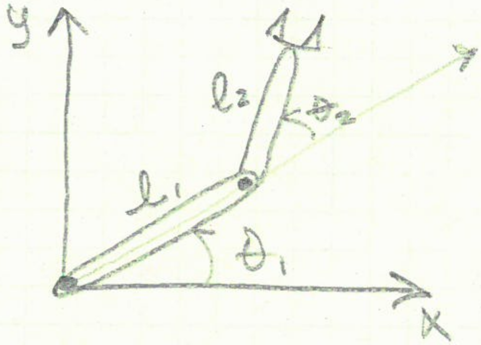
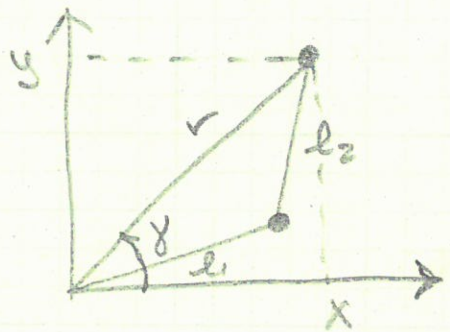
Forward Kinematics

Given joint angles, **find** end effector position



This serial manipulator exists in a plane, hense it is called a planar manipulator



$r \equiv$ 2D vector $= (x, y)$

$\bar{r} = \sqrt{x^2 + y^2}$ $\quad \gamma = \text{atan2}(y, x)$

$r = \bar{r}\cos(\gamma)\hat{i} + \bar{r}\sin(\gamma)\hat{j}$

* the z component $(\hat{k})$ is 0

Also: $r = [l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)]\hat{i} + [l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)]\hat{j}$

$$\left.\begin{array}{l} \cos\theta_1 = C_1 \\ \sin(\theta_1+\theta_2) = S_{12} \end{array}\right\} \text{Can write more compactly}$$

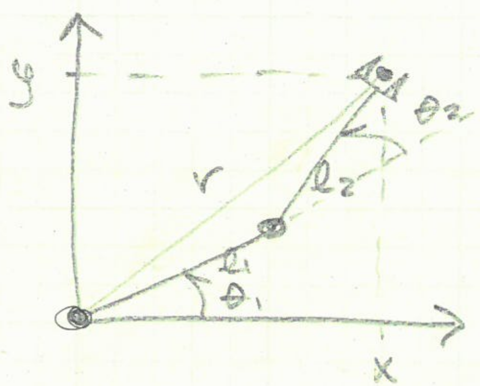
$$r = [l_1 C_1 + l_2 C_{12}]\hat{i} + [l_1 S_1 + l_2 S_{12}]\hat{j}$$

Put into a matrix:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 C_1 + l_2 C_{12} \\ l_1 S_1 + l_2 S_{12} \end{bmatrix} \equiv \text{position} = P$$

$$\left.\begin{array}{l} \text{Velocity} \equiv V = \frac{d}{dt}P \\ \text{accel} \equiv a = \frac{d}{dt}V \end{array}\right\} \text{dynamics}$$

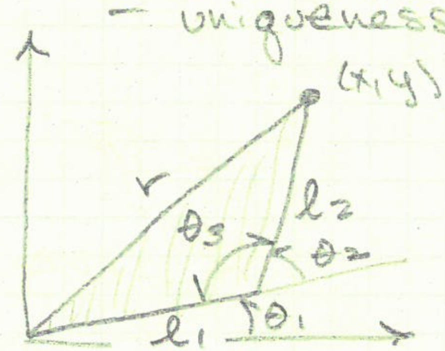
Inverse Kinematics



Given: end effector position

Find: joint angles

* This is very hard for most real robot arms:
- existance: is there a solution
- uniqueness: there are multiple solns



Given: x, y

Find: θ_1 θ_2

$\theta_3 = 180 - \theta_2$

Law cosines: $l_1^2 + l_2^2 - 2 l_1 l_2 \cos(\theta_3) = \bar{r}^2$

$\cos(180 - \theta_2) = -\cos(\theta_2)$

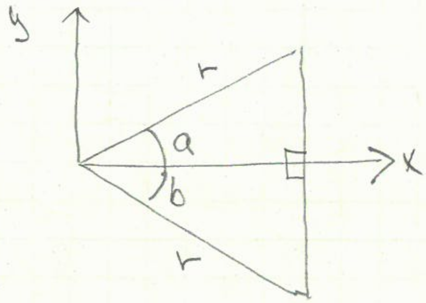
$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos(\theta_2)$

$$\cos(\theta_2) = \frac{(x^2+y^2) - (l_1^2 + l_2^2)}{2 l_1 l_2}$$

Existance: $-1 \leq RHS \leq 1$, if not, no soln

* the point is outside the work space

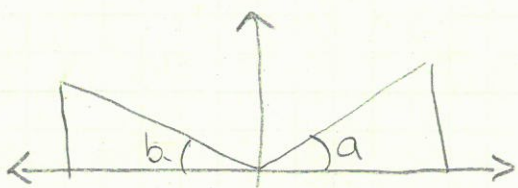
uniqueness :



$$a = -b$$

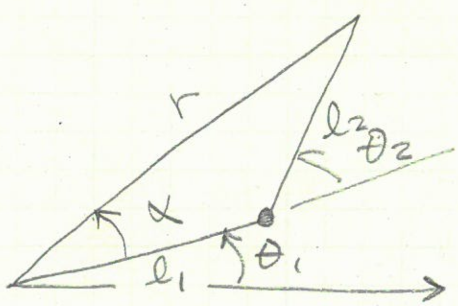
$$\cos(a) = \cos(b) = \cos\left(\frac{x}{r}\right)$$

$$r = \sqrt{x^2 + y^2}$$



$$\sin(a) = \sin(b) = \sin\left(\frac{y}{r}\right)$$

* You will <u>always</u> have 2 solutions and you, the engineer, need to figure out which one to use
* So θ_2 will have 2 answers: θ_2 $-\theta_2$



Reuse law cosines

$$\theta_1 = \gamma - \alpha$$

$$l_2^2 = l_1^2 + \bar{r}^2 - 2 l_1 \bar{r} \cos(\alpha)$$

$$\cos(\alpha) = \frac{l_1^2 - l_2^2 + x^2 + y^2}{2 l_1 \sqrt{x^2 + y^2}}$$

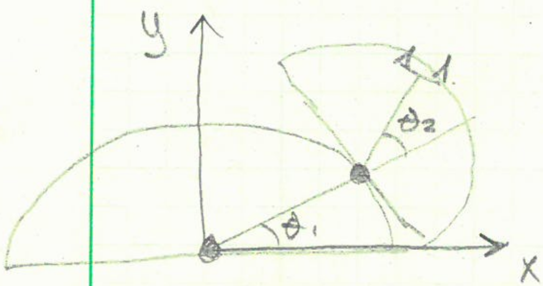
* still have existance an uniqueness issues

$$-1 \le RHS \le 1$$

Simplify:

we will assume: $0 \le \theta_1 \le 180$ floor
$-90 \le \theta_2 \le 90$

* this is also realistic since the servos only have 180° movement



θ_2: $0 \le \cos\theta_2 \le 1$

$$0 \le \underbrace{\frac{(x^2+y^2)-(l_1^2+l_2^2)}{2l_1l_2}}_{m} \le 1$$

$$\pm\theta_2 = \operatorname{acos}(m)$$

θ_1:

$$\gamma = \operatorname{atan2}(y,x)$$

$$\alpha = \operatorname{acos}\left(\frac{l_1^2-l_2^2+x^2+y^2}{2l_1\sqrt{x^2+y^2}}\right) \qquad -1 \le \alpha \le 1$$

$$\theta_1 = \gamma \mp \alpha$$

4 solutions:

θ_2	$\theta_1 = \gamma - \alpha$ $\theta_1 = \gamma + \alpha$
$-\theta_2$	$\theta_1 = \gamma - \alpha$ $\theta_1 = \gamma + \alpha$

$\Bigg\}$ need to test that each gets to (x,y) <u>and</u> doesn't hit anything