Models for Population Growth - Section 9.4

1. A tank contains 1,200 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 14 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 20 minutes?

2. Detectives found Mr. Smith's body at 2100. The room he was found in is kept at a constant 70 °F. At the time he was found his temperature was 94 °F. Five hours later, the coroner finds Mr. Smith's temperature to be 82 °F . Medical records indicate that Mr. Smith's normal body temperature was 99 °F. The rate at which his body cooled off is proportional to the difference between his body temperature and the surrounding temperature. When did Mr. Smith die?

3. What is/are the equilibrium solution(s) to $\frac{dP}{dt} = kP\left(2 - \frac{P}{M}\right)$?

Population Problems:

Populations often increase exponentially early on and level off as they approach their carrying capacity (a limit on the population due to resource constraints). In this case, we need a model where the relative growth rate decreases as the population increases and is negative if the population is larger than the carrying capacity. The simplest such model is given below and is called the Logistic Model.

4. You open a bank account with \$7500. The account pays 2% interest compounded continuously, and you use this account to make payments totaling \$100 each year. Construct a differential equation for the balance in this account at time t (in years). What are the equilibrium solutions?

5. Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in the lake) to be 10,000. The number of fish tripled in the first year.

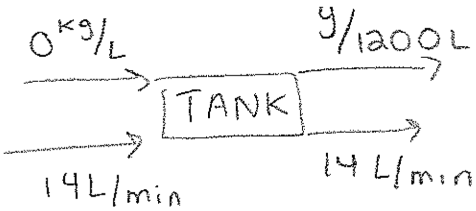
a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.

b)How long will it take for the population to increase to 5000?

6. In a certain part of the Pike National Forest, dead pine needles accumulate on the ground at a rate of 8 kilograms per square meter per year. They decompose at a continuous rate of 80% per year. If there are 7 kilograms of pine needles per square meter initially, how much will there be after 2 years? ~~What will happen in the long run~~?

7. A population of insects in a region will grow at a rate that is proportional to their current population. In the absence of any outside factors the population will triple in two weeks' time. On any given day there is a net migration into the area of 15 insects and 16 are eaten by the local bird population and 7 die of natural causes. ~~If there are initially 100 insects in the area will the population survive? If not, when do they die out?~~**(Hint: use the idea of rate in – rate out)** Solve to find the general solution

1.



$y(0) = 15$

$$\frac{dy}{dt} = 0 - \frac{14y}{1200}$$

$$\int \frac{1}{14y}\, dy = \int -\frac{1}{1200}\, dt$$

$$\frac{1}{14}\ln|y| = -\frac{1}{1200}t + C$$

$$-\frac{14}{1200}t + C = \ln|y|$$

$$y = \pm e^{-\frac{14}{1200}t + c}$$

$$A = \pm e^{c}$$

$$y = Ae^{-14/1200\, t}$$

$$15 = A$$

$$y = 15e^{-14/1200\, t}$$

$$\boxed{y(20) = 11.88 \text{ kg}}$$

2.

$$\frac{dT}{dt} = k(T - T_s)$$

$$\frac{dT}{dt} = k(T - 70)$$

$$\frac{1}{T-70}\,dT = k\,dt$$

$$\ln|T-70| = kt + C$$

$$T - 70 = \pm e^{kt+c}$$

$$A = \pm e^{c}$$

$$T = Ae^{kt} + 70$$

$$T(0) = 94$$

$$94 = A + 70$$

$$A = 24$$

$$T = 24e^{kt} + 70$$

$$T(5) = 82$$

$$82 = 24e^{5k} + 70$$

$$\ln\left(\frac{12}{24}\right) = 5k$$

$$k = \frac{1}{5}\ln\left(\frac{1}{2}\right)$$

$$T = 24e^{\frac{1}{5}\ln\left(\frac{1}{2}\right)t} + 70$$

$$99 = 24e^{\frac{1}{5}\ln\left(\frac{1}{2}\right)t} + 70$$

$$\frac{29}{24} = e^{\frac{1}{5}\ln\left(\frac{1}{2}\right)t}$$

$t = 0$ @ 2100

$$\ln\left(\frac{29}{24}\right) = \frac{1}{5}\ln\left(\frac{1}{2}\right)t$$

$$t = \frac{\ln\left(\frac{29}{24}\right)}{\frac{1}{5}\ln\left(\frac{1}{2}\right)} = -1.36 \text{ hrs}$$

$\approx$ 1 hr 18 min before 2100

$\approx$ $\boxed{1942}$

3. What are the equilibrium solutions?

$$\frac{dP}{dt} = kP\left(2 - \frac{P}{M}\right)$$

$$0 = kP\left(2 - \frac{P}{M}\right)$$

$$0 = P\left(2 - \frac{P}{M}\right) \qquad P = 0$$

$$2 - \frac{P}{M} = 0$$

$$\frac{P}{M} = 2$$

$$\boxed{P = 2M \text{ or } 0}$$

4 You open a bank account with \$7500. The account pays 2% interest compounded continuously, and you use this account to make payments totaling \$100/year. Construct a DE for the balance in this account at time t (in years). What are the equilibrium solutions?

B = \$ in the bank

B(0) = 7500

t = time (yrs)

$$\frac{dB}{dt} = .02B - 100$$

$$\int \frac{1}{.02B - 100}\, dB = \int dt$$

$$50 \ln|.02B - 100| = t + C$$

$$.02B - 100 = \pm e^{\frac{t}{50} + c} \qquad A = \pm e^{c}$$

$$.02B = Ae^{\frac{t}{50}} + 100$$

$$B = \frac{Ae^{\frac{t}{50}}}{.02} + \frac{100}{.02} \qquad D = \frac{A}{.02}$$

$$B = De^{\frac{t}{50}} + 5000$$

equilibrium Solutions:

B = 0 B = 5000

5. $P(0) = 400$

$M = 10{,}000$

$P(1) = 1200$

$$P(t) = \frac{M}{1 + Ae^{-kt}} \qquad A = \frac{M - P_0}{P_0}$$

$$A = \frac{10{,}000 - 400}{400} = 24$$

$$P(t) = \frac{10{,}000}{1 + 24e^{-kt}}$$

$$1200 = \frac{10{,}000}{1 + 24e^{-k}}$$

$$\frac{10{,}000}{1200} = 1 + 24e^{-k}$$

$$\frac{\frac{100}{12} - 1}{24} = e^{-k}$$

$$-\ln\left(\frac{\frac{100}{12} - 1}{24}\right) = k$$

$$k = 1.1856$$

$$P(t) = \frac{10{,}000}{1 + 24e^{-1.1856t}}$$

6. $\frac{dP}{dt} = $ rate in − rate out

$$= \frac{8 \text{ kg}/\text{m}^2}{\text{year}} - \frac{.8P \frac{\text{kg}}{\text{m}^2}}{\text{year}}$$

$$P(0) = 7 \text{ kg}/\text{m}^2$$

$$\frac{dP}{dt} = 8 - .8P$$

$$\int \frac{1}{8 - .8P}\, dP = \int dt$$

$$-\frac{5}{4} \ln|8 - .8P| + C_1 = t + C_2$$

$$C_3 = C_2 - C_1$$

$$-\frac{5}{4} \ln|8 - .8P| = t + C_3$$

$$C = -\frac{4}{5} C_3$$

$$\ln|8 - .8P| = -\frac{4}{5} t + C$$

$$8 - .8P = \pm e^{-4/5t} e^{c}$$

$$A = \pm e^{c}$$

$$8 - .8P = Ae^{-4/5t}$$

$$B = \frac{-A}{.8}$$

$$P = Be^{-4/5t} + 10$$

$$P(0) = 7$$

$$7 = Be^{0} + 10$$

$$B = -3$$

$$P = -3e^{-4/5t} + 10$$

$$\boxed{P(2) = 9.39 \text{ kg}/\text{m}^2}$$

u-substitution:

$$u = 8 - \frac{8}{10}P$$

$$du = -\frac{8}{10}\, dP$$

$$-\frac{5}{4}\, du = dP$$

$$\int \frac{1}{u} \left(-\frac{5}{4}\right) du$$

$$= -\frac{5}{4} \ln|8 - .8P|$$

7.

$$\frac{dP}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = \frac{kP + 15 \text{ insects}}{\text{day}} = kP + 15$$

$$\text{rate out} = \frac{16 + 7 \text{ insects}}{\text{day}} = 23$$

$$\frac{dP}{dt} = (kP+15) - 23 = kP - 8$$

$$\int \frac{1}{kP-8}\, dP = \int dt$$

$$\frac{1}{k} \ln|kP-8| + C_1 = t + C_2 \qquad C_3 = C_2 - C_1$$

$$C = kC_3$$

$$\ln|kP-8| = kt + C$$

$$|kP-8| = e^{kt+C} \qquad A = \pm e^{C}$$

$$kP - 8 = Ae^{kt}$$

$$A_2 = \frac{A}{k}$$

$$kP = Ae^{kt} + 8$$

$$\boxed{P = A_2 e^{kt} + \frac{8}{k}}$$