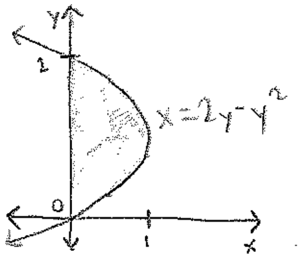
Sect. 5.4

49) The area of the region that lies to the right of the y-axis and to the left of the parabola $x = 2y - y^2$ (the shaded region in the figure) is given by the integral $\int_0^2 (2y - y^2)\,dy$. Find the area of the region.

1. Picture:



2. Goal: To find the area of the shaded region.

3. Setup: $A \approx \sum_{i=1}^{n} (2y - y^2)\Delta y$ from $y=0$ to $y=2$

4. Work: $A = \lim_{n\to\infty} \sum_{i=1}^{n} (2y - y^2)\Delta y$ from $y=0$ to $y=2$

$A = \int_0^2 (2y - y^2)\,dy$

$A = y^2 - \frac{1}{3}y^3 \Big|_0^2$

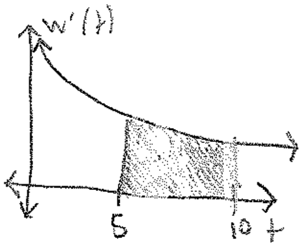
$A = (2)^2 - \frac{1}{3}(2)^3 - \left[(0)^2 - \frac{1}{3}(0)^3\right]$

$A = 4 - \frac{8}{3}$

5. Solution: $\boxed{A = \frac{4}{3} \text{ units}^2}$ ✓

#51.) If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\,dt$ represent?

1. Picture:



2. Goal: To determine what $\int_5^{10} w'(t)\,dt$ represents

3./4./5. Set-up/Work/Solution:

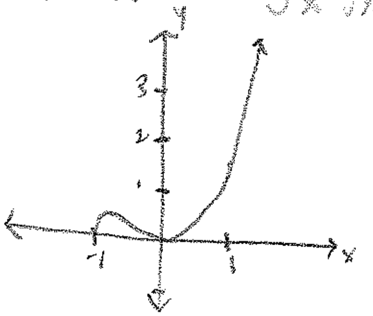
The integral of $\int_5^{10} w'(t)\,dt$ represents the section of area under the curve of $w'(t)$ from $t=5$ to 10. In terms of the problem, it is a value in pounds which represents the total amount of growth of a child (lbs) from $t=5$ years to $t=10$ years in the child's life.

✓

Sect. 5.5

#3.) Evaluate the integral by making the given substitution.

1. Picture: $\int x^2\sqrt{x^3+1}\,dx$, $u=x^3+1$



2. Goal: Evaluate the integral.

3. Set-up: $\int x^2\sqrt{x^3+1}\,dx$, $u=x^3+1$

4. Work:

$$\text{let } u = x^3+1$$

$$du = 3x^2\,dx$$

$$dx = \frac{du}{3x^2}$$

$$\int \cancel{x^2}\sqrt{u}\cdot\frac{du}{3\cancel{x^2}}$$

$$\tfrac{1}{3}\int\sqrt{u}\,du$$

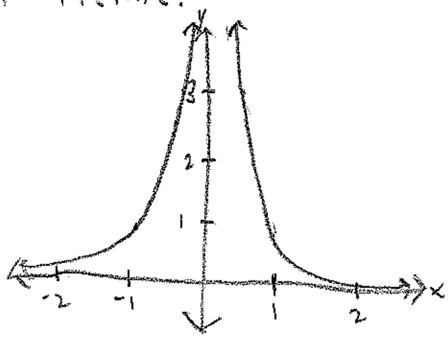
$$\tfrac{1}{3}\left(\tfrac{2}{3}\right)u^{3/2}+C$$

5. Solution: $\boxed{\tfrac{2}{9}\left(x^3+1\right)^{3/2}+C}$ ✓

#17.) Evaluate the indefinite integral.

$$\int \frac{e^u}{(1-e^u)^2}\,du$$

1. Picture:



2. Goal: Evaluate the integral.

3. Setup: $\displaystyle\int \frac{e^u}{(1-e^u)^2}\,du$

4. Work: let $v = 1-e^u$

$$dv = -e^u\,du$$

$$du = \frac{dv}{-e^u}$$

$$\int \frac{e^u}{v^2}\cdot\frac{dv}{-e^u}$$

$$\int \frac{-1}{v^2}\,dv$$

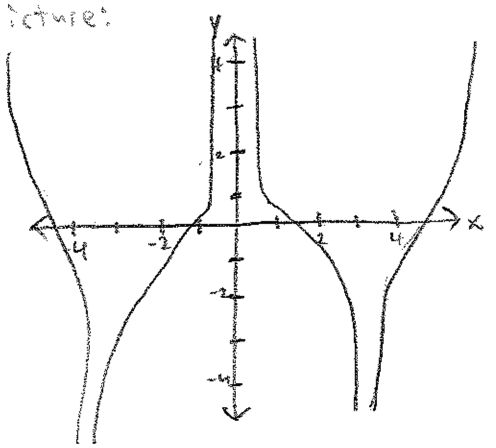
$$\frac{1}{v} + C$$

5. Solution: $\boxed{\dfrac{1}{(1-e^u)} + C}$ ✓

#33.) Evaluate the indefinite integral.

$$\int \frac{\cos x}{\sin^2 x}\,dx$$

1. Picture:



2. Goal: Evaluate the integral.

3. Set-up: $\int \frac{\cos(x)}{\sin^2(x)}\,dx$

4. Work:

$$\text{let } u = \sin(x)$$

$$du = \cos(x)\,dx$$

$$dx = \frac{du}{\cos(x)}$$

$$\int \frac{\cancel{\cos(x)}}{u^2} \cdot \frac{du}{\cancel{\cos(x)}}$$

$$\int \frac{1}{u^2}\,du$$

$$-\frac{1}{u} + C$$

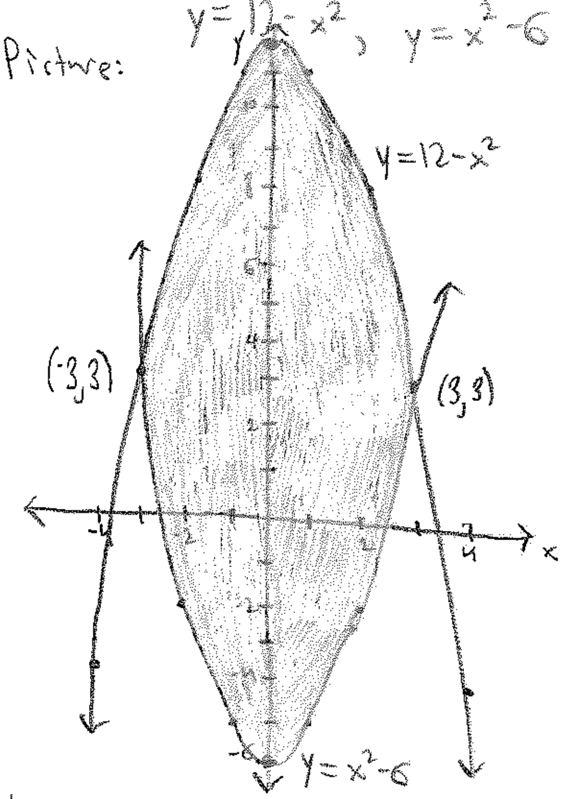
5. Solution: $\boxed{\frac{-1}{\sin(x)} + C}$ ✓

Sect. 6.1

#13.) Sketch the region enclosed by the given curves and find its area.

$y = 12 - x^2$, $y = x^2 - 6$

1. Picture:



2. Goal: Sketch and find the area between the curves.

3. Set-up: $y = 12 - x^2$, $y = x^2 - 6$

4. Work: $12 - x^2 = x^2 - 6$

$18 = 2x^2$

$\sqrt{x^2} = \sqrt{9}$

$x = \pm 3$ $\therefore$ integral will be from -3 to $+3$.

$$A = \int_{-3}^{3} (12 - x^2 - (x^2 - 6))\,dx$$

$$A = \int_{-3}^{3} (18 - 2x^2)\,dx$$

$$A = 18x - \tfrac{2}{3}x^3 \Big|_{-3}^{3}$$

$$A = 18(3) - \tfrac{2}{3}(3)^3 - \left[18(-3) - \tfrac{2}{3}(-3)^3\right]$$

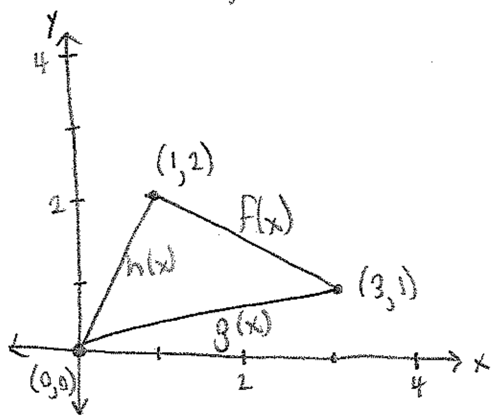
$$A = 54 - 18 + 54 - 18$$

Solution: $\boxed{A = 72 \text{ units}^2}$ ✓

#29.) Use calculus to find the area of the triangle with the given vertices.

$(0,0)$, $(3,1)$, $(1,2)$

1. Picture:



2. Find the area of the triangle.

3. Set-up:

$(0,0)$, $(1,2)$, $(3,1)$

$m_f = \frac{2-1}{1-3} = -\frac{1}{2}$ $\quad f(x)-2 = -\frac{1}{2}(x-1)$

$f(x) = -\frac{1}{2}x + \frac{5}{2}$

$m_g = \frac{1-0}{3-0} = \frac{1}{3}$ $\quad g(x)-0 = \frac{1}{3}(x-0)$

$g(x) = \frac{1}{3}x$

$m_h = \frac{2-0}{1-0} = 2$ $\quad h(x)-0 = 2(x-0)$

$h(x) = 2x$

4. Work: $A_\Delta = \int_0^1 (2x - \frac{1}{3}x)dx + \int_1^3 (-\frac{1}{2}x + \frac{5}{2} - \frac{1}{3}x)dx$

$A_\Delta = \frac{5}{3}(\frac{1}{2})x^2 \Big|_0^1 + \left[(-\frac{5}{6})(\frac{1}{2})x^2 + \frac{5}{2}x \Big|_1^3\right]$

$A_\Delta = \frac{5}{6}(1)^2 - \frac{5}{6}(0)^2 + \left[-\frac{5}{12}(3)^2 + \frac{5}{2}(3) - \left(-\frac{5}{12}(1)^2 + \frac{5}{2}(1)\right)\right]$

$A_\Delta = \frac{5}{6} - \frac{15}{4} + \frac{15}{2} + \frac{5}{12} - \frac{5}{2}$

$A_\Delta = \frac{15}{12} - \frac{45}{12} + \frac{60}{12}$

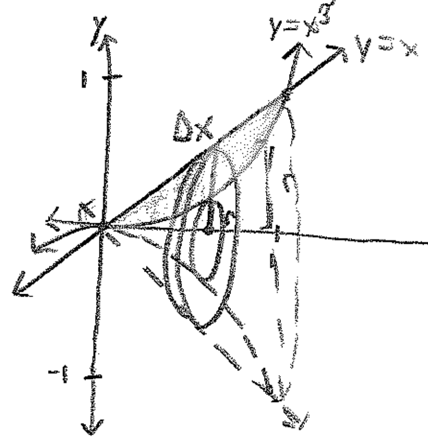
5. Solution: $\boxed{A_\Delta = \frac{5}{2} \text{ units}^2}$ ✓

Sect. 6.2

#7.) Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

$y = x^3$, $y = x$, $x \geq 0$; about the x-axis

1. Picture:



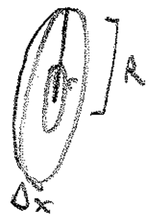
2. Goal: Find the volume of the shape formed by the curves.

3. Set-up: Use washer method.

$r = x^3$

$R = x$

thickness $= \Delta x$



4. Work:

$$V \approx \sum_{i=1}^{n} (\pi x^2 - \pi x^6)\Delta x \quad \text{from } x=0 \text{ to } x=1$$

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi(x^2 - x^6)\Delta x \quad \text{from } x=0 \text{ to } x=1$$

$$V = \pi \int_0^1 (x^2 - x^6)\,dx$$

$$V = \tfrac{\pi}{3}x^3 - \tfrac{\pi}{7}x^7 \Big|_0^1$$

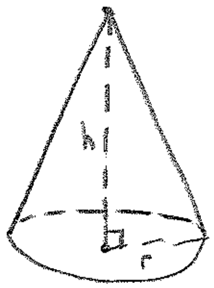
$$V = \tfrac{\pi}{3}(1)^3 - \tfrac{\pi}{7}(1)^7 - \left[\tfrac{\pi}{3}(0)^3 - \tfrac{\pi}{7}(0)^7\right]$$

5. Solution: $\boxed{V = \frac{4\pi}{21} \text{ units}^3}$ ✓

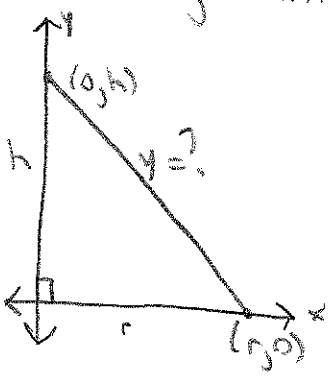
#47.) Find the volume of the described solid S.

A right circular cone with height h and base radius r.

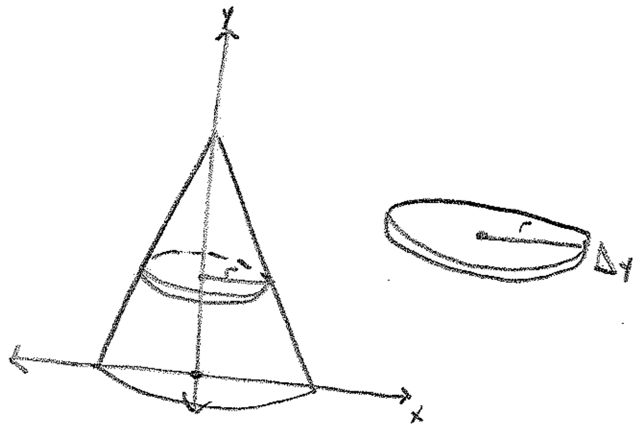
1. Picture:



2. Goal: Find the volume of the cone.
3. Set-up: Cone is a triangle with area $= \frac{1}{2}bh$ revolved around h.



Disc Method



4. Work:

$$m = \frac{h-0}{0-r}$$

$$m = -h/r$$

$$y - h = -h/r\,(x-0)$$

$$y = -h/r\,x + h$$

$$y - h = -h/r\,x$$

$$-r/h\,(y-h) = x = r$$

$$V_\Delta = \int_0^h \pi\left(\frac{-r}{h}(y-h)\right)^2 dy$$

$$V_\Delta = \int_0^h \pi\left(\frac{-r}{h}y + r\right)^2 dy$$

$$V_\Delta = \int_0^h \pi\left(\frac{r^2}{h^2}y^2 - \frac{2r^2}{h}y + r^2\right) dy$$

$$V_\Delta = \frac{\pi r^2}{3h^2}y^3 - \frac{\pi r^2}{h}y^2 + \pi r^2 y \Big|_0^h$$

$$V_\Delta = \frac{\pi r^2}{3\cancel{h^2}}\cdot h^{\cancel{3}} - \frac{\pi r^2}{\cancel{h}}\cdot h^{\cancel{2}} + \pi r^2 h - \left[\cancel{\frac{-\pi r^2}{3h^2}(0)^3} - \cancel{\frac{\pi r^2}{h}(0)^2} + \cancel{\pi r^2(0)}\right]$$

$$V_\Delta = \tfrac{1}{3}\pi r^2 h - \cancel{\pi r^2 h} + \cancel{\pi r^2 h}$$

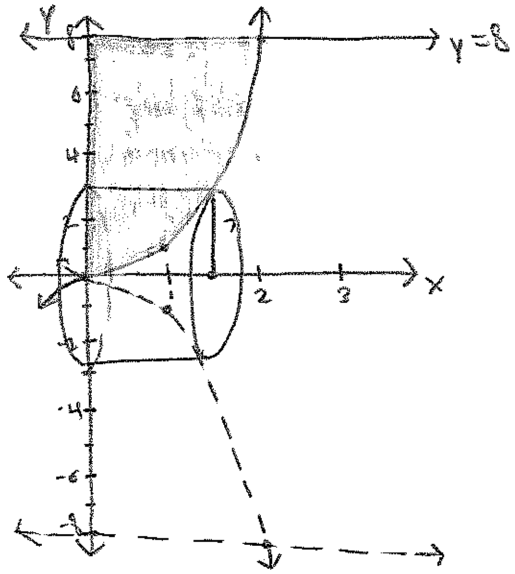
5. Solution: $\boxed{V_\Delta = \tfrac{1}{3}\pi r^2 h \text{ units}^3}$ ✓

Sect. 6.3

#11. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

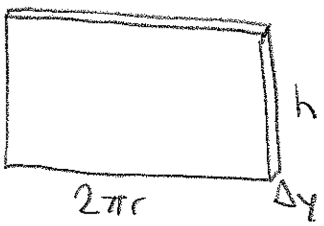
$$y = x^3, \quad y = 8, \quad x = 0$$

1. Picture:



2. Goal: Find the volume of the solid.

3. Set-up:



$r = x^3$

$h = x$

4. Work: $\sqrt[3]{y} = \sqrt[3]{x^3}$

$$x = \sqrt[3]{y} = h$$

$$r = x^3 = y$$

$$V = \int_0^8 (2\pi y \cdot \sqrt[3]{y})\, dy$$

$$V = \int_0^8 (2\pi y^{4/3})\, dy$$

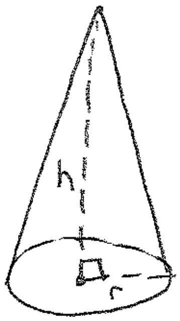
$$V = 2\pi (3/7)\, y^{7/3} \Big|_0^8$$

$$V = 6/7\, \pi (8)^{7/3} - \left[\cancel{6/7\, \pi (0)^{7/3}}\right]$$

5. Solution: $\boxed{V = \frac{768}{7}\pi \text{ units}^{2}}$ 3 ✓

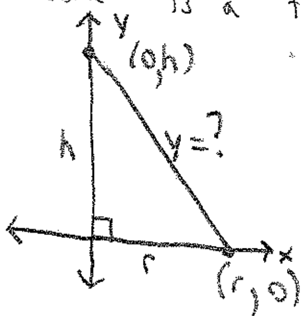
#47.) Use cylindrical shells to find the volume of the solid.
A right circular cone with height h and base radius r.

1. Picture:

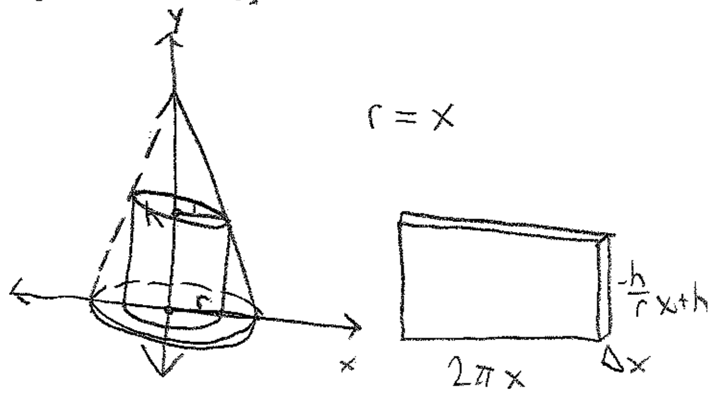


2. Goal: Find the volume of the cone.

3. Set-up: Cone is a triangle with area = $\frac{1}{2}bh$ revolved around line h.



Shell method



4. Work:

$$m = \frac{h-0}{0-r}$$

$$m = -h/r$$

$$y-h = -h/r\,(x-0)$$

$$y = -h/r\,x + h$$

$$V_\Delta = \int_0^r \left(2\pi x \cdot \left(-\frac{h}{r}x + h\right)\right)dx$$

$$V_\Delta = \int_0^r \left(-\frac{2\pi h}{r}x^2 + 2\pi h x\right)dx$$

$$V_\Delta = -\frac{2\pi h}{3r}x^3 + \pi h x^2 \Big|_0^r$$

$$V_\Delta = -\frac{2\pi h}{3r}r^3 + \pi h r^2 - \left[\cancel{-\frac{2\pi h}{3r}}(0)^3 + \pi h (0)^2\right]$$

$$V_\Delta = -\frac{2}{3}\pi h r^2 + \pi h r^2$$

Solution: $\boxed{V_\Delta = \frac{1}{3}\pi h r^2 \text{ units}^3}$ ✓