## 5. GEARING

Often the motor shaft is connected to the load (link) through a gear train. The gear train reduces the required motor torque at the expense of increasing its speed requirements. The effects of a gear train may be incorporated into our analysis by considering the simple, single stage, gear train depicted in Fig. 5-1. With reference to

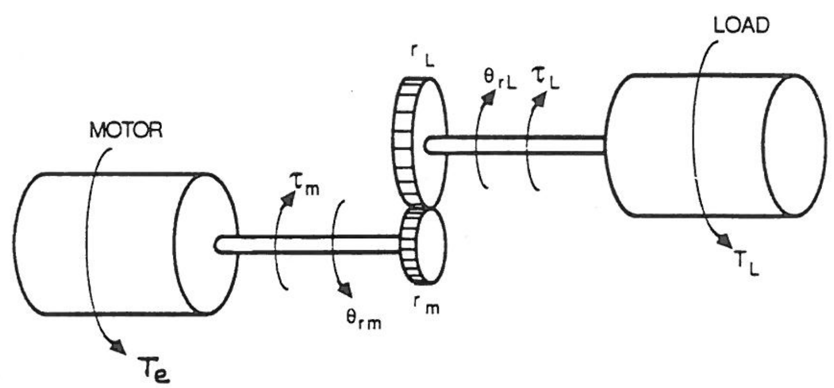


Fig. 5-1. Single stage gear train.

Fig. 5-1, the peripheral (linear) motion of each gear must be identical. This peripheral motion is equal to the angular rotation times the gear radius. Specifically,

$$r_m \, \theta_{rm} = r_L \, \theta_{rL} \tag{5.1}$$

where $r_m$ and $r_L$ represent, respectively, the radius of the input and output gears. Differentiating the previous equation with respect to time

$$r_m \, \omega_{rm} = r_L \, \omega_{rL} \tag{5.2}$$

Equivalently,

$$\omega_{rm} = \left(\frac{r_L}{r_m}\right) \omega_{rL} \tag{5.3}$$

If $r_L > r_m$, the motor speed will be greater than the load speed. It is convenient to define the quantity $r_L/r_m$ as the gear ratio, N. For a multi-stage gear train, the gear ratio may be defined as the ratio of motor speed to the load speed. In particular,

$$N = \frac{\omega_{rm}}{\omega_{rL}} \tag{5.4}$$

whereupon

$$\omega_{rm} = N\,\omega_{rL} \tag{5.5}$$

Neglecting frictional losses, the input mechanical power will be equal to the output power. Thus,

$$\tau_m\,\omega_{rm} = \tau_L\,\omega_{rL} \tag{5.6}$$

Expressing $\tau_m$ in terms of $\tau_L$ and the gear ratio, N

$$\tau_m = \left(\frac{1}{N}\right)\tau_L \tag{5.7}$$

Equations (5.5) and (5.7) confirm our earlier claim that the motor torque is reduced whereas the speed is proportionately increased.

Now, lets examine the inertial dynamics of the motor-gear train-load combination shown in Fig. 5-1. Applying Newtons law of rotation to the motor inertia,

$$T_e - \tau_m = J_m\,\frac{d\omega_{rm}}{dt} \tag{5.8}$$

where $J_m$ is the motor inertia. For the load inertia,

$$\tau_L - T_L = J_L\,\frac{d\omega_{rL}}{dt} \tag{5.9}$$

But $\tau_L = N\tau_m$ and $\omega_{rL} = \frac{1}{N}\omega_{rm}$. Substituting these relationships into (5.9)

$$N\tau_m - T_L = \frac{J_L}{N}\,\frac{d\omega_{rm}}{dt} \tag{5.10}$$

Solving (5.10) for $\tau_m$ and substituting into (5.8)

$$\boxed{T_e = (J_m + \frac{J_L}{N^2}) \frac{d\omega_{rm}}{dt} + \frac{1}{N} T_L} \tag{5.11}$$

This equation is of the general form

$$T_e = J_{eq} \frac{d\,\omega_{rm}}{dt} + T_{eq} \tag{5.12}$$

where $J_{eq}$ and $T_{eq}$ represent the equivalent inertia and load torque seen by the motor. Comparing (5.11) with (5.12)

$$J_{eq} = J_m + \frac{J_L}{N^2} \tag{5.13}$$

$$T_{eq} = \frac{1}{N} T_L \tag{5.14}$$

Thus, the equivalent load inertia seen by the motor is equal to the actual load inertia divided by $N^2$ whereas the equivalent load torque is $1/N$ times the actual load torque. The fact that a gear train can be used to reduce the equivalent inertia seen by the motor is significant since the mechanical time constant (hence the speed of response) is affected by the load inertia.

Equation (5.12) represents the dynamic equation of motion with all quantities referred to the motor side of the gear train assuming that the load inertia is constant. However, in the manipulator, the load inertia is not constant. Consequently, it is useful to express the analogous equation of motion with all of quantities referred to the load. It is not difficult to show that the following equation results

$$N\,T_e = (J_L + N^2\,J_m) \frac{d^2\,\omega_{rL}}{dt} + T_L \tag{5.15}$$

This implies that the accelerating torque applied to the load is N times the electromagnetic torque of the motor. Moreover, the motor inertia as seen by the load is equal to the actual motor inertia multiplied by $N^2$. This, in turn, implies that if we have a gear train coupling each motor to the manipulator, the applied torques, $\tau_1$ and $\tau_2$, are obtained by multiplying the electromagnetic torque by the appropriate gear ratio, i.e.

$$\tau_1 = N_1\, T_{e,1} \tag{5.16}$$

$$\tau_2 = N_2\, T_{e,2} \tag{5.17}$$

where the gear ratio, $N_1$, may or may not be equal to $N_2$. The inertia of each motor may be taken into account by adding, to each of the diagonal entries of the "mass" matrix in (4.1), the appropriate motor inertia multiplied by the square of the gear ratio.

In this preceding discussion, we have tacitly assumed that the inertia of the gears is negligible and that there is no compliance in the shafts interconnecting the motor and load to the gears or between the gear teeth themselves. Realistically, there will be some compliance, however, in many cases, the compliance can be neglected. If compliance is neglected, we can account for the added inertia of the gear train by adding the inertia of the low speed gear to the load and that of the high speed gear to the motor. In the end, we can combine all inertias into one, equivalent inertia as seen by the load and add that inertia to the appropriate diagonal entry of the mass matrix in (4.1).

## 6. CONTROL

The overall control problem is usually divided into two subproblems: (1) trajectory planning and (2) motion control. In the planning stage, we might be given the path specifications in a form similar to that given in Example 3A. Additionally, we may be given path constraints which relate to possible obstacles and information about the manipulator dynamic constraints. From this information, we must establish the joint angular positions, velocities and accelerations which provide the desired motion along the selected path. The general area of trajectory planning is a non-trivial one and we shall instead focus on the problem of motion control assuming that someone else has already planned the trajectory. That is, we shall assume that the *desired* angular displacement, velocity and acceleration, which will be denoted as $\overline{\theta}_d$, $\overline{\omega}_d$ and $\overline{\alpha}_d$, respectively, have been predetermined[1]. Our concern will be how to control the applied torque so that, for example, the actual displacement $\overline{\theta}(t)$ approximates the desired displacement, $\overline{\theta}_d(t)$. We shall consider two general approaches to the motion control problem.

## Open Loop Control

This is the simplest method of control; therefore, we shall consider it first even though it is seldom used. The basic idea with open loop control is to calculate $\overline{\tau}_d$, the desired torque, in terms of the desired angular displacement ($\overline{\theta}_d$), velocity ($\overline{\omega}_d$) and acceleration ($\overline{\alpha}_d$) using the general expression

$$\overline{\tau}_d = \mathbf{M}(\overline{\theta}_d)\ \overline{\alpha}_d + \mathbf{V}(\overline{\theta}_d,\overline{\omega}_d) + \mathbf{G}(\overline{\theta}_d) \tag{6.1}$$

which is identical to (4.2) with the addition of the subscript "d" to indicate that the torque is calculated on the basis of the desired trajectory.

In an open loop system, the joint actuators are controlled so that the output torques are equal to the calculated or desired torques (scalar components of $\overline{\tau}_d$). If the actuator consists of a conventional, permanent magnetic dc motor, the output torque is directly proportional to the armature current. Thus, given the proportionality constant ($k_v$), we would control the armature current so as to provide the desired torque.

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[1]Although, for example, $\overline{\theta}_d(t)$ is a vector consisting of several angular displacements, we shall refer to it as a singular instead of plural quantity.

If the actuator is a brushless dc motor, the current controlled drive system of Fig. 6.5 (class notes on Inverter Systems) may be used to supply the deisred torque. In this drive system, the developed electromagnetic torque, $T_e$, is proportional to the input or control signal, $I_{ref}$. Moreover, the dynamic response to a change in $I_{ref}$ is very fast and may be considered in many cases as instantaneous (see Fig. 6-11 in class notes on Inverter Systems).

An obvious disadvantage of open loop control is that inaccuracies in the mathematical model of the manipulator will give use to a difference (error) between the desired and actual trajectory. In particular, if the parameters associated with the mathematical model used to calculate $\overline{\tau}_d$ are in error and the calculated torque is applied to the manipulator, then the actual trajectory $\overline{\theta}$, $\overline{\omega}$ will differ from $\overline{\theta}_d$, $\overline{\omega}_d$. Moreover, the actuators themselves may be imperfect in the sense that the actual, or developed torque will not be identical to the commanded torque. We might be able to come close to our desired trajectory by carefully formulating a highly detailed mathematical model and using experimental data to verify or refine the parameters. In this regard, we might include friction in our mathematical model. Other effects which may warrant consideration included gear backlash, the possible bending of the links or the torsional compliance in the drive train. Unfortunately, each of these effects are extremely difficult to model and would make our model unwieldy. For this reason, closed loop control is more frquently used.

Another device which has potential application to manipulator control is the stepper motor. In the stepper motor, the angular velocity is directly proportional to the stepping rate which is easily controlled. Thus, given the desired angular velocity, $\omega_d(t)$, and the proportionality constant relating the motor speed to the stepping rate, the stepping rate may be controlled accordingly. We must be careful, however, that the load torque presented to the motor is less than the so called pullout torque. This can be checked during the planning stage by comparing the predicted torque-speed ($\tau_d - \omega_d$) trajectories (plotted in Fig. 3A-6 for the trajectory of Example 3A) with the pullout torque-speed characteristics of the motor. Clearly, the $\tau_d - \omega_d$ trajectory, with an appropriate change of scale to account for possible gearing, must fall underneath the pullout torque-speed characteristics of the motor. Otherwise the motor may fall out of step and the actual trajectory will differ from the desired trajectory.

### Closed Loop Control

Although there are many approaches that may be taken with regard to closed loop control, we will consider only one which is particularly well suited to the manipulator control problem. For reasons which will become apparent, this method is called the *computed torque method* [2-3].

The general structure of the control system is depicted in Fig. 6-1. As before, we shall assume that the trajectory has already been planned. In particular, the desired angular position, velocity and acceleration ($\overline{\theta}_d$, $\overline{\omega}_d$, $\overline{\alpha}_d$) are assumed to be known functions of time. We will also assume that the actual angular position, velocity and acceleration ($\overline{\theta}$, $\overline{\omega}$ and $\overline{\alpha}$) may be related to the applied torque, $\overline{\tau}$, by the differential equation

$$\overline{\tau} = \mathbf{M}(\overline{\theta})\overline{\alpha} + \mathbf{V}(\overline{\theta},\overline{\omega}) + \mathbf{G}(\overline{\theta}) \tag{6.2}$$

As stated previously, this equation relates the **actual** applied torques to the **actual** response. However, we must recognize that we are unable to establish the exact parameters associated with this equation.

From Fig. 6-1, the desired torque, $\overline{\tau}_d$, is calculated as follows

$$\overline{\tau}_d = \mathbf{M}(\overline{\theta}) \left[\overline{\alpha}_d + \mathbf{K}_v(\overline{\omega}_d - \overline{\omega}) + \mathbf{K}_p(\overline{\theta}_d - \overline{\theta})\right] + \left[\mathbf{V}(\overline{\theta},\overline{\omega}) + \mathbf{G}(\overline{\theta})\right] \tag{6.3}$$

where $\mathbf{K}_v$, $\mathbf{K}_p$ are 2x2 gain matrices which will be specified shortly. Here, $\overline{\tau}_d$

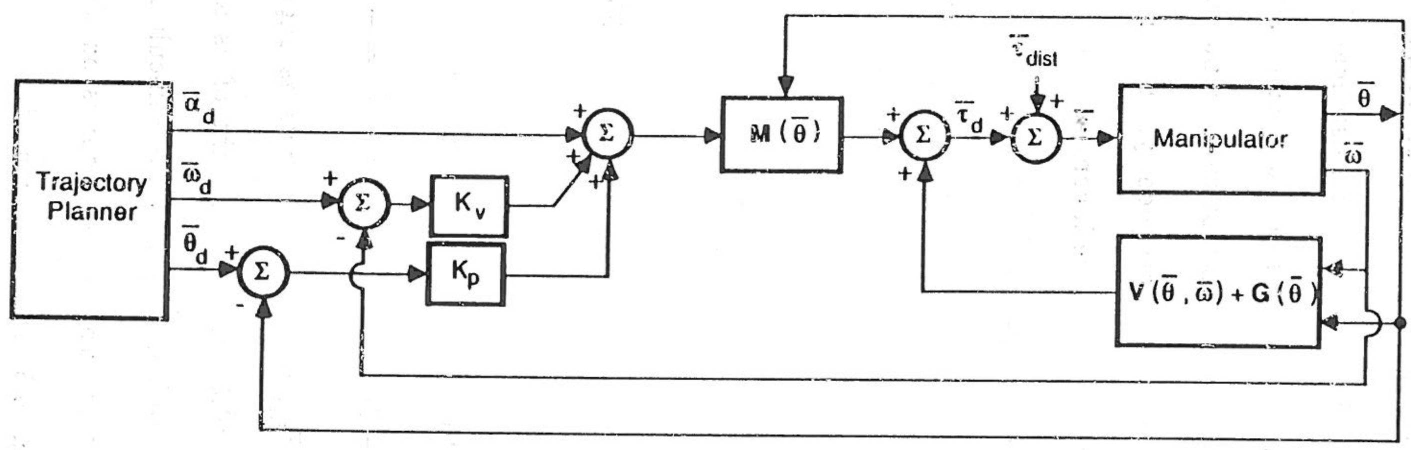


Fig. 6-1. Manipulator control by computed torque method.

represents the desired or reference torque. The scalar components of $\overline{\tau}_d$ are supplied to the individual actuator control systems and if the actuators are perfect, the actuator torques will be equal to respective components of $\overline{\tau}_d$.

The critical reader will find several inconsistencies in the development thus far. First, it was stated that the parameters of (6.2) cannot be established exactly. Yet, our control law (6.3) is written in terms of these same parameters. Also, the actuators themselves will be imperfect, consequently, the actual torque will differ (somewhat) from the desired torque, $\overline{\tau}_d$. These inconsistencies may be conveniently removed by adding a disturbance torque, $\overline{\tau}_{dist}$ as shown in Fig. 6-1, to account for the inaccuracies in the mathematical model of the manipulator and the imperfections of the actuators. Although, it is very difficult to quantitatively express $\overline{\tau}_{dist}$ as an explicit function of time, we may be able to establish reasonable upper bounds on the scalar components of $\overline{\tau}_{dist}$. We shall assume that this is the case.

From Fig. 6-1, the actual actuator torque, $\overline{\tau}$, is expressed as the sum of the desired torque, $\overline{\tau}_d$, and the disturbance torque, $\overline{\tau}_{dist}$. In particular,

$$\overline{\tau} = \overline{\tau}_d + \overline{\tau}_{dist} \tag{6.4}$$

Substituting (6.2) and (6.3) into (6.4) and simplifying

$$(\overline{\alpha}_d - \overline{\alpha}) + \mathbf{K}_v(\overline{\omega}_d - \overline{\omega}) + \mathbf{K}_p(\overline{\theta}_d - \overline{\theta}) = - \mathbf{M}^{-1}(\theta)\, \overline{\tau}_{dist} \tag{6.5}$$

Now, the product $\mathbf{M}^{-1}(\theta)\, \overline{\tau}_{dist}$ has the same units as angular acceleration. Thus, the term on the right side of (6.5) may be thought of as an "acceleration disturbance," denoted $\overline{\alpha}_{dist}$. Again, although it may be quite difficult to express $\overline{\alpha}_{dist}$ explicitly, it is reasonable to assume that suitable upper bounds on the scalar components of $\overline{\alpha}_{dist}$ can be found. In any case, (6.5) can be expressed

$$(\overline{\alpha}_d - \overline{\alpha}) + \mathbf{K}_v(\overline{\omega}_d - \overline{\omega}) + \mathbf{K}_p(\overline{\theta}_d - \overline{\theta}) = \overline{\alpha}_{dist} \tag{6.6}$$

where

$$\bar{\alpha}_{dist} = -M^{-1}(\bar{\theta})\,\bar{\tau}_{dist} \tag{6.7}$$

Let's define the positional error, $\bar{E}$, as $(\bar{\theta}_d - \bar{\theta})$. It is clear that $\dfrac{d\bar{E}}{dt} = (\bar{\omega}_d - \bar{\omega})$ and $\dfrac{d^2\bar{E}}{dt^2} = \bar{\alpha}_d - \bar{\alpha}$. Thus, (6.6) may be expressed

$$\frac{d^2\bar{E}}{dt^2} + K_v \frac{d\bar{E}}{dt} + K_p\bar{E} = \bar{\alpha}_{dist} \tag{6.8}$$

which represents a second order, linear, matrix differential equation. Methods of solving such equations are well known.

Let's take a moment to look at what has been accomplished. Starting with a highly complicated, non-linear system, we have found that the positional error, $\bar{E}$, satisfies a simple, second order, linear system of equations which is easily solved. This is the beauty of the computed torque method of control. Moreover, if we assume that our mathematical model of the manipulator is perfect and that there are no actuator errors, the disturbance torque $\bar{\tau}_{dist}$, hence the acceleration disturbance, $\bar{\alpha}_{dist}$, are both zero whereupon (6.8) represents a second order, linear, homogeneous, matrix differential equation. If the gain matrices, $K_v$ and $K_p$, are selected so that the eigenvalues associated with this system of equations lie in the open left half plane, the positional error will exponentially approach zero.

In order to gain appreciation for these results, let's consider the following scenario. Suppose that at $t=0$, the manipulator of Example 3A is located at point A in Fig. 3A-1. Moreover, suppose $\bar{\omega}_d(0) = \bar{\omega}(0)$ and $\bar{\theta}_d(0) = \bar{\theta}(0)$. Thus, at $t=0$ $\dfrac{d\bar{E}}{dt} = \bar{E} = 0$. Now suppose that for $t>0$, $\bar{\theta}_d$ and $\bar{\omega}_d$ are specified as in this example (Figs. 3A-3 and 3A-4). If we assume that $\bar{\alpha}_{dist}$ is zero for $t>0$ (implying a perfect model and perfect actuators), the second order equation (6.8) has no inputs. Since there are no initial conditions either, the error will remain equal to zero regardless of how $K_v$ and $K_p$ are selected. Practically, this will not be the case since we will have modeling errors giving rise to a non-zero $\overline{\alpha}_{dist}$. In this practical situation, we would like to select $K_p$ and $K_v$ so that $\overline{E}(t)$ is small even though $\overline{\alpha}_{dist}$ is non-zero. This, in general, requires that $K_v$ and $K_p$ are selected so that the eigenvalues associated with (6.8) lie in the left half plane.

A difficulty that we have when characterizing the position error $\overline{E}$ is that the exact form of $\overline{\alpha}_{dist}$ is not known although we might be able to estimate upper bounds of the elements of $\overline{\alpha}_{dist}$. The natural question arises: if the maximum value of the elements of $\overline{\alpha}_{dist}$ can be established, can we establish an upper bound on the error, $\overline{E}$? If so, how is the maximum error related to the choice of $K_p$ and $K_v$. In other words, how do we select $K_p$ and $K_v$ so that the error $\overline{E}$ is acceptable for a given estimate of the upper bounds associated with $\overline{\alpha}_{dist}$.

To address this issue, lets suppose that the matrices $K_p$ and $K_v$ are selected to be diagonal. This implies that, for the two link manipulator, (6.8) can be written as two, second order, scalar differential equations of the form

$$\frac{d^2E}{dt^2} + K_v \frac{dE}{dt} + K_p E = \alpha_{dist} \tag{6.9}$$

where we have dropped the bar notation to indicate that (6.9) is a scalar differential equation. Since we have two uncoupled differential equations, it appears that the joint errors are independent. This interpretation, however, is not correct since a torque disturbance, $\tau_{dist}$, on one joint affects $\alpha_{dist}$ at all joints via (6.7). However, (6.9) remains valid as long as we remember $\alpha_{dist}$ accounts for the effects of $\tau_{dist}$ at all joints.

The steady-state joint error is readily established from (6.9) by setting derivatives to zero. In particular,

$$E_{ss} = \frac{1}{K_p} \alpha_{dist} \qquad (6.10)$$

Clearly, for sufficiently large $K_p$, we can make the steady state error as small as we desire for a finite $\alpha_{dist}$. This is a typical characteristic of closed loop control.

But what about the tracking error, E(t), as the manipulator moves along some chosen trajectory? Well, since it is difficult to establish the exact form of $\alpha_{dist}(t)$, we have to be content with establishing an upper bound for E(t) assuming, of course, that we can establish an upper bound for $\alpha_{dist}(t)$ along the selected trajectory.

To accomplish this objective, suppose that at $t=0$, $\frac{dE}{dt}=E=0$. The error for $t>0$ may be expressed in terms of $\bar{\alpha}_{dist}(t)$ using the convolutional integral

$$E(t) = \int_0^t \alpha_{dist}(\tau)\, h(t-\tau)\, d\tau \qquad (6.11)$$

where h(t) represents the impulse response function corresponding to the second order differential equation of (6.9). From (6.11),

$$|E(t)| \leq \int_0^t |\alpha_{dist}(\tau)|\, |h(t-\tau)|\, d\tau \qquad (6.12)$$

Denoting $\alpha_M$ as the maximum value of $|\alpha_{dist}(\tau)|$ on the interval (0,t),

$$|E(t)| \leq \alpha_M \int_0^t |h(t-\tau)|\, d\tau \qquad (6.13)$$

Letting $g=t-\tau$ and expressing the previous integral in terms of g

$$|E(t)| \leq \alpha_M \int_0^t |h(g)|\, dg \leq \alpha_M \int_0^\infty |h(g)|\, dg \qquad (6.14)$$

which implies that the maximum, absolute position error is bounded provided $\int_0^\infty |h(g)| dg$ is finite. For simplicity, lets assume that $K_p$ and $K_v$ are selected so that the natural response of the second order system (6.9) is critically damped ($K_v = 2\sqrt{K_p}$). In this case, the impulse response function is equal to

$$h(t) = t\, e^{-at}. \tag{6.15}$$

where

$$a = \frac{K_v}{2} = \sqrt{K_p} \tag{6.16}$$

In this particular case,

$$\int_0^\infty |h(g)|\,dg = \frac{1}{a^2} = \frac{1}{K_p} \tag{6.17}$$

Thus, (6.14) becomes

$$|E(t)| \leq \frac{1}{K_p}\,\alpha_M \tag{6.18}$$

Interestingly, the expression for the maximum position error is nearly identical to our previous expression for the steady state error (6.10); however, these two errors have different interpretations. Again, we see that the maximum position error can be reduced by increasing $K_p$. If we do this, however, we must remember to increase $K_v$ in accordance with (6.16) since (6.18) applies only when $K_p$ and $K_v$ are selected to provide a critically damped response.

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