Models for Population Growth - Section 9.4

Problem. A bacteria culture which grows at a rate proportional to its size contains 100 cells initially and contains 420 cells after one hour. Model this scenario using a differential equation.

$$\frac{dP}{dt} = kP$$

$$\frac{1}{P}\,dP = k\,t$$

$$\ln|P| = kt + C_1$$

$$|P| = e^{kt}C_2$$

$$\boxed{P = C_3 e^{kt}}$$

$$P(0) = 100 = C_3 e^0$$

$$C_3 = 100$$

$$\boxed{P(t) = 100e^{kt}}$$

We can account for "harvesting" by modifying this model slightly.

Problem. You open a bank account with $7500 ($B(0)$). The account pays 2% interest compounded continuously ($.02(B)$), and you use this account to make payments totaling $100 ($-100$) each year. Construct a differential equation for the balance in this account at time t (in years). What are the equilibrium solutions?

$$\frac{dB}{dt} = .02(B) - 100$$

$$\int \frac{1}{.02B-100}\,dB = \int dt$$

$u = .02B - 100$

$du = .02\,dB$

$dB = \frac{du}{.02}$

$$\frac{1}{.02}\int \frac{1}{u}\,du = \frac{1}{.02}\ln|u|$$

$\frac{1}{2/100} = 50$

$$50\ln|.02B-100| = t + C_1$$

$$\ln|.02B-100| = \frac{1}{50}t + C_2$$

$$|.02B-100| = e^{t/50}C_3$$

$$.02B - 100 = C_4 e^{t/50}$$

$$\frac{2}{100}B = C_4 e^{t/50} + 100$$

$$B = C_5 e^{t/50} + 5000$$

$$B(0) = 7500 = C_5(1) + 5000$$

$$C_5 = 2500$$

$$\boxed{B(t) = 2500e^{t/50} + 5000}$$

$$\frac{dB}{dt} = 2500\,\frac{1}{50}e^{t/50} = 0$$

$$e^{t/50} = 0$$

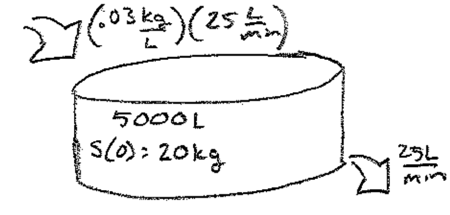
DNE

Populations often increase exponentially early on and level off as they approach their carrying capacity (a limit on the population due to resource constraints). In this case, we need a model where the relative growth rate decreases as the population increases and is negative if the population is larger than the carrying capacity. The simplest such model is given below and is called the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

M = carrying capacity

Problem. A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?



$$\frac{dS}{dt} = \left(\frac{.03\,kg}{L}\right)\left(\frac{25L}{min}\right) - \left(\frac{S(t)}{5000L}\right)\left(\frac{25L}{min}\right)$$

$$\frac{dS}{dt} = \left(.75 - \frac{1}{200}S\right) = \frac{150-S}{200}$$

$$\int \frac{1}{150-S}\,dS = \int \frac{1}{200}\,dt$$

$$-\ln|150-S| = \frac{t}{200} + C_1$$

$$e^{\ln|150-S|} = e^{C_2 - t/200}$$

$$|150-S| = C_3 e^{-t/200}$$

$$150-S = C_4 e^{-t/200}$$

$$\boxed{S = 150 - C_4 e^{-t/200}}$$

$$S(0) = 20 = 150 - C_4$$

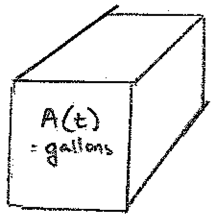
$$C_4 = 130$$

$$\boxed{S(t) = 150 - 130e^{-t/200}}$$

$$S(30) = 150 - 130e^{-\frac{30}{200}}$$

$$\boxed{S(30) = 38.1 \text{ kg NaCl}}$$

Problem. Let A be the number of gallons of water in Rhonda's aquarium t hours after she begins to fill it. Find a formula for A given that the aquarium is filled at a rate of $\frac{t}{A^2}$ gallons per minute and it initially contains 3 gallons of water.



$$\frac{dA}{dt} = \frac{t}{A^2}$$

$$\int A^2\,dA = \int t\,dt$$

$$\frac{1}{3}A^3 = \frac{t^2}{2} + C_1$$

$$A^3 = \frac{3}{2}t^2 + C_2$$

$$\boxed{A = \sqrt[3]{\frac{3t^2}{2} + C_2}}$$

$$A(0) = 3 = \sqrt[3]{0 + C_2}$$

$$27 = C_2$$

$$\boxed{A(t) = \sqrt[3]{\frac{3}{2}t^2 + 27}}$$

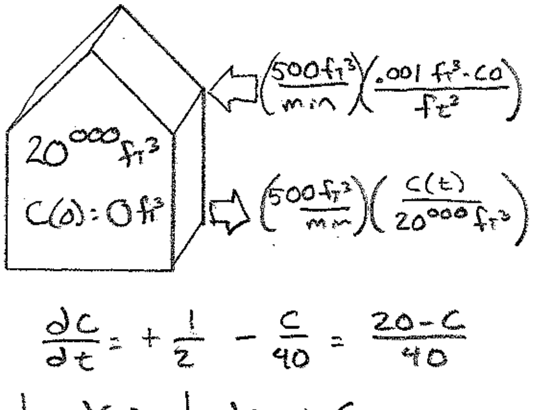
"Makes Sense" check:

$(20\,000)(.04\%) = 8\,ft^3\text{-}CO$

$500\frac{ft^3}{min}(.1\%) = .5\frac{ft^3\text{-}CO}{min}$

If no CO was leaving, it would take $8/.5 = 16$ minutes to reach $8\,ft^3$-CO

Problem. A house with a volume of 20,000 ft^3 initially contains no carbon monoxide. A gas heater introduces air into the house that contains 0.1% carbon monoxide at a rate of 500 ft^3/min. The carbon monoxide mixes with the rest of the air in the house instantaneously, and the mixed air flows out of the house at the same rate it enters. A carbon monoxide detector in the house is designed to sound an alarm when the air contains 0.04% carbon monoxide. How long will it take for the alarm to sound?



$$\frac{dC}{dt} = +\frac{1}{2} - \frac{C}{40} = \frac{20-C}{40}$$

$$\frac{1}{20-C}\,dC = \frac{1}{40}\,dt + C_1$$

$$-\ln|20-C| = \frac{t}{40} + C_2$$

$$\ln|20-C| = C_3 - \frac{t}{40}$$

$$|20-C| = C_4 e^{-t/40}$$

$$20 - C = C_5 e^{-t/40}$$

$$C(t) = 20 - C_5 e^{-t/40}$$

$$C(0) = 0 = 20 - C_5$$

$$C_5 = 20$$

$$\boxed{C(t) = 20\left(1 - e^{-t/40}\right) ft^3\text{-}CO}$$

$$.0004 = \frac{C(t)}{20000}$$

$$C(t) = 8 = 20\left(1 - e^{-t/40}\right)$$

$$\frac{2}{5} = 1 - e^{-t/40}$$

$$e^{-t/40} = \frac{3}{5}$$

$$-t/40 = \ln(3/5)$$

$$t = -40\ln(3/5)$$

$$\boxed{t = 20.4 \text{ min to reach } 0.04\%\ CO}$$

Problem. In a certain part of the Pike National Forest, dead pine needles accumulate on the ground at a rate of 8 kilograms per square meter per year. They decompose at a continuous rate of 80% per year. If there are 7 kilograms of pine needles per square meter initially, how much will there be after 2 years? What will happen in the long run?

$$\frac{dN}{dt} = +8 - .8N = 8\left(1 - \frac{N}{10}\right)$$

$$\int \frac{1}{1 - \frac{N}{10}}\, dN = \int 8\, dt$$

$u = 1 - \frac{N}{10}$

$du = -\frac{dN}{10}$

$$-10 \int \frac{1}{u}\, du$$

$$-10 \ln|u|$$

$$\boxed{-10 \ln\left|1 - \frac{N}{10}\right| = 8t + C_1}$$

$$\ln\left|1 - \frac{N}{10}\right| = -\frac{8t}{10} + C_2$$

$$\left|1 - \frac{N}{10}\right| = C_3 e^{-8t/10}$$

$$1 - \frac{N}{10} = C_4 e^{-4t/5}$$

$$1 - C_4 e^{-4t/5} = \frac{N}{10}$$

$$\boxed{N = 10 + C_5 e^{-4t/5}}$$

$$N(0) = 7 = 10 + C_5 e^0$$

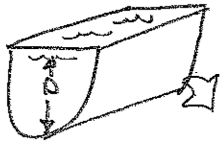
$$C_5 = -3$$

$$\boxed{N = 10 - 3e^{-4t/5}}$$

$$\lim_{t \to \infty} \left(10 - 3\underbrace{e^{-4t/5}}_{\frac{1}{\infty} \to 0}\right) = 10$$

$\boxed{\text{in the long run, we expect } 10\,\text{kg/m}^2 \text{ of pine needles}}$

Problem. A rancher's water trough leaks at a rate proportional to the square root of the depth of the water. If the water level is 25 inches at noon and 24 inches at 2:00 pm, when will the tank be empty?



$$\frac{dD}{dt} = k\sqrt{D}$$

$$D^{-\frac{1}{2}} dD = k\,dt$$

$$2D^{\frac{1}{2}} = kt + C_1$$

$$\boxed{D = \left(\tfrac{1}{2}kt + C\right)^2}$$

$$D(0) = 25 = \left(\tfrac{1}{2}k(0) + C\right)^2$$

$$\boxed{C = \pm 5}$$

$$D(2) = 24 = \left(\tfrac{1}{2}k(2) + 5\right)^2$$

$$\pm 2\sqrt{6} - 5 = k$$

$$D(t) = \left(\tfrac{1}{2}(\pm 2\sqrt{6} - 5)t + 5\right)^2$$

$$D(2) = 24 = \left(\tfrac{1}{2}k(2) - 5\right)^2$$

$$\pm 2\sqrt{6} + 5 = k$$

$$D(t) = \left(\tfrac{1}{2}(\pm 2\sqrt{6} + 5)t - 5\right)^2$$

use calc tool

$$\boxed{D(t) = \left(\tfrac{1}{2}(2\sqrt{6} - 5)t + 5\right)^2 \quad \text{or} \quad D(t) = \left(\tfrac{1}{2}(-2\sqrt{6} + 5)t - 5\right)^2}$$

Problem. During a chemical reaction, substance A is converted into substance B at a rate proportional to the square of the amount of substance A present. When $t = 0$, there are 60 grams of substance A, and after 1 hour ($t = 1$) only 10 grams of substance A remain. How much of substance A remains after two hours?

$$\frac{dA}{dt} = kA^2$$

$$\int A^{-2}\, dA = \int k\, dt$$

$$-A^{-1} = kt + C_1$$

$$\frac{1}{A} = C - kt$$

$$\boxed{\frac{1}{C - kt} = A}$$

$$A(0) = \frac{1}{C - 0} = 60$$

$$C = \frac{1}{60}$$

$$A(t) = \frac{1}{(1/60) - kt}$$

$$A(t) = \frac{60}{1 - 60kt}$$

$$A(1) = 10 = \frac{60}{1 - 60k}$$

$$1 - 60k = \frac{60}{10} = 6$$

$$1 - 6 = 60k$$

$$k = \frac{-5}{60} = -\frac{1}{12}$$

$$A(t) = \frac{60}{1 - 60\left(-\frac{1}{12}\right)t}$$

$$\boxed{A(t) = \frac{60}{1 + 5t}}$$