Sequences and Series

Sequence

A **sequence** is an ordered list of numbers (e.g., a_n); the numbers are called "elements" or "terms". Every convergent sequence is bounded, thus an unbounded sequence is divergent.

Sequence Test	Converge	Notes
Squeeze Theorem	$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$	$a_n \le b_n \le c_n$
Def 1, pg 692	$\lim_{n\to\infty} a_n = L$	
l'Hospital's Rule	$\lim_{n\to\infty} \frac{f(x)}{g(x)} \Rightarrow \lim_{n\to\infty} \frac{f'(x)}{g'(x)}$	where $f(x)$ = numerator and $g(x)$ = denominator
Theorem 3, pg 693	if $f(n) = a_n$ then $\lim_{n\to\infty} f(x) = L$	
Theorem 6, pg 694	$\lim_{n\to\infty} \lvert a_n\rvert = 0$ then a_n converges	
Theorem 9, pg 696	$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \end{cases}$	Divergent for all other values of r
Theorem 12, pg 698	Every bounded ($m \le a_n \le M$), monotonic sequence is convergent	The bounds exists for $n \ge 1$, also see Theorem 10 and 11

Series

A **series** is the sum of the terms of a sequence: $\sum_{n=1}^{\infty} a_n$.

Series Test	Converge	Diverge	Notes
Divergence	N/A	$\lim_{n\to\infty} a_n \ne 0$	Doesn't show convergence and the converse is not true
Integral	if $\int_1^{\infty} f(x)dx$ converges	if $\int_1^{\infty} f(x)dx$ diverges	$f(x)$ must be positive, decreasing, and continous, also $f(n) = a_n$ for all n
Root	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L > 1$ or ∞	inconclusive if $L = 1$
Ratio	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L > 1$ or ∞	inconclusive if $L = 1$
Direct Comparison	$0 \le a_n \le b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges	$0 \le b_n \le a_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges	$a_n, b_n > 0$
Limit Comparison	$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $\sum_{n=1}^{\infty} b_n$ diverges	$a_n, b_n > 0$ and L is a positive constant, if L is ∞ or 0, then pick a different b_n
Absolute	$\sum_{n=1}^{\infty} \lvert a_n\rvert = 0$		Definition of absolutely convergent, the sum is independent of the order in which the terms are summed
Conditional	$\sum_{n=1}^{\infty} \lvert a_n\rvert$ diverges but $\sum_{n=1}^{\infty} a_n$ converges		The sum is dependent of the order in which the terms are summed

Common Series

Series Test	Formula	Converge	Diverge	Notes
Alternating	$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$	$0 < a_{n+1} \leq a_n$ for all n and $\lim_{n\to\infty} a_n = 0$	N/A	
Geometric	$\sum_{n=1}^{\infty} ar^{n-1}$	$\|r\| < 1$ and converges to $\frac{a}{1-r}$	$\|r\| \geq 1$	finite sum of the first n terms: $= \frac{a(1-r^n)}{1-r}$
P-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	cannot calculate sum
Power	$\sum_{n=0}^{\infty} c_n(x-a)^n$	i, converge if $x = a$ ii, converge for all x iii, converge if $\|x-a\| < R$		R is the radius of convergence, you need to check the end points for convergence too. Typically use Ratio Test.
Taylor	$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$	$\|x-a\| < R$	$\|x-a\| > R$	Taylor series is centered about a. Same note as power series
Maclaurin	$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$	$\|x\| < R$	$\|x\| > R$	A Macluarin series is a Taylor series centered about 0. Same note as power series