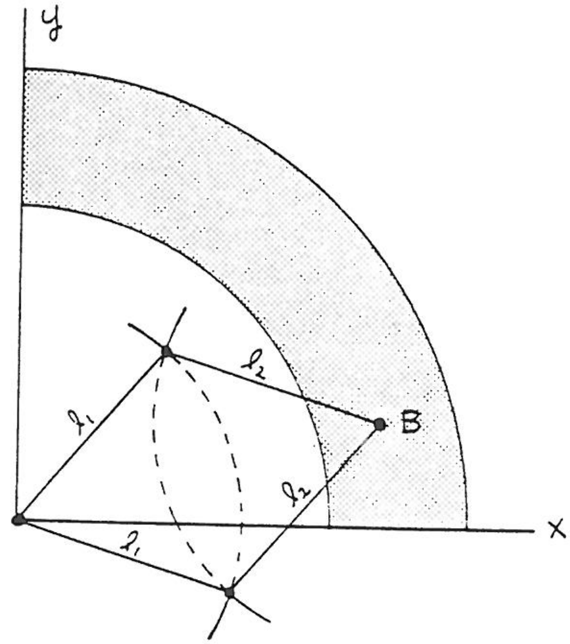

Fig. 2—4. Inverse kinematics problem with end effector close to boundary of workspace.

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} -\ell_1 s_1 - \ell_2 s_{12} & -\ell_2 s_{12} \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{bmatrix}^{-1} \begin{bmatrix} v_{2x} \\ v_{2y} \end{bmatrix} \tag{2.13}$$

It should be noted that the 2 by 2 matrix above is a function of the joint angles θ_1 and θ_2. It can be shown that this matrix is always invertible except when $\theta_2 = 0$ (link-1 and link-2 axes coincide). With the manipulator links stretched out thusly, the motion of the end effector must be along an arc of radius $\ell_1 + \ell_2$. In this case, the ratio of v_{2x} to v_{2y} must be equal to $-\tan\theta_1$. That is, we cannot choose v_{2x}, v_{2y} arbitrarily. The fact that the matrix in (2.13) cannot be inverted also implies that v_{2x} and v_{2y} cannot be independently specified for this (and only this) condition. This is something to keep in mind when planning the trajectory. In any case, we shall avoid this situation in subsequent analyses by assuming that θ_2 is limited to values *between* $\pm 90°$ wherein the matrix in (2.13) can always be inverted. Thus, we are always free to choose v_{2x} and v_{2y} arbitrarily. Finally, the joint angular accelerations, α_1, and α_2, may be related to the acceleration of the end effector using (1.5). In particular,

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -\ell_1 s_1 - \ell_2 s_{12} & -\ell_2 s_{12} \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} a_{2x} \\ a_{2y} \end{bmatrix} + \begin{bmatrix} \ell_1 c_1 \omega_1^2 + \ell_2 c_{12}(\omega_1 + \omega_2)^2 \\ \ell_1 s_1 \omega_1^2 + \ell_2 s_{12}(\omega_1 + \omega_2)^2 \end{bmatrix} \right\} \tag{2.14}$$

It should be noted that we first need to establish the joint angular positions, θ_1 and θ_2, as well as the joint angular velocities, ω_1 and ω_2, before we can establish the angular accelerations, α_1 and α_2. The solution procedure of the inverse kinematics problem is summarized in Fig. 2-5.

3. MANIPULATOR DYNAMICS

Our task here is to relate the motion of the manipulator to the torque that must be applied to the manipulator joints so as to achieve this motion. Let's start by stating what we know and what our assumptions are going to be. We will assume that we know the joint angles, θ_1 and θ_2, as well as their first and second derivatives (angular velocities and accelerations). Also, we will assume that the mass of each link can be represented by two equivalent masses, one located at joint-2 and the other at the end of the manipulator. This is the so called point mass assumption. The equivalent mass at joint-2, which will be denoted as m_1, may include the mass of the actuator used to position link-2. Likewise, the equivalent mass at the end of the manipulator, m_2, may include that of the end effector.

In accordance with Newton's Law of motion, a force of $\overline{F}_2(t)$ must act on m_2 to provide an acceleration $\overline{a}_2(t)$. In particular,

$$\overline{F}_2(t) = m_2\, \overline{a}_2(t) \tag{3.1}$$

where $\overline{a}_2(t)$ is given by (1.4). Wait a minute! We have neglected an important

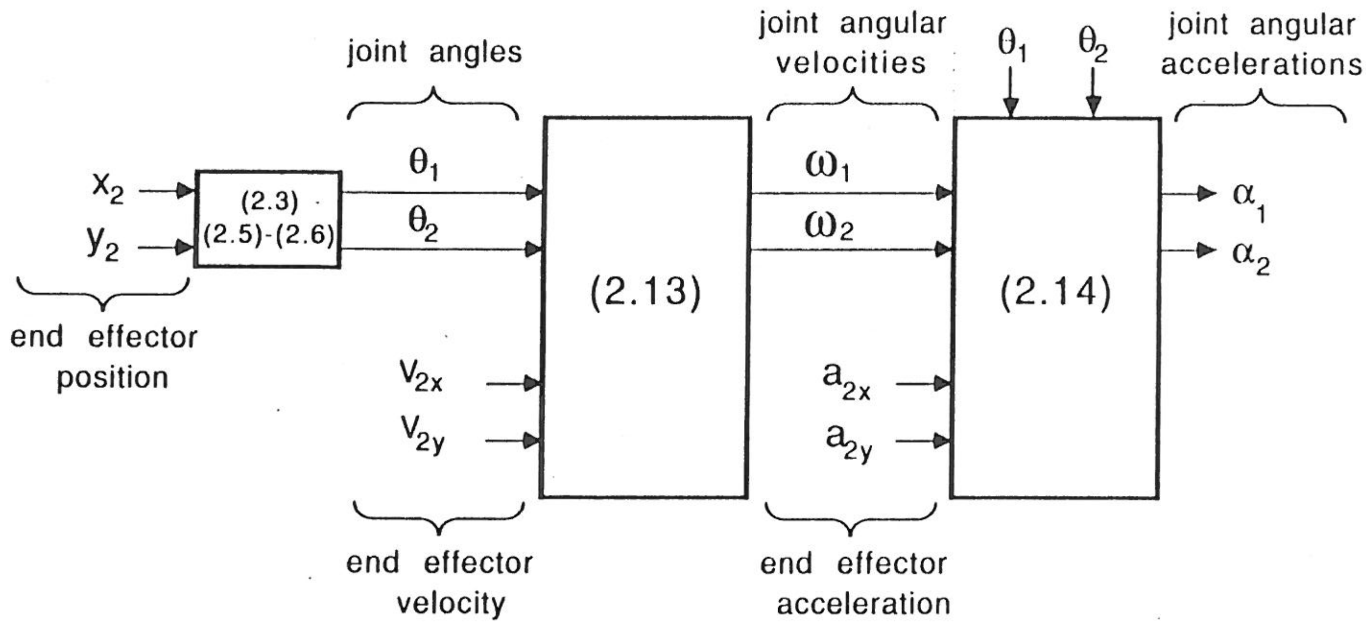

Fig. 2—5. Block diagram illustrating solution of the inverse kinematics problem. Solution proceeds from left to right.

component of force. Unless the manipulator is somewhere out in space, a gravitational force will exist in the $-\hat{a}_y$ direction. This force will be numerically equal to $m_2 g$ where g represents the acceleration constant (9.8 m/sec^2 on Earth). To counteract gravity, $\overline{F}_2(t)$ must include an additional component in the $\hat{a}_y$ direction numerically equal to $m_2 g$. In particular,

$$\begin{aligned} \overline{F}_2 &= m_2 \overline{a}_2 + m_2 g\ \hat{a}_y \\ &= (m_2\ a_{2x})\hat{a}_x + m_2(a_{2y} + g)\ \hat{a}_y \qquad (3.2) \end{aligned}$$

Likewise, a force $\overline{F}_1(t)$ must act on m_1 to produce an acceleration $\overline{a}_1(t)$. Including the gravitational component,

$$\begin{aligned} \overline{F}_1 &= m_1 \overline{a}_1 + m_1 g\ \hat{a}_y \\ &= (m_1 a_{1x})\hat{a}_x + m_1(a_{1y} + g)\hat{a}_y \qquad (3.3) \end{aligned}$$

where a_{1x}, a_{1y} represent the x and y components of $\overline{a}_1$ given in (1.7).

Thus, we have established the forces which must act on masses m_1 and m_2 to produce our desired motion. Our next task is to relate these forces to the resultant torque at the two joints of our manipulator. To do this, let us momentarily focus our attention on a simpler manipulator involving only a single link and joint as shown in Fig. 3-1.

We will assume that we have already established the force $\overline{F} = F_x\hat{a}_x + F_y\hat{a}_y$ which must be applied to mass m located at the end of the link so as to achieve the desired motion. We can express this force in terms of the standard unit direction vectors as we have done. Alternatively, we can define a new set of unit vectors, one oriented along the link axis ($\hat{a}_r$) the other oriented tangent to the link axis ($\hat{a}_\theta$) as shown in Fig. 3-2. The force $\overline{F}$ can be expressed in terms of these unit vectors as

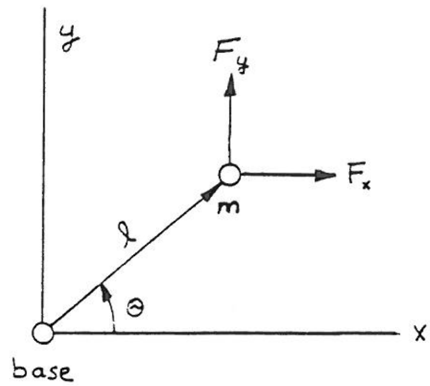

Fig. 3-1. Elementary one link manipulator.

$$\overline{F} = (F_x \cos\theta + F_y \sin\theta)\hat{a}_r + (F_y \cos\theta - F_x \sin\theta)\hat{a}_\theta \tag{3.4}$$

Now, the component of force along the $\hat{a}_r$ direction will not produce a torque. It will produce a *tensile* force on the link. On the other hand, the $\hat{a}_\theta$ component will produce a torque in the counterclockwise direction. This torque will be equal to the force in the $\hat{a}_\theta$ direction multiplied by the length, ℓ, of the given link. In particular

$$\tau = \ell\,(F_y \cos\theta - F_x \sin\theta) \tag{3.5}$$

Now, consider the vector cross product $\overline{r}\times\overline{F}$ where $\overline{r}$ is the vector specifying the position of the end effector. From Fig. 3-2,

$$\overline{r}\times\overline{F} = (\ell \cos\theta\ \hat{a}_x + \ell \sin\theta\ \hat{a}_y) \times (F_x\hat{a}_x + F_y\hat{a}_y) \tag{3.6}$$

Expanding the right side using the familiar distributive law of algebra yields

$$\overline{r}\times\overline{F} = \ell\cos\theta\ F_y(\hat{a}_x\times\hat{a}_y) + \ell\sin\theta\ F_x(\hat{a}_y\times\hat{a}_x) \tag{3.7}$$

where we have made use of the fact that $(\hat{a}_x\times\hat{a}_x)$ and $(\hat{a}_y\times\hat{a}_y)$ are both zero. Now $\hat{a}_x\times\hat{a}_y$ is another unit vector oriented perpendicular to both $\hat{a}_x$ and $\hat{a}_y$; pointing in a

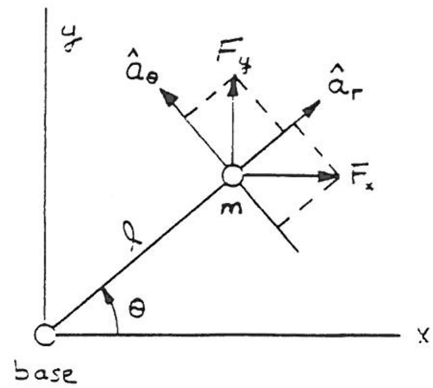

Fig. 3-2. Alternative unit direction vectors.

direction determined by the right hand rule, i.e. if we turn $\hat{a}_x$ into $\hat{a}_y$, the progression of an imaginary screw at the common origin of $\hat{a}_x$ and $\hat{a}_y$ determines the direction of $\hat{a}_x \times \hat{a}_y$. Here, it is pointed outward from the plane of the paper (in the $\hat{a}_z$ direction). In view of this

$$\bar{r} \times \bar{F} = (\ell \cos\theta \, F_y - \ell \sin\theta \, F_x) \, \hat{a}_z \tag{3.8}$$

Comparison with (3.5) reveals that the torque τ is the $\hat{a}_z$ component of $\bar{r} \times \bar{F}$, i.e.,

$$\tau = (\bar{r} \times \bar{F}) \cdot \hat{a}_z \tag{3.9}$$

When interpreting (3.9), it should be understood that taking the dot product of a given vector with the unit vector, $\hat{a}_z$, is the same as extracting the z component of that vector. The cross product $\bar{r} \times \bar{F}$ is called the moment of $\bar{F}$ about the joint axis (it is a vector) whereas the joint torque τ (a scalar) is the $\hat{a}_z$ component of this moment.

With these preliminary results out of the way, let's return to the 2 link planar manipulator of Fig. 3-3 where the ~~$\hat{a}_x$ and $\hat{a}_y$ components of the~~ forces $\bar{F}_1$ and $\bar{F}_2$ are indicated. Now, $\bar{F}_2$ will give rise to a moment about joint-2. The corresponding

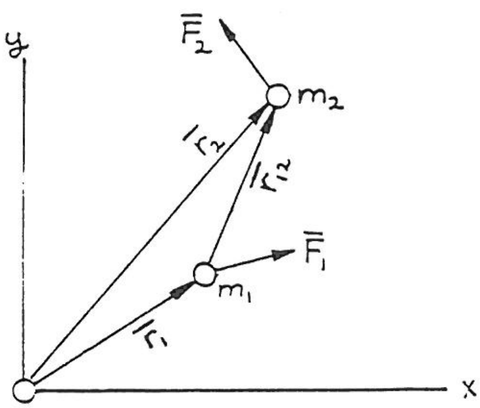

Fig. 3-3. Force components in the 2 link planar manipulator.

torque, τ_2, will be equal to the $\hat{a}_z$ component of this moment. In particular,

$$\tau_2 = (\overline{r}_{12} \times \overline{F}_2) \cdot \hat{a}_z \tag{3.10}$$

where $\overline{r}_{12}$ is the vector pointing from joint-2 to the end effector (point of application of $\overline{F}_2$). From Fig. 3-3,

$$\overline{r}_{12} = \ell_2 \cos(\theta_1 + \theta_2)\hat{a}_x + \ell_2 \sin(\theta_1 + \theta_2)\hat{a}_y \tag{3.11}$$

Evaluating the cross product in (3.10) and taking the z component gives

$$\tau_2 = \ell_2 \cos(\theta_1 + \theta_2)F_{2y} - \ell_2 \sin(\theta_1 + \theta_2)F_{2x} \tag{3.12}$$

Substituting (3.2) into (3.12) and using shorthand notation for sin θ and cos θ

$$\tau_2 = m_2\ell_2 c_{12}(a_{2y} + g) - m_2\ell_2 s_{12} a_{2x} \tag{3.13}$$

Finally, substituting a_{2x}, a_{2y} from (1.4) into (3.13) and simplifying the results gives

$$\tau_2 = m_2\ell_1\ell_2 c_2\alpha_1 + m_2\ell_1\ell_2 s_2\omega_1^2 + m_2\ell_2^2(\alpha_1 + \alpha_2) + m_2 g\ell_2 c_{12} \tag{3.14}$$

This represents the torque which must be applied to joint-2 so as to achieve the desired end effector acceleration in the presense of gravity. The component of torque needed to overcome gravity is readily identified in (3.14) (last term).

What about the torque which must be applied to joint-1 at the base of the manipulator? The net moment about joint-1 is obtained by summing the moments attributed to $\overline{F}_1$ and $\overline{F}_2$. The, torque, τ_1, is the $\hat{a}_z$ component of this moment. Specifically,

$$\tau_1 = \left[(\overline{r}_2 \times \overline{F}_2) + (\overline{r}_1 \times \overline{F}_1)\right] \cdot \hat{a}_z \tag{3.15}$$

where $\overline{r}_2$ is the vector drawn from the origin to the end effector (point where $\overline{F}_2$ is applied) and $\overline{r}_1$ is drawn from the origin to joint-2 (point where $\overline{F}_1$ is applied). Noting that $\overline{r}_2 = \overline{r}_1 + \overline{r}_{12}$,

$$\begin{aligned}\tau_1 &= \left[(\overline{r}_{12} \times \overline{F}_2) + (\overline{r}_1 \times \overline{F}_2) + (\overline{r}_1 \times \overline{F}_1)\right] \cdot \hat{a}_z \\ &= (\overline{r}_{12} \times \overline{F}_2) \cdot \hat{a}_z + \left[\overline{r}_1 \times (\overline{F}_1 + \overline{F}_2)\right] \cdot \hat{a}_z \end{aligned} \tag{3.16}$$

From (3.10), the first term in the right side of (3.16) is simply τ_2. Thus,

$$\tau_1 = \tau_2 + \left[\overline{r}_1 \times (\overline{F}_1 + \overline{F}_2)\right] \cdot \hat{a}_z \tag{3.17}$$

From this result, we see that in order to calculate τ_1, we can assume that both $\overline{F}_1$ and $\overline{F}_2$ are applied at the end of link-1 and calculate the resulting moment about the joint at the base of the manipulator. If we do this, however, we must add τ_2 to the resulting torque. We can easily generalize this result for an n link manipulator; however, we will not do so. In any case,

$$\begin{aligned}\overline{r}_1 \times (\overline{F}_1 + \overline{F}_2) &= \left[\ell_1 c_1 \hat{a}_x + \ell_1 s_1 \hat{a}_y\right] \times \left[(F_{1x} + F_{2x})\, \hat{a}_x + (F_{1y} + F_{2y})\, \hat{a}_y\right] \\ &= \left[\ell_1 c_1 (F_{1y} + F_{2y}) - \ell_1 s_1 (F_{1x} + F_{2x})\right] \hat{a}_z \end{aligned} \tag{3.18}$$

Thus,

S1

$$\tau_1 = \tau_2 + \ell_1 c_1 (F_{1y} + F_{2y}) - \ell_1 \cancel{c_1} (F_{1x} + F_{2x}) \tag{3.19}$$

Substituting F_{1x}, F_{1y}, F_{2x}, F_{2y} from (3.2)-(3.3) into (3.19) gives S1

$$\tau_1 = \tau_2 + \ell_1 c_1 \left[m_1 (a_{1y} + g) + m_2 (a_{2y} + g) \right] - \ell_1 \cancel{c_1} \left[m_1 a_{1x} + m_2 a_{2x} \right] \tag{3.20}$$

Lastly, substituting a_{2x}, a_{2y} from (1.4), a_{1x}, a_{1y} from (1.7) and τ_2 from (3.14) into the previous equation and spending a month of Sundays to "simplify" the results gives

$$\begin{aligned} \tau_1 = {} & m_2 \ell_2^2 (\alpha_1 + \alpha_2) + m_2 \ell_1 \ell_2 c_2 (2\alpha_1 + \alpha_2) \\ & + (m_1 + m_2) \ell_1^2 \alpha_1 - m_2 \ell_1 \ell_2 s_2 \omega_2^2 - 2 m_2 \ell_1 \ell_2 s_2 \omega_1 \omega_2 \\ & + (m_1 + m_2)\, g \ell_1 c_1 + m_2 g \ell_2 c_{12} \end{aligned} \tag{3.21}$$

The last two terms in (3.21) represent the components of torque needed to overcome gravitational effects and are called the gravitational torques. Terms which we proportional to the square of an angular velocity (ω^2 terms) are called *centrifugal* torques. Terms related to the product of two angular velocities (e.g. $-2m_2\ell_1\ell_2\omega_1\omega_2$) are called *Coriolis* torques. Finally, terms proportional to angular acceleration, α, will be referred to as the accelerating torques.

Let's take some time to look at what we have accomplished. Equations (3.14) and (3.21) give us expressions for the joint torques in terms of the joint angular positions, velocities and accelerations. The complexity of these equations is somewhat surprising since we have selected one of the simplest manipulators imaginable. However, this is a relatively minor inconvenience which we have to accept. In any case, if we are given the desired trajectory of the end effector, we can now solve the inverse kinematics problem to give us the required joint angular positions, velocities and accelerations. Then, using the previous results, we can establish the required joint torques.

Example 3A

Lets apply the previous results to a numerical example. In particular, suppose that for the given two link manipulator:

$$\ell_1 = \ell_2 = 1.0 \text{ m}$$
$$m_1 = 5 \text{ lb} = 2.268 \text{ kg}$$
$$m_2 = 10 \text{ lb} = 4.535 \text{ kg}$$

The initial position of the end effector is specified as (x=1.5, y=0) and the desired final position is (x=1.5, y=1.0). In addition, we also have to specify the path which the end effector is to follow and the rate at which this motion is to occur. We'll assume that the motion of the end effector is along a straight vertical path as shown in Fig. 3A-1 and that this motion occurs within a time period of 1 sec. During the

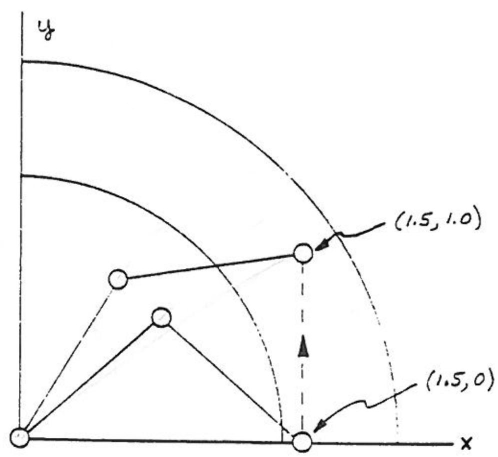

Fig. 3A-1. Selected manipulator trajectory.

interval from 0 to 0.5 sec, the acceleration in the vertical direction is assumed to be constant and equal to 4 m/sec^2. During the period from 0.5 to 1.0 sec (deceleration interval), the acceleration is constant and equal to -4 m/sec^2. The end effector velocity (position) may be obtained by integrating the acceleration (velocity) with respect to time. The results of this integration may be summarized as follows