# Time Response of Dynamic Systems

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**Multi-dimensional trajectories**
**Numerical integration**
**Linear and nonlinear systems**
**Linearization of nonlinear models**
**LTI System Response**
**Phase-plane plots**





# Multi-Dimensional Trajectories

**Position, velocity, and acceleration are vectors**

$$\boxed{\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}; \quad \mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}; \quad \mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}; \quad \mathbf{j} = \begin{bmatrix} j_x \\ j_y \end{bmatrix}; \quad \mathbf{s} = \begin{bmatrix} s_x \\ s_y \end{bmatrix}}$$

$$\begin{bmatrix} x(0) \\ x(t) \\ v_x(0) \\ v_x(t) \\ a_x(0) \\ a_x(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & t & t^2/2 & t^3/6 & t^4/24 & t^5/120 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \cdots \end{bmatrix} \begin{bmatrix} x(0) \\ v_x(0) \\ a_x(0) \\ \cdots \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(t) \\ v_y(0) \\ v_y(t) \\ a_y(0) \\ a_y(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & t & t^2/2 & t^3/6 & t^4/24 & t^5/120 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & t^2/2 & t^3/6 & t^4/24 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & t^2/2 & t^3/6 \end{bmatrix} \begin{bmatrix} y(0) \\ v_y(0) \\ a_y(0) \\ j_y(0) \\ s_y(0) \\ c_y(0) \end{bmatrix}$$

# Two-Dimensional Trajectory

## *Solve for Cartesian components separately*

$$\begin{bmatrix} j_x(0) \\ s_x(0) \\ c_x(0) \end{bmatrix} = \begin{bmatrix} -60/t^3 & 60/t^3 & -36/t^2 & -24/t^2 & -9/t & 3/t \\ 360/t^4 & -360/t^4 & 192/t^3 & 168/t^3 & 36/t^2 & -24/t^2 \\ -720/t^5 & 720/t^5 & -360/t^4 & -360/t^4 & -60/t^3 & 60/t^3 \end{bmatrix} \begin{bmatrix} x(0) \\ x(t) \\ v_x(0) \\ v_x(t) \\ a_x(0) \\ a_x(t) \end{bmatrix}$$

$$\begin{bmatrix} j_y(0) \\ s_y(0) \\ c_y(0) \end{bmatrix} = \begin{bmatrix} -60/t^3 & 60/t^3 & -36/t^2 & -24/t^2 & -9/t & 3/t \\ 360/t^4 & -360/t^4 & 192/t^3 & 168/t^3 & 36/t^2 & -24/t^2 \\ -720/t^5 & 720/t^5 & -360/t^4 & -360/t^4 & -60/t^3 & 60/t^3 \end{bmatrix} \begin{bmatrix} y(0) \\ y(t) \\ v_y(0) \\ v_y(t) \\ a_y(0) \\ a_y(t) \end{bmatrix}$$

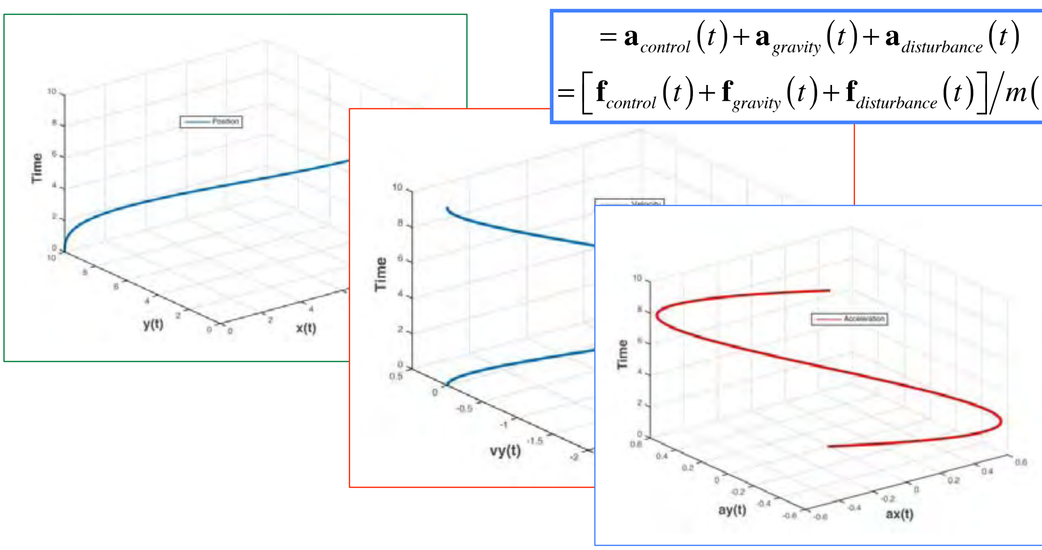


# Two-Dimensional Example

## Required acceleration <u>vector</u> is specified by

$$\mathbf{a}(t) = \mathbf{a}(0) + \mathbf{j}(0)t + \mathbf{s}(0)t^2/2 + \mathbf{c}t^3/6$$

$$= \mathbf{a}_{control}(t) + \mathbf{a}_{gravity}(t) + \mathbf{a}_{disturbance}(t)$$
$$= \left[\mathbf{f}_{control}(t) + \mathbf{f}_{gravity}(t) + \mathbf{f}_{disturbance}(t)\right] / m(t)$$



# Six-Degree-of-Freedom (Rigid Body) Equations of Motion

$$\dot{\mathbf{r}}_I = \mathbf{H}_B^I \mathbf{v}_B$$
$$\dot{\mathbf{v}}_B = \frac{1}{m}\mathbf{f}_B - \tilde{\boldsymbol{\omega}}_B \mathbf{v}_B$$

**Translational position and velocity**

$$\mathbf{r}_I = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad \mathbf{v}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\dot{\Theta} = \mathbf{L}_B^I \boldsymbol{\omega}_B$$
$$\dot{\boldsymbol{\omega}}_B = \boldsymbol{I}_B^{-1}\left(\mathbf{m}_B - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B\right)$$

**Rotational position and velocity**

$$\Theta = \begin{bmatrix} \varphi \\ \theta \\ \psi \end{bmatrix}; \quad \boldsymbol{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$



# … nonlinear, and complex

**Rate of change of Translational Position**

$$\dot{x}_I = (\cos\theta\cos\psi)u + (-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi)v + (\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi)w$$
$$\dot{y}_I = (\cos\theta\sin\psi)u + (\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)v + (-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi)w$$
$$\dot{z}_I = (-\sin\theta)u + (\sin\phi\cos\theta)v + (\cos\phi\cos\theta)w$$

**Rate of change of Translational Velocity**

$$\dot{u} = X/m - g\sin\theta + rv - qw$$
$$\dot{v} = Y/m + g\sin\phi\cos\theta - ru + pw$$
$$\dot{w} = Z/m + g\cos\phi\cos\theta + qu - pv$$

**Rate of change of Angular Position**

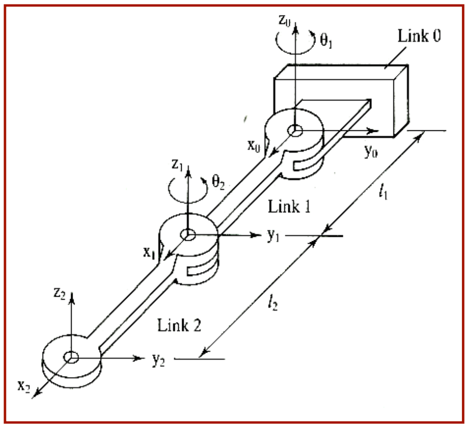
$$\dot{\phi} = p + (q\sin\phi + r\cos\phi)\tan\theta$$
$$\dot{\theta} = q\cos\phi - r\sin\phi$$
$$\dot{\psi} = (q\sin\phi + r\cos\phi)\sec\theta$$

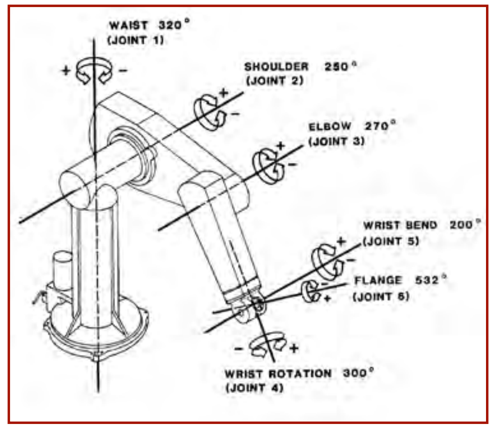
**Rate of change of Angular Velocity**

$$\dot{p} = \left(I_{zz}L + I_{xz}N - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)p + \left[I_{xz}^2 + I_{zz}\left(I_{zz} - I_{yy}\right)\right]r\right\}q\right) \Big/ \left(I_{xx}I_{zz} - I_{xz}^2\right)$$
$$\dot{q} = \left[M - \left(I_{xx} - I_{zz}\right)pr - I_{xz}\left(p^2 - r^2\right)\right] \Big/ I_{yy}$$
$$\dot{r} = \left(I_{xz}L + I_{xx}N - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)r + \left[I_{xz}^2 + I_{xx}\left(I_{xx} - I_{yy}\right)\right]p\right\}q\right) \Big/ \left(I_{xx}I_{zz} - I_{xz}^2\right)$$

# Multiple Rigid Links Lead to Multiple Constraints

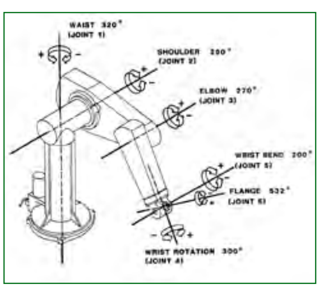
- **Each link is subject to the same 6-DOF rigid-body dynamic equations**
- **… but each link is constrained to have a single degree of freedom w.r.t. proximal link**






# Newton-Euler Link Dynamics

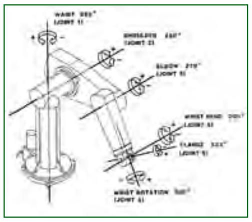


- **Link dynamics are coupled**
  - **Proximal-link loads are affected by distal-link positions and velocities**
  - **Distal-link accelerations are affected by proximal-link motions**
  - **Joints produce constraints on motions of links**

- **Net forces and torques at each joint related to velocities and accelerations of the centroids of the links**
- **Equations of motion derived directly for each link, with constraints**

$$\dot{\mathbf{v}}_B = \frac{1}{m}\mathbf{f}_B - \tilde{\omega}_B \mathbf{v}_B$$
$$\dot{\omega}_B = \boldsymbol{I}_B^{-1}\left(\mathbf{m}_B - \tilde{\omega}_B \boldsymbol{I}_B \omega_B\right)$$

$$\rightarrow \quad \dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t), \mathbf{p}(t), t\right]$$



# Lagrangian Link Dynamics

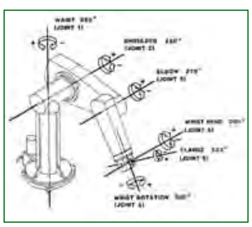
- **Lagrange's equation derived from Newton's Laws**
- **Dynamic behavior <u>described by work done and energy stored</u> in the system**
- **Equations of motion derived from *Lagrangian* function and Lagrange's equation**
  - $q_n$ = generalized coordinate
  - $F_n$ = generalized force

$$L(q_n,\dot{q}_n) = KE - PE$$

$$\frac{d}{dt}\left(\frac{dL(q_n,\dot{q}_n)}{d\dot{q}_n}\right) - \frac{\partial L(q_n,\dot{q}_n)}{\partial q_n} = \mathbf{F}_n$$

$\rightarrow$ $\boxed{\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t\right]}$





# Hamiltonian Link Dynamics

- **Hamilton's Principle: Lagrange's equation is a necessary condition for an extremum**
- ***Hamiltonian* function and Hamilton's equations**
  - $p_n$ = generalized momentum

$$\text{extremum } I = \int_{t_1}^{t_2} L(q_n,\dot{q}_n)\,dt$$

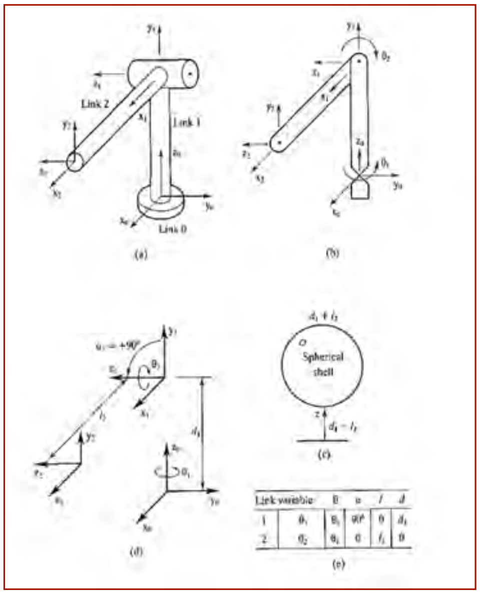
$$H(p,q) = \sum \dot{q}_n p_n - L(q_n,\dot{q}_n)$$

$$\dot{q}_n = \frac{\partial H(p,q)}{\partial p_n}; \quad \dot{p}_n = -\frac{\partial H(p,q)}{\partial q_n}$$

*Expressed as an optimization problem*

$\rightarrow$ $\boxed{\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t\right]}$

# Example: Two-Link Robot Equations of Motion



McKerrow, 1991

### State vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \text{Angle of } 1^{\text{st}} \text{ link}, \theta_1, rad \\ \text{Angular rate of } 1^{\text{st}} \text{ link}, \dot{\theta}_1, rad/\text{sec} \\ \text{Angle of } 2^{\text{nd}} \text{ link}, \theta_2, rad \\ \text{Angular rate of } 2^{\text{nd}} \text{ link}, \dot{\theta}_2, rad/\text{sec} \end{bmatrix}$$

### Control vector

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \text{, torque at } 1^{\text{st}} \text{ joint} \\ \tau_2 \text{, torque at } 2^{\text{nd}} \text{ joint} \end{bmatrix}$$



# When Possible, Simplify the Equations

### Two-link robot equations of motion

- Mass, ***m***, located at end of Link 2
- Inertias of Links 1 and 2 neglected

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{\cos^2 x_3}\left( x_2 x_4 \sin 2x_3 + \frac{u_1}{m l_2^2} \right)$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{g}{l_2}\cos x_3 - \frac{x_2}{2}\sin 2x_3 + \frac{u_2}{m l_2^2}$$

## Differential Equations Integrated to Produce Time Response

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t), \mathbf{p}(t), t\right], \quad \mathbf{x}(0) \text{ given}$$

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t), \mathbf{p}(t), t\right] dt$$

**Numerical integration is an approximation**



## Rectangular and Trapezoidal Integration of Differential Equations

**Rectangular (Euler) Integration**

$$\begin{aligned} \mathbf{x}(t_k) &= \mathbf{x}(t_{k-1}) + \Delta\mathbf{x}(t_{k-1}, t_k) \\ &\approx \mathbf{x}(t_{k-1}) + \mathbf{f}\left[\mathbf{x}(t_{k-1}), \mathbf{u}(t_{k-1}), \mathbf{w}(t_{k-1})\right] \Delta t, \quad \Delta t = t_k - t_{k-1} \end{aligned}$$

**Trapezoidal (modified Euler) Integration** *(ode23)*

$$\mathbf{x}(t_k) \approx \mathbf{x}(t_{k-1}) + \frac{1}{2}\left[\Delta\mathbf{x}_1 + \Delta\mathbf{x}_2\right]$$

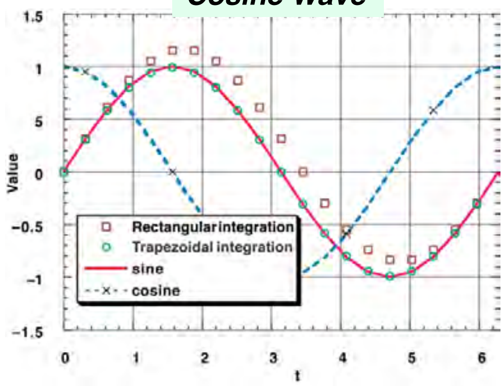
*where*

$$\Delta\mathbf{x}_1 = \mathbf{f}\left[\mathbf{x}(t_{k-1}), \mathbf{u}(t_{k-1}), \mathbf{w}(t_{k-1})\right] \Delta t$$

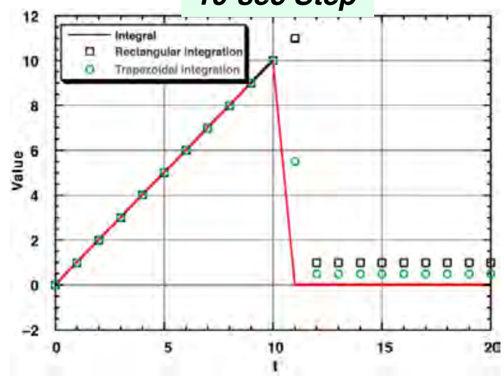
$$\Delta\mathbf{x}_2 = \mathbf{f}\left\{\left[\mathbf{x}(t_{k-1}) + \Delta\mathbf{x}_1\right], \mathbf{u}(t_k), \mathbf{w}(t_k)\right\} \Delta t$$

# Numerical Integration Examples

***Cosine Wave***




***10-sec Step***



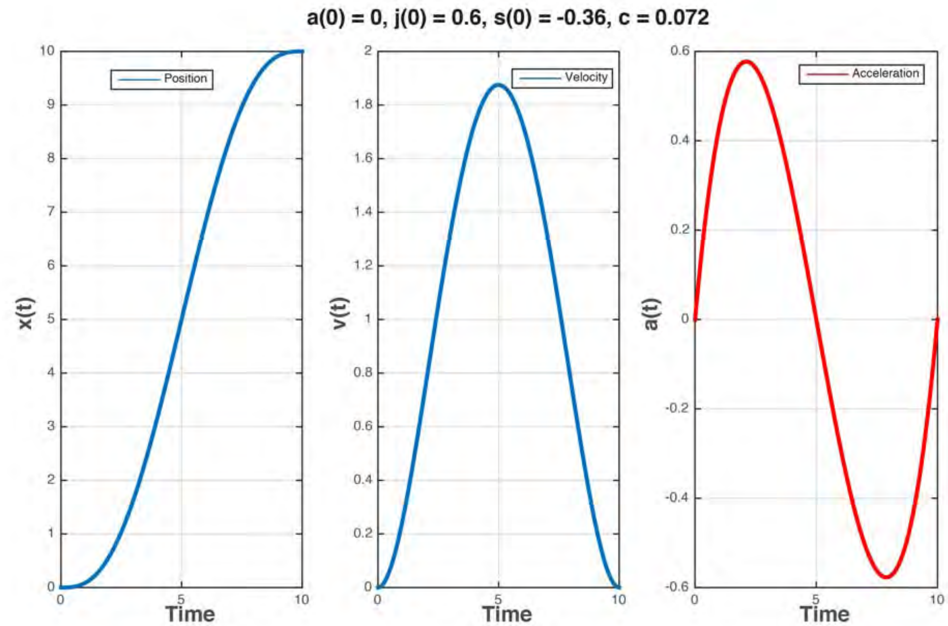

***How can approximation accuracy be improved?***



# More Complicated Algorithms
## (e.g., MATLAB)

| Solver | Problem Type | Order of Accuracy | When to Use |
|---|---|---|---|
| `ode45` | Nonstiff | Medium | Most of the time. This should be the first solver you try. |
| `ode23` | Nonstiff | Low | For problems with crude error tolerances or for solving moderately stiff problems. |
| `ode113` | Nonstiff | Low to high | For problems with stringent error tolerances or for solving computationally intensive problems. |
| `ode15s` | Stiff | Low to medium | If `ode45` is slow because the problem is stiff. |
| `ode23s` | Stiff | Low | If using crude error tolerances to solve stiff systems and the mass matrix is constant. |
| `ode23t` | Moderately Stiff | Low | For moderately stiff problems if you need a solution without numerical damping. |
| `ode23tb` | Stiff | Low | If using crude error tolerances to solve stiff systems. |

# 1-D Example




$$\boxed{a_{net}(t) = (0) + 0.6t - 0.36t^2/2 + 0.072t^3/6}$$



# Comparison of Exact and Numerically Integrated Trajectories

**Calculate trajectory, given constants for $t_f = 10$**

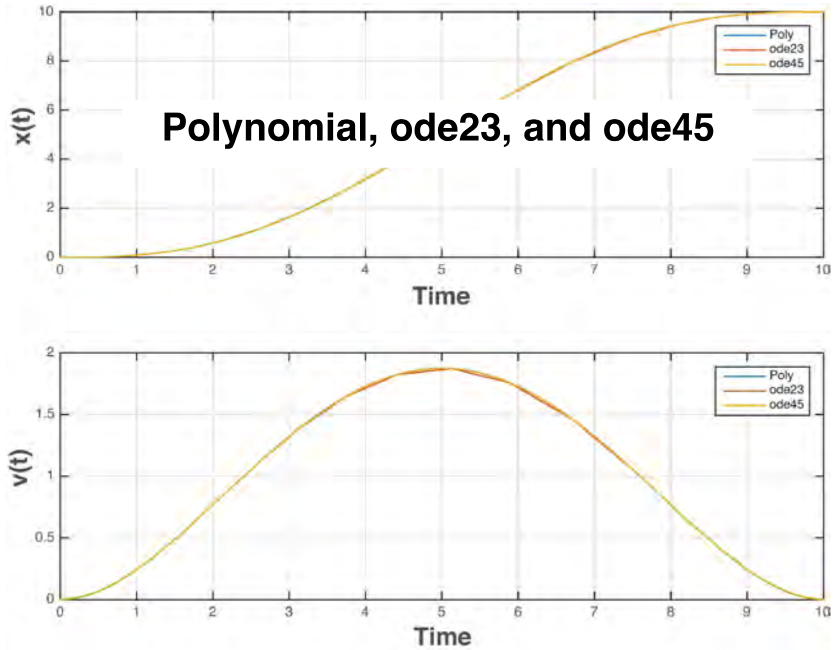
$$\begin{bmatrix} x(t) \\ v(t) \\ a(t) \end{bmatrix} = \begin{bmatrix} 1 & t & t^2/2 & t^3/6 & t^4/24 & t^5/120 \\ 0 & 1 & t & t^2/2 & t^3/6 & t^4/24 \\ 0 & 0 & 1 & t & t^2/2 & t^3/6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.6 \\ -0.36 \\ 0.072 \end{bmatrix}$$

**Calculate trajectory by numerical integration**

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = 0.6t - 0.36t^2/2 + 0.072t^3/6$$

# Comparison of Exact and Numerically Integrated Trajectories

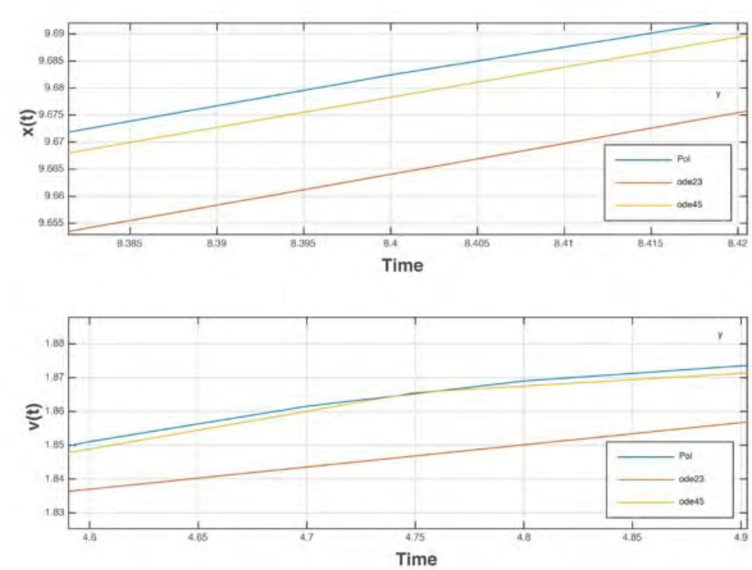
**Polynomial, ode23, and ode45**



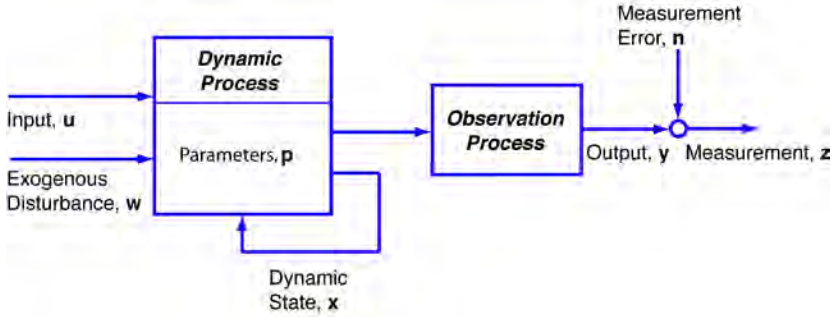



# Comparison of Exact and Numerically Integrated Trajectories *(Zoom)*

**Polynomial, ode23, and ode45**



# Generic Dynamic System




**Dynamic Process: Current state may depend on prior state**

x : state   *dim* = (*n* x 1)
u : input   *dim* = (*m* x 1)
w : disturbance   *dim* = (*s* x 1)
p : parameter   *dim* = ($\ell$ x 1)

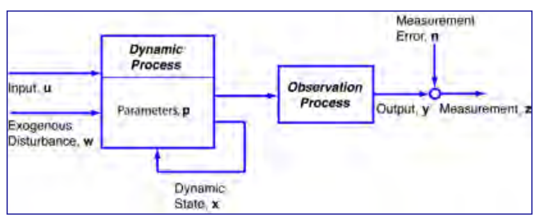
*t* : time (independent variable, 1 x 1)

**Observation Process: Measurement may contain error or be incomplete**

y : output (error-free)
*dim* = (*r* x 1)
n : measurement error
*dim* = (*r* x 1)
z : measurement
*dim* = (*r* x 1)



# Equations of the System




**Dynamic Equation**

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t]$$

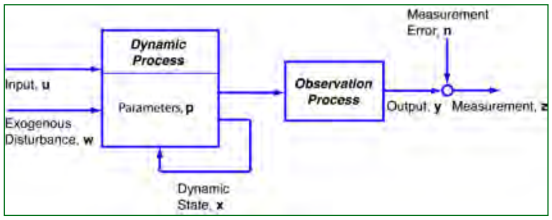
**Output Equation**

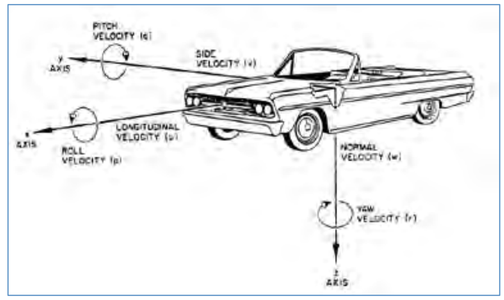
$$\mathbf{y}(t) = \mathbf{h}[\mathbf{x}(t),\mathbf{u}(t)]$$

**Measurement Equation**

$$\mathbf{z}(t) = \mathbf{y}(t) + \mathbf{n}(t)$$

# Dynamic System Example: Automotive Vehicle

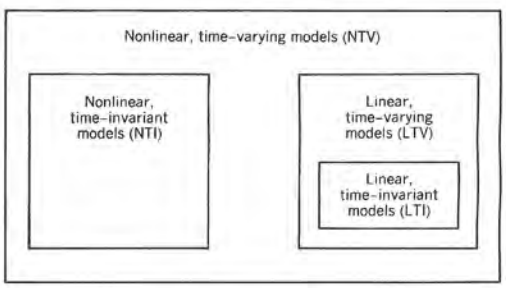


- **Dynamic Process**
  - **x : dynamic state**
    - Position, velocity, angle, angular rate
  - **u : input**
    - Steering, throttle, brakes
  - **w : disturbance**
    - Road surface, wind
  - **p : parameter**
    - Weight, moments of inertia, drag coefficient, spring constants
  - ***t* : time (independent variable)**

- **Observation Process**
  - **y : error-free output**
    - Speed, front-wheel angle, engine rpm, acceleration, yaw rate, throttle, brakes, GPS location
  - **n : measurement error**
    - Perturbations to y
  - **z : measurement**
    - Sum of y and n






# Nonlinearity and Time Variation in Dynamic Systems

**Nonlinear, time-varying dynamics**

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t\right]$$

**Nonlinear, time-invariant dynamics**

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t)\right]$$

**Linear, time-varying dynamics**

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t)$$

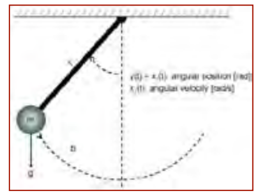
**Linear, time-invariant dynamics**

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{L}\mathbf{w}(t)$$

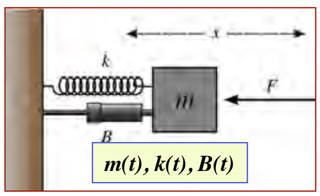
# Nonlinearity and Time Variation in Dynamic Systems

Nonlinear, time-varying dynamics

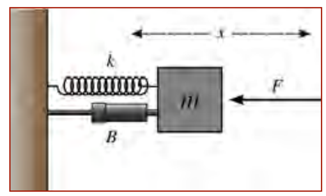
Nonlinear, time-invariant dynamics



Linear, time-varying dynamics




Linear, time-invariant dynamics






# Solutions of Ordinary Differential Dynamic Equations

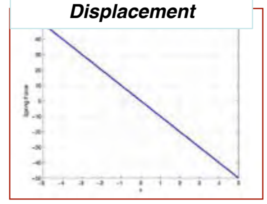
| Time-Domain Model (ODE) | Solution by Numerical Integration | Principle of Superposition | Frequency-Domain Model |
|---|---|---|---|
| . Nonlinear, time-varying | *Yes* | *No* | *No* |
| . Nonlinear, time-invariant | *Yes* | *No* | *Yes (amplitude-dependent, harmonics)* |
| . Linear, time-varying | *Yes* | *Yes* | *Approximate* |
| . Linear, time-invariant | *Yes* | *Yes* | *Yes* |

# Comparison of Damped Linear and Nonlinear Systems

**Linear Spring**

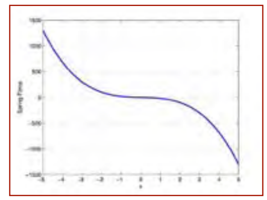
$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = \underbrace{-10x_1(t)}_{\text{Spring}} \underbrace{- x_2(t)}_{\text{Damper}}$$

*Spring Force vs. Displacement*



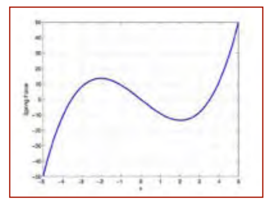
**Linear plus Stiffening Cubic Spring**

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) - 10x_1^3(t) - x_2(t)$$



**Linear plus Weakening Cubic Spring**

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) + 0.8x_1^3(t) - x_2(t)$$





# MATLAB Simulation of Linear and Nonlinear Dynamic Systems

**MATLAB Main Program**

```
% Nonlinear and Linear Examples
   clear
   tspan = [0 10];
   xo = [0, 10];
   [t1,x1 = ode23('NonLin',tspan,xo);
   xo = [0, 1];
   [t2,x2] = ode23('NonLin',tspan,xo);
   xo = [0, 10];
   [t3,x3] = ode23('Lin',tspan,xo);
   xo = [0, 1];
   [t4,x4] = ode23('Lin',tspan,xo);

   subplot(2,1,1)
   plot(t1,x1(:,1),'k',t2,x2(:,1),'b',t3,x3(:,1),'r',t4,x4(:,1),'g')
   ylabel('Position'), grid
   subplot(2,1,2)
   plot(t1,x1(:,2),'k',t2,x2(:,2),'b',t3,x3(:,2),'r',t4,x4(:,2),'g')
   xlabel('Time'), ylabel('Rate'), grid
```

**Linear Spring**

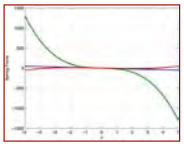
$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) - x_2(t)$$

```
function  xdot  =  Lin(t,x)
%  Linear Ordinary Differential Equation
% x(1) = Position
% x(2) = Rate
xdot = [x(2)
        -10*x(1) - x(2)];
```

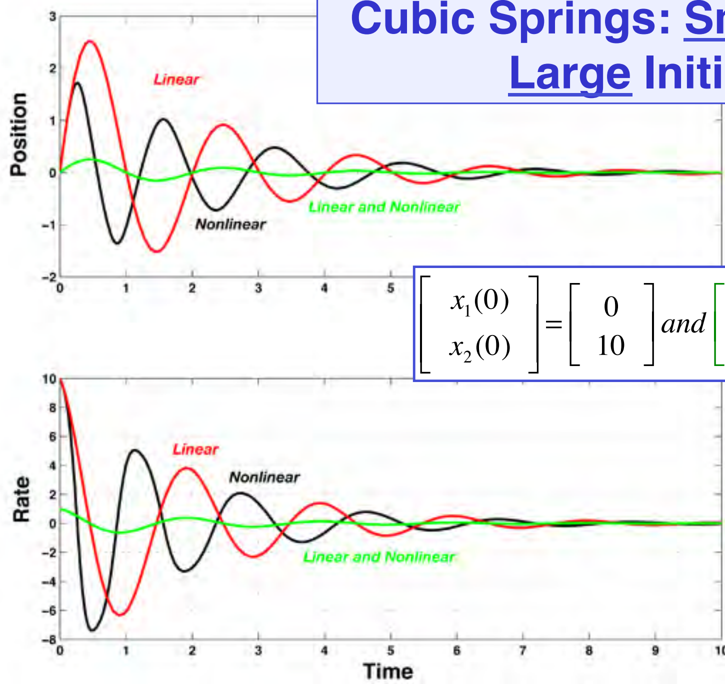
**Weakening Spring**

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) + 0.8x_1^{\,3}(t) - x_2(t)$$

```
function  xdot  =  NonLin(t,x)
%  Nonlinear Ordinary Differential Equation
% x(1) = Position
% x(2) = Rate
xdot = [x(2)
        -10*x(1) + 0.8*x(1)^3 - x(2)];
```



## Linear and Stiffening Cubic Springs: Small and Large Initial Rates

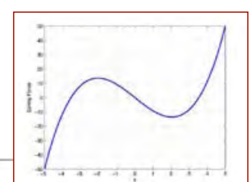


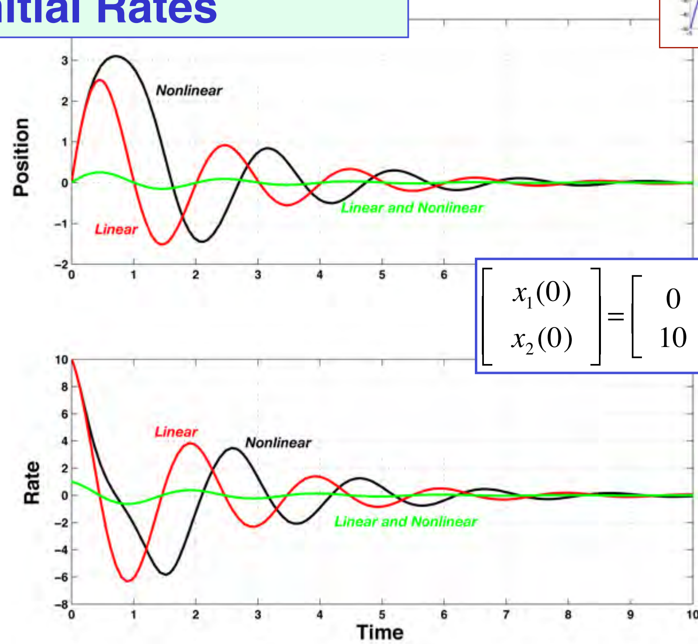

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \textit{and} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Linear and nonlinear responses are indistinguishable with small initial condition**



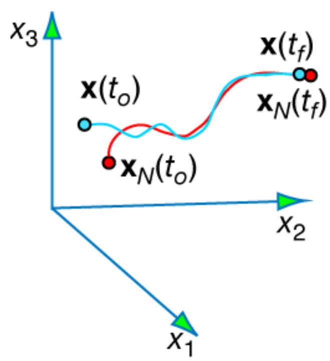
## Linear and Weakening Cubic Springs: Small and Large Initial Rates






$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \textit{and} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

# Linearization of Nonlinear Equations




$$\dim(\mathbf{x}) = n \times 1$$
$$\dim(\mathbf{u}) = m \times 1$$
$$\dim(\mathbf{w}) = s \times 1$$

- **Given**
  - **Nominal (or reference) robot trajectory, control, and disturbance histories**

$$\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t) \quad \text{for } t \text{ in } [t_o, t_f]$$

  - **Actual path, perturbed by**
    - **Initial condition variation**
    - **Control variation**
    - **Disturbance variation**

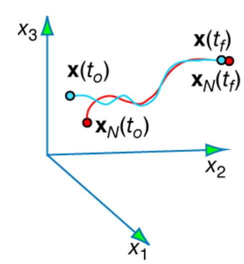
$$\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t) \quad \text{for } t \text{ in } [t_o, t_f]$$



# Linearization of Nonlinear Equations

**Difference between nominal and actual paths:**

$$\Delta\mathbf{x}(t_o) = \mathbf{x}(t_o) - \mathbf{x}_N(t_o)$$
$$\Delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_N(t)$$




**Difference between nominal and actual inputs:**

$$\Delta\mathbf{u}(t) = \mathbf{u}(t) - \mathbf{u}_N(t) \quad [\text{Control perturbation}]$$
$$\Delta\mathbf{w}(t) = \mathbf{w}(t) - \mathbf{w}_N(t) \quad [\text{Disturbance perturbation}]$$

# Expansion of All Terms to First Degree

**Both paths satisfy the nonlinear dynamic equations**

$$\boxed{\begin{aligned}\dot{\mathbf{x}}_N(t) &= \mathbf{f}[\mathbf{x}_N(t),\mathbf{u}_N(t),\mathbf{w}_N(t),t] \\ \dot{\mathbf{x}}(t) &= \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),t]\end{aligned}}$$

**Actual dynamics can be approximated by the sum of the nominal dynamics plus perturbation effects**

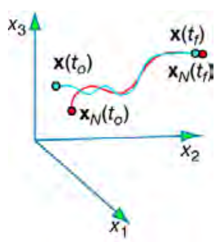
$$\boxed{\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t),\mathbf{u}_N(t),\mathbf{w}_N(t),t]}$$

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t) \\ &= \mathbf{f}\left\{[\mathbf{x}_N(t)+\Delta\mathbf{x}(t)],[\mathbf{u}_N(t)+\Delta\mathbf{u}(t)],[\mathbf{w}_N(t)+\Delta\mathbf{w}(t),t]\right\}\end{aligned}$$

$$\approx \mathbf{f}[\mathbf{x}_N(t),\mathbf{u}_N(t),\mathbf{w}_N(t),t] + \boxed{\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(t)}\Delta\mathbf{x}(t) + \boxed{\frac{\partial\mathbf{f}}{\partial\mathbf{u}}(t)}\Delta\mathbf{u}(t) + \boxed{\frac{\partial\mathbf{f}}{\partial\mathbf{w}}(t)}\Delta\mathbf{w}(t)$$

**The partial-derivative (*Jacobian*) matrices are evaluated along the nominal path**



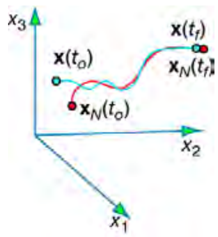



# Jacobian Matrices Express the Solution Sensitivity to Small Perturbations

$$\boxed{\mathbf{F}(t) = \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}} \quad ; \quad \mathbf{G}(t) = \left.\frac{\partial\mathbf{f}}{\partial\mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} \quad ; \quad \mathbf{L}(t) = \left.\frac{\partial\mathbf{f}}{\partial\mathbf{w}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

**Sensitivity to state perturbations: stability matrix**

$$\mathbf{F}(t) = \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \left[\begin{matrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \dots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \dots & \partial f_2/\partial x_n \\ \dots & \dots & \dots & \dots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \dots & \partial f_n/\partial x_n \end{matrix}\right]_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$




# Sensitivity to Small Control and Disturbance Perturbations

$$\mathbf{F}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} \quad ; \quad \boxed{\mathbf{G}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}} \quad ; \quad \boxed{\mathbf{L}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}$$

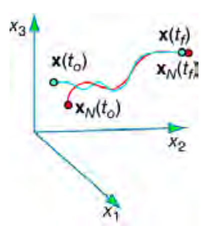
**Control-effect matrix**

$$\mathbf{G}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \left.\begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 & \cdots & \partial f_1/\partial u_m \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 & \cdots & \partial f_2/\partial u_m \\ \cdots & \cdots & \cdots & \cdots \\ \partial f_n/\partial u_1 & \partial f_n/\partial u_2 & \cdots & \partial f_n/\partial u_m \end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

**Disturbance-effect matrix**

$$\mathbf{L}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \left.\begin{bmatrix} \partial f_1/\partial w_1 & \partial f_1/\partial w_2 & \cdots & \partial f_1/\partial w_s \\ \partial f_2/\partial w_1 & \partial f_2/\partial w_2 & \cdots & \partial f_2/\partial w_s \\ \cdots & \cdots & \cdots & \cdots \\ \partial f_n/\partial w_1 & \partial f_n/\partial w_2 & \cdots & \partial f_n/\partial w_s \end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$






# Linearized Equation Approximates Perturbation Dynamics

**Solve the nominal and perturbation parts *separately***

**Nominal (nonlinear) equation**

$$\boxed{\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t], \quad \mathbf{x}_N(0)\ given}$$

**Perturbation (linear) equation**

$$\boxed{\Delta\dot{\mathbf{x}}(t) \approx \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t) + \mathbf{L}(t)\Delta\mathbf{w}(t), \quad \Delta\mathbf{x}(0)\ given}$$

**Approximate total solution**

$$\boxed{\mathbf{x}(t) \approx \mathbf{x}_N(t) + \Delta\mathbf{x}(t)}$$

# Stiffening Cubic Spring Example

## Nonlinear equation

$$\dot{x}_1(t) = f_1 = x_2(t)$$
$$\dot{x}_2(t) = f_2 = -10x_1(t) - 10x_1^3(t) - x_2(t)$$

**Integrate nonlinear equation to produce nominal path**

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \rightarrow \int_0^{t_f} \begin{bmatrix} f_{1_N} \\ f_{2_N} \end{bmatrix} dt \rightarrow \begin{bmatrix} x_{1_N}(t) \\ x_{2_N}(t) \end{bmatrix} \quad in \quad [0, t_f]$$



# Stiffening Cubic Spring Example

## Evaluate partial derivatives along the path

$$\mathbf{F}(t) = \left[\begin{array}{c|c} \dfrac{\partial f_1}{\partial x_1} = 0 & \dfrac{\partial f_1}{\partial x_2} = 1 \\ \hline \dfrac{\partial f_2}{\partial x_1} = -10 - 30x_{1_N}{}^2(t) & \dfrac{\partial f_2}{\partial x_2} = -1 \end{array}\right]$$

$$\mathbf{G}(t) = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} = 0 \\ \dfrac{\partial f_2}{\partial u} = 0 \end{bmatrix} \qquad \mathbf{L}(t) = \begin{bmatrix} \dfrac{\partial f_1}{\partial w} = 0 \\ \dfrac{\partial f_2}{\partial w} = 0 \end{bmatrix}$$

# Nominal and Perturbation Dynamic Equations

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t)], \quad \mathbf{x}_N(0) \textit{ given}$$
$$\dot{x}_{1_N}(t) = x_{2_N}(t)$$
$$\dot{x}_{2_N}(t) = -10x_{1_N}(t) - 10x_{1_N}^3(t) - x_{2_N}(t)$$

$$\begin{bmatrix} x_{1_N}(0) \\ x_{2_N}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}(t)\Delta\mathbf{x}(t), \quad \Delta\mathbf{x}(0) \textit{ given}$$
$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \left[\begin{array}{c|c} 0 & 1 \\ \hline -\left(10 + 30x_{1_N}{}^2(t)\right) & -1 \end{array}\right] \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_1(0) \\ \Delta x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

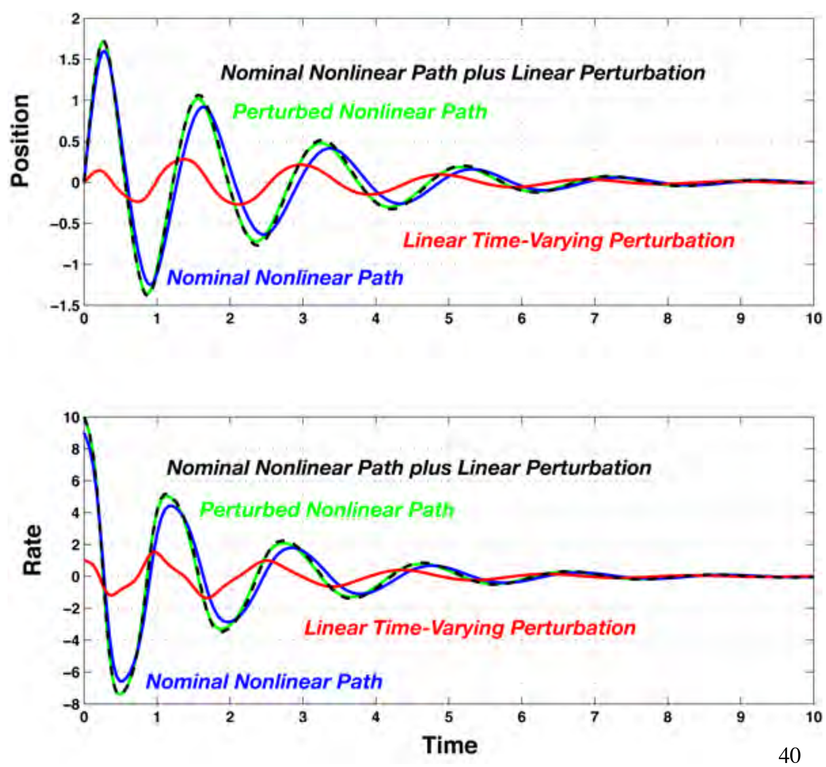


# Comparison of Approximate and Exact Solutions

$\mathbf{x}_N(t)$
$\Delta\mathbf{x}(t)$
$\mathbf{x}_N(t) + \Delta\mathbf{x}(t)$
$\mathbf{x}(t)$

$\dot{\mathbf{x}}_N(t)$
$\Delta\dot{\mathbf{x}}(t)$
$\dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t)$
$\dot{\mathbf{x}}(t)$

$x_{2_N}(0) = 9$
$\Delta x_2(0) = 1$
$x_{2_N}(t) + \Delta x_2(t) = 10$
$x_2(t) = 10$



# Nominal and Perturbation Dynamic Solutions for Cubic Spring Example with $\mathbf{x}_N(0) = 0$

## Nominal solution remains at equilibrium

**Nonlinear**

$$\boxed{\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t)], \quad \mathbf{x}_N(0) = 0, \quad \mathbf{x}_N(t) = 0 \text{ in } [0,\infty]}$$

**Linear**

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \left[\begin{array}{c|c} 0 & 1 \\ \hline -10 & -1 \end{array}\right] \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

$=$ Linear, Time-Invariant (LTI) System



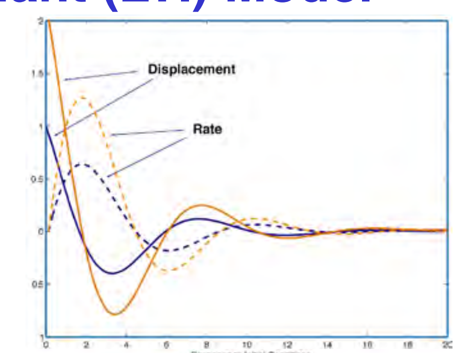
# Initial-Condition Response of a Linear, Time-Invariant (LTI) Model

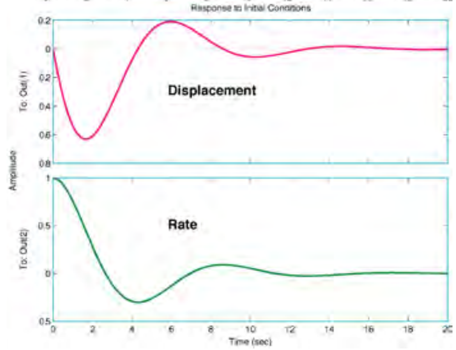
```
%   Linear Model - Initial Condition
    F       =  [-0.5572 -0.7814;0.7814 0];
    G       =  [1 -1;0 2];
    Hx      =  [1 0;0 1];
    sys     =  ss(F, G, Hx,0);

    xo          =  [1;0];
    [y1,t1,x1]  =  initial(sys, xo);

    xo          =  [2;0];
    [y2,t2,x2]  =  initial(sys, xo);
    plot(t1,y1,t2,y2)

    figure
    xo      =  [0;1];
    initial(sys, xo)
```

$$\mathbf{F} = \begin{bmatrix} -0.5572 & -0.7814 \\ 0.7814 & 0 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$\mathbf{H}_x = \mathbf{I}_2; \quad \mathbf{H}_u = \mathbf{0}$$

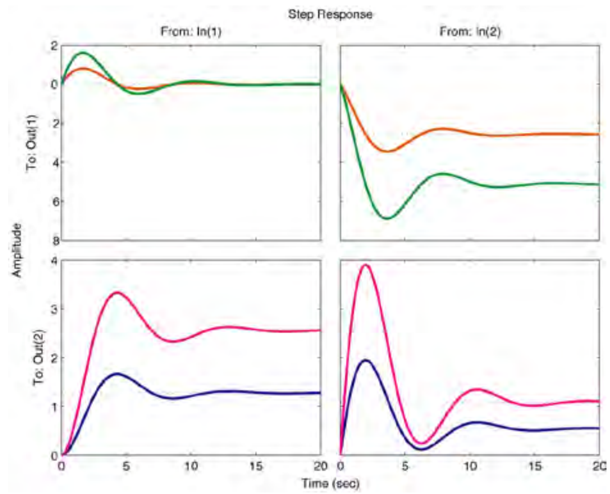



- **Doubling the initial condition doubles the output**
- **Stability, speed of response, and damping are independent of the initial condition**

# Step Response of a Linear, Time-Invariant Model

```
%   Linear Model - Step
    F   =   [-0.5572 -0.7814;0.7814 0];
    G   =   [1 -1;0 2];
    Hx  =   [1 0;0 1];
    sys =   ss(F, G, Hx,0);
    sys2  =   ss(F, 2*G, Hx,0);

%   Step response
    step(sys, sys2)
```

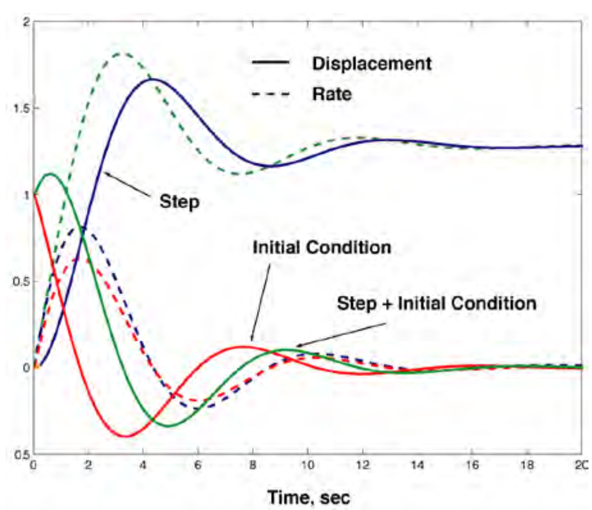



- **Doubling the step input doubles the output**
- **Stability, speed of response, and damping are independent of the input**



# Response to Combined Initial Condition and Step Input

```
%   Linear Model - Superposition
    F      =   [-0.5572 -0.7814;0.7814 0];
    G      =   [1;0];
    Hx     =   [1 0;0 1];
    sys    =   ss(F, G, Hx,0);

    xo     =   [1; 0];
    t      =   [0:0.2:20];
    u      =   ones(1,length(t));

    [y1,t1,x1]  =  lsim(sys,u,t,xo);
    [y2,t2,x2]  =  lsim(sys,u,t);

    u          =   zeros(1,length(t));
    [y3,t3,x3]  =  lsim(sys,u,t,xo);

    plot(t1,y1,t2,y2,t3,y3)
```




**Linear system responses are additive**

## Initial-Condition Responses of 1st-Order LTI Systems are Exponentials

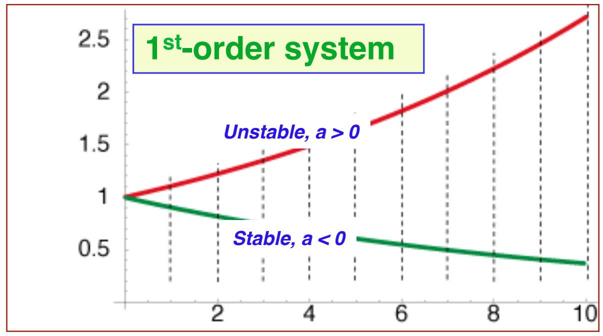
$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t), \quad \Delta\mathbf{x}(0)\ given$$

**State vector is a scalar**

$$\mathbf{F} = [a]$$

$$\Delta\dot{x}(t) = a\Delta x(t)$$
$$\Delta x(0)\ \text{given}$$

$$\Delta x(t) = \int_0^t \Delta\dot{x}(t)dt = \int_0^t a\Delta x(t)dt = e^{at}\Delta x(0)$$




*LTI system integral is a closed-form expression*

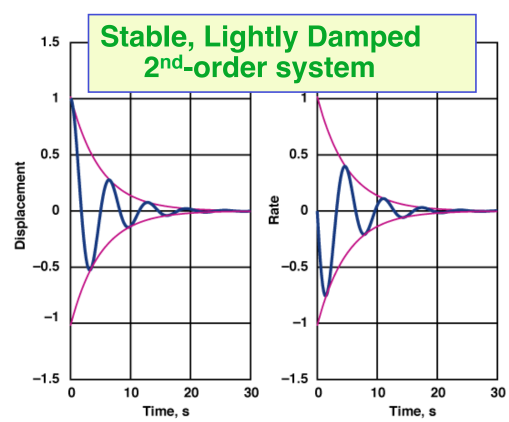


## Initial-Condition Responses of 2nd-Order LTI Systems
## Exponentials and Sinusoids

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t)$$
$$\Delta\mathbf{x}(0)\ given$$

$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

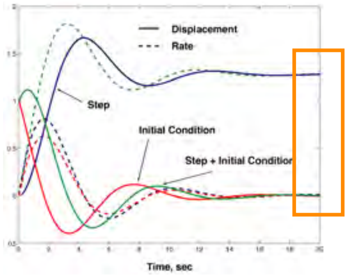
$$\Delta\dot{x}_1 = f_{11}\Delta x_1 + f_{12}\Delta x_2 + g_1\Delta u$$
$$\Delta\dot{x}_2 = f_{21}\Delta x_1 + f_{22}\Delta x_2 + g_2\Delta u$$




**Sinusoid with exponential envelope**

$$\Delta x_1(t) = A_1 e^{-\zeta\omega_n t}\cos\left[\omega_n\sqrt{1-\zeta^2}\,t + \varphi_1\right]$$
$$\Delta x_2(t) = A_2 e^{-\zeta\omega_n t}\cos\left[\omega_n\sqrt{1-\zeta^2}\,t + \varphi_2\right]$$

*LTI system integral is a closed-form expression*




# Equilibrium Response of Linear, Time-Invariant Models

- General equation

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t)$$

- At equilibrium,
  - Derivative goes to zero
  - State is unchanging

$$\mathbf{0} = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t)$$

- State at equilibrium

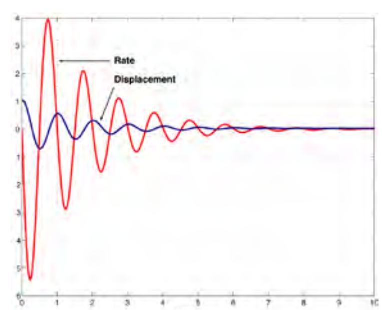
$$\Delta\mathbf{x}^* = -\mathbf{F}^{-1}\left(\mathbf{G}\Delta\mathbf{u}^* + \mathbf{L}\Delta\mathbf{w}^*\right)$$
$$= -\frac{Adj\left(\mathbf{F}\right)}{\det\left(\mathbf{F}\right)}\left(\mathbf{G}\Delta\mathbf{u}^* + \mathbf{L}\Delta\mathbf{w}^*\right)$$

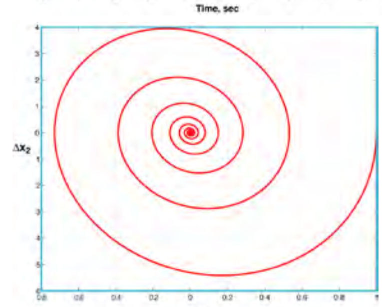


# State ("Phase")-Plane Plots

```
% 2nd-Order Model - Initial Condition Response
  clear
  z          =  0.1;  % Damping ratio
  wn         =  6.28; % Natural frequency, rad/s
  F          =  [0 1;-wn^2 -2*z*wn];
  G          =  [1 -1;0 2];
  Hx         =  [1 0;0 1];
  sys        =  ss(F, G, Hx,0);
  t          =  [0:0.01:10];
  xo         =  [1;0];
  [y1,t1,x1] =  initial(sys, xo, t);

  plot(t1,y1)
  grid on

  figure
  plot(y1(:,1),y1(:,2))
  grid on
```

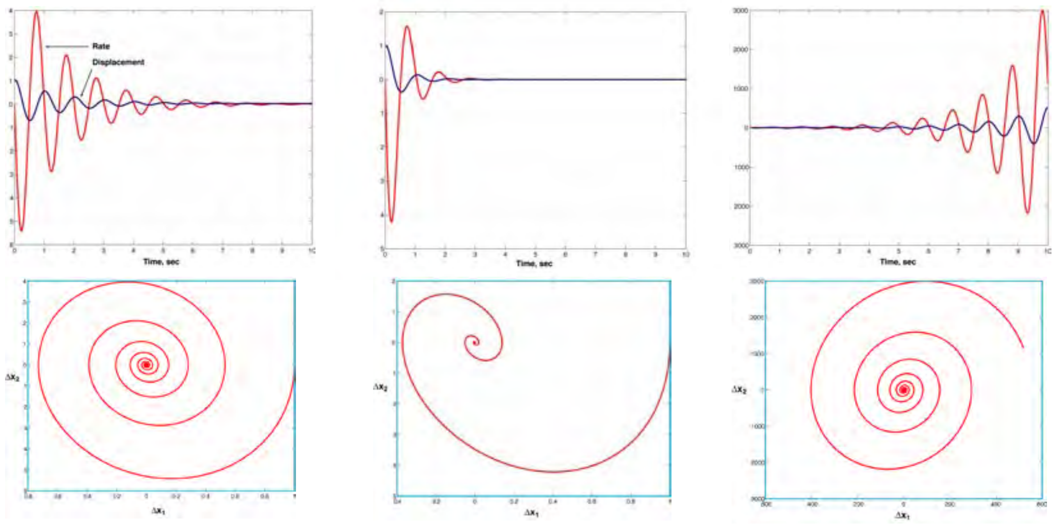
- Cross-plot of one component against another
- Time or frequency not shown explicitly in phase plane



# Effects of Damping Ratio on State-Plane Plots

- Damping ratio = 0.1
- Damping ratio = 0.3
- Damping ratio = –0.1



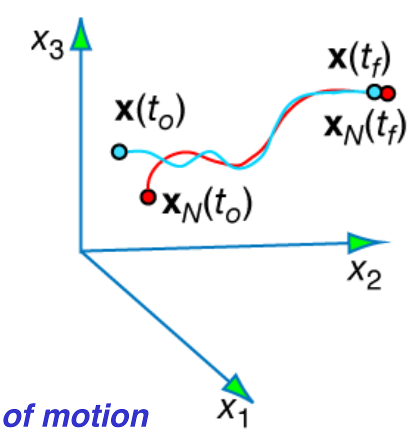



# Next Time: Dynamic Effects of Feedback Control

# *Supplemental Material*



# Perturbed Initial Conditions Produce Perturbed Path




- Given
  - Initial condition, control, and disturbance histories

$$\mathbf{x}(t_0), \mathbf{u}(t), \mathbf{w}(t) \quad \text{for } t \text{ in } [t_o, t_f]$$

  - Path (or *trajectory*) is approximated by executing a numerical algorithm
  - Perturbing the initial condition produces a new path

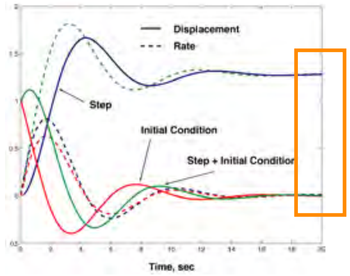
- Both paths satisfy the *same equations of motion*

$$\boxed{\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}(t), \mathbf{w}(t), t], \quad \mathbf{x}_N(t_0) \quad given}$$

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t), t], \quad \mathbf{x}(t_0) \quad given}$$

- $\mathbf{x}_N$: Nominal path
- $\mathbf{x}$: Perturbed path

# Equilibrium Response of Second-Order LTI System




## System description

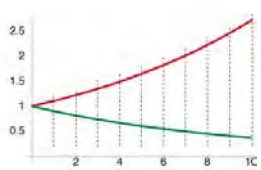
$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

## State equilibrium depends on constant input values

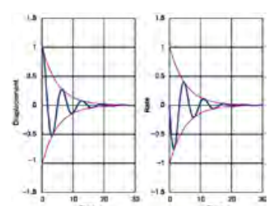
$$\begin{bmatrix} \Delta x_1 * \\ \Delta x_2 * \end{bmatrix} = -\frac{\begin{bmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{bmatrix}}{\left(f_{11}f_{22} - f_{12}f_{21}\right)} \left[ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \Delta u * + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \Delta w * \right]$$

$$|\mathbf{F}| = \left(f_{11}f_{22} - f_{12}f_{21}\right) \neq 0$$





# Response of Higher-Order LTI Systems is a Superposition of Sub-System Responses



$$\mathbf{F}_{System} = \begin{bmatrix} \left[\mathbf{F}_{System\ 1}\right] & \begin{bmatrix} \text{Effect of} \\ \text{\#2 on \#1} \end{bmatrix} \\ \begin{bmatrix} \text{Effect of} \\ \text{\#1 on \#2} \end{bmatrix} & \left[\mathbf{F}_{System\ 2}\right] \end{bmatrix}$$

- Third-order system with uncoupled 1st- and 2nd-order sub-systems

$$\mathbf{F} = \left[\begin{array}{c|cc} f_{11} & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & f_{32} & f_{33} \end{array}\right]$$

$$\Delta \mathbf{x} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix}$$

- Coupling in first row and first column

$$\mathbf{F} = \left[\begin{array}{c|cc} f_{11} & f_{12} & f_{13} \\ \hline f_{21} & 0 & 1 \\ f_{31} & f_{32} & f_{33} \end{array}\right]$$

# Examples of Coupled and Uncoupled Third-Order Systems

### Third-order system with uncoupled 1st- and 2nd-order sub-systems

$$\mathbf{F} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1.414 \end{bmatrix} \qquad \Delta\mathbf{x} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix}$$

**Coupling in first row and first column**

***Position Coupling, $\Delta x_3$***

$$\mathbf{F} = \begin{bmatrix} -1 & 0.1 & 0 \\ 0.1 & 0 & 1 \\ 0 & -1 & -1.414 \end{bmatrix}$$

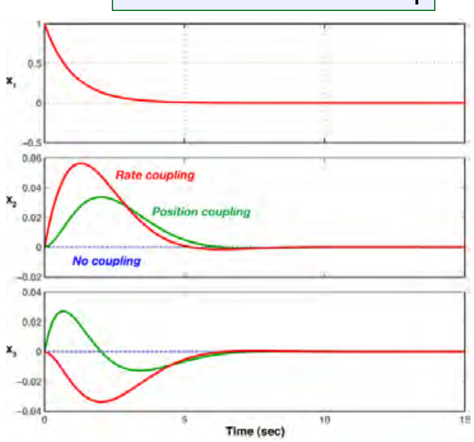
***Rate Coupling, $\Delta x_2$***

$$\mathbf{F} = \begin{bmatrix} -1 & 0 & 0.1 \\ 0 & 0 & 1 \\ 0.1 & -1 & -1.414 \end{bmatrix}$$

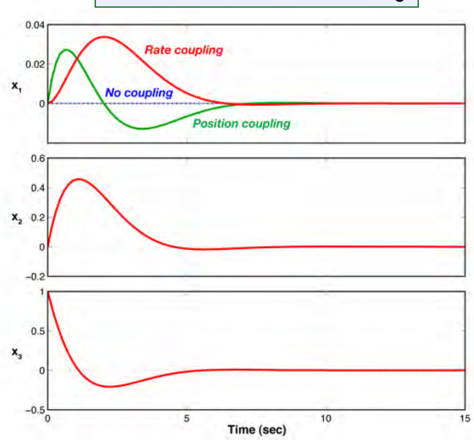


# 3rd-Order LTI Systems with Coupled Response

**Initial Condition on $x_1$**




**Initial Condition on $x_3$**




**With coupling, the two modes appear in all three components**