Improper Integrals

Type 1. Infinite Intervals

* Infinite limit(s) of integration
* "Long run" in a scenario

(Ex)

$$\int_1^{\infty} \frac{1}{(2x+1)^3}\,dx$$

u-sub

Side work

$u = 2x+1$

$du = 2\,dx$

$\frac{1}{2}du = dx$

$$= \lim_{b\to\infty} \int_1^{b} \frac{1}{(2x+1)^3}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2}\int_{x=1}^{x=b} \frac{1}{u^3}\,du$$

$$= \lim_{b\to\infty} \left.\frac{-1}{4u^2}\right|_{x=1}^{x=b} = \lim_{b\to\infty} \left.\frac{-1}{4(2x+1)^2}\right|_{x=1}^{x=b}$$

$$= \lim_{b\to\infty} \left[\frac{-1}{4(2b+1)^2} + \frac{1}{4(9)}\right]$$

$$= \lim_{b\to\infty} \left(\frac{-1}{4(2b+1)^2}\right) + \frac{1}{36}$$

$$= 0 + \frac{1}{36} = \frac{1}{36}$$

Converges

Improper Integrals

Type 2. Discontinuous Integrands

* At one or both limits of integration

(EX)
$$\int_0^3 z^2 \ln(z)\,dz$$

By Parts

Sidework

$u = \ln(z) \quad v = z^3/3$

$du = \frac{1}{z}dz \quad dv = z^2 dz$

$$= \lim_{a \to 0^+} \int_a^3 z^2 \ln(z)\,dz$$

$$= \lim_{a \to 0^+} \frac{z^3}{3}\ln(z)\Big|_{z=a}^{z=3} - \int_{z=a}^{z=3} \frac{z^2}{3}\,dz$$

$$= \lim_{a \to 0^+} \frac{z^3}{3}\ln(z) - \frac{z^3}{9}\Big|_{z=a}^{z=3}$$

$$= \lim_{a \to 0^+} (9\ln(3) - 3) - \left(\frac{a^3\ln(a)}{3} - \frac{a^3}{3}\right)$$

$$= 9\ln(3) - 3 - \lim_{a \to 0^+} \frac{\ln(a)}{3a^{-3}} + 0$$

$$\underset{L}{=} 9\ln(3) - 3 - \lim_{a \to 0^+} \frac{\frac{1}{a}}{-9a^{-4}}$$

L'Hospital's Rule

$$= 9\ln(3) - 3 - \lim_{a \to 0^+} \frac{-a^3}{9}$$

$$= 9\ln(3) - 3 - 0 = 9\ln(3) - 3$$

Converges

Improper Integrals

Type 2. Discontinuous Integrands

* Between limits of integration

(EX) $$\int_0^5 \frac{w}{w-2}\,dw = \int_0^2 \frac{w}{w-2}\,dw + \int_2^5 \frac{w}{w-2}\,dw$$

Start with left integral ...

Long Division

Sidework

$$\begin{array}{r} 1 \\ w-2\,\overline{)\,w} \\ -(w-2) \\ \hline 2 \end{array}$$

$$\lim_{b\to 2^-} \int_0^b \frac{w}{w-2}\,dw$$

$$= \lim_{b\to 2^-} \int_0^b 1 + \frac{2}{w-2}\,dw$$

$$= \lim_{b\to 2^-} w + 2\ln|w-2| \Big|_{w=0}^{w=b}$$

$$= \lim_{b\to 2^-} \left(b + 2\ln|b-2|\right) - \left(0 + 2\ln(2)\right)$$

$$= 2 + 2\lim_{b\to 2^-} \ln|b-2| - 2\ln(2)$$

$$= 2 - \infty - 2\ln(2)$$

Diverges

Since left integral diverges, the entire integral $\int_0^5 \frac{w}{w-2}\,dw$ diverges.