Section 9.1, #3

Picture:

N/A

Goal: The goal is to determine the values of r that will sastisfy the differential equation and to show that any member of $y = ae^{r_1 x} + be^{r_2 x}$ is also a solution.

Set up:

Given: $y = e^{rx}$ must sastisfy $2y'' + y' - y = 0$

Mathematical Model: Differential Equations

Work:

a) $y = e^{rx}$ $y' = re^{rx}$ $y'' = r^2 e^{rx}$

$2r^2 e^{rx} + re^{rx} - e^{rx} = 0$ $(2r^2 + r - 1)e^{rx} = 0$ $(2r-1)(r+1)e^{rx} = 0$ $\boxed{r = \frac{1}{2}, -1}$

b) $y = ae^{x/2} + be^{-x}$ $y' = \frac{1}{2}ae^{x/2} - be^{-x}$ $y'' = \frac{1}{4}ae^{x/2} + be^{-x}$

$2(\frac{1}{4}ae^{x/2} + be^{-x}) + \frac{1}{2}ae^{x/2} - be^{-x} - ae^{x/2} - be^{-x} = \cancel{\frac{1}{2}ae^{x/2}} + \cancel{2be^{-x}} + \cancel{\frac{1}{2}ae^{x/2}} - \cancel{be^{-x}} - \cancel{ae^{x/2}} - \cancel{be^{-x}} = \boxed{0}$

Conclusion:

In order for $y = e^{rx}$ to be a solution, r must take on a value of $\frac{1}{2}$ or -1. The first and second derivatives of the function had to be taken so that they could be substituted into the differential equation. For part b), the left side of the equation had to equal zero in order for the family of functions to be a solution to the differential equation.

Section 9.1, #5

Picture:

N/A

Goal: The goal is to determine which functions are solutions to the given differential equation.

Set up:

Mathematical Model: The functions must sastisfy $y''+y=\sin x$

Work:

a) $y=\sin x \quad y'=\cos x \quad y''=-\sin x$

$-\sin x+\sin x\neq\sin x$

b) $y=\cos x \quad y'=-\sin x \quad y''=-\cos x$

$-\cos x+\cos x\neq\sin x$

c) $y=\frac{1}{2}x\sin x \quad y'=\frac{1}{2}x\cos x+\frac{1}{2}\sin x \quad y''=-\frac{1}{2}x\sin x+\frac{1}{2}\cos x+\frac{1}{2}\cos x$

$\frac{1}{2}x\sin x-\frac{1}{2}x\sin x+\cos x\neq\sin x$

d) $y=-\frac{1}{2}x\cos x \quad y'=\frac{1}{2}x\sin x-\frac{1}{2}\cos x \quad y''=\frac{1}{2}x\cos x+\frac{1}{2}\sin x+\frac{1}{2}\sin x$

$-\frac{1}{2}x\cos x+\frac{1}{2}x\cos x+\sin x=\sin x$ ✓

Conclusion:

Since the function $y=-\frac{1}{2}x\cos x$ and its second derivative were the only pair to equal $\sin x$ when added, it is the only function that is a solution to the differential equation.

Section 9.3, #2

Picture:

N/A

Goal: The goal is to solve the separable differential equation.

Set up:

Mathematical Model: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ $h(y)dy = g(x)dx$ $\int h(y)dy = \int g(x)dx$

Work:

$\frac{dy}{dx} = \frac{x}{e^y}$ $e^y dy = x\,dx$ $\int e^y dy = \int x\,dx$ $e^y = \frac{1}{2}x^2 + C$

$\boxed{y = \ln\left|\frac{1}{2}x^2 + C\right|}$

Conclusion:

Since the differential equation was first order and could be separated as a function of x and a function of y, I was able to get all the y's on one side and x's on the other and integrate to solve the equation. Since both sides would have an arbitrary constant, I moved them both to the right and called the result "C".

Section 9.3, # 11

Picture: N/A

Goal: The goal is to find the specific solution of the differential equation that sastisfies the initial condition.

Set up:

Mathematical Model: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ $h(y)\,dy = g(x)\,dx$ $\int h(y)\,dy = g(x)\,dx$

Given: $y(0) = -3$

Work:

$\frac{dy}{dx} = \frac{x}{y}$ $y\,dy = x\,dx$ $\int y\,dy = \int x\,dx$ $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$

$y = \pm\sqrt{x^2 + K}$ $y(0) = -3 = \sqrt{K}$ $K = 9$ $\boxed{y = -\sqrt{x^2 + 9}}$

Conclusion:

By separating the equation into a function of x and of y and integrating, the function can be solved for y. Then the initial condition can be plugged in. Since I had to take the square root of both sides to solve for y, I had to put the ± sign in front. This allowed me to choose the negative sign for my final answer since $\sqrt{9}$ had to equal -3 instead of +3.

Section 9.3, # 43

Picture: N/A

Goal: The goal is to solve the differential equation at any time t and to interpret $\lim_{t\to\infty} C(t)$.

Set up:

Mathematical Model: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ $h(y)\,dy = g(x)\,dx$ $\int h(y)\,dy = \int g(x)\,dx$

Given: $C(0) = C_0$, K is a positive constant, $C_0 < r/K$

Work:

a) $\frac{dC}{dt} = r - KC$ $\frac{1}{r-KC}\,dC = dt$ $\int \frac{1}{r-KC}\,dC = \int dt$ $u = r - KC$ $-\frac{1}{K}du = dC$

$\int \frac{1}{u}\,du = \ln u + c$

$-\frac{1}{K}\ln|r-KC| = t + A$ $\ln|r-KC| = -Kt + A$

$r - KC = e^{-Kt+A}$ $r - KC = Ae^{-Kt}$ $-KC = Ae^{-Kt} - r$ $C = Ae^{-Kt} + r/K$

$C(0) = C_0$, $C_0 = A + r/K$ $A = C_0 - r/K$ $\boxed{C(t) = (C_0 - r/K)e^{-Kt} + r/K}$

b) $\lim_{t\to\infty}(C_0 - r/K)e^{-Kt} + r/K = \boxed{r/K}$ since e^{-Kt} approach zero since K is a positive constant

Conclusion:

By separating the differential equation and integrating, I was able to solve for $C(t)$. Since I knew the initial condition $C(0) = C_0$, I was able to solve for the arbitrary constant A in terms of C_0, r, and K so that I could find the limit as $t \to \infty$. Since the limit is r/K and $r/K > C_0$, this means that the concentration increases over time and eventually approaches the value r/K.

Section 9.3, # 46

Picture: N/A

Goal: The goal is to set up a differential equation that models the situation and find the limit as $t \to \infty$.

Setup:

Mathematical Model. $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ $h(y)dy = g(x)dx$ $\int h(y)\,dy = \int g(x)\,dx$

Given: $V = 180\ m^3$ initially $.15\%$ CO_2 $.05\%$ CO_2 flows in $2m^3/min$ and mixed flows out at $2m^3/min$

Work: a is amount and % is percentage

rate flowing in $= \frac{.05\%}{100} \cdot 2m^3/min = .001 m^3/min$

rate flowing out: $\frac{a\,m^3}{180\,m^3} \cdot 2m^3/min = \frac{a}{90}\ m^3/min$

$\frac{da}{dt} = .001 - \frac{a}{90} = \frac{9-100a}{9000}$

$\frac{da}{9-100a} = \frac{dt}{9000}$

$-\frac{1}{100}\ln|9-100a| = \frac{1}{9000}t + C$

$\ln|9-100a| = -\frac{1}{90}t + C$

$9 - 100a = e^{-t/90 + C} = Ce^{-t/90}$

$a = .09 + Ce^{-t/90}$

$a(0) = \frac{.15\%}{100} \cdot 180m^3 = .27m^3$

$.09 + C = .27$ $C = .18$

$a(t) = .09 + .18e^{-t/90}$

$\boxed{\%(t) = \frac{a(t)\,m^3}{180m^3} \cdot 100 = .05 + .1e^{-t/90}}$

$\boxed{\lim_{t\to\infty} .05 + .1e^{-t/90} = .05}$; as $t \to \infty$, $e^{-t/90}$ approaches zero

Conclusion:

A differential equation in terms of amount and time had to be set up and solved first with the given initial condition. Once it was solved, the amount at time t just had to be converted to a percentage at time t by dividing by the full volume and multiplying by 100. The limit of the percentage function as $t \to \infty$ (in the long run) was than .05% since $e^{-t/90}$ goes to zero, meaning that in the long run the percentage of CO_2 in the

Section 9.4, # 3

Picture: N/A

Goal: The goal is to determine the biomass in the ecosystem a year later and to find how much time it will take for the biomass to reach 4×10^7 Kg.

Set up:

Mathematical Model: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ $\int h(y)\,dy = \int g(x)\,dx$

Given: $\frac{dy}{dt} = Ky\left(1-\frac{y}{M}\right)$ $K = .71$ per year $M = 8\times10^7$ Kg

$y(0) = 2\times10^7$ Kg

Work:

$\frac{dy}{dt} = Ky\left(1-\frac{y}{M}\right)$ $\frac{dy}{y(1-y/m)} = Kdt$ $\frac{M\,dy}{y(M-y)} = Kdt$

$\frac{M}{y(M-y)} = \frac{A}{y} + \frac{B}{M-y}$ $M = A(M-y) + By$ $y=0,\ M = AM$ $A=1$

$y = M,\ M = BM$ $B=1$

$\left(\frac{1}{y} + \frac{1}{M-y}\right)dy = Kdt$ $\ln\left|\frac{M-y}{y}\right| = -Kt + C$ $\frac{M}{y} - 1 = Ae^{-Kt}$

$\frac{M}{y} = Ae^{-Kt} + 1$ $y = \frac{M}{1+Ae^{-Kt}}$ $y(0) = 2\times10^7 = \frac{8\times10^7}{1+A(1)}$ $A = 3$

a) $y(t) = \frac{M}{1+3e^{-Kt}}$ $y(1) = \frac{8\times10^7}{1+3e^{-(.71)(1)}} = \boxed{3\times10^7 \text{ Kg}}$

b) $4\times10^7 = \frac{8\times10^7}{1+3e^{-.71t}}$ $\boxed{t \approx 1.5 \text{ years}}$

Conclusion:

The model for the population growth was a separable differential equation. Once I was able to manipulate the left side so that it could be integrated (through partial fractions), I was able to solve for y in terms of the other variables. I had an initial condition which allowed me to solve for A, which then allowed me to solve for y(1) and the time when $y(t) = 4\times10^7$.

Section 9.4, #9

Picture: N/A

Goal: The goal is to write a differential equation that models the situation, solve the equation, and determine at what time 90% of a given population has heard the rumor.

Set up:

Mathematical Model: $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ $\int h(y)\,dy = \int g(x)\,dx$

Given: Small-town has 1000 people, $y(0) = \frac{80}{1000}$

$y(4) = \frac{1}{2}$

Work:

a) $\frac{dy}{dt} = Ky(1-y)$

b) $\frac{dy}{y(1-y)} = K\,dt$ $\quad \frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$ $\quad 1 = A(1-y) + By$ $\quad y=0,\ 1=A$ $\quad y=1,\ 1=B$

$\left(\frac{1}{y} + \frac{1}{1-y}\right)dy = K\,dt$ $\quad \ln\left|\frac{1-y}{y}\right| = -Kt + C$ $\quad \frac{1}{y} - 1 = Ae^{-Kt}$ $\quad \boxed{y = \frac{1}{1+Ae^{-Kt}}}$

c) $y(0) = \frac{80}{1000} = \frac{2}{25} = \frac{1}{1+A(1)}$ $\quad A = 11.5$

$y(4) = \frac{1}{2} = \frac{1}{1+11.5e^{-K(4)}}$ $\quad K = -.61059$ $\quad y(t) = \frac{9}{10} = \frac{1}{1+11.5e^{-.61059t}}$ $\quad t = 7.6$ hours

8 AM + 7.6 hours = $\boxed{3:36 \text{ PM}}$

Conclusion:

y(t) is the fraction of the population that has heard the rumor at time t. So for the differential equation, the product of the fraction of the population that has heard it times the fraction that hasn't is y(1-y). The equation can then be separated and the left side integrated through partial fractions. The first initial condition got rid of K and allowed me to solve for A, which in turn allowed me to use the second condition to solve for K. Once all of the variables were solved for, I was able to find the time at which the fraction was 9/10.