Exam 2: "Cheat Sheet"

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$
2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$
3. $\displaystyle\int e^x\,dx = e^x$
4. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a}$
5. $\displaystyle\int \sin x\,dx = -\cos x$
6. $\displaystyle\int \cos x\,dx = \sin x$
7. $\displaystyle\int \sec^2 x\,dx = \tan x$
8. $\displaystyle\int \csc^2 x\,dx = -\cot x$
9. $\displaystyle\int \sec x \tan x\,dx = \sec x$
10. $\displaystyle\int \csc x \cot x\,dx = -\csc x$
11. $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
12. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

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13. A sequence $\{a_n\}$ has **limit** L if for every $\epsilon > 0$ there is an integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.
14. $\lim_{n\to\infty} a_n = \infty$ means for every positive number M there is an integer N such that $a_n > M$ whenever $n > N$.
15. **Squeeze Theorem.** If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.
16. **Theorem.** If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
17. $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.
18. **Theorem.** Every bounded, monotonic sequence is convergent.
19. The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent if $|r| < 1$ and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If $|r| \ge 1$, the geometric series is divergent.
20. **Test for Divergence.** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
21. **The Integral Test.** Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. The the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.
22. **Remainder Estimate for the Integral Test.** If a_n converges by the Integral Test and $R_n = s - s_n$, then
$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$$
23. **The Comparison Test.** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
    (a) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is convergent.
    (b) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is divergent.
24. **Limit Comparison Test.** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
    (a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both diverge.
    (b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges.
    (c) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

25. **The Alternating Series Test.** If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies (a) $b_{n+1} \le b_n$ for all n and (b) $\lim_{n\to\infty} b_n = 0$ then the series is convergent.

26. **Alternating Series Estimation Theorem.** If $s = \sum(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies (a) and (b) above, then $|R_n| = |s - s_n| \le b_{n+1}$

27. A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent and **conditionally convergent** if $\sum a_n$ is convergent but not absolutely convergent.

28. **The Ratio Test**
(a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
(b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or is infinity, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

29. **The Root Test**
(a) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
(b) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or is infinity, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

30. **Theorem.** For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities: (i) The series converges only when $x = a$. (ii) The series converges for all x. (iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

31. **Theorem.** If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$ then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on $(a-R, a+R)$ and
(a) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$, each with radius of convergence R.

32. **Theorem.** If f has a power series representation at a, that is, if $f(x) = c_n(x-a)^n$ for $|x-a| < R$, then its coefficients are of the form $\dfrac{f^{(n)}(a)}{n!}$

33. **Taylor's Formula.** If f has $n+1$ derivatives in an interval I that contains the number a, then for x in I there is a number z strictly between x and a such that the remainder term in the Tayler series can be expressed as

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

34. $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$. IOC=$(-1, 1)$.

35. $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$. IOC=$(-\infty, \infty)$.

36. $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$. IOC=$(-\infty, \infty)$.

37. $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$. IOC=$(-\infty, \infty)$.

38. $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$. IOC=$[-1, 1]$.

39. **The Binomial Series** If k is any real number and $|x| < 1$, then $(1+x)^k = \displaystyle\sum_{n=0}^{\infty}\binom{k}{n}x^n$ where $\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$ for $n \ge 1$ and $\binom{k}{0} = 1$.