-7



$$\xrightarrow[Q]{P} C$$

3-8
$$\tau = I\alpha\ , \quad p_\tau = \int \tau\, dt = I\omega$$

$$I = \tfrac{1}{2} m R^2 = \tfrac{1}{2}\rho \cdot \pi R^2 t \cdot R^2$$

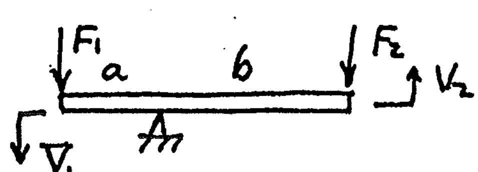
$$= \frac{0.28\ \pi \cdot 1 \cdot (5)^4}{2 \cdot 386} = 0.712\ \text{lb s}^2\text{in}$$

3-9
$$\xrightarrow[V]{F} TF \xrightarrow[Q]{P} \quad ; \quad \text{Area} = A$$

$$F = A P$$

$$A V = Q$$

3-10



(b)

$$\xrightarrow[V_1]{F_1} TF \xrightarrow[V_2]{F_2}$$

$$a F_1 = b F_2$$

$$\frac{V_1}{a} = \frac{V_2}{b}$$

(c) $\xrightarrow[\omega_1]{\tau_1} TF \xrightarrow[\omega_2]{\tau_2}$

$$\left.\begin{aligned} \omega_1 r_1 &= \omega_2 r_2 , \\ \tau_1 / r_1 &= \tau_2 / r_2 , \end{aligned}\right\} r_1, r_2 \text{ radii of gears}$$

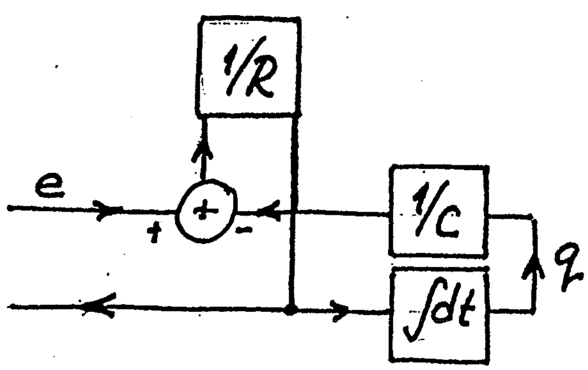
3-11 Ang. Momentum = $\bar{H}$

$|\bar{H}| \cong J\Omega$, torques associated with F_1 and F_2 cause change in <u>direction</u> of $\bar{H}$ not magnitude. Consider F_1 first; let shaft length be L. Torque is then $F_1 L$, $\bar{\tau} = \dot{\bar{H}}$ means that tip of $\bar{H}$ vector must move up with angular rate V_2/L. $|\dot{H}| = J\Omega \cdot V_2/L = F_1 L$

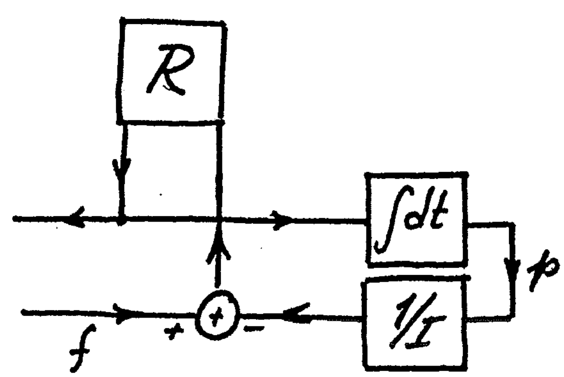
so

$$\left.\begin{aligned} F_1 &= \left(\frac{J\Omega}{L^2}\right) V_2 \\ \text{Similarly}\quad F_2 &= \left(\frac{J\Omega}{L^2}\right) V_1 \end{aligned}\right\} \quad \frac{F_1}{V_1}\ GY\ \frac{F_2}{V_2}$$

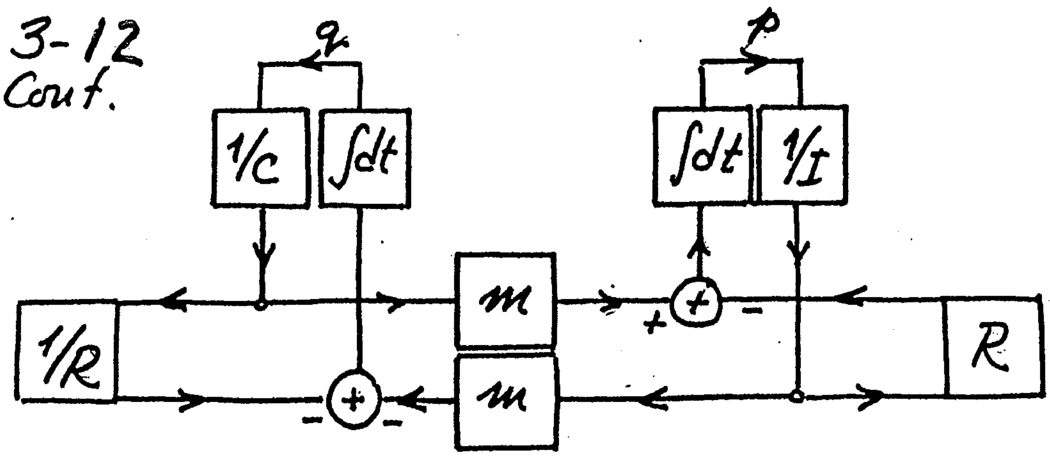
3-12



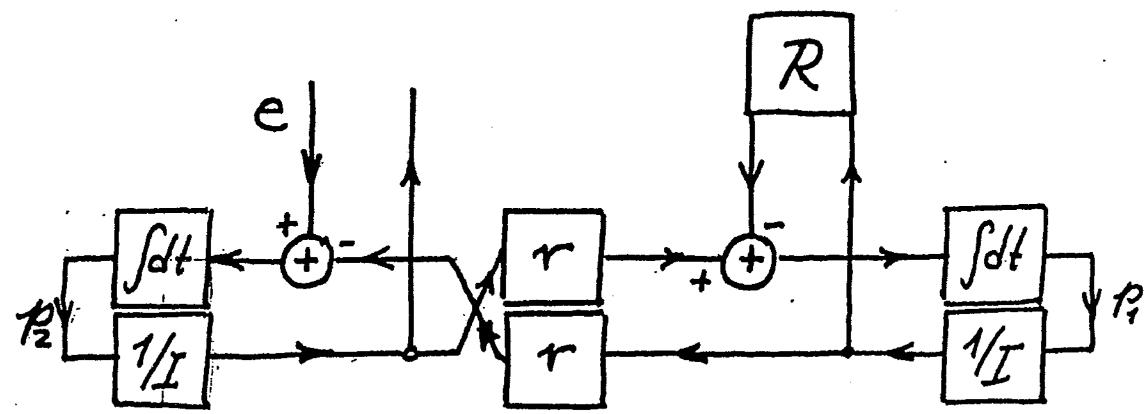
(a)



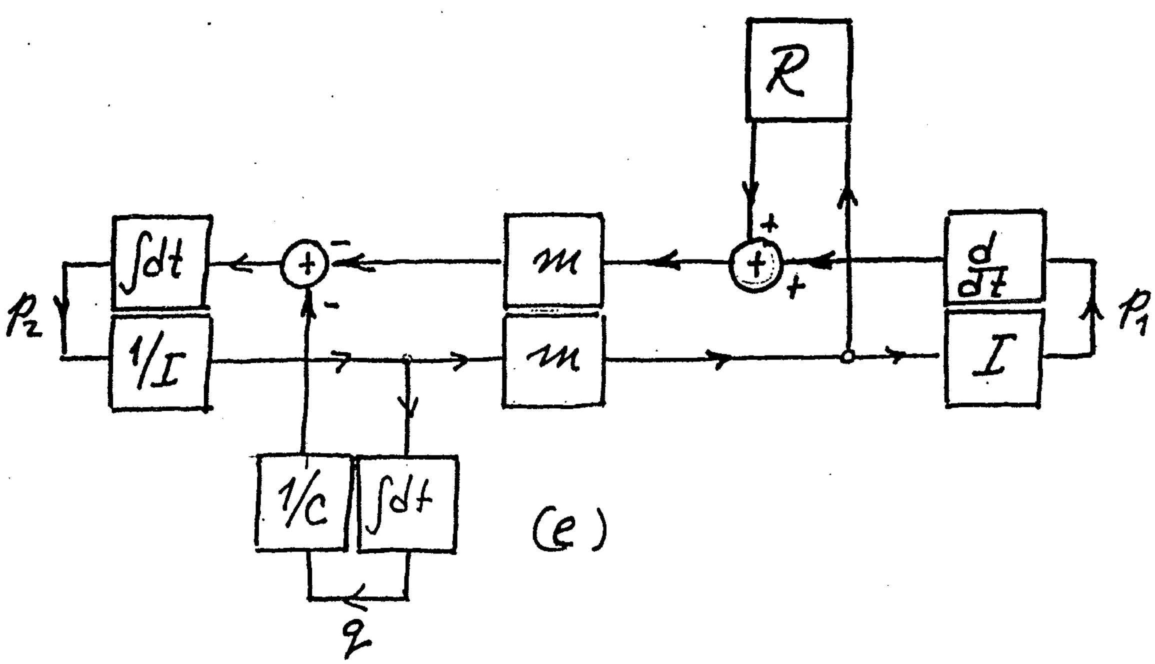
(b)


q
1/C
∫dt
p
∫dt
1/I
m
m
1/R
R
+
−
−
−
+

(c)

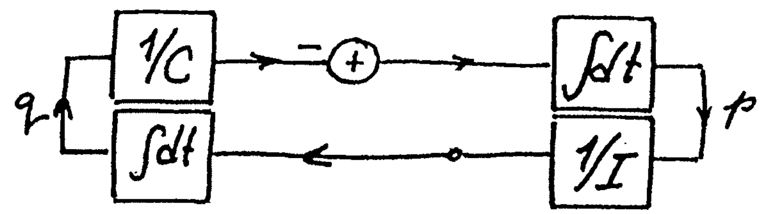

e
R
+
−
+
−
∫dt
1/I
r
r
∫dt
1/I
p2
p1

(d)


R
∫dt
1/I
m
m
d/dt
I
p2
p1
+
+
−
−
1/C
∫dt
q

(e)

3-13



3-14

$$\xrightarrow[\omega]{\tau} TF \xrightarrow[V]{F}$$

$$\tau = rF$$
$$r\omega = V$$

3-15 $\vdash\!\xrightarrow[i]{e} \overset{T}{GY} \xrightarrow[V]{F}\!\dashv I$

$$e = TV = T\frac{p}{I} = \frac{T\int F\,dt}{I}$$

$$= \frac{T}{I}\int Ti\,dt = \frac{T^2}{I}\int i\,dt = \frac{T^2}{I}\cdot q$$

so $\frac{1}{C} \longleftrightarrow \frac{T^2}{I}$

3.16

$$P = P_o \frac{V_o^\gamma}{V^\gamma} = \frac{P_o V_o^\gamma}{(V_o - A_p X)^\gamma} = \frac{P_o}{\left[1 - \frac{A_p X}{V_o}\right]^\gamma}$$

P is absolute pressure:

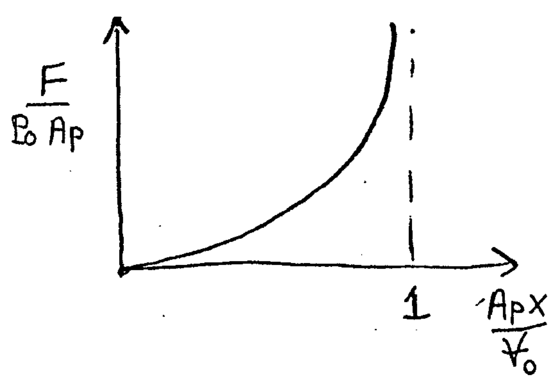
$P - P_o$ is gage pressure in cylinder,

$$(P - P_o) A_p = F$$

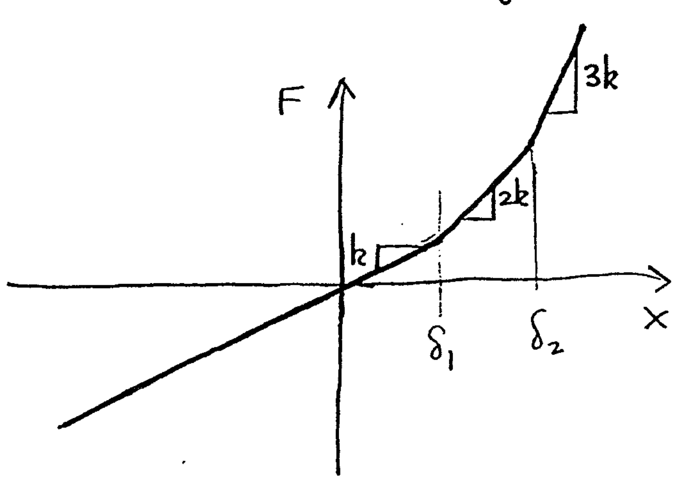
or

$$F = \left\{ \frac{P_o}{\left[1 - \frac{A_p X}{V_o}\right]^\gamma} - P_o \right\} A_p$$

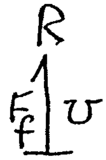
$$F = P_o A_p \left[\frac{1}{\left(1 - \frac{A_p X}{V_o}\right)^\gamma} - 1 \right]$$



3-17



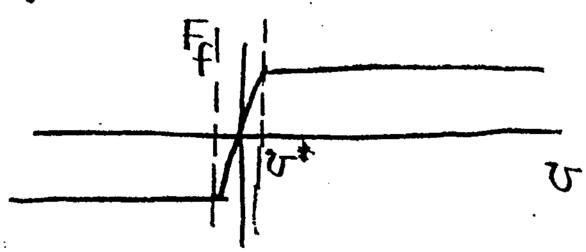
3-18

a) 

The only possible causality for the friction model shown in the problem is "effort" out, "flow" in.
For any specified velocity, F_f can be computed, but if F_f is specified, v is indeterminant.

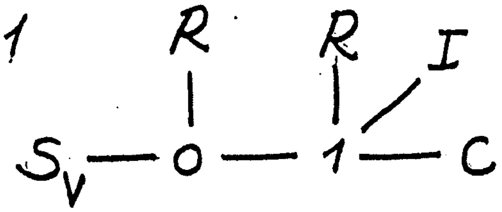
(b) If used simply as $F_f = \mu N$, then F_f will be applied when $v=0$, which is not correct. When $v=0$, the mass "sticks", and the friction force exactly balances all other forces on the mass. When the other forces exceed the "stick" force, then F_f returns to $F_f = \mu N$.

A possible change in the constitutive law might be,

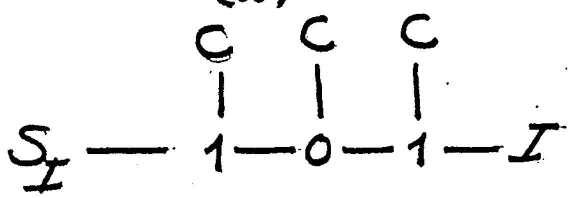


For v^* very small, the fundamental character of friction is maintained.

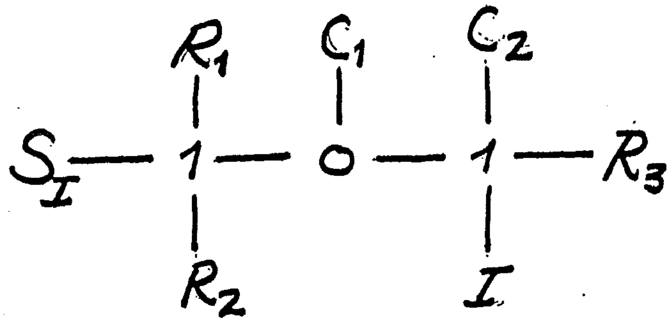
4-1



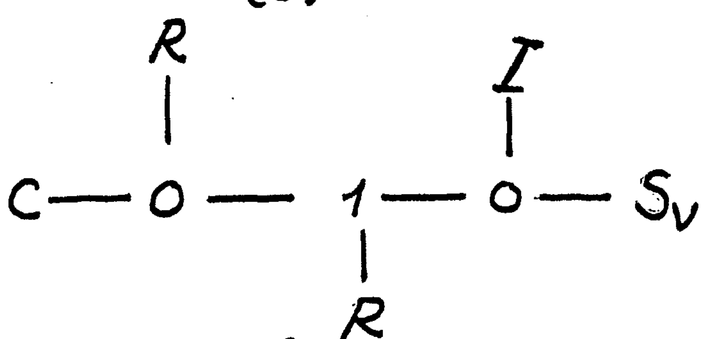
(a)



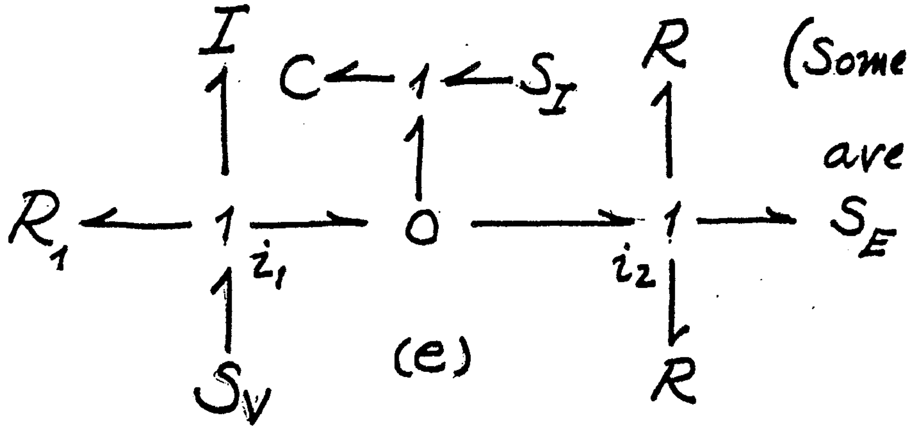
(b)



(c)



(d)

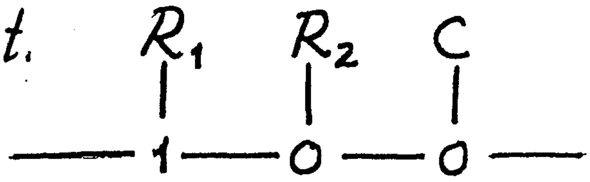


(e)

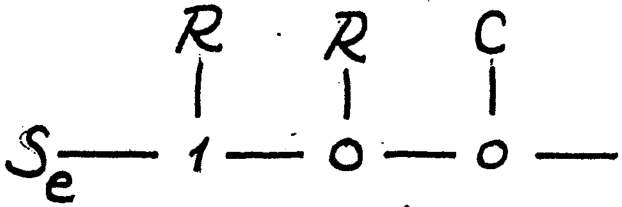
(Some signs are not defined in problem)

4-1 Cont.

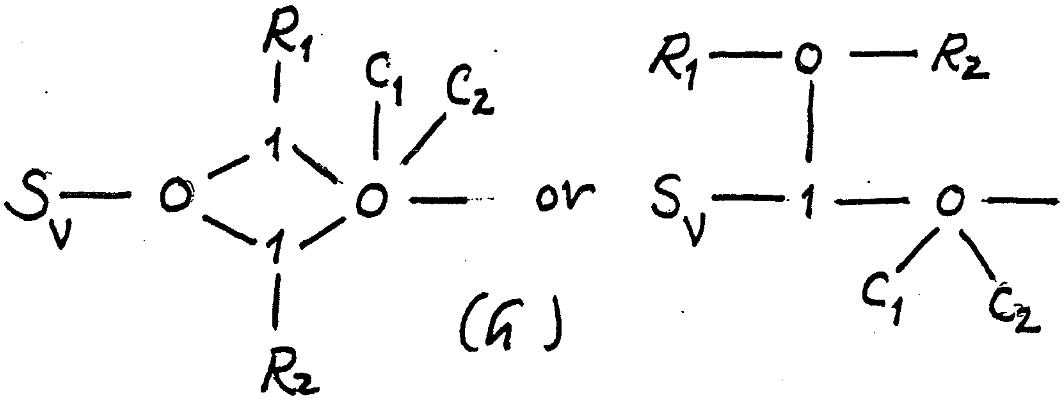
4-2



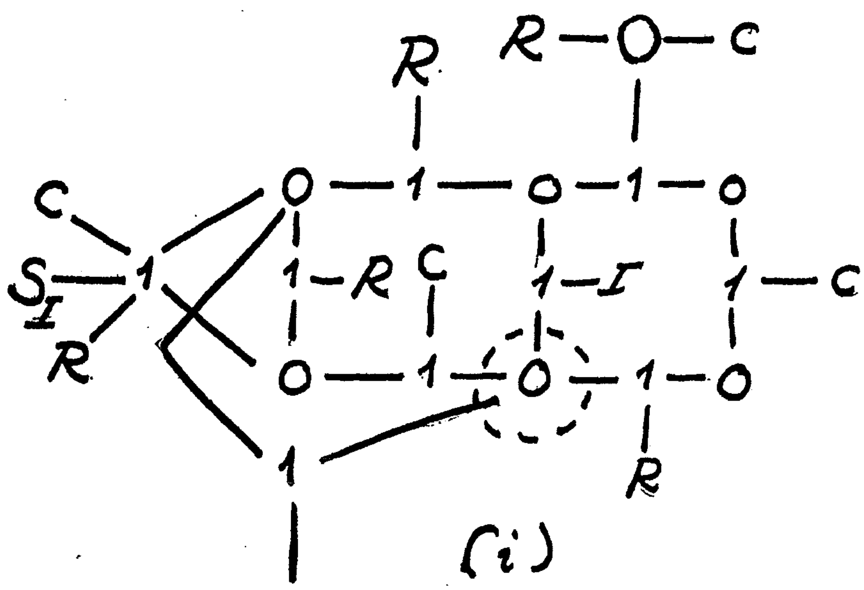
(f)



(g)



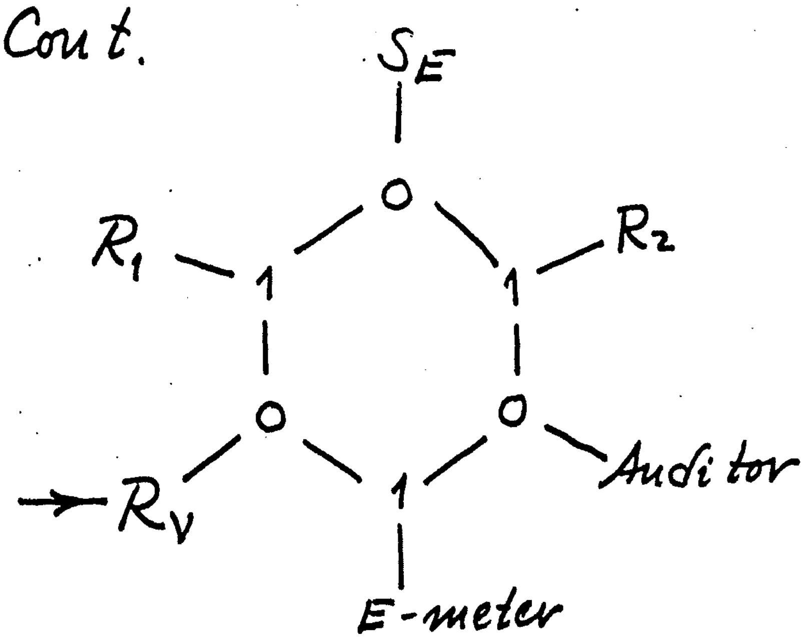
(h)



(i)

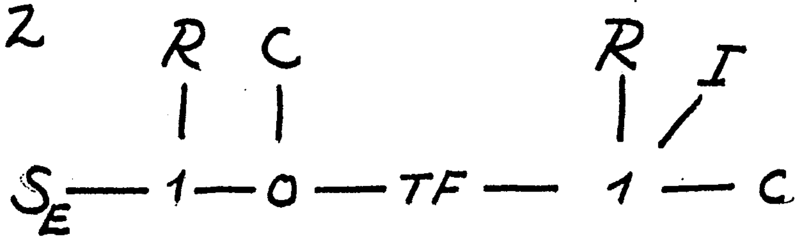
4-1 Cont.

4-3

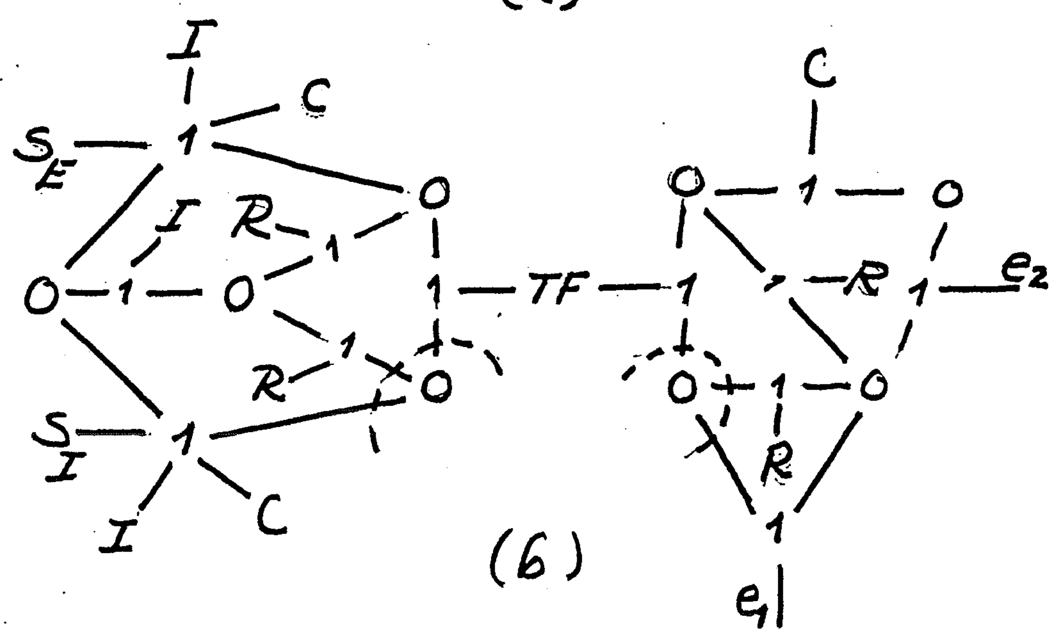


(j)

4-2

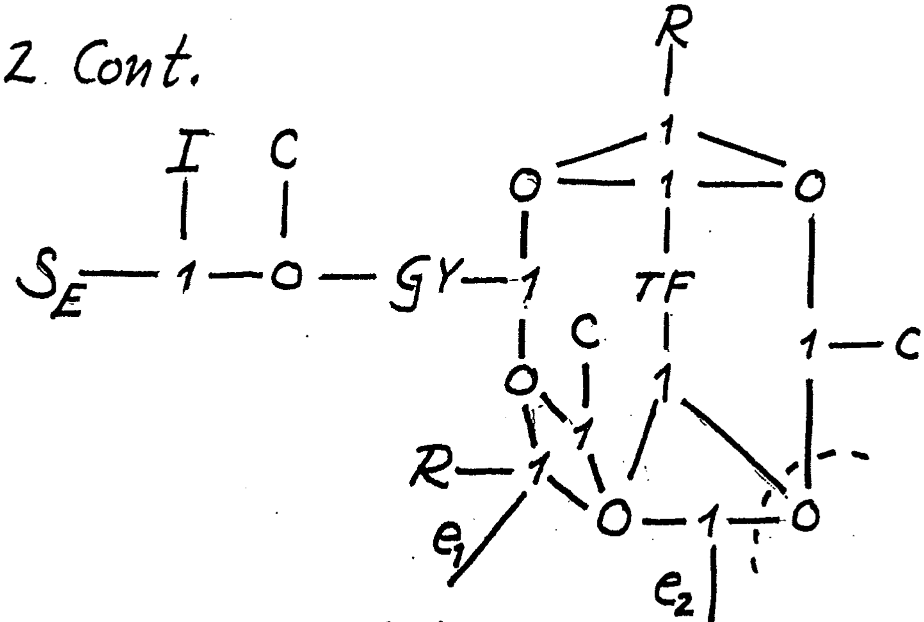


(a)



(b)

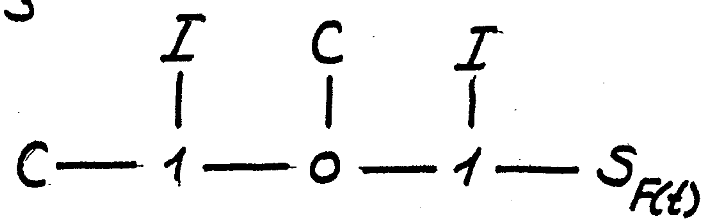
4-2 Cont.



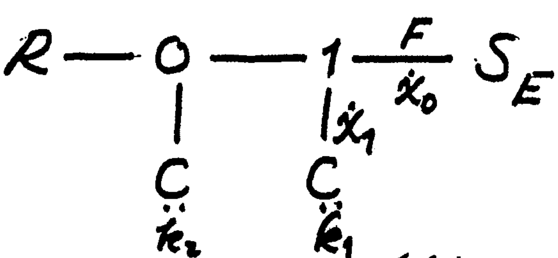
4-4

(c)

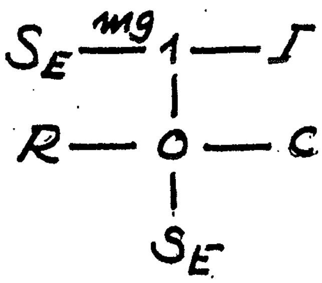
4-3



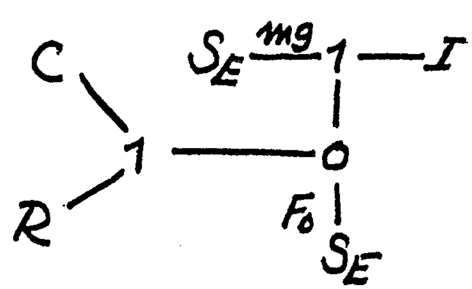
(a)

6:


(6)

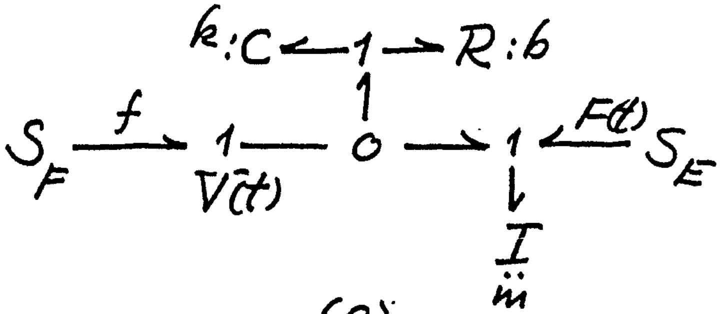


(c)

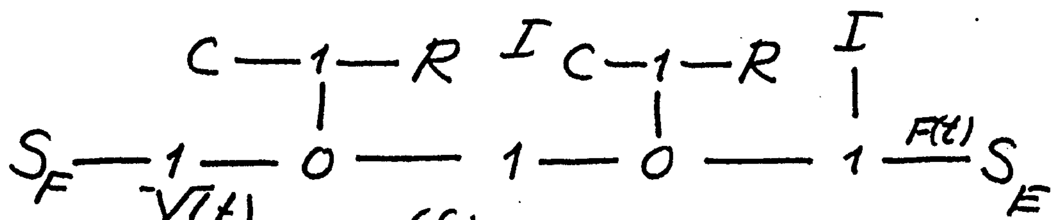


(d)

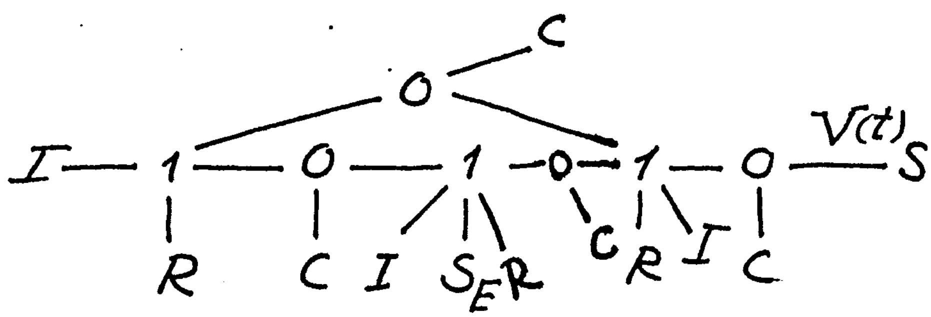
4-3 Cont.


k:C
R:b
f
S_F
V(t)
F(t)
S_E
I
m
1
0

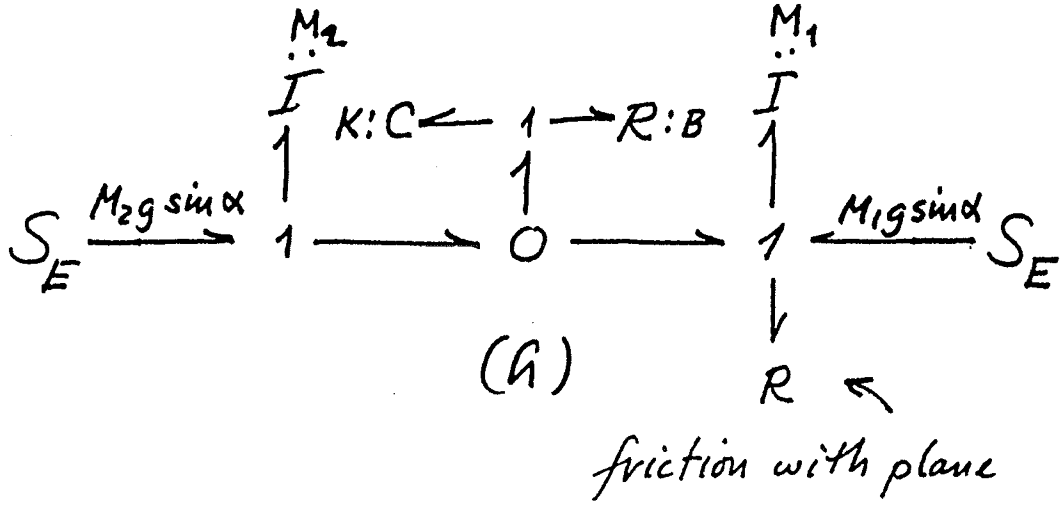
(e)


C
R
I
S_F
V(t)
F(t)
S_E
1
0

(f)


C
I
R
S_E
V(t)
S
1
0

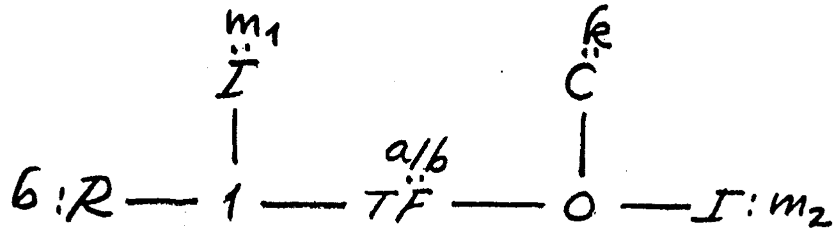
(g)


M_2
M_1
I
K:C
R:B
M_2 g sin α
M_1 g sin α
S_E
R
1
0
friction with plane

(h)

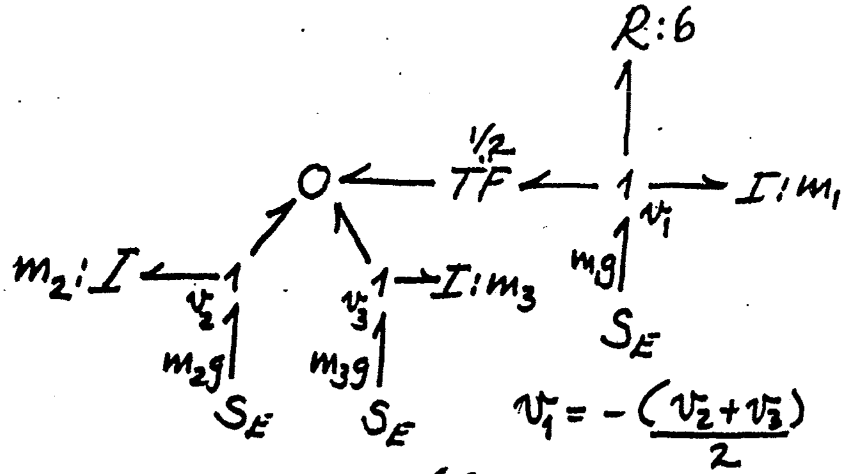
4-4

4-6



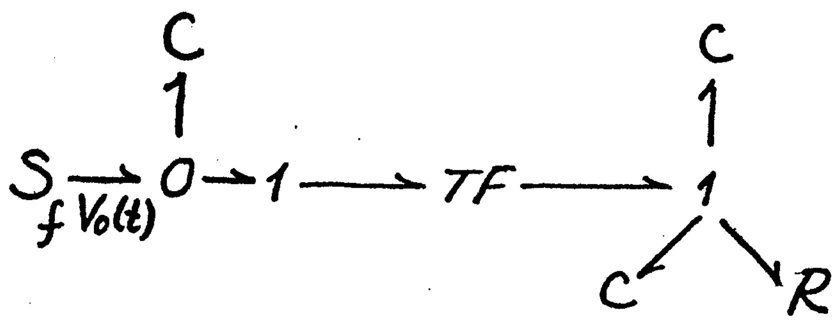
m_1
I
k
C
a/b
$b : R — 1 — TF — 0 — I : m_2$

(a)



$R : b$
$1/2$
TF
$I : m_1$
v_1
mg
S_E
$m_2 : I$
v_2
$m_2 g$
S_E
v_3
$I : m_3$
$m_3 g$
S_E
$v_1 = -\frac{(v_2 + v_3)}{2}$

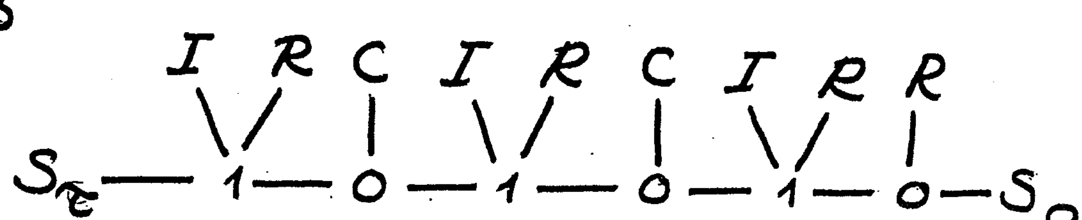
(b)



C
C
S_f
$V_0(t)$
0
1
TF
1
C
R

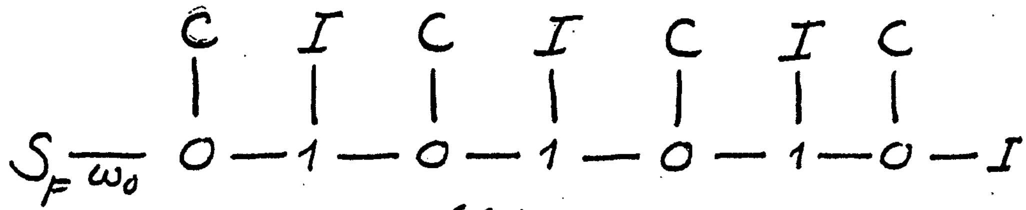
(c)

4-5

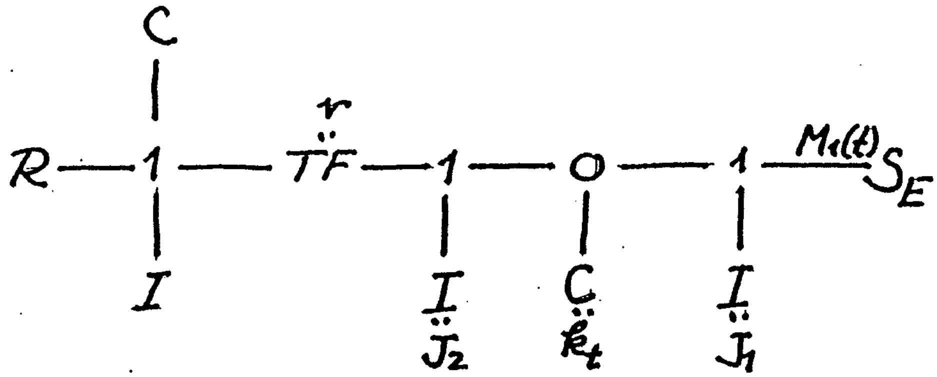


I R C I R C I R R
S_τ — 1 — 0 — 1 — 0 — 1 — 0 — S_Ω

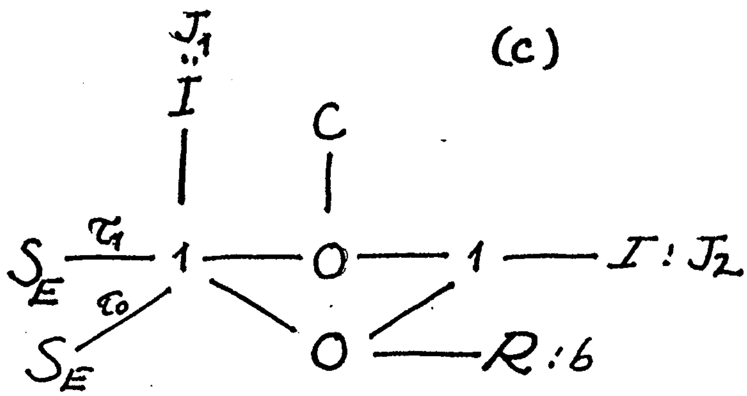
(a)


C I C I C I C
$S_F \overset{\omega_0}{—}$ 0 — 1 — 0 — 1 — 0 — 1 — 0 — I

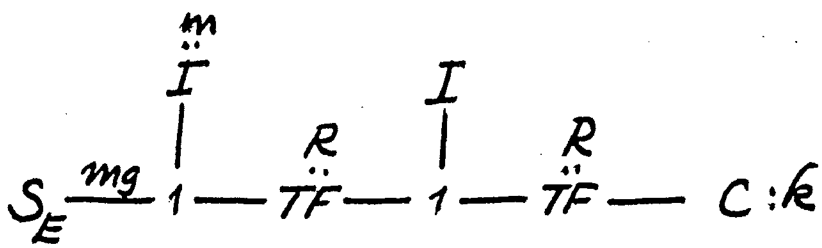
(b)


C
r
R — 1 — TF — 1 — 0 — 1 $\overset{M_1(t)}{—}$ S_E
I
$I : J_2$
$C : k_t$
$I : J_1$

(c)

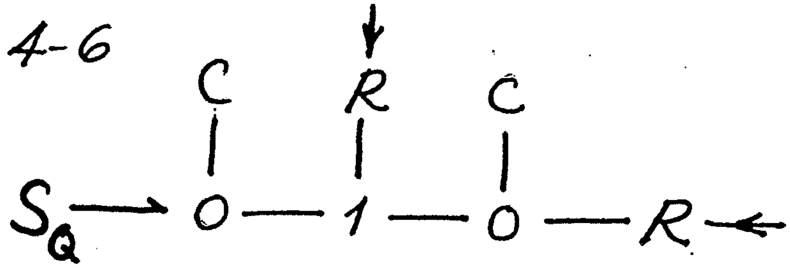

J_1
I
C
$S_E \overset{\tau_1}{—}$ 1 — 0 — 1 — $I : J_2$
τ_0
S_E
0 — $R : b$

(d)

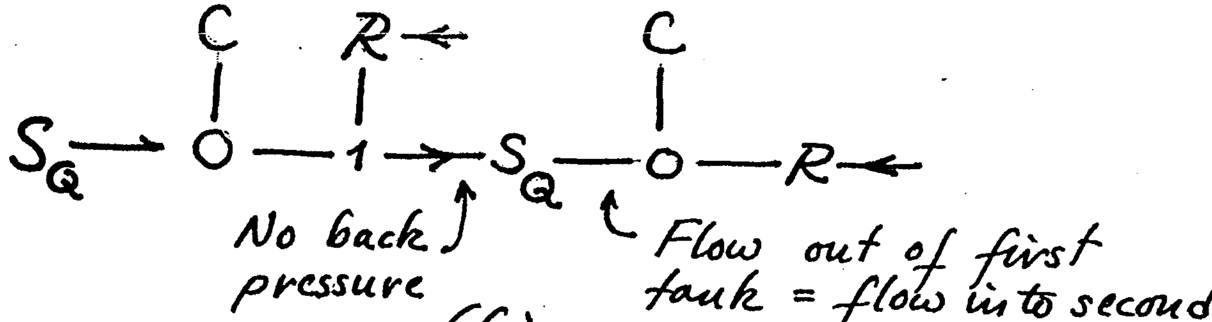

m
I
I
R
R
$S_E \overset{mg}{—}$ 1 — TF — 1 — TF — C : k

(e)

4-6

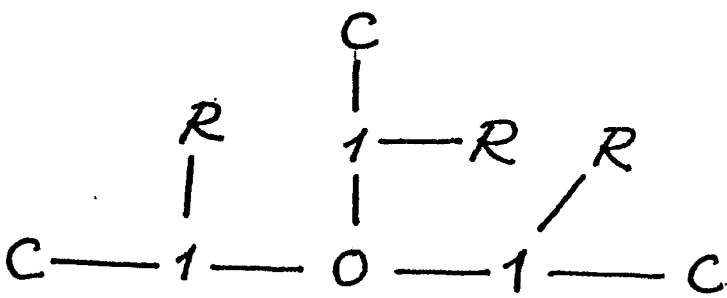

C
R
C
S_Q
0
1
0
R

(a)

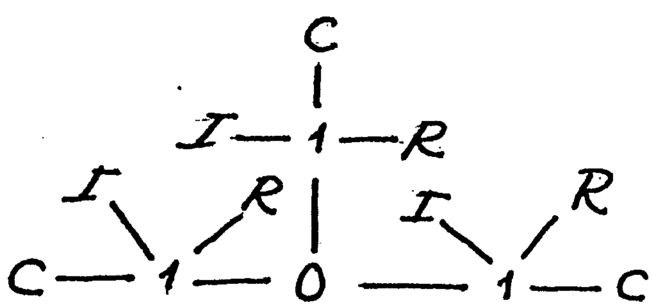

C
R
C
S_Q
0
1
S_Q
0
R
No back pressure
Flow out of first tank = flow into second

(b)

4-7


C
R
1
R
R
C
1
0
1
C

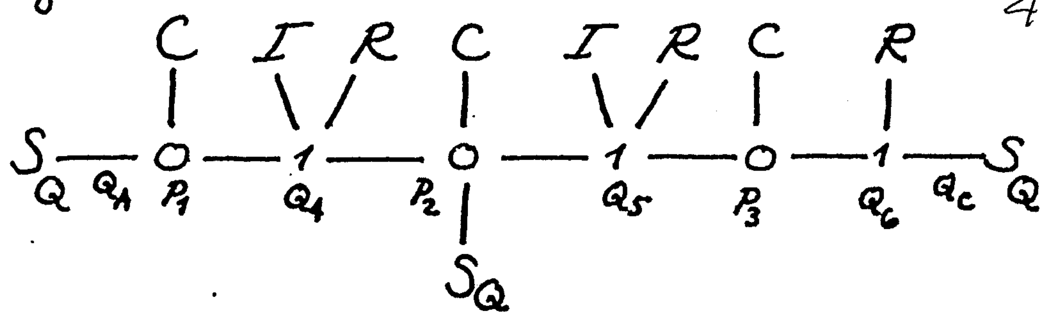
(a)


C
I
1
R
I
R
I
R
C
1
0
1
C

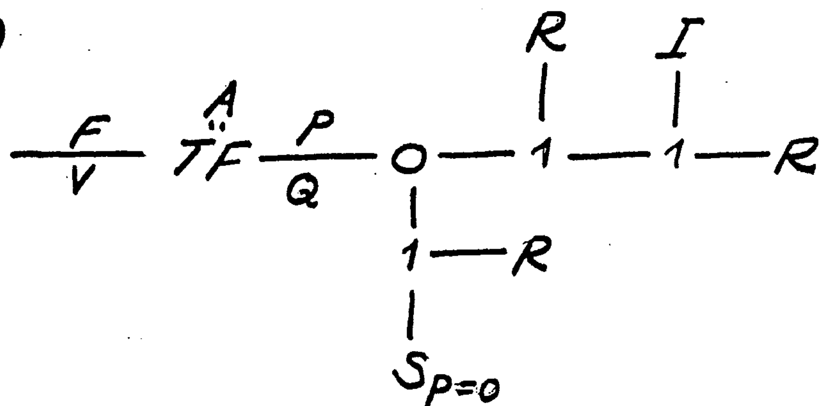
(b)

4-8

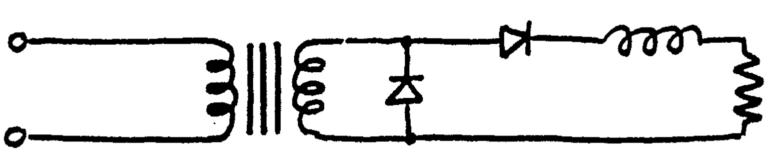
4-9



4-9



(a)



(6)

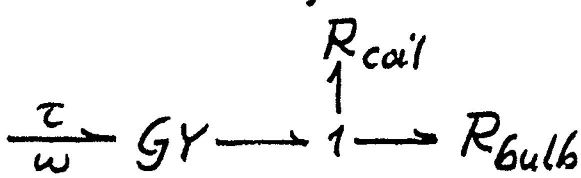
4-10

$$\xrightarrow[\omega]{\tau} GY \xrightarrow[i]{e} R$$

(a)

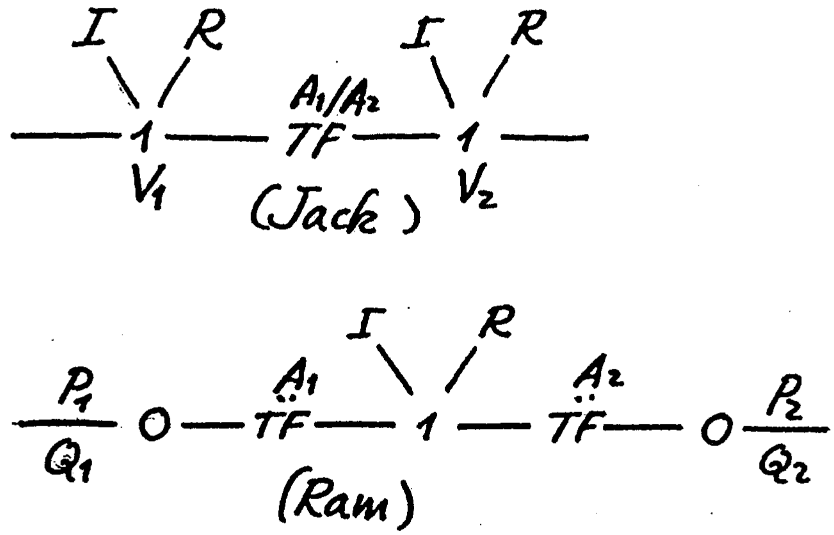
(b) If $R \to 0$, $e \to 0$ but $e = T\omega$ so $\omega \to 0$

If bulb shorts, there still is coil resistance

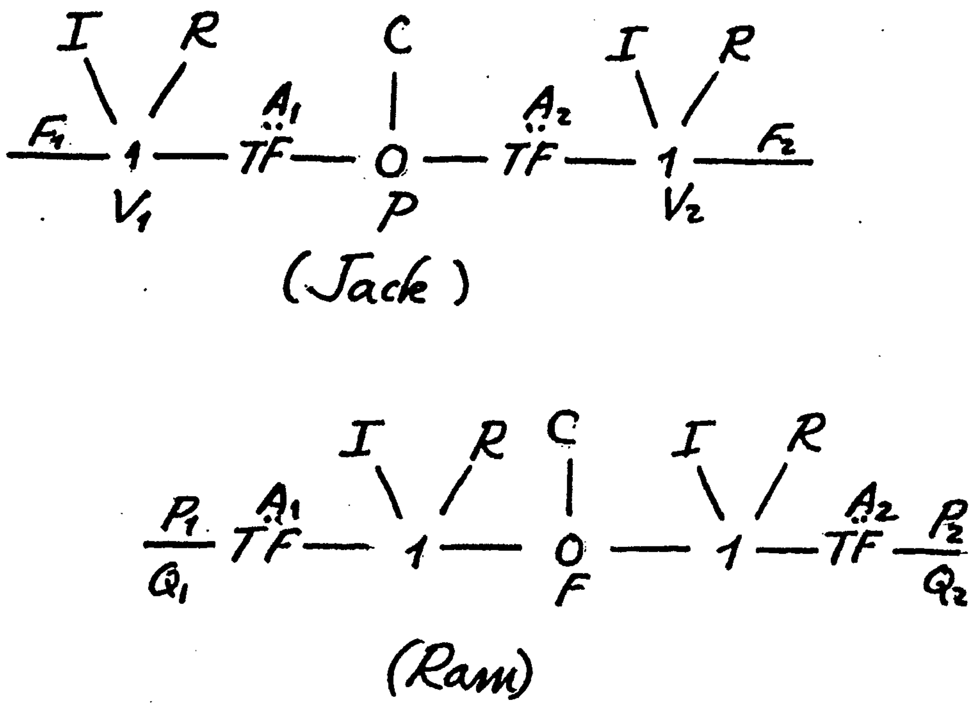


4-11

4-10

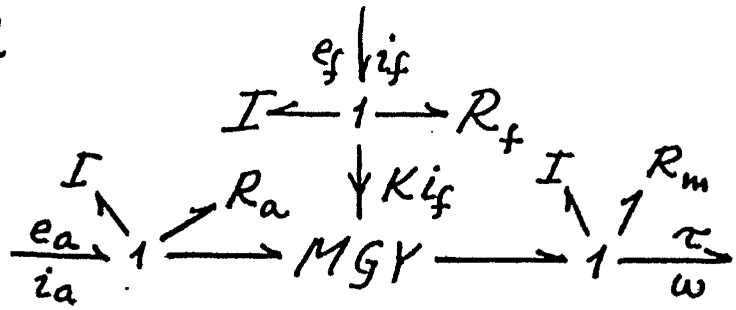

I
R
A_1/A_2
TF
1
V_1
V_2
(Jack)
P_1
Q_1
0
A_1
A_2
P_2
Q_2
(Ram)

(a)


I
R
C
F_1
F_2
A_1
A_2
TF
0
1
V_1
V_2
P
(Jack)
P_1
Q_1
F
P_2
Q_2
(Ram)

(b)

4-12



(a, b)

(c) No change.

(d) $$\eta_{motor} = \frac{\tau\omega}{e_a i_a + e_f i_f}$$

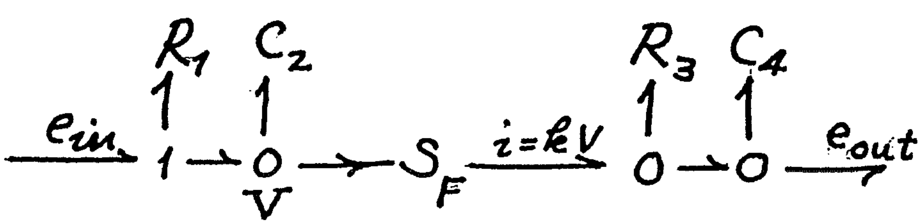
$$\eta_{generator} = \frac{e_a i_a}{\tau\omega + e_f i_f}$$

For steady operation, I's play no role

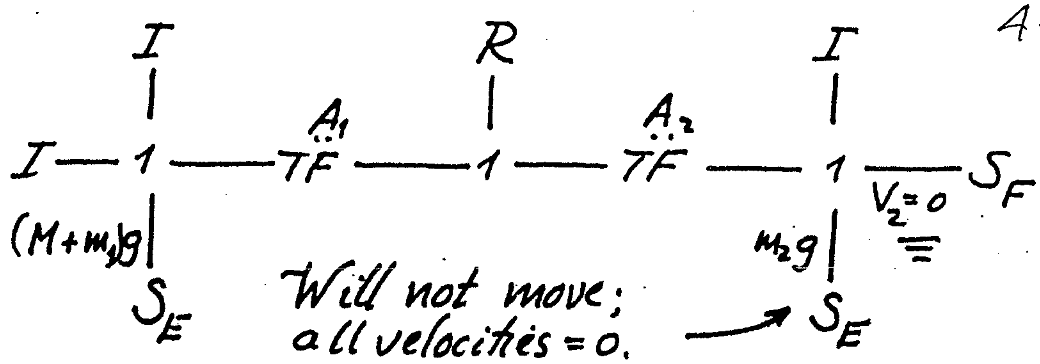
$$i_a = \frac{e_a - K i_f \omega}{R_a}, \quad i_f = e_f / R_f$$

$$\tau = K i_f i_a - R_m \omega$$

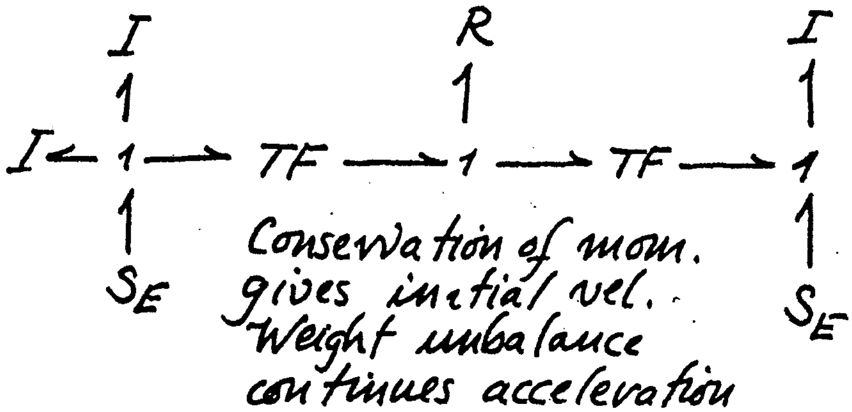
4-13



4-14

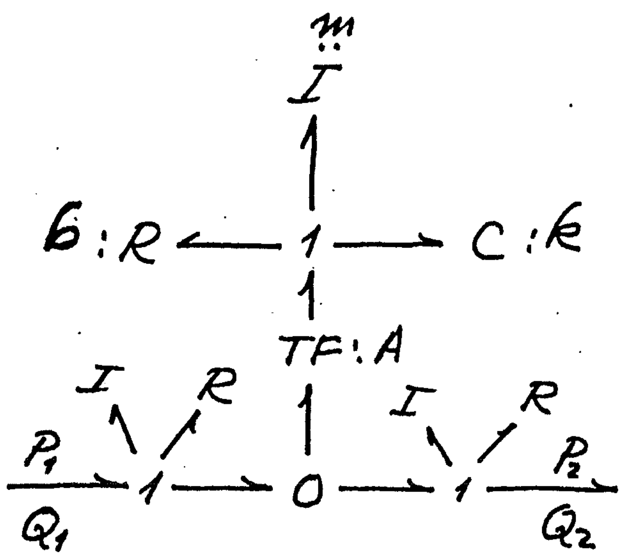

I
R
I
A_1
A_2
I — 1 — TF — 1 — TF — 1 — S_F
$V_2 = 0$
$(M+m_1)g$
$m_2 g$
S_E
Will not move;
all velocities = 0.
S_E

(a)

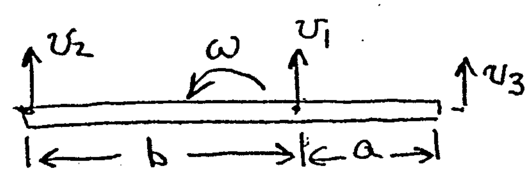

I
R
I
I ← 1 → TF → 1 → TF → 1
S_E
Conservation of mom.
gives initial vel.
Weight unbalance
continues acceleration
S_E

(b)

4-15


m
I
b : R ← 1 → C : k
TF : A
I
R
I
R
P_1
Q_1
1 → 0 → 1
P_2
Q_2

4-16 look at the lever,

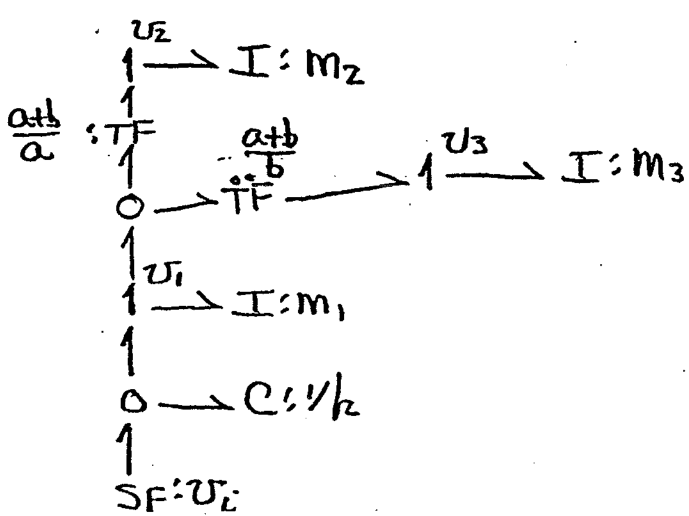


kinematics:

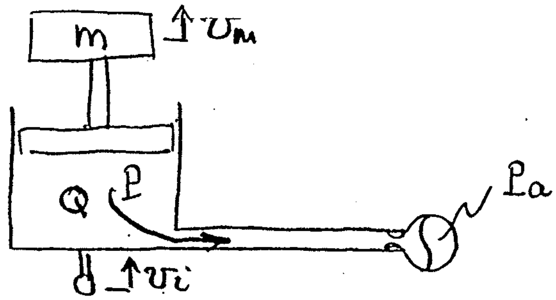
$$\omega = \frac{v_3 - v_2}{(a+b)}$$

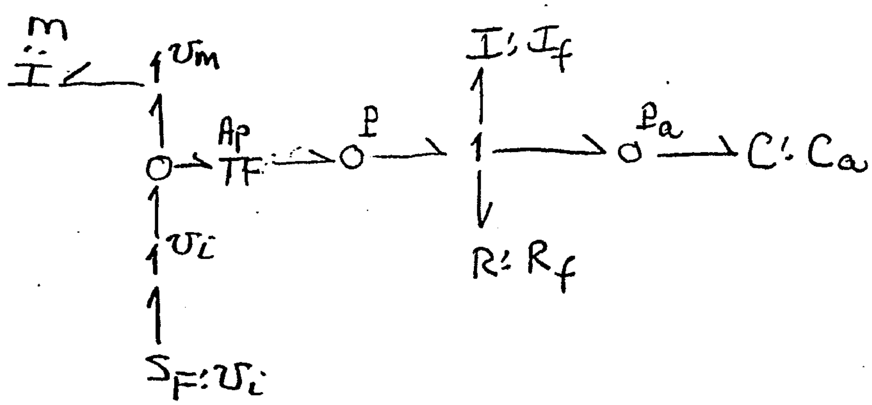
$$v_3 = v_1 + a\left[\frac{v_3 - v_2}{(a+b)}\right]$$

$$v_1 = \frac{b}{a+b}\, v_3 + \frac{a}{a+b}\, v_2$$ kinematic constraint

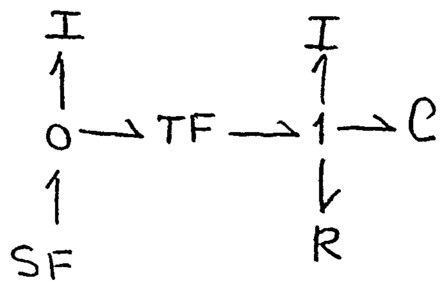


4-17

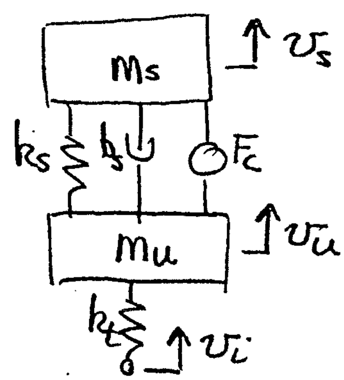




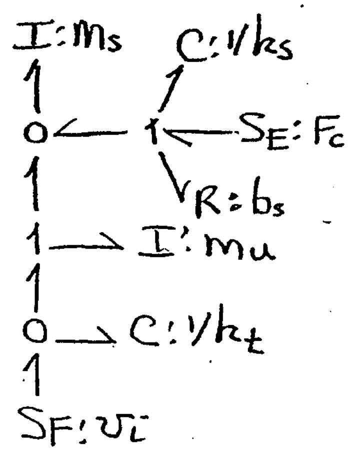
can be simplified to:



4-18



k_s, b_s positive in tension

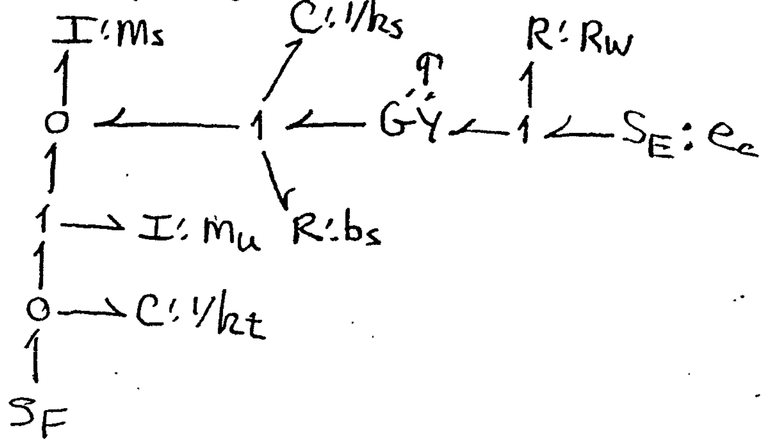


(a)

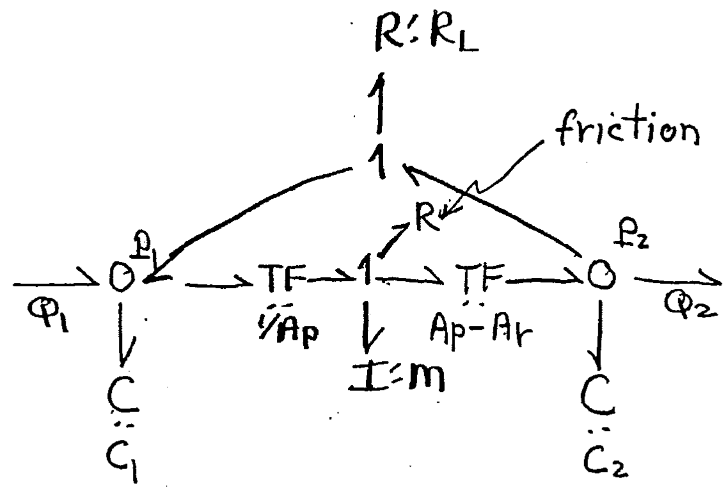
(continued)

4-18 4-15

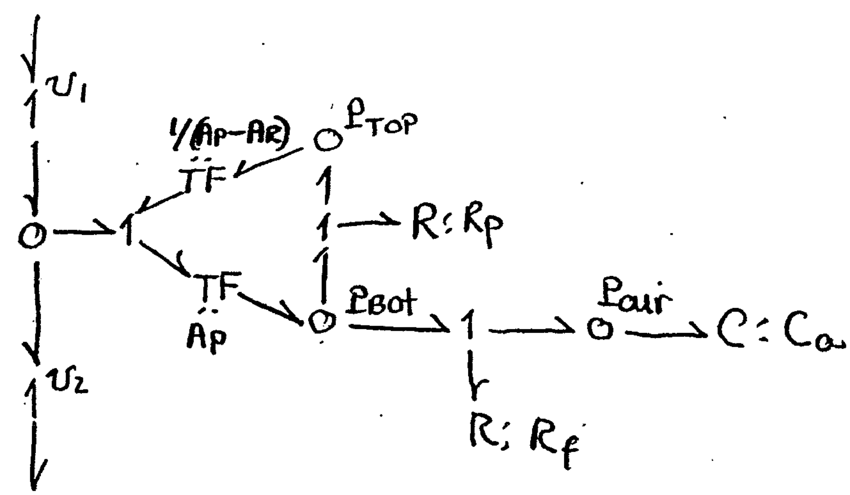
(b)



4-19

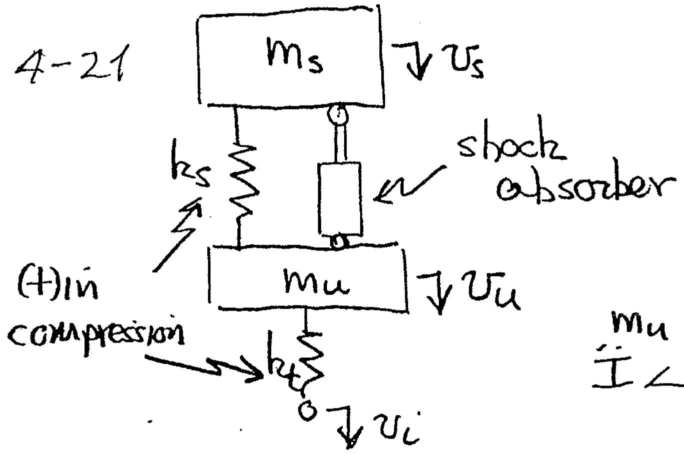


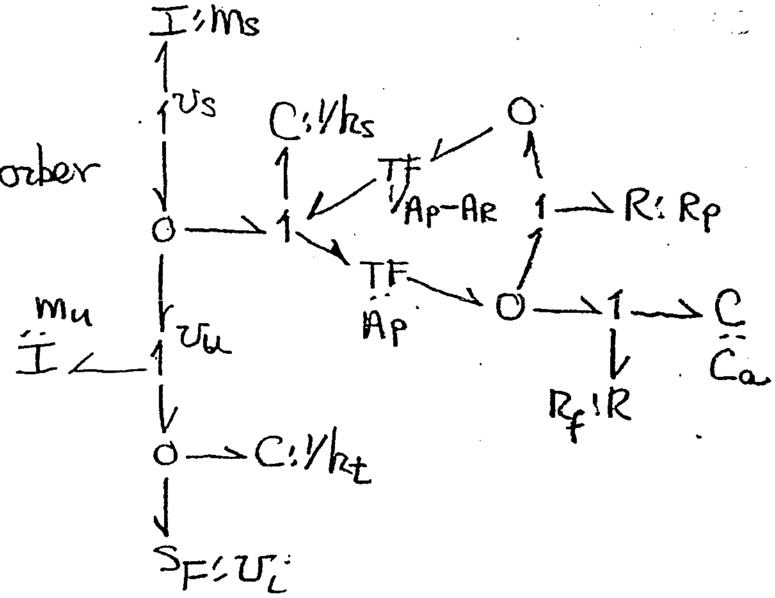
4-20



Realistic that might be included:
(1) compliance of the top + bottom chambers
(2) nonlinear resistances for the piston and foot value
(3) seal friction
(4) leakage

4-21


m_s
v_s
k_s
shock absorber
(+) in compression
m_u
v_u
k_t
v_i


I: m_s
v_s
C: $1/k_s$
TF $A_p - A_R$
R: R_p
TF A_p
C C_a
R_f : R
m_u I
v_u
C: $1/k_t$
S_F: v_i

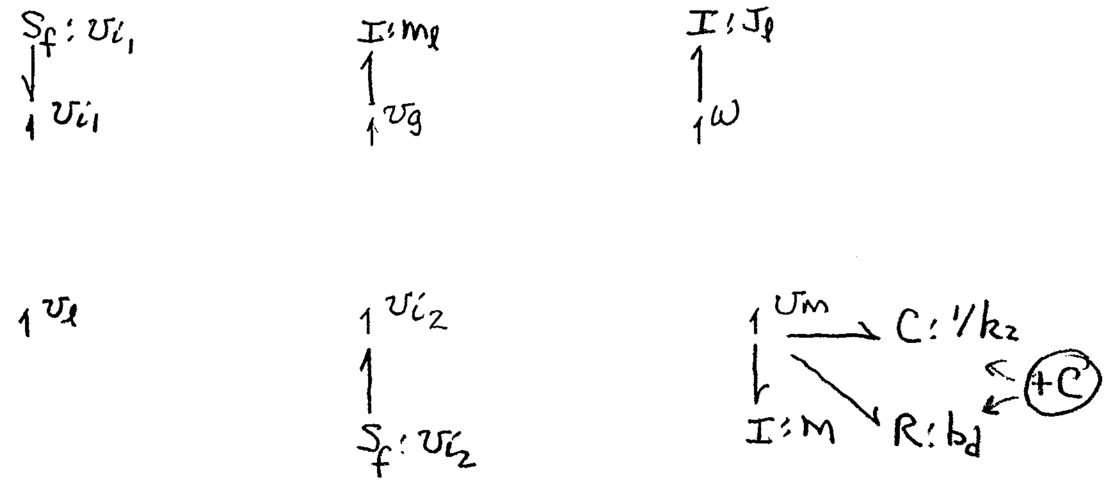
using the rigid body kinematic formula, $\underline{v}_P = \underline{v}_o + \underline{\omega} \times \underline{r}_{P/o}$

(a) $v_\ell = v_g + (a-g)\omega$

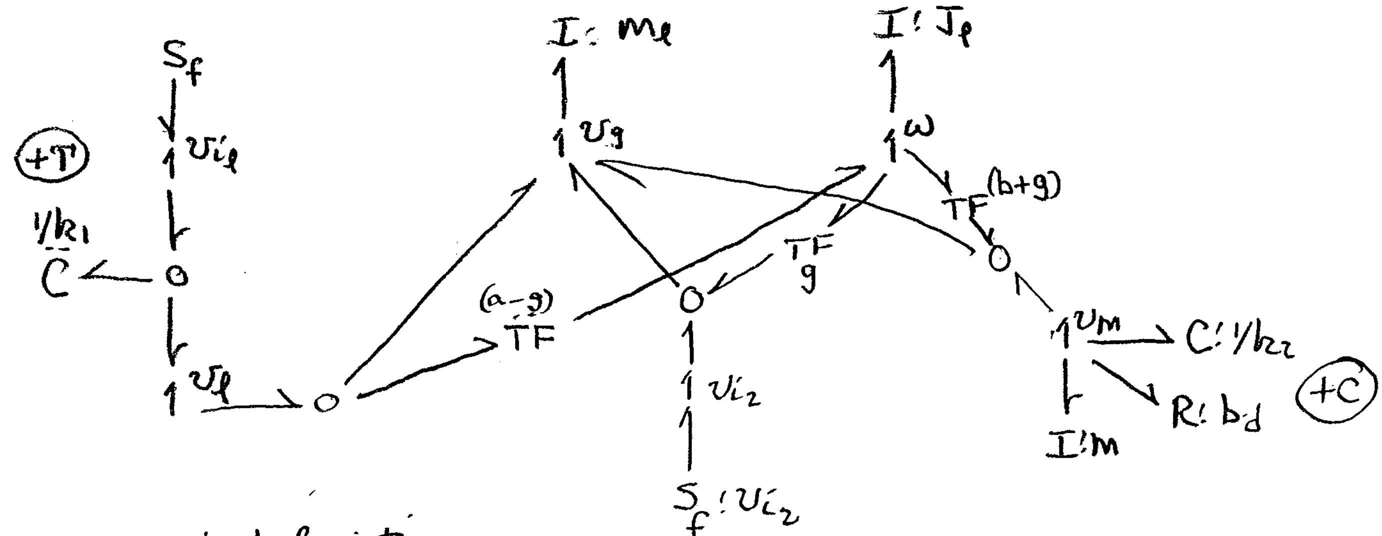
$v_{i2} = v_g - g\omega$

$v_M = v_g - (b+g)\omega$

(b)



Now enforce constraints using 0-junctions



some simplifications could be performed.

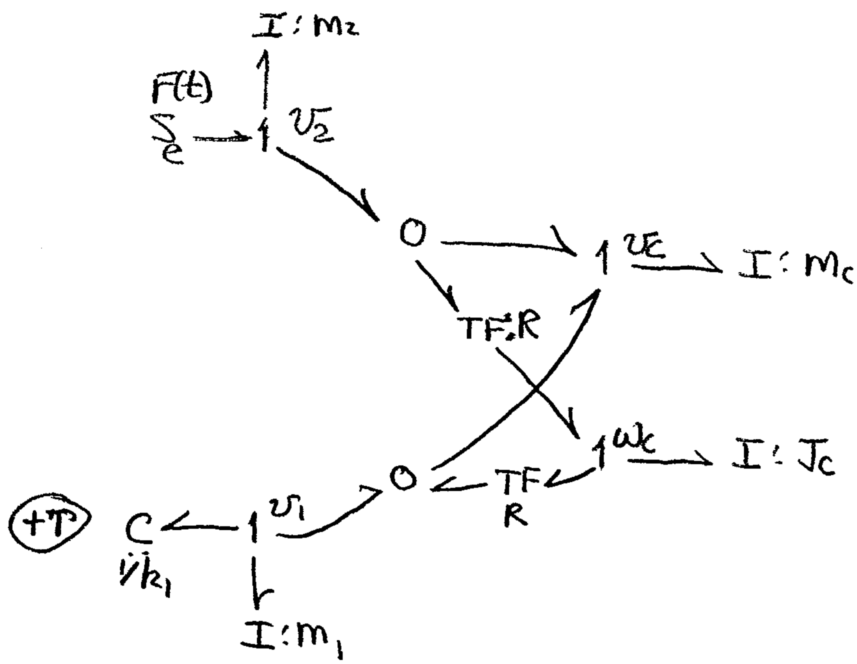
4-23

(a) kinematic constraints using $\underline{v}_P = \underline{v}_O + \underline{\omega} \times \underline{r}_{P/O}$

$$v_2 = v_c + R\omega$$

$$v_1 = v_c - R\omega$$

(b)



4-19

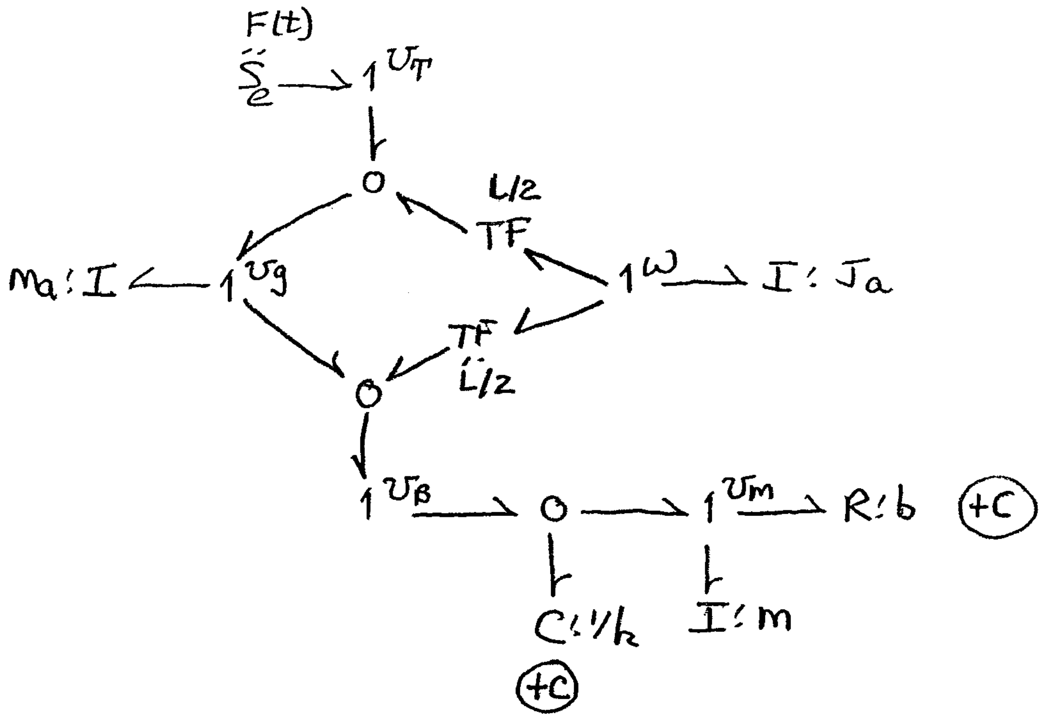
4-24

(a) kinematic constraints using $\underline{v}_P = \underline{v}_0 + \underline{\omega} \times \underline{r}_{P/0}$

$$v_T = v_g - \frac{L}{2}\omega$$

$$v_B = v_g + \frac{L}{2}\omega$$

(b)



(a) $\omega_n = dc / D(IL)^{1/2}$, $f_n = dc / 2\pi D(IL)^{1/2}$

4-25 (b) $f_n = 343/2\pi = 54.6Hz$ This frequency is near the lower limit of audibility for humans.

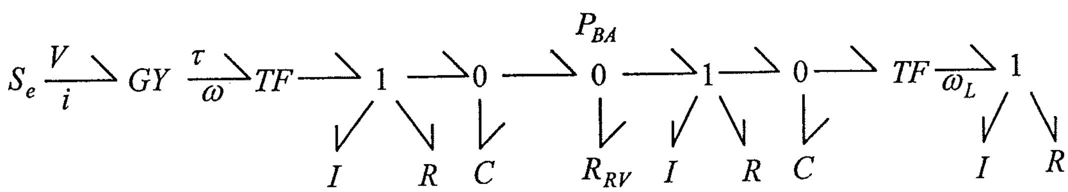
$C = V_o(1/B + 2r_0/t_w E)$; $B = 1.52 \times 10^9$; $E = 2.3 \times 10^9$;

4-26

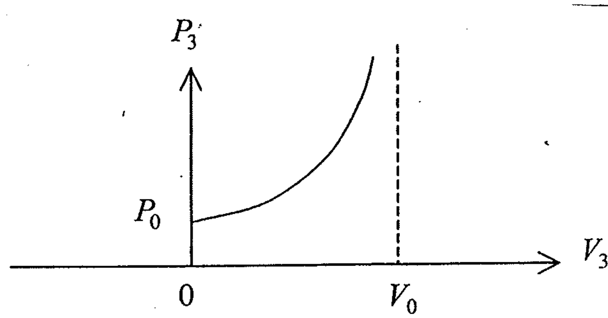
$V_0 / B = 6.58 \times 10^{-10} V_0$ $[m^5/N]$; $V_0(2r_0/t_w E) = 2.61 \times 10^{-9} V_0$ $[m^5/N]$

In this example, the hose flexibility is the major contributor to the hydraulic compliance.

4-27

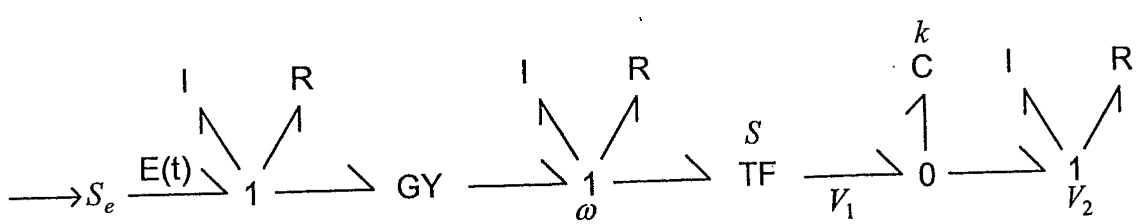


4-28

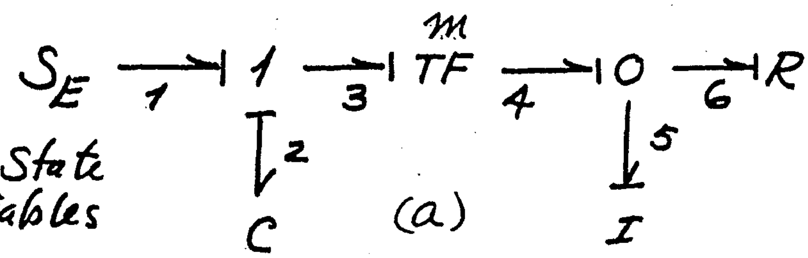


4-29 (a) $I_{eq} = (m + \rho l A^2 / a)$, (b) $a = \rho l A^2 / m$, $r = 15[mm]$.

4-30



5-1 5-1



(a)

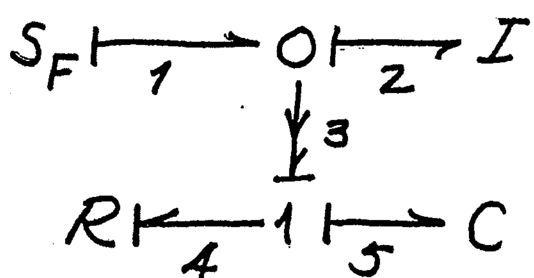
Two State Variables

$$\dot{p}_5 = e_4 = m e_3 = m\left(E_1(t) - e_2\right) = \underline{m\left(E_1(t) - q_2/C_2\right)}$$

$$\dot{q}_2 = f_3 = m f_4 = m\left(f_5 + f_6\right) = m\left(p_5/I_5 + e_6/R_6\right)$$

$$= m\left(p_5/I_5 + m e_3/R_6\right) = m\left[\frac{p_5}{I_5} + \frac{m(E_1 - e_2)}{R_6}\right]$$

$$= \underline{m\left[\frac{p_5}{I_5} + \frac{m\left(E_1(t) - q_2/C_2\right)}{R_6}\right].}$$



← Deriv. Caus.

← Int. Caus.

(6)

One State Vbl.

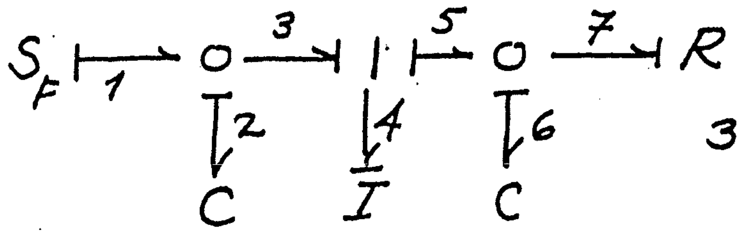
$$\dot{q}_5 = f_4 = e_4/R_4 = (e_3 - e_5)/R_4$$

$$= \frac{e_2 - q_5/C_5}{R_4} = \frac{dp_2/dt - q_5/C_5}{R_4}$$

$$= \frac{d(I_2 f_2)/dt - q_5/C_5}{R_4}$$

$$\dot{q}_5 = \frac{I_2\, dF_1(t)/dt - q_5/C_5}{R_4}$$

5-1 Cont.



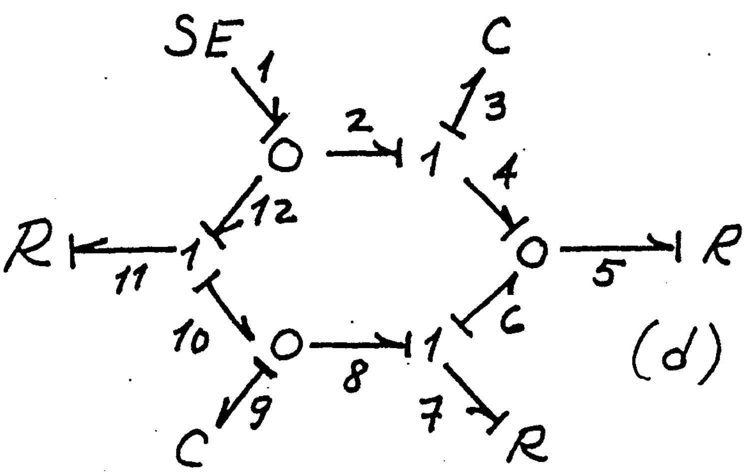
3 State Vbls.

(c)

$$\dot{q}_2 = F_1(t) - f_3 = F_1(t) - f_4 = \underline{F_1(t) - p_4/I_4}$$

$$\dot{p}_4 = e_3 - e_5 = e_2 - e_6 = \underline{q_2/C_2 - q_6/C_6}$$

$$\dot{q}_6 = f_5 - f_7 = f_4 - e_7/R_7 = p_4/I_4 - \frac{e_6}{R_7} = \underline{p_4/I_4 - \frac{q_6/C_6}{R_7}}$$



Two State Variables

(d)

$$\dot{q}_3 = f_4 = f_5 - f_6 = \frac{e_5}{R_5} - f_7 = \frac{e_4}{R_5} - \frac{e_7}{R_7} = \frac{e_2 - e_3}{R_5} - \left(\frac{e_8 - e_6}{R_7}\right)$$

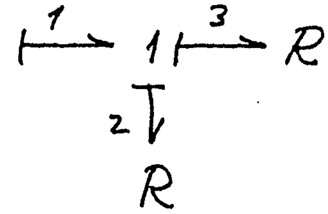
$$= \frac{E_1(t) - q_3/C_3}{R_5} - \left(\frac{e_9 - e_4}{R_7}\right) = \frac{E_1 - q_3/C_3}{R_5} - \left(\frac{q_9/C_9 - e_2 + e_3}{R_7}\right)$$

$$= \underline{\frac{E_1(t) - q_3/C_3}{R_5} - \left(\frac{q_9/C_9 - E_1(t) + q_3/C_3}{R_7}\right)}$$

$$\dot{q}_9 = \underline{\frac{E_1(t) - q_9/C_9}{R_{11}} - \left(\frac{q_9/C_9 - E_1(t) + q_3/C_3}{R_7}\right)}$$

5-2

(a)



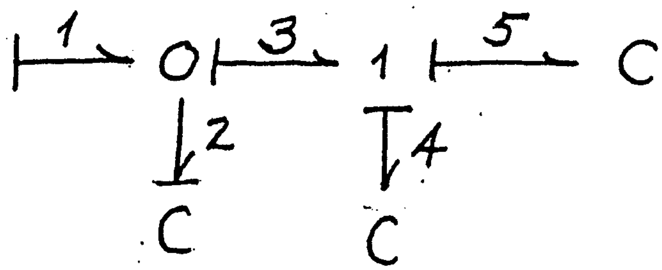
$$e_1 = e_2 + e_3 = R_2 f_2 + R_3 f_3$$
$$= (R_2 + R_3) f_1$$

5-3

$$R_{eq} = R_2 + R_3$$

or $e_1 = \Phi_{eq}(f_1)$, $\Phi_{eq}(f_1) = \Phi_{R_1}(f_1) + \Phi_{R_2}(f_1)$

(b)



$$e_1 = e_3 = e_4 + e_5 = q_4/C_4 + q_5/C_5 \qquad (1)$$

now find q_4, q_5 in terms of q_1

$$\dot{q}_4 = f_3 = f_1 - f_2 = f_1 - \frac{d}{dt} C_2 e_2 = f_1 - \frac{d}{dt} C_2 e_3$$
$$= f_1 - C_2 \frac{d}{dt}\left(\frac{q_4}{C_4} + \frac{q_5}{C_5}\right)$$

$\dot{q}_5 = \dot{q}_4$; integrating

$$q_4 = q_1 - C_2\left(\frac{q_4}{C_4} + \frac{q_5}{C_5}\right) + \cancelto{0}{\text{const.}}$$
$$q_5 = q_1 - C_2\left(\frac{q_4}{C_4} + \frac{q_5}{C_5}\right) + \cancelto{0}{\text{const}}$$

solving,
$$q_4 = q_5 = \frac{C_4 C_5}{C_4 C_5 + C_2 C_4 + C_2 C_5} q_1$$

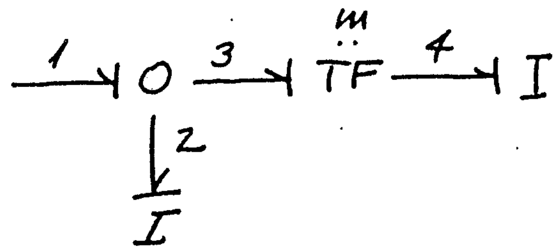
then, from (1) we eventually conclude

$$e_1 = \underbrace{\frac{C_4 + C_5}{C_4 C_5 + C_4 C_2 + C_5 C_2}}_{1/C_{eq}} \cdot q_1$$

Wow!

5-2

Continued



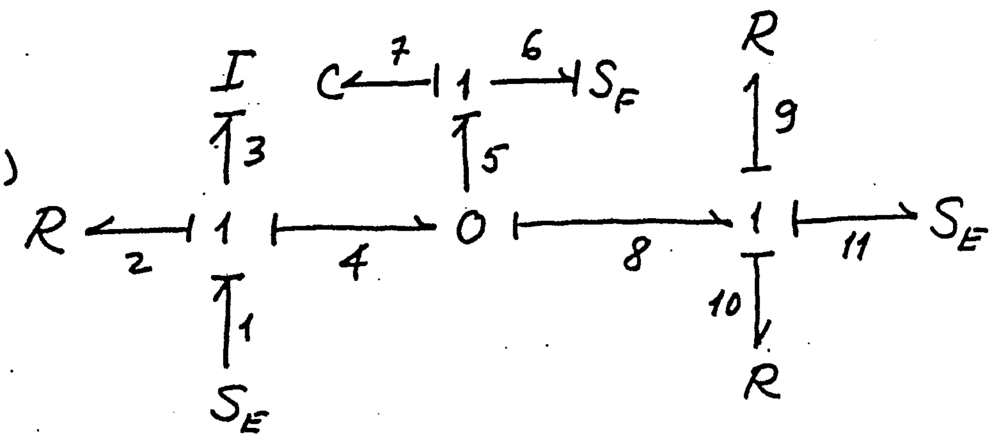
$$f_1 = f_3 + f_2 = \frac{p_2}{I_2} + m\,\frac{p_4}{I_4}$$

$$\left.\begin{aligned} \dot{p}_2 &= e_1 \\ \dot{p}_4 &= m e_1 \end{aligned}\right\} \quad \begin{aligned} p_2 &= p_1 \\ p_4 &= m p_1 \end{aligned} \qquad \text{so } f_1 = \frac{p_1}{I_2} + m m \frac{p_1}{I_4}$$

$$I_{eq} = \left[\frac{1}{I_2} + \frac{m^2}{I_4}\right]^{-1}$$

5-3

(a)

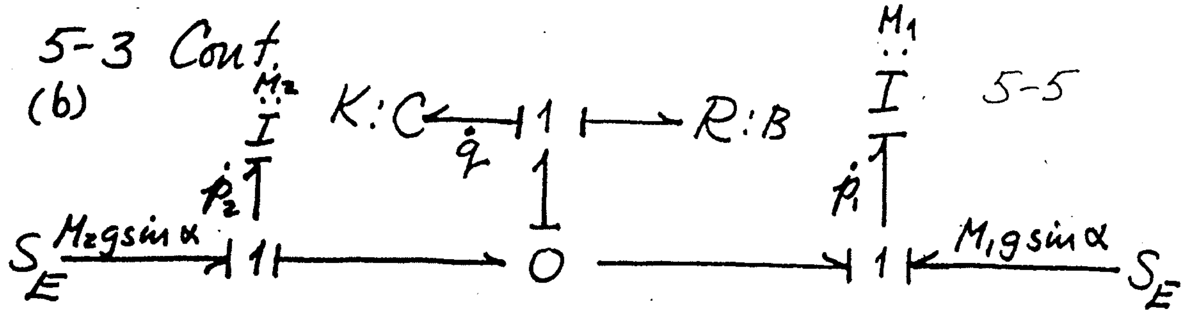


$$\dot{p}_3 = -\frac{R_2}{I_3}\,p_3 + E_1(t) - E_{11}(t) + (R_9 + R_{10})\left(F_6(t) - \frac{p_3}{I_3}\right)$$

$$\dot{q}_7 = F_6(t)$$

5-3 Cont.

(b)



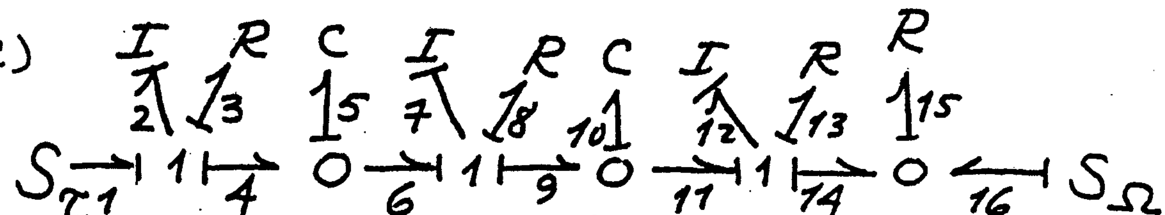
(6)

$$\dot{p}_1 = M_1 g \sin\alpha + Kq + B\left(\frac{p_2}{M_2} - \frac{p_1}{M_1}\right)$$

$$\dot{q} = \frac{p_2}{M_2} - \frac{p_1}{M_1}$$

$$\dot{p}_2 = M_1 g \sin\alpha - Kq - B\left(\frac{p_2}{M_2} - \frac{p_1}{M_1}\right)$$

(c)



(c)

$$\dot{p}_2 = \tau_1(t) - R_3 \frac{p_2}{J_2} - K_5 q_5$$

$$\dot{q}_5 = \frac{p_2}{J_2} - \frac{p_7}{J_7}$$

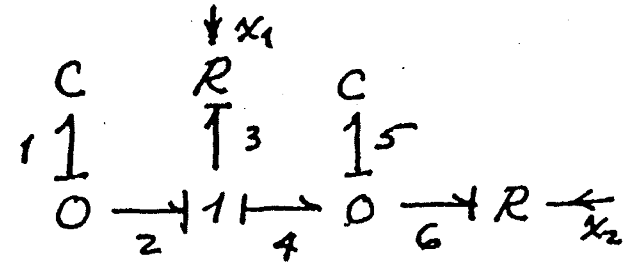
$$\dot{p}_7 = K_5 q_5 - R_8 \frac{p_7}{J_7} - K_{10} q_{10}$$

$$\dot{q}_{10} = \frac{p_7}{J_7} - \frac{p_{12}}{J_{12}}$$

$$\dot{p}_{12} = K_{10} q_{10} - R_{13} \frac{p_{12}}{J_{12}} - R_{15}\left(\frac{p_{12}}{J_{12}} + \Omega_{16}(t)\right)$$

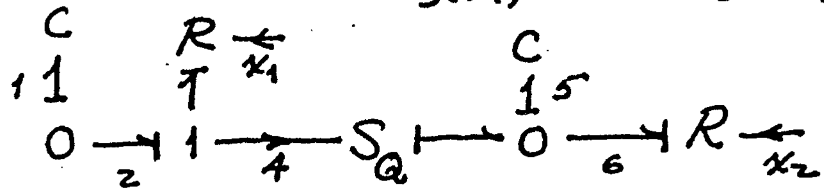
5-3 Cont.

(d)



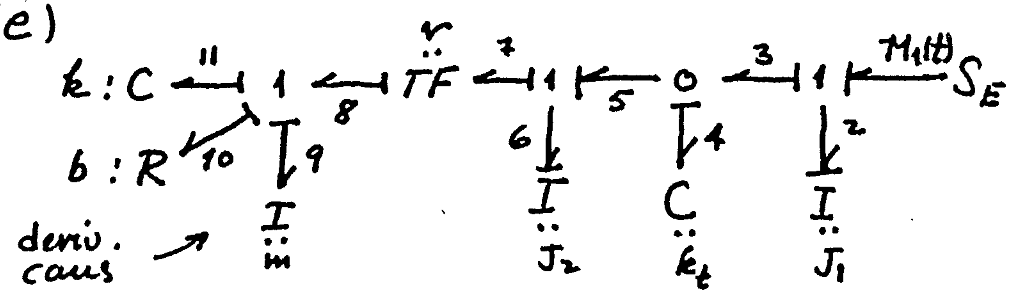
$$\dot{q}_1 = -\frac{(q_1/C_1 - q_5/C_5)}{R_3(x_1)}$$

$$\dot{q}_5 = \frac{q_1/C_1 - q_5/C_5}{R_3(x_1)} - \frac{q_5/C_5}{R_6(x_2)}$$



$$\dot{q}_1 = -\frac{q_1/C_1}{R_3(x_1)} \quad ; \quad \dot{q}_5 = \frac{q_1/C_1}{R_3(x_1)} - \frac{q_5/C_5}{R_6(x_2)}$$

(e)



$$\underline{\dot{p}_2 = M_1(t) - k_t q_4}; \quad \underline{\dot{q}_4 = p_2/J_1 - p_6/J_2}$$

$$\underline{\dot{q}_{11} = r p_6/J_2}$$

$$\dot{p}_6 = k_t q_4 - r(e_{11} + e_{10} + e_9) = k_t q_4 - r[k q_{11} + b r p_6/J_2 + \dot{p}_9]$$

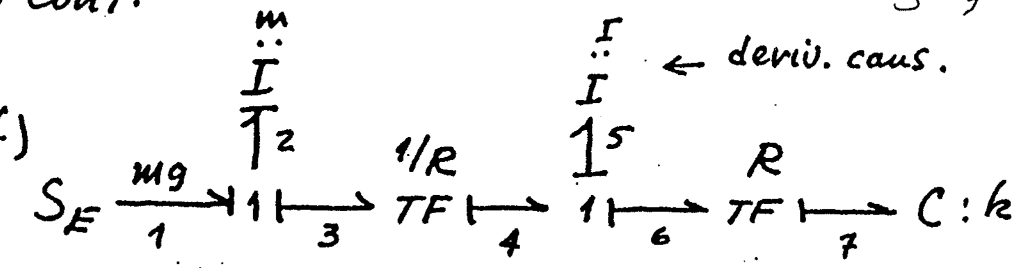
but $p_9 = m f_9 = m f_8 = m r f_7 = m r f_6 = m r p_6/J_2$

sub. in last eq., solve for $\dot{p}_6$

$$\underline{\dot{p}_6 = \left[\frac{1}{1 + mr^2/J_2}\right]\left[k_t q_4 - r k q_{11} - \frac{r^2 b}{J_2} p_6\right]}$$

5-3 Cont.

(f)



$$\dot q_7 = R \cdot \frac{1}{R} \cdot p_2/m = \underline{p_2/m}$$

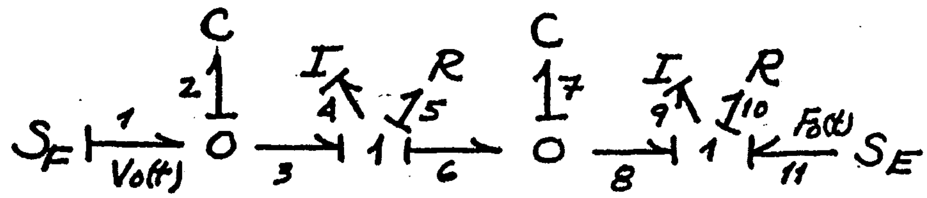
$$\dot p_2 = mg - e_3 = mg - \frac{1}{R}(e_5 + e_6) = mg - \frac{1}{R}(\dot p_5 + Rkq_7)$$

but $p_5 = I f_5 = I \cdot \frac{1}{R} \cdot \frac{p_2}{m}$ or $\dot p_5 = \frac{I}{Rm}\dot p_2$

so $\dot p_2 = \underline{[1 + I/R^2 m]^{-1}[mg - kq_7]}$

5-4
$$\dot q_1 = I(t) - \lambda_1/L_1$$
$$\dot\lambda_1 = -R_1\lambda_1/L_1 + q_1/C_1 - q_2/C_2$$
$$\dot q_2 = \lambda_1/L_1 - \lambda_2/L_2$$
$$\dot\lambda_2 = -R_2\lambda_2/L_2 + q_2/C_2 - R_L\lambda_2/L_2$$
$$e = R_L\lambda_2/L_2$$

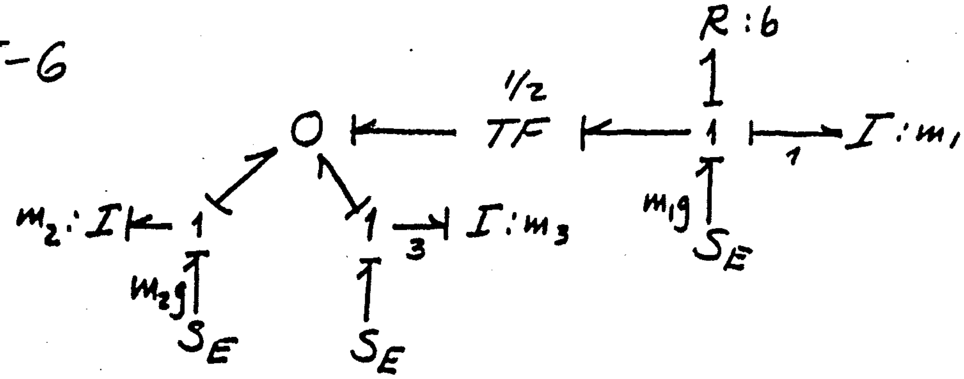
5-5



$$\dot q_2 = V_0(t) - p_4/m \; ; \; \dot p_4 = Aq_2^3 - B\,\mathrm{sgn}\left(\frac{p_4}{m}\right) - Aq_7^3 \; ;$$

$$\dot q_7 = p_4/m - p_9/m \; ; \; \dot p_9 = Aq_7^3 - B\,\mathrm{sgn}\left(\frac{p_9}{m}\right) + F_0(t).$$

5-6



5-8

$$\dot{p}_2 = m_2 g - \tfrac{1}{2}\left(m_1 g - b\left(\tfrac{1}{2}\left(-\tfrac{p_2}{m_2} - \tfrac{p_3}{m_3}\right) - \dot{p}_1\right)\right)$$

$$= m_2 g - \frac{m_1 g}{2} - \frac{b}{4}\left(\frac{p_2}{m_2} + \frac{p_3}{m_3}\right) - \frac{\dot{p}_1}{2}$$

$$\dot{p}_3 = m_3 g - \frac{m_1 g}{2} - \frac{b}{4}\left(\frac{p_2}{m_2} + \frac{p_3}{m_3}\right) - \frac{\dot{p}_1}{2}$$

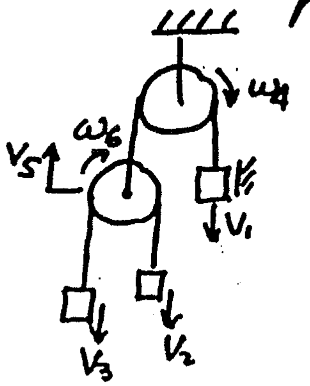
but $p_1 = m_1 f_1 = m_1\left(-\frac{1}{2}\left(\frac{p_2}{m_2} + \frac{p_3}{m_3}\right)\right)$

$$\dot{p}_1 = -\frac{m_1}{2}\left(\frac{\dot{p}_2}{m_2} + \frac{\dot{p}_3}{m_3}\right)$$

$$\begin{bmatrix} 1 + \frac{m_1}{4m_2} & \frac{m_1}{4m_3} \\ \frac{m_1}{4m_2} & 1 + \frac{m_1}{4m_3} \end{bmatrix} \begin{bmatrix} \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} \frac{-b}{4m_2} & \frac{-b}{4m_3} \\ \frac{-b}{4m_2} & \frac{-b}{4m_3} \end{bmatrix} \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} m_2 g - \frac{m_1 g}{2} \\ m_3 g - \frac{m_1 g}{2} \end{bmatrix}$$

invert to get standard form.

With pulley inertias and pin friction, the system becomes quite complex:

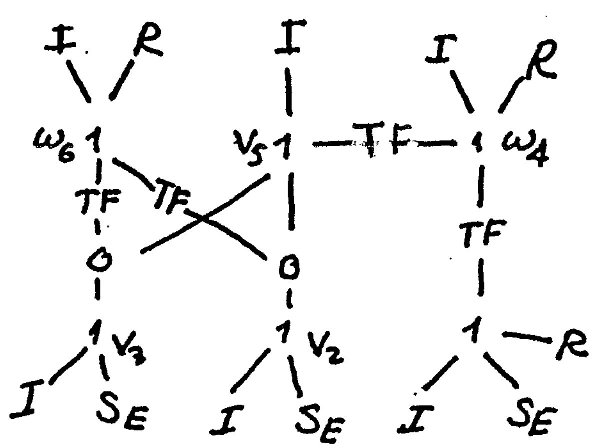


$V_1 = R\omega_4$

$V_5 = V_1$

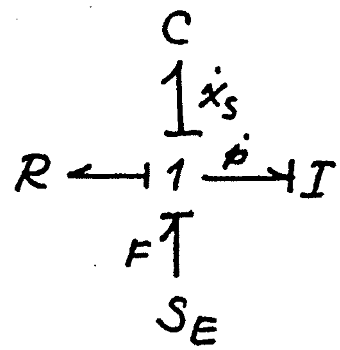
$V_2 = -V_5 + R\omega_6$

$V_3 = -V_5 - R\omega_6$



Lots of derivative causality!

5-7



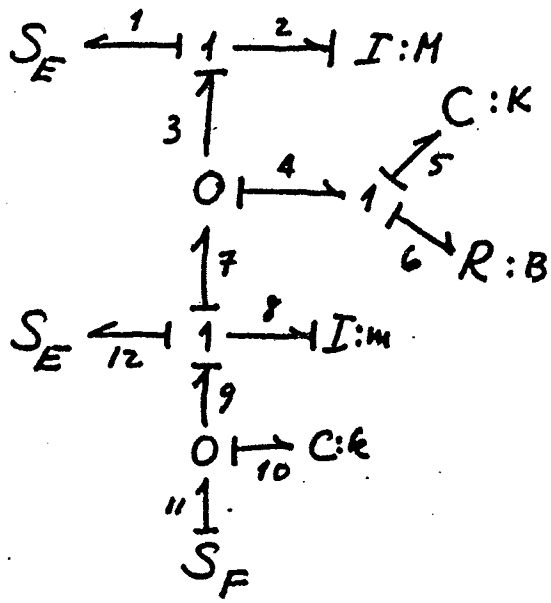
$\dot{x}_s = p/m$ 5-9

$\dot{p} = -\phi_s(x_s) - \frac{b}{m^2}|p|p + F(t)$

or $\dot{x}_s = v$

$\dot{v} = -\frac{\Phi(x_s)}{m} - \frac{b}{m}|v|v + \frac{F(t)}{m}$

5-8

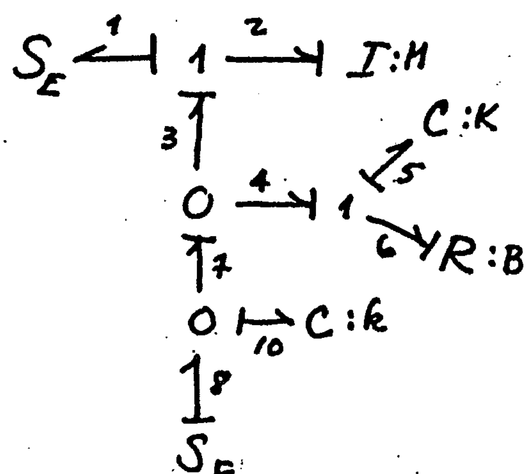


$\dot{p}_2 = -Mg + Kq_5 + B\left(\frac{p_8}{m} - \frac{p_2}{M}\right)$

$\dot{q}_5 = p_8/m - p_2/M$

$\dot{p}_8 = -mg + kq_{10} - Kq_5$
$\qquad -B\left(\frac{p_8}{m} - \frac{p_2}{M}\right)$

$\dot{q}_{10} = V(t) - p_8/m$

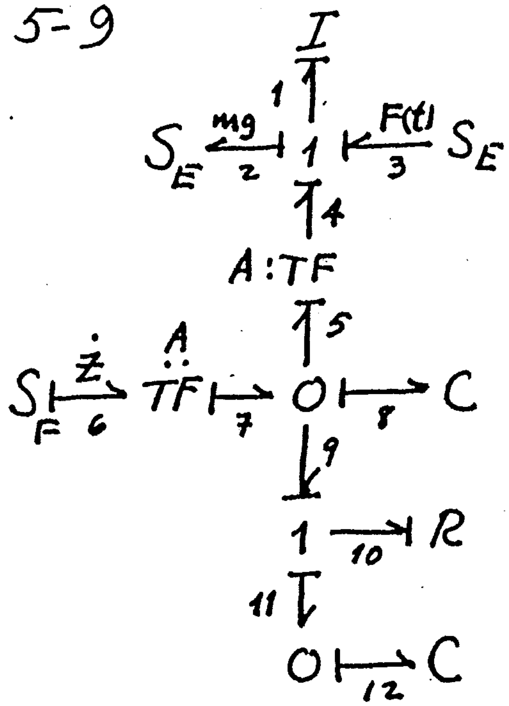


$\dot{p}_2 = -Mg + kq_{10}$

$\dot{q}_5 = \frac{1}{B}\left(-Kq_5 + kq_{10}\right)$

$\dot{q}_{10} = V_0(t) - \frac{1}{B}(-Kq_5 + kq_{10})$
$\qquad - p_2/M$

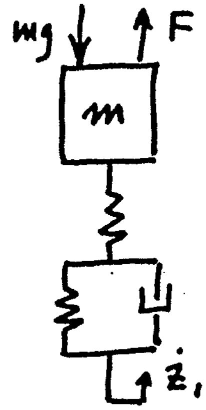
5-9



$$\dot{p}_1 = F(t) - mg + A\,\Phi_{C8}^{-1}(q_8),$$

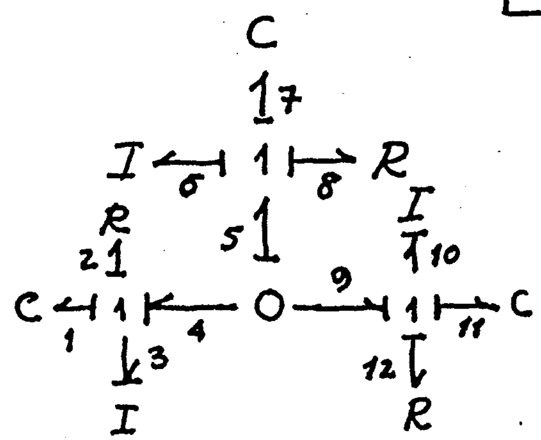
$$\dot{q}_8 = A\,\dot{z}_1(t) - A\,p/m$$
$$+\Phi_{R10}^{-1}\left(\Phi_{C8}^{-1}(q_8) - \Phi_{C12}^{-1}(q_{12})\right),$$

$$\dot{q}_{12} = +\Phi_{R10}^{-1}\left(\Phi_{C8}^{-1}(q_8) - \Phi_{C12}^{-1}(q_{12})\right).$$



Transformers not shown here.

5-10



Note: these equations are for linear capacitors.

$$\dot{p}_3 = -q_1/C_1 - R_2 p_3/I_3 + \left(\dot{p}_6 + q_7/C_7 + R_8(-p_3/I_3 - p_{10}/I_{10})\right)$$

$$\dot{p}_{10} = -q_{11}/C_1 - R_{12} p_{10}/I_{10} + \left(\dot{p}_6 + q_7/C_7 + R_8(-p_3/I_3 - p_{10}/I_{10})\right)$$

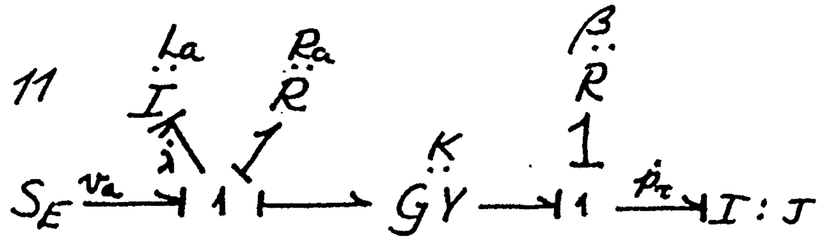
$$\dot{q}_1 = p_3/I_3 \;;\; \dot{q}_{11} = p_{10}/I_{10} \;;\; \dot{q}_7 = -p_3/I_3 - p_{10}/I_{10}$$

Diff. Caus. $p_6 = I_6 f_6 = I_6\left(-p_3/I_3 - p_{10}/I_{10}\right)$

$$\dot{p}_6 = \frac{-I_6}{I_3}\dot{p}_3 - \frac{I_6}{I_{10}}\cdot\dot{p}_{10}$$ ← Sub. into state eq.

2x2 matrix inversion is req'd.

5-11



5-11

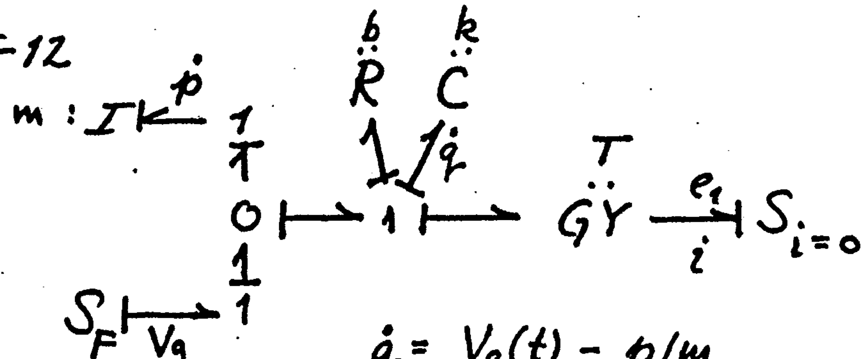
$$\dot\lambda = v_a(t) - R_a\lambda/L_a - K p_\tau/J$$

$$\dot p_\tau = K\lambda/L_a - \beta p_\tau/J$$

Notes: $\lambda = Li$, $p_\tau = J\dot\theta_0$, $K_T = K_V = K$ in SI units.

Bond graph is second order unless θ_0 is desired in which case $\dot\theta_0 = p_\tau/J$ is added to above equ's.

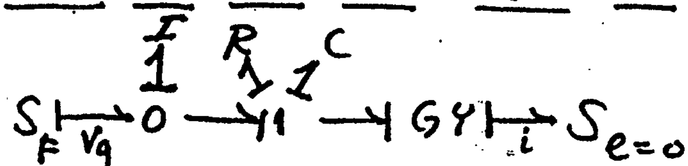
5-12



$$\dot q = V_g(t) - p/m$$

$$\dot p = kq + b(V_g(t) - p/m) + T\cdot(0)$$

$$e_1 = T\,(V_g(t) - p/m)$$



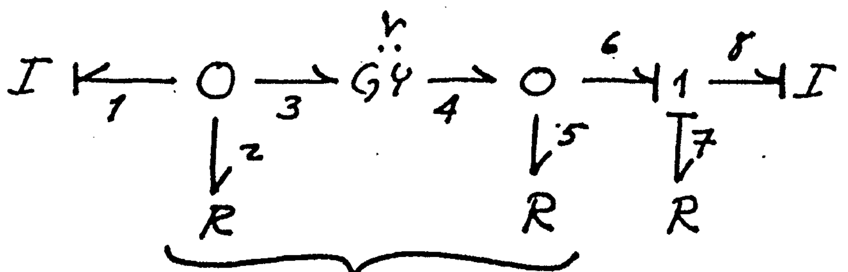
no state eqns. $\dot q = 0$, $p/m = V_g(t)$

$$i = \frac{1}{T}\left(-k\cancel{q}^{\,0} - b\cancel{\dot q}^{\,0} + \frac{d}{dt}(V_g - \cancel{\dot q}^{\,0})m\right) = \frac{m\dot V_g}{T}$$

With coil resistance $\vdash GY \dashv \overset{R}{\underset{\top}{1}} \vdash S_{e=0}$, second order again

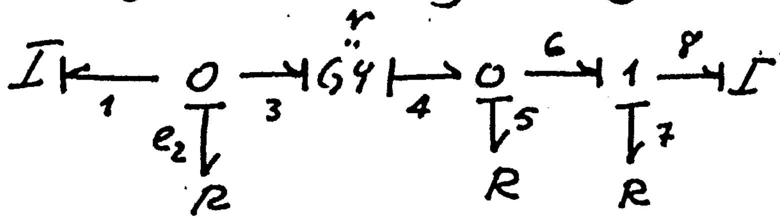
5-13

(a)



5-12

Causality in this R-field not determined by I_1 and I_8. Some bond must be assigned causality arbitrarily. Try bond 2:



$$\dot{p}_1 = e_2 \; ; \; \dot{p}_8 = -R_7 \frac{p_8}{I_8} + R_5\left(-\frac{p_8}{I_8} + \frac{1}{r} e_2\right)$$

now find e_2 in terms of itself, p_1 and p_8

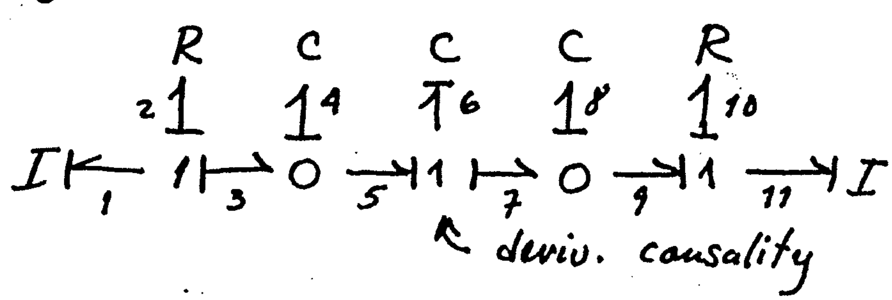
$$e_2 = R_2\left(-p_1/I_1 - \frac{R_5}{r}\left(-p_8/I_8 + \frac{e_2}{r}\right)\right)$$

Solve for e_2 and use in state equations.

(b) System has a causal conflict of type

$S_E - GY - S_F$ or $S_E - S_E$ i.e. a nonsense system.

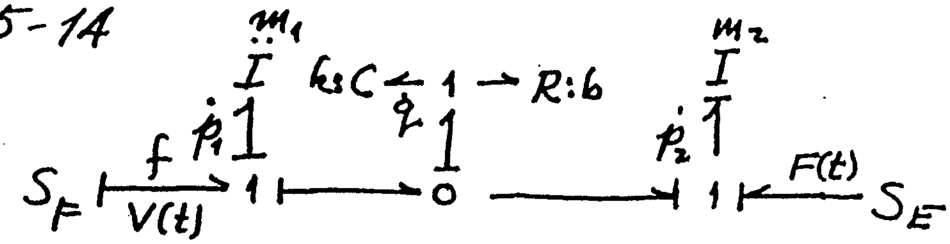
(c)



$$q_6 = C_6 e_6 = C_6\left(\frac{q_4}{C_4} - \frac{q_8}{C_8}\right)$$

$\dot{q}_6$ appears in state eqn's. – may be replaced by expression in $\dot{q}_4$, $\dot{q}_8$, 2x2 matrix inversion req'd.

5-14

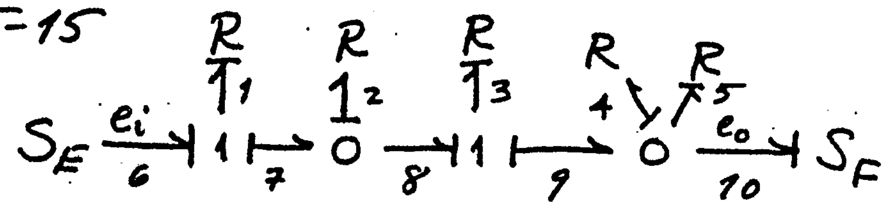


5-13

(a) $\dot p_2 = F(t) - kq - b\,(V(t) - p_2/m_2)$
$\dot q = V(t) - p_2/m_2$
complete State Equ's.

(b) $f = \dot p_1 + kq + b\,(V(t) - p_2/m_2)$
$= m_1 \dot V_1(t) + b(V(t)) + kq - b\,p_2/m_2$

5-15



causality chosen arbitrarily on bonds 1, 2, 3, 4. We use f_1, e_2, f_3, e_4 as auxiliary variables, e_i and f_{10} as inputs. Equ's are

$e_o = e_4 = R_4\,(f_3 - e_4/R_5 - f_{10})$

$f_3 = (e_2 - e_4)/R_3$

$e_2 = R_2\,(f_1 - f_3)$

$f_1 = (e_{in} - e_2)/R_1$

4 equs may be solved for aux. var in terms of e_{in} and f_{10}

If there is a load resistance instead of $\underset{10}{S_F}$, use $f_{10} = e_4/R_L$ instead of f_{10} as input quantity.

5.16 state variables p_5, q_7, p_{10}, q_{12}

$$\dot{p}_5 = \frac{T}{R_2}\left(e - \frac{T}{I_5} p_5\right) - \frac{q_7}{C_7}$$

$$\dot{q}_7 = \frac{p_5}{I_5} - \frac{1}{R}\frac{p_{10}}{I_{10}}$$

$$\dot{p}_{10} = -\frac{q_{12}}{C_{12}} - \frac{R_{11}}{I_{10}} p_{10} + \frac{1}{R}\frac{q_7}{C_7}$$

$$\dot{q}_{12} = p_{10}/I_{10}$$

$$\frac{d}{dt}\begin{bmatrix} p_5 \\ q_7 \\ p_{10} \\ q_{12} \end{bmatrix} = \begin{bmatrix} -T^2/R_2 I_5 & -1/C_7 & 0 & 0 \\ 1/I_5 & 0 & -1/R I_{10} & 0 \\ 0 & 1/R C_7 & -R_{11}/I_{10} & -1/C_{12} \\ 0 & 0 & 1/I_{10} & 0 \end{bmatrix}\begin{bmatrix} p_5 \\ q_7 \\ p_{10} \\ q_{12} \end{bmatrix} + \begin{bmatrix} T/R_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} e$$

output eqns:

$$e_4 = \frac{T}{R_2}\left(e - \frac{T}{I_5} p_5\right)$$

$$f_7 = \frac{p_5}{I_5} - \frac{1}{R}\frac{p_{10}}{I_{10}}$$

$$e_2 = e - \frac{T}{I_5} p_5$$

5.17 state variables p_5, p_{10}, q_{12}

To handle derivative causality, $p_8 = I_8\left[\frac{R}{I_5}p_5 + \frac{p_{10}}{I_{10}}\right]$

$$\dot{p}_8 = e_8 = R\frac{I_8}{I_5}\dot{p}_5 + \frac{I_8}{I_{10}}\dot{p}_{10}$$

state eqns:

$$\dot{p}_5 = \frac{T}{R_2}\left(e - \frac{T}{I_5}p_5\right) - R\,\dot{p}_8$$

$$\dot{p}_{10} = \frac{q_{12}}{C_{12}} - \dot{p}_8$$

$$\dot{q}_{12} = v_i - \frac{p_{10}}{I_{10}}$$

substitute for $\dot{p}_8$

$$\dot{p}_5\left[1 + R^2\frac{I_8}{I_5}\right] + R\frac{I_8}{I_{10}}\dot{p}_{10} = \frac{T}{R_2}e - \frac{T^2}{R_2 I_5}p_5$$

$$R\frac{I_8}{I_5}\dot{p}_5 + \dot{p}_{10}\left[1 + \frac{I_8}{I_{10}}\right] = q_{12}/C_{12}$$

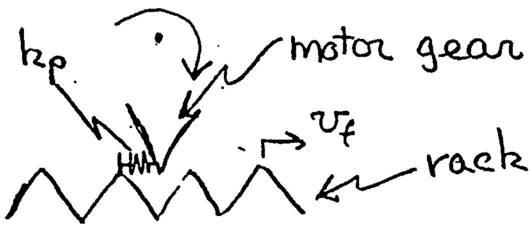
$$\dot{q}_{12} = v_i - \frac{p_{10}}{I_{10}}$$

can write as,

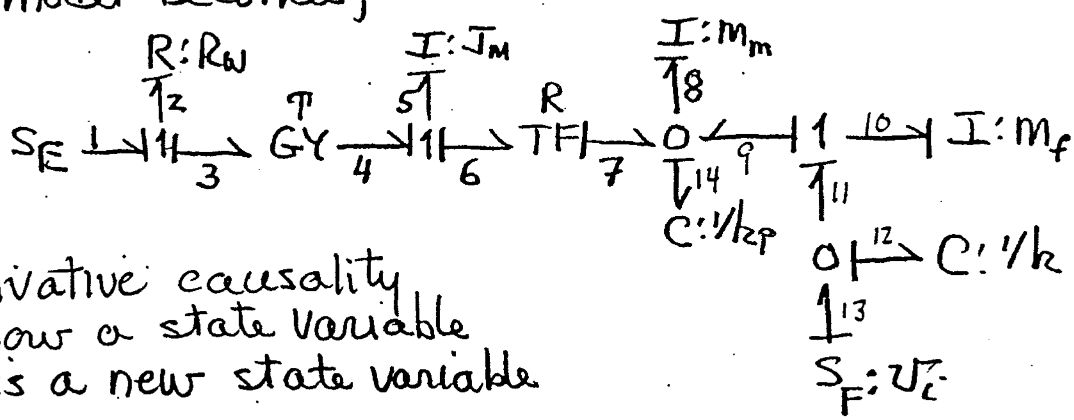
$$\begin{bmatrix} 1+R^2\frac{I_8}{I_5} & R\frac{I_8}{I_{10}} & 0 \\ R I_8/I_5 & 1+\frac{I_8}{I_{10}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{p}_5 \\ \dot{p}_{10} \\ \dot{q}_{12} \end{bmatrix} = \begin{bmatrix} -T^2/R_2 I_5 & 0 & 0 \\ 0 & 0 & 1/C_{12} \\ 0 & -1/I_{10} & 0 \end{bmatrix}\begin{bmatrix} p_5 \\ p_{10} \\ q_{12} \end{bmatrix} + \begin{bmatrix} T/R_2 \\ 0 \\ 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v_i$$

final eqns. requires inversion of this matrix

5.18 If we imagine that a very stiff spring is between the gear tooth of the motor and the tooth of the rack, such as



the model becomes,



No derivative causality
p_8 now a state variable
q_{14} is a new state variable

$$\dot{p}_5 = \frac{T}{R_2}\left(e - \frac{T}{I_5} p_5\right) - \frac{R}{C_{14}} q_{14}$$

$$\dot{p}_8 = q_{14}/C_{14}$$

$$\dot{p}_{10} = \frac{q_{12}}{C_{12}} - \frac{q_{14}}{C_{14}}$$

$$\dot{q}_{12} = v_i - p_{10}/I_{10}$$

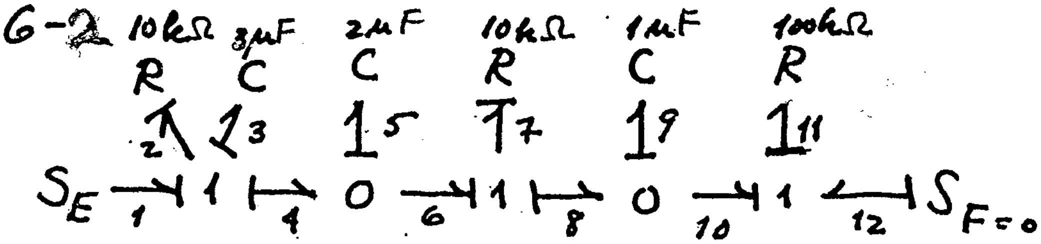
$$\dot{q}_{14} = \frac{R}{I_5} p_5 - \frac{p_8}{I_8} + \frac{p_{10}}{I_{10}}$$

6-1. $x = \frac{V_0}{\omega_n} \sin \omega_n t$; $\omega_n = 10 \text{ rad/sec} = \sqrt{\frac{k}{m}}$

Period $= T = \frac{2\pi}{\omega_n}$, Δt should be a relatively small fraction of T say 1/10 or less

$\Delta t \cong \frac{1}{10} \cdot \frac{2\pi}{10}$ sec.

6-2



$$\begin{bmatrix} \dot q_3 \\ \dot q_5 \\ \dot q_9 \end{bmatrix} = \begin{bmatrix} \frac{-1}{R_2C_3} & \frac{-1}{R_2C_5} & 0 \\ -\frac{1}{R_2C_3} & \frac{-1}{R_2C_5} - \frac{1}{R_7C_5} & \frac{1}{R_7C_9} \\ 0 & \frac{1}{R_7C_5} & -\frac{1}{R_7C_9} \end{bmatrix} \begin{bmatrix} q_3 \\ q_5 \\ q_9 \end{bmatrix} + \begin{bmatrix} 1/R_2 \\ 1/R_2 \\ 0 \end{bmatrix} E_1$$

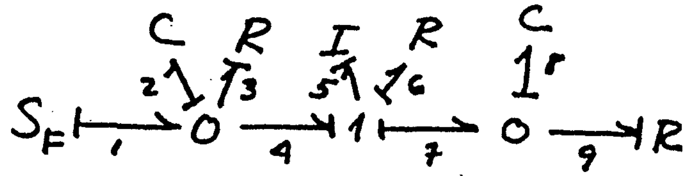
$$e_{12} = q_9 / C_9$$

$R_2 C_3 = 10 \cdot 10^3 \cdot 3 \times 10^{-6} = 30 \times 10^{-3}\, s$ etc.

Characteristic Eqn. $\det [sI - A] = 0$

third order in this case.

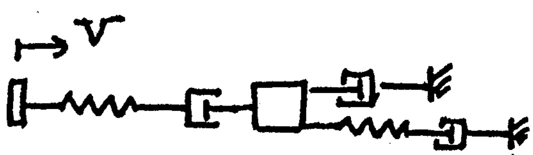
6-3

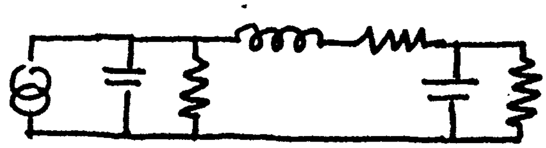


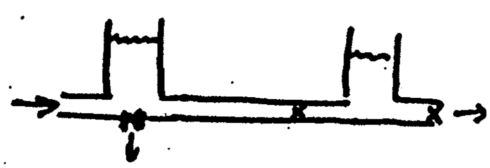
6-2

$$\begin{bmatrix}\dot q_2\\ \dot p_5\\ \dot q_8\end{bmatrix}=\underbrace{\begin{bmatrix}-1/R_3C_2 & -1/I_5 & 0\\ 1/C_2 & -R_6/I_5 & -1/C_8\\ 0 & 1/I_5 & -1/R_9C_8\end{bmatrix}}_{A}\begin{bmatrix}q_2\\ p_5\\ q_8\end{bmatrix}+\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{B}F_1(t)$$

$$\begin{bmatrix}f_4\\ e_7\end{bmatrix}=\underbrace{\begin{bmatrix}0 & 1/I_5 & 0\\ 0 & 0 & 1/C_8\end{bmatrix}}_{C}\begin{bmatrix}q_2\\ p_5\\ q_8\end{bmatrix}+\underbrace{\begin{bmatrix}0\\0\end{bmatrix}}_{D}F_1(t)$$







6-4 $\quad Re\, e^{j\omega t} = \cos\omega t$

$$\frac{d}{dt} Re\, e^{j\omega t} = -\omega \sin\omega t$$

$$\frac{d}{dt} e^{j\omega t} = j\omega e^{j\omega t} = \omega e^{j(\omega t + \pi/2)}$$

$$Re \frac{d}{dt} e^{j\omega t} = \omega \cos(\omega t + \pi/2) = -\omega \sin\omega t$$

6-5 for 1 d.o.f system, $\frac{b}{m} = 2\zeta\omega_n$

for 4 dampers $\quad b = \frac{1}{4} \cdot 2 m \zeta \omega_n$

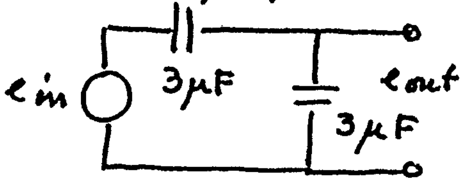
$$b = \frac{3000}{2 \cdot 386} \cdot 0.707 \cdot 2\pi \cdot 1 \quad \frac{lb}{in/sec}$$

6-6 System has three real roots –

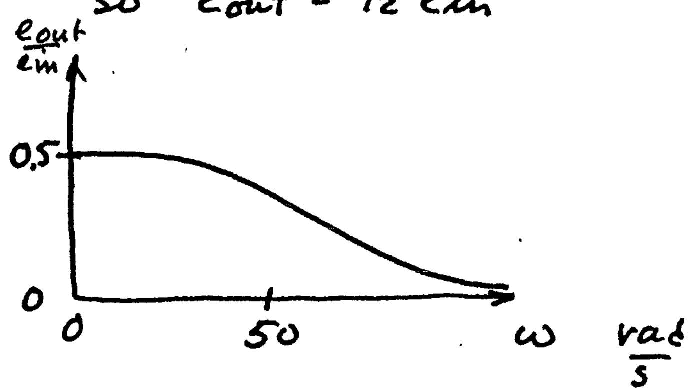
time constants are in the neighborhood of

$(1-3) \times 10^{-2}$ s or $\omega = 1/\tau \cong 30$ to 100 rad/s

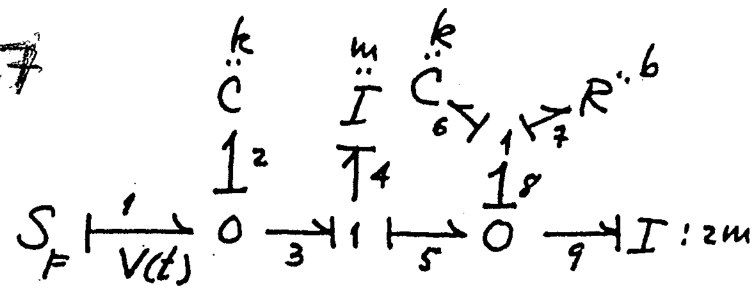
at zero freq., system looks like



so $e_{out} = 1/2\, e_{in}$



6-7



(Gravity forces left out so deflections are measured from equilibrium.)

State equations

$$\begin{bmatrix}\dot q_2\\ \dot p_4\\ \dot q_6\\ \dot p_9\end{bmatrix}=\underbrace{\begin{bmatrix}0 & -1/m & 0 & 0\\ k & -b/m & -k & b/2m\\ 0 & 1/m & 0 & -1/2m\\ 0 & b/m & k & -b/2m\end{bmatrix}}_{A}\begin{bmatrix}q_2\\ p_4\\ q_6\\ p_9\end{bmatrix}+\underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{B}V(t)$$

Acceleration

$$a=\frac{1}{2m}\cdot\dot p_9=\frac{1}{2m}\left(kq_6+\frac{b}{m}p_4-\frac{b}{2m}p_9\right)$$

$$a=\underbrace{\begin{bmatrix}0 & \frac{b}{2m^2} & \frac{k}{2m} & \frac{-b}{4m^2}\end{bmatrix}}_{C}\begin{bmatrix}q_2\\ p_4\\ q_6\\ p_9\end{bmatrix}$$

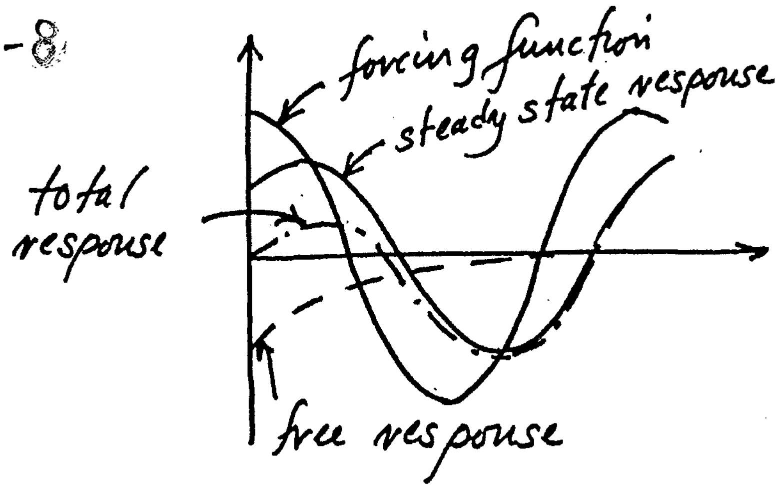
if $V(t)=1\cdot e^{i\omega t}$, Assume $q_2=Q_2e^{i\omega t}$ etc.

then

$$\begin{bmatrix}i\omega & 1/m & 0 & 0\\ -k & i\omega+b/m & k & -b/2m\\ 0 & -1/m & i\omega & 1/2m\\ 0 & -b/m & -k & i\omega+b/2m\end{bmatrix}\begin{bmatrix}Q_2\\ P_4\\ Q_6\\ P_9\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

must be solved, for example, by Cramer's rule. Then a can be found from P_4, Q_6 and P_9. The algebra is a bit lengthy, it must be admitted

6-8

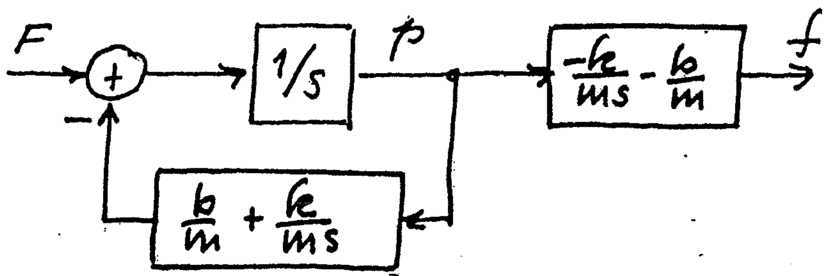


6-9 Both Fig. 6.24 (a) and (b) contain errors in sign. The transfer function in Fig. 6.24 (a) should be $\frac{-(bs+k)}{ms^2+bs+k}$ as in Eq. 6-117 and in Fig. 6.24 (b) f should be computed as $kx - \frac{b}{m}p$ as in Eq. 6.24.

From 6-24 (a) we can relate p and x to themselves as follows: $x = -\frac{1}{ms}p$, $p = \frac{1}{s}\left(F + kx - \frac{b}{m}p\right)$. substituting for x we find $p = \frac{msF}{ms^2+bs+k}$ and then $x = \frac{-F}{ms^2+bs+k}$

using the correct expression for f (above) the correct transfer function is found.

6-10



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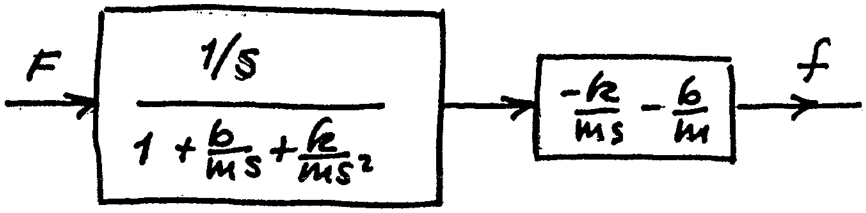
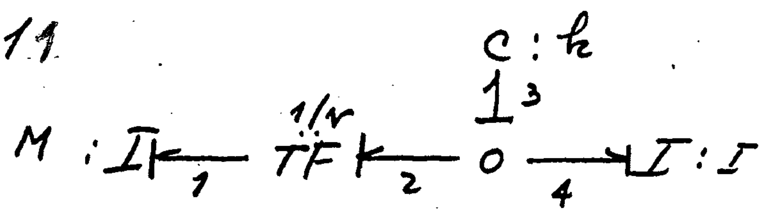


6-11



$$\begin{bmatrix} \dot p_1 \\ \dot q_3 \\ \dot p_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & k/r & 0 \\ -1/rm & 0 & -1/I \\ 0 & k & 0 \end{bmatrix}}_{A} \begin{bmatrix} p_1 \\ q_3 \\ p_4 \end{bmatrix}$$

$$\det[sI - A] = s\left[s^2 + \frac{k}{I}\right] + \frac{k}{r}\cdot\frac{s}{rm} = 0$$

or $s\left(s^2 + \dfrac{k}{I} + \dfrac{k}{r^2 M}\right) = 0$

$s = 0 \leftarrow$ steady spin mode

$s = \pm j\omega_n = \pm j\sqrt{k\left(\dfrac{1}{I} + \dfrac{1}{r^2 M}\right)}$

undamped oscillator mode

6-12

$$\begin{array}{ccccccc} & & R & & C & & \\ & & \uparrow 2 & & \uparrow 4 & & \\ S_E & \xrightarrow[1]{E} & 1 & \xrightarrow[3]{} & 0 & \xrightarrow{e_a} & S_{I=0} \end{array} \; ; \; \begin{array}{ccccccccccc} & & R & & C & & R & & C & & \\ & & \uparrow 2 & & \uparrow 4 & & \uparrow 6 & & \uparrow 8 & & \\ S_E & \xrightarrow[1]{E} & 1 & \xrightarrow[3]{} & 0 & \xrightarrow[5]{e_a} & 1 & \xrightarrow[7]{} & 0 & \xrightarrow[9]{e_b} & S_{I=0} \end{array}$$

(a)

$$\dot{q}_4 = -\frac{q_4}{RC} + \frac{E(t)}{R}$$

$$e_a = q_4/C$$

$$\dot{q}_4 = -\frac{2q_4}{RC} + \frac{q_8}{RC} + \frac{E(t)}{R}$$

$$\dot{q}_8 = \frac{q_4}{RC} - \frac{q_8}{RC}$$

$$e_a = q_4/C \,, \quad e_b = q_8/C$$

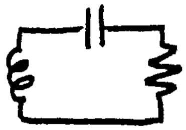
(b) $H_{e_a/E} = \dfrac{1}{RCs+1}$

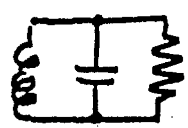
$$H_{e_a/E} = \frac{RCs+1}{R^2C^2s^2+3RCs+1}$$

$$H_{e_b/E} = \frac{1}{R^2C^2s^2+3RCs+1}$$

(c) Loading effects preclude affirmative answers.

6-13

 (a)

(b) 

$$A = \begin{bmatrix} -R/L & -1/c \\ 1/L & c \end{bmatrix}$$

$$s^2 + s\frac{R}{L} + \frac{1}{LC} = 0$$

$R = 0$, undamped

$R \to \infty$, overdamped

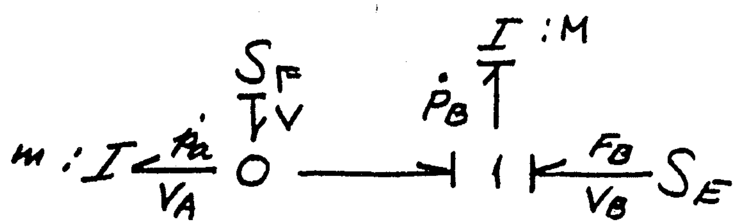
$$A = \begin{bmatrix} 0 & 1/C \\ -1/L & -1/RC \end{bmatrix}$$

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

$R \to 0$, overdamped

$R \to \infty$, undamped

6-14 6-8

(a) 

(b) $\dot p_B = F_B(t) + \dot p_a$, but

$p_a = m V_A = m(V(t) - p_B/M)$

so $\dot p_a = F_B(t) + m(\dot V(t) - \dot p_B/M)$

or $\dot p_B = \frac{M}{m+M}\left(F_B(t) + m\dot V(t)\right)$ ← state eqn.

$$\left.\begin{aligned} V_B &= p_B/M \\ V_A &= V(t) - p_B/M \end{aligned}\right\}$$ output eqn's.

(c) i. $\frac{V_B}{F_B} = \frac{p_B}{M F_B} = \frac{M}{i\omega(m+M} \frac{F_B}{F_B}$ ✓

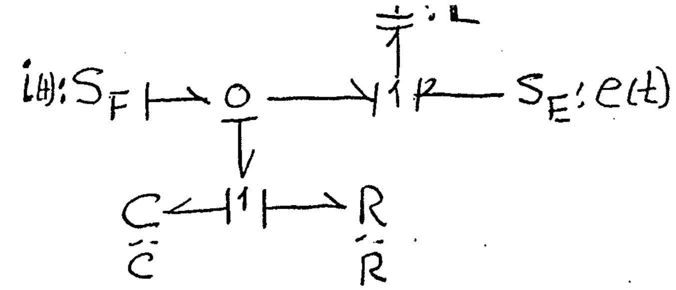
ii. $p_B \equiv 0 = F_B - m\dot V$, $F_B = mj\omega V$ ✓

iii. $\cancel{j\omega}\, p_B = \frac{M}{m+M}(m \cancel{j\omega} V)$

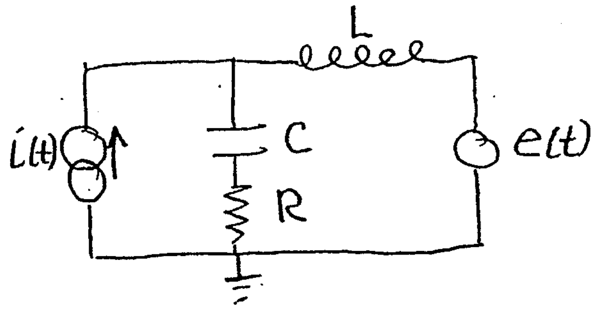
$V_B = \frac{1}{M}\cdot\frac{M m}{(m+M)} V$ ✓

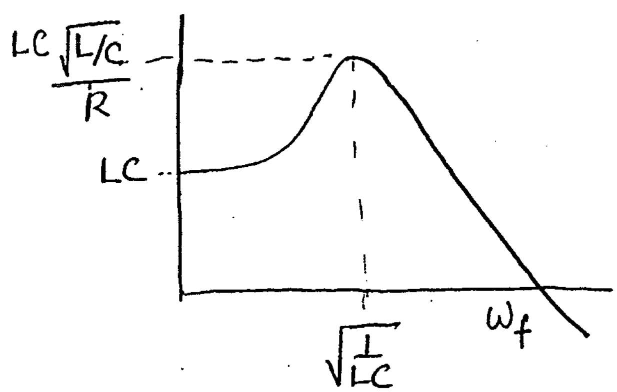
iv. Son of a gun!

6.15

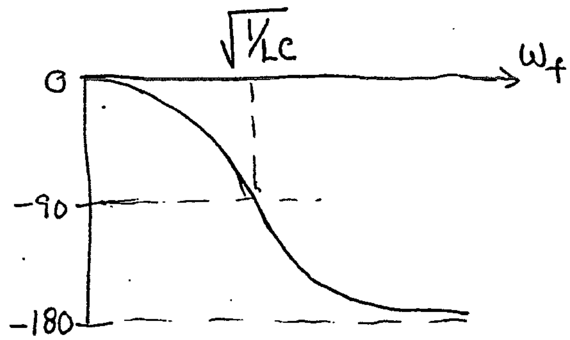


circuit:





mag.



Phase

magnitude and phase of $\dfrac{1}{\frac{1}{LC} - \omega_f^2 + j\,\frac{R}{L}\,\omega_f}$

6.16 (a) $\dfrac{Y}{X}(j\omega) = \dfrac{G\,(\omega_0^2 - \omega^2)}{\omega_n^2 - \omega^2 + j\,2\zeta\omega_n\omega}$

(b) $\left|\dfrac{Y}{X}\right| = \dfrac{G\left[(\omega_0^2 - \omega^2)^2\right]^{1/2}}{\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\right]^{1/2}}$

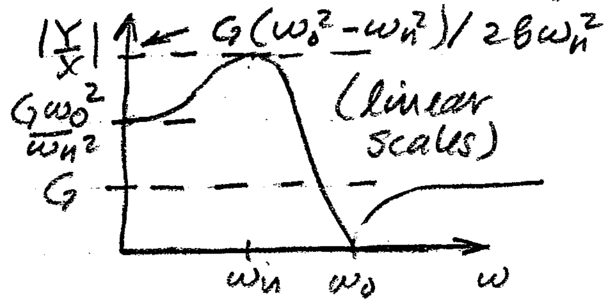
6.16 (Continued)

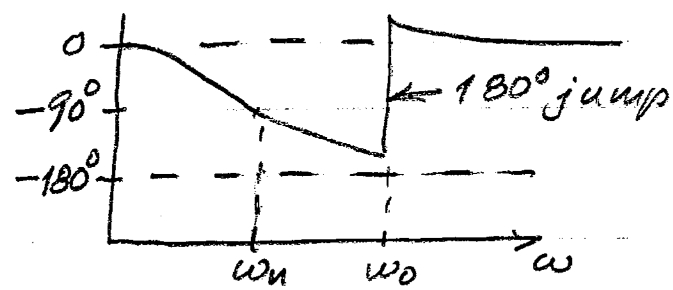
$\omega \to 0 \quad |\frac{Y}{X}| \to \frac{G\omega_o^2}{\omega_n^2}\,; \quad \angle Y/X \to 0$

$\omega = \omega_n, \quad |Y/X| = \frac{G(\omega_o^2-\omega_n^2)}{2\zeta\omega_n}\,; \quad \angle Y/X = -\pi/2, -90^\circ$

$\omega = \omega_o, \quad |Y/X| = 0\,; \quad \angle Y/X$ jumps by π, 180°
for $\omega<\omega_o$ to $\omega>\omega_o$
(sign change for numerator)

$\omega \to \infty, \quad |Y/X| \to G, \quad \angle Y/X \to 0$





6.17 a) $p_f = I_f A_p (v_i - \frac{p_m}{M})$ dealing with derivative causality.

$$\dot{p}_M = A_p\left[\frac{q_a}{C_a} + R_f A_p\left(v_i - \frac{p_M}{M}\right) + \dot{p}_f\right]$$

$$= \frac{A_p}{C_a} q_a + R_f A_p^2 v_i - R_f A_p^2 \frac{p_M}{M} + A_p\left[I_f A_p\left(\dot{v}_i - \frac{\dot{p}_M}{M}\right)\right]$$

(1) $$\dot{p}_M = \frac{A_p/C_a}{1+\frac{m_e}{M}} q_a + \frac{R_f A_p^2}{1+\frac{m_e}{M}} v_i - \frac{R_f A_p^2}{1+\frac{m_e}{M}} \frac{p_M}{M} + \frac{m_e}{1+\frac{m_e}{M}} \dot{v}_i$$

(2) $$\dot{q}_a = A_p\left(v_i - \frac{p_M}{M}\right)$$

where $m_e = A_p^2 I_f$

6.17 (continued)

(b) In the s-domain,

$$\begin{bmatrix} s+\dfrac{R_f A_p^2}{1+m_e/m}\dfrac{1}{m} & -\dfrac{A_p/C_a}{1+m_e/m} \\ A_p/m & s \end{bmatrix} \begin{bmatrix} p_m \\ q_a \end{bmatrix} = \begin{bmatrix} \dfrac{R_f A_p^2 + m_e s}{1+m_e/m} \\ A_p \end{bmatrix} v_i$$

$$D = s^2 + \frac{R_f A_p^2/m}{1+m_e/m} s + \frac{A_p^2/mC_a}{1+m_e/m}$$

solve for p_m using Cramer's rule:

$$\frac{p_m}{v_i} = \frac{\dfrac{(R_f A_p^2 + m_e s)s}{1+m_e/m} + \dfrac{A_p^2/C_a}{1+m_e/m}}{D}$$

and

$$\frac{v_m}{v_i} = \frac{\dfrac{m_e/m}{1+m_e/m} s^2 + \dfrac{R_f A_p^2/m}{1+m_e/m} s + \dfrac{A_p^2/mC_a}{1+m_e/m}}{s^2 + \dfrac{R_f A_p^2/m}{1+m_e/m} s + \dfrac{A_p^2/mC_a}{1+m_e/m}}$$

or

$$\boxed{\frac{v_m}{v_i} = \frac{G\left[s^2 + 2\zeta_o\omega_o s + \omega_o^2\right]}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

where

$$G = \frac{m_e/m}{1+m_e/m} \qquad 2\zeta_o\omega_o = \frac{R_f A_p^2/m}{m_e/m} \qquad \omega_o^2 = \frac{A_p^2/mC_a}{m_e/m}$$

$$2\zeta\omega_n = \frac{R_f A_p^2/m}{1+m_e/m} \qquad \omega_n^2 = \frac{A_p^2/mC_a}{1+m_e/m}$$

(C) Comparing to Prob 6.17 we see that the transfer functions are the same except for the damping, ζ_0, in the numerator of (b). This damping will compromise the notch in the frequency response at $\omega = \omega_0$. Still, this is an excellent isolator but only over a narrow frequency range near ω_0. The high frequency isolation is poor.

(d) The high frequency asymptote is,

$$G = \frac{m_e/M}{1 + m_e/M} \quad \text{where } m_e = A_p^2 I_f$$

I_f is the fluid inertia and would calculated from,

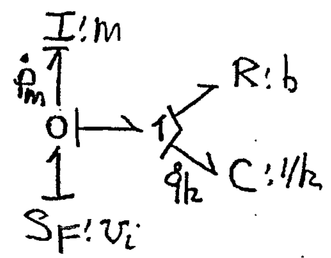
$$I_f = \frac{\rho L}{A_t}$$

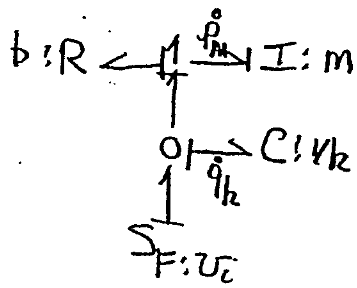
and

$$m_e = A_p^2 \frac{\rho L}{A_t}$$

To improve the high frequency isolation we need to reduce m_e. This could be done by increasing A_t.

6.18





$$\dot p_m = k q_k + b(v_i - p_m/m)$$
$$\dot q_k = v_i - p_m/m$$

$$\dot p_m = -\frac{b}{m} p_m + k q_k$$
$$\dot q_k = v_i - p_m/m$$

s-domain:

$$\begin{bmatrix} s+\frac{b}{m} & -k \\ 1/m & s \end{bmatrix}\begin{bmatrix} p_m \\ q_k \end{bmatrix} = \begin{bmatrix} b \\ 1 \end{bmatrix} v_i \qquad \begin{bmatrix} s+\frac{b}{m} & -k \\ 1/m & s \end{bmatrix}\begin{bmatrix} p_m \\ q_k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_i$$

use Cramer's rule: solve for p_m, then

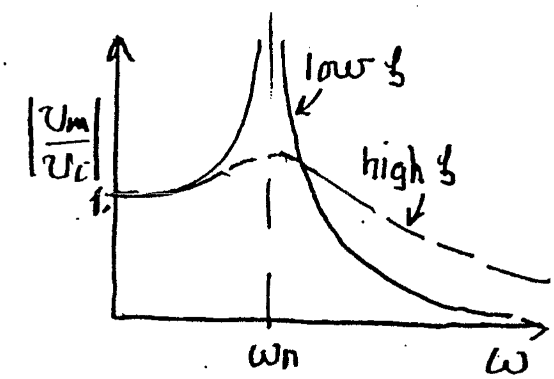
$$\frac{v_m}{v_i} = \frac{\frac{b}{m}s + \frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \qquad \frac{v_m}{v_i} = \frac{k/m}{s^2 + \frac{b}{m}s + k/m}$$

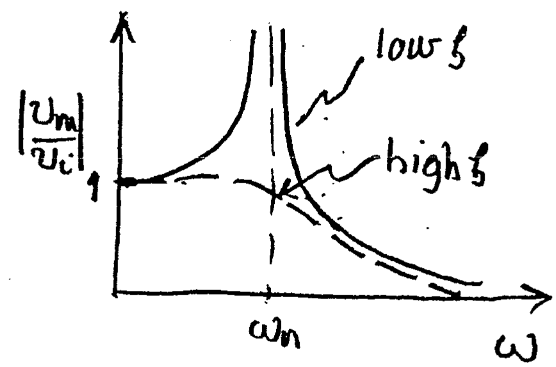
For frequency response

$$\left|\frac{v_m}{v_i}(\omega)\right| = \frac{\left[(\omega_n^2)^2 + (2\zeta\omega_n\omega)^2\right]^{1/2}}{\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\right]^{1/2}} \qquad \left|\frac{v_m}{v_i}\right| = \frac{\omega_n^2}{\left[(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2\right]^{1/2}}$$

$$\omega_n^2 = \frac{k}{m} \qquad 2\zeta\omega_n = \frac{b}{m}$$

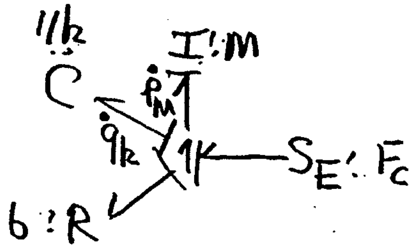
6.18 (continued)





The system with the damper attached to inertial ground has much better high frequency isolation without large response at the resonant frequency, ω_n.

6.19 (a)



$$\dot{p}_M = -k q_k - \frac{b}{M} p_M + F_c$$

$$\dot{q}_k = p_M / M$$

s-domain;

$$\begin{bmatrix} s + \frac{b}{M} & k \\ -1/m & s \end{bmatrix} \begin{bmatrix} p_M \\ q_k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_c$$

use Cramers rule, solve for $q_k = x$

$$\frac{x}{F_c} = \frac{1/M}{s^2 + \frac{b}{m}s + \frac{k}{m}} = G_p(s)$$

(b) $$\frac{x}{x_{des}} = \frac{G_c G_p}{1 + G_c G_p}$$

$$G_c = K_p + K_D s + \frac{K_I}{s} = \frac{K_D s^2 + K_p s + K_I}{s}$$

$$= \frac{(K_D s^2 + K_p s + K_I)\, 1/M}{s\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right) + (K_D s^2 + K_p s + K_I)\, 1/M}$$

$$\frac{X}{X_{des}} = \frac{\frac{K_D}{m}s^2 + \frac{K_P}{m}s + \frac{K_I}{m}}{s^3 + \left(\frac{b}{m} + \frac{K_D}{m}\right)s^2 + \left(\frac{k}{m} + \frac{K_P}{m}\right)s + \frac{K_I}{m}}$$

(c) The eigenvalues are solutions of,

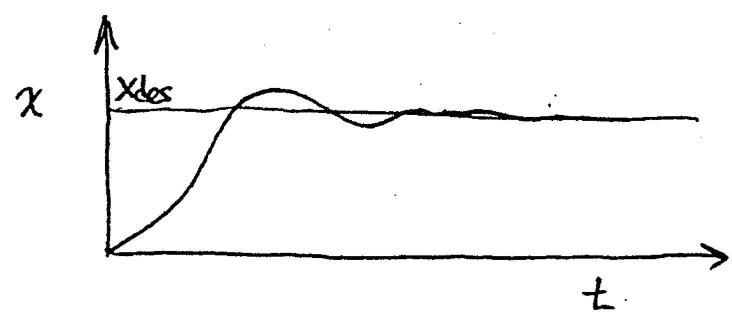
$$s^3 + \left(\frac{b}{m} + \frac{K_D}{m}\right)s^2 + \left(\frac{k}{m} + \frac{K_P}{m}\right)s + \frac{K_I}{m} = 0$$

without control

$$s^2 + \frac{b}{m}s + \frac{k}{m} = 0$$

The control allows independent adjustment of all coefficients of s in the characteristic eqn. Thus we can place the closed loop eigenvalues anywhere we desire.

(d)



Final value theorem shows that,

$$\left.\frac{X}{X_{des}}\right|_{s\to 0} \longrightarrow 1$$

so the controller will move the mass to the proper final location. And with gain adjust we can probably get the response to be acceptable.

Notice that without integral control, $K_I = 0$, the system will not reach x_{des}.

6-20

a. $$\begin{bmatrix} \dot{p}_4 \\ \dot{q}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1/C_5 \\ -1/I_4 & -(1/R_2 + 1/R_6)/C_5 \end{bmatrix} \begin{bmatrix} p_4 \\ q_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/R_2 \end{bmatrix} E_1$$

b.
$$\begin{vmatrix} s & -1/C_5 \\ 1/I_4 & s + (1/R_2 + 1/R_6)/C_5 \end{vmatrix} = s^2 + \frac{1}{C_5}\left(\frac{1}{R_2} + \frac{1}{R_6}\right)s + \frac{1}{I_4 C_5} = s^2 + \frac{1}{C}\left(\frac{1}{R_a} + \frac{1}{R_b}\right)s + \frac{1}{LC} = 0$$

c. $\omega_n = (1/LC)^{1/2}$, $2\varsigma\omega_n = \dfrac{1}{C}\left(\dfrac{1}{R_a} + \dfrac{1}{R_b}\right)$, $\varsigma = \dfrac{1}{2}\left(\dfrac{L}{C}\right)^{1/2}\left(\dfrac{1}{R_a} + \dfrac{1}{R_b}\right)$.

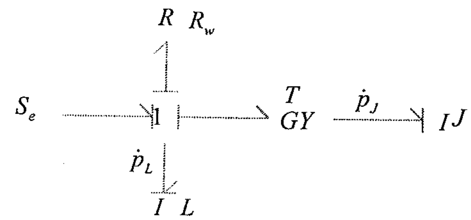
6-21.

a. $$\dot{p}_1 = \frac{-(R_2 + R_3 + R_5)p_1}{I_1} + R_5 F_6(t), \quad \tau = \frac{I_1}{(R_2 + R_3 + R_5)} = \frac{M}{(B_2 + B_3 + B_5)}$$

b. $$\dot{p}_1 = \frac{-(R_2 + R_3)p_1}{I_1} + E_6(t), \quad \tau = \frac{I_1}{(R_2 + R_3)} = \frac{M}{(B_2 + B_3)}$$

Problem 6-22

(a) The bond graph is shown below.
The state variables are the momentum variables p_L and p_J



The state equations are:

$$\frac{d}{dt}\begin{bmatrix} p_L \\ p_J \end{bmatrix} = \begin{bmatrix} -\frac{R_w}{L} & -\frac{T}{J} \\ \frac{T}{L} & 0 \end{bmatrix}\begin{bmatrix} p_L \\ p_J \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_c \qquad \text{(SP6-22-1)}$$

(b) In the s-domain these become,

$$\begin{bmatrix} s+\frac{R_w}{L} & \frac{T}{J} \\ -\frac{T}{L} & s \end{bmatrix}\begin{bmatrix} p_L(s) \\ p_J(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_c(s) \qquad \text{(SP6-22-2)}$$

The characteristic equation is $\det[sI - A] = 0$ or

$$s^2 + \frac{R_w}{L}s + \frac{T^2}{JL} = 0 \qquad \text{(SP6-22-3)}$$

The eigenvalues are,

$$s_{1,2} = -\frac{1}{2}\frac{R_w}{L} \pm \sqrt{\left(\frac{1}{2}\frac{R_w}{L}\right)^2 - \frac{T^2}{JL}} \qquad \text{(SP6-22-4)}$$

If the resistance is set to zero, the undamped eigenvalues are,

$$s_{1,2} = \pm j\sqrt{\frac{T^2}{JL}} = \pm j\omega_n$$

thus the natural frequency is,

$$\omega_n = \sqrt{\frac{T^2}{JL}} \qquad \text{(SP6-22-5)}$$

To obtain the damping ratio, set,

$$\varsigma\omega_n = \frac{1}{2}\frac{R_w}{L}$$

thus,

$$\varsigma = \frac{1}{2}\frac{R_w}{L}\frac{1}{\omega_n} \qquad \text{(SP6-22-6)}$$

(c) Using Cramer's Rule we obtain the transfer function relating the angular momentum $p_J(s)$ to the input voltage $e_c(s)$, thus,

$$\frac{p_J}{e_c}(s) = \frac{\begin{vmatrix} s+\frac{R_w}{L} & 1 \\ -\frac{T}{L} & 0 \end{vmatrix}}{s^2+\frac{R_w}{L}s+\frac{T^2}{JL}} = \frac{\frac{T}{L}}{s^2+\frac{R_w}{L}s+\frac{T^2}{JL}} \qquad \text{(SP6-22-7)}$$

and we obtain the angular velocity ω by dividing the angular momentum by the moment of inertia, thus,

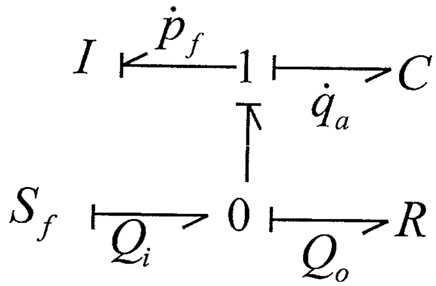
$$\frac{\omega}{e_c}(s) = \frac{\frac{T}{JL}}{s^2+\frac{R_w}{L}s+\frac{T^2}{JL}} \qquad \text{(SP6-22-8)}$$

(d) The transfer function for angular position is simply obtained from the transfer function for angular velocity by dividing by s (i. e. integration in the s-domain)

$$\frac{\theta}{e_c}(s) = \frac{\frac{T}{JL}}{s\left(s^2+\frac{R_w}{L}s+\frac{T^2}{JL}\right)} \qquad \text{(SP6-22-9)}$$

Problem 6.23

The bond graph is given in the problem statement and repeated here.



(a) The state equations come directly from the bond graph by following the causality indicated, thus,

$$\frac{d}{dt}p_f = -\frac{q_a}{C_a} + R_f(Q_i - \frac{p_f}{I_f})$$
$$\frac{d}{dt}q_a = \frac{p_f}{I_f} \qquad \text{(SP6-23-1)}$$

or

$$\frac{d}{dt}\begin{bmatrix} p_f \\ q_a \end{bmatrix} = \begin{bmatrix} -\frac{R_f}{I_f} & -\frac{1}{C_a} \\ \frac{1}{I_f} & 0 \end{bmatrix}\begin{bmatrix} p_f \\ q_a \end{bmatrix} + \begin{bmatrix} R_f \\ 0 \end{bmatrix} Q_i \qquad \text{(SP6-23-2)}$$

(b) In the s-domain,

$$\begin{bmatrix} s + \frac{R_f}{I_f} & \frac{1}{C_a} \\ -\frac{1}{I_f} & s \end{bmatrix}\begin{bmatrix} p_f(s) \\ q_a(s) \end{bmatrix} = \begin{bmatrix} R_f \\ 0 \end{bmatrix} Q_i(s) \qquad \text{(SP6-23-3)}$$

The characteristic equation is the determinant of the left side,

$$s^2 + \frac{R_f}{I_f}s + \frac{1}{I_f C_a} = 0 \qquad \text{(SP6-23-4)}$$

and the eigenvalues are the solutions to the characteristic equation,

$$s_{1,2} = -\frac{1}{2}\frac{R_f}{I_f} \pm \sqrt{(\frac{1}{2}\frac{R_f}{I_f})^2 - \frac{1}{I_f C_a}} \qquad \text{(SP6-23-5)}$$

It is convenient to let

$$\omega_0^2 = \frac{1}{I_f C_a}$$

$$\varsigma\omega_0 = \frac{1}{2}\frac{R_f}{I_f}$$

such that the eigenvalues become,

$$s_{1,2} = -\varsigma\omega_0 \pm \sqrt{(\varsigma\omega_0)^2 - \omega_0^2} \qquad \text{(SP6-23-6)}$$

(c) The output we are after is Q_0 and we can write the output equation for Q_0 by following the causality on the bond graph with the result,

$$Q_0 = Q_i - \frac{p_f}{I_f}$$

Thus,

$$\frac{Q_0}{Q_i} = 1 - \frac{1}{I_f}\frac{p_f}{Q_i} \qquad \text{(SP6-23-7)}$$

We obtain the desired transfer function by using Cramer's rule and deriving first the transfer function for $\frac{p_f}{Q_i}(s)$ and then follow the operations above to obtain the desired transfer function.

$$\frac{p_f}{Q_i}(s) = \frac{R_f s}{s^2 + \frac{R_f}{I_f}s + \frac{1}{I_f C_a}} \qquad \text{(SP6-23-8)}$$

and

$$\frac{Q_o}{Q_i}(s) = 1 - \frac{\frac{R_f}{I_f}s}{s^2 + \frac{R_f}{I_f}s + \frac{1}{I_f C_a}} = \frac{s^2 + \frac{1}{I_f C_a}}{s^2 + \frac{R_f}{I_f}s + \frac{1}{I_f C_a}} \qquad \text{(SP6-23-9)}$$

or using the parameter definitions above,

$$\frac{Q_o}{Q_i}(s) = \frac{s^2 + \omega_0^2}{s^2 + 2\varsigma\omega_0 s + \omega_0^2} \qquad \text{(SP6-23-10)}$$

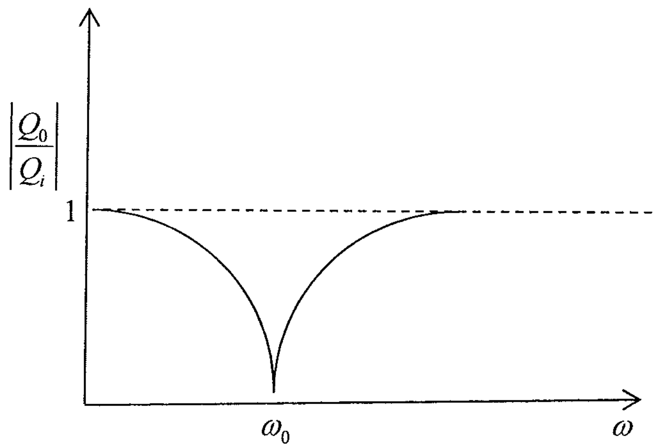
(d) The complex frequency response function comes from letting $s = j\omega$ in the transfer function, thus

$$\frac{Q_o}{Q_i}(j\omega) = \frac{\omega_0^2 - \omega^2}{\omega_0^2 - \omega^2 + j2\varsigma\omega_0\omega} \qquad \text{(SP6-23-11)}$$

and the magnitude of the frequency response is the magnitude of the complex frequency response function, thus

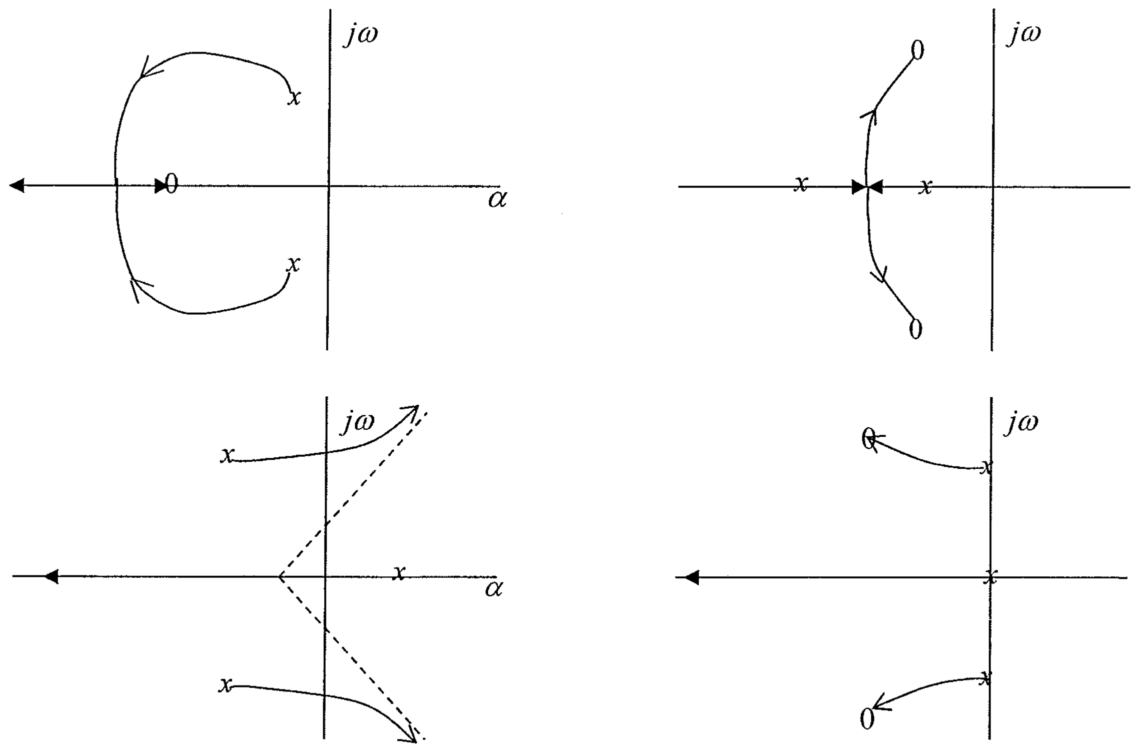
$$\left|\frac{Q_o}{Q_i}\right| = \frac{[(\omega_0^2 - \omega^2)^2]^{1/2}}{[(\omega_0^2 - \omega^2)^2 + (2\varsigma\omega_0\omega)^2]^{1/2}} \qquad \text{(SP6-23-12)}$$

This frequency response is interesting in that at very low input frequencies the response is unity (i. e. set $\omega = 0$ and evaluate) and at very high input frequencies the response is unity(i. e. imagine that the input frequency is much larger than ω_0). However when the input frequency equals ω_0 the response is ZERO! The frequency response would look something like that shown below.

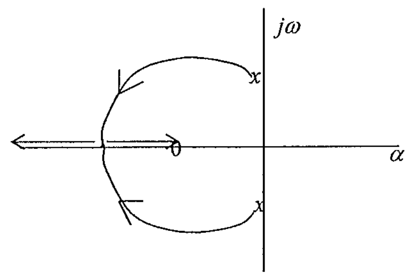


The side branch accumulator shown in the problem statement for Problem 6-24 is very useful in hydraulic systems where the system pressure is generated by a multi-piston positive displacement pump. For such pumps the output flow has a steady state component and an oscillatory component due to the motion of the pistons. The side branch accumulator removes much of the oscillation and the downstream flow is very smooth.

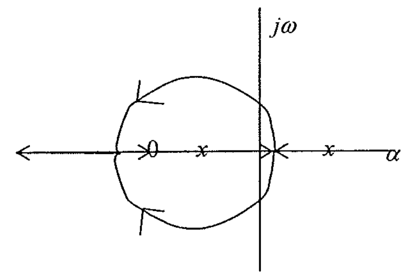
Problem 6-24



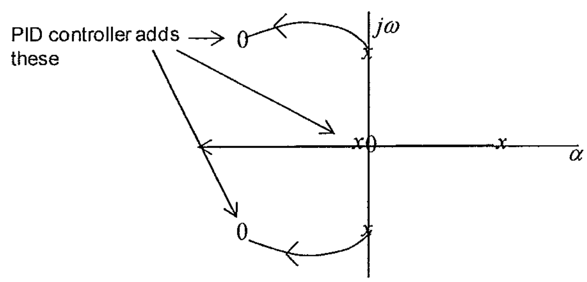
Problem 6.25



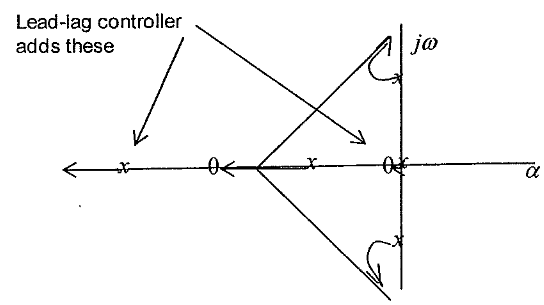
PD control yields stable closed loop behavior and will produce nicely located closed loop eigenvalues



PD control stabilizes the system and potentially produces nicely located closed loop eigenvalues



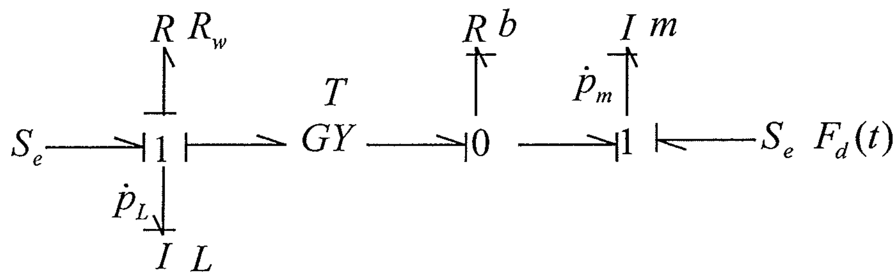
PID adds the 2 arbitrary zeros and the pole at the origin. This pole effectively cancels the system zero and yields a stable closed loop system



Lead-lag control pulls the centroid to the left allowing stable closed loop behavior for some range of system gain. The performance needs to be tested to determine if acceptable response can be achieved

Problem 6-26

The bond graph from the problem statement is repeated here with the state variables identified



(a) The state equations come directly from the bond graph as,

$$\frac{d}{dt}p_L = e_c - \frac{R_w}{L}p_L - T[\frac{p_m}{m} + \frac{1}{b}T\frac{p_L}{L}]$$
$$\frac{d}{dt}p_m = F_d + T\frac{p_L}{L} \qquad \text{(SP6-26-1)}$$

In matrix form these become,

$$\frac{d}{dt}\begin{bmatrix} p_L \\ p_m \end{bmatrix} = \begin{bmatrix} -(\frac{R_w}{L} + \frac{T^2}{bL}) & -\frac{T}{m} \\ \frac{T}{L} & 0 \end{bmatrix}\begin{bmatrix} p_L \\ p_m \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F_d \qquad \text{(SP6-26-2)}$$

(b)

In the s -domain we get,

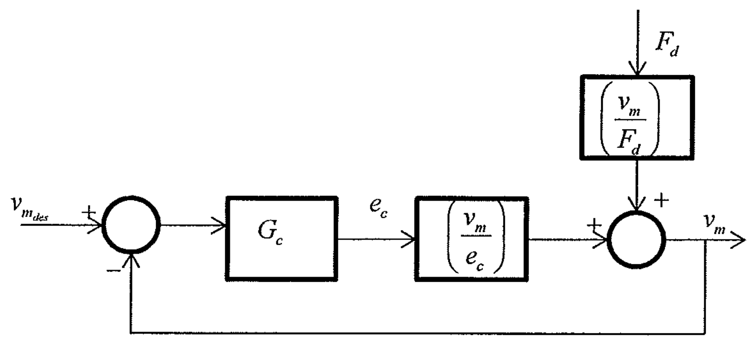
$$\begin{bmatrix} s + (\frac{R_w}{L} + \frac{T^2}{bL}) & \frac{T}{m} \\ -\frac{T}{L} & s \end{bmatrix}\begin{bmatrix} p_L(s) \\ p_m(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F_d \qquad \text{(SP6-26-3)}$$

Use Cramer's Rule to obtain the transfer functions. The easiest way to obtain the velocity transfer functions $\frac{v_m}{e_c}(s)$ and $\frac{v_m}{F_d}(s)$ is to derive the transfer functions for the momentum p_m and divide the result by the mass m . Thus,

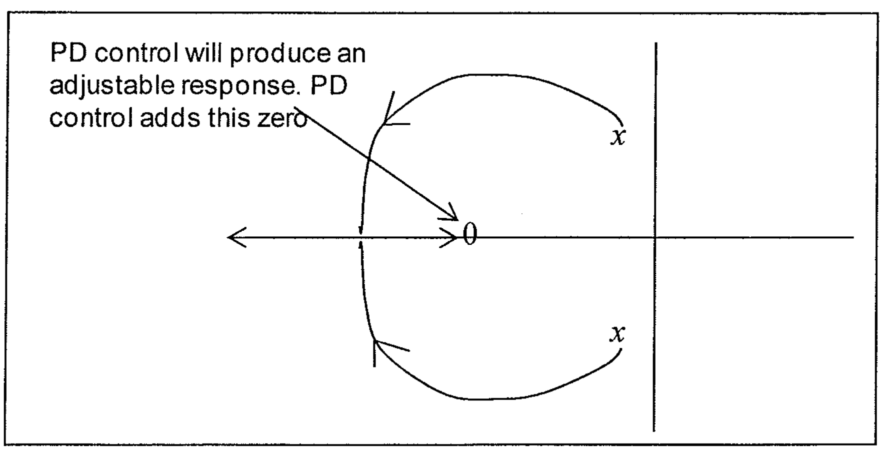
$$\frac{v_m}{e_c}(s) = \frac{\frac{1}{m}Det\begin{bmatrix} s + (\frac{R_w}{L} + \frac{T^2}{bL}) & 1 \\ -\frac{T}{L} & 0 \end{bmatrix}}{s^2 + (\frac{R_w}{L} + \frac{T^2}{bL})s + \frac{T^2}{mL}} = \frac{\frac{T}{mL}}{s^2 + (\frac{R_w}{L} + \frac{T^2}{bL})s + \frac{T^2}{mL}} \qquad \text{(SP6-26-4)}$$

$$\frac{v_m}{F_d}(s)=\frac{\frac{1}{m}Det\begin{bmatrix} s+(\frac{R_w}{L}+\frac{T^2}{bL}) & 0 \\ -\frac{T}{L} & 1 \end{bmatrix}}{s^2+(\frac{R_w}{L}+\frac{T^2}{bL})s+\frac{T^2}{mL}}=\frac{\frac{1}{m}\left[s+(\frac{R_w}{L}+\frac{T^2}{bL})\right]}{s^2+(\frac{R_w}{L}+\frac{T^2}{bL})s+\frac{T^2}{mL}} \qquad \text{(SP6-26-5)}$$

(c)



(d) PD control produces a stable closed loop system with potential for good overall response. The control would be as shown in the figure below.



(e) Representing PD control as $G_c(s)=K(s+z)$, the closed loop response of the velocity to the disturbance force is derived as,

$$\frac{v_m}{F_d}\Big|_{CL}=\frac{(\frac{v_m}{F_d})}{1+G_c(\frac{v_m}{e_c})}=\frac{\frac{1}{m}\left[s+(\frac{R_w}{L}+\frac{T^2}{bL})\right]}{s^2+(\frac{R_w}{L}+\frac{T^2}{bL})s+\frac{T^2}{mL}+K(s+z)\frac{T}{mL}} \qquad \text{(SP6-26-6)}$$

or

$$\frac{v_m}{F_d}\bigg|_{CL} = \frac{\dfrac{1}{m}\left[s + (\dfrac{R_w}{L} + \dfrac{T^2}{bL})\right]}{s^2 + (\dfrac{R_w}{L} + \dfrac{T^2}{bL} + \dfrac{KT}{mL})s + \dfrac{T^2}{mL} + Kz\dfrac{T}{mL}} \qquad \text{(SP6-26-7)}$$

If we let $s = 0$ in this transfer function to test for the final value to a step input, we obtain,

$$\frac{v_m}{F_d}\bigg|_{CL_{s\to 0}} = \frac{\dfrac{1}{m}(\dfrac{R_w}{L} + \dfrac{T^2}{bL})}{\dfrac{T^2}{mL} + Kz\dfrac{T}{mL}} \qquad \text{(SP6-26-8)}$$

In order to keep the response to the disturbance force small, it is necessary to have a large system gain K and/or have the location of the controller zero far out on the negative real axis of the s-plane. This will affect the response of the system to a desired velocity. Thus there will exist a tradeoff between desired response and disturbance rejection.

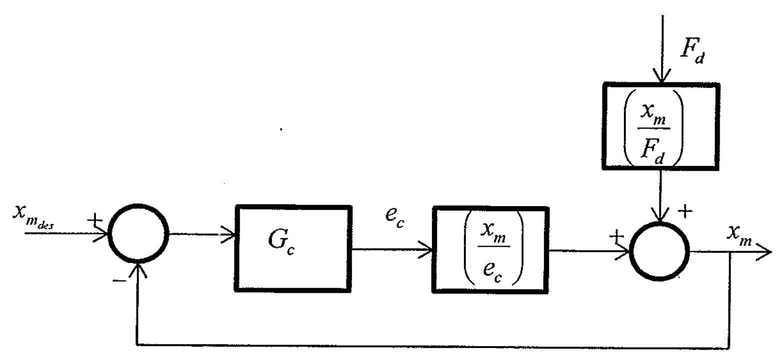
Problem 6.27

The transfer functions from Problem 6-26 are repeated here only for position as output rather than velocity as output.

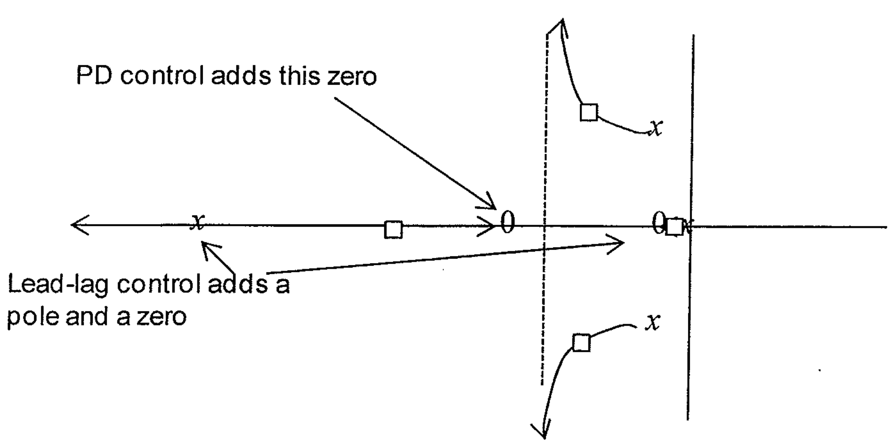
$$\frac{x_m}{e_c}(s) = \frac{\frac{T}{mL}}{s(s^2 + (\frac{R_w}{L} + \frac{T^2}{bL})s + \frac{T^2}{mL})} \qquad \text{(SP6-27-1)}$$

$$\frac{x_m}{F_d}(s) = \frac{\frac{1}{m}\left[s + (\frac{R_w}{L} + \frac{T^2}{bL})\right]}{s(s^2 + (\frac{R_w}{L} + \frac{T^2}{bL})s + \frac{T^2}{mL})} \qquad \text{(SP6-27-2)}$$

(c) The general block diagram for the control system is shown in the figure below.



(d) We now have 3 open loop zeros and PD control alone produces a pole-zero excess of 2 thus creating 2, 90 degree asymptotes that very likely will be in the unstable region of the s-plane. A lead-lag compensator is added to the PD controller and this is shown in the Root Locus below. The lead-lag compensator allows us to "pull" the asymptotes to the left leading to the possibility of a stable, well behaved closed loop system.



(e) Representing PD plus lead-lag control as $G_c(s) = K(s+z)\dfrac{(s+z_1)}{(s+p_1)}$, the closed loop response of the velocity to the disturbance force is derived as,

$$\frac{x_m}{F_d}\Big|_{CL} = \frac{\frac{1}{m}\left[s+(\frac{R_w}{L}+\frac{T^2}{bL})\right](s+p_1)}{s^4+[p_1+\frac{R_w}{L}+\frac{T^2}{bL}]s^3+[\frac{T^2}{mL}+(\frac{R_w}{L}+\frac{T^2}{bL})p_1+K\frac{T}{mL}]s^2+[\frac{T^2}{mL}p_1+K\frac{T}{mL}(z+z_1)]s+K\frac{T}{mL}zz_1}$$

(SP6-27-3)

If we let $s = 0$ in this transfer function to test for the final value to a step input, we obtain,

$$\frac{x_m}{F_d}\Big|_{CL_{s\to 0}} = \frac{\frac{1}{m}(\frac{R_w}{L}+\frac{T^2}{bL})p_1}{K\frac{T}{mL}zz_1} \qquad \text{(SP6-27-4)}$$

This result indicates that it will be a problem having good disturbance rejection because the basic structure of the lead-lag controller is such that $\frac{p_1}{z_1}$ is a large value perhaps 10 or 100. Thus a very large overall gain K is required to keep the response small. This needs to be investigated further.

Problem 6.28

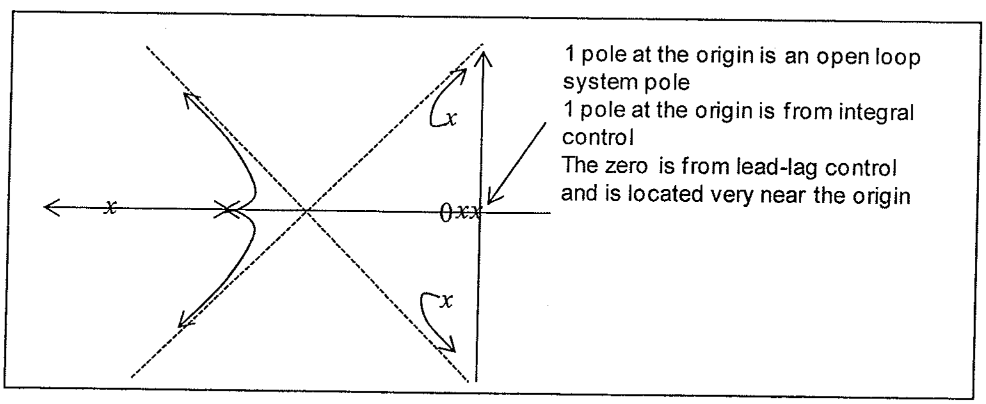
(a) The state equations are put into the s-domain with the result,

$$\begin{bmatrix} s+\frac{b_s}{m_s} & -k & 0 \\ \frac{1}{m_s} & s & 0 \\ -\frac{1}{m_s} & 0 & s \end{bmatrix}\begin{bmatrix} P_s \\ Q_s \\ X \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} V_c + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} F_d \qquad \text{(SP6-28-1)}$$

To derive the transfer functions $\frac{X}{V_c}(s)$ and $\frac{X}{F_d}(s)$, use Cramer's Rule and substitute the appropriate forcing vector into the 3[rd] column of the matrix $[sI-A]$. Take the appropriate determinants with the result,

$$\frac{X}{V_c}(s) = \frac{\frac{k_s}{m_s}}{s(s^2+\frac{b_s}{m_s}s+\frac{k_s}{m_s})}$$
$$\frac{X}{F_d}(s) = \frac{\frac{1}{m_s}s}{s(s^2+\frac{b_s}{m_s}s+\frac{k_s}{m_s})} \qquad \text{(SP6-28-2)}$$

(b) In the Root Locus below the 3 open loop poles are shown along with poles and zeros from a proposed controller. For a load leveling application we desire that the response to a disturbance force return to the original desired position. To insure that this occurs it is proposed to use integral control. In the Root Locus below, the second pole at the origin is from the integral control action, $G_c(s) = \frac{K_i}{s}$. Integral control alone would yield 4 open loop poles with 4 asymptotes at $\pm 45^0$ and $\pm 135^0$. The centroid of such a configuration would be on the real axis somewhere near the 4 poles. Such a control system would be unstable for virtually any value of gain, depending on the amount of passive damping in the system. It is further proposed to add a lead-lag controller which can "pull" the centroid to the left and yield the closed loop system shown in the Root Locus below.



This control action would be,

$$G_c(s) = \frac{K}{s}\frac{(s+z)}{(s+p)}$$

and it is possible to obtain stable closed loop behavior.

(c) Block diagram reduction yields the closed loop transfer function for response to a disturbance as,

$$\frac{X}{F_d}\bigg|_{CL} = \frac{\left(\frac{X}{F_d}\right)}{1+G_c\left(\frac{X}{V_c}\right)} = \frac{\frac{1}{m_s}s^2(s+p)}{s^5+(\frac{b_s}{m_s}+p)s^4+(\frac{b_s}{m_s}p+\frac{k_s}{m_s})s^3+(\frac{k_s}{m_s}p)s^2+K\frac{k_s}{m_s}s+Kz\frac{k_s}{m_s}} \quad \text{(SP6-28-3)}$$

Letting $s \to 0$ shows that the final value of response to a step change in disturbance force is zero indicating that the system returns to the original position.

Problem 6.29

The s-domain representation from problem 6.28 is repeated here for convenience.

$$\begin{bmatrix} s+\dfrac{b_s}{m_s} & -k & 0 \\ \dfrac{1}{m_s} & s & 0 \\ -\dfrac{1}{m_s} & 0 & s \end{bmatrix}\begin{bmatrix} P_s \\ Q_s \\ X \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}V_c+\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}F_d \qquad \text{(SP6-29-1)}$$

This time the control is state variable feedback with the form,

$$V_c = K\left(x_{ref} - [k_{p_s}\ k_{q_s}\ k_x]\begin{bmatrix} P_s \\ Q_s \\ X \end{bmatrix}\right) \qquad \text{(SP6-29-2)}$$

Substituting into the matrix equation,

$$\begin{bmatrix} s+\dfrac{b_s}{m_s} & -k_s & 0 \\ \dfrac{1}{m_s}+Kk_{p_s} & s+Kk_{q_s} & Kk_x \\ -\dfrac{1}{m_s} & 0 & s \end{bmatrix}\begin{bmatrix} P_s \\ Q_s \\ X \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}Kx_{ref}+\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}F_d \qquad \text{(SP6-29-3)}$$

From this representation we use Cramer's rule to derive the closed loop transfer function for $\dfrac{X}{F_d}(s)$,

$$\frac{X}{F_d}(s)=\frac{\dfrac{1}{m_s}(s+Kk_{q_s})}{s^3+(\dfrac{b_s}{m_s}+Kk_{q_s})s^2+(\dfrac{b_s}{m_s}Kk_{q_s}+\dfrac{k_s}{m_s}+k_sKk_{p_s})s+\dfrac{k_s}{m_s}Kk_x} \qquad \text{(SP6-29-4)}$$

As guaranteed with state variable feedback, the closed loop eigenvalues can be arbitrarily placed. There are sufficient gains to affect each coefficient of s in the closed loop denominator.

The final value of the response to a step change in disturbance force is not zero as can be seen by letting $s \to 0$ in the transfer function. The result is,

$$\frac{X}{F_d}\bigg|_{s\to 0}=\frac{Kk_{q_s}}{k_sKk_x} \qquad \text{(SP6-29-5)}$$

It may not be practical to let k_{q_s} be zero due to its effect on the closed loop eigenvalues. So what is done in this situation is add an additional integral control action which will drive the error to zero. One must be careful when doing this to not make the integral control too fast as this will drive the system unstable.

Problem 6.30

The equations of motion from the bond graph and put into matrix format are,

(a)
$$\frac{d}{dt}\begin{bmatrix} p_s \\ p_a \\ q_s \\ q_a \end{bmatrix} = \begin{bmatrix} -\frac{b}{m}-\frac{b_a}{m} & \frac{b_a}{m_a} & k & -k_a \\ \frac{b_a}{m} & -\frac{b_a}{m_a} & 0 & k_a \\ -\frac{1}{m} & 0 & 0 & 0 \\ \frac{1}{m} & -\frac{1}{m_a} & 0 & 0 \end{bmatrix}\begin{bmatrix} p_s \\ p_a \\ q_s \\ q_a \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ 1 \\ 0 \end{bmatrix} v_{in} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} F_c \qquad \text{(SP-30-1)}$$

(b) In the s-domain these become,

$$\begin{bmatrix} s+\frac{b}{m}+\frac{b_a}{m} & -\frac{b_a}{m_a} & -k & k_a \\ -\frac{b_a}{m} & s+\frac{b_a}{m_a} & 0 & -k_a \\ \frac{1}{m} & 0 & s & 0 \\ -\frac{1}{m} & \frac{1}{m_a} & 0 & s \end{bmatrix}\begin{bmatrix} p_s \\ p_a \\ q_s \\ q_a \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 1 \\ 0 \end{bmatrix} v_{in} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} F_c \qquad \text{(SP-30-2)}$$

It is bit tedious to carry out the steps to derive the transfer functions but with a little patience these become,

$$\frac{v_m}{v_{in}}(s) = \frac{(\frac{b}{m}s+\frac{k}{m})(s^2+\frac{b_a}{m_a}s+\frac{k_a}{m_a})}{D(s)}$$

$$\frac{v_m}{F_c}(s) = \frac{-\frac{1}{m}s^3}{D(s)} \qquad \text{(SP-30-3)}$$

where,

$$D(s) = s^4 + [\frac{b}{m}+\frac{b_a}{m_a}+\frac{b_a}{m}]s^3 + [\frac{k_a}{m_a}+\frac{k}{m}+\frac{k_a}{m}+\frac{b}{m}\frac{b_a}{m_a}]s^2$$
$$+[\frac{b}{m}\frac{k_a}{m_a}+\frac{b_a}{m_a}\frac{k}{m}]s+\frac{k}{m}\frac{k_a}{m_a} \qquad \text{(SP-30-4)}$$

(c) The complex frequency response comes from substituting $s = j\omega$ in the appropriate transfer function with the result,

$$\frac{v_m}{v_{in}}(j\omega) = \frac{(\frac{k}{m} + j\frac{b}{m}\omega)(\frac{k_a}{m_a} - \omega^2 + j\frac{b_a}{m_a}\omega)}{\omega^4 - [\frac{k_a}{m_a} + \frac{k}{m} + \frac{k_a}{m} + \frac{b}{m}\frac{b_a}{m_a}]\omega^2 + \frac{k}{m}\frac{k_a}{m_a} + j\omega[(\frac{b}{m}\frac{k_a}{m_a} + \frac{b_a}{m_a}\frac{k}{m}) - (\frac{b}{m} + \frac{b_a}{m_a} + \frac{b_a}{m})\omega^2]}$$

(SP-30-5)

The magnitude response comes from the taking the magnitude of the complex frequency response function with the result,

$$\left|\frac{v_m}{v_{in}}(j\omega)\right| = \frac{[(\frac{k}{m})^2 + (\frac{b}{m}\omega)^2]^{1/2}[(\frac{k_a}{m_a} - \omega^2)^2 + (\frac{b_a}{m_a}\omega)^2]^{1/2}}{[\{\omega^4 - (\frac{k_a}{m_a} + \frac{k}{m} + \frac{k_a}{m} + \frac{b}{m}\frac{b_a}{m_a})\omega^2 + \frac{k}{m}\frac{k_a}{m_a}\}^2 + \omega^2\{\frac{b}{m}\frac{k_a}{m_a} + \frac{b_a}{m_a}\frac{k}{m} - (\frac{b}{m} + \frac{b_a}{m_a} + \frac{b_a}{m})\omega^2\}^2]^{1/2}}$$

(SP-30-6)

Notice that if the damper b_a is very small then the numerator will be near zero at the operating frequency $\omega^2 = \frac{k_a}{m_a}$.

(d) In order to use control to cancel the effect of the actuator damping it is proposed to use,

$$F_c = -Kb_a(\frac{p_s}{m} - \frac{p_a}{m_a}) \qquad \text{(SP-30-7)}$$

The gain K is included to facilitate testing the effect of the control force being too large or too small. If $K = 1$ then the control force perfectly cancels the actuator damping force. If we substitute Eq. (SP-30-7) into (SP-30-1), we can carry out the steps and derive the closed loop transfer function relating the structure mass velocity to the input velocity. It is convenient to write the control force as shown below before substituting into (SP-30-1).

$$F_c = [\ -K\frac{b_a}{m} \quad K\frac{b_a}{m_a} \quad 0 \quad 0]\begin{bmatrix} p_s \\ p_a \\ q_s \\ q_a \end{bmatrix} \qquad \text{(SP-30-8)}$$

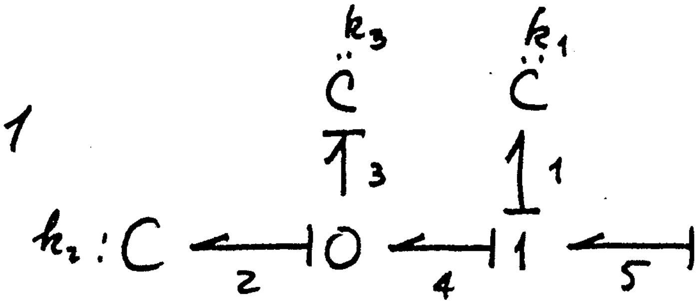
The result of the algebra is the closed loop transfer function,

$$\frac{v_m}{v_{in}}(s) = \frac{(\frac{b}{m}s + \frac{k}{m})(s^2 + \frac{b_a}{m_a}(1-K)s + \frac{k_a}{m_a})}{s^4 + [\frac{b}{m} + \frac{b_a}{m_a}(1-K) + \frac{b_a}{m}(1-K)]s^3 + [\frac{k_a}{m_a} + \frac{k}{m} + \frac{k_a}{m} + \frac{b}{m}\frac{b_a}{m_a}(1-K)]s^2 + [\frac{b}{m}\frac{k_a}{m_a} + \frac{b_a}{m_a}\frac{k}{m}(1-K)]s + \frac{k}{m}\frac{k_a}{m_a}} \quad \text{(SP-30-9)}$$

You can turn this into a complex frequency response and then into a magnitude response if you wish. However, from the transfer function itself we see that if $K = 1$, the actuator damping term in the numerator will vanish and we will obtain a perfect zero response at the operating frequency $\omega^2 = \frac{k_a}{m_a}$.

We also can observe that if K is positive and made too large, then some terms in the denominator could become negative and this would yield unstable closed loop eigenvalues.

7-1



State Equs: $\dot{q}_1 = f_5 \rightarrow \underline{q_1 = q_5} + \cancel{const}^{\,0}$

$\dot{q}_2 = -f_3 + f_4 = -\dot{q}_3 + f_5 \rightarrow q_2 = -q_3 + q_5 + \cancel{const}^{\,0}.$

Deriv. Caus.

$$q_3 = \frac{e_3}{k_3} = \frac{e_2}{k_2} = \frac{k_2 q_2}{k_3}$$

so $$q_2 = -\frac{k_2}{k_3} q_2 + q_5$$

or $$\underline{q_2 = \frac{k_3}{k_2 + k_3} q_5}$$

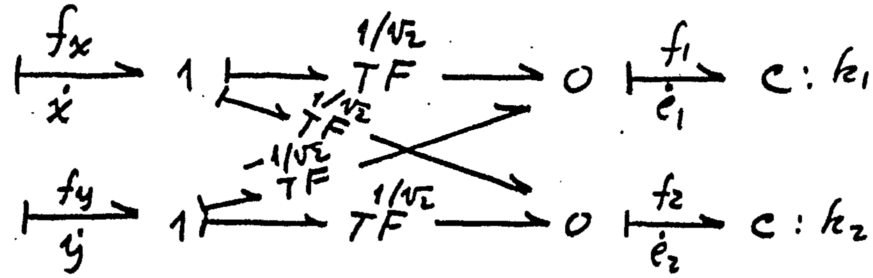
Output Equ: $e_5 = e_1 + e_4 = k_1 q_1 + k_2 q_2$

$$e_5 = k_1 q_5 + \frac{k_2 k_3}{k_2 + k_3} q_5$$

$$e_5 = \underbrace{\left[k_1 + \frac{k_2 k_3}{k_2 + k_3} \right]}_{k_{eq}} q_5$$

7-2 (in this case, f's are forces, e's deflections)

$$e_1 = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \; ; \; e_2 = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \qquad 7\text{-}2$$



$$f_x = \frac{1}{\sqrt{2}}f_1 + \frac{1}{\sqrt{2}}f_2 = \frac{k_1 e_1}{\sqrt{2}} + \frac{k_2 e_2}{\sqrt{2}}$$

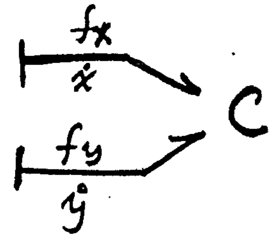
$$= \frac{k_1}{\sqrt{2}}\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right) + \frac{k_2}{\sqrt{2}}\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)$$

$$= \left(\frac{k_1+k_2}{2}\right)x + \left(\frac{k_2-k_1}{2}\right)y$$

$$f_y = -\frac{1}{\sqrt{2}}f_1 + \frac{1}{\sqrt{2}}f_2 = -\frac{k_1 e_1}{\sqrt{2}} + \frac{k_2 e_2}{\sqrt{2}}$$

$$= -\frac{k_1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y\right) + \frac{k_2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)$$

$$= \left(\frac{k_2-k_1}{2}\right)x + \left(\frac{k_1+k_2}{2}\right)y$$



7-3 $\qquad \ell = a + b$, velocity of middle mass $\equiv V_3$

$$V_3 = \frac{b}{\ell} V_1 + \frac{a}{\ell} V_2$$

$$\begin{tikzcd}
 & I:m & & I:2m & & I:m & \\
{} \arrow[r, "F_1", "V_1"'] & 1 \arrow[u, "4"] \arrow[r] & \overset{b/\ell}{TF} \arrow[r] & 0 \arrow[u, "3"] & \overset{a/\ell}{TF} \arrow[l] & 1 \arrow[u, "5"] \arrow[l] & {} \arrow[l, "F_2"', "V_2"]
\end{tikzcd}$$

$$V_1 = p_4/m \quad , \quad V_2 = p_5/m$$

$$\dot p_4 = F_1 - \frac{b}{\ell}\dot p_3 \;;\; \dot p_5 = F_2 - \frac{a}{\ell}\dot p_3$$

but $p_3 = 2m V_3 = 2m\left(\frac{b}{\ell}V_1 + \frac{a}{\ell}V_2\right)$

$$= 2m\left(\frac{b}{\ell}\frac{p_4}{m} + \frac{a}{\ell}\frac{p_5}{m}\right)$$

combining

$$\left(1 + \frac{2b^2}{\ell^2}\right)\dot p_4 + \frac{2ab}{\ell^2}\dot p_5 = F_1$$

$$\frac{2ab}{\ell^2}\dot p_4 + \left(1 + \frac{2a^2}{\ell^2}\right)\dot p_5 = F_2$$

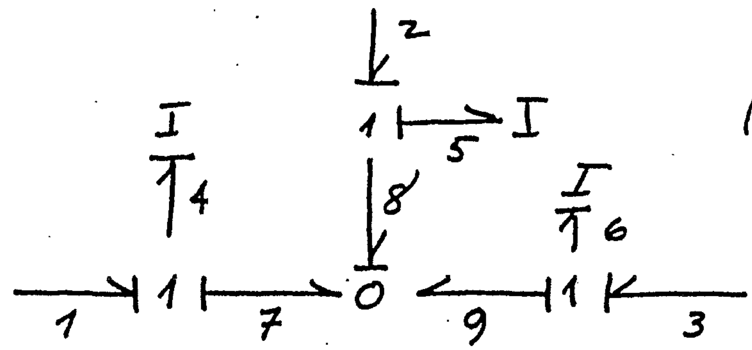
integrate in time and use first eqn's above

$$\left(1 + \frac{2b^2}{\ell^2}\right) m V_1 + \frac{2ab}{\ell^2} m V_2 = P_1$$

$$\frac{2ab}{\ell^2} m V_1 + \left(1 + \frac{2a^2}{\ell^2}\right) m V_2 = P_2$$

I-field in mass matrix form

7-4



$$\frac{\rho l}{A} = I$$

$$\dot{p}_4 = e_1 - e_7 = e_1 - e_8 = e_1 - (e_2 - \dot{p}_5)$$

$$\dot{p}_6 = e_3 - e_9 = e_3 - e_8 = e_3 - (e_2 - \dot{p}_5)$$

but $p_5 = I f_5 = I f_8 = -I(f_7 + f_9) = -I\left(\frac{p_4}{I} + \frac{p_6}{I}\right)$

$$\dot{p}_5 = -\dot{p}_4 - \dot{p}_6$$

Combining and solving, the result is

$$\left.\begin{aligned} \dot{p}_6 &= \tfrac{2}{3}\dot{p}_3 - \tfrac{1}{3}\dot{p}_1 - \tfrac{1}{3}\dot{p}_2 \\ \dot{p}_4 &= \tfrac{2}{3}\dot{p}_1 - \tfrac{1}{3}\dot{p}_2 - \tfrac{1}{3}\dot{p}_3 \end{aligned}\right\} \quad \begin{aligned} p_6 &= \tfrac{2}{3}p_3 - \tfrac{1}{3}p_1 - \tfrac{1}{3}p_2 \\ p_4 &= \tfrac{2}{3}p_1 - \tfrac{1}{3}p_2 - \tfrac{1}{3}p_3 \end{aligned}$$

Finally

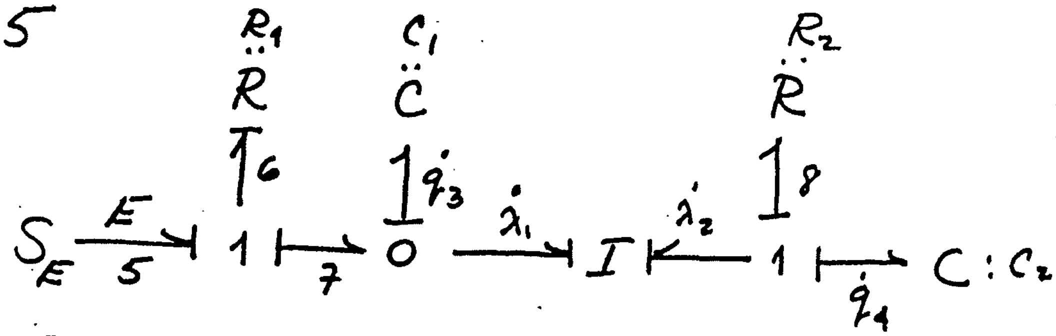
$$f_1 = \frac{p_4}{I} = \frac{1}{I}\left(\frac{2}{3}p_1 - \frac{1}{3}p_2 - \frac{1}{3}p_3\right)$$

$$f_3 = \frac{p_6}{I} = \frac{1}{I}\left(\frac{2}{3}p_3 - \frac{1}{3}p_1 - \frac{1}{3}p_2\right)$$

$$f_2 = -\frac{p_4}{I} - \frac{p_6}{I} = \frac{1}{I}\left(\frac{2}{3}p_2 - \frac{1}{3}p_1 - \frac{1}{3}p_3\right)$$

Final I-field relations in inverse mass matrix form

7-5



7-5

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}L_1 & M_{12}\\ M_{12} & L_2\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}i_1\\ i_2\end{bmatrix} = \frac{1}{L_1L_2 - M_{12}^2}\begin{bmatrix}L_2 & -M_{12}\\ -M_{12} & L_1\end{bmatrix}$$

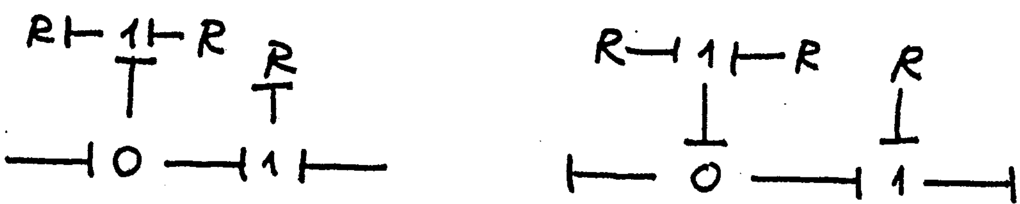
$$\dot{q}_3 = \left(E - \frac{q_3}{C_1}\right)\frac{1}{R_1} - \frac{L_2\lambda_1 - M_{12}\lambda_2}{L_1L_2 - M_{12}^2}$$

$$\dot{\lambda}_1 = q_3/C_3$$

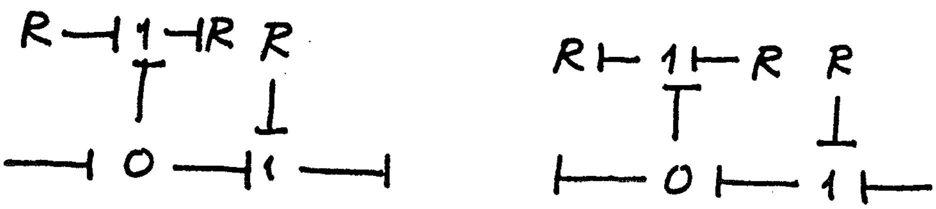
$$\dot{\lambda}_2 = -q_4/C_2 - R_2\left(\frac{-M_{12}\lambda_1 + L_1\lambda_2}{L_1L_2 - M_{12}}\right)$$

$$\dot{q}_4 = \frac{-M_{12}\lambda_1 + L_1\lambda_2}{L_1L_2 - M_{12}^2}$$

7-6

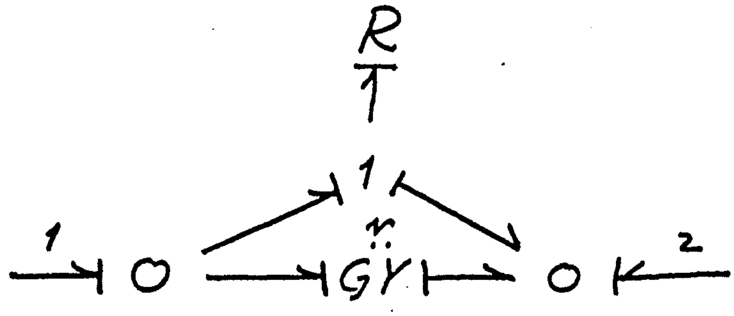


Ousager Forms



Casimir Forms

7-7



$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1/R & 1/r - 1/R \\ -1/R - 1/r & 1/R \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

7-8 $[e] = [R][f]$; Power $= \mathbb{P} = [f]^T[e]$

$\mathbb{P} = [f]^T[R][f]$

but $[e]^T = [f]^T[R]^T$ so another expression for $\mathbb{P}$ is $\mathbb{P} = [e]^T[f] = [f]^T[R]^T[f]$

if R is antisymmetric, $[R]^T = -[R]$

so $\mathbb{P} = -\mathbb{P}$ which means $\mathbb{P} = 0$

7-9

$$e_1 = m_1 e_3 = m_1\, C_3^{-1}(q_3) = m_1\, C_3^{-1}(m_1 q_1 + m_2 q_2) \qquad 7\text{-}7$$

$$e_2 = m_2 e_3 = m_2\, C_3^{-1}(q_3) = m_2\, C_3^{-1}(m_1 q_1 + m_2 q_2)$$

$$\frac{\partial e_1}{\partial q_2} = m_1 \frac{\partial C_3^{-1}}{\partial q_3} \cdot \frac{\partial q_3}{\partial q_2} = m_1 \frac{\partial C_3^{-1}}{\partial q_3} \cdot m_2$$

$$\frac{\partial e_2}{\partial q_1} = m_2 \frac{\partial C_3^{-1}}{\partial q_3} \cdot \frac{\partial q_3}{\partial q_1} = m_2 \frac{\partial C_3^{-1}}{\partial q_3} \cdot m_1 = \frac{\partial e_1}{\partial q_2}$$

---

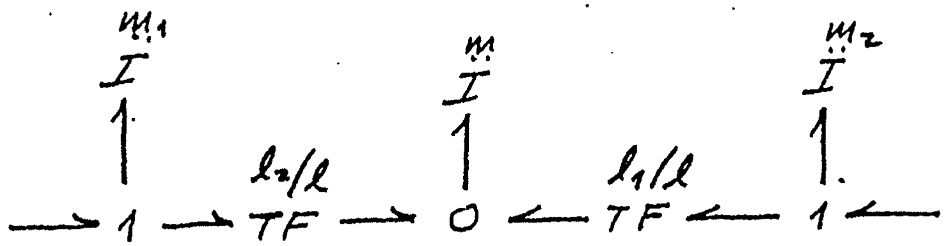
if $q_3 = q_3(q_1, q_2)$, $\dot q_3 = \frac{\partial q_3}{\partial q_1} \cdot \dot q_1 + \frac{\partial q_3}{\partial q_2} \dot q_2$

$$\begin{tikzcd}
 & & C & & \\
 \arrow[r, "e_1"', "\dot q_1"] & \overset{\frac{\partial q_3}{\partial q_1}}{MTF} \arrow[r] & 0 \arrow[u, "e_3", "\dot q_3"'] & \overset{\frac{\partial q_3}{\partial q_2}}{MTF} \arrow[l] & \arrow[l, "e_2"', "\dot q_2"]
\end{tikzcd}$$

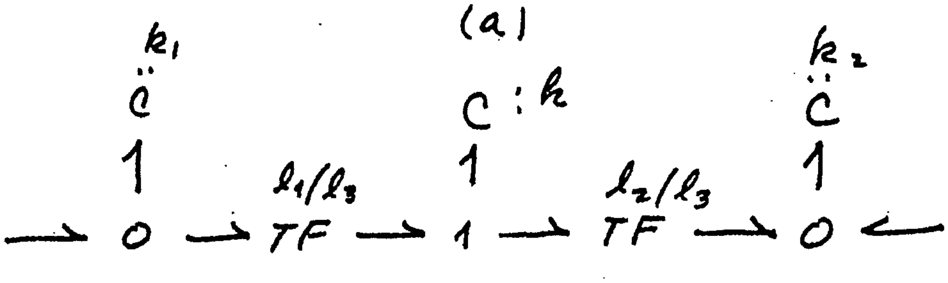
$$e_1 = \frac{\partial q_3}{\partial q_1} \cdot C_3^{-1}(q_3); \quad \frac{\partial e_1}{\partial q_2} = \frac{\partial^2 q_3}{\partial q_2 \partial q_1} \cdot C_3^{-1}(q_3) + \frac{\partial q_3}{\partial q_1} \frac{\partial C_3^{-1}}{\partial q_3} \cdot \frac{\partial q_3}{\partial q_2}$$

$$e_2 = \frac{\partial q_3}{\partial q_2} C_3^{-1}(q_3); \quad \frac{\partial e_2}{\partial q_1} = \frac{\partial^2 q_3}{\partial q_1 \partial q_2} C_3^{-1}(q_3) + \frac{\partial q_3}{\partial q_2} \frac{\partial C_3^{-1}}{\partial q_3} \frac{\partial q_3}{\partial q_1}$$

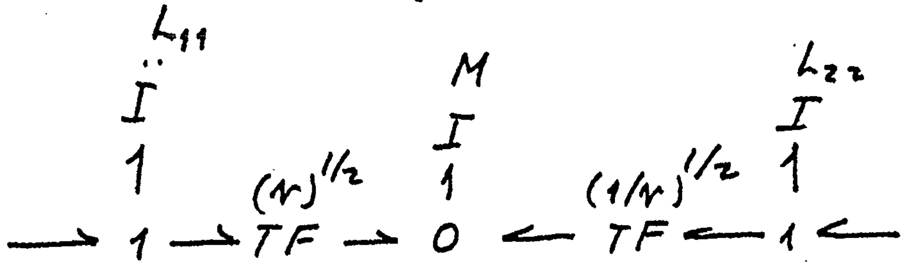
$$\therefore \quad \frac{\partial e_1}{\partial q_2} = \frac{\partial e_2}{\partial q_1}$$



(a)



(b)

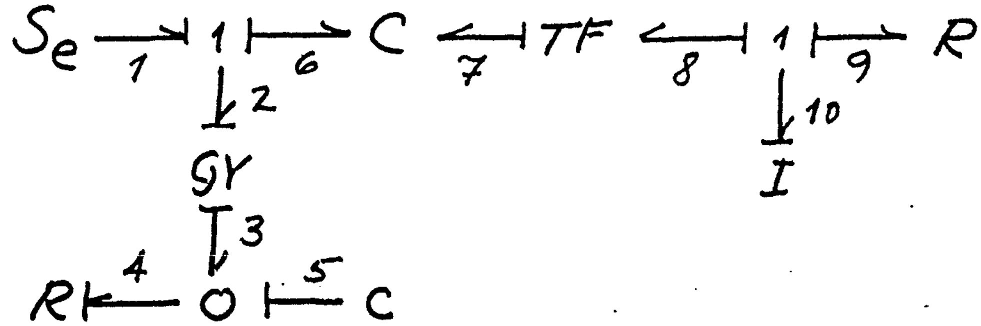


(c)

Note: in part (3) some signs do not correspond to the figure. For part (a) you can show that $v_1 = \underline{-}(l_1/l_2)v_2 + (l^2/ml_2^2)p_2$. For part (b), $(\underline{-}l_2/l_1)x_1 = x_2$. It might be nicer to show V_2 to be positive upwards in both Fig.(a) and Fig.(b) so that in the limiting cases, a simple →TF→ with <u>through</u> power convention results.

7-11

7-9



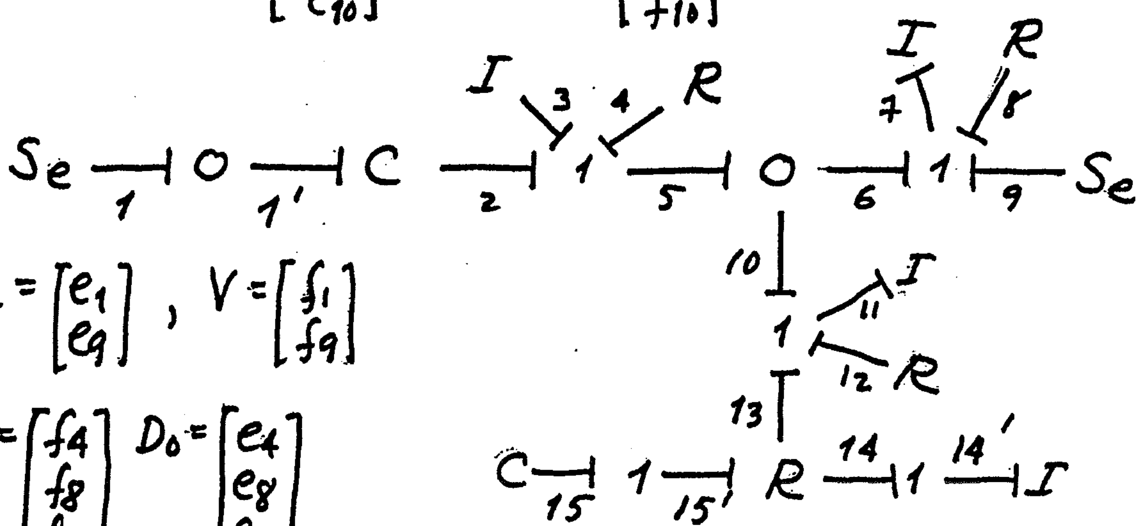
external, 1, 4, 5, 6, 7, 9, 10; internal, 2, 3, 8;

Storage, 5, 6, 7, 10; Source, 1; diss., 4, 9.

$$U = [e_1] \qquad V = [f_1]$$

$$D_i = \begin{bmatrix} e_4 \\ f_9 \end{bmatrix} \qquad D_o = \begin{bmatrix} f_4 \\ e_9 \end{bmatrix}$$

$$\dot{X}_i = \begin{bmatrix} f_5 \\ f_6 \\ f_7 \\ e_{10} \end{bmatrix} \qquad Z_i = \begin{bmatrix} e_5 \\ e_6 \\ e_7 \\ f_{10} \end{bmatrix}$$



$$U = \begin{bmatrix} e_1 \\ e_9 \end{bmatrix}, \quad V = \begin{bmatrix} f_1 \\ f_9 \end{bmatrix}$$

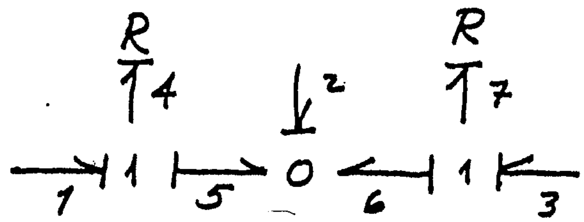
$$D_i = \begin{bmatrix} f_4 \\ f_8 \\ f_{12} \\ e_{15'} \\ f_{14} \end{bmatrix} \quad D_o = \begin{bmatrix} e_4 \\ e_8 \\ e_{12} \\ f_{15'} \\ e_{14} \end{bmatrix} \quad \dot{X}_i = \begin{bmatrix} f_2 \\ e_7 \\ e_{11} \\ f_{15} \\ e_{14'} \end{bmatrix} \qquad \dot{X}_d = \begin{bmatrix} f_1' \\ e_3 \end{bmatrix} \quad Z_d = \begin{bmatrix} e_1' \\ f_3 \end{bmatrix}$$

internal 5, 6, 10.

7-13

$$K = \begin{bmatrix} R_4 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & R_7 \end{bmatrix}$$

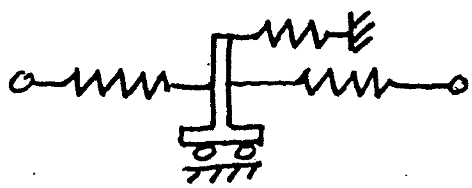
Casimir form



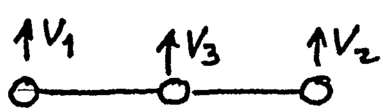
$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1/R_4 & -1/R_4 & 0 \\ -1/R_4 & 1/R_4 + 1/R_7 & -1/R_7 \\ 0 & -1/R_7 & 1/R_7 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

Onsager form

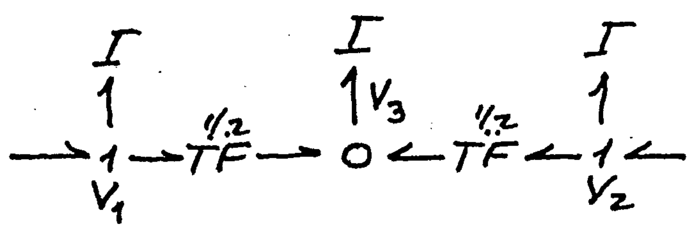
7-14



7-15



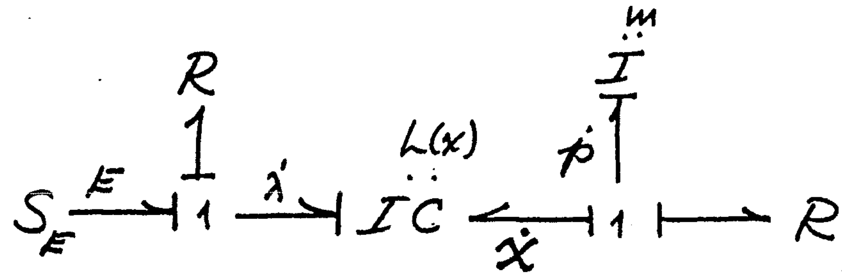
$$V_3 = \frac{V_1 + V_2}{2}$$



8-1 See Problem 5-11.

8-2 See Problem 5-12.

8-3



$$\dot{\lambda} = E(t) - R\lambda/L(x)$$

$$\dot{p} = -\left(-\frac{\lambda^2}{2}\frac{L'(x)}{L^2(x)}\right) - F_0\,\mathrm{sgn}(p/m)$$

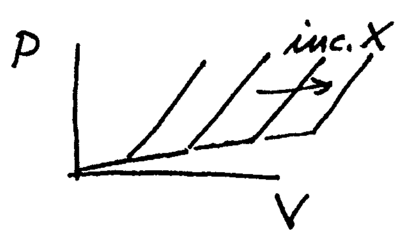
$$\dot{x} = p/m$$

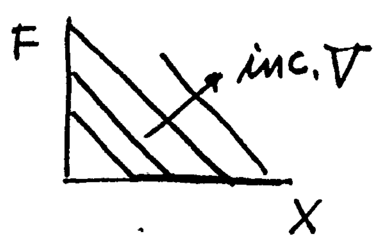
8-4

$$\xrightarrow[\dot{V}]{P} C \xrightarrow[\dot{x}]{F}$$

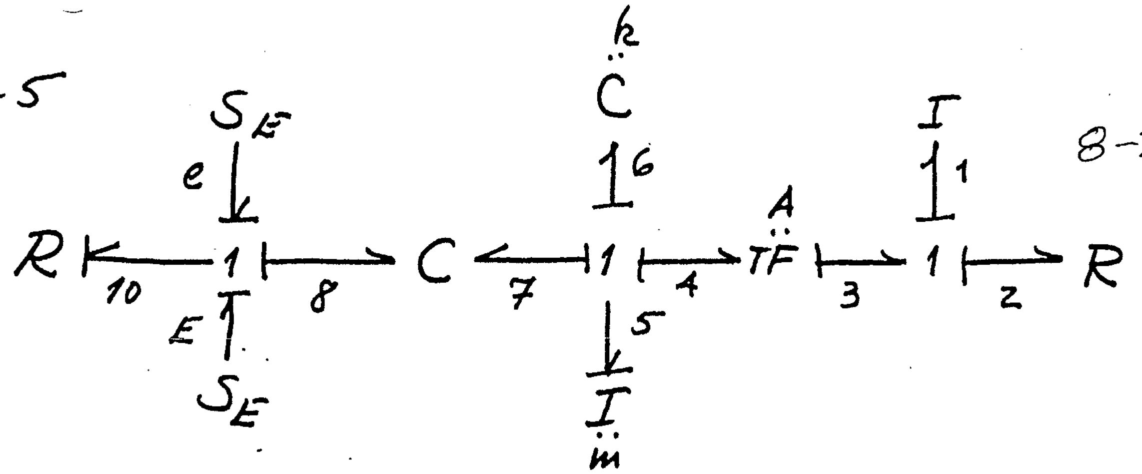
$$\mathcal{U} = \int P\,dV - F\,dx$$

$$P = \frac{\partial \mathcal{U}}{\partial V}, \quad -F = \frac{\partial \mathcal{U}}{\partial x} \quad \text{so} \quad \frac{\partial P}{\partial x} = -\frac{\partial F}{\partial V}$$





8-5



8-2

$$C(q_7) = \frac{\epsilon A}{q_7} \; ; \; \mathcal{U} = \frac{q_8^2 q_7}{2\epsilon A}$$

$$\therefore \; e_8 = \frac{q_8 q_7}{\epsilon A} \; , \quad e_7 = \frac{q_8^2}{2\epsilon A}$$

$$\dot{q}_8 = \frac{1}{R_{10}}\left(E + e(t) - \frac{q_8 q_7}{\epsilon A}\right)$$

$$\left.\begin{aligned} \dot{q}_7 &= \frac{p_5}{m} \\ \dot{q}_6 &= \frac{p_5}{m} \end{aligned}\right\} \; q_7 = q_6 + \text{const}$$

$$\dot{p}_5 = -\frac{q_8^2}{2\epsilon A} - k q_6 - A\left(\dot{p}_1 + R_2\left(A \frac{p_5}{m}\right)\right)$$

but $p_1 = I_1 f_1 = I_1 A \frac{p_5}{m} \rightarrow \dot{p}_1 = \frac{I_1 A}{m} \dot{p}_5$

so $\left(1 + \frac{IA^2}{m}\right)\dot{p}_5 = \frac{-q_8}{2\epsilon A} - k q_6 - \frac{RA^2}{m} p_5$

↑ divide through to get final form.

8-6

$$\dot{\lambda}_1 \xrightarrow[i_1]{} IC \xleftarrow[\dot{\theta}]{\tau}, \quad \dot{\lambda}_2 \Big\downarrow i_2$$

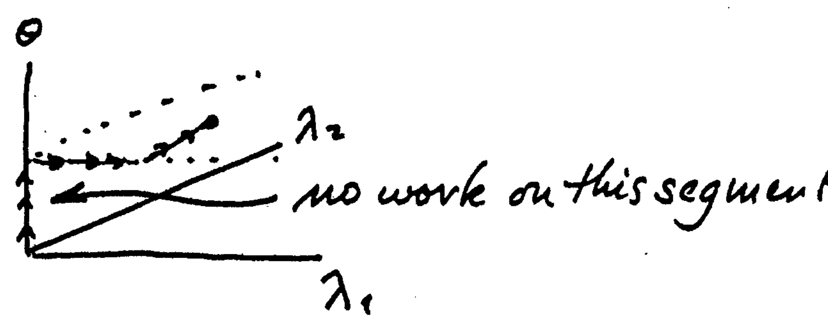
$$[L(\theta)] = \begin{bmatrix} L_1 & L_0\cos\theta \\ L_0\cos\theta & L_2 \end{bmatrix}$$

$$[\Gamma(\theta)] = [L(\theta)]^{-1} = \frac{\begin{bmatrix} L_2 & -L_0\cos\theta \\ -L_0\cos\theta & L_1 \end{bmatrix}}{L_1L_2 - L_0^2\cos^2\theta}$$

so $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = [\Gamma(\theta)]\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

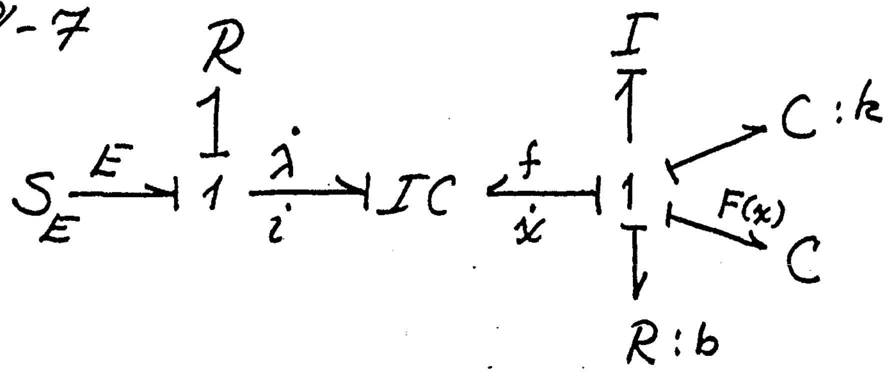
then $W_m(\lambda_1, \lambda_2, \theta) = \frac{1}{2}[\lambda_1\ \lambda_2][\Gamma(\theta)]\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

where W_m is the stored energy evaluated along this special path:



$$\tau = \frac{\partial W_m(\lambda_1, \lambda_2, \theta)}{\partial \theta}$$

8-7



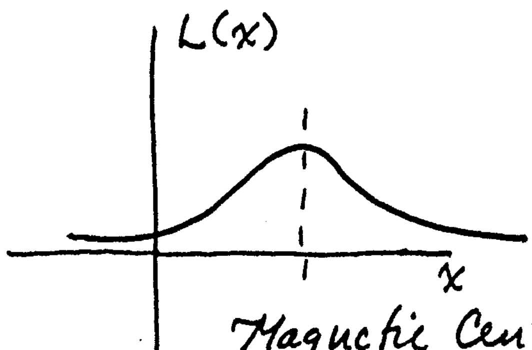
$x = 0$ is no-current rest position

$$\mathcal{U} = \lambda^2 / 2L(x); \quad i = \frac{\lambda}{L(x)}, \; f = -\frac{\lambda^2}{2} \frac{L'(x)}{L^2(x)}$$

$$\dot{\lambda} = E(t) - \frac{R\lambda}{L(x)}$$

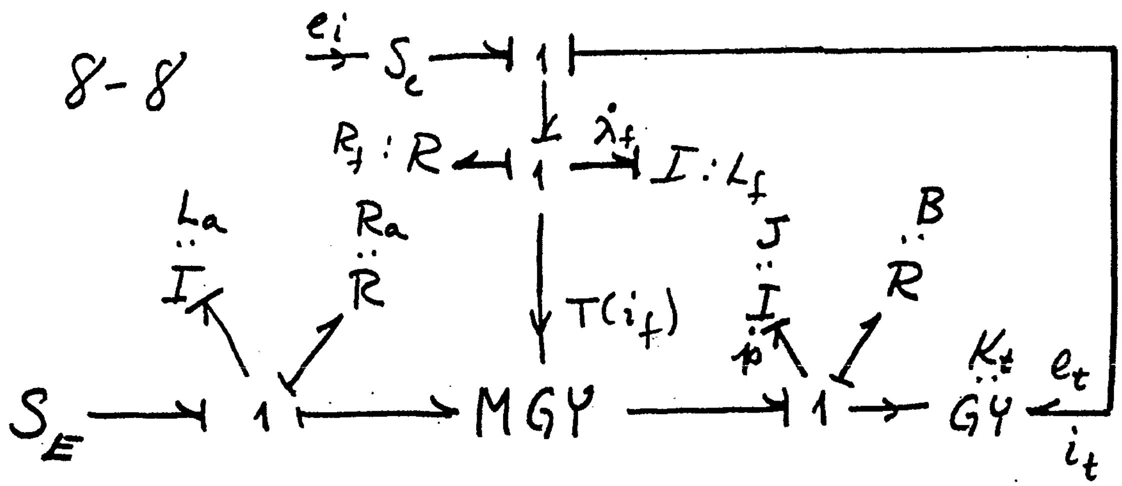
$$\dot{x} = p/m$$

$$\dot{p} = \frac{\lambda^2}{2} \frac{L'(x)}{L^2(x)} - \frac{b\,p}{m} - kx - F(x)$$



Magnetic Center

8-8



$$\dot{\lambda}_a = E - \frac{R_a \lambda_a}{L_a} - T\left(\frac{\lambda_f}{L_f}\right)\cdot \frac{p}{J}$$

$$\dot{\lambda}_f = G e_i - K_t \frac{p}{J} - R_f \frac{\lambda_f}{L_f}$$

$$\dot{p} = T\left(\frac{\lambda_f}{L_f}\right)\cdot \frac{\lambda_a}{L_a} - \frac{B p}{J}$$

---

if $i_a = \text{const}$ and $T(i_f) = k \cdot i_f$, $\tau = k \cdot i_f \cdot i_a$

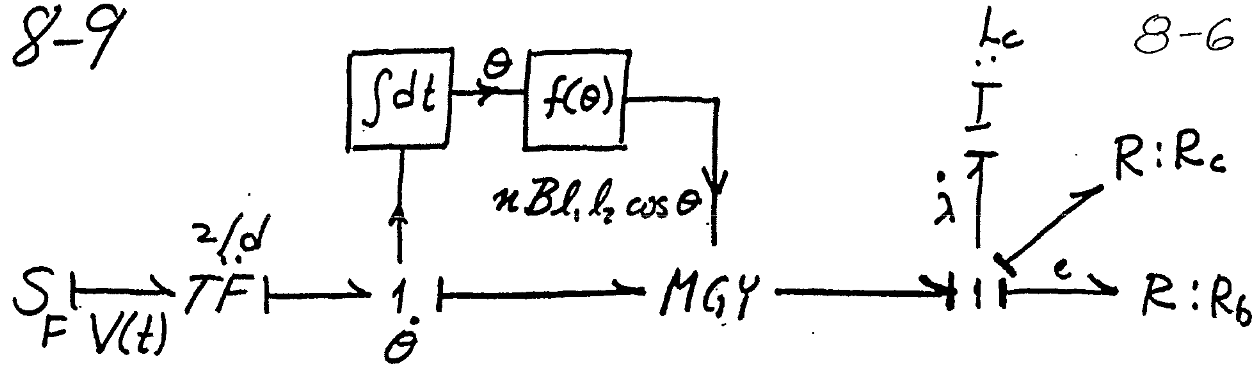
$$\begin{bmatrix} \dot{\lambda}_f \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -R_f/L_f & -K_t/J \\ k i_a/L_f & -B/J \end{bmatrix} \begin{bmatrix} \lambda_f \\ p \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix} e_i$$

$$\begin{bmatrix} s + \frac{R_f}{L_f} & K_t/J \\ -\frac{k i_a}{L_f} & s + \frac{B}{J} \end{bmatrix} \begin{bmatrix} \lambda_f \\ p \end{bmatrix} = \begin{bmatrix} G e_i \\ 0 \end{bmatrix}$$

$$p = \frac{\frac{k i_a}{L_f} G e_i}{s^2 + \left(\frac{R_f}{L_f} + \frac{B}{J}\right)s + \frac{k i_a K_t}{L_f J}} ; \quad \frac{\omega}{e_i} = \frac{\frac{1}{J} \frac{k i_a}{L_f} G}{s^2 + \left(\frac{R_f}{L_f} + \frac{B}{J}\right)s + \frac{k i_a K_t}{L_f J}}$$

8-9

8-6



$$\dot\theta = \frac{2}{d} V(t)$$

$$\dot\lambda = nBl_1l_2\cos\theta\cdot\frac{2}{d}V(t) - \left(\frac{R_c+R_b}{L_c}\right)\lambda$$

$$e = \frac{R_b}{L_c}\lambda$$

---

if $V = \text{const}$, $\theta = \frac{2V}{d}t$

$$\dot\lambda + \frac{R_c+R_b}{L_c}\lambda = \frac{2V}{d}\cdot nBl_1l_2\cos\left(\frac{2V}{d}\right)t$$

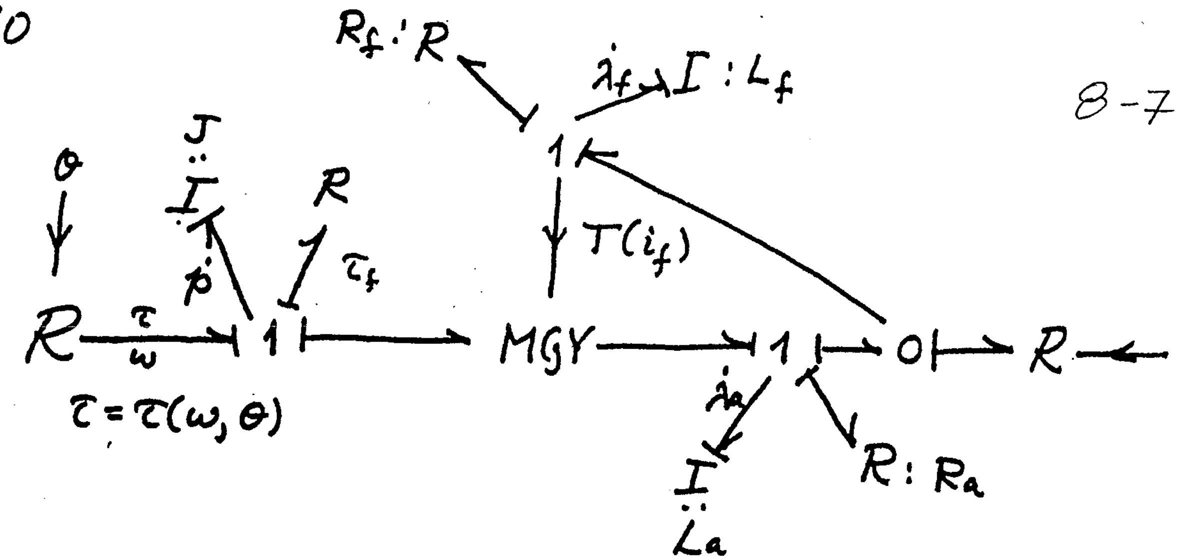
amplitude increases with V

but frequency does also so λ (and e) do not increase so fast with V when

$$\frac{2V}{d} > \frac{R_c+R_b}{L_c}.$$

8-10

8-7

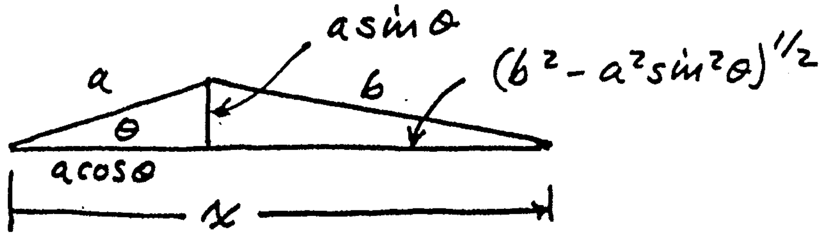


$$\dot{p} = \tau\left(\frac{p}{J}, \theta(t)\right) - \tau_f(p/J) - T\left(\frac{\lambda_f}{L_f}\right)\frac{\lambda_a}{L_a}$$

$$\dot{\lambda}_a = T\left(\frac{\lambda_f}{L_f}\right)\frac{p}{J} - \frac{R_a \lambda_a}{L_a} - R(t)\left(\frac{\lambda_a}{L_a} - \frac{\lambda_f}{L_f}\right)$$

$$\dot{\lambda}_f = R(t)\left(\frac{\lambda_a}{L_a} - \frac{\lambda_f}{L_f}\right) - R_f\frac{\lambda_f}{L_f}$$

8-11



$$x = a\cos\theta + (b^2 - a^2\sin^2\theta)^{1/2}$$

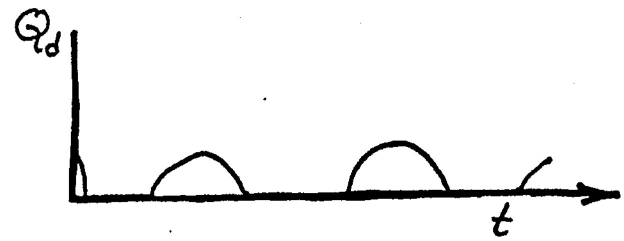
$$\dot{x} = \left[-a\sin\theta + \tfrac{1}{2}(b^2 - a^2\sin^2\theta)^{-1/2}(-a^2\sin\theta\cdot 2\cdot\cos\theta)\right]\dot{\theta}$$

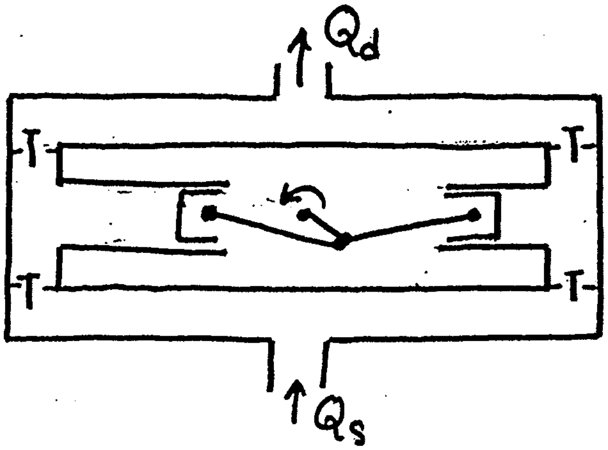
$Q = A\dot{x}$ area

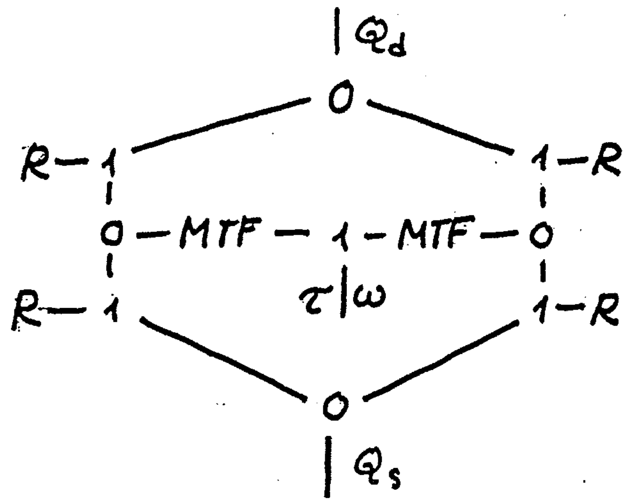
$$\xrightarrow{\dot{\theta}} \overset{f(\theta)}{MTF} \xrightarrow{Q}$$

(a) $f(\theta) = A\left[-a\sin\theta - (b^2 - a^2\sin^2\theta)^{-1/2}(a^2\sin\theta\cos\theta)\right]$

(b)

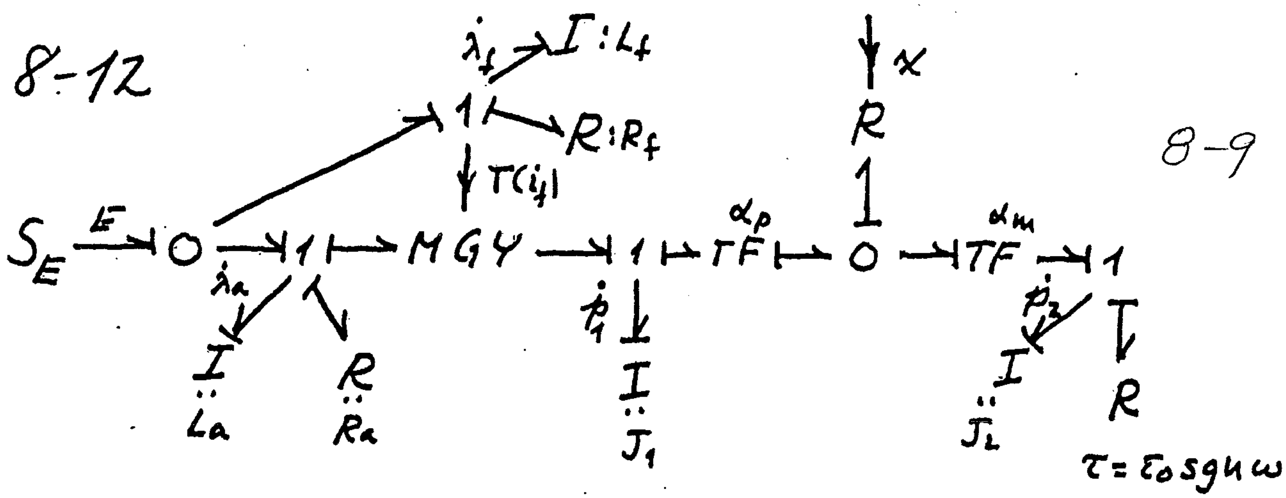






8-12

8-9



$$\dot\lambda_f = E - \frac{R_f}{L_f}\lambda_f$$

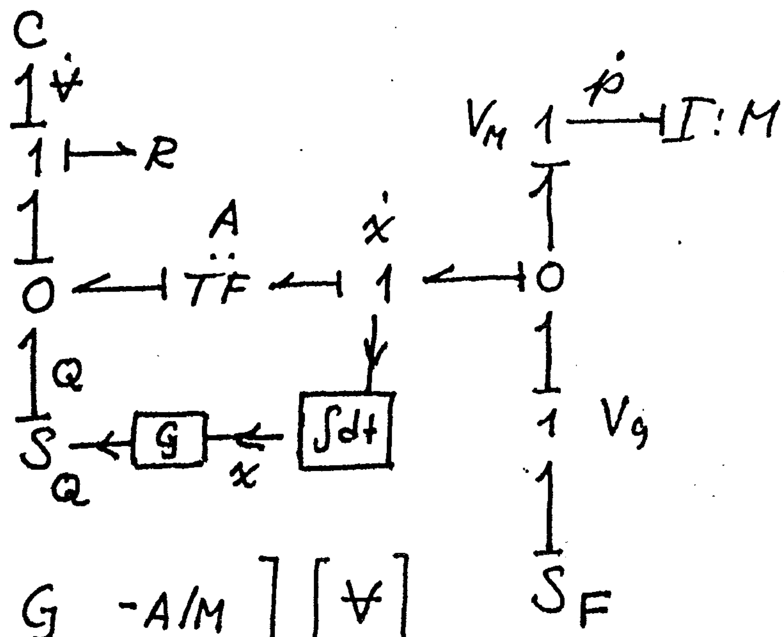
$$\dot\lambda_a = E - \frac{R_a \lambda_a}{L_a} - T\left(\frac{\lambda_f}{L_f}\right)\cdot\frac{p_1}{J_1}$$

$$\dot p_1 = T\left(\frac{\lambda_f}{L_f}\right)\cdot\frac{\lambda_a}{L_a} - \alpha_p A(x)\left(\alpha_p \frac{p_1}{J_1} - \alpha_m \frac{p_2}{J_2}\right)\left|\alpha_p \frac{p_1}{J_1} - \alpha_m \frac{p_2}{J_2}\right|$$

$$\dot p_2 = \alpha_m A(x)\left(\alpha_p \frac{p_1}{J_1} - \alpha_m \frac{p_2}{J_2}\right)\left|\alpha_p \frac{p_1}{J_1} - \alpha_m \frac{p_2}{J_2}\right|$$

$$- \tau_0 \operatorname{sgn} \frac{p_2}{J_2}$$

8-13



8-10

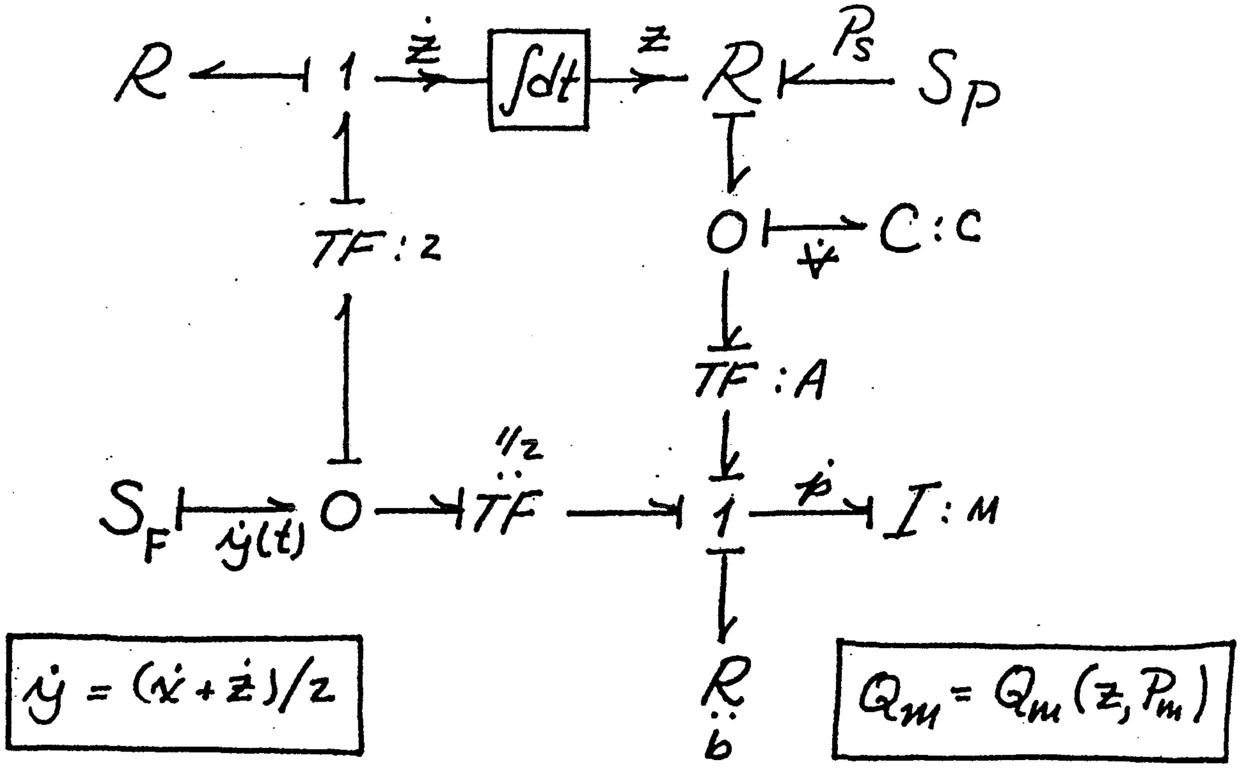
$$\begin{bmatrix} \dot{\forall} \\ \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & G & -A/M \\ 0 & 0 & -1/M \\ A/C & 0 & -RA^2/M \end{bmatrix} \begin{bmatrix} \forall \\ x \\ p \end{bmatrix}$$

$$+ \begin{bmatrix} A \\ 1 \\ RA^2 \end{bmatrix} V_g(t)$$

$$\det \begin{bmatrix} s & -G & A/M \\ 0 & s & 1/M \\ -A/C & 0 & s+R^2A/M \end{bmatrix} = 0$$

(for eigenvalues)

8-14



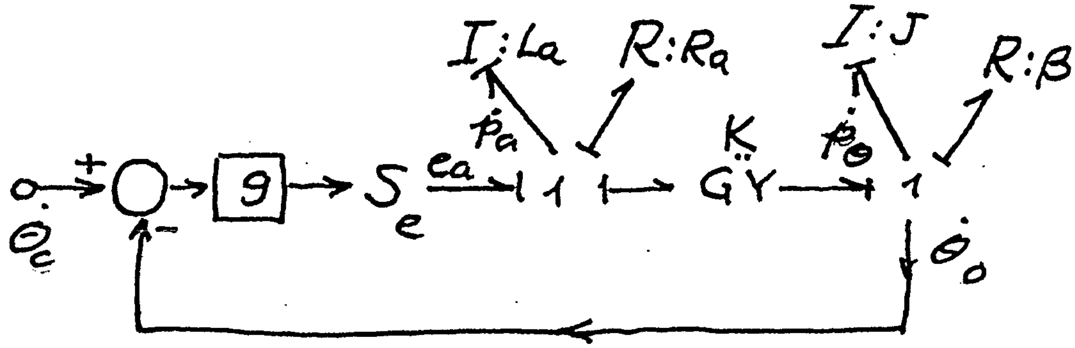
$$\dot{z} = 2\left(\dot{y}(t) - 1/2 \cdot p/m\right)$$

$$\dot{\forall} = Q_m(z, \forall/C) - A\,p/m$$

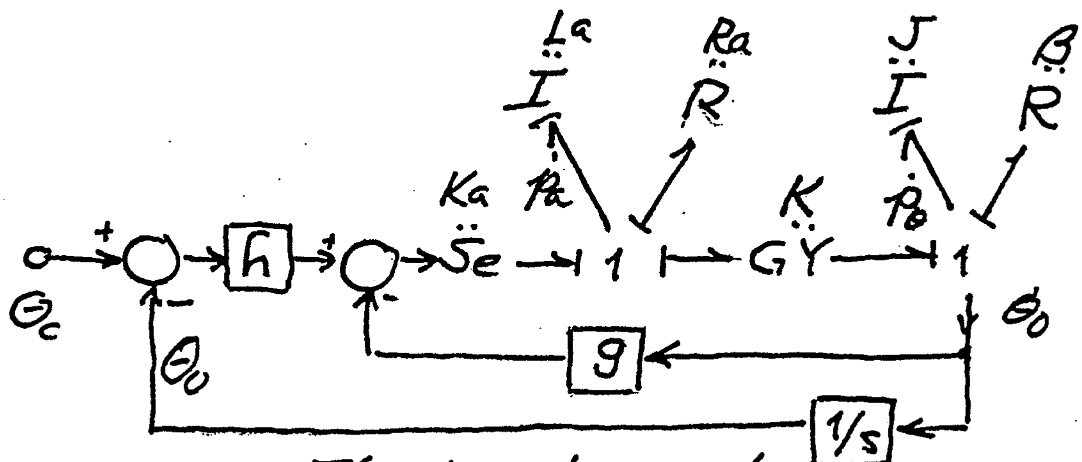
$$\dot{p} = -bp/M + A\forall/C + \tfrac{1}{2}\left(2 F_0 \operatorname{sgn}(2\dot{y}(t) - p/M)\right)$$

$$V_{max} = Q_{max}/A$$

$$F_z = 2F_r = 2F_0 \operatorname{sgn} \dot{z}$$



Second order system



Third order system

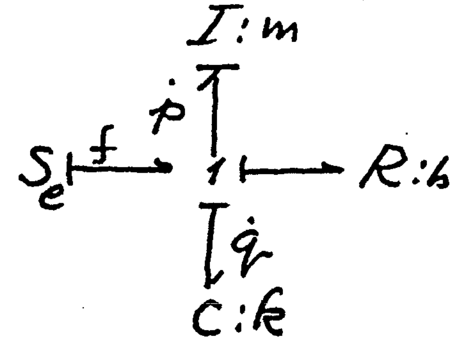
$$\dot{p}_a = K_a[-g p_\theta / J + h(\theta_c - \theta_0)] - R_a[p_a / L_a - K p_\theta / J]$$

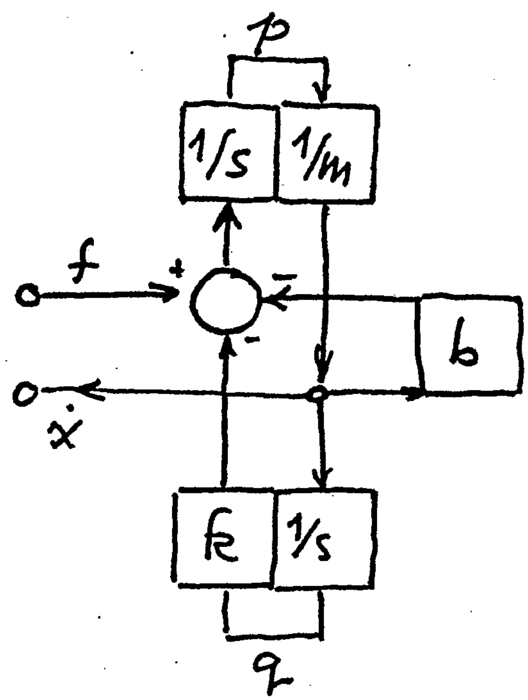
$$\dot{p}_\theta = K p_a / L_a - \beta p_\theta / J$$

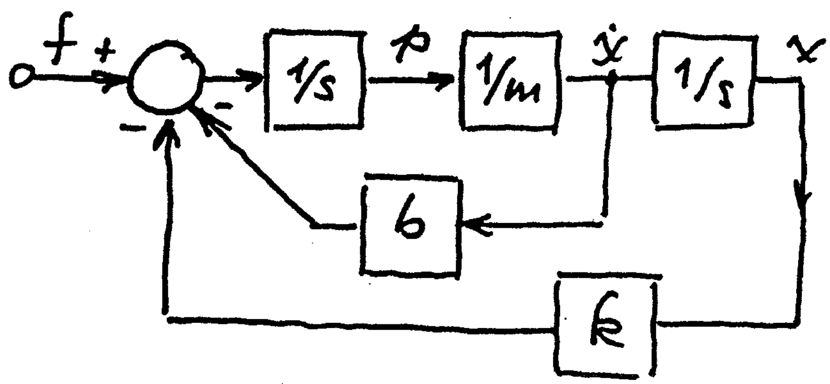
$$\dot{\theta}_0 = p_\theta / J$$

8-16

8-13







If linear operators 1/s and 1/m are interchanged, original diagram results

8-17

$$F(q,X) = \frac{q^2}{2} \cdot \frac{d}{dX}\left(\frac{1}{C}\right) = -\frac{q^2}{2C^2} \cdot \frac{dC}{dX}$$

a) if $e_0 = \frac{q}{C(X)}$, $F = -\frac{e_0^2 C^2(X)}{2C^2(X)} \cdot \frac{dC}{dX}$

$$\boxed{F(e_0, X) = -\frac{e_0^2}{2}\frac{dC}{dX}}$$

b) if $C(X) = \epsilon A/X$, $1/C = \frac{X}{\epsilon A}$

$$\boxed{F(q,X) = +\frac{q^2}{2} \cdot \frac{d}{dX}\left(\frac{X}{\epsilon A}\right) = \frac{q^2}{2\epsilon A}}$$

$$\boxed{F(e_0, X) = -\frac{e_0^2}{2}\frac{d}{dX}\left(\frac{\epsilon A}{X}\right) = +\frac{\epsilon A e_0^2}{2X^2}}$$

8-18

$$\tau(e_0, \theta) = -\frac{e_0^2}{2} \cdot \frac{dC}{d\theta}; \quad C = C_0 + C_1 \cos 2\theta$$

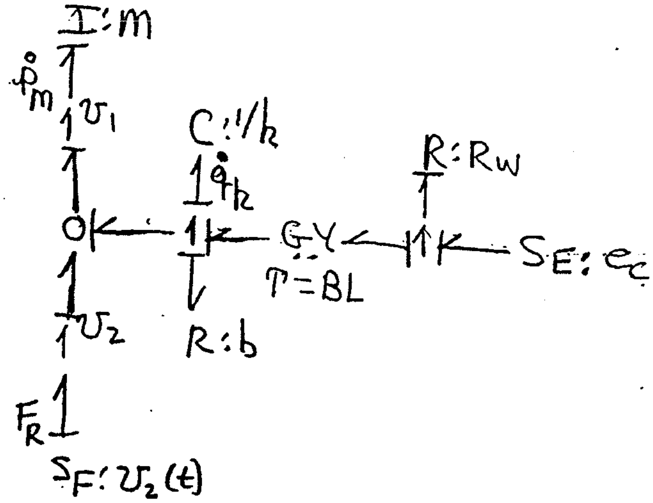
$$\tau(e_0, \theta) = -\frac{e_0^2}{2}\left(-C_1(\sin 2\theta) \cdot 2\right)$$

$$= e_0^2 C_1 \sin 2\theta$$

$$\tau_{max} = (1000)^2 \cdot 10 \times 10^{-12} \cdot 1$$

$$= 10^{-5}\ Nm$$

8.19



$$\dot{p}_M = -kq_k - b\left(\frac{p_M}{M} - v_2(t)\right) + \frac{T}{R_W}\left[e_c - T\left(\frac{p_M}{M} - v_2(t)\right)\right]$$

$$\dot{q}_k = \frac{p_M}{M} - v_2(t)$$

$$F_R = -kq_k - b\left(\frac{p_M}{M} - v_2(t)\right) + \frac{T}{R_W}\left[e_c - T\left(\frac{p_M}{M} - v_2(t)\right)\right]$$

8.20 In the s-domain

$$\begin{bmatrix} s + \frac{b}{m} + \frac{T^2}{R_W m} & k \\ -1/m & s \end{bmatrix} \begin{bmatrix} p_M \\ q_k \end{bmatrix} = \begin{bmatrix} b + \frac{T^2}{R_W} \\ -1 \end{bmatrix} v_2 + \begin{bmatrix} T/R_W \\ 0 \end{bmatrix} e_c$$

$$F_R = -k\,q_k - \left[\frac{b}{m} + \frac{T^3}{R_W m}\right] p_M + \left[b + \frac{T^2}{R_W}\right] v_2 + \frac{T}{R_W} e_c$$

Derive

$$\frac{p_m}{v_2} = \frac{\left(b + \frac{T^2}{R_W}\right)s + k}{D} \qquad \frac{q_k}{v_2} = \frac{-\left(s + \frac{b}{m} + \frac{T^2}{R_W m}\right) + \frac{b}{m} + \frac{T^2}{R_W M}}{D}$$

$$\frac{p_m}{e_c} = \frac{\frac{T}{RW}s}{D} \qquad \frac{q_k}{e_c} = \frac{\frac{T}{R_W m}}{D}$$

$$D = s^2 + \left(\frac{b}{m} + \frac{T^2}{R_W m}\right)s + k/m$$

8.20 (continued)

Get output eqn:

$$\frac{F_R}{v_2}(s) = -k\,\frac{q_k}{v_2}(s) - \left[\frac{b}{m}+\frac{T^2}{R_w m}\right]\frac{p_m}{v_2} + \left[b+\frac{T^2}{R_w}\right]$$

$$= +\frac{k\,s}{D} - \left[\frac{b}{m}+\frac{T^2}{R_w m}\right]\frac{\left[\left(b+\frac{T^2}{R_w}\right)s+k\right]}{D} + \left(b+\frac{T^2}{R_w}\right)$$

$$= \frac{\left[k - \left(\frac{b}{m}+\frac{T^2}{R_w m}\right)\left(b+\frac{T^2}{R_w}\right)\right]s - \left(\frac{b}{m}+\frac{T^2}{R_w m}\right)k + \left(b+\frac{T^2}{R_w}\right)\left(s^2+\left(\frac{b}{m}+\frac{T^2}{R_w m}\right)s+\frac{k}{m}\right)}{D}$$

$$= \frac{\left(k+\frac{T^2}{R_w}\right)s^2 + k\,s}{D}$$

$$\boxed{\frac{F_R}{v_2}(s) = \frac{s\left[\left(b+\frac{T^2}{R_w}\right)s+k\right]}{s^2+\left(\frac{b}{m}+\frac{T^2}{R_w m}\right)s+\frac{k}{m}} = G_{Fv}(s)}$$

and

$$\frac{F_R}{e_c}(s) = -k\,\frac{q_k}{e_c}(s) - \left[\frac{b}{m}+\frac{T^2}{R_w m}\right]\frac{p_m}{e_c} + \frac{T}{R_w}$$

$$= -\frac{k\,T/R_w m}{D} - \left[\frac{b}{m}+\frac{T^2}{R_w m}\right]\frac{\frac{T}{R_w}s}{D} + \frac{T}{R_w}$$

$$= \frac{-\frac{kT}{R_w m} - \left[\frac{b}{m}+\frac{T^2}{R_w m}\right]\frac{T}{R_w}s + \frac{T}{R_w}\left(s^2+\left(\frac{b}{m}+\frac{T^2}{R_w m}\right)s+\frac{k}{m}\right)}{D}$$

$$\boxed{\frac{F_R}{e_c}(s) = \frac{\frac{T}{R_w}s^2}{s^2+\left(\frac{b}{m}+\frac{T^2}{R_w m}\right)s+\frac{k}{m}} = G_{F_c}(s)}$$

8.21

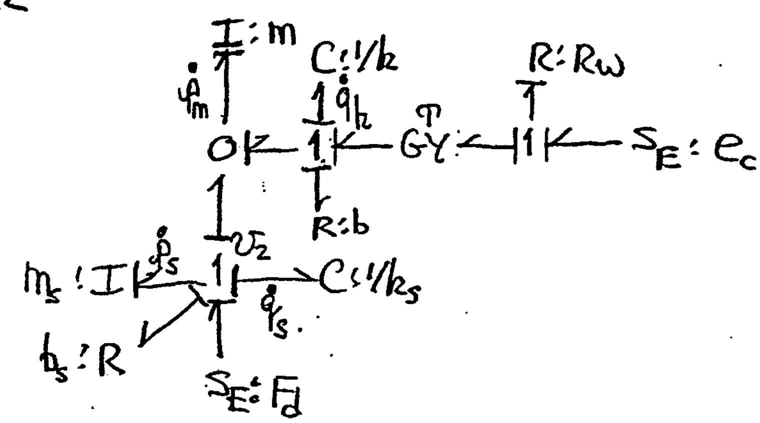
$$F_R = G_{F v} v_z + G_{Fc} e_c = b_c v_z$$

$$e_c = \frac{(b_c - G_{F v})\, v_z}{G_{FC}}$$

$$= \frac{(b_C D - N_{F v})}{N_{FC}} v_z = \frac{\left[b_c\left(s^2 + \left(\frac{b}{m} + \frac{T^2}{R_\omega m}\right)s + \frac{k}{m}\right) - s\left[\left(b + \frac{T^2}{R_\omega}\right)s + k\right]\right] v_z}{\frac{T}{R_\omega} s^2}$$

$$e_c = \left\{ \frac{\left[b_c - \left(b + \frac{T^2}{R_\omega}\right)\right] s^2 + \left[b_c\left(\frac{b}{m} + \frac{T^2}{R_\omega m}\right) - k\right] s + b_c \frac{k}{m}}{\frac{T}{R_\omega} s^2} \right\} v_z$$

8.22



$$\dot p_m = -k q_k - b\left(\frac{p_m}{m} - \frac{p_s}{m_s}\right) + \frac{T}{R_\omega}\left(e_c - T\left(\frac{p_m}{m} - \frac{p_s}{m_s}\right)\right)$$

$$\dot p_s = -\frac{b_s}{m_s} p_s - k_s q_s + F_d - \dot p_M \longleftarrow$$ little trick to save some time.

$$\dot q_k = \frac{p_M}{m} - \frac{p_s}{m_s}$$

$$\dot q_s = \frac{p_s}{m_s}$$

8.22 (continued)

$$\begin{bmatrix} s+\frac{b}{m}+\frac{T^2}{mR_w} & -\left(\frac{b}{m_s}+\frac{T^2}{R_w m_s}\right) & k & 0 \\ s & s+\frac{b_s}{m_s} & 0 & k_s \\ -1/m & 1/m_s & s & 0 \\ 0 & -1/m_s & 0 & s \end{bmatrix} \begin{bmatrix} p_m \\ p_s \\ q_k \\ q_s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} F_d + \begin{bmatrix} T/R_w \\ 0 \\ 0 \\ 0 \end{bmatrix} e_c$$

$$D = \left(s+\frac{b}{m}+\frac{T^2}{mR_w}\right)s\left(s^2+\frac{b_s}{m_s}s+\frac{k_s}{m_s}\right) + \left(\frac{b}{m_s}+\frac{T^2}{R_w m_s}\right)s^3 + k\left\{\frac{s^2}{m_s}+\frac{1}{m}\left(s^2+\frac{b_s}{m_s}s+\frac{k_s}{m_s}\right)\right\}$$

$$D = s^4+\left[\frac{b_s}{m_s}+\frac{b}{m}+\frac{T^2}{R_w m}+\frac{b}{m_s}+\frac{T^2}{R_w m_s}\right]s^3+\left[\frac{k_s}{m_s}+\frac{b_s}{m_s}\left(\frac{b}{m}+\frac{T^2}{R_w m}\right)+\frac{k}{m_s}+\frac{k}{m}\right]s^2$$
$$+\left[\frac{k_s}{m_s}\left(\frac{b}{m}+\frac{T^2}{mR_w}\right)+\frac{k}{m}\frac{b_s}{m_s}\right]s+\frac{k}{m}\frac{k_s}{m_s}$$

solve for p_s using Cramer's rule.

$$\frac{p_s}{F_d}(s) = \frac{s\left(s^2+\left(\frac{b}{m}+\frac{T^2}{mR_w}\right)s+\frac{k}{m}\right)}{D}$$

$$\frac{v_2}{F_d} = \frac{\frac{1}{m_s}\,s\left[s^2+\left(\frac{b}{m}+\frac{T^2}{R_w m}\right)s+\frac{k}{m}\right]}{D} = G_{vd} \text{ open loop TF.}$$

8.22 (continued)

Also need v_2/e_c,

$$\frac{p_s}{e_c}(s) = \frac{-\mathcal{T}/R_w \; s^3}{D}$$

$$\frac{v_2}{e_c}(s) = \frac{-\frac{\mathcal{T}}{R_w m_s} \; s^3}{D} = G_{vc}$$

Then

$$v_2 = G_{vd} F_d + G_{vc} e_c \qquad \text{but } e_c = G_c v_2 \leftarrow \text{Prob 8.21}$$

$$\left.\frac{v_2}{F_d}\right|_{cl} = \frac{G_{vd}}{1 - G_{vc} G_c} = \frac{N_{vd}}{D - N_{vc} G_c}$$

$$= \frac{\frac{1}{m_s} s\left[s^2 + \left(\frac{b}{m} + \frac{\mathcal{T}^2}{R_w m}\right)s + \frac{k}{m}\right]}{D + \frac{\mathcal{T}}{R_w m_s} \cancel{s^3} \left\{\frac{\left[b_c - \left(b + \frac{\mathcal{T}^2}{R_w}\right)\right]s^2 + \left[b_c\left(\frac{b}{m} + \frac{\mathcal{T}^2}{R_w m}\right) - k\right]s + b_c \frac{k}{m}}{\frac{\mathcal{T}}{R_w} \cancel{s^2}}\right\}}$$

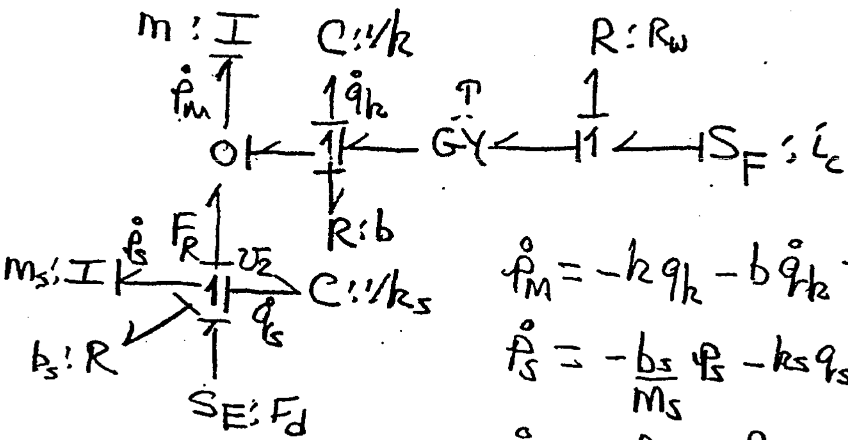
$$\left.\frac{v_2}{F_d}\right|_{c.l.} = \frac{\frac{1}{m_s} s\left[s^2 + \left(\frac{b}{m} + \frac{\mathcal{T}^2}{R_w m}\right)s + \frac{k}{m}\right]}{s^4 + \left[\frac{b_c}{m_s} + \frac{b_s}{m_s} + \frac{b}{m} + \frac{\mathcal{T}^2}{R_w m}\right]s^3 + \left[\left(\frac{b_c}{m_s} + \frac{b_s}{m_s}\right)\left(\frac{b}{m} + \frac{\mathcal{T}^2}{R_w m}\right) + \frac{k_s}{m_s} + \frac{k}{m}\right]s^2 + \left[\left(\frac{b_c}{m_s} + \frac{b_s}{m_s}\right)\frac{k}{m} + \frac{k_s}{m_s}\left(\frac{b}{m} + \frac{\mathcal{T}^2}{m R_w}\right)\right]s + \frac{k}{m}\frac{k_s}{m_s}}$$

closed loop TF.

From the Figure for Prob 8.22, we see that b_s is a damper to ground. From the c.l. transfer function we see that b_c ended up functioning identically to b_s. Thus the control did create a damper to ground.

8.23

complete system:



$$\dot{p}_M = -k q_k - b\dot{q}_k + T i_c$$

$$\dot{p}_s = -\frac{b_s}{m_s} p_s - k_s q_s + F_d - \dot{p}_M$$

$$\dot{q}_k = \frac{p_M}{M} - \frac{p_s}{m_s}$$

$$\dot{q}_s = p_s / m_s$$

system eqns

Actuator eqns:

$$\left.\begin{aligned} \dot{p}_M &= -k q_k - b\dot{q}_k + T i_c \\ \dot{q}_k &= \frac{p_M}{M} - v_2 \end{aligned}\right\} \quad \begin{bmatrix} s & bs+k \\ -1/M & s \end{bmatrix} \begin{bmatrix} p_M \\ q_k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} v_2 + \begin{bmatrix} T \\ 0 \end{bmatrix} i_c$$

$$F_r = \dot{p}_M$$

derive $\dfrac{p_M}{v_2} = \dfrac{bs+k}{s^2 + \frac{b}{M}s + \frac{k}{M}}$

$$\therefore \boxed{\frac{F_r(s)}{v_2} = \frac{s(bs+k)}{s^2 + \frac{b}{M}s + \frac{k}{M}}}$$

derive $\dfrac{p_M}{i_c} = \dfrac{Ts}{s^2 + \frac{b}{M}s + \frac{k}{M}}$

$$\therefore \boxed{\frac{F_r(s)}{i_c} = \frac{Ts^2}{s^2 + \frac{b}{M}s + \frac{k}{M}}}$$

8.23 (continued)

Derive control filter that yields $F_r = b_c v_z$ 8-21

i.e. $F_r = \left(\frac{F_r}{v_z}\right) v_z + \left(\frac{F_r}{i_c}\right) i_c = b_c v_z$

$$\therefore\ i_c = \frac{\left[b_c - \left(\frac{F_r}{v_z}\right)\right]}{(F_r/i_c)} v_z = \frac{b_c\left[s^2+\frac{b}{M}s+\frac{k}{M}\right] - s(bs+k)}{T s^2} v_z$$

$$\boxed{i_c = \frac{[(b_c - b)]s^2 + \left[b_c\frac{b}{M} - k\right]s + b_c\frac{k}{M}}{T s^2} v_z}$$

$G_c(s)$ needed below

Return to system eqns:

$$\begin{bmatrix} s & 0 & bs+k & 0 \\ s & s+b_s/M_s & 0 & k_s \\ -1/M & 1/M_s & s & 0 \\ 0 & -1/M_s & 0 & s \end{bmatrix} \begin{bmatrix} p_M \\ p_s \\ q_k \\ q_s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} F_d + \begin{bmatrix} T \\ 0 \\ 0 \\ 0 \end{bmatrix} i_c$$

$$D = s^2\left(s^2+\frac{b_s}{M_s}s+\frac{k_s}{M_s}\right) + (bs+k)\left\{\frac{s^2}{M_s} + \frac{1}{M}\left(s^2+\frac{b_s}{M_s}s+\frac{k_s}{M_s}\right)\right\}$$

$$D = s^4 + \left[\frac{b_s}{M_s}+\frac{b}{M_s}+\frac{b}{M}\right]s^3 + \left[\frac{k_s}{M_s}+\frac{b}{M}\frac{b_s}{M_s}+\frac{k}{M_s}+\frac{k}{M}\right]s^2 + \left[\frac{b}{M}\frac{k_s}{M_s}+\frac{b_s}{M_s}\frac{k}{M}\right]s + \frac{k}{M}\frac{k_s}{M_s}$$

8.23 (continued)

solve for $\frac{P_s}{F_d}(s)$ using Cramers rule:

$$\frac{P_s(s)}{F_d} = \frac{s\left(s^2 + \frac{b}{M}s + \frac{k}{M}\right)}{D} \quad \therefore \quad \frac{v_2}{F_d} = \frac{\frac{1}{M_s}s\left(s^2 + \frac{b}{M}s + \frac{k}{M}\right)}{D}$$

solve for $\frac{P_s}{i_c}(s)$:

$$\frac{P_s}{i_c} = \frac{-T' s^3}{D} \quad \therefore \quad \frac{v_2}{i_c}(s) = \frac{-\frac{T'}{M_s}s^3}{D}$$

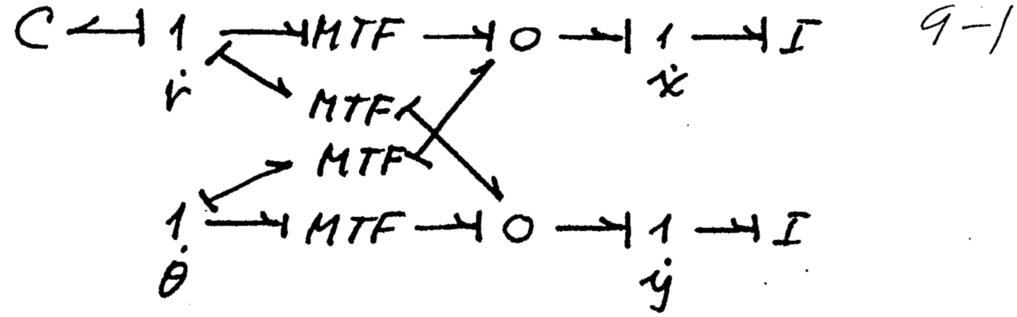
Derive closed loop response:

$$v_2 = \left(\frac{v_2}{F_d}\right)F_d + \left(\frac{v_2}{i_c}\right)i_c \qquad i_c = G_c v_2$$

$$\left.\frac{v_2}{F_d}\right|_{cl} = \frac{(v_2/F_d)}{1 - \left(\frac{v_2}{i_c}\right)G_c} = \frac{\frac{1}{M_s}s\left(s^2 + \frac{b}{M}s + \frac{k}{M}\right)}{D + \frac{\cancel{T'}}{M_s}\cancel{s^3}\left[\frac{(b_c - b)s^2 + \left(b_c\frac{b}{M} - k\right)s + b_c\frac{k}{M}}{\cancel{T'}\cancel{s^2}}\right]}$$

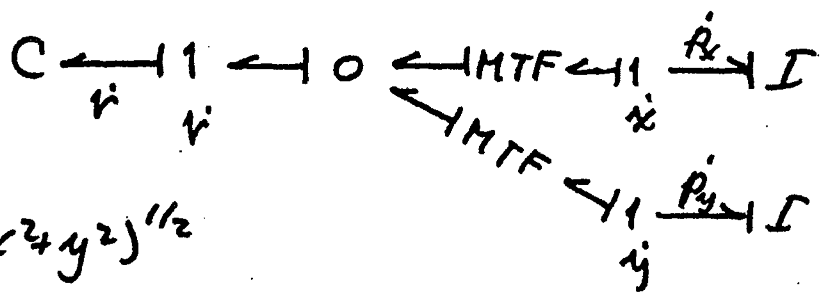
$$\boxed{\left.\frac{v_2}{F_d}\right|_{cl} = \frac{\frac{1}{M_s}s\left(s^2 + \frac{b}{M}s + \frac{k}{M}\right)}{s^4 + \left[\frac{b_c}{M_s} + \frac{b_s}{M_s} + \frac{b}{M}\right]s^3 + \left[\frac{b_c}{M_s}\frac{b}{M} + \frac{k_s}{M_s} + \frac{b}{M}\frac{b_s}{M_s} + \frac{k}{M}\right]s^2 + \left[\frac{b_c}{M_s}\frac{k}{M} + \frac{b_s}{M_s}\frac{k}{M} + \frac{b}{M}\frac{k_s}{M_s}\right]s + \frac{k}{M}\frac{k_s}{M_s}}}$$

9-1



$x = r\cos\theta \rightarrow \dot{x} = (\cos\theta)\dot{r} - (r\sin\theta)\dot{\theta}$

$y = r\sin\theta \rightarrow \dot{y} = (\sin\theta)\dot{r} + (r\cos\theta)\dot{\theta}$



$r = (x^2+y^2)^{1/2}$

$\dot{r} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x\dot{x} + 2y\dot{y})$

$= [(x^2+y^2)^{-1/2}x]\,\dot{x} + [(x^2+y^2)^{-1/2}y]\,\dot{y}$

$\dot{x} = p_x/m, \quad \dot{y} = p_y/m, \quad \dot{r} = [(x^2+y^2)^{1/2}x]\,p_x/m$

$+ [(x^2+y^2)^{1/2}y]\,p_y/m$

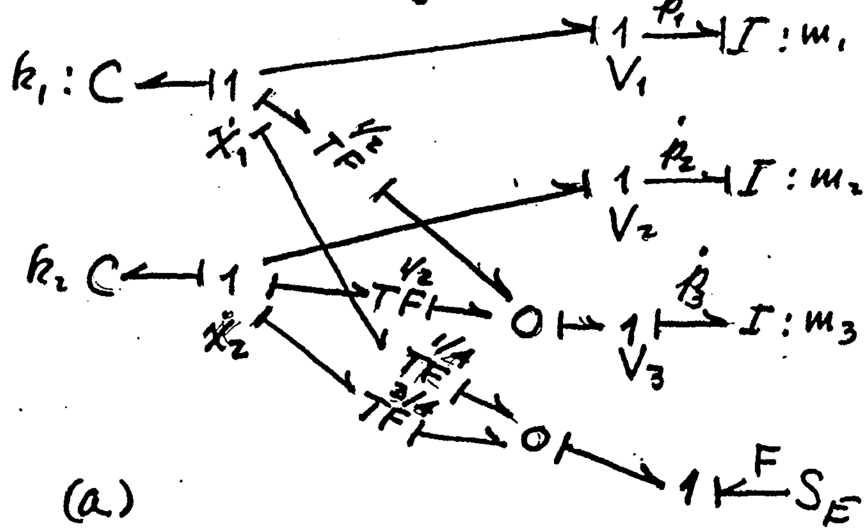
$\dot{p}_x = -[(x^2+y^2)^{-1/2}x]\,k(r-l_0)$

$\dot{p}_y = -[(x^2+y^2)^{-1/2}y]\,k(r-l_0)$

five state variables - this could be reduced to four by using $r = (x^2+y^2)^{1/2}$.

9-2 Interchange m_2 and m_3 in Figure.

9-2



(a)

$\dot{x}_1 = p_1/m_1$

$\dot{x}_2 = p_2/m_2$

$\dot{p}_1 = \frac{F(t)}{4} - \frac{\dot{p}_3}{2} - k_1 x_1$

$\dot{p}_2 = \frac{3}{4}F(t) - \frac{\dot{p}_3}{2} - k_2 x$

diff. caus

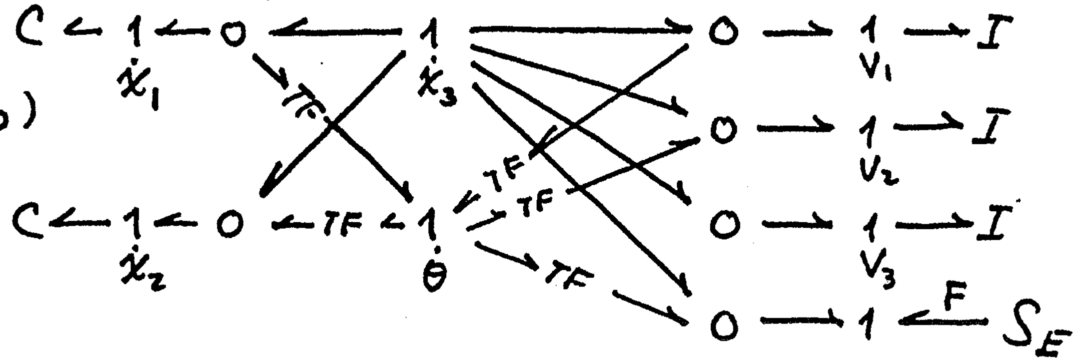
$p_3 = m_3\left(\frac{p_1}{m_1} + \frac{p_2}{m_2}\right)$

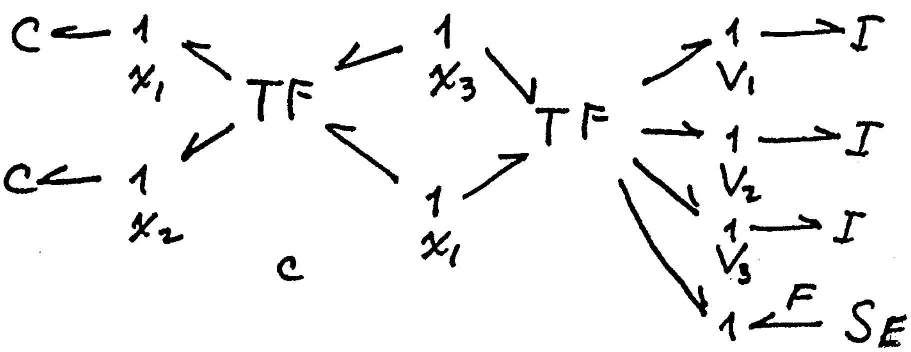
so

$\dot{p}_3 = \frac{m_3}{m_1}\dot{p}_1 + \frac{m_3}{m_2}\dot{p}_2$

Substitute in last two equations.

(b)

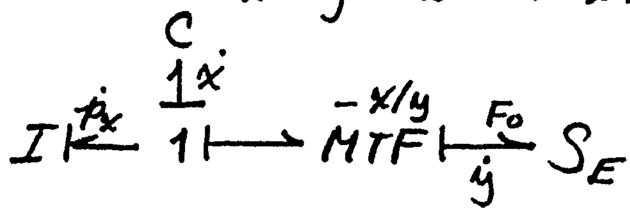




c

9-3

9-3 1. $x^2+y^2=\ell^2 \rightarrow x\dot{x}+y\dot{y}=0$



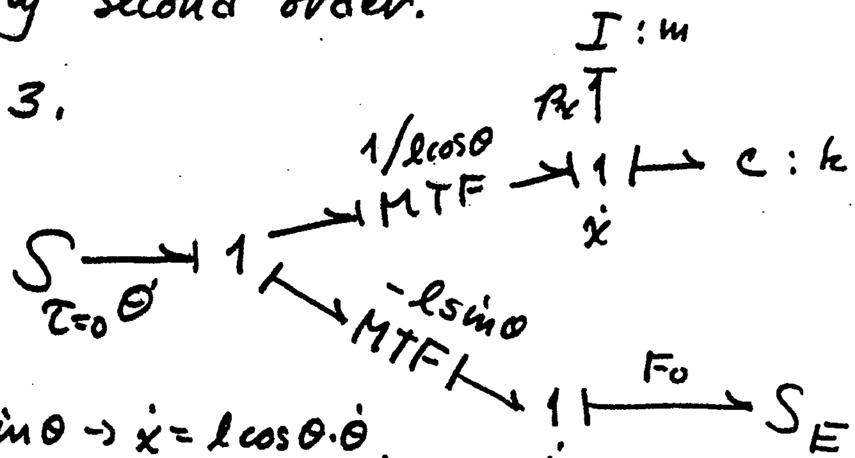
$\dot{x} = p_x/m$

$\dot{p}_x = -kx - \left(-\frac{x}{y}\right)F_0$

$\dot{y} = -\frac{x}{y} \cdot \frac{p_x}{m}$ {or use $x^2+y^2=\ell^2$ to eliminat
{y in equation above

2. Essentially same as above except y has been eliminated. Only second order.

3.



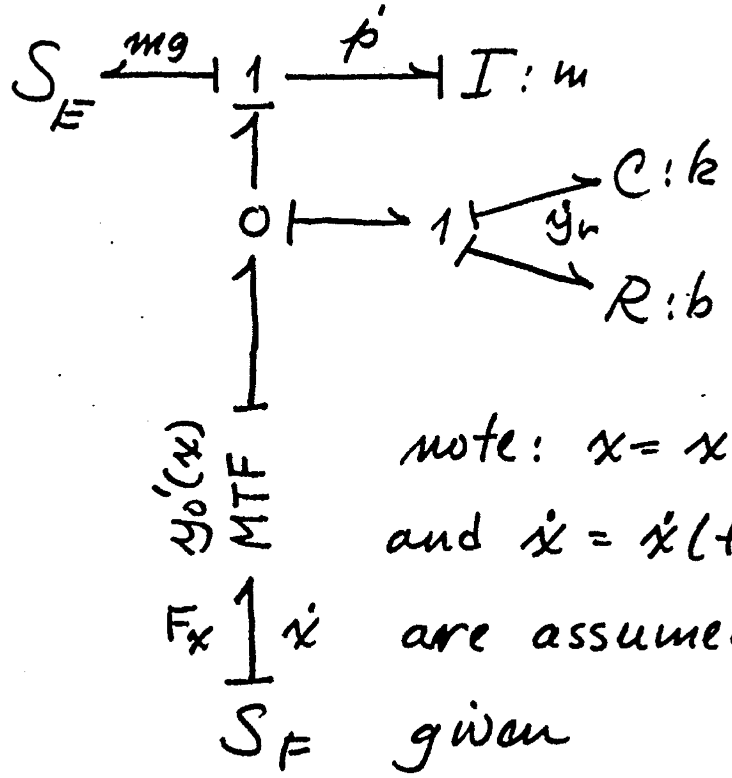
$x = \ell\sin\theta \rightarrow \dot{x} = \ell\cos\theta \cdot \dot{\theta}$

$y = \ell\cos\theta \rightarrow \dot{y} = -\ell\sin\theta \cdot \dot{\theta}$

$\dot{x} = p_x/m \; ; \; \dot{p}_x = \frac{\ell\sin\theta}{\ell\cos\theta} F_0 - kx$

$\dot{\theta} = \frac{1}{\ell\cos\theta} \cdot \frac{p_x}{m}$ or use $\theta = \sin^{-1}\frac{x}{\ell}$

9-4



9-4

note: $x = x(t)$
and $\dot{x} = \dot{x}(t)$
are assumed
given

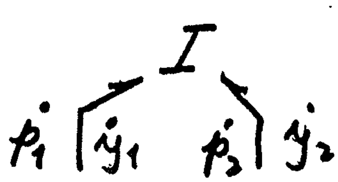
$$\dot{p} = -mg + k y_r + b\left((y_0'(x)) \cdot \dot{x}(t) - p/m \right)$$

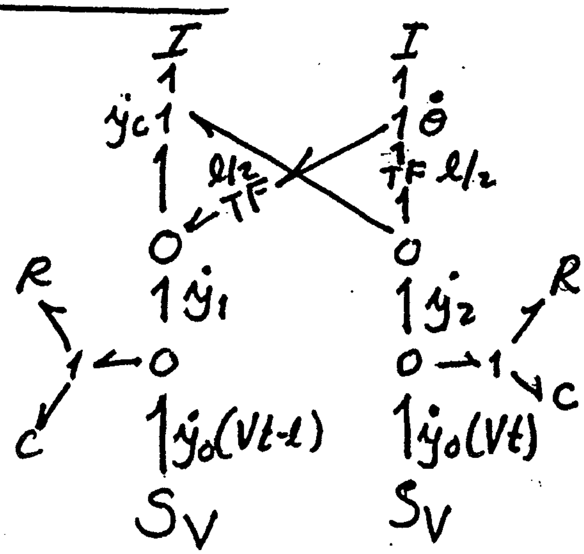
$$\dot{y}_r = y_0'(x) \cdot \dot{x}(t) - p/m$$

9-5

$$y_2 = y_c + \frac{l}{2}\theta$$

$$y_1 = y_c - \frac{l}{2}\theta$$

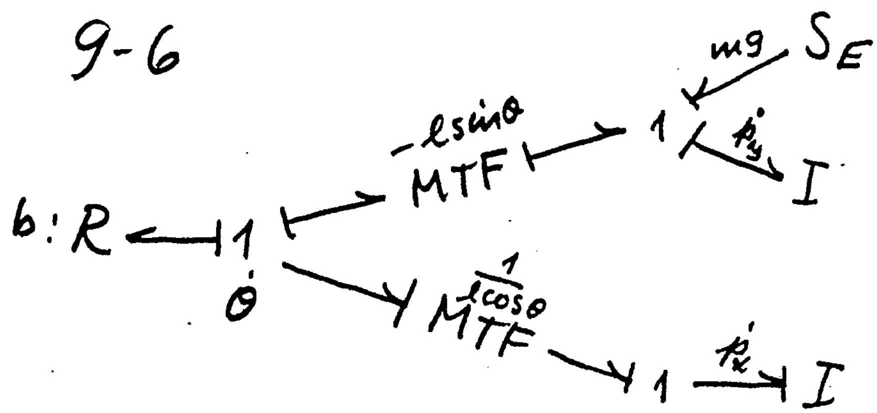




$$\begin{bmatrix} \frac{M}{4} + \frac{I_c}{l^2} & \frac{M}{4} - \frac{I_c}{l^2} \\ \frac{M}{4} - \frac{I_c}{l^2} & \frac{M}{4} + \frac{I_c}{l^2} \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

9-6

9-5



$$\boxed{\dot\theta = \frac{1}{l\cos\theta}\,\frac{p_x}{m}} \quad (1)$$

$$\dot p_x = \frac{1}{l\cos\theta}\left[-b\left(\frac{p_x}{ml\cos\theta} - (-l\sin\theta)\left[-mg+\dot p_y\right]\right]\right.$$

but $p_y = m\left(-l\sin\theta\cdot\frac{1}{l\cos\theta}\cdot\frac{p_x}{m}\right)$

$$\dot p_y = \frac{d}{dt}\left(\frac{\sin\theta}{\cos\theta}\cdot p_x\right) = \frac{-1}{\cos^2\theta}\dot\theta p_x - \frac{\sin\theta}{\cos\theta}\dot p_x$$

Substituting,

$$\dot p_x = \frac{-b p_x}{ml^2\cos^2\theta} - \frac{\sin\theta}{\cos\theta}mg + \frac{\sin\theta}{\cos\theta}\left(-\frac{\dot\theta p_x}{\cos^2\theta} - \frac{\sin\theta}{\cos\theta}\dot p_x\right)$$

or

$$\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)\dot p_x = \frac{-b p_x}{ml^2\cos^2\theta} - \frac{\sin\theta}{\cos\theta}mg - \frac{\sin\theta}{\cos^3\theta}p_x\cdot\frac{p_x}{ml\cos\theta}$$

$$\boxed{\dot p_x = -\frac{b}{ml^2}p_x - \sin\theta\cos\theta\, mg - \frac{\sin\theta\, p_x^2}{ml\cos^2\theta}} \quad (2)$$

from (1), $p_x = ml\cos\theta\,\dot\theta$, $p_x^2 = m^2l^2\cos^2\theta\,\dot\theta^2$

$$\dot p_x = \ddot\theta ml\cos\theta - ml\sin\theta\,\dot\theta^2$$

Substitute these into (2) to get standard eqn.

9-7 (a) $x_1 = l_1 \sin\theta_1$

$y_1 = l_1 \cos\theta_1$

$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$

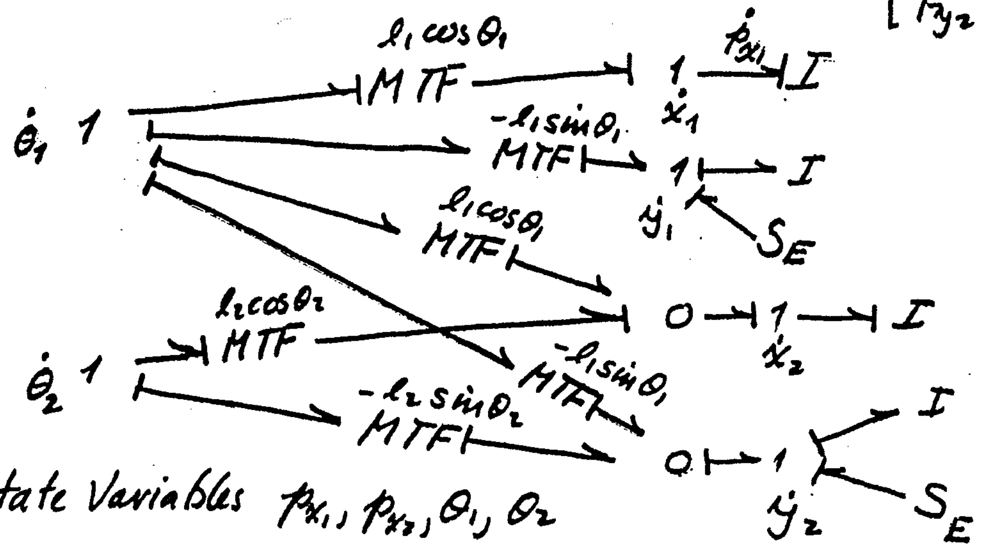
$y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$

9-6

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 & 0 \\ -l_1 \sin\theta_1 & 0 \\ l_1 \cos\theta_1 & l_2 \cos\theta_2 \\ -l_1 \sin\theta_1 & -l_2 \sin\theta_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

(a)

$$\begin{bmatrix} l_1 \cos\theta_1 & -l_1 \sin\theta_1 & l_1 \cos\theta_1 & -l_1 \sin\theta_1 \\ 0 & 0 & l_2 \cos\theta_2 & -l_2 \sin\theta_2 \end{bmatrix} \begin{bmatrix} F_{x_1} \\ F_{y_1} \\ F_{x_2} \\ F_{y_2} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$



(b)

State Variables $p_{x_1}, p_{x_2}, \theta_1, \theta_2$

$$T^* = \frac{m_1}{2}\left[(l_1 \cos\theta_1 \dot{\theta}_1)^2 + (l_1 \sin\theta_1 \dot{\theta}_1)^2 \right]$$

$$+ \frac{m_2}{2}\left[(l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)^2 + (l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2)^2 \right]$$

(c)

$$\delta W = (m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1)\delta\theta_1$$

$$+ (m_2 g l_2 \sin\theta_2)\delta\theta_2$$

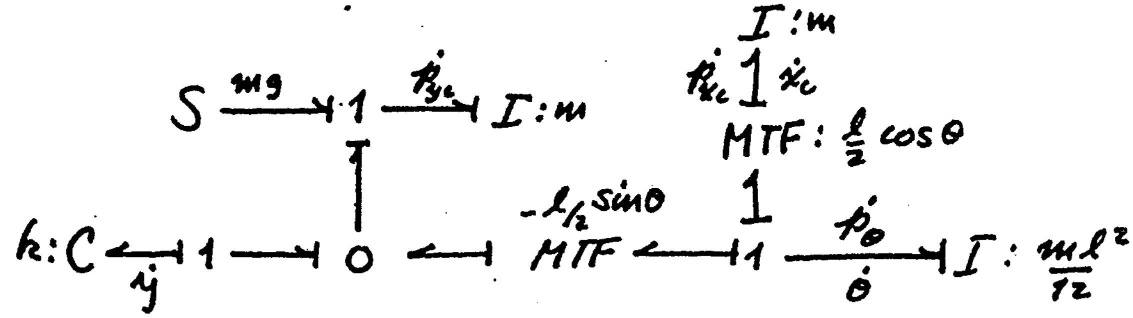
(T^* can be simplified using trigonometric identities.)

State Variables $\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2$

9-8

$$y_c = y + \frac{l}{2}\cos\theta \qquad \dot y_c = \dot y - \frac{l}{2}\sin\theta\,\dot\theta$$

$$x_c = \frac{l}{2}\sin\theta \qquad \dot x_c = \frac{l}{2}\cos\theta\,\dot\theta$$ 9-7



$$\dot\theta = p_\theta \cdot \frac{12}{ml^2}$$

$$\dot p_{y_c} = mg - ky$$

$$\dot y = p_{y_c}/m + \frac{l}{2}\sin\theta\, p_\theta \cdot \frac{12}{ml^2}$$

$$\dot p_\theta = -\frac{l}{2}\sin\theta\cdot ky - \frac{l}{2}\cos\theta\,\dot p_{x_c}$$

but $p_{x_c} = m\dot x_c = m\frac{l}{2}\cos\theta\cdot\dot\theta = m\frac{l}{2}\cos\theta\, p_\theta\cdot\frac{12}{ml^2}$

$$= \frac{6}{l}\cos\theta\, p_\theta$$

so

$$\dot p_{x_c} = \frac{6}{l}\cos\theta\cdot\dot p_\theta + \frac{6}{l}p_\theta(-\sin\theta)\dot\theta$$

$$= \frac{6}{l}\cos\theta\,\dot p_\theta - \frac{6}{l}\sin\theta\cdot p_\theta^2\cdot\frac{12}{ml^2}$$

After substitution in last state equation,

$$(1+3\cos^2\theta)\,\dot p_\theta = -\frac{l}{2}\sin\theta\cdot ky + \frac{36\sin\theta\cos\theta\, p_\theta^2}{ml^2}$$

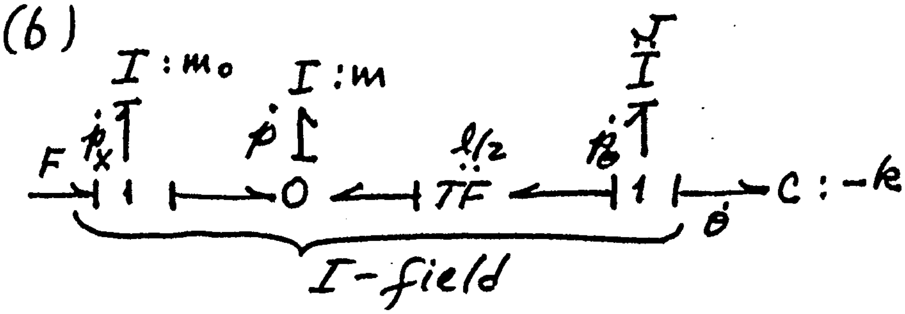
which is a short step from the final form.

9-9 (a) $T^* = \frac{1}{2} J\dot{\theta}^2 + \frac{1}{2} m v^2 + \frac{1}{2} m_0 \dot{x}^2$

$v \cong \dot{x} + \frac{\ell}{2}\dot{\theta}$; $V = mg\frac{\ell}{2}\cos\theta \cong mg\frac{\ell}{2}\left(1 - \frac{\theta^2}{2}\right)$

$V = -\frac{k\theta^2}{2} + \text{const}$, so $k = mg\ell/2$

(b)



$\dot{\theta} = p_\theta / J$

$\dot{p}_x = F(t) - \dot{p}$; $\dot{p}_\theta = -(-k)\theta - \frac{\ell}{2}\dot{p}$

but $p = m\left(\frac{p_x}{m_0} + \frac{\ell}{2}\frac{p_\theta}{J}\right)$, so $\dot{p} = \frac{m}{m_0}\dot{p}_x + \frac{m\ell}{2J}\dot{p}_\theta$

then

$$\left[\begin{array}{c|c} 1 + m/m_0 & m\ell/2J \\ \hline m\ell/2m_0 & 1 + m\ell^2/4J \end{array}\right]\begin{bmatrix}\dot{p}_x \\ \dot{p}_\theta\end{bmatrix} = \begin{bmatrix} F(t) \\ k\theta \end{bmatrix}$$

(invert to get standard form.

(c) We can write T^* as follows

$$T^* = \frac{1}{2}[\dot{x} \,|\, \dot{\theta}]\underbrace{\left[\begin{array}{c|c} J + m\ell^2/4 & m\ell/2 \\ \hline m\ell/2 & m_0 + m \end{array}\right]}_{\text{Mass matrix of I-field}}\begin{bmatrix}\dot{x} \\ \dot{\theta}\end{bmatrix}$$

$$[I]^{-1} = \frac{1}{(J + \frac{m\ell^2}{4})(m_0 + m) - \frac{m^2\ell^2}{4}}\left[\begin{array}{c|c} m_0 + m & -m\ell/2 \\ \hline -m\ell/2 & J + m\ell^2/4 \end{array}\right]$$

$$= \begin{bmatrix}\Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22}\end{bmatrix} \text{ and } \begin{bmatrix}\dot{x} \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22}\end{bmatrix}\begin{bmatrix}p_x \\ p_\theta\end{bmatrix}$$

so state equ's are

$\dot{p}_x = F$; $\dot{p}_\theta = k\theta$; $\dot{\theta} = \Gamma_{21} p_x + \Gamma_{22} p_\theta$

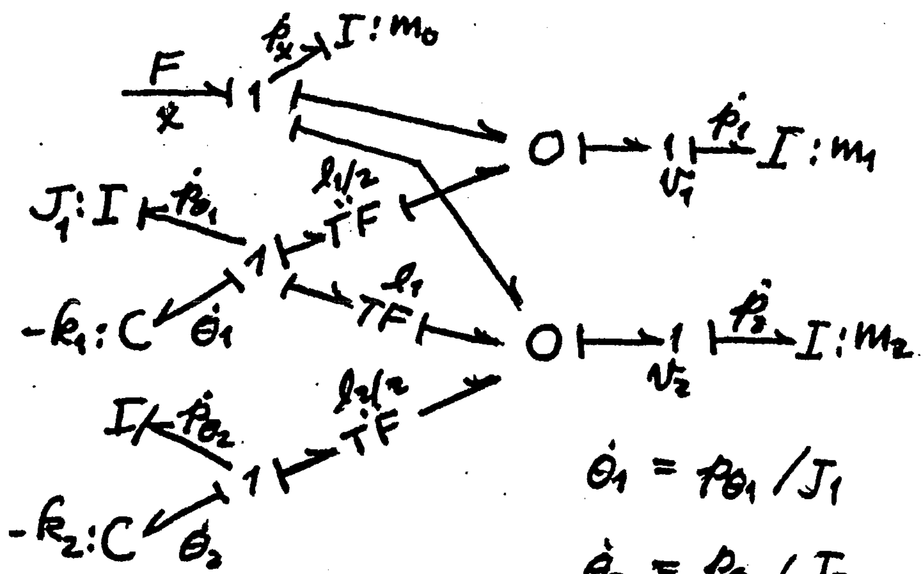
9-10 $T^* = \frac{1}{2} m_0 \dot{x}^2 + \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

$v_1 \cong \dot{x} + \frac{l_1}{2}\dot{\theta}_1, \quad v_2 = \dot{x} + l_1\dot{\theta}_1 + \frac{l_2}{2}\dot{\theta}_2$

$V = m_1 g \frac{l_1}{2}\cos\theta_1 + m_2 g\left(l_1\cos\theta_1 + \frac{l_2}{2}\cos\theta_2\right)$ 9-9

$\cong \left(m_1 g \frac{l_1}{2} + m_2 g l_1\right)\left(-\frac{\theta_1^2}{2}\right) + m_2 g \frac{l_2}{2}\left(-\frac{\theta_2^2}{2}\right) + const$

$\cong -k_1 \frac{\theta_1^2}{2} - k_2 \frac{\theta_2^2}{2}; \quad k_1 = m_1 g \frac{l_1}{2} + m_2 g; \quad k_2 = m_2 g \frac{l_2}{2}.$



$$\left.\begin{aligned}
\dot{\theta}_1 &= p_{\theta_1}/J_1 \\
\dot{\theta}_2 &= p_{\theta_2}/J_2 \\
\dot{p}_x &= F(t) - \dot{p}_1 - \dot{p}_2 \\
\dot{p}_{\theta_1} &= k_1\theta_1 - \frac{l_1}{2}\dot{p}_1 - l_1\dot{p}_2 \\
\dot{p}_{\theta_2} &= k_2\theta_2 - \frac{l_2}{2}\dot{p}_2
\end{aligned}\right\} \text{Basic State Equations}$$

Derivative Causality Calculations

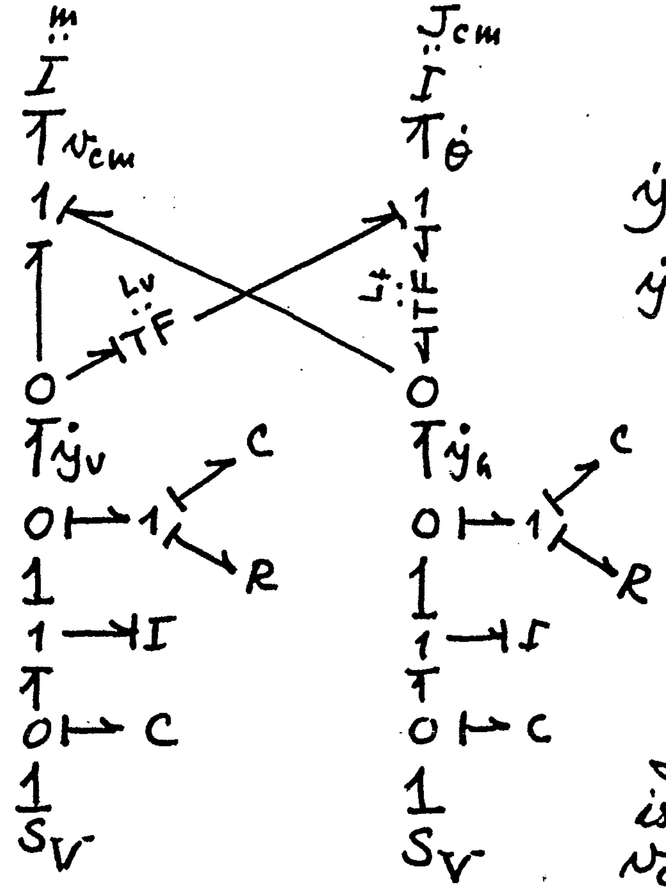
$$p_1 = m_1 v_1 = m_1\left(\frac{p_x}{m_0} + \frac{l_1}{2}\frac{p_{\theta_1}}{J_1}\right)$$

$$p_2 = m_2 v_2 = m_2\left(\frac{p_x}{m_0} + l_1\frac{p_{\theta_1}}{J_1} + \frac{l_2}{2}\frac{p_{\theta_2}}{J_2}\right)$$

Substitute in for $\dot{p}_1$ and $\dot{p}_2$ and do a 3x3 matrix inversion to get standard form.

9-11

9-10



$$\dot{y}_v = \dot{y}_{cm} + L_v \dot{\theta}$$

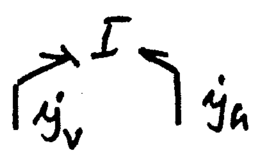
$$\dot{y}_h = \dot{y}_{cm} - L_f \dot{\theta}$$

With y_{cm} and θ as components of the q_k vector,

all integral causality is possible as shown.

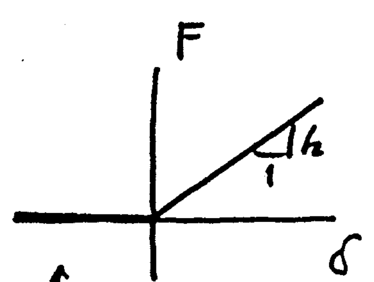
If the MTF structure is set up to compute v_{cm} and $\dot{\theta}$ from $\dot{y}_v$ and $\dot{y}_h$, then an algebraic loop results as the bond graph tries to invert the transformation.

The I-field form is just



where the I-matrix may be found directly from the transformer structure, or by evaluating T^* and then differentiating to get mass matrix elements.

Nonlinear constitutive laws for the tire C-elements can be used to model loss of contact.



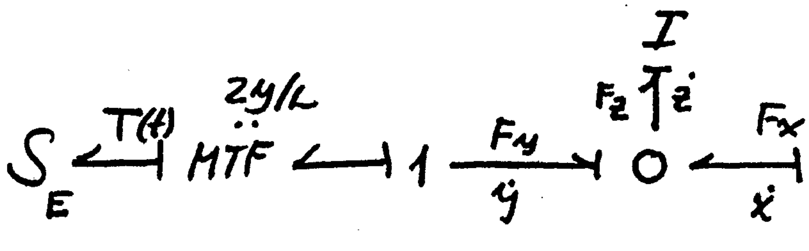
zero force for negative deflection.

9-12 $\delta = 2(y^2 - L^2)^{1/2} - 2L$

$$= 2L\left(1+\left(\frac{y}{L}\right)^2\right)^{1/2} - 2L$$

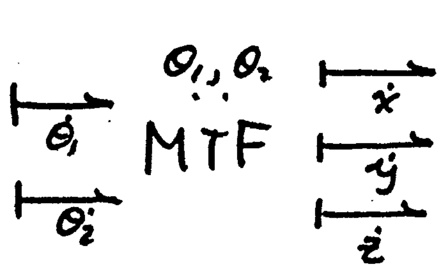
$$\cong 2L\left(1+\tfrac{1}{2}(y/L)^2\right) - 2L = y^2/L$$

$$\dot{\delta} = \frac{2y}{L}\,\dot{y} \qquad F_y = -2\frac{y}{L}\,T(t)$$



Note that the force source fixes causality on all bonds. This means that at any instant given $T(t)$ and y one can compute F_y. The graph of (a) does not show this necessary causality. Given F_y, it is not possible to compute y since $T(t)$ may even jump to zero at any time.

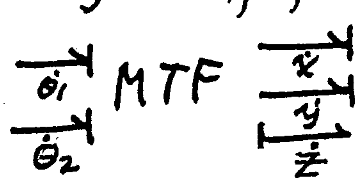
9-13
$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \left[\begin{array}{c|c} L\cos\theta_1\cos\theta_2 & -L\sin\theta_1\sin\theta_2 \\ L\cos\theta_1\sin\theta_2 & L\sin\theta_1\cos\theta_2 \\ L\sin\theta_2 & 0\end{array}\right]\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$$



Corresponds to this causality

Integral causality on x, y, z bonds would mean inversion of a non-square matrix – clearly impossible

There are just two geometric degrees of freedom so this would work:



for example.

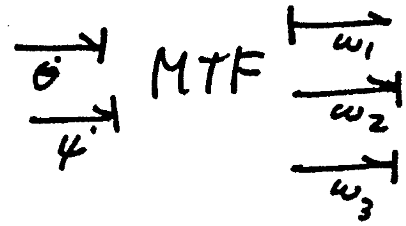
9-15 The MTF law is

9-12

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 0 & \sin\theta \\ 0 & \cos\theta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

which shows immediately that $\omega_1, \omega_2, \omega_3$ cannot be chosen independently. It would, however, be possible to solve for $\dot{\theta}$ and $\dot{\psi}$ in terms of ω_2 and ω_3 and then to find ω_1 also interms of ω_2 and ω_3.

Thus, this causality is possible



The rest of the MTF laws are

$$\begin{bmatrix} \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

This MTF has two geometric degrees of freedom and 3 "force" degrees of freedom. Because of the 0's, not all sets of two flows will serve as inputs, however.

9-16

$$v_{x_1} = v_x - \omega_y h + \omega_z a$$
$$v_{y_1} = v_y + \omega_x h + \omega_z b$$
$$v_{z_1} = v_z - \omega_x a - \omega_y b$$

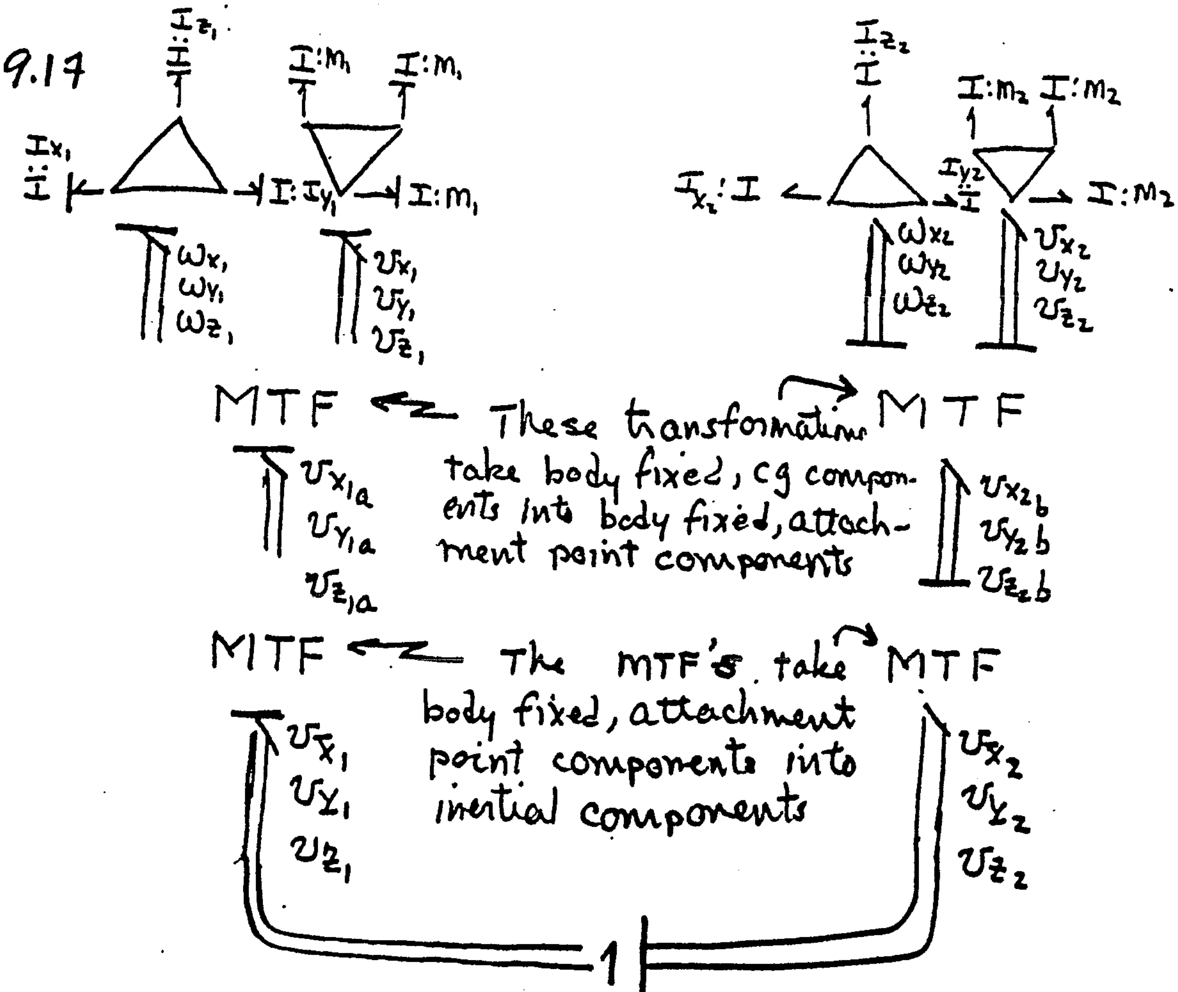
$$v_{x_2} = v_x - \omega_y h + \omega_z a$$
$$v_{y_2} = v_y + \omega_x h - \omega_z c$$
$$v_{z_2} = v_z - \omega_x a + \omega_y c$$

$$\begin{bmatrix} v_{x_1} \\ v_{y_1} \\ v_{z_1} \\ v_{x_2} \\ v_{y_2} \\ v_{z_2} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & -h & a \\ 0 & 1 & 0 & h & 0 & b \\ 0 & 0 & 1 & -a & -b & 0 \\ 1 & 0 & 0 & 0 & -h & a \\ 0 & 1 & 0 & h & 0 & -c \\ 0 & 0 & 1 & -a & c & 0 \end{bmatrix}}_{\underline{M}} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

and

$$\begin{bmatrix} F_x \\ F_y \\ F_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \underline{M}^t \begin{bmatrix} F_{x_1} \\ F_{y_1} \\ F_{z_1} \\ F_{x_2} \\ F_{y_2} \\ F_{z_2} \end{bmatrix} \longleftarrow \text{amazing}$$

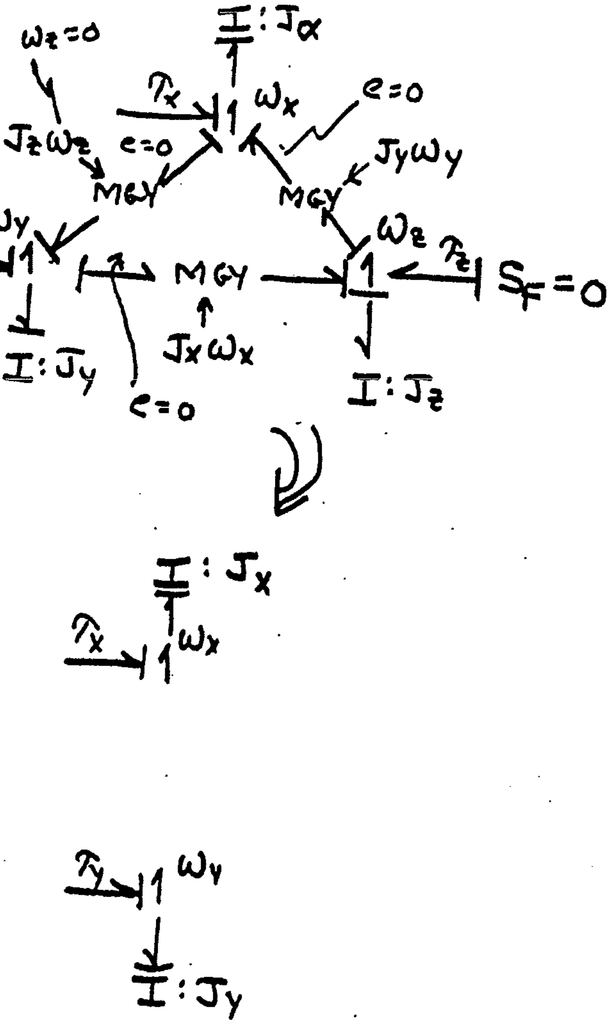
9.17



This 1-junction enforces constraint that inertial direction components at attachment point must be equal.

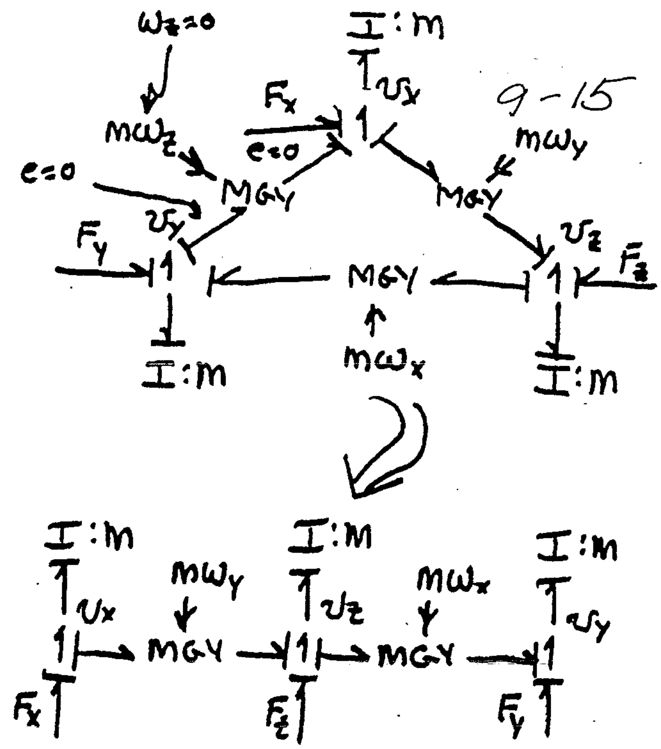
IF causality in body 1 is all integral, then some inertia element in body 2 must be in derivative causality, thus formulation is greatly complicated.

9.18



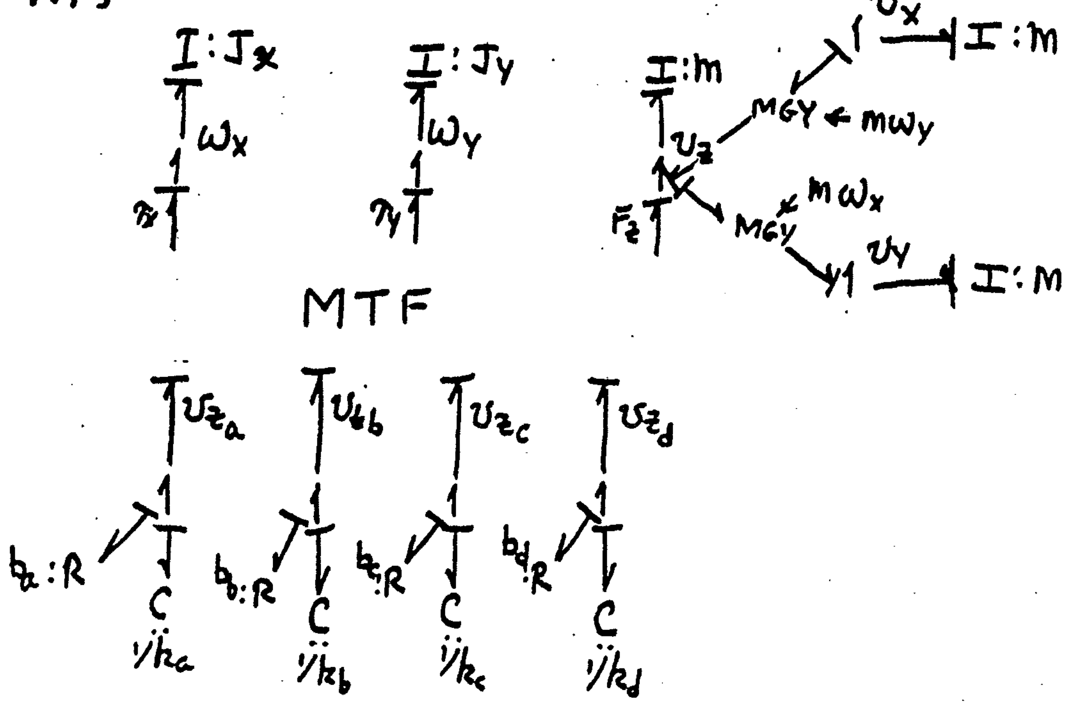
$\omega_z = 0$
$I : J_x$
τ_x
ω_x
$e = 0$
$e=0$
$J_z\omega_z$
$J_y\omega_y$
MGY
MGY
ω_y
τ_y
MGY
ω_z
τ_z
$S_F = 0$
$I : J_y$
$J_x\omega_x$
$I : J_z$
$e = 0$
$I : J_x$
τ_x
ω_x
τ_y
ω_y
$I : J_y$

9-15



$\omega_z = 0$
$I : m$
v_x
F_x
$m\omega_z$
$e=0$
$m\omega_y$
$e = 0$
MGY
MGY
v_y
v_z
F_y
MGY
F_z
$I : m$
$m\omega_x$
$I : m$
$I : m$
$I : M$
$I : m$
$m\omega_y$
$m\omega_x$
v_x
v_z
v_y
MGY
MGY
F_x
F_z
F_y

9.19



$I : J_x$
$I : J_y$
ω_x
ω_y
τ_x
τ_y
v_x
$I : m$
$I : m$
MGY $\leftarrow m\omega_y$
v_z
$m\omega_x$
F_z
MGY
v_y
$I : m$
MTF
v_{z_a}
v_{z_b}
v_{z_c}
v_{z_d}
$b_a : R$
$b_b : R$
$b_c : R$
$b_d : R$
C
C
C
C
$1/k_a$
$1/k_b$
$1/k_c$
$1/k_d$

where

9-16

$$v_{z_a} = v_z + c_f \omega_x - w\,\omega_y$$
$$v_{zb} = v_z + c_f \omega_x + w\,\omega_y$$
$$v_{z_c} = v_z - c_r \omega_x + w\,\omega_y$$
$$v_{z_d} = v_z - c_r \omega_x - w\,\omega_y$$

or

$$\begin{bmatrix} v_{z_a} \\ v_{z_b} \\ v_{z_c} \\ v_{z_d} \end{bmatrix} = \left[\begin{array}{c|c|c} 1 & c_f & -w \\ 1 & c_f & w \\ 1 & -c_r & w \\ 1 & -c_r & -w \end{array}\right] \begin{bmatrix} v_z \\ \omega_x \\ \omega_y \end{bmatrix}$$

9-20

From the bond graph

$$\dot{p}_x = F_x + m\omega_z v_y$$
$$\dot{p}_y = F_y - m\omega_z v_x$$

or

$$m(\dot{v}_x - \omega_z v_y) = F_x$$
$$m(\dot{v}_y + \omega_z v_x) = F_y$$

$\therefore$

$$a_x = \dot{v}_x - \omega_z v_y$$
$$a_y = \dot{v}_y + \omega_z v_x$$

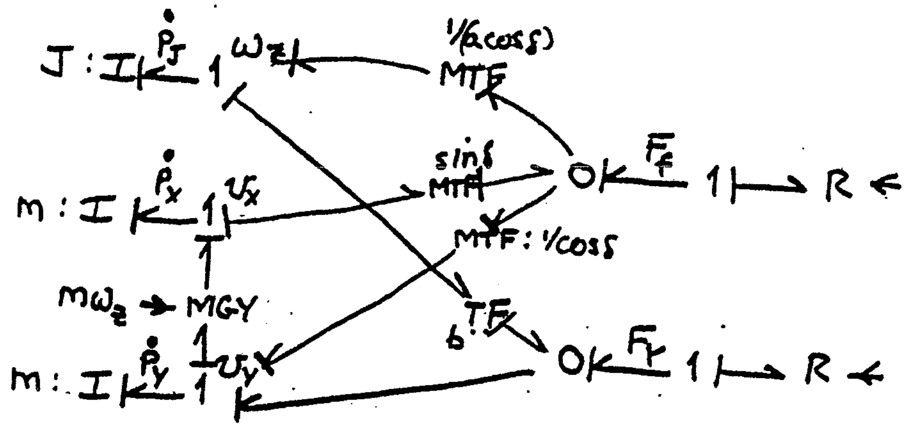
but $\quad v_x = \dot{r} \qquad \omega_z = \dot{\theta}$

$\qquad v_y = r\dot{\theta}$

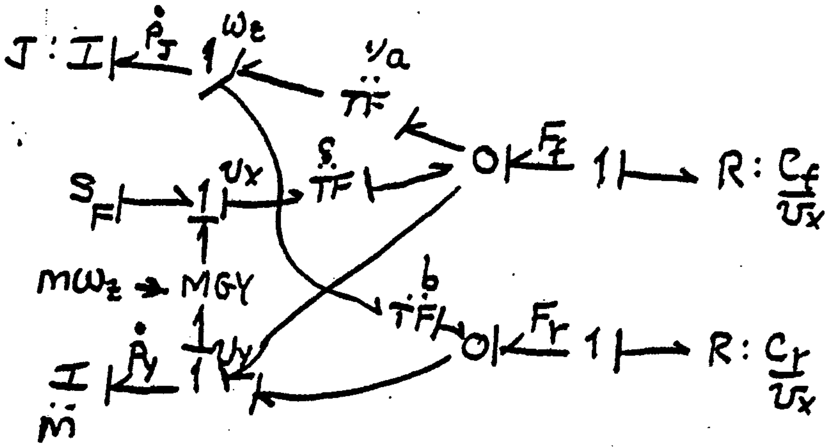
$\therefore\ a_x = \ddot{r} - \dot{\theta} r \dot{\theta}$ $\Leftarrow$ same as a_r

$a_y = r\ddot{\theta} + \dot{r}\dot{\theta} + \dot{\theta}\dot{r}$ $\Leftarrow$ same as a_θ

9-21



9-22



state vbls. p_J, p_y

9-23

$$\dot p_J = -a\frac{C_f}{v_x}\left(a\frac{p_J}{J}+\frac{p_y}{m}-v_x\delta\right)+b\frac{C_r}{v_x}\left(\frac{p_y}{m}-b\frac{p_J}{J}\right)$$

$$\dot p_y = -\frac{C_r}{v_x}\left(\frac{p_y}{m}-b\frac{p_J}{J}\right)-\frac{C_f}{v_x}\left(a\frac{p_J}{J}+\frac{p_y}{m}-\delta v_x\right)-m\frac{p_J}{J}v_x$$

$$\frac{d}{dt}\begin{bmatrix} p_y \\ p_J \end{bmatrix} = \begin{bmatrix} \frac{-(C_r+C_f)}{m v_x} & \frac{(bC_r-aC_f)}{J v_x}-\frac{m v_x}{J} \\ \frac{bC_r-aC_f}{m v_x} & \frac{-(a^2C_f+b^2C_r)}{J v_x} \end{bmatrix}\begin{bmatrix} p_y \\ p_J \end{bmatrix}+\begin{bmatrix} C_f \\ aC_f \end{bmatrix}\delta$$

9-18

$$9\text{-}24 \quad \begin{bmatrix} s + \dfrac{C_r + C_f}{m v_x} & \dfrac{aC_f - bC_r}{J v_x} + \dfrac{m v_x}{J} \\[2ex] \dfrac{aC_f - bC_r}{m v_x} & s + \dfrac{a^2 C_f + b^2 C_r}{J v_x} \end{bmatrix} \begin{bmatrix} \rho_y \\[2ex] \rho_J \end{bmatrix} = 0$$

$$s^2 + \left[\frac{a^2 C_f + b^2 C_r}{J v_x} + \frac{C_r + C_f}{m v_x} \right] s + \frac{C_r + C_f}{m v_x}\left(\frac{a^2 C_f + b^2 C_r}{J v_x}\right) - \frac{(aC_f - bC_r)^2}{m J v_x^2} - \frac{(aC_f - bC_r)}{\cancel{m v_x}}\frac{\cancel{m v_x}}{J} =$$

look at the constant term,

$$\frac{1}{m J v_x^2}\left\{(a+b)^2 C_f C_r\right\} - \frac{(aC_f - bC_r)}{J} = \frac{1}{m J v_x^2}\left\{(a+b)^2 C_f C_r + (bC_r - aC_f)m\right.$$

thus

$$s^2 + \left[\frac{a^2 C_f + b^2 C_r}{J v_x} + \frac{C_r + C_f}{m v_x} \right] s + \frac{1}{m J v_x^2}\left[(a+b)^2 C_f C_r + (bC_r - aC_f) m v_x^2\right] = 0$$

This equation will always have stable eigenvalues if all coefficients are positive. Thus if $bC_r > aC_f$, the system is stable. If

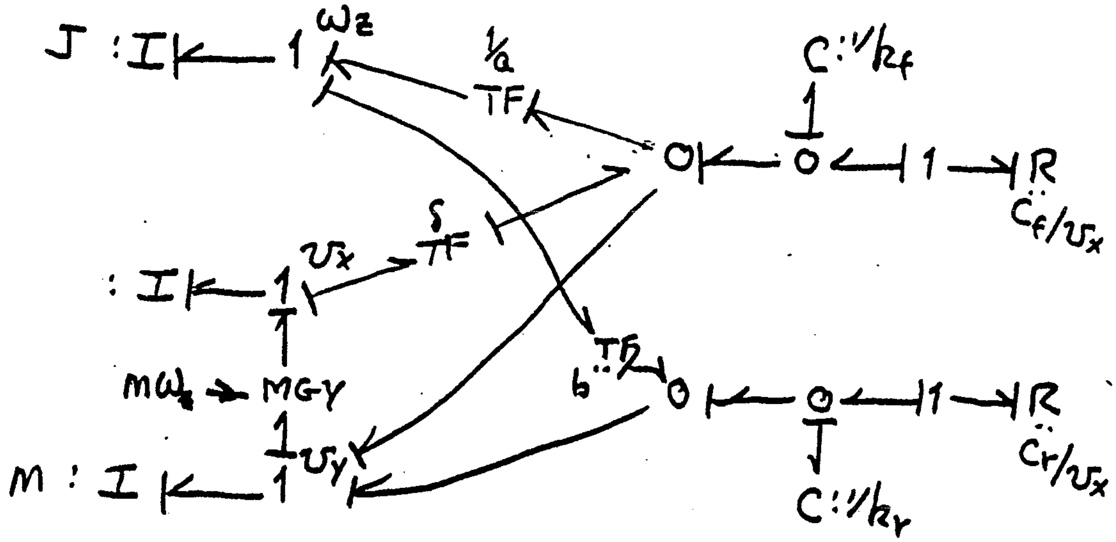
$$bC_r < aC_f$$

then last term is positive only if,

$$v_x^2 < \frac{(a+b)^2 C_f C_r}{m(aC_f - bC_r)} \qquad \text{a critical speed.}$$

9-25

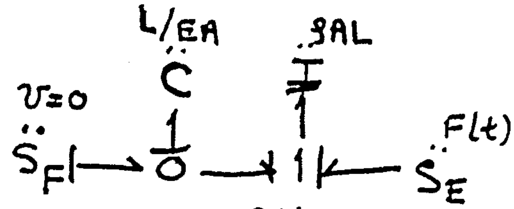
9-19



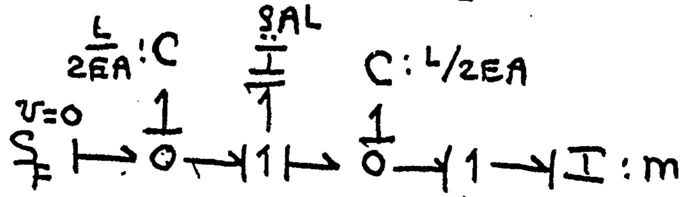
Note that the cornering forces are generated with the opposite causality from the previous problem.

10.1

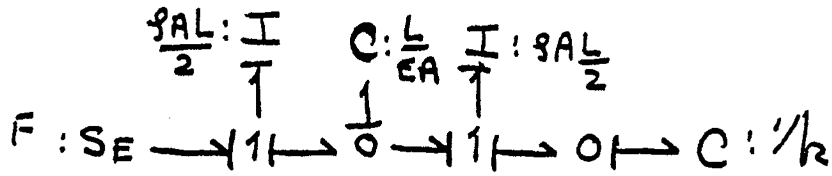
(a)



(b)

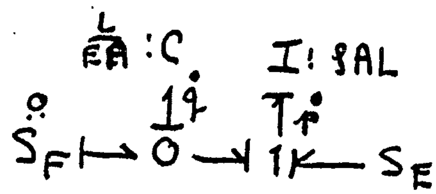


(c)



10.2

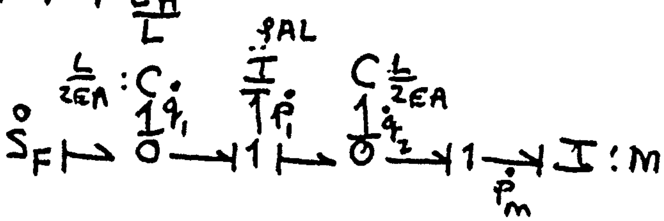
a)



$$\dot{q} = -\frac{p}{\rho A L}$$

$$\dot{p} = F + q\,\frac{EA}{L}$$

b)



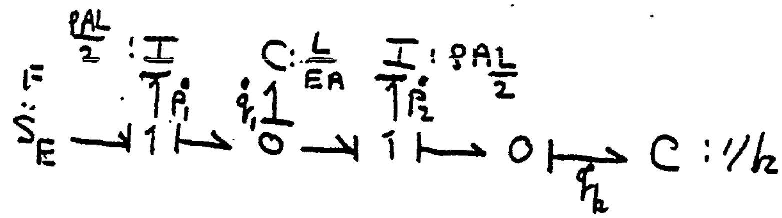
$$\dot{q}_1 = -\frac{p_1}{\rho A L}$$

$$\dot{p}_1 = \frac{2EA}{L}\,q_1 - \frac{2EA}{L}\,q_2$$

$$\dot{q}_2 = \frac{p_1}{\rho A L} - \frac{p_m}{m}$$

$$\dot{p}_m = \frac{2EA}{L}\,q_2$$

(c)



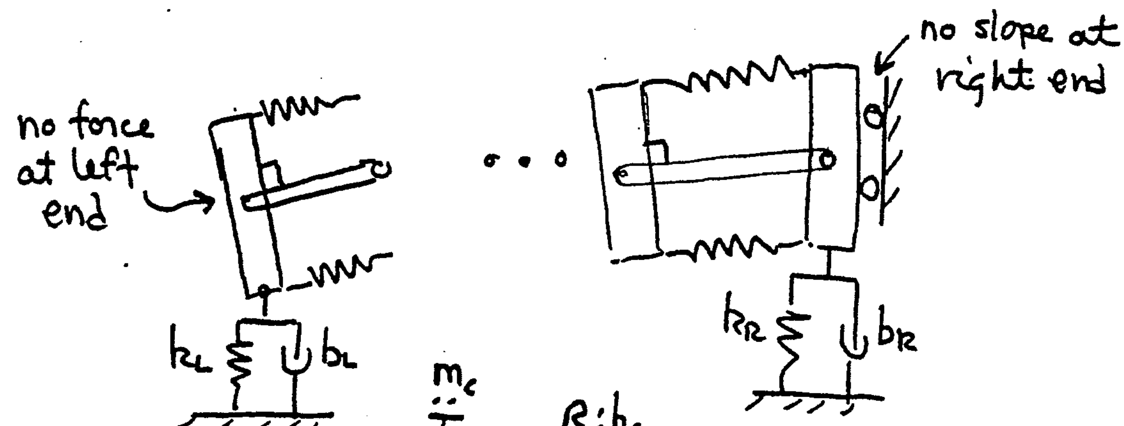
$$\dot p_1 = F - \frac{EA}{L} q_1$$

$$\dot q_1 = \frac{2 p_1}{\rho AL} - \frac{2}{\rho AL} p_2$$

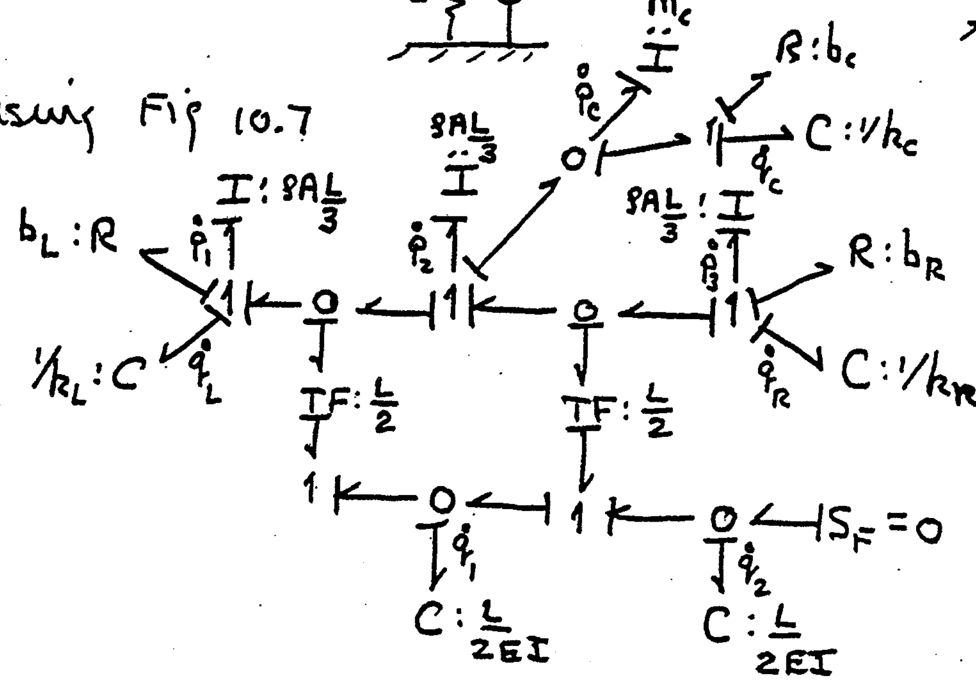
$$\dot p_2 = \frac{EA}{L} q_1 - k q_k$$

$$\dot q_k = \frac{2 p_2}{\rho AL}$$

10.3 It helps to draw the boundary lumps, so that modelling decisions are clear.



using Fig 10.7



10.4 From soln. to P10.3,

$$\dot{p}_1 = -b_L \frac{p_1}{I} - k_L q_L + \frac{2}{L}(-1)\frac{q_1}{C}$$

$$\dot{p}_2 = +\frac{2}{L}\frac{q_1}{C} + \frac{2}{L}\left(\frac{q_1}{C} - \frac{q_2}{C}\right) - \left(k_c q_c + b_c\left(\frac{p_2}{I} - \frac{p_c}{m_c}\right)\right)$$

$$\dot{p}_3 = -b_R \frac{p_3}{I} - k_R q_R - \frac{2}{L}\left(\frac{q_1}{C} - \frac{q_2}{C}\right)$$

$$\dot{q}_1 = \frac{2}{L}\left(\frac{p_3}{I} - \frac{p_2}{I}\right) - \frac{2}{L}\left(\frac{p_2}{I} - \frac{p_1}{I}\right) \qquad I = \rho A \frac{L}{3}$$

$$\dot{q}_2 = -\frac{2}{L}\left(\frac{p_3}{I} - \frac{p_2}{I}\right) \qquad C = \frac{L}{2EI}$$

$$\dot{q}_L = \frac{p_1}{I} \qquad \dot{q}_R = \frac{p_3}{I} \qquad \dot{q}_c = \frac{p_2}{I} - \frac{p_c}{m_c}$$

10.5 $w(x,t) = Y(x) \cdot f(t)$ and $F = 0$ for now

$$\therefore \quad \frac{T \frac{d^2Y}{dx^2} \cancel{f}}{\cancel{\rho} Y \cancel{f}} = \frac{\cancel{\rho} \cancel{Y} \frac{d^2f}{dt^2}}{\cancel{\rho} \cancel{Y} f} = -\omega^2$$

$$T \frac{d^2Y}{dx^2} + \rho\omega^2 Y = 0$$

or

$$\frac{d^2Y}{dx^2} + \frac{\rho}{T}\omega^2 Y = 0$$

$$\therefore \quad Y = A\cos kx + B\sin kx \qquad k^2 = \frac{\rho}{T}\omega^2$$

$$\frac{dY}{dx}(0) = 0 = Bk \qquad \therefore B = 0$$

$$\frac{dY}{dx}(L) = 0 = -Ak\sin kL = 0$$

$\therefore$ $k = 0$ which is ok for this system

or $\sin kL = 0$ $\quad k_n L = n\pi \quad n = 0, 1,$

10-4

and
$$\boxed{\begin{aligned} \omega_n^2 &= \frac{T}{\rho}\left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \ldots \\ Y_n(x) &= \cos n\pi \frac{x}{L} \end{aligned}}$$

for $\omega_n = 0$

$$\frac{d^2 Y_0}{dx^2} = 0$$
$$Y_0 = ax + b$$

but $\frac{dY_0}{dx}(0) = 0 = a \leftarrow$ since Y_0 must satisfy boundary conditions

$\therefore$ $Y_0 = b$ where b is arbitrary, typically set to unity

$\therefore$ $\boxed{Y_0 = 1 \qquad \omega_0 = 0}$

10.6 $\quad w(x,t) = \sum_n Y_n \eta_n(t)$

substitute

$$-\sum_{n=1}^{\infty} T \frac{d^2 Y_n}{dx^2}\eta_n + \sum_{n=1}^{\infty} \rho Y_n \frac{d^2\eta_n}{dt^2} = F\delta(x - x_1)$$

but $\frac{d^2 Y_n}{dx^2} = -\frac{\rho}{T}\omega_n^2 Y_n$

$$\therefore \sum_{n=1}^{\infty} \rho\omega_n^2 Y_n \eta_n + \sum_{n=1}^{\infty} \rho Y_n \frac{d^2\eta_n}{dt^2} = F\delta(x - x_1)$$

multiply by $Y_m(x)$ and $\int_0^L dx$

$$m_m \ddot{\eta}_m + m_m \omega_m^2 \eta_m = F Y_m(x_1)$$

$$m_m = \int_0^L \rho Y_m^2\, dx = \int_0^L \rho \cos^2 m\pi \frac{x}{L}\, dx = \frac{\rho L}{2} \text{ for each mode.}$$

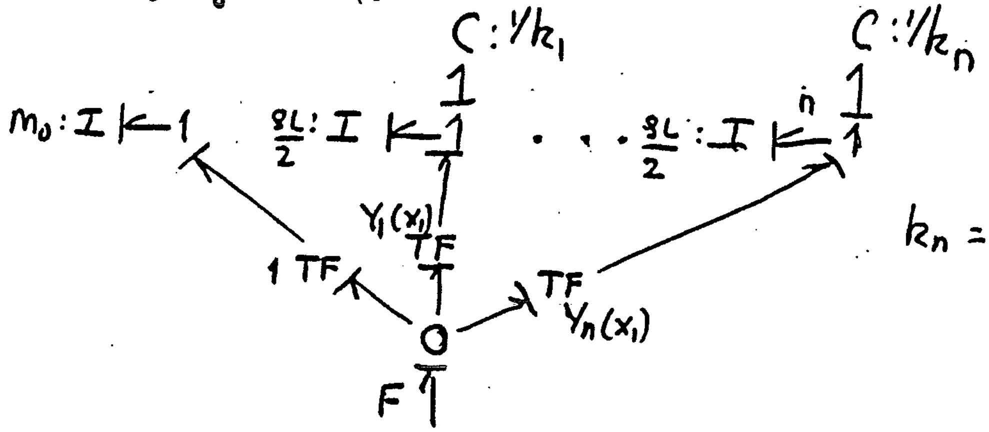
for zero freq. mode

$$m_0 = \int_0^L \rho Y_0^2 \, dx = \rho L = \text{mass of string}$$ 10-5

and

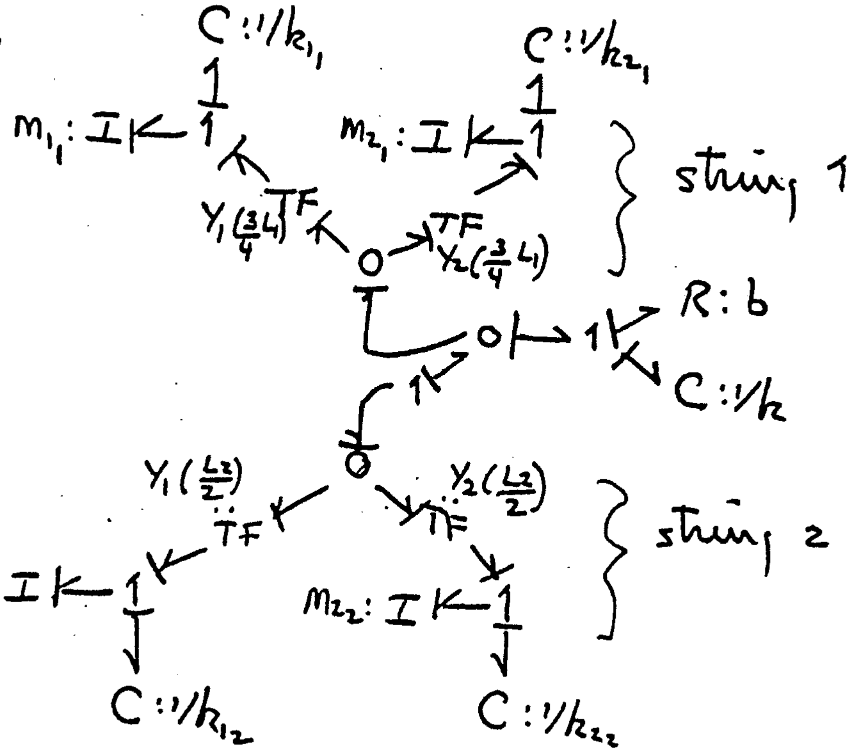
$$m_0 \ddot{\eta}_0 = F(1)$$

Thus

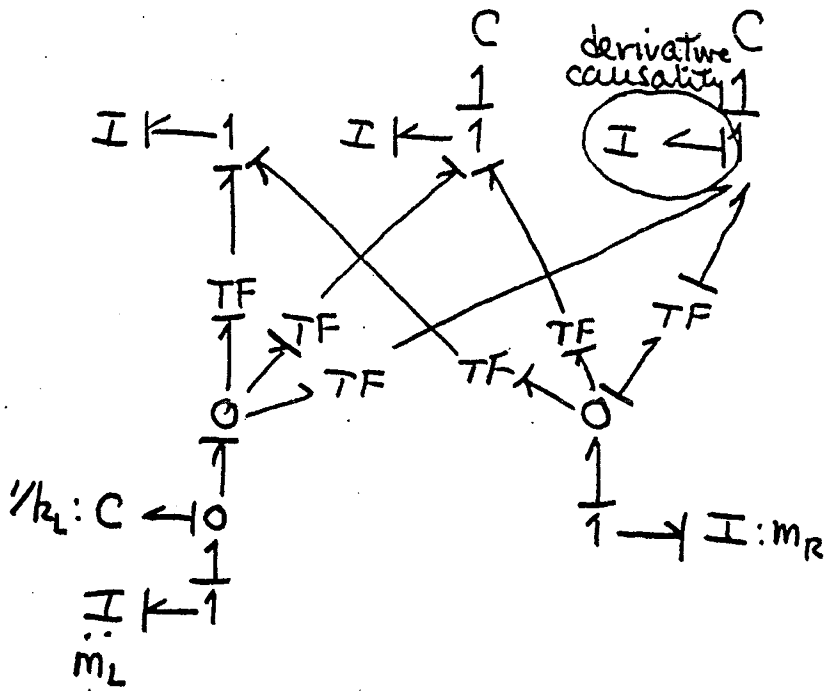


$$k_n = m_n \omega_n^2$$

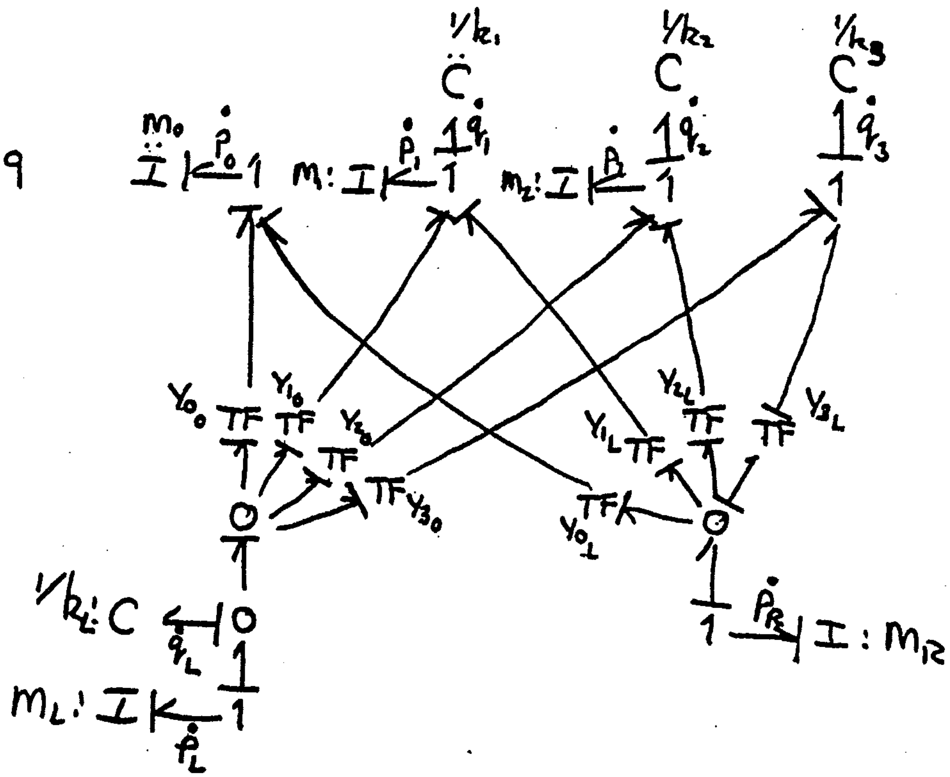
10.7



10.8



10.9



$$\mathring{p}_0 = Y_{0_0} k_L q_L + Y_{0_L} \frac{1}{Y_{3_L}} \left(k_3 q_3 - Y_{3_0} k_L q_L \right)$$

$$\mathring{p}_1 = -k_1 q_1 + Y_{1_0} k_L q_3 + \frac{Y_{1_L}}{Y_{3_L}} \left(k_3 q_3 - Y_{3_0} k_L q_L \right)$$

$$\mathring{p}_2 = -k_2 q_2 + Y_{2_0} k_L q_L + \frac{Y_{2_L}}{Y_{3_L}} \left(k_3 q_3 - Y_{3_0} k_L q_L \right)$$

$$\dot q_1 = \frac{p_1}{m_1} \qquad \dot q_2 = \frac{p_2}{m_2} \qquad \dot q_3 = \frac{1}{Y_{3L}}\left[\frac{p_R}{m_R} - Y_{0L}\frac{p_0}{m_0} - Y_{1L}\frac{p_1}{m_1} - Y_{2L}\frac{p_2}{m_2}\right]$$

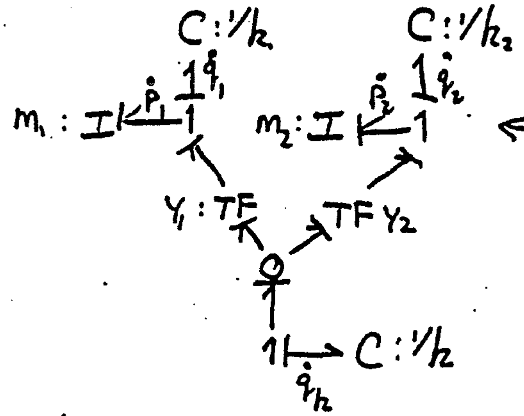
$$\dot q_L = \frac{p_L}{m_L} - Y_{0_0}\frac{p_0}{m_0} - Y_{1_0}\frac{p_1}{m_1} - Y_{2_0}\frac{p_2}{m_2} - \frac{Y_{3_0}}{Y_{3L}}\left[\frac{p_R}{m_R} - Y_{0L}\frac{p_0}{m_0} - Y_{1L}\frac{p_1}{m_1} - Y_{2L}\frac{p_2}{m_2}\right]$$

$$\dot p_L = -k_L q_L$$

$$\dot p_R = (-1)\frac{1}{Y_{3L}}\left[k_3 q_3 - Y_{3_0} k_L q_L\right]$$

10.10

A.



fixed-free modes

where $m_1 = m_2 = \rho\frac{AL}{2}$

$k_i = m_i\,\omega_i^2 \qquad \omega_i^2 = \frac{E}{\rho}(2i-1)^2\left(\frac{\pi}{2L}\right)^2$

$Y_i = \sin(2i-1)\frac{\pi}{2}$ (at $x=L$)

$$\dot p_1 = -k_1 q_1 + Y_1(-k q_k)$$

$$\dot p_2 = -k_2 q_2 + Y_2(-k q_k)$$

$$\dot q_1 = p_1/m_1 \qquad \dot q_2 = p_2/m_2$$

$$\dot q_k = Y_1\frac{p_1}{m_1} + Y_2\frac{p_2}{m_2}$$

$$\Rightarrow \begin{bmatrix} s & 0 & k_1 & 0 & Y_1 k \\ 0 & s & 0 & k_2 & Y_2 k \\ -1/m_1 & 0 & s & 0 & 0 \\ 0 & -1/m_2 & 0 & s & 0 \\ -Y_1/m_1 & -Y_2/m_2 & 0 & 0 & s \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \\ q_k \end{bmatrix} = 0$$

expand determinant in terms of cofactors (looks bad, but requires only a little patience)

$$s\begin{bmatrix} s & 0 & k_2 & Y_2 k \\ 0 & s & 0 & 0 \\ -1/m_2 & 0 & s & 0 \\ -Y_2/m_2 & 0 & 0 & s \end{bmatrix} + k_1\begin{bmatrix} 0 & s & k_2 & Y_2 k \\ -1/m_1 & 0 & 0 & 0 \\ 0 & -1/m_2 & s & 0 \\ -\frac{Y_1}{m_1} & -\frac{Y_2}{m_2} & 0 & s \end{bmatrix} + Y_1 k\begin{bmatrix} 0 & s & 0 & k_2 \\ -1/m_1 & 0 & s & 0 \\ 0 & -1/m_2 & 0 & s \\ -Y_1/m_1 & -Y_2/m_2 & 0 & 0 \end{bmatrix}$$

again

$$s^2 \begin{bmatrix} s & k_2 & Y_2 k \\ -1/m_2 & s & 0 \\ -\frac{Y_2}{m_2} & 0 & s \end{bmatrix} + \frac{k_1}{M_1}\begin{bmatrix} s & k_2 & Y_2 k \\ -1/m_2 & s & 0 \\ -\frac{Y_2}{m_2} & 0 & s \end{bmatrix} - Y_1 k s \begin{bmatrix} 0 & s & k_2 \\ 0 & -1/m_2 & s \\ -Y_1/m_1 & -Y_2/m_2 & 0 \end{bmatrix}$$

again

$$s^3 s^2 + \frac{k_2}{M_2} s^2 s + \frac{Y_2^2 k}{m_2} s^2 s + \frac{k_1}{M_1} s s^2 + \frac{k_1}{M_1}\frac{k_2}{M_2} s + \frac{k_1}{M_1}\frac{Y_2^2 k}{m_2} s + \frac{Y_1^2 k s}{M_1}\left(s^2 + \frac{k_2}{M_2}\right) = 0$$

now let $k \to \infty$ and retain

$$k\left[\frac{Y_2^2}{m_2} s^3 + \frac{k_1}{M_1}\frac{Y_2^2}{M_2} s + \frac{Y_1^2}{m_1} s\left(s^2 + \frac{k_2}{M_2}\right)\right] = 0$$

$$ks\left[\left(\frac{Y_2^2}{M_2} + \frac{Y_1^2}{M_1}\right)s^2 + \frac{k_1}{M_1}\frac{Y_2^2}{M_2} + \frac{k_2}{m_2}\frac{Y_1^2}{M_1}\right] = 0$$

Thus, the remaining frequency is

$$\omega_n^2 = \frac{\frac{k_1}{m_1}\frac{Y_2^2}{m_2} + \frac{k_2}{M_2}\frac{Y_1^2}{M_1}}{\frac{Y_2^2}{m_2} + \frac{Y_1^2}{M_1}} = \frac{\omega_2^2 + \omega_1^2\left(\frac{Y_2}{Y_1}\right)^2\frac{M_1}{M_2}}{1 + \left(\frac{Y_2}{Y_1}\right)^2\frac{M_1}{M_2}}$$

using the fixed-free modal parameters

$$\omega_1^2 = \frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2 \qquad \omega_2^2 = 9\frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2 \qquad Y_1 = \sin\frac{\pi}{2} = 1 \qquad Y_2 = \sin 3\frac{\pi}{2} = -1$$

$$M_1 = M_2$$

$$\omega_n^2 = \frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2\left[\frac{9+1}{1+1}\right] = 5\frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2 = \frac{5}{4}\frac{E}{\rho}\left(\frac{\pi}{L}\right)^2$$

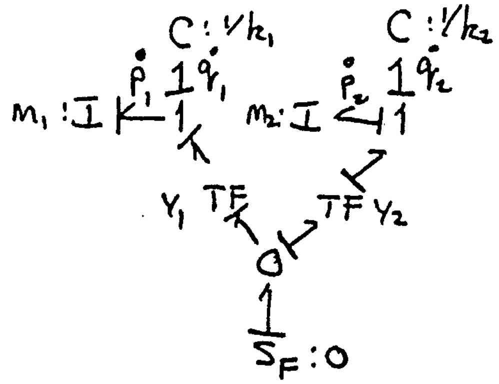
The actual fixed-fixed mode frequency is

$$\omega_{act}^2 = \frac{E}{\rho}\left(\frac{\pi}{L}\right)^2$$

thus our 2 mode model, based upon fixed-free modes yields

$$\omega_n = 1.12\sqrt{\frac{E}{\rho}}\,\frac{\pi}{L} \qquad \text{while } \omega_{Act} = \sqrt{\frac{E}{\rho}}\,\frac{\pi}{L} \qquad (12\% \text{ error})$$

B)



$$P_2 = \cancel{M_2}\,\frac{1}{Y_2}(-1)\,Y_1\,\frac{P_1}{\cancel{M_1}} = -\frac{Y_1}{Y_2}\,P_1$$ 10-9

$$\dot{p}_1 = -k_1 q_1 + \frac{Y_1}{Y_2}\left(k_2 q_2 + \dot{p}_2\right)$$

$$= -k_1 q_1 + \frac{Y_1}{Y_2} k_2 q_2 - \left(\frac{Y_1}{Y_2}\right)^2 \dot{p}_1$$

(1) $$\dot{p}_1 = \frac{-k_1}{1+\left(\frac{Y_1}{Y_2}\right)^2}\,q_1 + \frac{Y_1/Y_2}{1+\left(\frac{Y_1}{Y_2}\right)^2}\,k_2 q_2$$

(2) $\dot{q}_1 = +\frac{p_1}{M_1}$ (3) $\dot{q}_2 = \frac{1}{Y_2}(-1)\,Y_1\,\frac{p_1}{M_1}$

in Laplace domain

$$\begin{bmatrix} s & \frac{k_1}{1+\left(\frac{Y_1}{Y_2}\right)^2} & -\frac{k_2 Y_1/Y_2}{1+\left(\frac{Y_1}{Y_2}\right)^2} \\ -1/M_1 & s & 0 \\ \frac{Y_1}{Y_2}\frac{1}{M_1} & 0 & s \end{bmatrix} \begin{bmatrix} p_1 \\ q_1 \\ q_2 \end{bmatrix} = 0$$

expand

$$s^3 + \frac{k_1}{1+\left(\frac{Y_1}{Y_2}\right)^2}\frac{s}{M_1} + \frac{\left(\frac{Y_1}{Y_2}\right)^2}{1+\left(\frac{Y_1}{Y_2}\right)^2}\frac{k_2}{M_1}\,s = 0$$

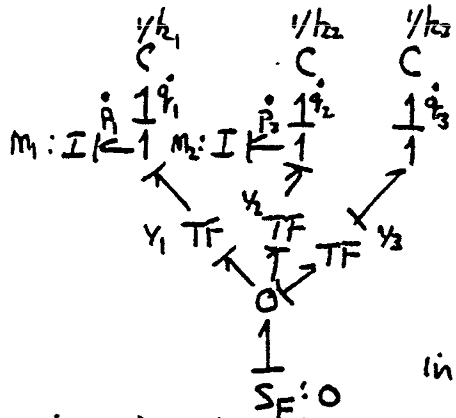
$$s\left[s^2 + \frac{k_1/M_1}{1+\left(\frac{Y_1}{Y_2}\right)^2} + \frac{k_2}{M_1}\,\frac{(Y_1/Y_2)^2}{1+(Y_1/Y_2)^2}\right] = 0$$

thus

$$\omega_n^2 = \frac{\omega_1^2 + \left(\frac{Y_1}{Y_2}\right)^2 \frac{M_2}{M_1}\omega_2^2}{1+\left(\frac{Y_1}{Y_2}\right)^2}$$

← this result is identical to part A for $M_1/M_2 = 1$

(C)



$$\dot{p}_1 = -k_1 q_1 + \frac{Y_1}{Y_3} k_3 q_3$$

10-10

$$\dot{p}_2 = -k_2 q_2 + \frac{Y_2}{Y_3} k_3 q_3$$

$$\dot{q}_1 = \frac{p_1}{m_1} \quad \dot{q}_2 = \frac{p_2}{m_2} \quad \dot{q}_3 = \frac{1}{Y_3}\left[-Y_1 \frac{p_1}{m_1} - Y_2 \frac{p_2}{m_2}\right]$$

in Laplace domain

$$\begin{bmatrix} s & 0 & k_1 & 0 & -\frac{Y_1}{Y_3}k_3 \\ 0 & s & 0 & k_2 & -\frac{Y_2}{Y_3}k_3 \\ -1/m_1 & 0 & s & 0 & 0 \\ 0 & -1/m_2 & 0 & s & 0 \\ \frac{Y_1}{Y_3}\frac{1}{m_1} & \frac{Y_2}{Y_3}\frac{1}{m_2} & 0 & 0 & s \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

here we go again

$$s \begin{vmatrix} s & 0 & k_2 & -\frac{Y_2}{Y_3}k_3 \\ 0 & s & 0 & 0 \\ -1/m_2 & 0 & s & 0 \\ \frac{Y_2}{Y_3}\frac{1}{m_2} & 0 & 0 & s \end{vmatrix} + k_1 \begin{vmatrix} 0 & s & k_2 & -\frac{Y_2}{Y_3}k_3 \\ -1/m_1 & 0 & 0 & 0 \\ 0 & -1/m_2 & s & 0 \\ \frac{Y_1}{Y_3}\frac{1}{m_1} & \frac{Y_2}{Y_3}\frac{1}{m_2} & 0 & s \end{vmatrix} - \frac{Y_1}{Y_3}k_3 \begin{vmatrix} 0 & s & 0 & k \\ -1/m_1 & 0 & s & c \\ 0 & -1/m_2 & 0 & s \\ \frac{Y_1}{Y_3}\frac{1}{m_1} & \frac{Y_2}{Y_3}\frac{1}{m_2} & 0 & 0 \end{vmatrix}$$

again

$$s^2 \begin{vmatrix} s & k_2 & -\frac{Y_2}{Y_3}k_3 \\ -1/m_2 & s & 0 \\ \frac{Y_2}{Y_3}\frac{1}{m_2} & 0 & s \end{vmatrix} + \frac{k_1}{m_1} \begin{vmatrix} s & k_2 & -\frac{Y_2}{Y_3}k_3 \\ -1/m_2 & s & 0 \\ \frac{Y_2}{Y_3}\frac{1}{m_2} & 0 & s \end{vmatrix} + \frac{Y_1}{Y_3}k_3 s \begin{vmatrix} 0 & s & k_2 \\ 0 & -1/m_2 & s \\ \frac{Y_1}{Y_3}\frac{1}{m_1} & \frac{Y_2}{Y_3}\frac{1}{m_2} & 0 \end{vmatrix}$$

again

$$\left[s^5 + s^3\frac{k_2}{m_2} + \left(\frac{Y_2}{Y_3}\right)^2 \frac{k_3}{m_2} s^3\right] + \left[\frac{k_1}{m_1}s^3 + \frac{k_1}{m_1}\frac{k_2}{m_2}s + \frac{k_1}{m_1}\left(\frac{Y_2}{Y_3}\right)^2\frac{k_3}{m_2}s\right] + \left(\frac{Y_1}{Y_3}\right)^2 \frac{k_3}{m_1} s\left(s^2 + \frac{k_2}{m_2}\right) = 1$$

or $$s\left[s^4 + \left(\frac{k_2}{m_2} + \left(\frac{Y_2}{Y_3}\right)^2\frac{k_3}{m_2} + \frac{k_1}{m_1} + \left(\frac{Y_1}{Y_3}\right)^2\frac{k_3}{m_1}\right)s^2 + \frac{k_1}{m_1}\frac{k_2}{m_2} + \left(\frac{Y_2}{Y_3}\right)^2\frac{k_1}{m_1}\frac{k_3}{m_2} + \left(\frac{Y_1}{Y_3}\right)^2\frac{k_3}{m_1}\frac{k_2}{m_2}\right] = 0$$

to obtain the predicted natural frequencies, let $s \to j\omega$
and noting that $M_1 = M_2 = M_3$

$$\omega^4 - \left[\omega_2^2 + \omega_1^2 + \omega_3^2\left(\left(\frac{Y_2}{Y_3}\right)^2 + \left(\frac{Y_1}{Y_3}\right)^2\right)\right]\omega^2 + \omega_1^2\omega_2^2 + \omega_3^2\left(\left(\frac{Y_2}{Y_3}\right)^2\omega_1^2 + \left(\frac{Y_1}{Y_3}\right)^2\omega_2^2\right) = 0$$

For fixed-free modes, $Y_1 = \sin\frac{\pi}{2} = 1 \quad Y_2 = \sin 3\frac{\pi}{2} = -1 \quad Y_3 = \sin 5\frac{\pi}{2} = 1$

$$\omega_1^2 = \frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2 \qquad \omega_2^2 = 9\frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2 \qquad \omega_3^2 = 25\frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2$$

use in above,

$$\omega^4 - \left[1 + 9 + 25(1+1)\right]\frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2\omega^2 + \left[9 + 25(1+9)\right]\left(\frac{E}{\rho}\right)^2\left(\frac{\pi}{2L}\right)^4 = 0$$

$$\omega^4 - 60\frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2\omega^2 + 259\left(\frac{E}{\rho}\right)^2\left(\frac{\pi}{2L}\right)^4 = 0$$

$$\omega_{1,2}^2 = \left[30 \pm \sqrt{900 - 259}\right]\frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2$$

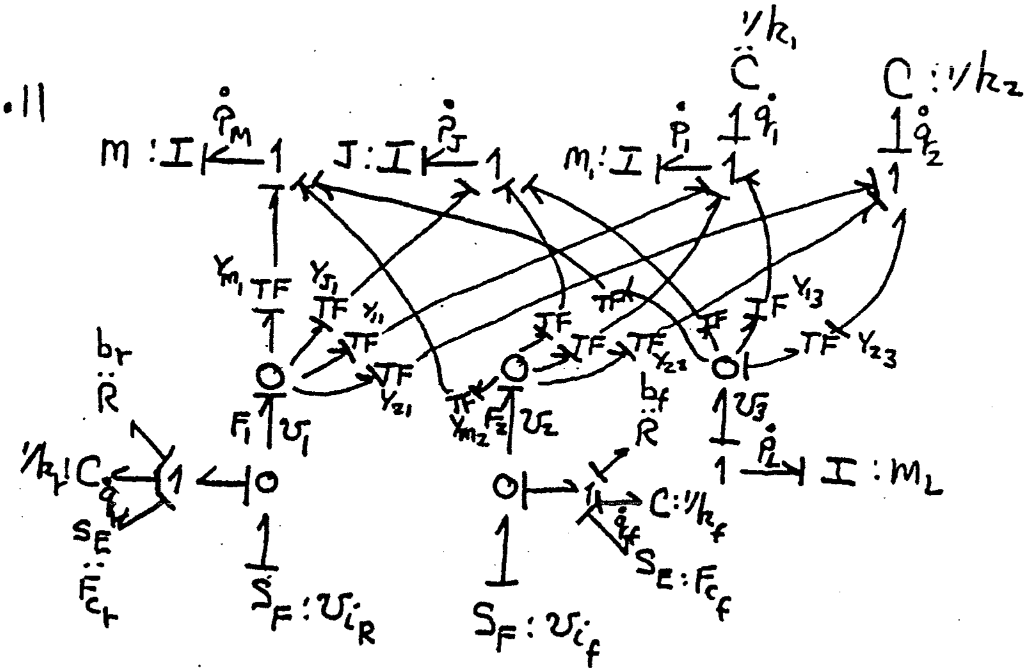
$$\left.\begin{aligned} \omega_1^2 &= 4.68\frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2 & \omega_1 &= 1.08\sqrt{\frac{E}{\rho}}\frac{\pi}{L} \\ \omega_2^2 &= 55.32\frac{E}{\rho}\left(\frac{\pi}{2L}\right)^2 & \omega_2 &= 3.72\sqrt{\frac{E}{\rho}}\frac{\pi}{L} \end{aligned}\right\} \text{predicted from fixed-free modes.}$$

actual fixed-fixed frequencies

$\omega_{1_a} = \sqrt{\frac{E}{\rho}}\frac{\pi}{L}$ 8% error in first freq.

$\omega_{2_a} = 2\sqrt{\frac{E}{\rho}}\frac{\pi}{L}$ 86% error in second freq.

10.11 10-12



$$\mathring{p}_m = Y_{m_1}\left(F_{c_r} + k_r q_r + b_r(v_{i_r} - v_1)\right) + Y_{m_2}\left(F_{c_f} + k_f q_f + b_f(v_{i_f} - v_2)\right) + \frac{Y_{13}}{Y_{23}}\left(k_2 q_2 - Y_{22}F_2 - Y_{21}F_1\right)$$

$$\mathring{p}_J = Y_{J_1}F_1 + Y_{J_2}F_2 + \frac{Y_{J_3}}{Y_{23}}\left(k_2 q_2 - Y_{22}F_2 - Y_{21}F_1\right)$$

$$\mathring{p}_1 = k_1 q_1 + Y_{11}F_1 + Y_{12}F_2 + \frac{Y_{13}}{Y_{23}}\left(k_2 q_2 - Y_{22}F_2 - Y_{21}F_1\right)$$

$$\mathring{p}_L = (-1)\frac{1}{Y_{23}}\left(k_2 q_2 - Y_{22}F_2 - Y_{21}F_1\right)$$

$$\mathring{q}_1 = p_1/m_1 \qquad \mathring{q}_2 = \frac{1}{Y_{23}}\left(\frac{p_L}{m_L} - Y_{13}\frac{p_1}{m_1} - Y_{J_3}\frac{p_J}{J} - Y_{m_3}\frac{p_m}{m}\right)$$

$$\mathring{q}_r = v_{i_R} - v_1$$

$$\mathring{q}_f = v_{i_f} - v_2$$

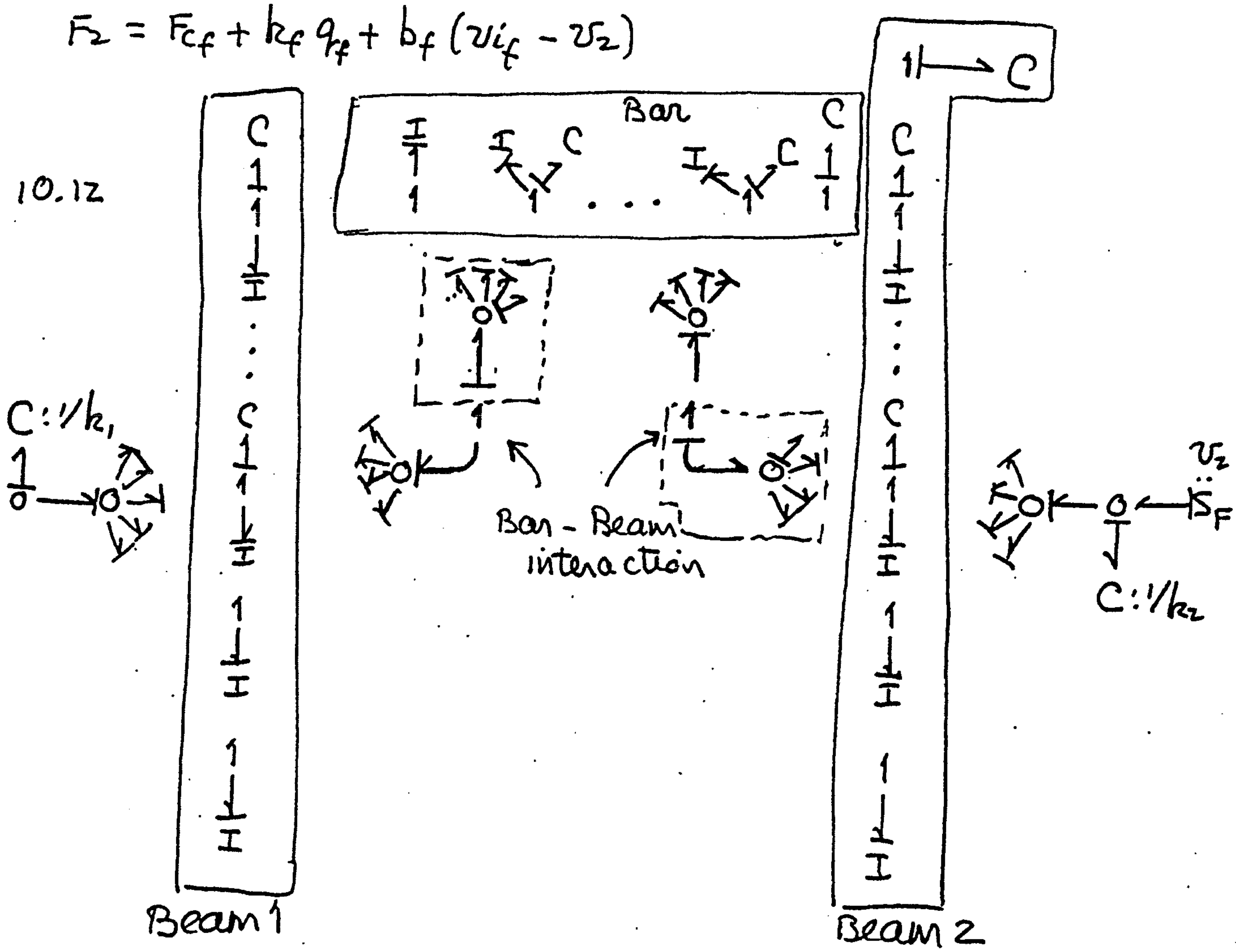
$$v_1 = Y_{m_1}\frac{p_m}{m} + Y_{J_1}\frac{p_J}{J} + Y_{11}\frac{p_1}{m_1} + \frac{Y_{21}}{Y_{23}}\left(\frac{p_L}{m_L} - Y_{13}\frac{p_1}{m_1} - Y_{J3}\frac{p_J}{J} - Y_{m_3}\frac{p_m}{m}\right)$$

$$v_2 = Y_{m_2}\frac{p_m}{m} + Y_{J_2}\frac{p_J}{J} + Y_{12}\frac{p_1}{m_1} + \frac{Y_{22}}{Y_{23}}\left(\frac{p_L}{m_L} - Y_{13}\frac{p_1}{m_1} - Y_{J_3}\frac{p_J}{J} - Y_{m_3}\frac{p_m}{m}\right)$$

$$F_1 = F_{c_r} + k_r q_r + b_r (v_{i_r} - v_1)$$

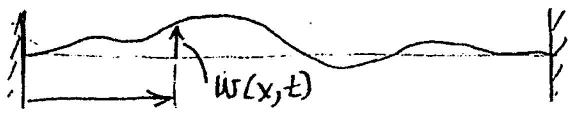
$$F_2 = F_{c_f} + k_f q_f + b_f (v_{i_f} - v_2)$$

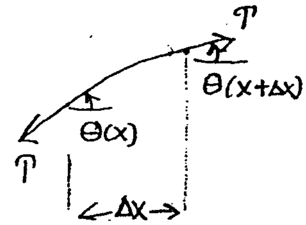
10.12



causal flow input at left end of bar dictates additional C- element in bar modes. Causal flow input into beam 2 dictates additional C-element in beam modes.

10.13





$$T\,\theta(x+\Delta x) - T\,\theta(x) = \rho \Delta x \frac{\partial^2 w}{\partial t^2}$$

$$\theta(x+\Delta x) = \theta(x) + \frac{\partial \theta}{\partial x}\Delta x \quad \text{and} \quad \theta \sim \frac{\partial w}{\partial x}$$

$\therefore$

$$T\frac{\partial^2 w}{\partial x^2} = \rho \frac{\partial^2 w}{\partial t^2} \quad \leftarrow \text{eqn. of motion}$$

Let

$$w(x,t) = Y(x)\cdot f(t)$$

$$T\frac{d^2 Y}{dx^2} f = \rho \frac{d^2 f}{dt^2} Y \quad \text{or} \quad \frac{T}{\rho}\frac{d^2 Y}{dx^2}\frac{1}{Y} = \frac{d^2 f}{dt^2}\frac{1}{f} = -\omega^2$$

or $$\frac{d^2 Y}{dx^2} + \frac{\rho \omega^2}{T} Y = 0$$

or $$\frac{d^2 Y}{dx^2} + \beta^2 Y = 0 \qquad \beta^2 = \frac{\rho}{T}\omega^2$$

$$Y(x) = A\cos\beta x + B\sin\beta x$$

apply boundary conditions

$$\left.\begin{array}{l} 0 = A \\ 0 = B\sin\beta L \end{array}\right\} \quad \sin\beta_n L = 0 \qquad \beta_n L = n\pi \quad n = 1, 2, 3, \dots$$

$$\therefore \quad \omega_n^2 = \frac{T}{\rho}\beta_n^2 = \frac{T}{\rho}\left(\frac{n\pi}{L}\right)^2$$

$$Y_n(x) = B_n \sin n\pi \frac{x}{L}$$

10.13 (continued)

The modal mass, m_n, is

$$m_n = \int_0^L \rho Y_n^2(x)\,dx = \rho \int_0^L B_n^2 \sin^2 \frac{n\pi x}{L}\,dx = \frac{\rho L B_n^2}{2}$$

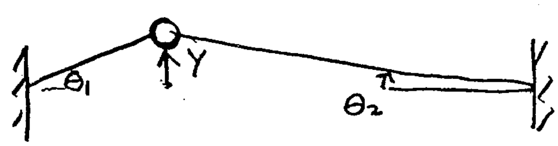
$$k_n = m_n \omega_n^2 = \frac{\rho L}{2} B_n^2 \, \frac{T}{\rho}\left(\frac{n\pi}{L}\right)^2$$

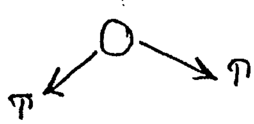
B_n is arbitrary. We usually choose $B_n = 1$, or perform some normalizing operation such as,

$$m_n = 1 = \frac{\rho L}{2} B_n^2 \quad \leftarrow \text{let all modal masses} = 1$$

$$B_n = \sqrt{\frac{2}{\rho L}}$$

10.14





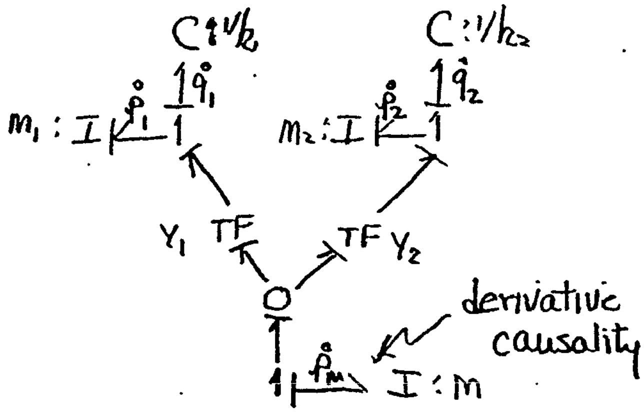
$$-T\theta_1 - T\theta_2 = m\ddot{y}$$

$$\theta_1 \sim \frac{y}{\frac{1}{4}L} \qquad \theta_2 \sim \frac{y}{\frac{3}{4}L}$$

$$\therefore \; -T\left[4\frac{y}{L} + \frac{4y}{3L}\right] = m\ddot{y}$$

$$m\ddot{y} + \frac{16}{3}\frac{T}{L}y = 0 \qquad \therefore \; \omega_n^2 = \frac{16}{3}\frac{T}{L}\frac{1}{m}$$

10.15



$$p_m = m\left[Y_1 \frac{p_1}{m_1} + Y_2 \frac{p_2}{m_2}\right]$$

$$\dot{p}_1 = -k_1 q_1 - Y_1 \dot{p}_M = -k_1 q_1 - Y_1\left[\frac{m}{m_1} Y_1 \dot{p}_1 + \frac{m}{m_2} Y_2 \dot{p}_2\right]$$

$$\dot{p}_2 = -k_2 q_2 - Y_2 \dot{p}_M = -k_2 q_2 - Y_2\left[\frac{m}{m_1} Y_1 \dot{p}_1 + \frac{m}{m_2} Y_2 \dot{p}_2\right]$$

$$\dot{q}_1 = p_1/m_1$$

$$\dot{q}_2 = p_2/m_2$$

in the s-domain

$$\begin{bmatrix} \left[1+\frac{m}{m_1}Y_1^2\right]s & \frac{m}{m_2}Y_1Y_2 s & k_1 & 0 \\ \frac{m}{m_1}Y_1Y_2 s & \left[1+\frac{m}{m_2}Y_2^2\right]s & 0 & k_2 \\ -1/m_1 & 0 & s & 0 \\ 0 & -1/m_2 & 0 & s \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{bmatrix} = 0$$

derive characteristic eqn:

$$\left(1+\frac{m}{m_1}Y_1^2\right)s^2\left[\left(1+\frac{m}{m_2}Y_2^2\right)s^2 + \frac{k_2}{m_2}\right] - \frac{m}{m_1}(Y_1Y_2)^2 s^4 \frac{m}{m_2} + \frac{k_1}{m_1}\left[\left(1+\frac{m}{m_2}Y_2^2\right)s^2 + \frac{k_2}{m_2}\right] = 0$$

$$\left[\left(1+\frac{m}{m_1}Y_1^2\right)\left(1+\frac{m}{m_2}Y_2^2\right) - \frac{m}{m_1}Y_1^2Y_2^2\frac{m}{m_2}\right]s^4 + \left[\left(1+\frac{m}{m_1}Y_1^2\right)\frac{k_2}{m_2} + \left(1+\frac{m}{m_2}Y_2^2\right)\frac{k_1}{m_1}\right]s^2 + \frac{k_1}{m_1}\frac{k_2}{m_2} = 0$$

$$\left[1+\frac{m}{m_1}Y_1^2 + \frac{m}{m_2}Y_2^2\right]s^4 + \left[\left(1+\frac{m}{m_1}Y_1^2\right)\frac{k_2}{m_2} + \left(1+\frac{m}{m_2}Y_2^2\right)\frac{k_1}{m_1}\right]s^2 + \frac{k_1}{m_1}\frac{k_2}{m_2} = 0$$

10.15 (continued)

let $\frac{k_1}{m_1} = \omega_1^2$, $\frac{k_2}{m_2} = \omega_2^2$, $m_1 = m_2 = \frac{m_s}{2} = m_m$, m_s = mass of string $= \rho L$

let $s = j\omega$ to introduce frequency eqn:

$$\omega^4 - \overbrace{\frac{\left(1+\frac{M}{M_M}Y_1^2\right)\omega_2^2 + \left(1+\frac{M}{M_M}Y_2^2\right)\omega_1^2}{1+\frac{M}{M_M}Y_1^2+\frac{M}{M_M}Y_2^2}}^{b}\,\omega^2 + \overbrace{\frac{\omega_1^2\omega_2^2}{1+\frac{M}{M_M}Y_1^2+\frac{M}{M_M}Y_2^2}}^{c} = 0$$

$$\omega^4 - b\omega^2 + c = 0$$

$$\omega^2 = +\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c} = \frac{b}{2} \pm \frac{b}{2}\sqrt{1-4\frac{c}{b^2}} = \frac{b}{2}\left\{1 \pm \sqrt{1-4\frac{c}{b^2}}\right\}$$

Lets do a little trick here. It is likely that $\frac{M}{M_M} \gg 1$, ie the mass element is more massive than the string. Then, as long as Y_1 and Y_2 are not zero (which they are not at $x=\frac{L}{4}$), it is reasonable to have,

$$b = \frac{Y_1^2\omega_2^2 + Y_2^2\omega_1^2}{Y_1^2+Y_2^2} \qquad c = \frac{\omega_1^2\omega_2^2}{\frac{M}{M_M}\left(Y_1^2+Y_2^2\right)}$$

Then,

$$\omega^2 = \frac{b}{2}\left\{1 \pm \sqrt{1 - \underbrace{\frac{4\,\omega_1^2\omega_2^2 \quad \left(Y_1^2+Y_2^2\right)^2}{\frac{M}{M_M}\left(Y_1^2+Y_2^2\right)\left(Y_1^2\omega_2^2+Y_2^2\omega_1^2\right)^2}}_{\epsilon}}\right\}$$

The next trick is that if $\frac{M}{M_M} \gg 1$, the radical looks like

$$\sqrt{1-\epsilon} \approx 1 - \frac{\epsilon}{2} \quad \text{for } \epsilon \ll 1$$

10.15 (continued)

Then

$$\omega^2 = \frac{b}{2}\left\{1 \pm \left(1 - \frac{\epsilon}{2}\right)\right\}$$

and the lowest frequency would be,

$$\omega^2 = \frac{b}{2}\,\frac{\epsilon}{2} \doteq \frac{1}{\cancel{4}}\,\frac{Y_1^2\omega_2^2 + Y_2^2\omega_1^2}{\cancel{Y_1^2+Y_2^2}} \cdot \frac{\cancel{4}\,\omega_1^2\omega_2^2\,\cancel{(Y_1^2+Y_2^2)}}{\frac{M}{M_M}\,(Y_1^2\omega_2^2+Y_2^2\omega_1^2)^2}$$

$$\omega^2 = \frac{1}{\frac{M}{M_m}}\,\frac{\omega_1^2\omega_2^2}{(Y_1^2\omega_2^2+Y_2^2\omega_1^2)}$$

$$\omega_1^2 = \frac{T}{\rho}\frac{\pi^2}{L^2} \qquad \omega_2^2 = \frac{T}{\rho}\frac{4\pi^2}{L^2} \qquad Y_1\left(\frac{L}{4}\right) = \sin\frac{\pi}{4} = 0.71 \qquad M_M = \frac{\rho L}{2}$$

(see soln to Prob 10.13)

$$Y_2\left(\frac{L}{4}\right) = \sin 2\frac{\pi}{4} = 1.0$$

Then

$$\omega^2 = \frac{1}{M}\,\frac{\rho L}{2}\left(\frac{T}{\rho}\right)^2 4\frac{\pi^4}{L^4}\;\frac{1}{(0.71)^2\frac{T}{\rho}4\frac{\pi^2}{L^2} + \frac{T}{\rho}\frac{\pi^2}{L^2}}$$

$$= \frac{1}{M}\,2\,\cancel{\rho}\,\cancel{L}\,\frac{T}{\cancel{\rho}}\,\frac{\pi^2}{L^{\cancel{2}}}\;\frac{1}{1+4(0.71)^2}$$

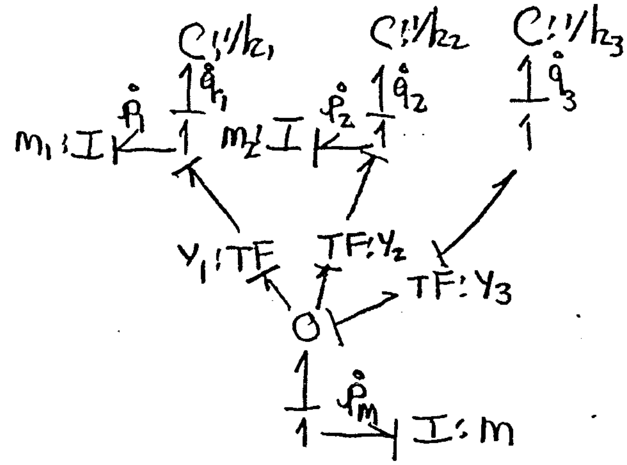
$$= 2\frac{T}{ML}\,\frac{\pi^2}{1+4(0.71)^2} = 6.54\,\frac{T}{ML}$$

$$\therefore\quad \omega = 2.56\sqrt{\frac{T}{ML}}$$

From Prob 10.14

$$\omega_n = 2.31\sqrt{\frac{T}{ML}}$$

10.16



no derivative causality

$$\dot{p}_1 = -k_1 q_1 + \frac{Y_1}{Y_3} k_3 q_3$$

$$\dot{p}_2 = -k_2 q_2 + \frac{Y_2}{Y_3} k_3 q_3$$

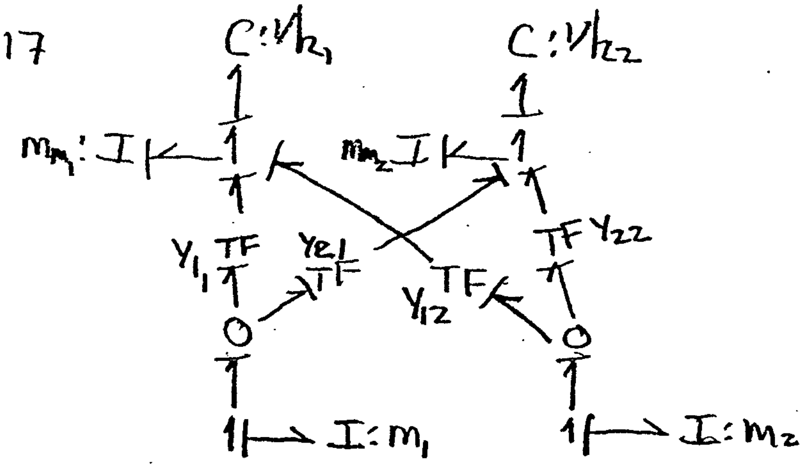
$$\dot{p}_M = -\frac{1}{Y_3} k_3 q_3$$

$$\dot{q}_1 = p_1/m_1$$

$$\dot{q}_2 = p_2/m_2$$

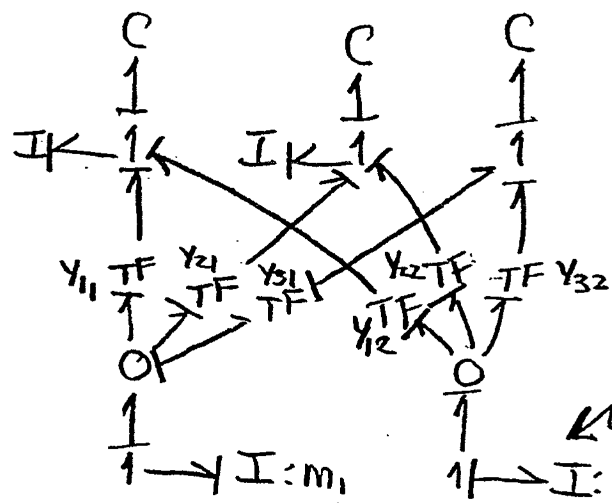
$$\dot{q}_3 = \frac{1}{Y_3}\left[\frac{p_M}{m} - Y_1 \frac{p_1}{m_1} - Y_2 \frac{p_2}{m_2}\right]$$

10.17



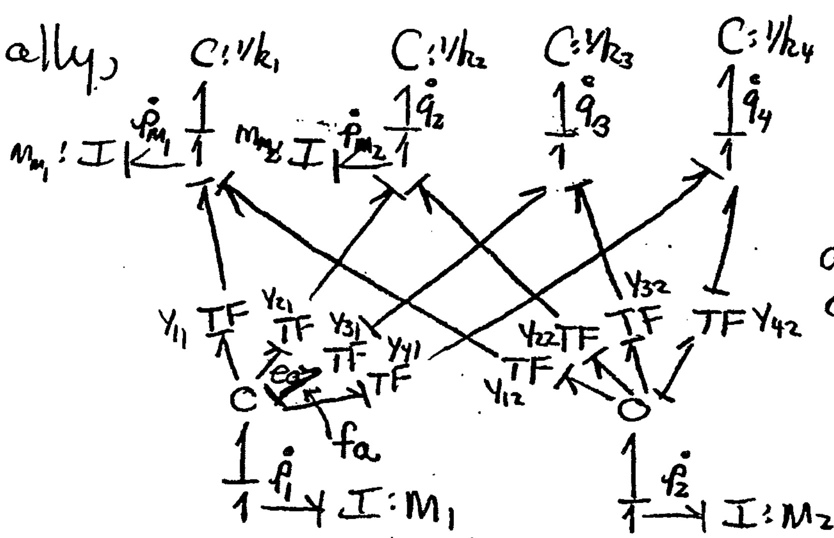
Both masses in derivative causality

Append additional modal compliance



only 1 mass in derivative causality.

Finally,



bond with e_a was arbitrarily assigned, thus, algebraic loop

10.17 (continued)

can relate e_a to itself, and f_a to itself

$$e_a = \frac{1}{Y_{31}}\left[k_3 q_3 - \frac{Y_{32}}{Y_{42}}(k_4 q_4 - Y_{41} e_a)\right].$$

$$e_a\left[1 - \frac{Y_{32}Y_{41}}{Y_{31}Y_{42}}\right] = \frac{1}{Y_{31}} k_3 q_3 - \frac{Y_{32}}{Y_{31}Y_{42}} k_4 q_4$$

$$f_a = \frac{p_1}{M_1} - Y_{11}\frac{p_{M1}}{M_{M1}} - Y_{21}\frac{p_{M2}}{M_{M2}} - \frac{Y_{41}}{Y_{42}}\left(\frac{p_2}{M_2} - Y_{12}\frac{p_{M1}}{M_{M1}} - Y_{22}\frac{p_{M2}}{M_{M2}} - \frac{Y_{32}}{Y_{31}} f_a\right)$$

solve for f_a

Derive eqns.

$$\dot{p}_{M_1} = -k_1 q_1 + Y_{11} e_a + \frac{Y_{12}}{Y_{42}}(k_4 q_4 - Y_{41} e_a)$$

$$\dot{p}_{M_2} = -k_2 q_2 + Y_{21} e_a + \frac{Y_{22}}{Y_{42}}(k_4 q_4 - Y_{41} e_a)$$

$$\dot{p}_1 = -e_a$$

$$\dot{p}_2 = -\frac{1}{Y_{42}}(k_4 q_4 - Y_{41} e_a)$$

$$\dot{q}_1 = p_{M1}/M_{M_1}$$

$$\dot{q}_2 = p_{M2}/M_{M2}$$

$$\dot{q}_3 = \frac{1}{Y_{31}} f_a$$

$$\dot{q}_4 = \frac{1}{Y_{42}}\left[\frac{p_2}{M_2} - Y_{12}\frac{p_{M1}}{M_{M_1}} - Y_{22}\frac{p_{M2}}{M_{M2}} - \frac{Y_{32}}{Y_{31}} f_a\right]$$

substitute for e_a + f_a and a complete state representation exists.

11-1 $L = \dfrac{N^2}{R}$; $R = \left(\dfrac{l}{\mu A}\right)_{core} + \left(\dfrac{l}{\mu_0 A}\right)_{gap}$

11-2
$L = \dfrac{N^2}{R_{Total}}$ at 1-junctions R's add
at 0-junctions $\dfrac{1}{R\text{'s}} = p$'s add

$$R_{TOTAL} = R_1 + R_5 + \cfrac{1}{\cfrac{1}{R_3} + \cfrac{1}{R_2 + R_4 + R_6}}$$

where all R's are of the form $\left(\dfrac{l}{\mu A}\right)$.

11-3

$$\begin{tikzcd} & & C & & \\ {} \arrow[r] & GY \arrow[r] & 1 \arrow[u] \arrow[r, "M", "\dot{\varphi}"'] & C & {} \arrow[l, "F"', "\dot{x}"] \end{tikzcd}$$

In this case, $R = R(x)$ because the area for the flux changes with x rather than because the length of an air gap as in Fig. 11.9 This is a more gentle change in R so from Eq. (11.20), F will be smaller, but will act through a longer stroke. $R(x) \cong \dfrac{l_0}{\mu A(x)} = \dfrac{l_0}{\mu(A_0 - \frac{x}{w})}$
for $x < w$ = width of core.

11.4

$$l_g = 2\times10^{-3}\ m$$
$$A_g = \pi \cdot 30\times10^{-3} \cdot 10\times10^{-3}\ m^2$$
$$A_m = \frac{\pi\,(30\times10^{-3})^2}{4}\ m^2$$

$$l_m = \frac{A_m}{A_g} l_g \cdot \frac{B_m}{\mu_0 H_m}$$

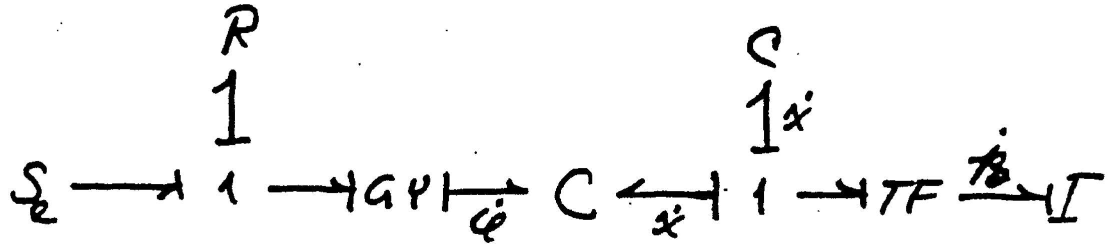
$$= \frac{\cancel{\pi}\,(30\times10^{-3})^{\cancel{2}}\,(2\times10^{-3})\ 1}{4\,\cancel{\pi}\,(\cancel{30}\times10^{-3})\times10\times10^{-3}}$$

$$= \frac{6}{4}\times10^{-3}\ m = \underline{\underline{1.5\ mm}}$$

$$B_g \cong \frac{A_m}{A_g}\cdot B_m = \frac{\pi\,(30\times10^{-3})^2\,(0.5)}{4\,\pi\,(30\times10^{-3})(10\times10^{-3})}\ T$$

$$\cong \frac{3}{4}\cdot(0.5) = \underline{\underline{0.375}}\ T$$

11.5



$$\dot\varphi = \frac{1}{N}\left[V_0(t) - R\cdot\frac{1}{N}\left(R_0+\frac{x}{\mu_0 A}\right)\varphi\right]$$

$$\dot x = l\, p_\theta / J_0$$

$$\dot p_\theta = l\left[-k(x-x_0) - \frac{\varphi^2}{2\mu_0 A}\right]$$

11.6

$$F=\frac{\varphi^2}{2\mu_0 A}, \quad \Delta M = NI = \mathbb{R}\varphi = \frac{x}{\mu_0 A}\varphi$$

$$\varphi = \frac{NI\mu_0 A}{x}, \quad F(x,I)=\frac{\mu_0 N^2 A}{2}\frac{I^2}{x^2}$$

11.7

Fig. 11.10 $\quad \xrightarrow[i]{e} GY \longrightarrow C \xleftarrow[\omega]{\tau}$

No permanent magnet - only variable reluctance effect.

Fig. 11.12 $\quad \xrightarrow[i]{e} MGY \longrightarrow \overset{\overset{C}{\uparrow}}{1} \xrightarrow[\dot{\theta}]{\tau}$

For round rotor, variable reluctance effect would be small.

Fig. 11.13a $\quad \xrightarrow[i]{e} GY \longrightarrow C \xleftarrow[V]{F}$

No permanent magnet

Fig. 11.13b $\quad \xrightarrow[i]{e} GY \xrightarrow[\dot{x}]{F}$

Permanent magnet doesn't move, reluctance constant, MGY necessary only for large motions.

Fig. 11.14 $\quad \xrightarrow[i]{e} MGY \longrightarrow \overset{\overset{C}{\uparrow}}{1} \xrightarrow[\dot{x}]{F}$

Reluctance does not change much with position. For moderate excursions MGY $\Rightarrow$ GY

11.8 $F = \frac{\varphi^2}{2\mu_0 A}$, $\varphi = BA$

$$\frac{F}{A} = \frac{B^2}{2\mu_0}$$

if $B = 0.5$, $A = 10^{-4}\,m^2$, $\mu_0 = 4\pi \times 10^{-7}\ Tm/A$

$$F = \frac{(0.5)^2 \times 10^{-4}}{2 \cdot 4\pi \times 10^{-7}} = 9.95\ N$$

$m = 1.02$ kg could be lifted

11.9 Maximum current

$$i_{max} = V_{max}/R = 10/8 = 1.25\ A$$

$$A_w = i_{max}/J = 1.25/20\times10^6$$

$$= 6.25\times10^{-8}\ m^2$$

$$l = \frac{R A_w}{\rho_{cu}} = \frac{8\cdot 6.25\times10^{-8}}{1.72\times10^{-8}} = 29.07\ m$$

$$F_{max} = B l i_{max} = 0.5\cdot 29.07\cdot 1.25 = 18.17\ N$$

or $$\boxed{F_m = \frac{BV^2}{R\rho_{cu}J} = 18.17\ N}$$

11.10 $\quad \mathbb{E} = \frac{1}{2}\mathbb{R}(\theta)\varphi^2$

$$M_2 = \frac{\partial \mathbb{E}}{\partial \varphi} = \mathbb{R}(\theta)\varphi$$

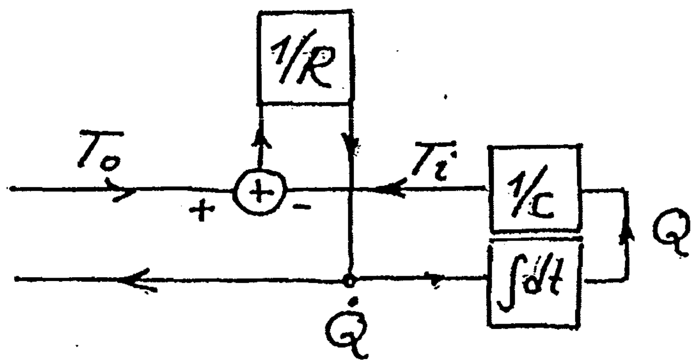
$$\tau = \frac{\partial \mathbb{E}}{\partial \theta} = \frac{1}{2}\varphi^2 \frac{d\mathbb{R}(\theta)}{d\theta}$$

$$M_1 = -M_2 = \left(\frac{\varphi}{\varphi_0} - 1\right)M_0 = -\mathbb{R}(\theta)\varphi$$

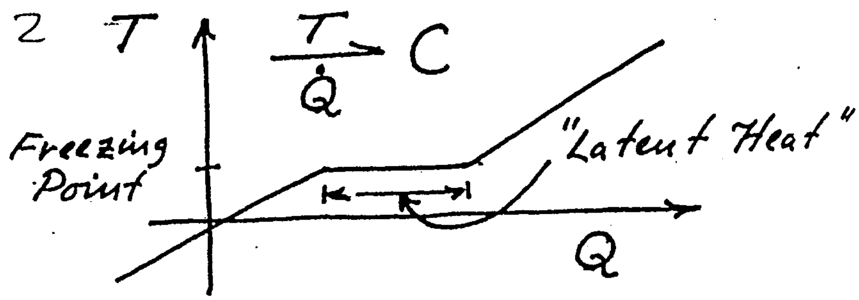
so $\quad \varphi = \dfrac{M_0 \varphi_0}{\varphi_0 \mathbb{R}(\theta) + M_0}$

$$\tau = \frac{1}{2}\frac{M_0^2 \varphi_0^2}{(\varphi_0 \mathbb{R}(\theta) + M_0)^2}\frac{d\mathbb{R}(\theta)}{d\theta}$$

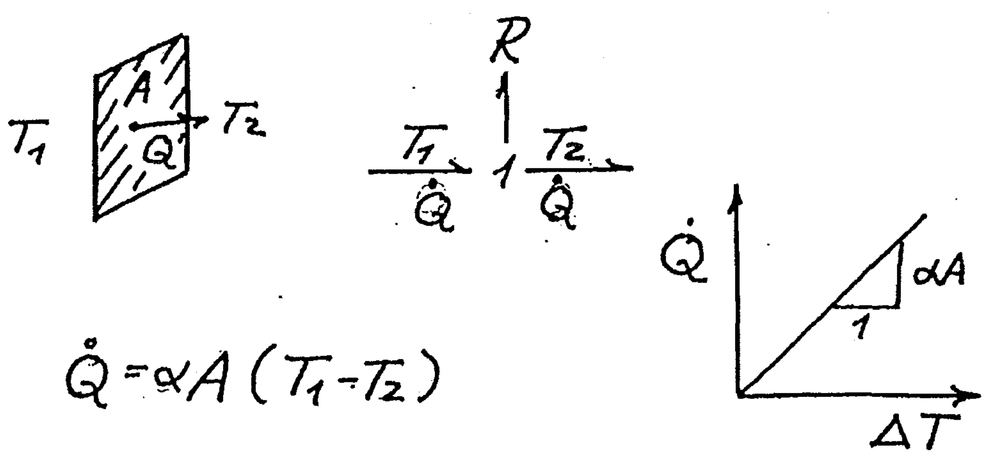
12-1



12-2



12-3



$$\dot{Q} = \alpha A (T_1 - T_2)$$

12- 4

$$C \xleftarrow[Q_3]{T_3}\!\!| 0 \xrightarrow[\dot{Q}_1]{T_1}\!| \overset{\overset{R}{\uparrow}}{1} |\!\xrightarrow[\dot{Q}_2]{T_2} C$$

Assume $\dot{Q}_1 = \dot{Q}_2 = -\dot{Q}_3 = H(T_1 - T_2)$ as in Eq. (9.92)

$T_1 = T_3 = T_{30} + \frac{Q_3}{C_3}$; $T_2 = T_{20} + \frac{Q_2}{C_2}$ as in Eq (9.98)

State Equ's. $\dot{Q}_2 = H(T_3 - T_2)$

$$= H\left(T_{30} + \frac{Q_3}{C_3} - T_{20} - \frac{Q_2}{C_2}\right)$$

$$\dot{Q}_3 = -H\left(T_{30} + \frac{Q_3}{C_3} - T_{20} - \frac{Q_2}{C_2}\right)$$

$$C \xleftarrow[\dot{S}_3]{T_3}\!\!| 0 \xrightarrow[\dot{S}_1]{T_1}\!| R |\!\xrightarrow[\dot{S}_2]{T_2} C$$

Corresponding assumptions: $\left.\begin{aligned}\dot{S}_2 &= \frac{(T_3 - T_2)\cdot H}{T_2} \\ \dot{S}_1 &= \frac{(T_3 - T_2)H}{T_1}\end{aligned}\right\}$ Eqn. (12.35)

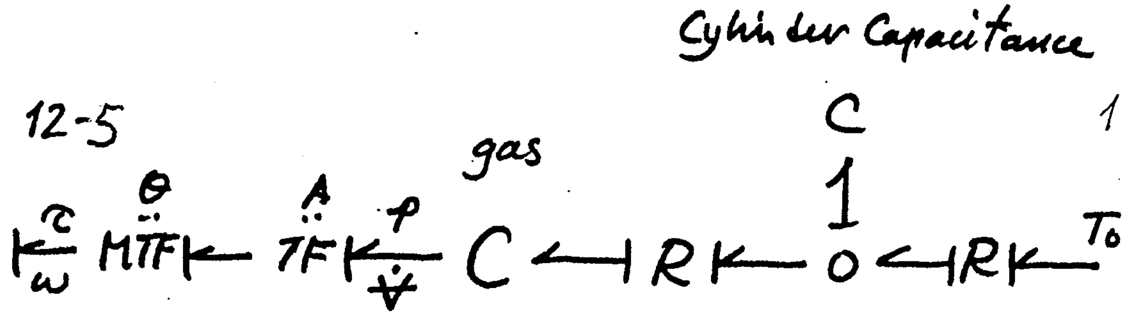
$\left.\begin{aligned}T_1 = T_3 &= T_{30}\, e^{S_3/C_3} \\ T_2 &= T_{20}\, e^{S_2/C_2}\end{aligned}\right\}$ Eqn. (9.100)

State Equs. $\dot{S}_2 = H\dfrac{(T_{30}\,e^{S_3/C_3} - T_{20}\,e^{S_2/C_2})}{T_{20}\,e^{S_2/C_3}}$

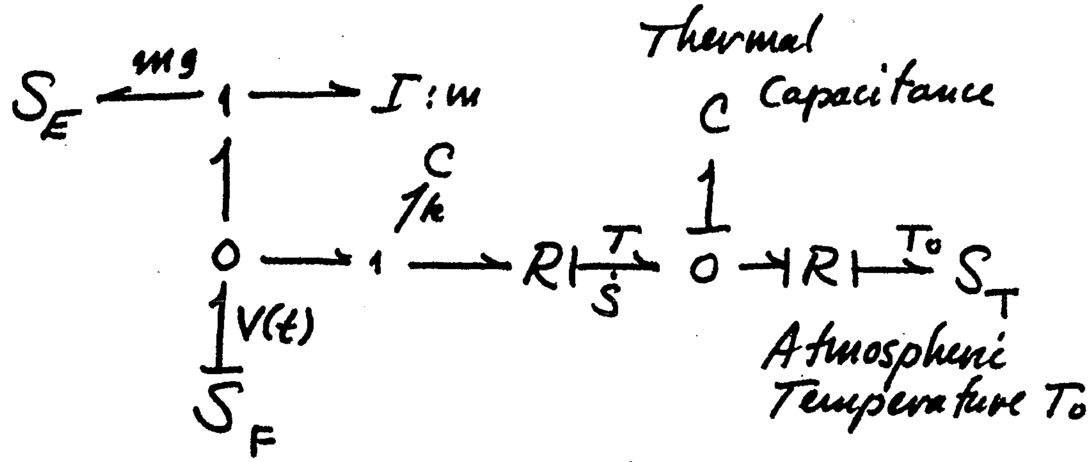
$$\dot{S}_3 = -H\frac{(T_{30}\,e^{S_3/C_3} - T_{20}\,e^{S_2/C_2})}{T_{30}\,e^{S_3/C_3}}$$

$\dot{S}_2 + \dot{S}_3 = \dot{S}_2 - \dot{S}_1 = \dfrac{(T_2 - T_3)^2 H}{T_2 T_3} \geqslant 0$ for any T_2, T_3

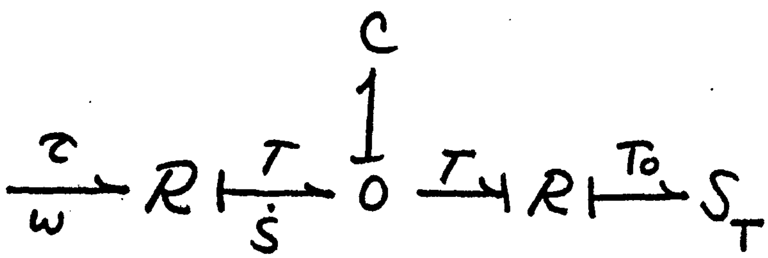
12-5



12-6



12-7



12-8 (a) $\rho V = \text{const.}$ $\quad \rho \Delta V + V \Delta \rho = 0$

$$\frac{\Delta \rho}{\rho} = -\frac{\Delta V}{V} \quad \text{so} \quad \Delta p = -B \frac{\Delta V}{V}$$

(b) $I = \rho A \ell, \quad C = A\ell / B$

(c) λ should be $> \ell$

or $f = \frac{c}{\lambda}$ should be $< \frac{c}{\ell}$

$\omega < 2\pi \frac{c}{\ell}$.

12-9 As Fig. 12.10 shows the inclusion of a Bernoulli resistor with $A_2 > A_1$ for inflow will give static power <u>recovery</u> i.e. the pressure acting on the mass would be greater than $P_0(t)$. To model complete dynamic pressure loss, simply leave the Bernoulli resistor out.

$$
\begin{tikzcd}
 & & I \\
\arrow[r, "P_0"] & \overset{A}{TF} \arrow[r] & 1 \arrow[u] \arrow[r, "F(t)"] & {}
\end{tikzcd}
$$

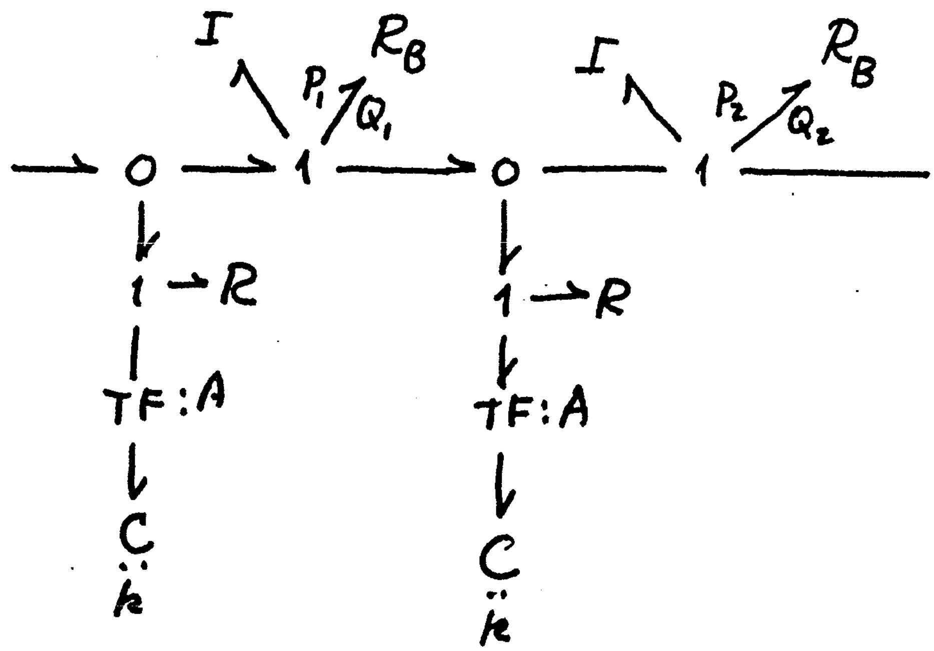
12-10

$$
\begin{tikzcd}
 & C & 1 \arrow[l] \arrow[r] & I & C & 1 \arrow[l] \arrow[r] & I & & R \\
{} \arrow[rr, "P_0"] & & 0 \arrow[u] \arrow[rrr] & & & 0 \arrow[u] \arrow[rrr] & & & 1 \arrow[u] \arrow[r, "P_L"] & {}
\end{tikzcd}
$$

$$I_i = \frac{\rho_0 l_i}{A_i} \; ; \quad C_i = \frac{V_i}{B} = \frac{V_i}{\rho_0 c^2}$$

12-11

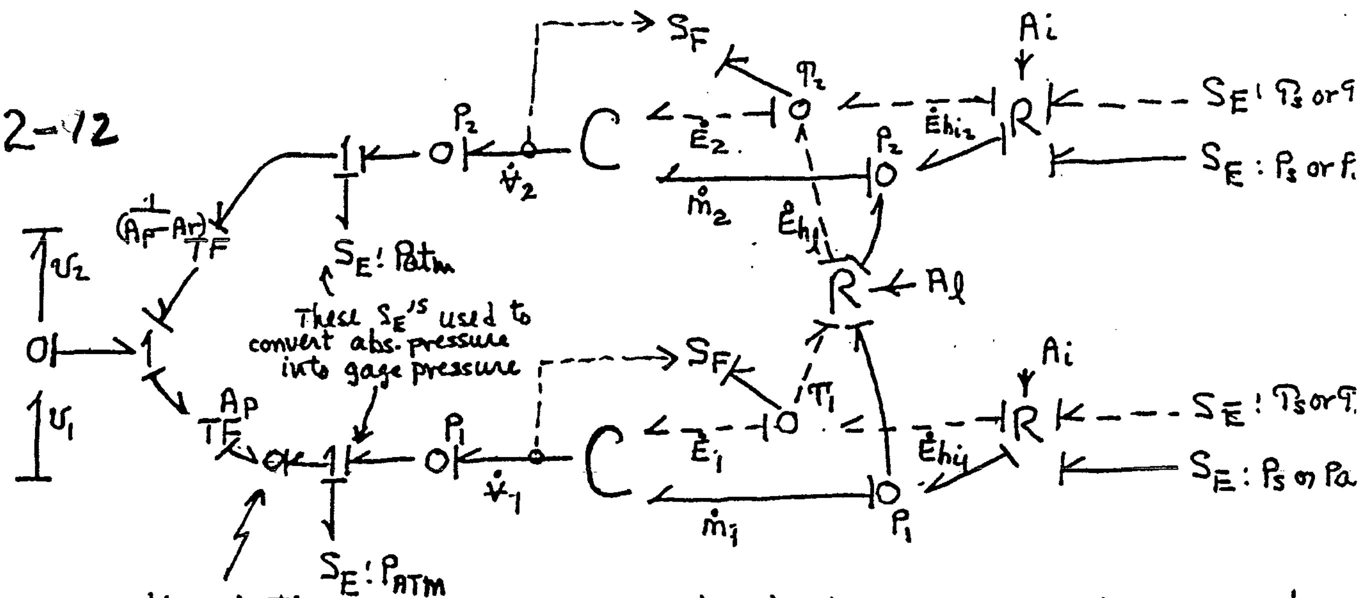
12-5



$$P_1 = \rho \frac{Q_1^2}{2}\left(\frac{1}{A_v^2} - \frac{1}{A_m^2}\right) \; ; \; P_2 = \rho \frac{Q_2^2}{2}\left(\frac{1}{A_m^2} - \frac{1}{A_v^2}\right)$$

when $Q_1 = Q_2$ $P_1 = -P_2$ and all dynamic pressure is recovered. For steady flow, the static pressure is less in the venturi than in the main pipe so the differing piston displacements are a measure of the flow.

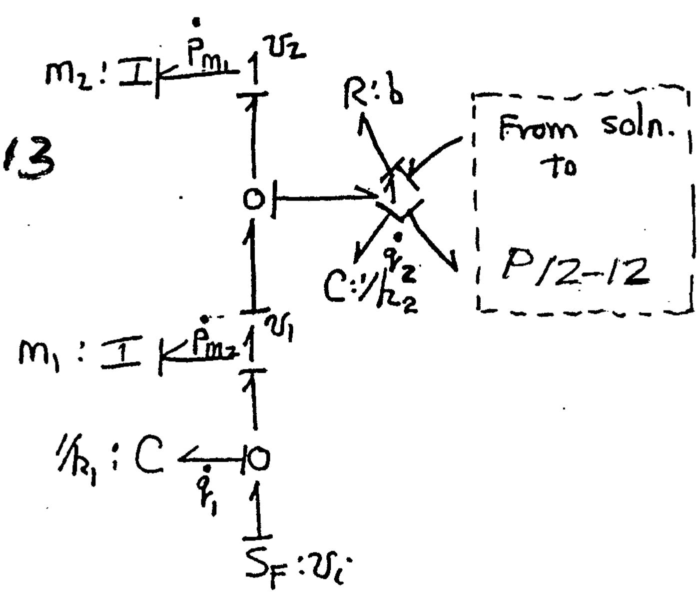
12-12



This o-junction used to change sign of volume compression in chamber 1

Note that we can only accept velocity inputs from the mechanical side

12-13



state eqns.

$$\dot{p}_{m_1} = k_2 q_2 + b\left(\frac{p_{m_1}}{m_1} - \frac{p_{m_2}}{m_2}\right) - (P_2 - P_{atm})(A_p - A_r) + (P_1 - P_{atm})A_p$$

$$\dot{p}_{m_2} = k_1 q_1 - k_2 q_2 - b\left(\frac{p_{m_1}}{m_1} - \frac{p_{m_2}}{m_2}\right) + (P_2 - P_{atm})(A_p - A_r) - (P_1 - P_{atm})A_p$$

$$\dot{q}_1 = v_i - p_{m_2}/m_2$$

$$\dot{q}_2 = \frac{p_{m_2}}{m_2} - \frac{p_{m_1}}{m_1}$$

where

$$P_2 = \frac{R}{C_v}\left(\frac{E_2}{\forall_2}\right)$$

$$P_1 = \frac{R}{C_v}\left(\frac{E_1}{\forall_1}\right)$$

and

$$\dot{\forall}_1 = -A_P\left(\frac{P\dot{m}_1}{m_1} - \frac{P\dot{m}_2}{m_2}\right)$$

$$\dot{\forall}_2 = (A_P - A_r)\left(\frac{P\dot{m}_1}{m_1} - \frac{P\dot{m}_2}{m_2}\right)$$

$$\dot{E}_1 = -P_1\dot{\forall}_1 + \dot{E}h_{i_1} - \dot{E}h_\ell$$

$$\dot{E}_2 = -P_2\dot{\forall}_2 + \dot{E}h_\ell + \dot{E}h_{i_2}$$

where $\dot{E}h_{i_1}$ and $\dot{E}h_{i_2}$ come from isentropic nozzle relations for intake

and $\dot{E}h_\ell$ comes from isentropic nozzle, 4 port R-element for leakage.

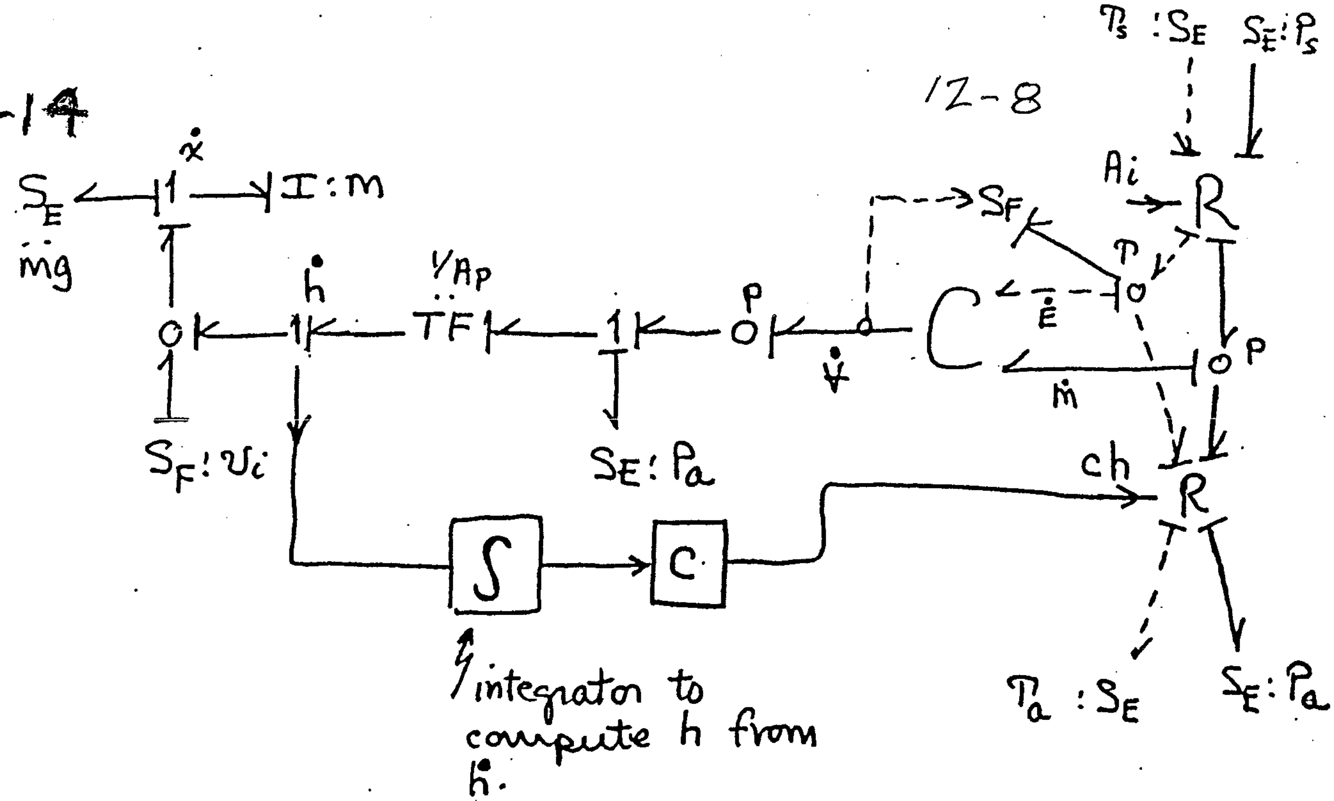
Isentropic nozzle relations need T_1 and T_2 where

$$T_1 = \frac{1}{C_v}\frac{E_1}{m_1} \qquad T_2 = \frac{1}{C_v}\frac{E_2}{m_2}$$

and $\dot{m}_1$ = [from 4 port R for intake 1]

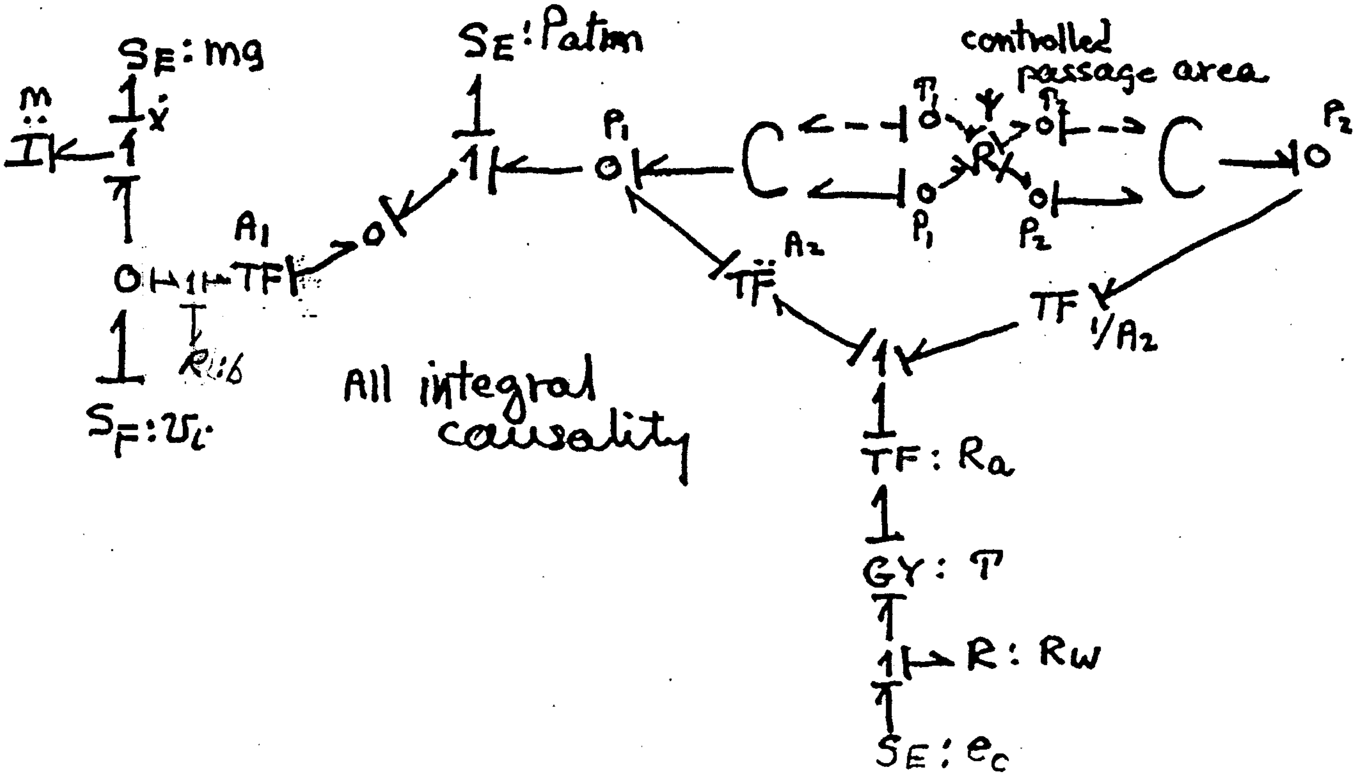
$\dot{m}_2$ = [from 4 port R for intake 2]

12-14



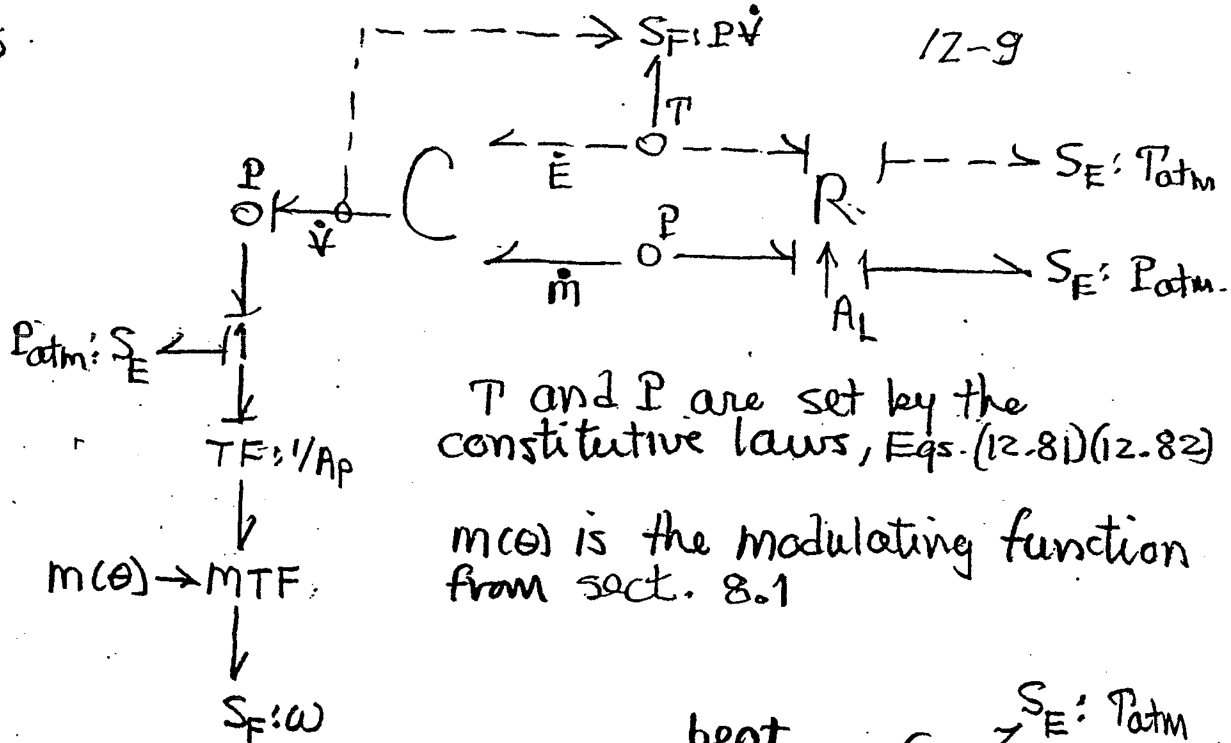
12-8
S_E
$\dot{m}g$
$\dot{x}$
1
I : m
0
$S_F : v_i$
$\dot{h}$
1
1/Ap
TF
1
$S_E : P_a$
P
0
$\dot{V}$
C
$\dot{E}$
$\dot{m}$
T
0
P
0
$T_s : S_E$
$S_E : P_s$
Ai
R
ch
R
$T_a : S_E$
$S_E : P_a$
S
C
integrator to compute h from $\dot{h}$.

12-15



S_E: mg
m
I
$\dot{x}$
1
0
1
$S_F : v_i$
Rub
A_1
TF
0
S_E: Patm
1
P_1
0
C
controlled passage area
T_1
T_2
0
R
0
P_1
P_2
C
P_2
0
A_2
TF
TF
1/A_2
1
TF : Ra
GY : T
1
R : Rw
$S_E : e_c$
All integral causality

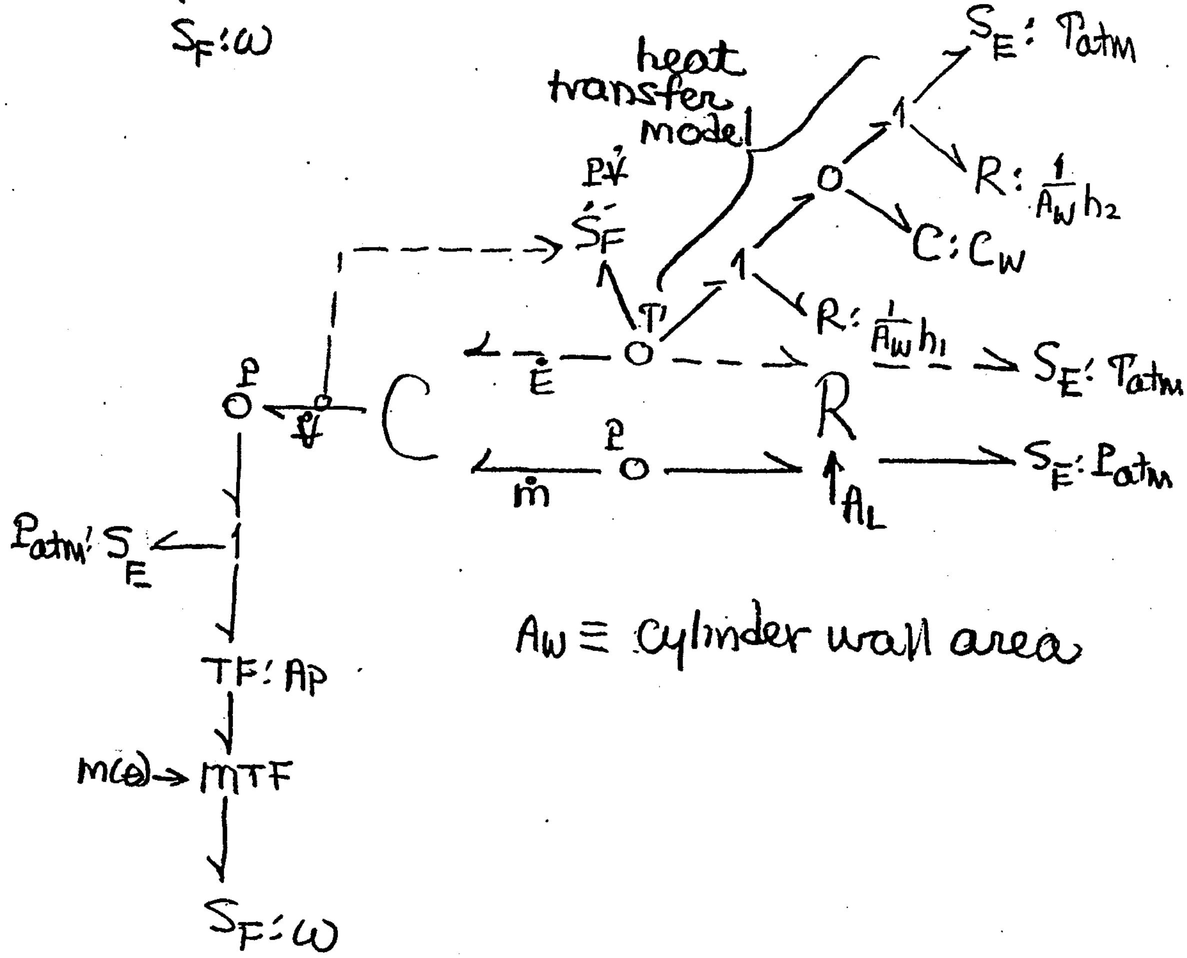
12-16



T and P are set by the constitutive laws, Eqs. (12.81)(12.82)

m(θ) is the modulating function from sect. 8.1

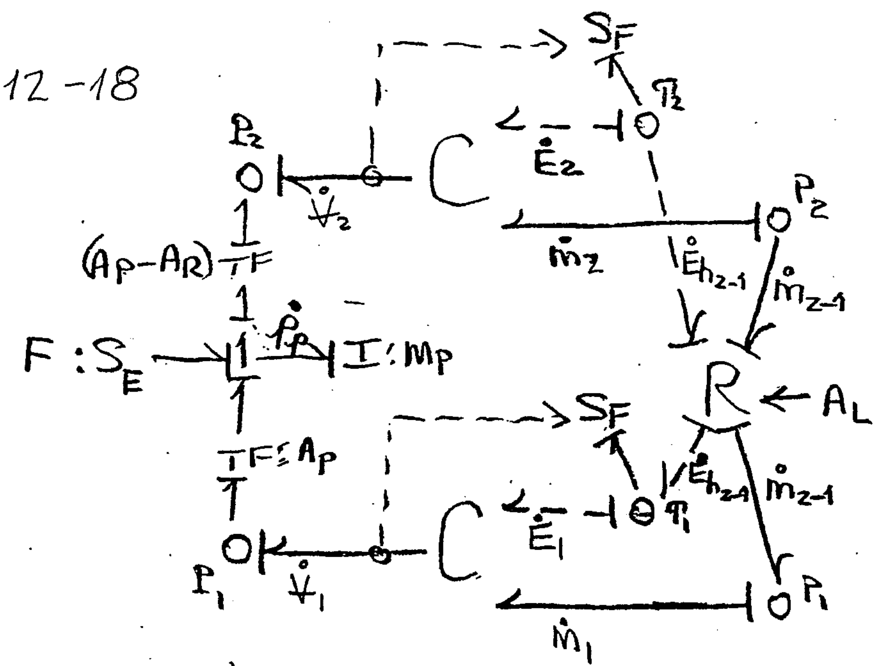
12-17



$A_W \equiv$ cylinder wall area

12-18

12-10



state variables

$E_1, m_1, V_1, E_2, m_2, V_2, p_p$

$$\dot{E}_1 = -P_1\dot{V}_1 + \dot{E}_{h_{2-1}}$$

$$\dot{m}_1 = \dot{m}_{2-1}$$

$$\dot{V}_1 = A_p \frac{p_p}{m_p}$$

$$\dot{E}_2 = -P_2\dot{V}_2 - \dot{E}_{h_{2-1}}$$

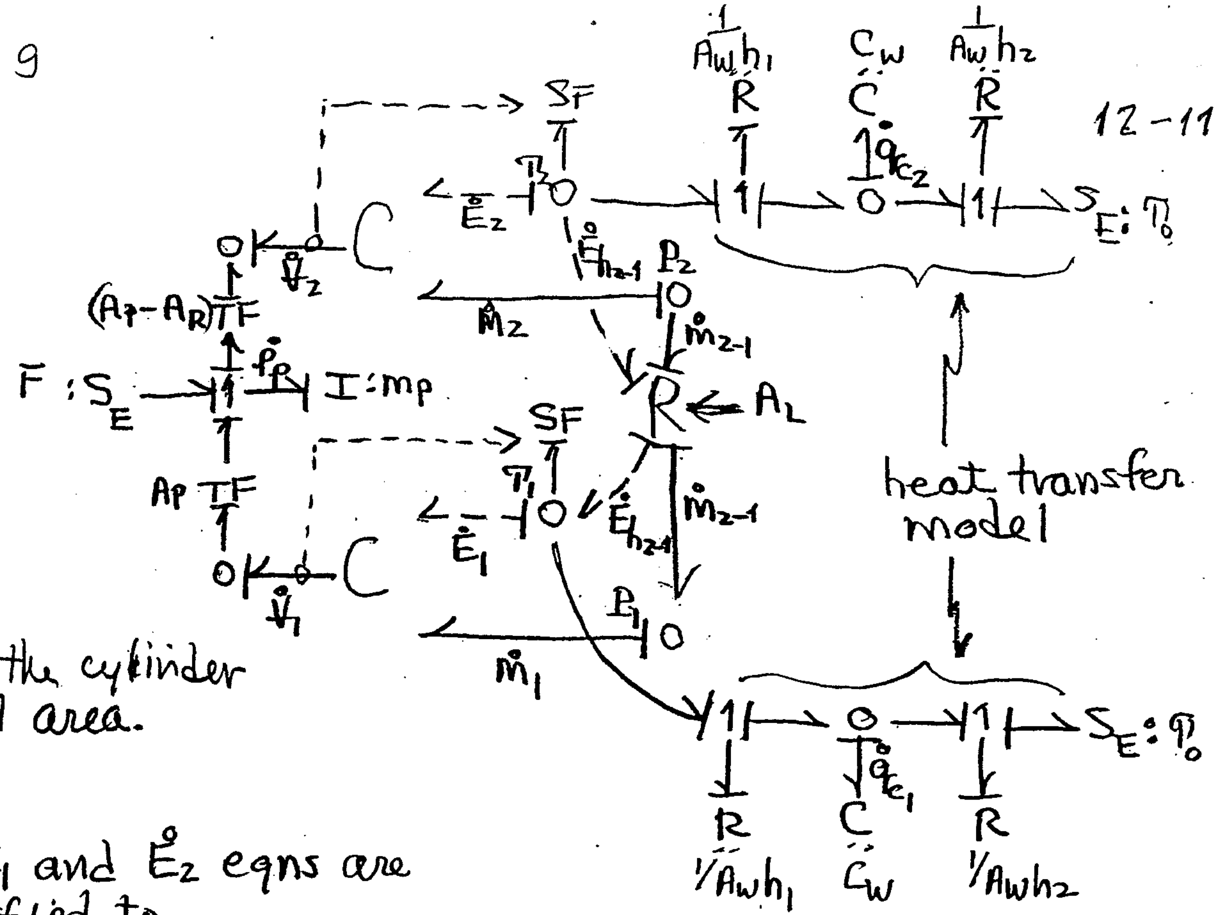
$$\dot{m}_2 = -\dot{m}_{2-1}$$

$$\dot{V}_2 = -(A_P - A_R)\frac{p_p}{m_p}$$

$$\left.\begin{aligned}\dot{E}_{h_{2-1}} &= \dot{E}_{h_{2-1}}(\mathcal{T}_1, P_1, \mathcal{T}_2, P_2)\\ \dot{m}_{h_{2-1}} &= \dot{m}_{h_{2-1}}(\mathcal{T}_1, P_1, \mathcal{T}_2, P_2)\end{aligned}\right\}\text{see sect. 12.5.2}$$

$$\left.\begin{aligned}\mathcal{T}_1 &= \frac{1}{C_v}\frac{E_1}{m_1} \qquad & P_1 &= \frac{R}{C_v}\frac{E_1}{V_1}\\ \mathcal{T}_2 &= \frac{1}{C_v}\frac{E_2}{m_2} \qquad & P_2 &= \frac{R}{C_v}\frac{E_2}{V_2}\end{aligned}\right\}\text{Eqs. (12.78) and (12.79)}$$

12-19



A_W is the cylinder wall area.

The $\dot{E}_1$ and $\dot{E}_2$ eqns are modified to,

$$\dot{E}_1 = -P_1\dot{V}_1 + \dot{E}_{h_{2-1}} - A_W h_1\left(T_1 - \frac{q_{c_1}}{C_W}\right)$$

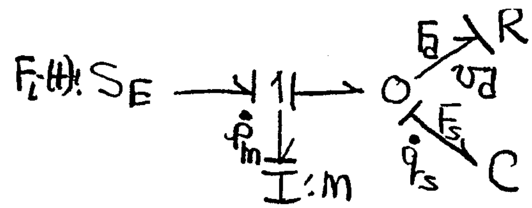
$$\dot{E}_2 = -P_2\dot{V}_2 - \dot{E}_{h_{2-1}} - A_W h_1\left(T_2 - \frac{q_{c_2}}{C_W}\right)$$

plus new state eqns.,

$$\dot{q}_{c_1} = A_W h_1\left(T_1 - \frac{q_{c_1}}{C_W}\right) - A_W h_2\left(\frac{q_{c_1}}{C_W} - T_o\right)$$

$$\dot{q}_{c_2} = A_W h_1\left(T_2 - \frac{q_{c_2}}{C_W}\right) - A_W h_2\left(\frac{q_{c_2}}{C_W} - T_o\right)$$

13.1



state variables
p_m , q_s

$$\dot{p}_m = F_i - F_s \qquad F_s = g_s\, q_s^3$$

$$\dot{q}_s = \frac{p_m}{m} - v_d \qquad v_d = \left\{\frac{|F_d|}{g_d}\right\}^{1/3} \cdot \operatorname{sign}(F_d)$$

$$F_d = F_s$$

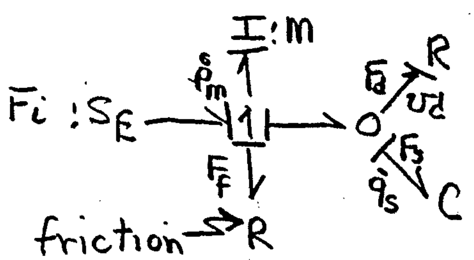
Note that causality on $\rightharpoondown$R requires v_d as output and F_d as input. Also, care must be used in inverting the damper constitutive law.

13.2

For the spring, $F_s = g_s\, q_s\, |q_s|$ to handle sign of F_s

For the damper, $v_d = \left\{\frac{|F_d|}{g_d}\right\}^{1/2} \cdot \operatorname{sign}(F_d)$

13.3



$$\dot{p}_m = F_i - F_f - F_s \qquad F_s = g_s\, q_s^3$$

$$\dot{q}_s = \frac{p_m}{m} - v_d \qquad F_d = F_s \ , \quad v_d = \left\{\frac{|F_d|}{g_d}\right\}^{1/3} \cdot \operatorname{sign}(F_d)$$

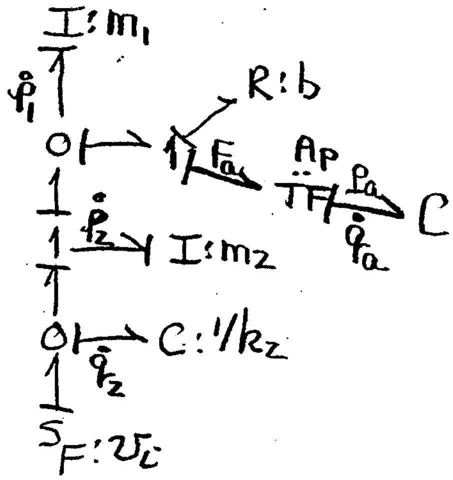
For friction characterized as $F_f = \mu N$ where μ = friction coefficient and N is the weight of m, it is usually sufficient to use,

$$F_f = \mu N \frac{v_m}{|v_m + \epsilon|}$$

$v_m = p_m / m$

ϵ is a "small" number (10^{-4}, 10^{-5}) which allows $v_m = 0$

13.4



state variables
p_1, p_2, q_2, q_a

← air spring

$$\dot{p}_1 = b\left(\frac{p_2}{m_2} - \frac{p_1}{m_1}\right) + F_a$$

$$\dot{p}_2 = k_2 q_2 - b\left(\frac{p_2}{m_2} - \frac{p_1}{m_1}\right) - F_a$$

$$\dot{q}_2 = v_i - p_2/m_2$$

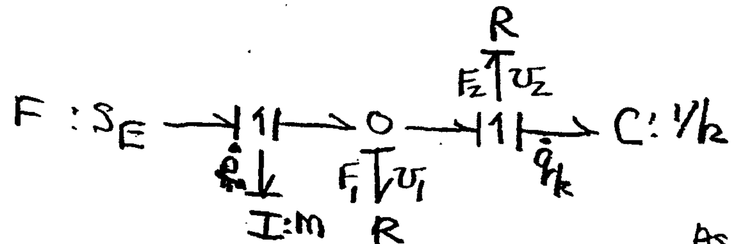
$$\dot{q}_a = A_p\left[\frac{p_2}{m_2} - \frac{p_1}{m_1}\right]$$

For the airspring
$\delta = \delta_{in} + q_a$
if $\delta \leq 0$ $\delta = 0$

$$F_a = P_0 A_p \left[\frac{1 - \left(\frac{A_p \delta}{V_0}\right)^{\gamma}}{\left(\frac{A_p\delta}{V_0} + \epsilon\right)^{\gamma}}\right]$$

ϵ is small number $(10^{-4}, 10^{-5})$ that allows for $\delta = 0$

13.5



Incomplete causality indicates algebraic loop. Causality was completed with the result shown.

As indicated by causality,
$F_1 = g_1 v_1^3$, $v_2 = \left\{\frac{|F_2|}{g_2}\right\}^{1/2} \operatorname{sign}(F_2)$

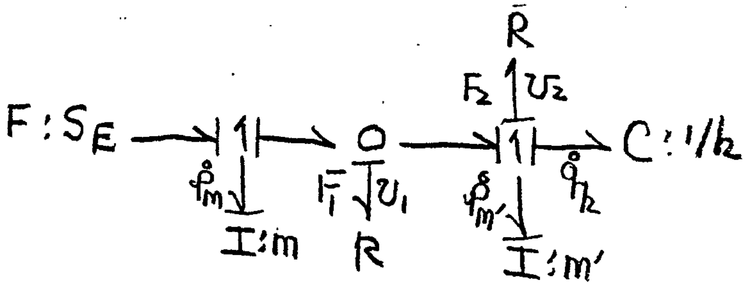
Algebraic loop involves,

$$F_1 = g_1\left[\frac{p_m}{m} - v_2\right]^3$$

$$v_2 = \left\{\frac{|F_1 - k q_k|}{g_2}\right\}^{1/2} \operatorname{sign}(F_1 - k q_k)$$

← complicated to solve for F_1 or v_2

Introduce some mass, m′, at the junction of the dampers,



no algebraic loop

13.5 (continued)

$$\mathring{p}_m = F - F_1$$

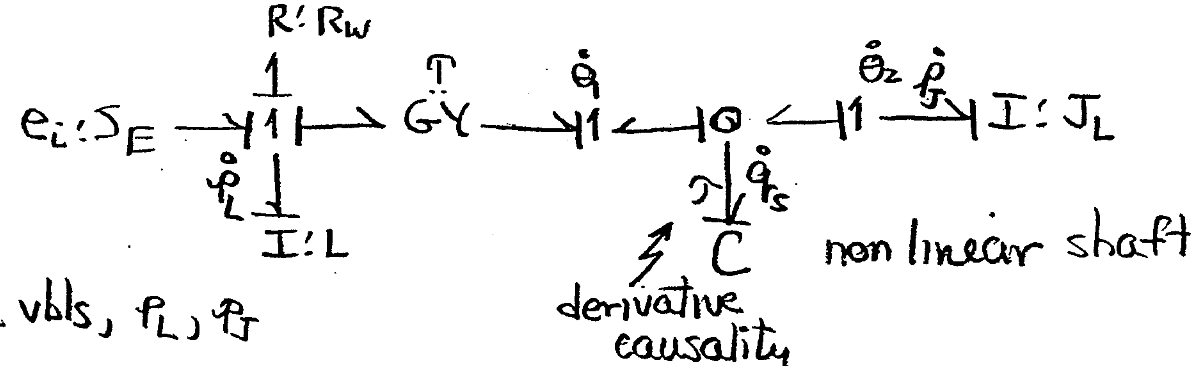
$$\mathring{q}_k = \frac{p_{m'}}{m'}$$

$$\mathring{p}_{m'} = -kq_k + F_1 - F_2$$

$$F_1 = g_1 \Big(\overbrace{\frac{p_m}{m} - \frac{p_{m'}}{m'}}^{v_1}\Big)^3$$

$$F_2 = g_2 \underbrace{\frac{p_{m'}}{m'}}_{v_2}\left|\frac{p_{m'}}{m'}\right|$$

13.6



state vbls, p_L, p_J

procedure from chap 5 suggests

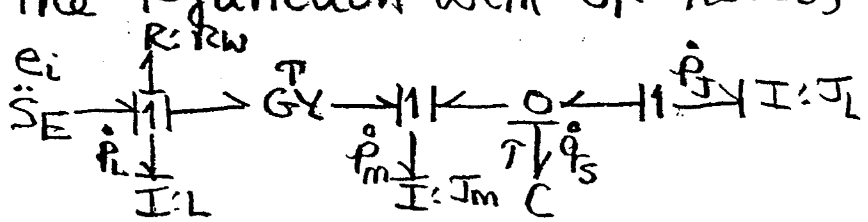
$$q_s = \left\{\frac{|\tau|}{g}\right\}^{1/3} \text{sign}(\tau) \qquad \tau = (-1)\frac{T}{L}p_L$$

$$\mathring{p}_L = e_i - \frac{R_W}{L}p_L - T(-1)\left[\frac{p_J}{J_L} - \mathring{q}_s\right]$$

$$\mathring{p}_J = \frac{T}{L}p_L$$

we need $\mathring{q}_s$ which comes from differentiating q_s

This could be done for this problem, but instead, append the rotary inertia of the motor, J_m, to the 1-junction with $\dot{\theta}_1$. yields,



$$\mathring{p}_L = e_i - \frac{R_W}{L}p_L - \frac{T}{J_m}p_m$$

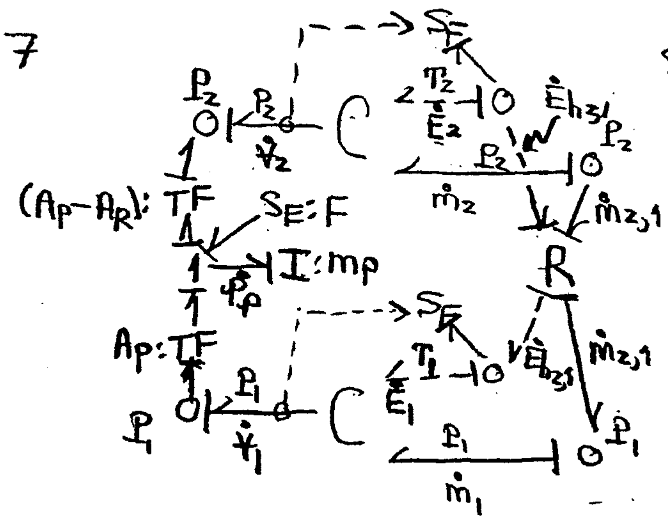
$$\mathring{p}_J = (-1)\,\tau$$

$$\mathring{p}_m = \frac{T}{L}p_L + \tau$$

$$\mathring{q}_s = \frac{p_J}{J_L} - \frac{p_m}{J_m}$$

$$\tau = g q_s^3$$

13.7



state variables
$p_p, V_1, V_2, E_2, m_2, E_1, m_1$

$$\dot p_p = F + A_P P_1 - (A_P - A_R) P_2$$

$$\dot V_1 = A_P \frac{p_p}{m_p} \qquad \dot V_2 = -(A_P - A_R)\frac{p_p}{m_p}$$

$$\dot E_2 = -P_2 \dot V_2 - \dot E_{h_{2,1}}$$

$$\dot m_2 = -\dot m_{2,1}$$

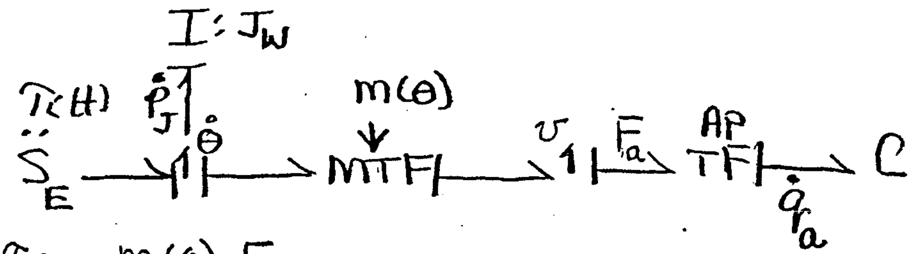
$$\dot E_1 = -P_1 \dot V_1 + \dot E_{h_{2,1}}$$

$$\dot m_1 = \dot m_{2,1}$$

$\dot m_{2,1}$, $\dot E_{h_{2,1}}$ are dependent upon P_1, T_1, P_2, T_2, and are calculated as shown in sect. 12.4.2

P_1, T_1, P_2, T_2 come from constitutive laws, Eqs. (12.81) and (12.82).

13.8



$$\dot{p}_J = \tau_i - m(\theta)\, F_a$$

$$\dot{q}_a = A_p\, m(\theta) \frac{p_J}{J_W}$$

$m(\theta)$ comes from eq. 13.59, but we need θ. Expand the state space with,

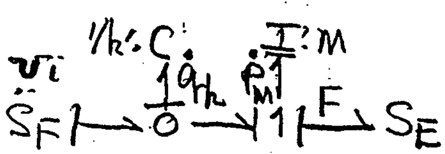
$$\dot{\theta} = \frac{p_J}{J_W}$$

F_a comes from prob. 13.4. Since q_a is positive if the air volume is decreasing, it is convenient to write,

$$\dot{S} = -\dot{q}_a .$$

This way we can assign an independent initial condition to S, and use the formula from prob 13.4.

13.9



$$\dot{p}_M = k q_k - F$$

$$\dot{q}_k = v_i - \frac{p_M}{M}$$

Let v be the velocity just before the wall is reached. The momentum at that instant is $p_i = mv$. We desire the momentum just after impact to be $p_f = -mv$. Therefore the $\Delta P = p_f - p_i = -2mv$. And the force required is

$$F = -\frac{2mv}{\Delta t}$$

For the sign of the force used in the bond graph and resulting equations,

$$F = \frac{2mv}{\Delta t} = \frac{2\,p_m}{\Delta t}$$

We need to bookkeep the location of the mass, so we expand the state space with,

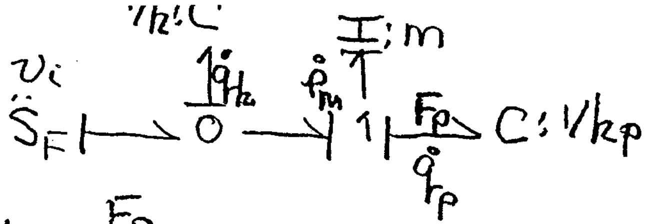
$$\dot{x} = \frac{p_m}{m},$$

then add,

IF $x < d$ $KK = 0,\ F = 0$
IF $x \geq d$ and $KK = 0$ then
$p_{m_0} = p_m,\ t_0 = t,\ KK = 1,\ F = 2\,p_{m_0}/\Delta t$
else
IF $t < t_0 + \Delta t$ $F = 2\,p_{m_0}/\Delta t$

These steps will apply a force for almost exactly Δt seconds or until the mass is moving away from the wall.

13.10



$$\dot{p}_m = k\,q_k - F_p$$

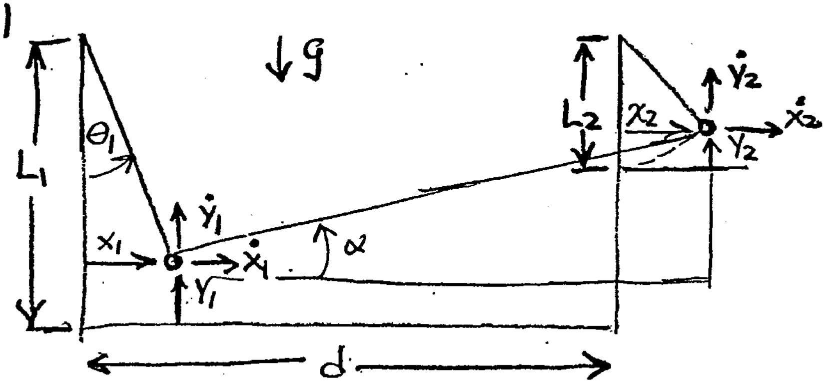
$$\dot{q}_k = v_i - \frac{p_m}{m}$$

$$\dot{q}_p = \frac{p_m}{m}$$

then add

If $q_p > d$ $\quad F_p = k_p(q_p - d)$

13.11



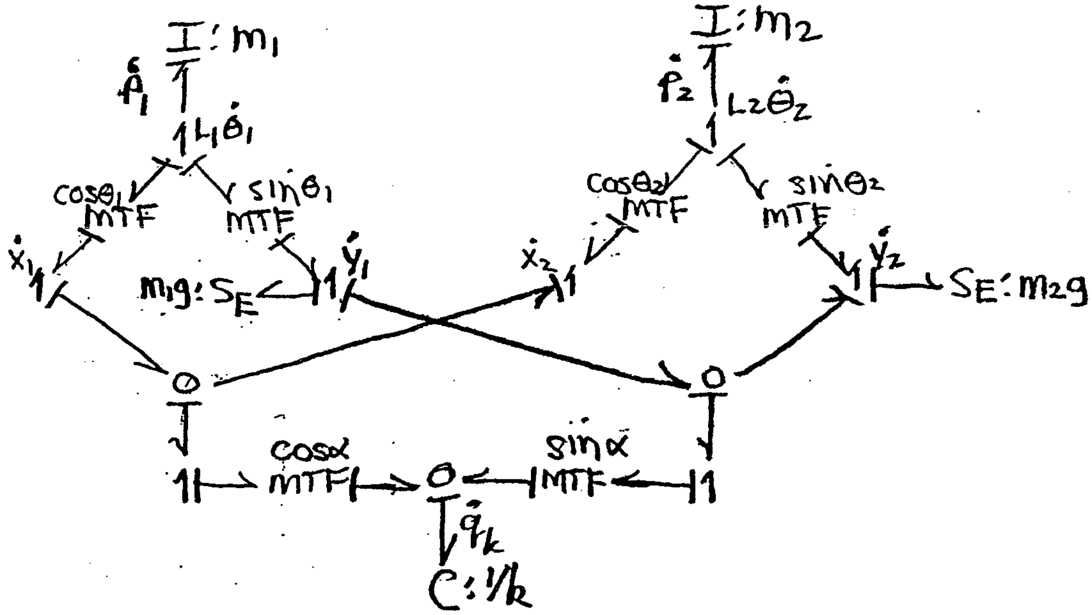
some kinematics

$\dot{x}_1 = L_1\dot{\theta}_1\cos\theta_1 \qquad \dot{x}_2 = L_2\dot{\theta}_2\cos\theta_2 \qquad \tan\alpha = \dfrac{L_1 - L_2 + y_2 - y_1}{d + x_2 - x_1}$

$\dot{y}_1 = L_1\dot{\theta}_1\sin\theta_1 \qquad \dot{y}_2 = L_2\dot{\theta}_2\sin\theta_2$

$v_s \equiv$ relative velocity across spring $= (\dot{x}_1 - \dot{x}_2)\cos\alpha + (\dot{y}_1 - \dot{y}_2)\sin\alpha$

13.11 (continued)



$$\dot{p}_1 = -\cos\theta_1 \cos\alpha\, k q_k - \sin\theta_1 \left[m_1 g + \sin\alpha\, k q_k\right]$$

$$\dot{p}_2 = +\cos\theta_2 \cos\alpha\, k q_k - \sin\theta_2 \left[m_2 g - \sin\alpha\, k q_k\right]$$

$$\dot{q}_k = \cos\alpha \left[\cos\theta_1 \frac{p_1}{m_1} - \cos\theta_2 \frac{p_2}{m_2}\right] + \sin\alpha \left[\sin\theta_1 \frac{p_1}{m_1} - \sin\theta_2 \frac{p_2}{m_2}\right]$$

Plus the additional eqns,

$$\dot{\theta}_1 = \frac{1}{L_1} \frac{p_1}{m_1} \qquad \dot{\theta}_2 = \frac{1}{L_2} \frac{p_2}{m_2}$$

$$\dot{x}_1 = \cos\theta_1 \frac{p_1}{m_1} \qquad \dot{x}_2 = \cos\theta_2 \frac{p_2}{m_2}$$

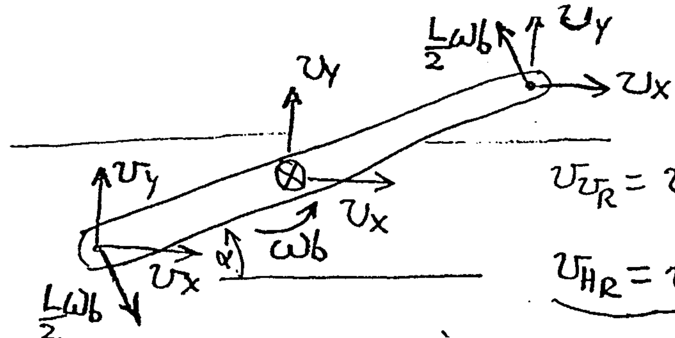
$$\dot{y}_1 = \sin\theta_1 \frac{p_2}{m_2} \qquad \dot{y}_2 = \sin\theta_2 \frac{p_2}{m_2}$$

These provide availability of $\theta_1, \theta_2, x_1, y_1, x_2, y_2$

$$\alpha = \tan^{-1}\left(\frac{L_1 - L_2 + y_2 - y_1}{d + x_2 - x_1}\right)$$

13.12

13-9



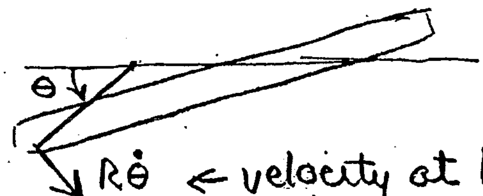
$v_{v_R} = v_y + \frac{L}{2}\omega_b \cos\alpha$

$\underbrace{v_{H_R} = v_x - \frac{L}{2}\omega_b \sin\alpha}$

$v_{v_L} = v_y - \frac{L}{2}\omega_b \cos\alpha$

$v_{H_L} = v_x + \frac{L}{2}\omega_b \sin\alpha$

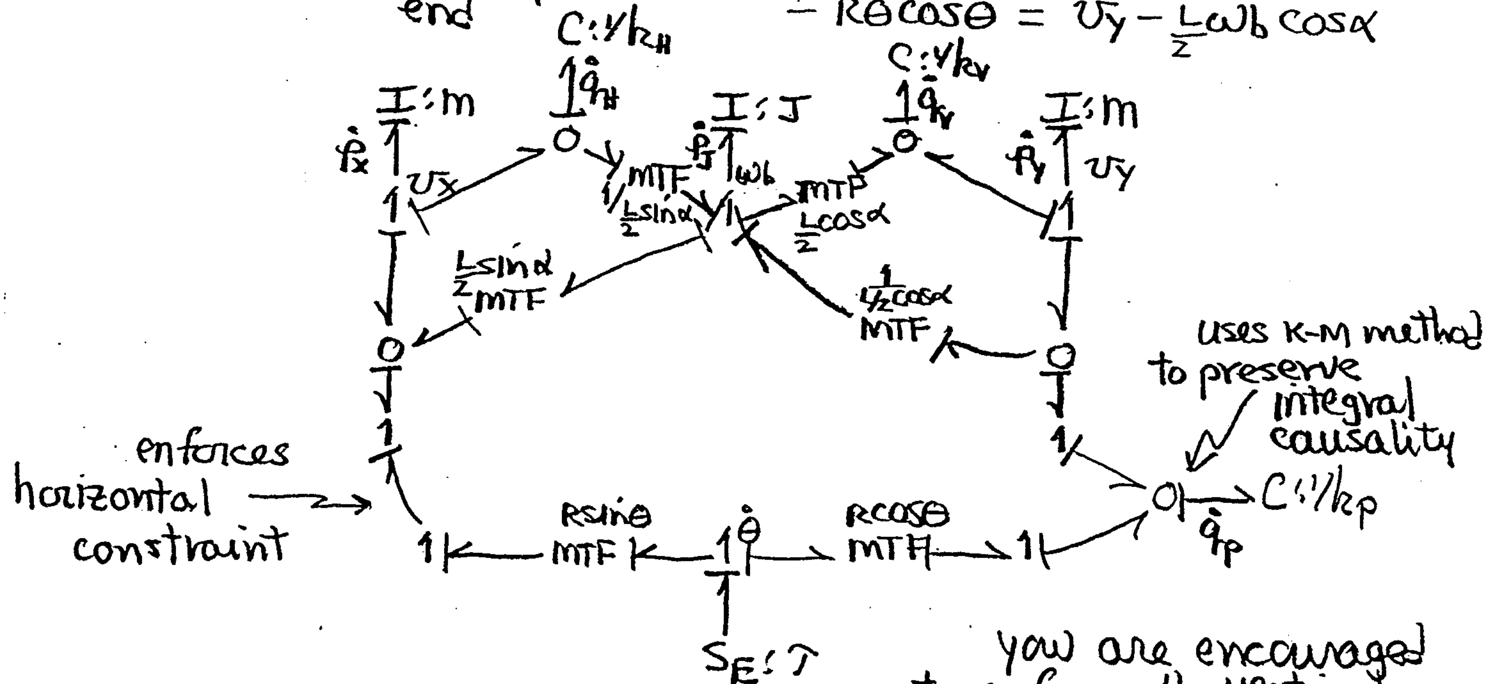
vertical and horizontal velocities at the 2 ends



constraint at left end

$R\dot{\theta}\sin\theta = v_x + \frac{L}{2}\omega_b \sin\alpha$

$-R\dot{\theta}\cos\theta = v_y - \frac{L}{2}\omega_b \cos\alpha$



you are encouraged to enforce the vertical constraint and see that derivative causality exists.

13.12 (continued)

$$\dot{p}_x = -k_H q_H + \frac{1}{R\sin\theta}\left(\tau - R\cos\theta\, k_p q_p\right)$$

$$\dot{p}_y = -k_V q_V - k_p q_p$$

$$\dot{p}_J = \frac{L}{2}\sin\alpha\, k_H q_H - \frac{L}{2}\cos\alpha\, k_V q_V + \frac{L}{2}\sin\alpha \frac{1}{R\sin\theta}\left(\tau - R\cos\theta\, k_p q_p\right)$$

$$\dot{q}_H = \frac{p_x}{m} - \frac{L}{2}\sin\alpha \frac{p_J}{J}$$

$$\dot{q}_V = \frac{L}{2}\cos\alpha \frac{p_J}{J} + \frac{p_y}{m}$$

$$\dot{q}_p = \frac{p_y}{m} - \frac{L}{2}\cos\alpha \frac{p_J}{J} + \frac{R\cos\theta}{R\sin\theta}\left(\frac{p_x}{m} + \frac{L}{2}\sin\alpha \frac{p_J}{J}\right)$$

We also need

$$\dot{\theta} = \frac{1}{R\sin\theta}\left(\frac{p_x}{m} + \frac{L}{2}\sin\alpha \frac{p_J}{J}\right)$$

$$\dot{\alpha} = \frac{p_J}{J}$$

This is a complete state representation. The parameters k_p and k_V must be "stiff" to simulate the original slider-crank device.

Also, care must be taken during simulation to avoid $\theta = 0$ (exactly). Notice that this formulation requires division by $\sin\theta$. This can be avoided by using the K-M method to relieve the horizontal constraint at the left end.