S1

$$\tau_1 = \tau_2 + \ell_1 c_1 (F_{1y} + F_{2y}) - \ell_1 \cancel{c_1} (F_{1x} + F_{2x}) \tag{3.19}$$

Substituting F_{1x}, F_{1y}, F_{2x}, F_{2y} from (3.2)-(3.3) into (3.19) gives S1

$$\tau_1 = \tau_2 + \ell_1 c_1 \left[m_1(a_{1y} + g) + m_2(a_{2y} + g) \right] - \ell_1 \cancel{c_1} \left[m_1 a_{1x} + m_2 a_{2x} \right] \tag{3.20}$$

Lastly, substituting a_{2x}, a_{2y} from (1.4), a_{1x}, a_{1y} from (1.7) and τ_2 from (3.14) into the previous equation and spending a month of Sundays to "simplify" the results gives

$$\begin{aligned} \tau_1 = {} & m_2 \ell_2^2 (\alpha_1 + \alpha_2) + m_2 \ell_1 \ell_2 c_2 (2\alpha_1 + \alpha_2) \\ & + (m_1 + m_2) \ell_1^2 \alpha_1 - m_2 \ell_1 \ell_2 s_2 \omega_2^2 - 2 m_2 \ell_1 \ell_2 s_2 \omega_1 \omega_2 \\ & + (m_1 + m_2)\, g \ell_1 c_1 + m_2 g \ell_2 c_{12} \end{aligned} \tag{3.21}$$

The last two terms in (3.21) represent the components of torque needed to overcome gravitational effects and are called the gravitational torques. Terms which we proportional to the square of an angular velocity (ω^2 terms) are called *centrifugal* torques. Terms related to the product of two angular velocities (e.g. $-2m_2\ell_1\ell_2\omega_1\omega_2$) are called *Coriolis* torques. Finally, terms proportional to angular acceleration, α, will be referred to as the accelerating torques.

Let's take some time to look at what we have accomplished. Equations (3.14) and (3.21) give us expressions for the joint torques in terms of the joint angular positions, velocities and accelerations. The complexity of these equations is somewhat surprising since we have selected one of the simplest manipulators imaginable. However, this is a relatively minor inconvenience which we have to accept. In any case, if we are given the desired trajectory of the end effector, we can now solve the inverse kinematics problem to give us the required joint angular positions, velocities and accelerations. Then, using the previous results, we can establish the required joint torques.

Example 3A

Lets apply the previous results to a numerical example. In particular, suppose that for the given two link manipulator:

$$\ell_1 = \ell_2 = 1.0 \text{ m}$$
$$m_1 = 5 \text{ lb} = 2.268 \text{ kg}$$
$$m_2 = 10 \text{ lb} = 4.535 \text{ kg}$$

The initial position of the end effector is specified as (x=1.5, y=0) and the desired final position is (x=1.5, y=1.0). In addition, we also have to specify the path which the end effector is to follow and the rate at which this motion is to occur. We'll assume that the motion of the end effector is along a straight vertical path as shown in Fig. 3A-1 and that this motion occurs within a time period of 1 sec. During the

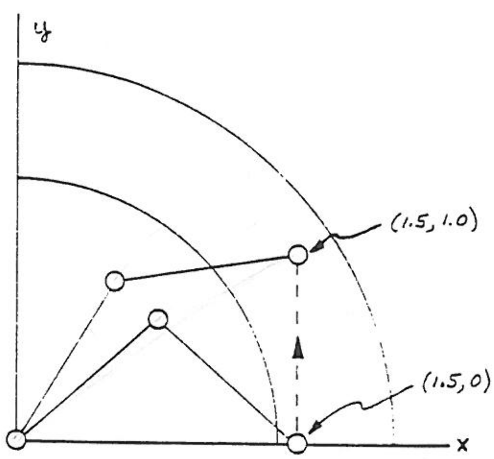


Fig. 3A-1. Selected manipulator trajectory.

interval from 0 to 0.5 sec, the acceleration in the vertical direction is assumed to be constant and equal to 4 m/sec^2. During the period from 0.5 to 1.0 sec (deceleration interval), the acceleration is constant and equal to -4 m/sec^2. The end effector velocity (position) may be obtained by integrating the acceleration (velocity) with respect to time. The results of this integration may be summarized as follows

$0 < t < 0.5$	$0.5 < t < 1.0$
$a_{2y} = 4$	$a_{2y} = -4$
$v_{2y}(t) = 4\ t$	$v_{2y}(t) = 4(1 - t)$
$y_2(t) = 2t^2$	$y_2(t) = -2t^2 + 4t - 1$

The vertical component of the end effector acceleration, velocity and position are plotted in Fig. 3A-2. It should be noted that we could have selected a different acceleration profile while reaching our final destination within the allotted time. Unfortunately, it is no simple matter to determine what the best (optimum) profile is as far as the actuators are concerned. This represents a fundamental problem in trajectory planning which is better covered in a specialized course on robotic systems. In any case, let's go with the given trajectory, and proceed to calculate the joint angular positions, velocities, accelerations and torques.

The joint angles may be determined from the x and y coordinates of the end effector using (2.3) and (2.5)-(2.6). These equations may be repeatedly applied at uniformly selected time instants to provide a smooth plot of $\theta_1(t)$ and $\theta_2(t)$ throughout the selected time interval. Once θ_1 and θ_2 are established, the joint angular velocities and accelerations may be determined by repeatedly applying (2.13) and (2.14) at the selected time instants. Note that the elements of the matrices in (2.13)-(2.14) are functions of the previously calculated joint angles. Given the joint angles, velocities and accelerations, the required joint torques are calculated using (3.14) and (3.21). The final results are plotted in Fig. 3A-3 (joint-1 variables) and Fig. 3A-4 (joint-2 variables). In Figs. 3A-3 and 3A-4, all variables have been previously defined with the exception of τ_{1g} and τ_{2g}. These represent the components of τ_1 and τ_2 needed to overcome gravity and are calculated by including only those terms proportional to g in (3.14) and (3.21). Interestingly, τ_{1g} is nearly constant (independent of time) and

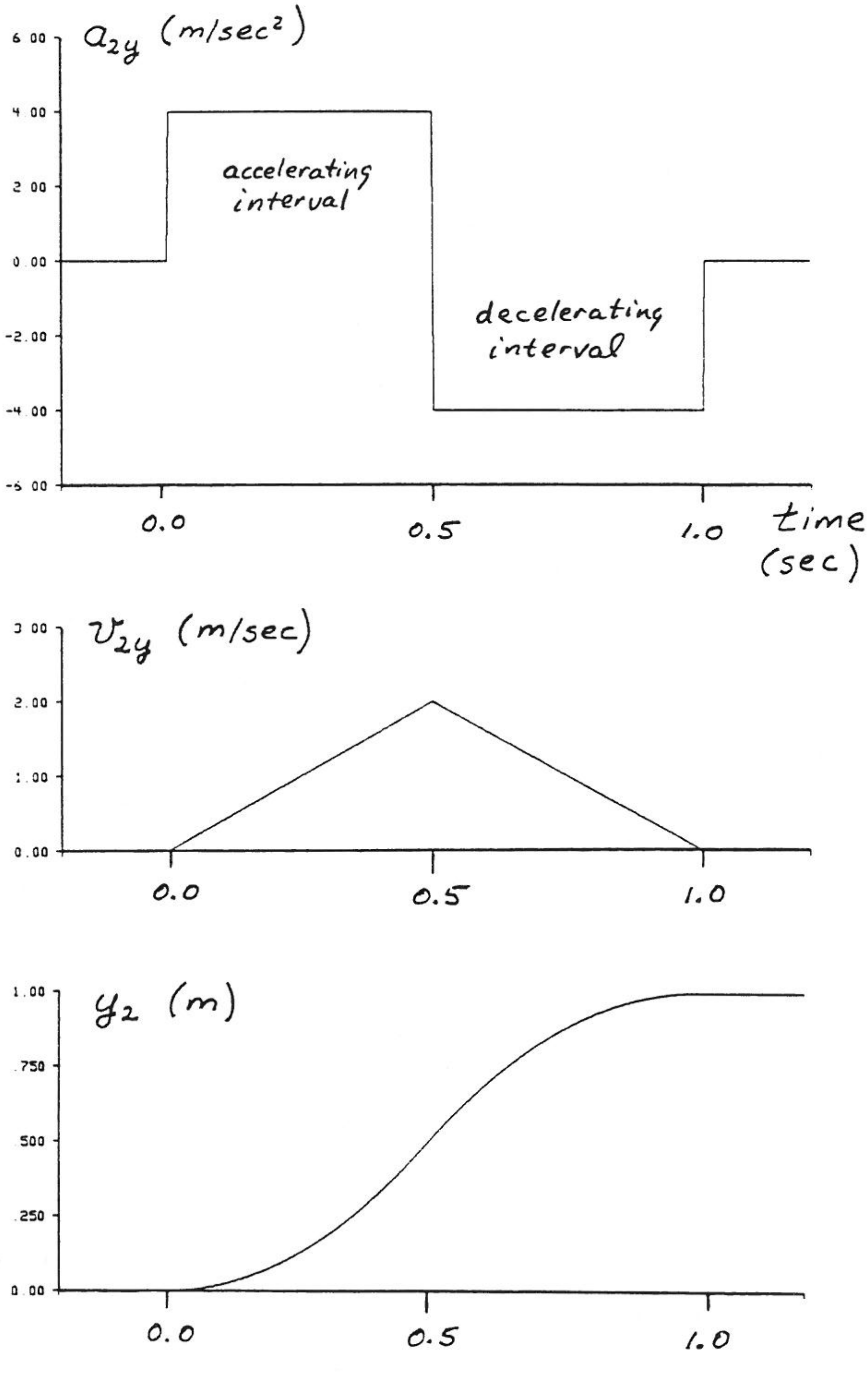


Fig. 3A-2. End effector vertical acceleration, velocity and position.

although τ_{2g} varies somewhat, it does not vary near as much as τ_2 implying that the change in τ_2 is primarily due to the accelerating, centrifugal and Coriolis terms in our torque equation. It is interesting to observe that the general form of τ_1 and τ_2 is similar to the assumed vertical acceleration of the end effector (Fig. 3A-2). That is, we have an initial, positive jump at $t=0$, followed by a negative jump at $t=0.5$ and another positive jump at $t=1$. In between jumps, the joint torques vary somewhat due, in part, to the change in gravitational loading and, more significantly, because of the centrifugal, Coriolis and accelerating components of torque.

The instantaneous power supplied by an actuator is the product of its torque, τ, and speed, ω. For purposes of illustration, these are plotted in Fig. 3A-5 where p_1 and p_2 represent, respectively, the power supplied by the actuator of joint-1 and joint-2. As shown, the power associated with the joint-1 actuator immediately begins to increase at $t=0$ as might be expected. At $t=0.5$, the required power suddenly decreases since the actuator torque suddenly decreases when we enter the deceleration interval (see τ_1 in Fig. 3A-3). Also, it is seen that very little power is initially required of the joint-2 actuator since ω_2 is quite small during the initial part of the trajectory, i.e. θ_2 does not change significantly (Fig. 3A-4). However, as time progresses ω_2 (and p_2) increase followed by a sudden decrease at $t=0.5$ for apparent reasons. No power supplied is supplied by either actuator for $t \geq 1.0$ since manipulator is at rest. It should be noted that the combined area bounded by the two curves in Fig. 3A-5 represents the total energy (in joules) needed to lift masses m_1 and m_2 from their initial to their final destinations. In particular, the total energy may be expressed

$$E = m_1 g(y_1^B - y_1^A) + m_2 g(y_2^B - y_2^A) \tag{3A-1}$$

where y_1^A, y_1^B represent the initial and final vertical coordinates of joint-2 and y_2^A, y_2^B represent the initial and final vertical coordinates of the end effector.

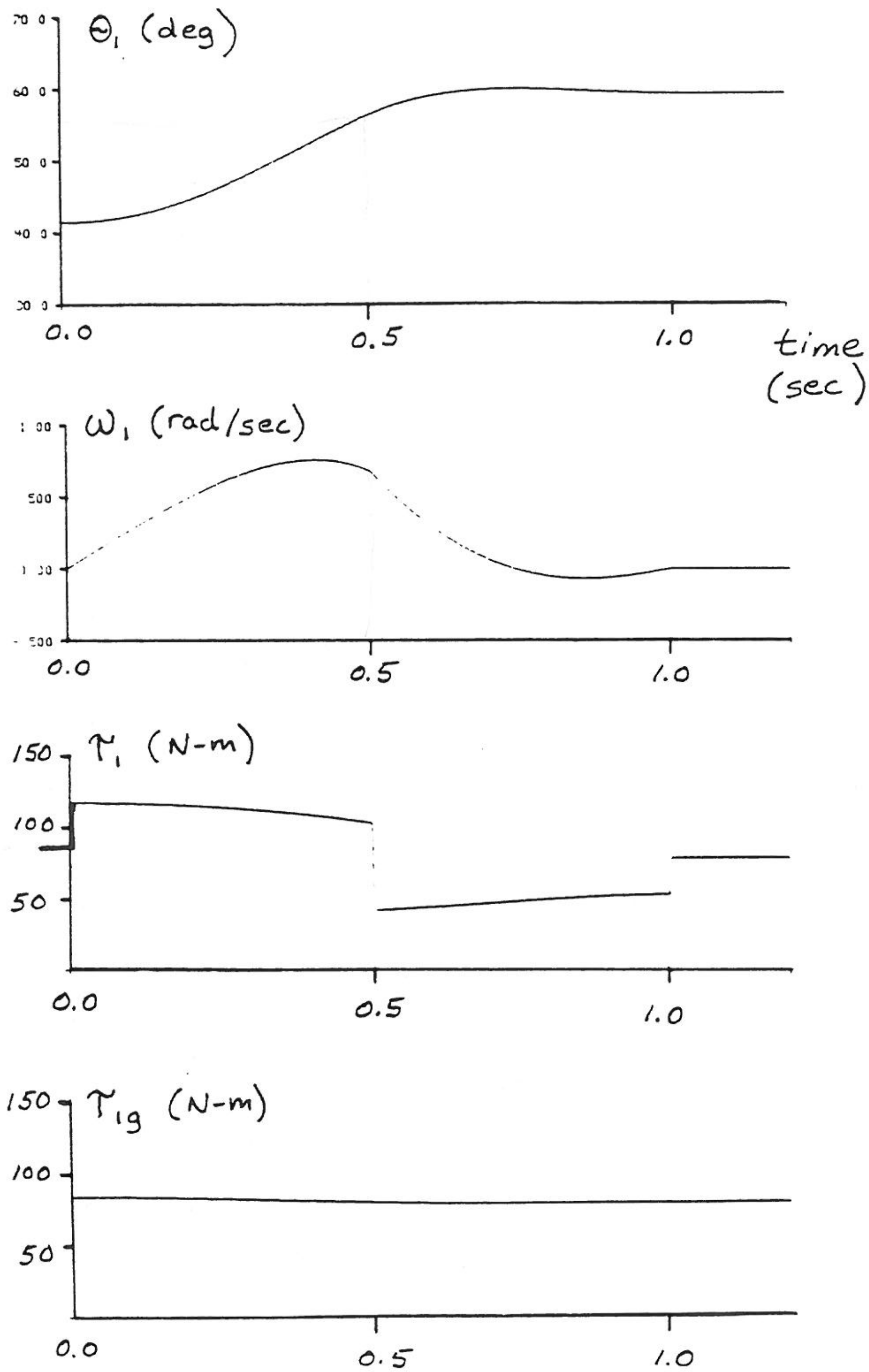


Fig. 3A-3. Joint-1 angular position, velocity and torque for selected trajectory.

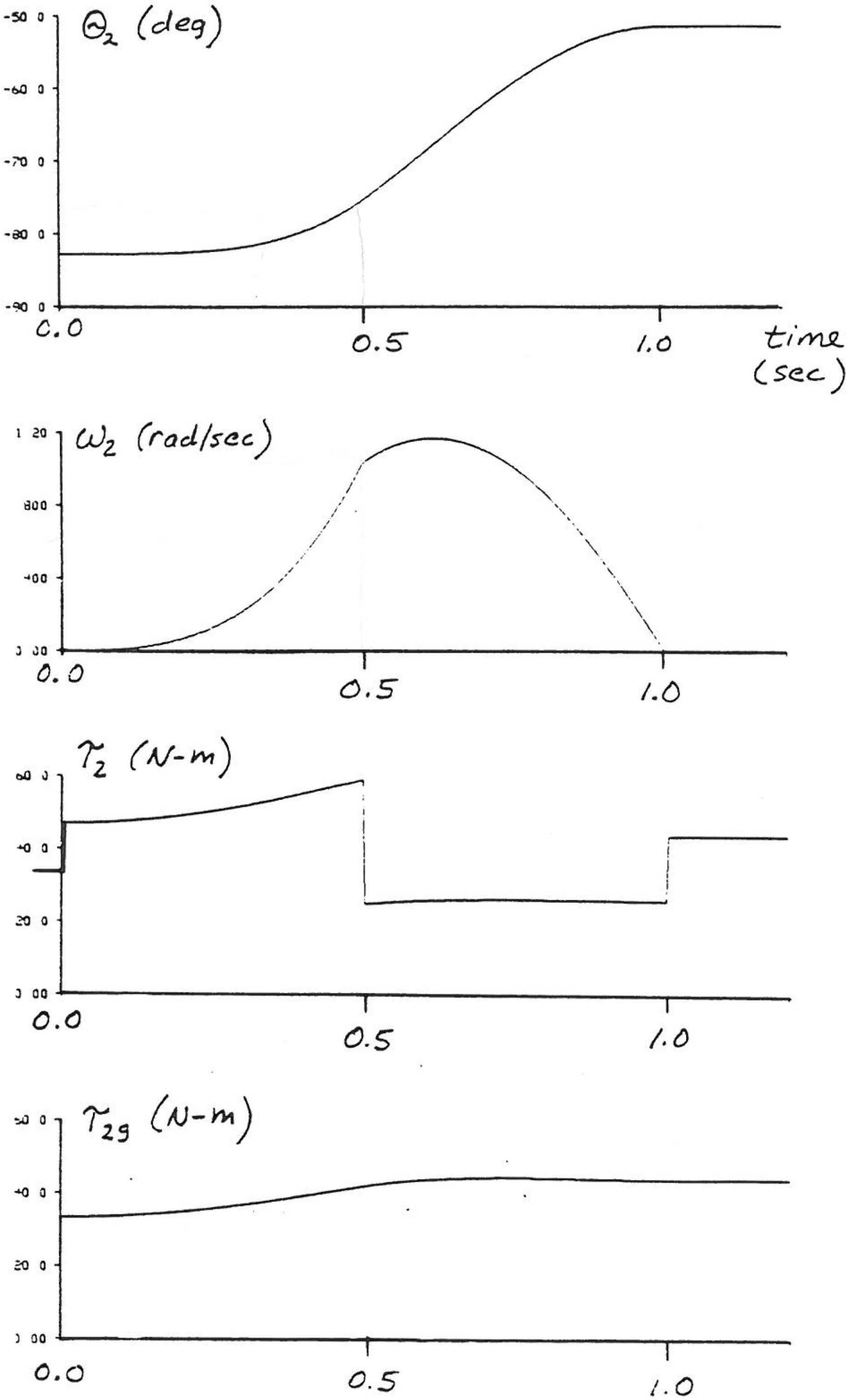
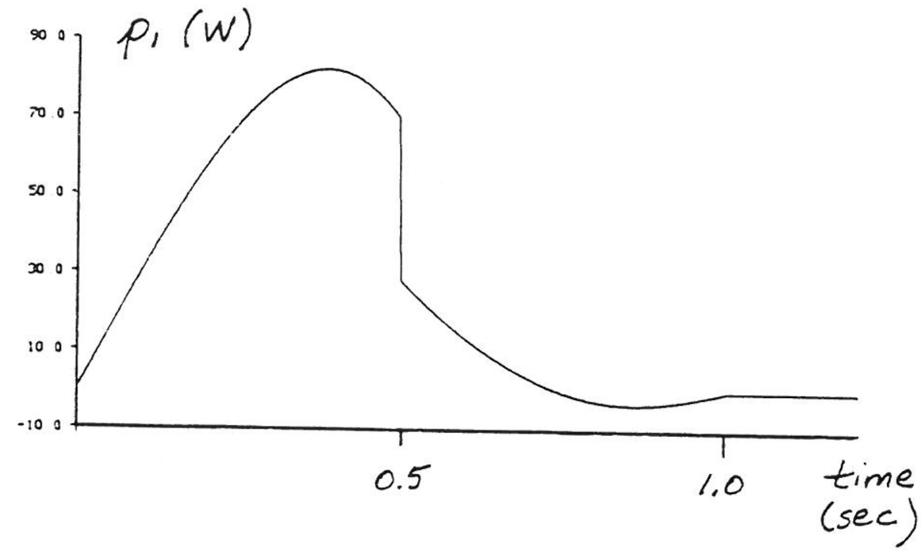


Fig. 3A-4. Joint-2 angular position, velocity and torque for selected trajectory.



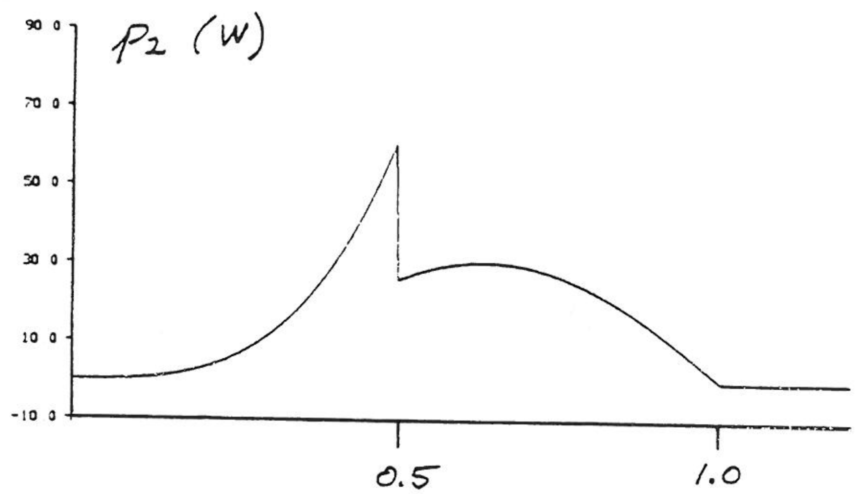


Fig. 3A-5. Mechanical power supplied by joint actuators for selected trajectory.

It is also instructive to plot the actuator torque, τ, versus its speed, ω. The torque-speed plots for both actuators are illustrated in Fig. 3A-6. In Fig. 3A-6, point A corresponds to $t=0^+$ and the arrows indicate the direction of the trajectory as time progresses.[1] In both cases, the speed increases from point A while the torques (τ_1 and τ_2) change only by a little. At $t=0.5$, the joint torques suddenly decrease but the speed cannot change instantaneously explaining the sudden downward jump at $t=0.5$ in Fig. 3A-6. Thereafter, the speed of joint-1 decreases while the torque remains more or less constant. In the case of joint-2, the speed momentarily increases during the interval beginning at $t=0.5^+$ before eventually decreasing. During this interval, the torque, τ_2, changes by only a little. At $t=1$, there is another vertical jump in both torque-speed trajectories since the joint torques suddenly increase (whereas speed remains continuous) when we reach our final destination. For $t \geq 1.0^+$, the manipulator is at rest.

From the given torque-speed plots, it is seen that the required joint torques are relatively large; approximately 115 N—m (16,300 oz—in) for joint-1 and 60 N—m (8,500 oz—in) for joint-2. However, the speed requirements are very low; less than 0.8 rad/sec for joint-1 and 1.2 rad/sec for joint-2. Although, we could find a motor which will supply the given torques, a motor-gear train combination may result in a more compact drive system. As will be seen, a gear train can be used to reduce the torque that needs to be supplied by the motor; however, the angular velocity of the motor will be proportionately larger. For example, suppose we select a gear train with a high speed to low speed (gear) ratio of N. The torque-speed characteristics

[1]Since the angular accelerations and joint torques, τ_1 and τ_2 are discontinuous at $t=0$, $t=0.5$ and $t=1$, it becomes necessary, for example, to distinguish $t=0^+$ from $t=0^-$. In particular, at $t=0^-$, the manipulator is at rest (no motion whatsoever). At $t=0^+$, the acceleration is nonzero (although the velocities remain equal to zero since velocities are, in general, continuous).

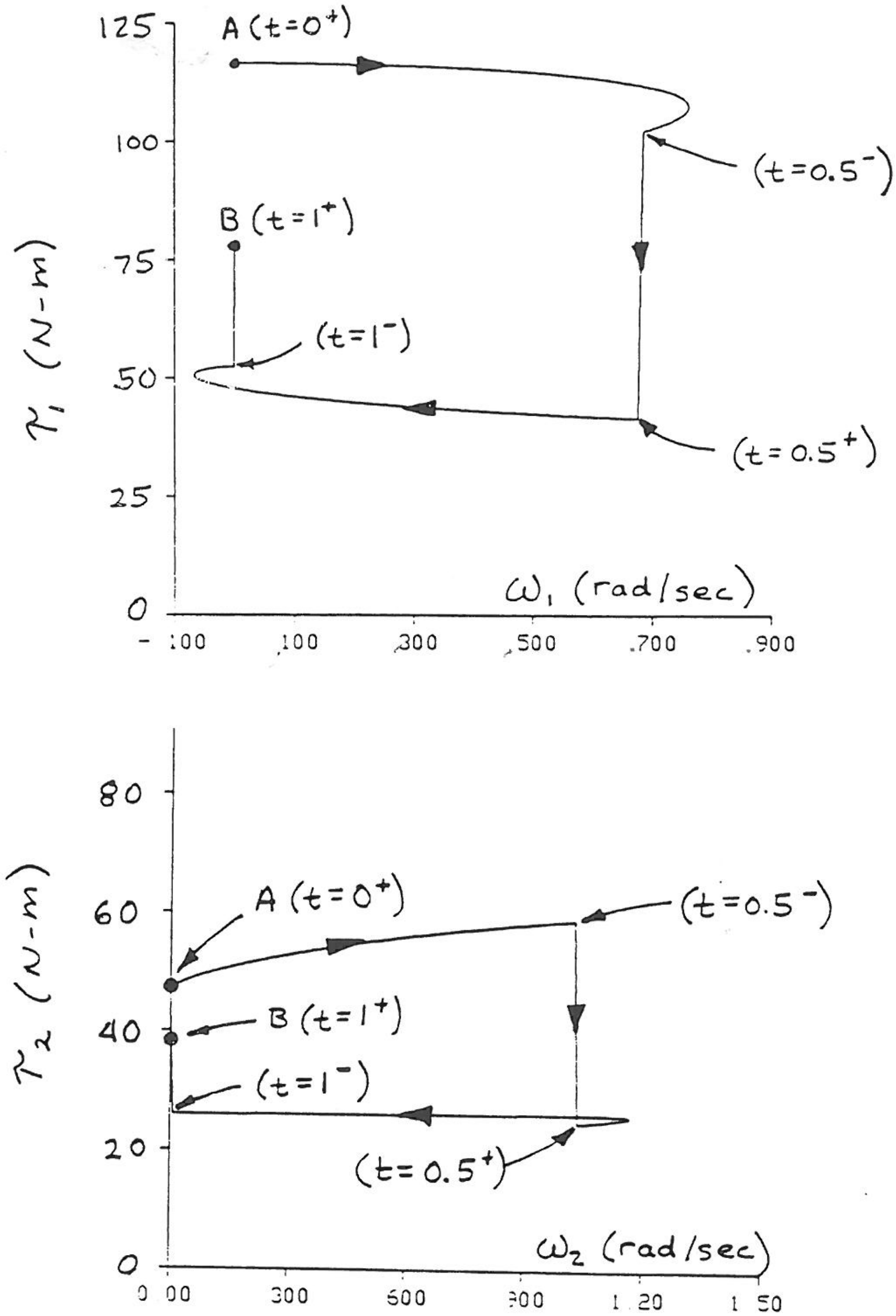


Fig. 3A-6. Torque-speed trajectories of joint actuators.

$T_e = I_a K_v$ $\qquad \left(\frac{V_s - k_v \omega_r}{R_m}\right) k_v = T_e$ $\qquad N = \frac{-1.306 \pm \sqrt{1.306^2 - 4 \cdots}}{2(1.306)}$ .0130

$1.306 = .01627\, \omega_r + T_e$ $\qquad \frac{V_a k_v}{R_m} - \frac{k_v^2 \omega_r}{R_m} = T_e$

$1.306 = (.01627)(.8)N + \frac{115}{N}$ $\qquad 1.306N - .01301N^2 - 115.0$

that will be seen by each motor may be established from Fig. 3A-6 by dividing the units of torque by N and multiplying the units of speed by N. The resulting plots must fall underneath the full load torque-speed characteristics of the motor (with room to spare since there will be losses associated with the gear train). From Fig. 3A-6, a gear ratio of 100 will reduce the peak motor torque to less than 1.15 N-m (163 oz-in) for joint-1 and 0.6 N-m (85 oz-in) for joint-2. Since the joint angular velocities are very small to begin with, the fact that the motor speed will be 100 times larger than in Fig. 3A-6 is of little consequence. The reduced torque requirements will result in a much smaller motor. However, this is offset by the fact that we have added a gear train. Clearly, there are tradeoff's involved.

4. SIMULATION

Given the joint angles and their first and second derivatives, we can establish the corresponding joint torques using (3.14) and (3.21). Sometimes, however, we are interested in the reverse problem; given the joint torques, what is the corresponding manipulator response? In this case, we have to solve the system of differential equations given by (3.14) and (3.21) for $\theta_1(t)$ and $\theta_2(t)$ given $\tau_1(t)$, $\tau_2(t)$ and the initial conditions $\theta_1(0)$, $\theta_2(0)$, $\omega_1(0)$, $\omega_2(0)$. To demonstrate how this can be accomplished, let's rewrite (3.14) and (3.21) in matrix form as

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \left[\begin{array}{c|c} m_2\ell_2^2 + 2m_2\ell_1\ell_2c_2 + (m_1+m_2)\ell_1^2 & m_2\ell_2^2 + m_2\ell_1\ell_2c_2 \\ \hline m_2\ell_1\ell_2c_2 + m_2\ell_2^2 & m_2\ell_2^2 \end{array}\right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -m_2\ell_1\ell_2s_2\omega_2^2 - 2m_2\ell_1\ell_2s_2\omega_1\omega_2 \\ m_2\ell_1\ell_2s_2\omega_1^2 \end{bmatrix} + g \begin{bmatrix} (m_1+m_2)\ell_1c_1 + m_2\ell_2c_{12} \\ m_2\ell_2c_{12} \end{bmatrix} \qquad (4.1)$$

This equation can be expressed symbolically as

$$\bar{\tau} = \mathbf{M}(\bar{\theta})\bar{\alpha} + \mathbf{V}(\bar{\theta},\bar{\omega}) + \mathbf{G}(\bar{\theta}) \qquad (4.2)$$

where $\bar{\tau}$ is a 2x1 matrix[1] consisting of the joint torques; $\bar{\alpha}$, $\bar{\omega}$ and $\bar{\theta}$ are each 2x1 vectors consisting of, respectively, the joint angular accelerations, velocities and displacements. Also, $\mathbf{M}(\bar{\theta})$ is a 2x2 "mass" matrix, $\mathbf{V}(\bar{\theta},\bar{\omega})$ is a 2x1 vector of Coriolis and centrifugal terms and $\mathbf{G}(\bar{\theta})$ is a vector of gravitational torques. Recalling that $\omega = \frac{d\theta}{dt}$, $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$, the previous equation can be expressed

$$\bar{\tau} = \mathbf{M}(\bar{\theta})\,\frac{d^2\bar{\theta}}{dt^2} + \mathbf{V}(\bar{\theta},\,\frac{d\bar{\theta}}{dt}) + \mathbf{G}(\bar{\theta}) \qquad (4.3)$$

Solving for $\frac{d^2\bar{\theta}}{dt^2}$,

[1]Matrices with only column are commonly called vectors. Consequently, the same bar notation will be used here. However, unlike the vectors considered previously, we do not attempt to associate a physical direction with the vector $\bar{\tau}$. General matrices or vector valued functions (vectors which are functions of other vectors) will be symbolized using bold type.

$$\frac{d^2\overline{\theta}}{dt^2} = \mathbf{M}^{-1}(\overline{\theta}) \left[\overline{\tau} - \mathbf{V}(\overline{\theta}, \frac{d\overline{\theta}}{dt}) - \mathbf{G}(\overline{\theta}) \right] \tag{4.4}$$

This second order differential equation may be expressed as two, first order, matrix differential equations involving only the first derivatives of $\overline{\omega}$ and $\overline{\theta}$. In particular,

$$\frac{d\overline{\theta}}{dt} = \overline{\omega} \tag{4.5}$$

$$\frac{d\overline{\omega}}{dt} = -\mathbf{M}^{-1}(\overline{\theta}) \left[\mathbf{V}(\overline{\theta},\overline{\omega}) + \mathbf{G}(\overline{\theta}) \right] + \mathbf{M}^{-1}(\overline{\theta})\, \overline{\tau} \tag{4.6}$$

This represents the so called state model of the manipulator. The angular displacements, $\overline{\theta}$, and velocities, $\overline{\omega}$, are called *state variables.* In a state model, the first derivatives of the state variables are expressed as functions of the state variables and of the input variables which, in this case, consist of the applied torques, $\overline{\tau}$. This system of equations implies the block diagram of Fig. 4-1. In Fig. 4-1, the blocks containing the term "1/s" represent integrators. The historical motivation behind this type of block diagram representation stems from the days of analog computers wherein integrators are easily constructed using operational amplifier circuitry. The non-linear, scalar functions which are implicit to the matrix equation (4.6) can also be realized using operational amplifier circuits which are capable of generating, for example, the standard, sinusoidal functions. Although analog computers have for the most part been displaced by digital computers, block diagrams remain a useful aid in visualizing the dynamic relationships that exist between our state variables. In fact, many digital computer programs, in which system data are entered using interactive graphics, contain a "menu" which often includes the standard computational elements of an analog computer (integrators, summers, multipliers, sinusoidal function generators, etc.) The main difference is that integrations are carried out using some numerical approximation which, in general, can be made as accurate as we desire at the

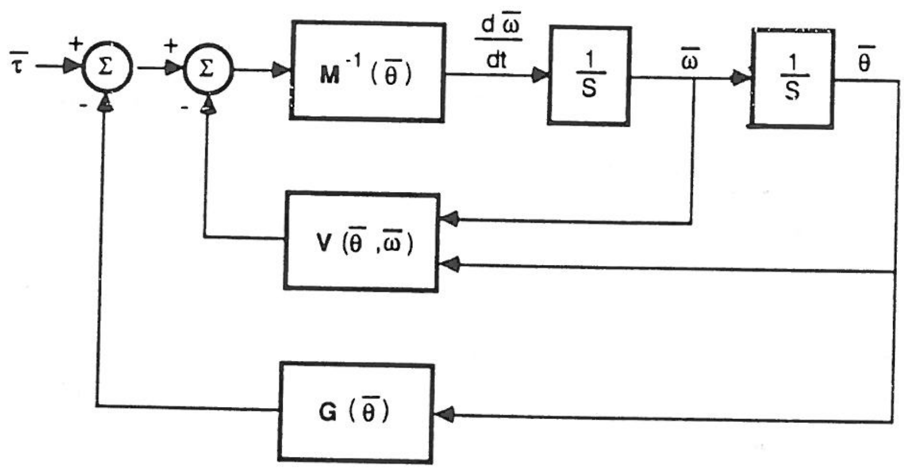


Fig. 4-1. Simulation block diagram of manipulator.

1. Initialize

$$\overline{\theta}(t) = \overline{\theta}(0)$$
$$\overline{\omega}(t) = \overline{\omega}(0)$$

2. Euler predictor:

$$\overline{\theta}(t+\Delta t) = \overline{\theta}(t) + \overline{\omega}(t)\ \Delta t$$
$$\overline{\omega}(t+\Delta t) = \overline{\omega}(t) - \mathbf{M}^{-1}(\overline{\theta}) \left[\mathbf{V}(\overline{\theta},\overline{\omega}) + \mathbf{G}(\overline{\theta})\right]\Delta t + \left[\mathbf{M}^{-1}(\overline{\theta})\,\overline{\tau}\right]\Delta t$$

3. Update:

$$t \longleftarrow t + \Delta t$$

4. Check. If $t < t_{max}$, repeat from 2.

An issue which has yet to be addressed is the selection of the time step Δt. Intuitively, if too large a time step is selected, we might expect inaccurate results since the forward difference approximations of $\frac{d\theta}{dt}$ and $\frac{d\omega}{dt}$ may no longer be reasonable. In fact, the numerical solution may actually "blow up". That is, the calculated response will appear as if the system were unstable even though the true solution corresponds to a stable response. Generally, this situation can be avoided by selecting the time step, Δt, small enough. We might expect that the accuracy of the solution will increase as we make our time step smaller and smaller. To a certain extent, this is true. However, if too small a time step is selected, numerical roundoff effects may become quite important limiting the accuracy of the solution. Selecting too small a time step may also require an excessive amount of computer time. Various methods of selecting a reasonable time step have been developed which are based upon the desired accuracy of the solution. In addition, highly advanced integration algorithms employing variable order approximations of derivatives, variable time stepping to improve simulations speed and error checking are in existence and may be used to solve systems of differential equations written in state variable form [1].

expense of computational speed.

Perhaps the simplest numerical integration technique is called the forward Euler algorithm which is sometimes called the Euler predictor algorithm. In this approach, the derivatives in (4.5)-(4.6) are replaced by their forward difference approximations

$$\left.\frac{d\overline{\theta}}{dt}\right|_t = \frac{\overline{\theta}(t+\Delta t)-\overline{\theta}(t)}{\Delta t} \tag{4.7}$$

$$\left.\frac{d\overline{\omega}}{dt}\right|_t = \frac{\overline{\omega}(t+\Delta t)-\overline{\omega}(t)}{\Delta t} \tag{4.8}$$

where Δt represents the *time step*. Replacing the derivatives in (4.5) - (4.6) with their forward difference approximations and solving for $\theta(t+\Delta t)$, $\omega(t+\Delta t)$ gives

$$\overline{\theta}(t+\Delta t) = \overline{\theta}(t) + \overline{\omega}(t)\ \Delta t \tag{4.9}$$

$$\overline{\omega}(t+\Delta t) = \overline{\omega}(t) - \mathbf{M}^{-1}(\overline{\theta})\ \Big[\mathbf{V}(\overline{\theta},\overline{\omega}) + \mathbf{G}(\overline{\theta})\Big]\ \Delta t + \Big[\mathbf{M}^{-1}(\overline{\theta})\ \overline{\tau}\Big]\ \Delta t \tag{4.10}$$

In the preceding equations, the value of $\overline{\theta}$ and $\overline{\omega}$ at a subsequent time instant, $t+\Delta t$, are calculated in terms of the present values of $\overline{\theta}$, $\overline{\omega}$ and the applied torque $\overline{\tau}$. Thus, given the time zero or initial values of $\overline{\theta}$ and $\overline{\omega}$, and the applied torques, $\overline{\tau}$, for $t>0$, the manipulator response can be calculated as follows: