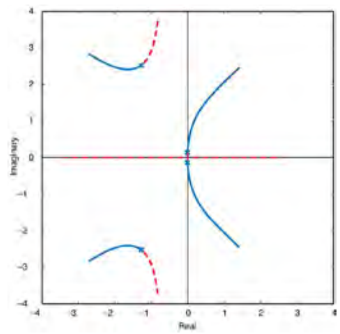
# Root Locus Analysis of Parameter Variations and Feedback Control

**Robert Stengel, Aircraft Flight Dynamics**
**MAE 331, 2014**

- **Effects of system parameter variations on modes of motion**
- **Root locus analysis**
  - Evans' s rules for construction
  - Application to longitudinal dynamic models




**Reading:**
*Flight Dynamics*
357-361, 465-467, 488-490, 509-514

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http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html



# Characteristic Equation: A Critical Component of the Response's Laplace Transform

$$\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\Delta\mathbf{x}(0) + \mathbf{G}\,\Delta\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)\right]$$

$$\left[s\mathbf{I} - \mathbf{F}\right]^{-1} = \frac{Adj\left(s\mathbf{I} - \mathbf{F}\right)}{\left|s\mathbf{I} - \mathbf{F}\right|} = \frac{\mathbf{C}^T\left(s\right)}{\left|s\mathbf{I} - \mathbf{F}\right|} \quad \frac{(n \times n)}{(1 \times 1)}$$

- **Characteristic equation defines the modes of motion**

$$\begin{aligned}\left|s\mathbf{I} - \mathbf{F}\right| &= \Delta(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \\ &= \left(s - \lambda_1\right)\left(s - \lambda_2\right)\left(\ldots\right)\left(s - \lambda_n\right) = 0\end{aligned}$$

- **Recall: *s* is a complex variable** $s = \sigma + j\omega$



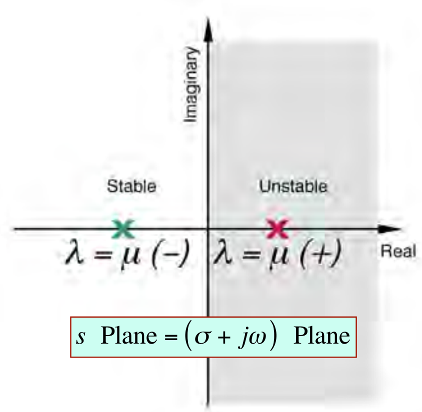
# Real Roots of the Dynamic System

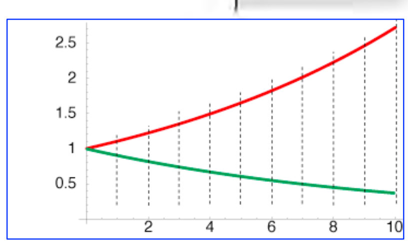
- **Roots are solutions of the characteristic equation**

$$\Delta(s) = \left(s - \lambda_1\right)\left(s - \lambda_2\right)\left(\ldots\right)\left(s - \lambda_n\right) = 0$$

- **Real roots**
  - **are confined to the real axis**
  - **represent convergent or divergent time response**
  - **time constant, $\tau = -1/\lambda = -1/\mu$, sec**

$$\lambda_i = \mu_i \quad \text{(Real number)}$$
$$x\left(t\right) = x\left(0\right)e^{\mu t}$$



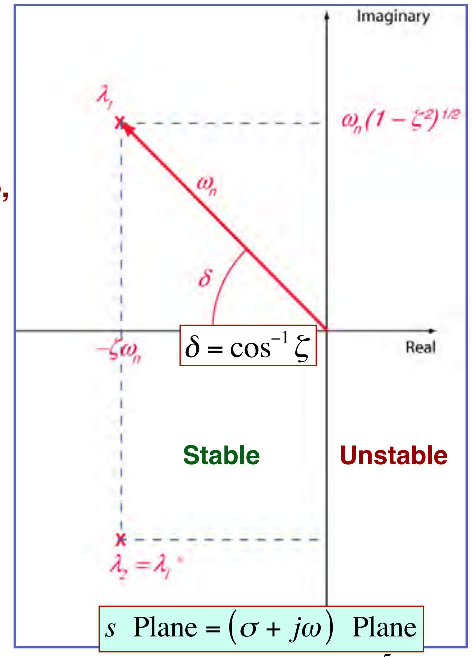
# Complex Roots of the Dynamic System

- **Complex roots**
  - **occur only in complex-conjugate pairs**
  - **represent oscillatory modes**
  - **natural frequency, $\omega_n$, and damping ratio, $\zeta$, as shown**

$$\lambda_1 = \mu_1 + j\nu_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

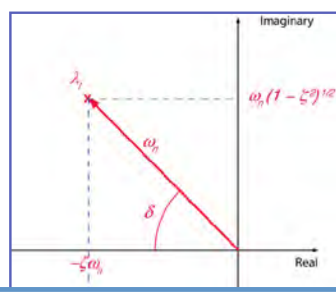
$$\lambda_2 = \mu_2 + j\nu_2 = \mu_1 - j\nu_1 = \lambda_1^*$$
$$= -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

  - **time constant = $-1/\mu = 1/\zeta\omega_n$**
  - **decay of exponential time-response envelope**






# Complex Roots, Damping Ratio, and Damped Natural Frequency



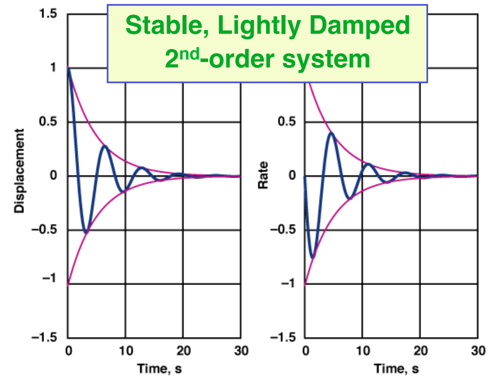

$$\left(s-\lambda_1\right)\left(s-\lambda_1^*\right) = \left[s-\left(\mu_1+j\nu_1\right)\right]\left[s-\left(\mu_1-j\nu_1\right)\right]$$
$$= s^2 - \left[\left(\mu_1-j\nu_1\right)+\left(\mu_1+j\nu_1\right)\right]s + \left(\mu_1-j\nu_1\right)\left(\mu_1+j\nu_1\right)$$
$$= s^2 - 2\mu_1 s + \left(\mu_1^2+\nu_1^2\right) \triangleq s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\mu_1 = -\zeta\omega_n = -1/\text{Time constant}$$
$$\nu_1 = \omega_n\sqrt{1-\zeta^2} \triangleq \omega_{n_{damped}} = \text{Damped natural frequency}$$

# Corresponding 2nd-Order Initial Condition Response

Identical exponentially decaying envelopes for both displacement and rate

General form of response

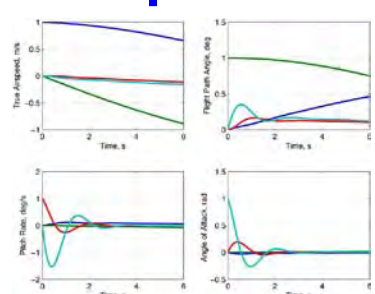



$$x_1(t) = A\color{red}{e^{-\zeta\omega_n t}} \sin\left[\omega_n\sqrt{1-\zeta^2}\,t + \varphi\right]$$

$$x_2(t) = A\color{red}{e^{-\zeta\omega_n t}} \left[\omega_n\sqrt{1-\zeta^2}\right]\cos\left[\omega_n\sqrt{1-\zeta^2}\,t + \varphi\right]$$

# Multi-Modal LTI Responses Superpose Individual Modal Responses

- With distinct roots, ($n = 4$) for example, *partial fraction expansion* for each state element is



$$\Delta x_i(s) = \frac{d_{1_i}}{(s-\lambda_1)} + \frac{d_{2_i}}{(s-\lambda_2)} + \frac{d_{2_i}}{(s-\lambda_3)} + \frac{d_{2_i}}{(s-\lambda_4)}$$

Corresponding 4th-order time response is

$$\Delta x_i(t) = d_{1_i} e^{\lambda_1 t} + d_{2_i} e^{\lambda_2 t} + d_{3_i} e^{\lambda_3 t} + d_{4_i} e^{\lambda_4 t}$$

# Evans's Rules for Root Locus Analysis



## Root Locus Example: $4^{th}$-Order Longitudinal Characteristic Equation

$$\begin{aligned}
\Delta_{Lon}(s) &= s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \\
&= s^4 + \left(D_V + L_\alpha / V_N - M_q\right) s^3 \\
&+ \left[\left(g - D_\alpha\right) L_V / V_N + D_V \left(L_\alpha / V_N - M_q\right) - M_q L_\alpha / V_N - M_\alpha\right] s^2 \\
&+ \left\{M_q \left[\left(D_\alpha - g\right) L_V / V_N - D_V L_\alpha / V_N\right] + D_\alpha M_V - D_V M_\alpha\right\} s \\
&+ g\left(M_V L_\alpha / V_N - M_\alpha L_V / V_N\right) = 0
\end{aligned}$$

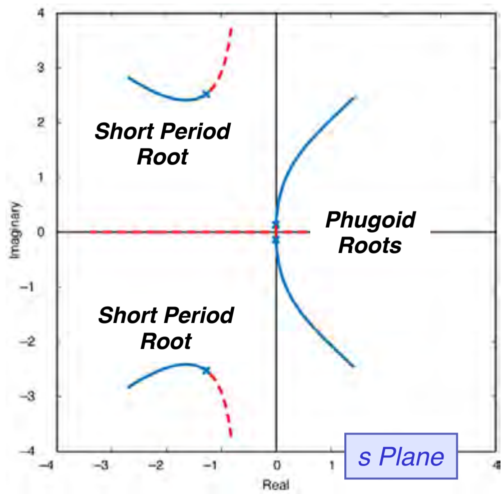
$\boxed{\text{with } L_q = D_q = 0}$

Typically factors into oscillatory phugoid and short-period modes

$$\Delta_{Lon}(s) = \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{Ph}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{SP}$$

# Root Locus Analysis of Parametric Effects on Aircraft Dynamics

- Parametric variations alter eigenvalues of **F**
- Graphical technique for finding the roots with varying parameter values




**Locus**: "the set of all points whose location is determined by stated conditions"



## Example: How do the roots vary when we change pitch-rate damping, $M_q$?

$$
\begin{aligned}
\Delta_{Lon}(s) = {} & s^4 + \left(D_V + \frac{L_\alpha}{V_N} - M_q\right)s^3 \\
& + \left[(g - D_\alpha)\frac{L_V}{V_N} + D_V\left(\frac{L_\alpha}{V_N} - M_q\right) - M_q\frac{L_\alpha}{V_N} - M_\alpha\right]s^2 \\
& + \left\{M_q\left[(D_\alpha - g)\frac{L_V}{V_N} - D_V\frac{L_\alpha}{V_N}\right] + D_\alpha M_V - D_V M_\alpha\right\}s \\
& + g\left(M_V\frac{L_\alpha}{V_N} - M_\alpha\frac{L_V}{V_N}\right) = 0
\end{aligned}
$$

- $M_q$ could be changed by
  - Variation in *aircraft aerodynamic configuration*
  - Effect of *feedback control*, i.e., control of pitching moment (via elevator) that is proportional to pitch rate

# Effect of Parameter Variations on Root Location


$$\Delta_{Lon}(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$
$$= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4)$$
$$= (s-\lambda_1)(s-\lambda_1^*)(s-\lambda_3)(s-\lambda_3^*)$$
$$= \left(s^2 + 2\zeta_P \omega_{n_P} s + \omega_{n_P}^2\right)\left(s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega_{n_{SP}}^2\right) = 0$$

- **Let "root locus gain" = *k* = $a_i$** (just a notation change)
  - **Option 1:** Vary ***k*** and calculate roots for each new value
  - **Option 2:** Apply Evans's Rules of Root Locus Construction



# Effect of $a_0$ Variation on Longitudinal Root Location

- **<u>Example</u>: Root locus gain, *k* = $a_0$**

$$\Delta_{Lon}(s) = \left[s^4 + a_3 s^3 + a_2 s^2 + a_1 s\right] + \left[k\right] \equiv d(s) + kn(s)$$
$$= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) = 0$$

$d(s)$: Polynomial in $s$, degree = $n$

$n(s)$: Polynomial in $s$, degree = $q$

*where*

$$d(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s$$
$$= (s-\lambda'_1)(s-\lambda'_2)(s-\lambda'_3)(s-\lambda'_4)$$
$$n(s) = 1$$

Degree?

# Effect of $a_1$ Variation on Longitudinal Root Location

- **Example**: Root locus gain, $k = a_1$

$$\Delta_{Lon}(s) = s^4 + a_3 s^3 + a_2 s^2 + ks + a_0 \equiv d(s) + kn(s)$$
$$= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) = 0$$

*where*

$$d(s) = s^4 + a_3 s^3 + a_2 s^2 + a_0$$
$$= (s-\lambda'_1)(s-\lambda'_2)(s-\lambda'_3)(s-\lambda'_4)$$
$$n(s) = s$$

Degree?



# Three Equivalent Equations for Defining Roots

$$d(s) + k\,n(s) = 0$$

$$1 + k\frac{n(s)}{d(s)} = 0$$

$$k\frac{n(s)}{d(s)} = -1 = (1)e^{-j\pi(rad)} = (1)e^{-j180(\deg)}$$

# Longitudinal Equation Example

### Original 4th-order polynomial

$$\Delta_{Lon}(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$
$$= (s-\lambda_1)(s-\lambda_1^*)(s-\lambda_3)(s-\lambda_3^*)$$
$$= \left(s^2 + 2\zeta_P \omega_{n_P} s + \omega_{n_P}^2\right)\left(s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega_{n_{SP}}^2\right) = 0$$

### Typical flight condition

$$\Delta_{Lon}(s) = s^4 + 2.57s^3 + 9.68s^2 + 0.202s + 0.145$$
$$= \left[s^2 + 2(0.0678)0.124s + (0.124)^2\right]\left[s^2 + 2(0.411)3.1s + (3.1)^2\right] = 0$$

Phugoid

Short Period



# Example: Effect of $a_0$ Variation

### Original 4th-order polynomial

$$\Delta_{Lon}(s) = s^4 + 2.57s^3 + 9.68s^2 + 0.202s + 0.145 = 0$$

### Example: $k = a_0$

$$\Delta(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$
$$= \left(s^4 + a_3 s^3 + a_2 s^2 + a_1 s\right) + k$$
$$= s\left(s^3 + a_3 s^2 + a_2 s + a_1\right) + k$$
$$= s(s+0.21)\left[s^2 + 2.55s + 9.62\right] + k$$

### Rearrange:

$$\frac{k}{s(s+0.21)\left[s^2 + 2.55s + 9.62\right]} = -1$$

# Example: Effect of $a_1$ Variation

## Example: $k = a_1$

$$\begin{aligned}\Delta(s) &= s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \\ &= s^4 + a_3 s^3 + a_2 s^2 + ks + a_0 \\ &= \left(s^4 + a_3 s^3 + a_2 s^2 + a_0\right) + ks \\ &= \left[s^2 - 0.00041s + 0.015\right]\left[s^2 + 2.57s + 9.67\right] + ks\end{aligned}$$

### Rearrange:

$$\frac{ks}{\left[s^2 - 0.00041s + 0.015\right]\left[s^2 + 2.57s + 9.67\right]} = -1$$



# Locations of Roots for Large and Small *k*

**Origins of the roots are the *n* poles of *d(s)***

$$\Delta(s) = d(s) + kn(s) \xrightarrow[k \to 0]{} d(s)$$

**Two destinations for the roots as *k* becomes large**

**1) *q* roots go to the zeros of *n(s)***

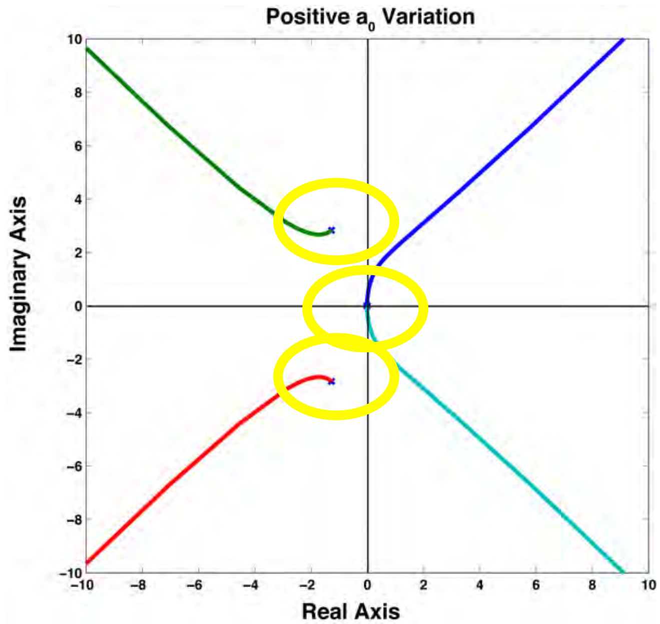
$$\frac{d(s) + kn(s)}{k} = \frac{d(s)}{k} + n(s) \xrightarrow[k \to \pm\infty]{} n(s) = \left(s - z_1\right)\left(s - z_2\right)\cdots$$

**2) *(n – q)* roots go to infinity**

$$\left[\frac{d(s) + kn(s)}{n(s)}\right] = \left[\frac{d(s)}{n(s)} + k\right] \xrightarrow[k \to \pm R \to \pm\infty]{} \left[\frac{s^n}{s^q} + k\right] \to s^{(n-q)} \pm R \to \pm\infty$$

## Origins of the roots are the *n* poles of *d(s)*

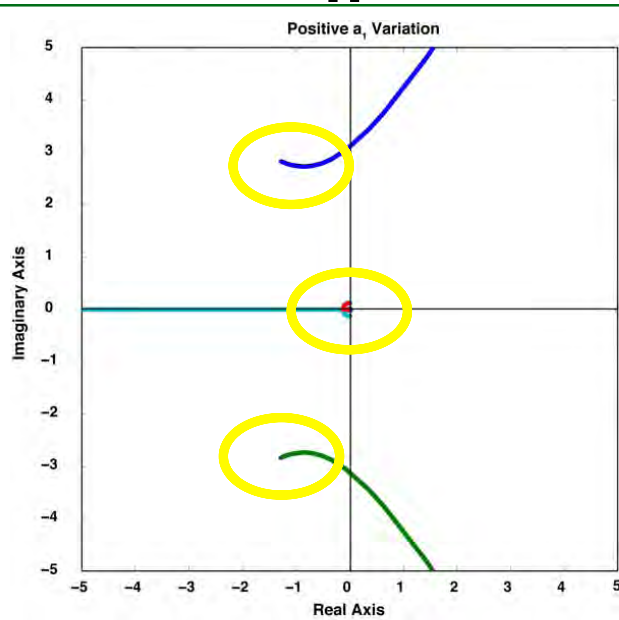
$$\frac{k}{s(s+0.21)\left[s^2+2.55s+9.62\right]}=-1$$






## Origins of the roots are the *n* poles of *d(s)*

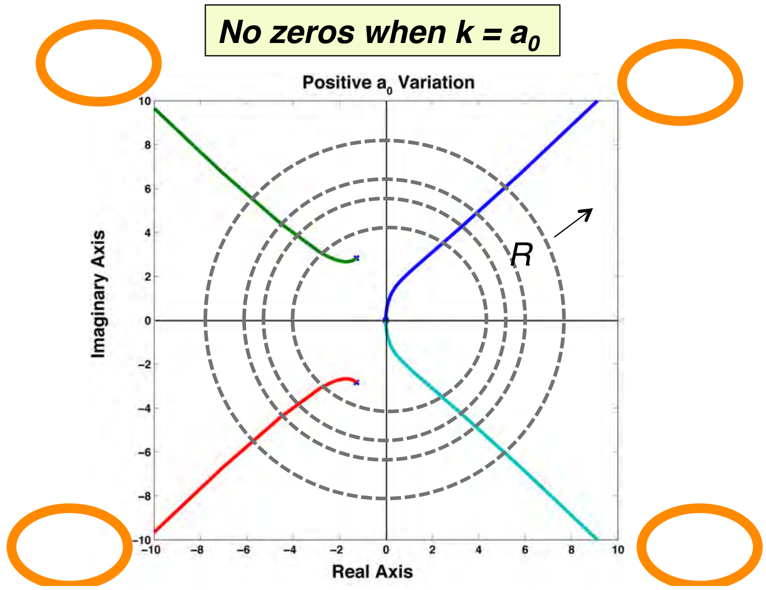
$$\frac{ks}{\left[s^2-0.00041s+0.015\right]\left[s^2+2.57s+9.67\right]}=-1$$



## Destination infinity for the roots as *k* becomes large

$$\frac{k}{s(s+0.21)\left[s^2+2.55s+9.62\right]}=-1$$

***No zeros when k = $a_0$***

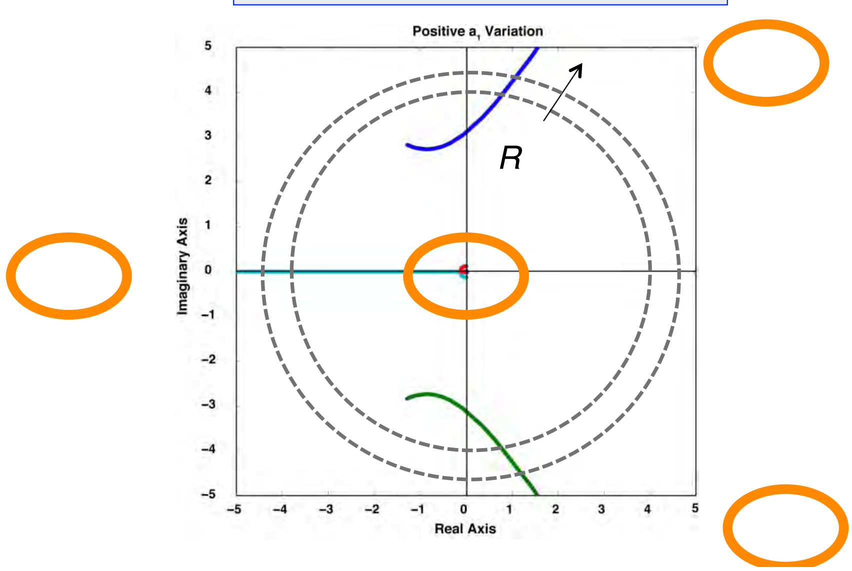





## Destinations for the roots as *k* becomes large

$$\frac{ks}{\left[s^2-0.00041s+0.015\right]\left[s^2+2.57s+9.67\right]}=-1$$

***One zero at origin when k = $a_1$***



## Asymptotes of the root loci are described by

*R(+)*

$$s^{(n-q)} = R e^{-j180°} \to \infty \quad or \quad R e^{-j360°} \to -\infty$$

*R(–)*

$$s = R^{1/(n-q)} e^{-j180°/(n-q)} \to \infty$$
$$or \quad R^{1/(n-q)} e^{-j360°/(n-q)} \to -\infty$$

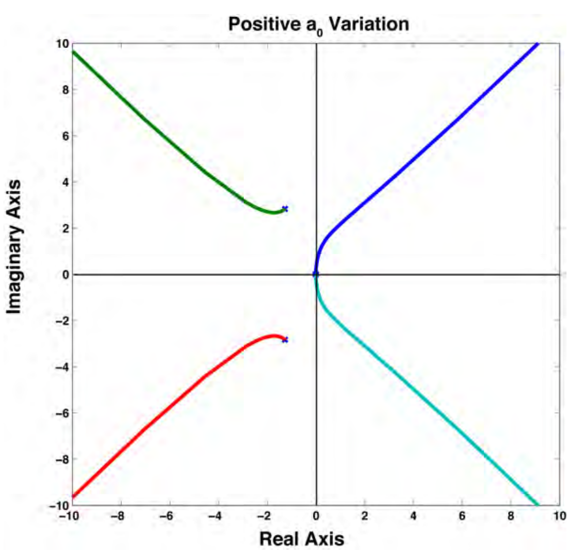
**Magnitudes of roots are the same for given *k***
**Angles from the origin are different**

***http://www.wolframalpha.com***



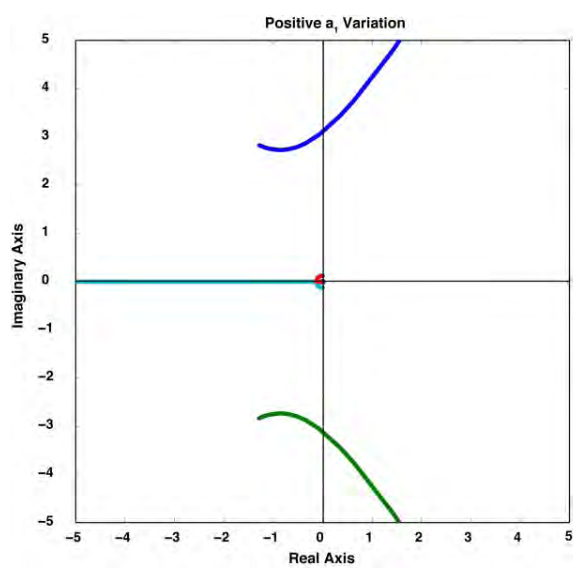
# Asymptotes of Roots (for *k* -> ±∞)

***4 roots to infinite radius***




**Asymptotes = ±45°, ±135°**

***3 roots to infinite radius***




**Asymptotes = ±60°, –180°**

# $(n - q)$ Roots Approach Asymptotes as $k \to \pm\infty$

**Asymptote angles for positive $k$**

$$\theta(rad) = \frac{\pi + 2m\pi}{n - q}, \quad m = 0,1,\ldots,(n-q)-1$$

**Asymptote angles for negative $k$**

$$\theta(rad) = \frac{2m\pi}{n - q}, \quad m = 0,1,\ldots,(n-q)-1$$

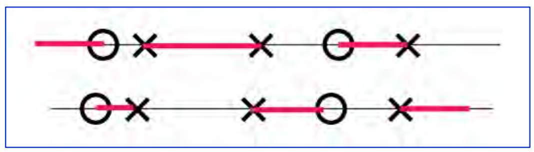


# Origin of Asymptotes = "Center of Gravity"

$$"c.g." = \frac{\sum_{i=1}^{n} \sigma_{\lambda_i} - \sum_{j=1}^{q} \sigma_{z_j}}{n - q}$$

# Root Locus on Real Axis

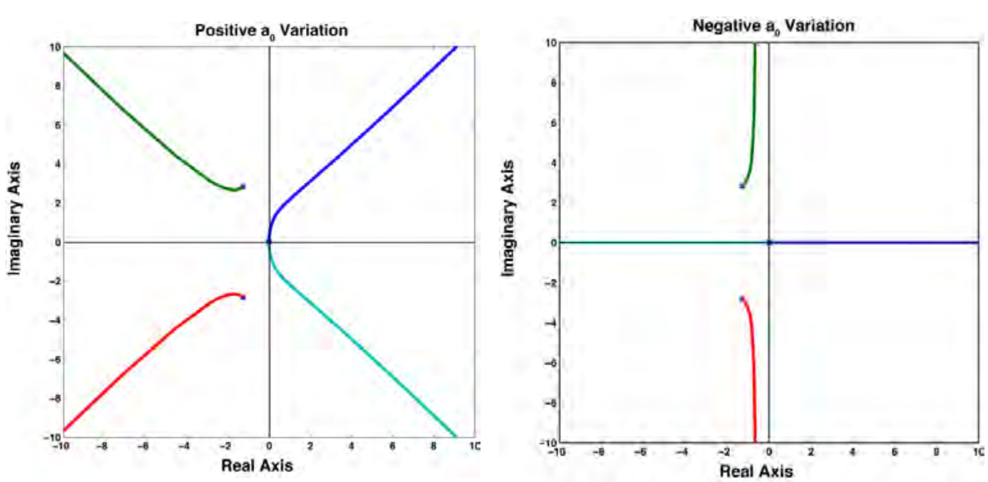
- **Locus on real axis**
  - **k > 0**: Any segment with **odd** number of poles and zeros to the right on the axis
  - **k < 0**: Any segment with **even** number of poles and zeros to the right on the axis





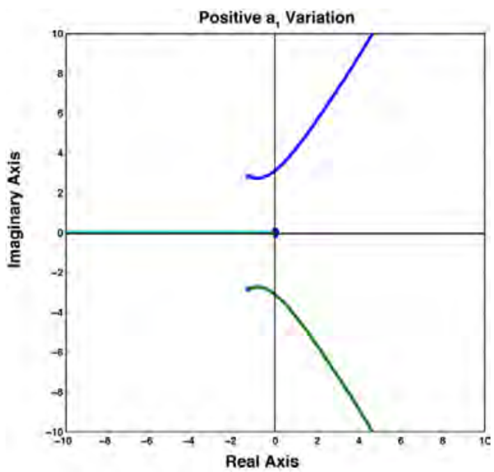
# First Example: Positive and Negative Variations of $k = a_0$

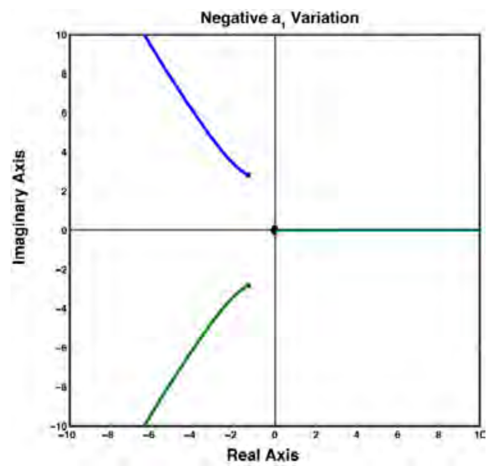
$$\boxed{\frac{k}{s(s+0.21)\left[s^2+2.55s+9.62\right]} = -1}$$



# Second Example: Positive and Negative Variations of $k = a_1$

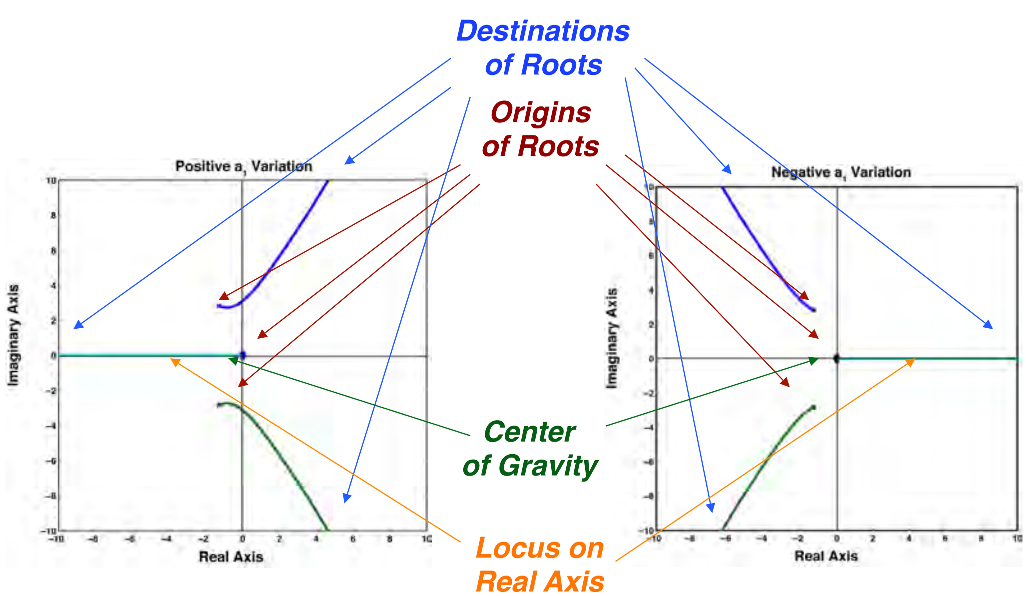
$$\frac{ks}{\left[s^2 - 0.00041s + 0.015\right]\left[s^2 + 2.57s + 9.67\right]} = -1$$





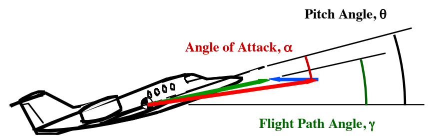

# Summary of Root Locus Concepts

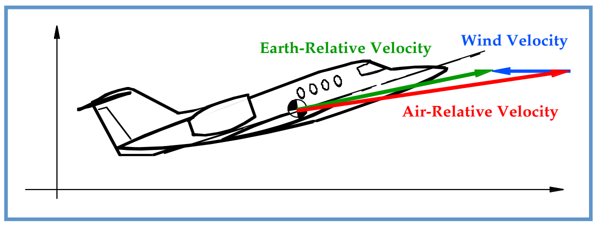


# Weather Hazard Factoids

## Wind Shear Encounter

- **Inertial Frames**
  - Earth-Relative
  - Wind-Relative (Constant Wind)
- **Non-Inertial Frames**
  - Body-Relative
  - Wind-Relative (Varying Wind)

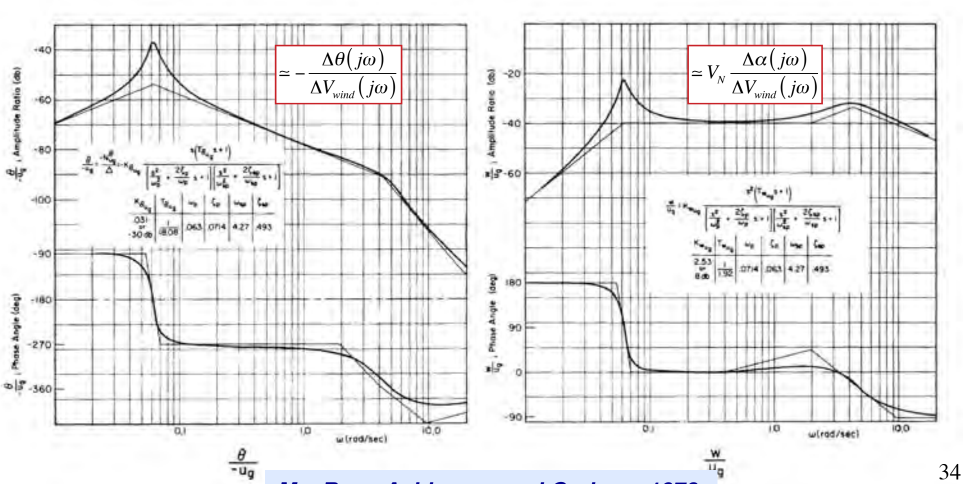





## Pitch Angle and Normal Velocity Frequency Response to Axial Wind

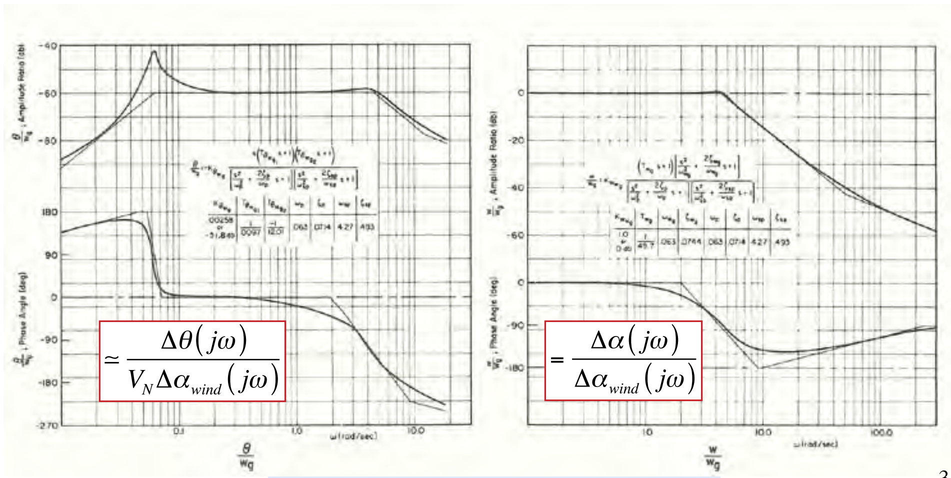
- Pitch angle resonance at phugoid natural frequency
- Normal velocity (~ angle of attack) resonance at phugoid and short period natural frequencies




*MacRuer, Ashkenas, and Graham, 1973*

# Pitch Angle and Normal Velocity Frequency Response to Vertical Wind

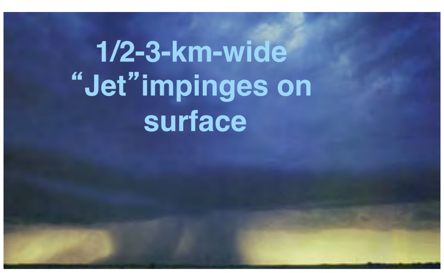
- **Pitch angle resonance at phugoid and short period natural frequencies**
- **Normal velocity (~ angle of attack) resonance**

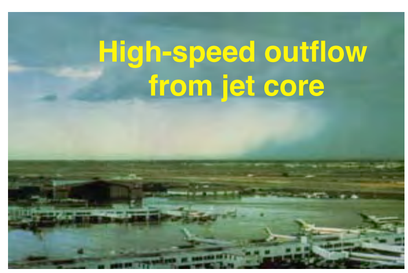



*MacRuer, Ashkenas, and Graham, 1973*

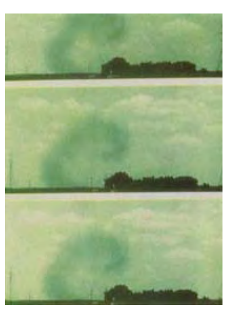


# Microbursts




**Ring vortex forms in outlow**

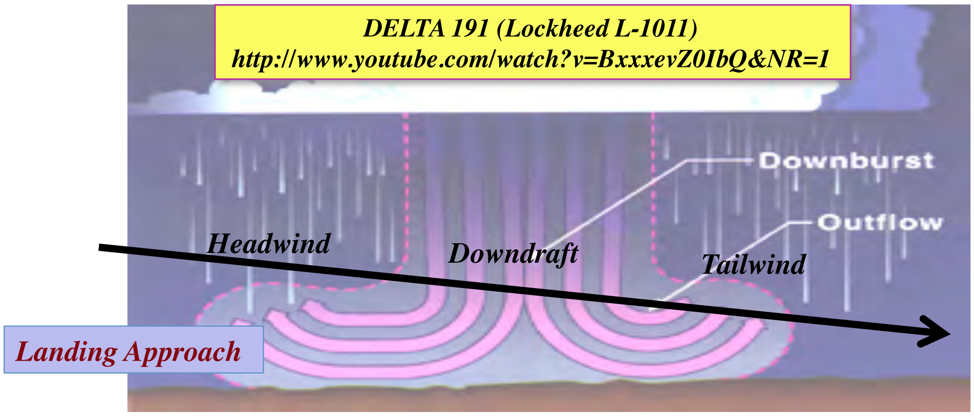





*http://en.wikipedia.org/wiki/Microburst*

# The Insidious Nature of Microburst Encounter

**The wavelength of the phugoid mode and the disturbance input are comparable**




http://en.wikipedia.org/wiki/Delta_Air_Lines_Flight_191



# Importance of Proper Response to Microburst Encounter

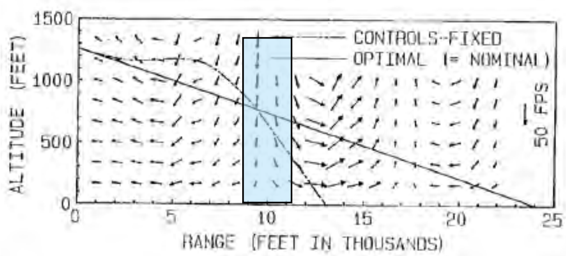
- Stormy evening July 2, 1994
- USAir Flight 1016, *Douglas DC-9*, Charlotte
- Windshear alert issued as 1016 began descent along glideslope

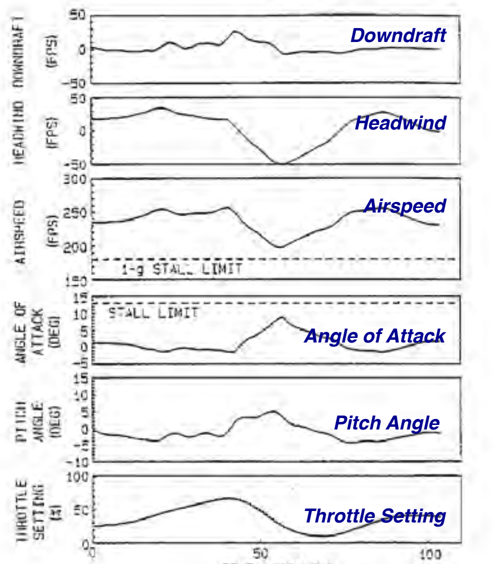
- DC-9 encountered 61-kt windshear, executed missed approach
- Plane continued to descend, striking trees and telephone poles before impact
- Go-around procedure begun correctly -- aircraft's nose rotated up -- but power was not advanced
- Together with increasing tailwind aircraft stalled
- Crew lowered nose to eliminate stall, but descent rate increased, causing ground impact

http://en.wikipedia.org/wiki/US_Airways_Flight_1016

# Optimal Flight Path Through Worst *JAWS* Profile

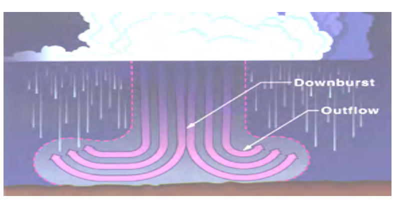
- Graduate research of *Mark Psiaki*
- Joint Aviation Weather Study (*JAWS*) measurements of microbursts (Colorado High Plains, 1983)
- Negligible deviation from intended path using available controllability
- Aircraft has sufficient performance margins to stay on the flight path



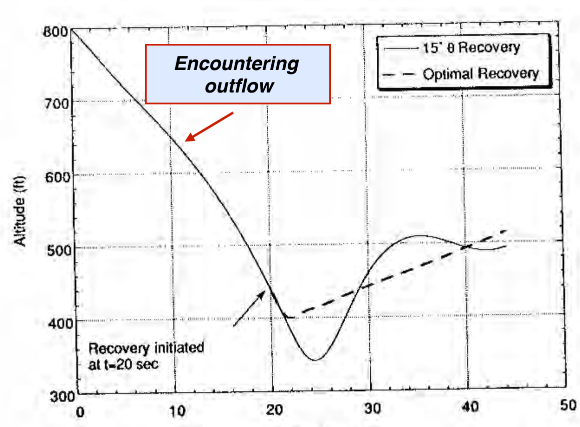



# Optimal and 15° Pitch Angle Recovery during Microburst Encounter

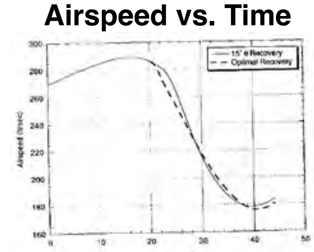



## Graduate Research of *Sandeep Mulgund*

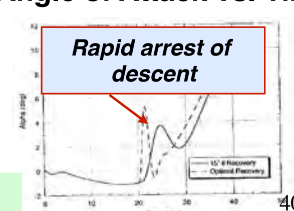
Altitude vs. Time




Airspeed vs. Time



Angle of Attack vs. Time




***FAA Windshear Training Aid*, 1987, addresses proper operating procedures for suspected windshear**

# *Tactical Airplane Maneuverability*

## Chapter 10, *Airplane Stability and Control*, Abzug and Larrabee

- What are the principal subject and scope of the chapter?
- What technical ideas are needed to understand the chapter?
- During what time period did the events covered in the chapter take place?
- What are the three main "takeaway" points or conclusions from the reading?
- What are the three most surprising or remarkable facts that you found in the reading?



# Root Locus Analysis of Simplified Longitudinal Modes

# Approximate Phugoid Model

## 2nd-order equation

$$\Delta\dot{\mathbf{x}}_{Ph} = \begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\gamma} \end{bmatrix} \approx \begin{bmatrix} -D_V & -g \\ L_V / V_N & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix} + \begin{bmatrix} T_{\delta T} \\ L_{\delta T} / V_N \end{bmatrix} \Delta\delta T$$

## Characteristic polynomial

$$\left|s\mathbf{I} - \mathbf{F}_{Ph}\right| = \det\left(s\mathbf{I} - \mathbf{F}_{Ph}\right) \equiv$$
$$\Delta(s) = s^2 + D_V s + gL_V / V_N$$
$$= s^2 + 2\zeta\omega_n s + \omega_n{}^2$$

## Parameters

$$gL_V / V_N, \quad D_V$$



# Approximate Phugoid Roots

## Approximate Phugoid Equation ($\gamma_N = 0$)

$$\Delta\dot{\mathbf{x}}_{Ph} = \begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\gamma} \end{bmatrix} \approx \begin{bmatrix} -D_V & -g \\ L_V / V_N & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix} + \begin{bmatrix} T_{\delta T} \\ L_{\delta T} / V_N \end{bmatrix} \Delta\delta T$$

## Characteristic polynomial

$$\left|s\mathbf{I} - \mathbf{F}_{Ph}\right| = \det\left(s\mathbf{I} - \mathbf{F}_{Ph}\right) \equiv \Delta(s) = s^2 + D_V s + gL_V / V_N$$
$$= s^2 + 2\zeta\omega_n s + \omega_n{}^2$$

## Natural frequency and damping ratio

$$\omega_n = \sqrt{gL_V / V_N}$$

$$\zeta = \frac{D_V}{2\sqrt{gL_V / V_N}}$$

$$\frac{L_V}{V_N} \approx \frac{1}{mV_N}\left[C_{L_V}\frac{\rho_N V_N^2}{2}S + C_{L_N}\rho_N V_N S\right]$$

$$D_V \approx \frac{1}{m}\left[C_{D_V}\frac{\rho_N V_N^2}{2}S + C_{D_N}\rho_N V_N S\right]$$

## Effect of Airspeed on Approximate Phugoid Natural Frequency and Period

**Neglecting compressibility effects**

$$g\frac{L_V}{V_N} \approx \frac{g}{m}\left[C_{L_N}\rho_N S\right]$$

$$= \frac{2g}{mV_N^2}\left[C_{L_N}\frac{1}{2}\rho_N V_N^2 S\right] = \frac{2g}{mV_N^2}\left[mg\right] = \frac{2g^2}{V_N^2}$$

$$\omega_n \approx \sqrt{2}\, {}^{g}\!/_{V_N} \approx \frac{13.87}{V_N}(m/s)$$

$$Period, T = 2\pi/\omega_n \approx 0.45 V_N \text{ sec}$$

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## Effect of L/D on Approximate Phugoid Damping Ratio

**Neglecting compressibility effects**

$$D_V \approx \frac{1}{m}\left[C_{D_N}\rho_N V_N S\right]$$

$$\zeta = \frac{D_V}{2\sqrt{gL_V/V_N}}$$

$$\approx \frac{C_{D_N}\rho_N V_N S/m}{2\sqrt{2}\, g/V_N} = \frac{C_{D_N}\rho_N V_N^2 S/2}{\sqrt{2}mg} = \frac{1}{\sqrt{2}}\left(\frac{C_{D_N}}{C_{L_N}}\right)$$

$$\zeta \approx \frac{1}{\sqrt{2}(L/D)_N}$$

| Velocity m/s | Natural Frequency rad/s | Period sec |
|---|---|---|
| 50 | 0.28 | 23 |
| 100 | 0.14 | 45 |
| 200 | 0.07 | 90 |
| 400 | 0.035 | 180 |

| L/D | Damping Ratio |
|---|---|
| 5 | 0.14 |
| 10 | 0.07 |
| 20 | 0.035 |
| 40 | 0.018 |

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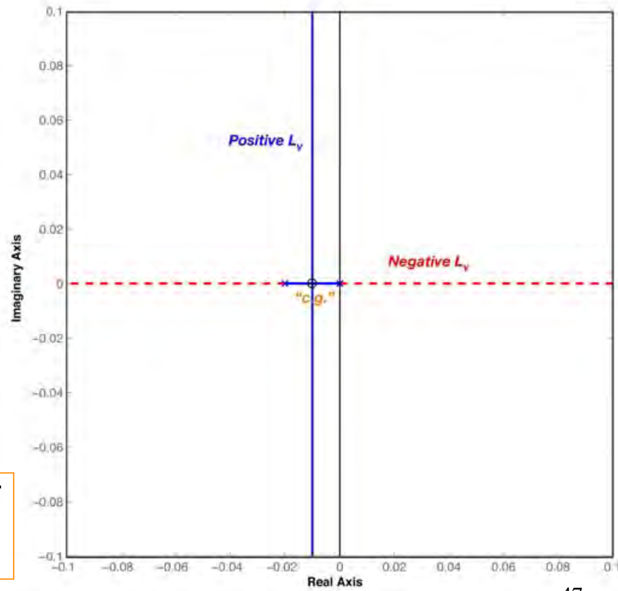
# Effect of $L_V/V_N$ Variation on Approximate Phugoid Roots

$$k = gL_V/V_N$$

$$\Delta(s) = \left(s^2 + D_V s\right) + k$$
$$= s\left(s + D_V\right) + k$$

**Change in damped natural frequency**

$$\omega_{n_{damped}} \triangleq \omega_n \sqrt{1-\zeta^2}$$





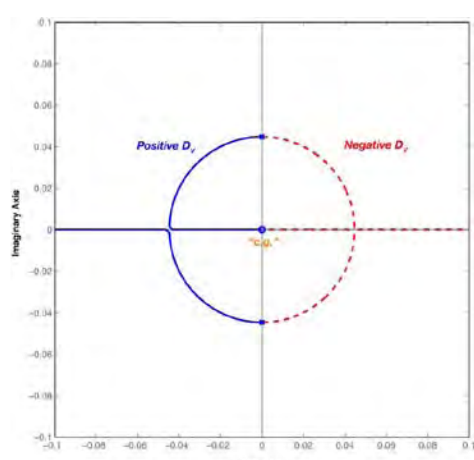

# Effect of $D_V$ Variation on Approximate Phugoid Roots

$$k = D_V$$

$$\Delta(s) = \left(s^2 + gL_V/V_N\right) + ks$$
$$= \left(s + j\sqrt{gL_V/V_N}\right)\left(s - j\sqrt{gL_V/V_N}\right) + ks$$

**Change in damping ratio**

$$\zeta$$



# Approximate Short-Period Model

## Approximate Short-Period Equation ($L_q$ = 0)

$$\Delta \dot{\mathbf{x}}_{SP} = \begin{bmatrix} \Delta \dot{q} \\ \Delta \dot{\alpha} \end{bmatrix} \approx \begin{bmatrix} M_q & M_\alpha \\ 1 & -L_\alpha / V_N \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta \alpha \end{bmatrix} + \begin{bmatrix} M_{\delta E} \\ -L_{\delta E} / V_N \end{bmatrix} \Delta \delta E$$

## Characteristic polynomial

$$\Delta(s) = s^2 + \left( \frac{L_\alpha}{V_N} - M_q \right) s - \left( M_\alpha + M_q \frac{L_\alpha}{V_N} \right)$$
$$= s^2 + 2\zeta\omega_n s + \omega_n{}^2$$

## Parameters

$$M_\alpha, \quad M_q, \quad \frac{L_\alpha}{V_N}$$



# Approximate Short-Period Roots

## Approximate Short-Period Equation ($L_q$ = 0)

$$\Delta \dot{\mathbf{x}}_{SP} = \begin{bmatrix} \Delta \dot{q} \\ \Delta \dot{\alpha} \end{bmatrix} \approx \begin{bmatrix} M_q & M_\alpha \\ 1 & -L_\alpha / V_N \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta \alpha \end{bmatrix} + \begin{bmatrix} M_{\delta E} \\ -L_{\delta E} / V_N \end{bmatrix} \Delta \delta E$$

## Characteristic polynomial

$$\Delta(s) = s^2 + \left( \frac{L_\alpha}{V_N} - M_q \right) s - \left( M_\alpha + M_q \frac{L_\alpha}{V_N} \right)$$
$$= s^2 + 2\zeta\omega_n s + \omega_n^2$$

Generally,
$L_\alpha > 0$
$M_\alpha < 0$
$M_q < 0$

## Natural frequency and damping ratio

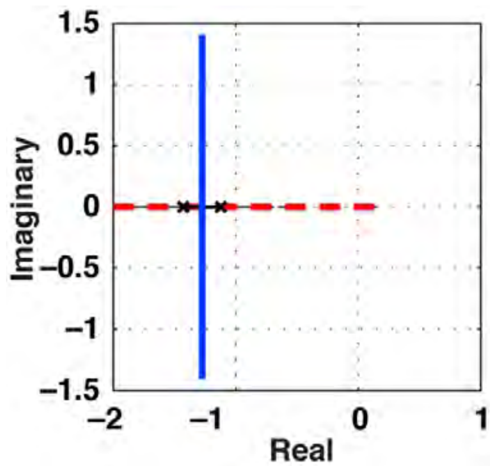
$$\omega_n = \sqrt{-\left( M_\alpha + M_q \frac{L_\alpha}{V_N} \right)}; \quad \zeta = \frac{\left( \frac{L_\alpha}{V_N} - M_q \right)}{2\sqrt{-\left( M_\alpha + M_q \frac{L_\alpha}{V_N} \right)}}$$

# Effect of $M_\alpha$ on Approximate Short-Period Roots

**$k = M_\alpha$**

$$\Delta(s) = s^2 + \left(\frac{L_\alpha}{V_N} - M_q\right)s - \left(M_q\frac{L_\alpha}{V_N}\right) - k = 0$$

$$= \left(s + \frac{L_\alpha}{V_N}\right)\left(s - M_q\right) - k = 0$$

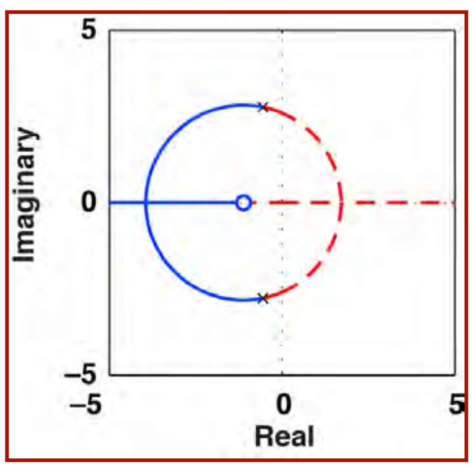



**Change in damped natural frequency**



# Effect of $M_q$ on Approximate Short-Period Roots

**$k = M_q$**




**Change primarily in damping ratio**

$$\Delta(s) = s^2 + \frac{L_\alpha}{V_N}s - M_\alpha - k\left(s + \frac{L_\alpha}{V_N}\right)$$

$$\Delta(s) = \left\{s - \left[\frac{L_\alpha}{2V_N} + \sqrt{\left(\frac{L_\alpha}{2V_N}\right)^2 + M_\alpha}\right]\right\}\left\{s - \left[\frac{L_\alpha}{2V_N} - \sqrt{\left(\frac{L_\alpha}{2V_N}\right)^2 + M_\alpha}\right]\right\} - k\left(s + \frac{L_\alpha}{V_N}\right) = 0$$

# Effects of Airspeed, Altitude, Mass, and Moment of Inertia on Fighter Aircraft Short Period

*Airspeed variation at constant altitude*

| Airspeed | Dynamic Pressure | Angle of Attack | Natural Frequency | Period | Damping Ratio |
|---|---|---|---|---|---|
| m/s | P | deg | rad/s | sec | - |
| 91 | 2540 | 14.6 | 1.34 | 4.7 | 0.3 |
| 152 | 7040 | 5.8 | 2.3 | 2.74 | 0.31 |
| 213 | 13790 | 3.2 | 3.21 | 1.96 | 0.3 |
| 274 | 22790 | 2.2 | 3.84 | 1.64 | 0.3 |

*Mass variation at constant altitude*

| Mass Variation | Natural Frequency | Period | Damping Ratio |
|---|---|---|---|
| % | rad/s | sec | - |
| -50 | 2.4 | 2.62 | 0.44 |
| 0 | 2.3 | 2.74 | 0.31 |
| 50 | 2.26 | 2.78 | 0.26 |

*Altitude variation with constant dynamic pressure*

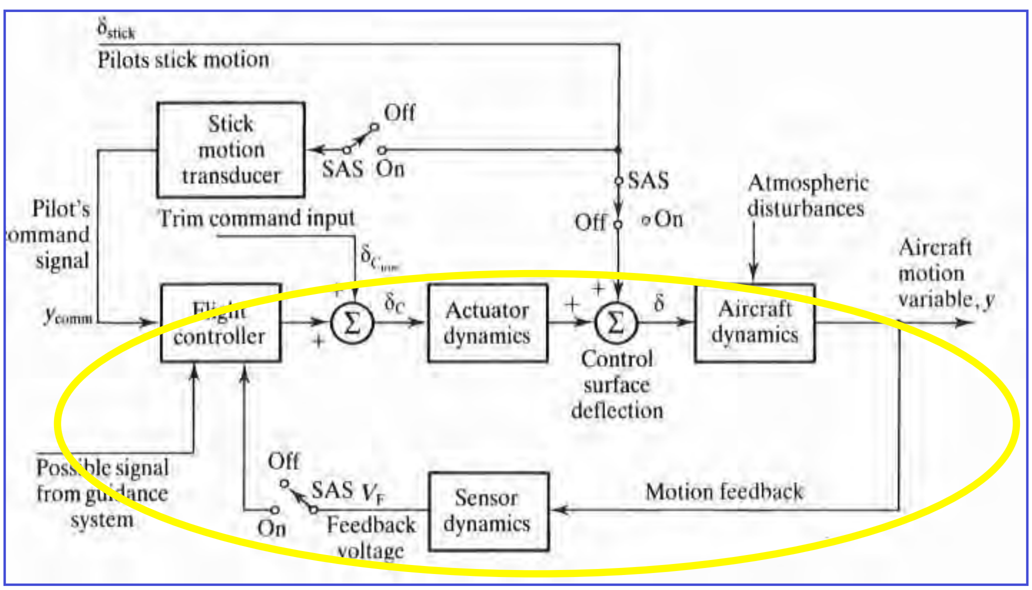
| Airspeed | Altitude | Natural Frequency | Period | Damping Ratio |
|---|---|---|---|---|
| m/s | m | rad/s | sec | - |
| 122 | 2235 | 2.36 | 2.67 | 0.39 |
| 152 | 6095 | 2.3 | 2.74 | 0.31 |
| 213 | 11915 | 2.24 | 2.8 | 0.23 |
| 274 | 16260 | 2.18 | 2.88 | 0.18 |

*Moment of inertia variation at constant altitude*

| Moment of Inertia Variation | Natural Frequency | Period | Damping Ratio |
|---|---|---|---|
| % | rad/s | sec | - |
| -50 | 3.25 | 1.94 | 0.33 |
| 0 | 2.3 | 2.74 | 0.31 |
| 50 | 1.87 | 3.35 | 0.31 |

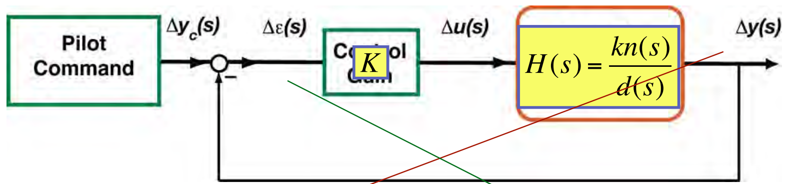


# Flight Control Systems




SAS = Stability Augmentation System

# Effect of Scalar Feedback Control on Roots of the System




**"Block diagram algebra"**

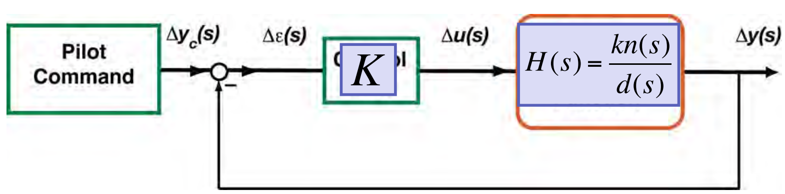
$$\Delta y(s) = H(s)\Delta u(s) = \frac{kn(s)}{d(s)}\Delta u(s) = \frac{kn(s)}{d(s)}K\Delta\varepsilon(s)$$

$$= KH(s)\left[\Delta y_c(s) - \Delta y(s)\right]$$

$$\Delta y(s) = KH(s)\Delta y_c(s) - KH(s)\Delta y(s)$$



# Scalar Closed-Loop Transfer Function




$$\left[1 + KH(s)\right]\Delta y(s) = KH(s)\Delta y_c(s)$$

$$\frac{\Delta y(s)}{\Delta y_c(s)} = \frac{KH(s)}{\left[1 + KH(s)\right]}$$

# Roots of the Closed-Loop Control System

$$\frac{\Delta y(s)}{\Delta y_c(s)} = \frac{K\dfrac{kn(s)}{d(s)}}{\left[1+K\dfrac{kn(s)}{d(s)}\right]} = \frac{Kkn(s)}{\left[d(s)+Kkn(s)\right]} = \frac{Kkn(s)}{\Delta_{\substack{closed\\loop}}(s)}$$

**Closed-loop roots are solutions to**

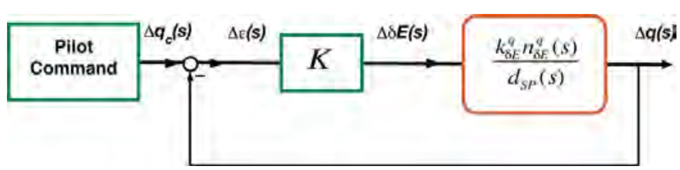
$$\Delta_{\substack{closed\\loop}}(s) = d(s) + Kkn(s) = 0$$

*or*

$$K\frac{kn(s)}{d(s)} = -1$$



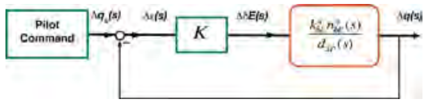
# Root Locus Analysis of Pitch Rate Feedback to Elevator (2nd-Order Approximation)




$$KH(s) = K\frac{\Delta q(s)}{\Delta\delta E(s)} = K\frac{k_q\left(s-z_q\right)}{s^2+2\zeta_{SP}\omega_{n_{SP}}s+\omega_{n_{SP}}^2} = -1$$

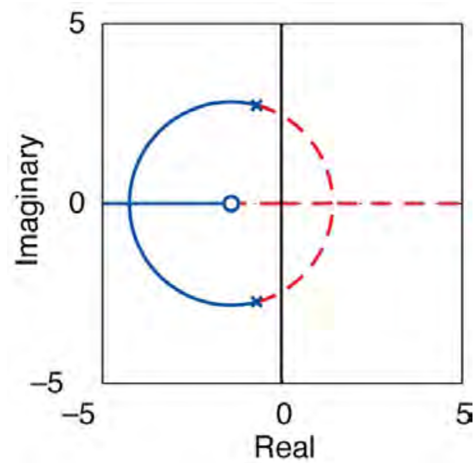
- # of roots = 2
- # of zeros = 1
- Destinations of roots (for $k = \pm\infty$):
  - *1 root* goes to zero of *n(s)*
  - *1 root* goes to infinite radius
- Angles of asymptotes, $\theta$, for the roots going to $\infty$
  - $K \to +\infty$: –180 deg
  - $K \to -\infty$: 0 deg

# Root Locus Analysis of Pitch Rate Feedback to Elevator ($2^{nd}$-Order Approximation)




- "Center of gravity" on real axis
- Locus on real axis
  - $K > 0$: Segment to the left of the zero
  - $K < 0$: Segment to the right of the zero

**Feedback effect is analogous to changing $M_q$**





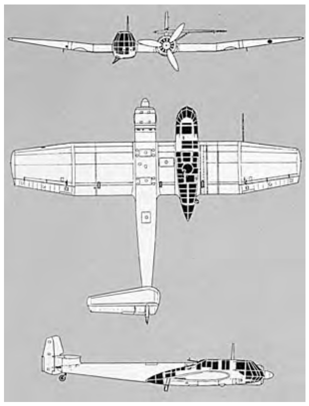

# *Unusual Aircraft Factoids*

## Asymmetrical Aircraft: DC-2-1/2

*DC-3* with *DC-2* right wing
Quick fix to fly aircraft out of harm's way during WWII

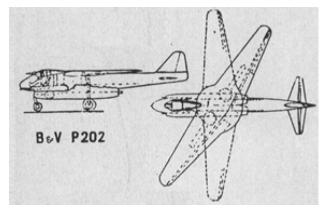
# Asymmetric Aircraft - WWII

**Blohm und Voss, BV 141**



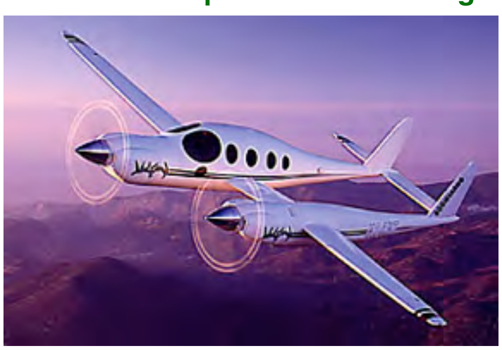
**B + V 141 derivatives**

**B + V P.202**



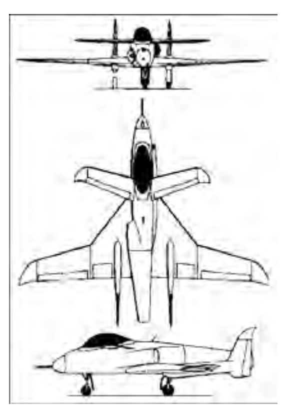
# Recent Asymmetric Aircraft

**Scaled Composites Boomerang**



**NASA AD-1**

**Scaled Composites Ares**



# Next Time:
# Advanced Longitudinal Dynamics

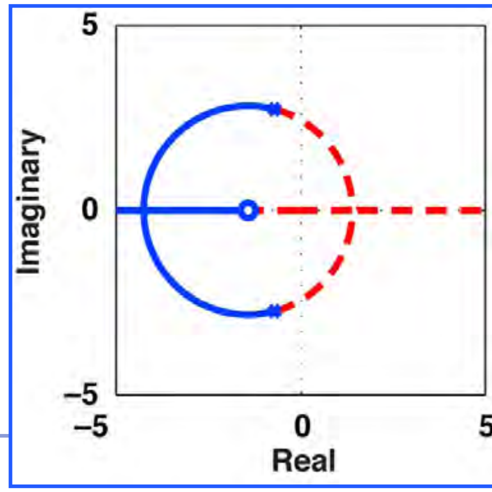


# Supplemental Material

# Effect of $L_\alpha/V_N$ on Approximate Short-Period Roots

**$k = L_\alpha/V_N$**

- Change primarily in damping ratio




$$\Delta(s) = s^2 - M_q s - M_\alpha + k\left(s - M_q\right)$$

$$= \left\{s + \left[\frac{M_q}{2} - \sqrt{\left(\frac{M_q}{2}\right)^2 + M_\alpha}\right]\right\}\left\{s + \left[\frac{M_q}{2} - \sqrt{\left(\frac{M_q}{2}\right)^2 + M_\alpha}\right]\right\} + k\left(s - M_q\right) = 0$$



# Root Locus Criterion

- All points on the locus of roots must satisfy the equation ***k[n(s)/d(s)]* = –1**
- i.e., all points on the root locus must have a phase angle(–1) = ±180 deg