TRAJECTORY CONTROL IN ROBOT MANIPULATORS

Class Notes for E495D

P. C. Krause

and

O. Wasynczuk

April 1988

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Introduction

In this section, an introduction to the control of robot manipulators is given. What constitutes a robot is philosoptical question which is often debated. For our purposes, it suffices to define a robot as a *manipulator* plus an *end effector*. The manipulator generally consists of a set of nearly rigid links connected by a set of joints. The joints may be *rotary* or *linear* (prismatic). The end effector may be a gripper, a welding torch, a paint sprayer or some other device.

The study of robots is often divided into several interrelated areas. Each area focuses upon a specific problem which must be addressed somewhere in the design stage. For example, in the study of *forward kinematics,* we are concerned with the following problem: given the joint angles, what is the position and orientation of the end effector? Sometimes, the problem is stated in reverse terms: given the desired position and orientation, what are the required joint angles? The area of *dynamics* addresses the problem: given a desired trajectory, what are the torques and/or forces which must be supplied so as to provide the desired motion. An *exact* a priori solution to this problem requires that we know *precisely* how the robot will respond to a set of joint torques/forces. Such absolute knowledge is never attained. A primary concern in the area of *robot control* is to compensate for errors in knowledge of the parameters of the system and to suppress external disturbances which may defy mathematical modeling. Typically, position and velocity sensors are used and a *control law* determines the proper torques/forces which yield the desired motion.

In our introduction to robot control, we will address each of these areas in the order listed. Since this is intended to be an introductory rather than an in depth treatment of the subject, we will narrow our focus in several respects. First, we will focus our attention on the manipulator itself forgetting about the end effector.

Second, the manipulator which we will consider will be one of the simplest imaginable. Even so, we will find the algebra to be quite involved at times. Occasionally, the generalization of the analysis to include more complex manipulators will become apparent. However, we will not spend the time to do so. The generalizations are more conveniently handled by using multiple frames of reference fixed in the various links of the robot. The body of mathematics which goes along with the multiple reference frame approach is best covered in a specialized coarse on robot dynamics and control.

1. FORWARD KINEMATICS

Kinematics involves the study of motion without regard to the forces which cause the motion. The basic question that we would like to answer is: given the joint angles and their time derivatives, what is the position, velocity and acceleration of the end effector.

Although we can approach the subject of kinematics from a completely general perspective, it is convenient in this introductory treatment to focus upon a specific manipulator design. In particular, we will consider the simple two link manipulator depicted in Fig. 1-1. As shown, there are two links and two rotary joints: joint-1 connects link-1 to the *base* of the manipulator and joint-2 connects link-2 to link-1. Also, ℓ_1 and ℓ_2 represent, respectively, the lengths of links 1 and 2; θ_1 is the angle between link-1 and the horizontal axis and θ_2 is the angle between link-2 and the axis of link-1. The end effector is assumed to be located at the end of link-2. It is convenient to select the base of the manipulator to be the origin of a standard cartesian coordinate system. The x axis is horizontal, the y axis is vertical. In later discussions, we will make use of the z axis which is assumed to be pointed outward from the xy plane in accordance with the right hand rule (turn x into y, the progression of a screw at the origin determines the direction of the z axis).

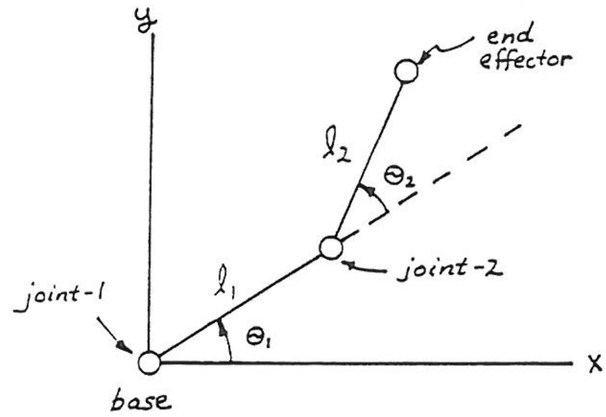

Fig. 1-1. A simple 2-link manipulator.

This type of manipulator is called a planar manipulator since the motion of the end effector is restricted to a single plane. Clearly, this is one of the simplest, non-trivial manipulators imaginable. Although we might have selected an even simpler (one link) manipulator, the only trajectory that we would be able to generate is a radial arc. Moreover, the analysis of a one link manipulator provides little challenge and consequently little insight into the problems likely to be encountered in an actual system.

In any case, for the given manipulator, the forward kinematics problem reduces to the following: given the joint angles θ_1 and θ_2 and their time derivatives, what is the position, velocity and acceleration of the end effector? Lets proceed to answer this question.

The position of the end effector may be described by a vector[1], $\bar{r}_2(t)$, pointing from the origin of our selected coordinate system to the position of the end effector as

[1]Vector quantities will be distinguished from scalar quantities using bar notation. The unit vector oriented along the "n" direction is given the special symbol $\hat{a}_n$.

shown in Fig. 1-2. The x and y components of this vector are simply the x and y coordinates of the end effector. From Fig. 1-2,

$$\overline{r}_2(t) = \Big[\ell_1\cos\theta_1 + \ell_2\cos(\theta_1 + \theta_2)\Big]\,\hat{a}_x + \Big[\ell_1\sin\theta_1 + \ell_2\sin(\theta_1 + \theta_2)\Big]\,\hat{a}_y \qquad (1.1)$$

where $\hat{a}_x$ and $\hat{a}_y$ represent unit vectors oriented along the x and y axes. Even for the two link manipulator, we can observe a "pattern" developing in our expression for the position of the end effector. With a little thought we can express the end effector position for a 3, 4 or n link planar manipulator involving rotary joints. In a planar manipulator with a prismatic (linear) joint, the length of the associated link will be a variable instead of a constant as is assumed here. However, we'll not go into this.

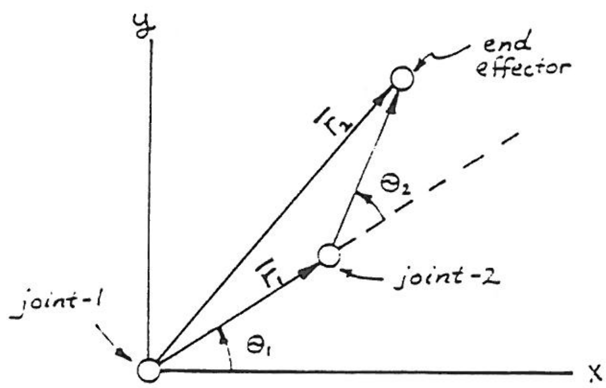

Fig. 1-2. Position vectors of 2-link manipulator.

The velocity of the end effector, which will be denoted as $\overline{v}_2(t)$, may be established by differentiating $\overline{r}_2(t)$ with respect to time. Since our unit vectors, $\hat{a}_x$ and $\hat{a}_y$, are fixed in space (therefore independent of time), the $\hat{a}_x$ and $\hat{a}_y$ components of $\overline{v}_2(t)$

are simply the time derivatives of the corresponding components of $\bar{r}_2(t)$.

$$\bar{v}_2(t) = \frac{d}{dt}\,\bar{r}_2(t) = -\left[\ell_1\omega_1\sin\theta_1 + \ell_2(\omega_1+\omega_2)\sin(\theta_1+\theta_2)\right]\hat{a}_x$$

$$+\left[\ell_1\omega_1\cos\theta_1 + \ell_2(\omega_1+\omega_2)\cos(\theta_1+\theta_2)\right]\hat{a}_y\,. \qquad (1.2)$$

where $\omega_1 = \dfrac{d\theta_1}{dt}$ and $\omega_2 = \dfrac{d\theta_2}{dt}$ (joint angular velocities). The previous expression can be expressed much more compactly if $\sin\theta_1$ is abbreviated as s_1, $\cos\theta_1$ as c_1, $\sin(\theta_1+\theta_2)$ as s_{12}, and so forth (a convention commonly used in the literature). We have to be careful, however, not to use c's and s's to represent other variables. In any case, (1.2) reduces to

$$\bar{v}_2(t) = -\left[\ell_1\omega_1 s_1 + \ell_2(\omega_1+\omega_2)s_{12}\right]\hat{a}_x + \left[\ell_1\omega_1 c_1 + \ell_2(\omega_1+\omega_2)c_{12}\right]\hat{a}_y \qquad (1.3)$$

We can differentiate once again to establish the acceleration of the end effector, denoted $\bar{a}_2(t)$

$$\bar{a}_2(t) = -\left[\ell_1\alpha_1 s_1 + \ell_1\omega_1^2 c_1 + \ell_2(\alpha_1+\alpha_2)s_{12} + \ell_2(\omega_1+\omega_2)^2 c_{12}\right]\hat{a}_x$$

$$+\left[\ell_1\alpha_1 c_1 - \ell_1\omega_1^2 s_1 + \ell_2(\alpha_1+\alpha_2)c_{12} - \ell_2(\omega_1+\omega_2)^2 s_{12}\right]\hat{a}_y \qquad (1.4)$$

where $\alpha_1 = \dfrac{d\omega_1}{dt} = \dfrac{d^2\theta_1}{dt^2}$ and $\alpha_2 = \dfrac{d\omega_2}{dt} = \dfrac{d^2\theta_2}{dt^2}$ (joint angular accelerations). Thus, we have established the position, velocity and acceleration of the end effector in terms of the joint angles, θ_1 and θ_2, the angular velocities, ω_1 and ω_2, and the angular accelerations, α_1 and α_2. We can follow a similar approach to calculate the position, velocity and acceleration of joint-2 in cartesian coordinates. Denoting these as $\bar{r}_1(t)$, $\bar{v}_1(t)$ and $\bar{a}_1(t)$, respectively, we have[2]

[2]Although it appears to be more natural to symbolize the position of joint-2 as $\bar{r}_2$ while using $\bar{r}_3$ to denote the position of the end effector, this will cause problems later on.

$$\overline{r}_1(t) = [\ell_1\ c_1]\ \hat{a}_x + [\ell_1\ s_1]\ \hat{a}_y \tag{1.5}$$

$$\overline{v}_1(t) = -[\ell_1\ \omega_1\ s_1]\ \hat{a}_x + [\ell_1\ \omega_1\ c_1]\ \hat{a}_y \tag{1.6}$$

$$\overline{a}_1(t) = -[\ell_1\ \alpha_1\ s_1 + \ell_1\ \omega_1^2\ c_1]\ \hat{a}_x + [\ell_1\ \alpha_1\ c_1 - \ell_1\ \omega_1^2\ s_1]\ \hat{a}_y \tag{1.7}$$

These expressions will be useful when we consider manipulator dynamics.

The relationships between the end effector position, velocity and acceleration and the joint angular positions, velocities and accelerations may also be expressed in matrix form. In particular, if we consider the expression of the position of the end effector given by (1.1), then it is apparent that

$$\begin{bmatrix} x_2 \\ \hline y_2 \end{bmatrix} = \begin{bmatrix} \ell_1\ c_1 + \ell_2\ c_{12} \\ \hline \ell_1\ s_1 + \ell_2\ s_{12} \end{bmatrix} \tag{1.8}$$

where x_2 and y_2 represent, respectively, the $\hat{a}_x$ and $\hat{a}_y$ components of $\overline{r}_2$. From (1.3), we can express the x and y components of the end effector velocity as

$$\begin{bmatrix} v_{2x} \\ v_{2y} \end{bmatrix} = \left[\begin{array}{c|c} -\ell_1\ s_1 - \ell_2\ s_{12} & -\ell_2\ s_{12} \\ \hline \ell_1\ c_1 + \ell_2\ c_{12} & \ell_2\ c_{12} \end{array}\right] \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \tag{1.9}$$

where v_{2x}, v_{2y} represent, respectively, the $\hat{a}_x$ and $\hat{a}_y$ components of $\overline{v}_2$. Finally, from (1.4) we can express the acceleration of the end effector as follows

$$\begin{bmatrix} a_{2x} \\ a_{2y} \end{bmatrix} = \left[\begin{array}{c|c} -\ell_1 s_1 - \ell_2 s_{12} & -\ell_2 s_{12} \\ \hline \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{array}\right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} -\ell_1 c_1 \omega_2^2 - \ell_2 c_{12}(\omega_1 + \omega_2)^2 \\ \hline -\ell_1 s_1 \omega_1^2 - \ell_2 s_{12}(\omega_1 + \omega_2)^2 \end{bmatrix} \tag{1.5}$$

These matrix equations will be useful when we consider the inverse kinematics problem (next on the agenda).

2. INVERSE KINEMATICS

In a typical control system hierarchy, we first specify the trajectory which the end effector is to follow. The trajectory information might include the desired position, velocity and acceleration of the end effector expressed as specific functions of time. We then have to determine the required joint **angular** positions, velocities and accelerations which will provide the desired motion. This represents the inverse kinematics problem.

Unfortunately, the solution of the inverse kinematics problem can be substantially more difficult than the forward kinematics problem. We have to worry about the *existence* as well as the *uniqueness* of the solution (things we need not be concerned about in the forward case). It is very well possible that depending upon the parameters of the system, no solution exists for the inverse problem. This leads us to the definition of the term *workspace* which represents the volume of space which the end effector can reach. Within the workspace, a solution to the inverse kinematics problem can be found. However, there may be several solutions depending upon the number of so called *degrees of freedom* available to us.

Let us momentarily narrow our attention to a subset of the inverse kinematics problem. In particular, suppose we are given the position of the end effector and we want to establish the required joint angles. It turns out that if we can solve this problem, it is easy to establish the joint angular velocities and accelerations from the end effector acceleration and velocity. Here, we shall try to establish a closed form solution to the inverse kinematics problem (one that does not require iterative, numerical calculations) given the two link, planar manipulator considered previously. Fortunately, a closed form solution exists for this case (not always true).

To establish a closed form solution, consider the situation depicted in Fig. 2-1. Let us focus our attention upon the triangle formed by the base, joint-2, and the end

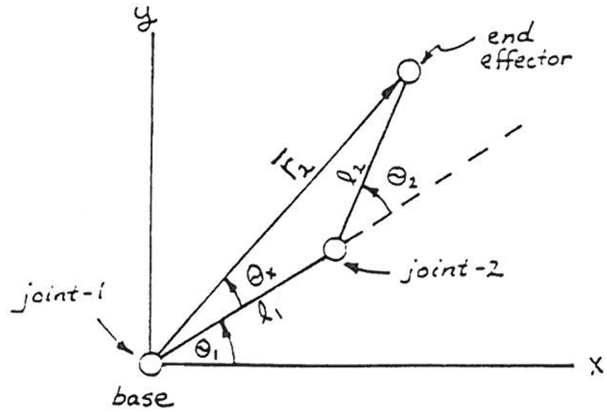

Fig. 2-1. Angular displacements in the two link planar manipulator.

effector. The obtuse angle at joint-2 measured inside the given triangle is $180° - \theta_2$. We can apply the law of cosines to this triangle obtaining

$$|\overline{r}_2|^2 = \ell_1^2 + \ell_2^2 - 2\ell_1\ell_2 \cos(180° - \theta_2) \tag{2.1}$$

where we have momentarily reverted back to ~~usin g~~ using the long way of writing $\sin\theta$ and $\cos\theta$. Also, $|\overline{r}_2|$ represents the length of the vector $\overline{r}_2$. Now $|\overline{r}_2|$ can be related to x_2 and y_2 which represent the coordinates of the end effector. In addition, $\cos(180° - \theta_2) = -\cos\theta_2$. Thus, (2.1) becomes

$$x_2^2 + y_2^2 = \ell_1^2 + \ell_2^2 + 2\ell_1\ell_2 \cos\theta_2 \tag{2.2}$$

From which

$$\cos\theta_2 = \frac{(x_2^2 + y_2^2) - (\ell_1^2 + \ell_2^2)}{2\ell_1\ell_2} \tag{2.3}$$

Thus, we can readily establish the angle, θ_2, given the lengths, ℓ_1 and ℓ_2, of the two links and the coordinates of the end effector. It should be noted that if the expression on the right side of (2.3) gives us a value less than -1 or greater than +1, no solution to this equation exists implying that the selected coordinates, x_2 and y_2, correspond to a point outside of the workspace of the manipulator. On the other hand, if x_2 and y_2 fall inside the workspace, two solutions for θ_2 exist both having the same magnitude but of opposite sign. We shall discuss the existence and uniqueness aspects shortly. But before we do so let's find θ_1.

To establish θ_1, we define θ_x to be the angle between $\overline{r}_2$ and the link-1 axis. It is the acute angle in the triangle of Fig. 2-1 closest to the base. Applying the law of cosines once again

$$\ell_2^2 = \ell_1^2 + |\overline{r}_2|^2 - 2\ell_1 |\overline{r}_2| \cos \theta_x \tag{2.4}$$

Solving for $\cos \theta_x$ and replacing $|\overline{r}_2|^2$ by $(x_2^2 + y_2^2)$ gives

$$\cos \theta_x = \frac{(\ell_1^2 - \ell_2^2) + (x_2^2 + y_2^2)}{2\ell_1 \sqrt{x_2^2 + y_2^2}} \tag{2.5}$$

Again, we have the possibility of two solutions for θ_x (one positive and one negative) provided that at least one solution exists. However, in this case, only one solution is permitted. Specifically, the positive solution for θ_x applies if $\theta_2 > 0$ (see Fig. 2-1). On the other hand, if $\theta_2 < 0$, then the negative solution of θ_x applies. If $\theta_2 = 0$, then θ_x can only be zero. In any case, from Fig. 2-1, $\theta_1 + \theta_x = \tan^{-1}(y_2/x_2)$. Thus

$$\theta_1 = \tan^{-1}\left(\frac{y_2}{x_2}\right) - \theta_x \tag{2.6}$$

where θ_x may be established using (2.5). Equations (2.3) and (2.5) - (2.6) provide us with a non-iterative means of establishing θ_1 and θ_2 given the desired position of the end effector (x_2 and y_2). Let's now examine the existence and uniqueness of these solutions. First, lets establish requirements for the existence of a solution. From (2.3), in order for a solution for θ_2 to exist, we must have

$$-1 \leq \frac{(x_2^2+y_2^2)-(\ell_1^2+\ell_2^2)}{2\ell_1\ell_2} \leq 1 \tag{2.7}$$

Here, we have assumed that θ_2 can take on any value between $\pm 180°$. However, mechanical constraints may limit the maximum and minimum values of θ_2. If this is the case, the previous equation has to be modified accordingly. For example, if θ_2 is limited to values between $\pm 90°$ then $0 < \cos\theta_2 < 1$, and the following constraint equation results

$$0 < \frac{(x_2^2+y_2^2)-(\ell_1^2+\ell_2^2)}{2\ell_1\ell_2} < 1 \tag{2.8}$$

Now, (2.8) actually gives us two constraint equations corresponding to each of the two inequalities. Expressing each inequality individually and after simplification

$$x_2^2 + y_2^2 > \ell_1^2 + \ell_2^2 \tag{2.9}$$

$$x_2^2 + y_2^2 < (\ell_1 + \ell_2)^2 \tag{2.10}$$

The boundaries implied by (2.9)-(2.10) represent circles in the x-y plane. The circle corresponding to (2.9) possesses a smaller radius than that of (2.10). This implies that the workspace consists of an annular ring (doughnut). However, we may have other constraints further limiting the workspace. For example, the workspace may be limited to points inside the first quadrant of the x-y plane giving us the additional constraints that

$$x_2 > 0 \tag{2.11}$$

$$y_2 > 0 \tag{2.12}$$

As an example, suppose that $\ell_1 = \ell_2 = 1$ m. The workspace defined by the constraint equations (2.9)-(2.12) is illustrated in Fig. 2-2.

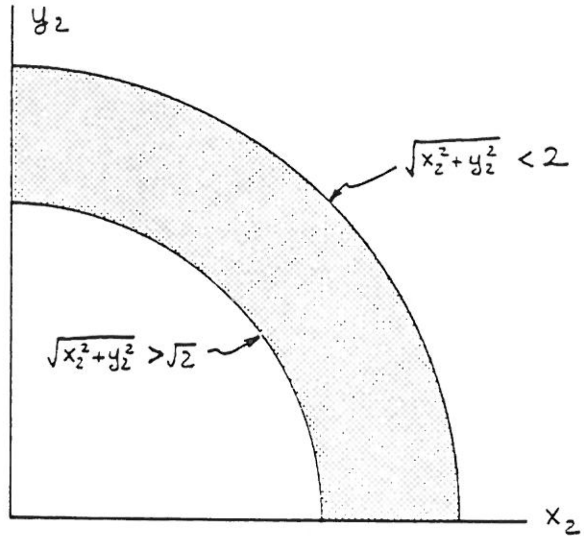

Fig. 2-2. Workspace of selected 2-link planar manipulator.

Of course, we have been somewhat arbitrary in the selection of our constraints. We have purposely selected constraints which result in a workspace that is easily described mathematically. Unfortunately, this is not always the case.

Everywhere within the workspace, at least one solution to the inverse kinematics problem exists. However, there may be several solutions. To illustrate this, lets first show a graphical method of obtaining a solution to the inverse kinematics problem. First, lets select some point inside the workspace of Fig. 2-2 representing the desired position of the end effector. Then, lets draw an arc with radius ℓ_2 centered at this point as shown in Fig. 2-3. Finally, we draw an arc centered about the origin (base) with radius ℓ_1. The intersections of these two arcs define the possible locations of joint-2 in our planar manipulator. Clearly, there are two such intersections implying that there are two link orientations which correspond to our selected end effector position. Which is the correction solution? It may be that either solution is permitted. However, when

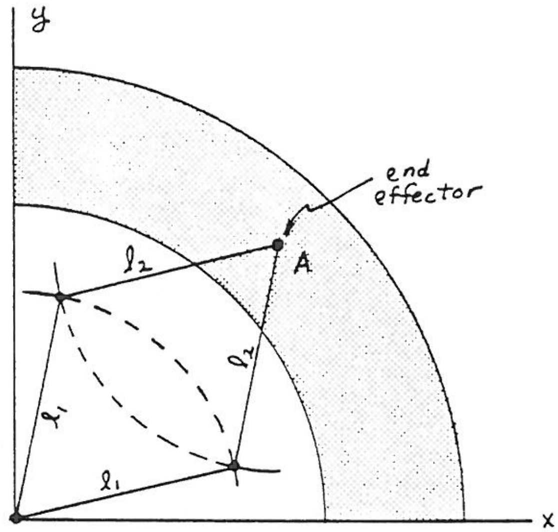

Fig. 2-3. Graphical solution of the inverse kinematics problem.

the end effector is close to the boundary of the workspace, constraints typically force us to choose one solution over the other. For example, if our desired end effector position is at point B in Fig. 2-4, one of the solutions corresponds to a value of θ_1 which is negative. If θ_1 is constrained to positive values, only one viable solution exists. When we consider the motion along a trajectory, we must be careful in the case that there are multiple solutions to select the one solution which provides a "smooth" motion in our workspace. That is, if we move from point B in Fig. 2-4 to point A in Fig. 2-3, we should be careful to always choose that solution which provides the smoothest possible motion. This is a problem in *trajectory planning*.

After having established the joint angles, θ_1 and θ_2, given the position of the end effector, our next task is to relate the joint angular velocities to the velocity of the manipulator. This is readily accomplished using (1.9). In particular,

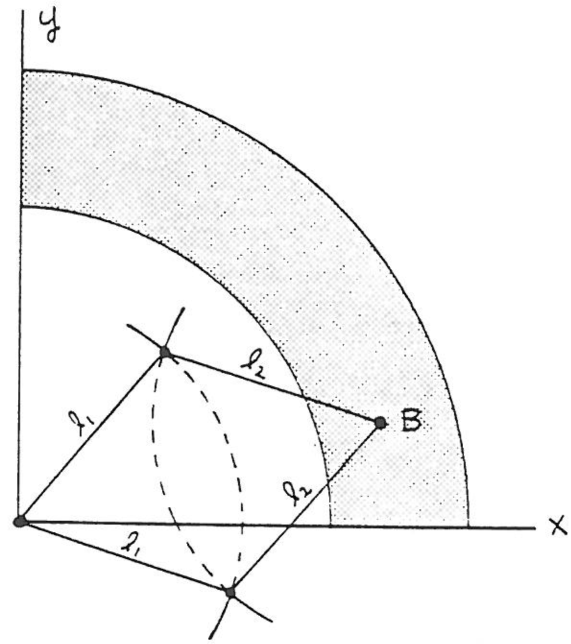

Fig. 2—4. Inverse kinematics problem with end effector close to boundary of workspace.

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} -\ell_1 s_1 - \ell_2 s_{12} & -\ell_2 s_{12} \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{bmatrix}^{-1} \begin{bmatrix} v_{2x} \\ v_{2y} \end{bmatrix} \tag{2.13}$$

It should be noted that the 2 by 2 matrix above is a function of the joint angles θ_1 and θ_2. It can be shown that this matrix is always invertible except when $\theta_2 = 0$ (link-1 and link-2 axes coincide). With the manipulator links stretched out thusly, the motion of the end effector must be along an arc of radius $\ell_1 + \ell_2$. In this case, the ratio of v_{2x} to v_{2y} must be equal to $-\tan\theta_1$. That is, we cannot choose v_{2x}, v_{2y} arbitrarily. The fact that the matrix in (2.13) cannot be inverted also implies that v_{2x} and v_{2y} cannot be independently specified for this (and only this) condition. This is something to keep in mind when planning the trajectory. In any case, we shall avoid this situation in subsequent analyses by assuming that θ_2 is limited to values *between* $\pm 90°$ wherein the matrix in (2.13) can always be inverted. Thus, we are always free to choose v_{2x} and v_{2y} arbitrarily. Finally, the joint angular accelerations, α_1, and α_2, may be related to the acceleration of the end effector using (1.5). In particular,

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -\ell_1 s_1 - \ell_2 s_{12} & -\ell_2 s_{12} \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} a_{2x} \\ a_{2y} \end{bmatrix} + \begin{bmatrix} \ell_1 c_1 \omega_1^2 + \ell_2 c_{12}(\omega_1 + \omega_2)^2 \\ \ell_1 s_1 \omega_1^2 + \ell_2 s_{12}(\omega_1 + \omega_2)^2 \end{bmatrix} \right\} \tag{2.14}$$

It should be noted that we first need to establish the joint angular positions, θ_1 and θ_2, as well as the joint angular velocities, ω_1 and ω_2, before we can establish the angular accelerations, α_1 and α_2. The solution procedure of the inverse kinematics problem is summarized in Fig. 2-5.

3. MANIPULATOR DYNAMICS

Our task here is to relate the motion of the manipulator to the torque that must be applied to the manipulator joints so as to achieve this motion. Let's start by stating what we know and what our assumptions are going to be. We will assume that we know the joint angles, θ_1 and θ_2, as well as their first and second derivatives (angular velocities and accelerations). Also, we will assume that the mass of each link can be represented by two equivalent masses, one located at joint-2 and the other at the end of the manipulator. This is the so called point mass assumption. The equivalent mass at joint-2, which will be denoted as m_1, may include the mass of the actuator used to position link-2. Likewise, the equivalent mass at the end of the manipulator, m_2, may include that of the end effector.

In accordance with Newton's Law of motion, a force of $\bar{F}_2(t)$ must act on m_2 to provide an acceleration $\bar{a}_2(t)$. In particular,

$$\bar{F}_2(t) = m_2\, \bar{a}_2(t) \tag{3.1}$$

where $\bar{a}_2(t)$ is given by (1.4). Wait a minute! We have neglected an important