Stability of Dynamic Systems

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Optimal Control and Estimation, MAE 546
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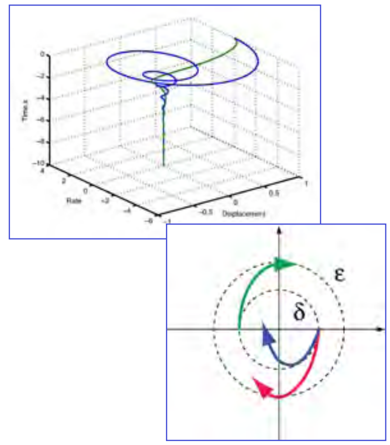
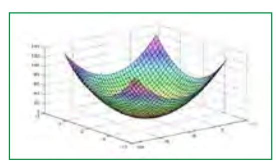
- **Bounds on the system norm**
- **Lyapunov criteria for stability**
- **Eigenvalues**
- **Transfer functions**
- **Continuous- and discrete-time systems**



Vector Norms for Real Variables

- **"Norm" = Measure of length or magnitude of a vector, x**
- **Euclidean or Quadratic Norm**



$$L^2\ norm = \|\mathbf{x}\|_2 = \left(\mathbf{x}^T\mathbf{x}\right)^{1/2} = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$$

- **Weighted Euclidean Norm**

$$\|\mathbf{y}\|_2 = \left(\mathbf{y}^T\mathbf{y}\right)^{1/2} = \left(y_1{}^2 + y_2{}^2 + \cdots + y_m{}^2\right)^{1/2}$$
$$= \left(\mathbf{x}^T\mathbf{D}^T\mathbf{D}\mathbf{x}\right)^{1/2} = \|\mathbf{D}\mathbf{x}\|_2$$

$\mathbf{x}^T\mathbf{D}^T\mathbf{D}\mathbf{x} \triangleq \mathbf{x}^T\mathbf{Q}\mathbf{x}$
$\mathbf{Q} \triangleq \mathbf{D}^T\mathbf{D} =$ Defining matrix

Uniform Stability

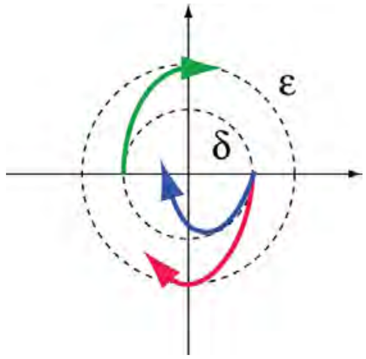
- **Autonomous dynamic system**
  - **Time-invariant**
  - **No forcing input**

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t)]}$$

- **Uniform stability about x = 0**

$$\boxed{\|\mathbf{x}(t_o)\| \le \delta, \quad \delta > 0}$$

Let $\delta = \delta(\varepsilon)$
If, for every $\varepsilon \ge 0$,
$\|\mathbf{x}(t)\| \le \varepsilon, \quad \varepsilon \ge \delta > 0, \quad t \ge t_0$
Then the system is uniformly stable

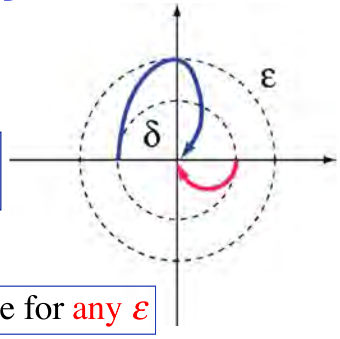


- **If system response is bounded, then the system possesses uniform stability**

Local and Global Asymptotic Stability

- **Local asymptotic stability**
  - **Uniform stability plus**

$$\boxed{\|\mathbf{x}(t)\| \xrightarrow[t \to \infty]{} 0}$$



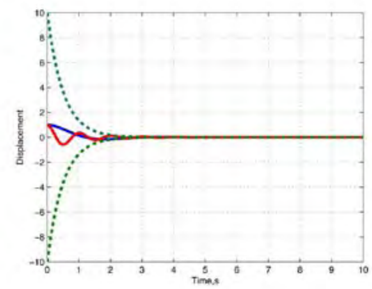
- **Global asymptotic stability**

$$\boxed{\text{System is asymptotically stable for any } \varepsilon}$$

- **If a linear system has uniform asymptotic stability, it also is globally stable**

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{x}(t)}$$

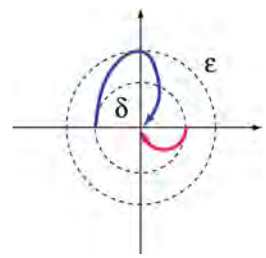
Exponential Asymptotic Stability



- **Uniform stability about x = 0 plus**

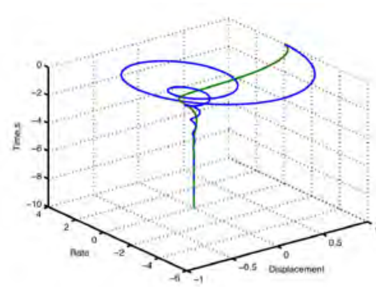
$$\boxed{\left\|\mathbf{x}(t)\right\| \le ke^{-\alpha t}\left\|\mathbf{x}(0)\right\|; \quad k, \alpha \ge 0}$$

- **If norm of x(*t*) is contained within an exponentially decaying envelope with convergence, system is *exponentially asymptotically stable (EAS)***
- **Linear system that is stable is EAS**



Exponential Asymptotic Stability

$$\boxed{k\int_0^\infty e^{-\alpha t}\,dt = -\left(\frac{k}{\alpha}\right)e^{-\alpha t}\Big|_0^\infty = \frac{k}{\alpha}}$$

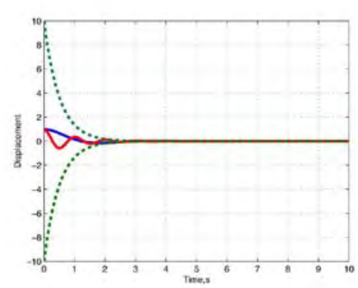


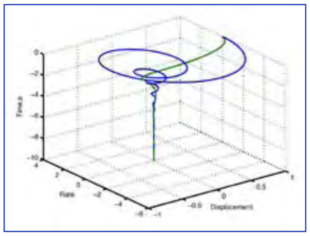
Therefore, time integrals of the norm of an EAS system are bounded

$$\int_0^\infty \left\|\mathbf{x}(t)\right\| dt = \int_0^\infty \left[\mathbf{x}^T(t)\mathbf{x}(t)\right]^{1/2} dt \le \left(\frac{k}{\alpha}\right)\left\|\mathbf{x}(0)\right\|$$

and

$$\int_0^\infty \left\|\mathbf{x}(t)\right\|^2 dt \text{ is bounded}$$





Exponential Asymptotic Stability

Weighted Euclidean norm and its square are bounded if system is EAS

$$\int_0^\infty \left\|\mathbf{Dx}(t)\right\| dt = \int_0^\infty \left[\mathbf{x}^T(t)\mathbf{D}^T\mathbf{Dx}(t)\right]^{1/2} dt \quad \text{is bounded}$$

$$\text{with } \infty > \mathbf{Q} = \mathbf{D}^T\mathbf{D} > 0$$

$$\int_0^\infty \left[\mathbf{x}^T(t)\mathbf{Qx}(t)\right] dt \quad \text{is bounded}$$

Conversely, if the weighted Euclidean norm is bounded, the system is EAS

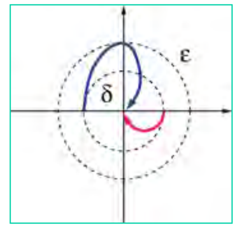
Initial-Condition Response of an EAS Linear System

$$\mathbf{x}(t) = \mathbf{\Phi}(t,0)\mathbf{x}(0) = e^{\mathbf{F}(t)}\mathbf{x}(0)$$

$$\left\|\mathbf{x}(t)\right\|^2 = \mathbf{x}^T(0)\mathbf{\Phi}^T(t,0)\mathbf{\Phi}(t,0)\mathbf{x}(0) \text{ is bounded}$$

- To be shown
  - Continuous-time LTI system is stable if all of its eigenvalues have negative real parts
  - Discrete-time LTI system is stable if all of its eigenvalues lie within the unit circle

Lyapunov's First Theorem



- **A nonlinear system is asymptotically stable at the origin if its linear approximation is <u>stable at the origin</u>, i.e.,**
  - **for all trajectories that start "close enough"**
  - **within a stable manifold**

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t)] \text{ is stable at } \mathrm{x}_o = 0 \text{ if}$$
$$\Delta\dot{\mathbf{x}}(t) = \left.\frac{\partial \mathbf{f}[\mathbf{x}(t)]}{\partial \mathbf{x}}\right|_{\mathbf{x}_o = 0} \Delta\mathbf{x}(t) \text{ is stable}$$

"At the origin" is a fuzzy concept

Lyapunov's Second Theorem*

Define a scalar <u>Lyapunov function</u>, a positive definite function of the state in the region of interest

$$V^*\left[\mathbf{x}^*(t)\right] \geq 0$$

Examples

$$V = E = \frac{mV^2}{2} + mgh; \quad \frac{E}{mg} = \frac{E}{weight} = \frac{V^2}{2g} + h$$
$$V = \frac{1}{2}\mathbf{x}^T\mathbf{x}; \quad V = \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x}$$

**** Who was Lyapunov? see http://en.wikipedia.org/wiki/Aleksandr_Lyapunov***

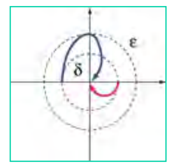
$$\boxed{V^*\left[\mathbf{x}^*(t)\right] \geq 0}$$

Lyapunov's Second Theorem

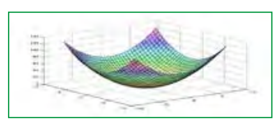
Evaluate the time derivative of the Lyapunov function

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}\dot{\mathbf{x}} \\ &= \frac{\partial V}{\partial \mathbf{x}}\dot{\mathbf{x}} \text{ for autonomous systems}\end{aligned}$$

- **If** $\frac{dV}{dt} < 0$ **in the neighborhood of the origin,** **the origin is asymptotically stable**



Quadratic Lyapunov Function



Lyapunov function

$$V\left[\mathbf{x}(t)\right] = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$$

Linear, Time-Invariant System

$$\dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{x}(t)$$

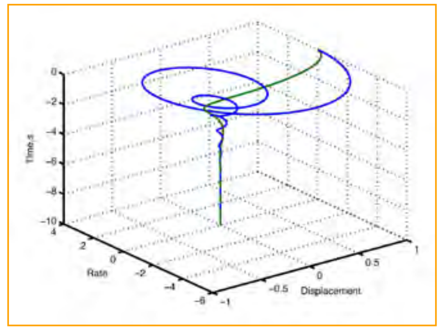
Rate of change for quadratic Lyapunov function

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial \mathbf{x}}\dot{\mathbf{x}} = \mathbf{x}^T(t)\mathbf{P}\dot{\mathbf{x}}(t) + \dot{\mathbf{x}}^T(t)\mathbf{P}\mathbf{x}(t) \\ &= \mathbf{x}^T(t)\left(\mathbf{PF} + \mathbf{F}^T\mathbf{P}\right)\mathbf{x}(t) \triangleq -\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t)\end{aligned}$$

Lyapunov Equation

The LTI system is stable if the Lyapunov equation is satisfied with positive-definite P and Q

$$\mathbf{PF} + \mathbf{F}^T\mathbf{P} = -\mathbf{Q}$$
$$\text{with}$$
$$\mathbf{P} > 0, \quad \mathbf{Q} > 0$$

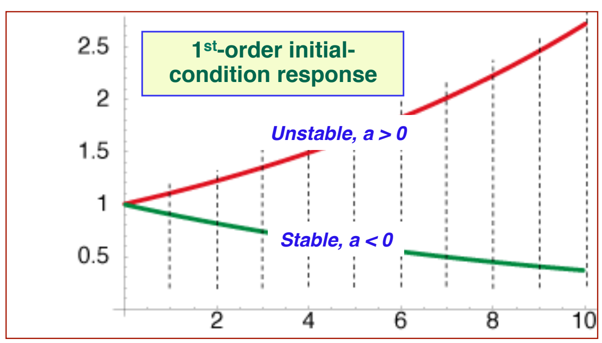


Lyapunov Stability: 1^{st}-Order Example

$$\Delta\dot{x}(t) = a\Delta x(t)\,,\ \Delta x(0)\ \text{given}$$

$$\mathbf{F} = a, \mathbf{P} = p, \mathbf{Q} = q$$

$$\Delta x(t) = \int_0^t \Delta\dot{x}(t)\,dt = \int_0^t a\Delta x(t)\,dt$$
$$= e^{at}\Delta x(0)$$



$$\mathbf{PF} + \mathbf{F}^T\mathbf{P} = -\mathbf{Q}$$
with $p > 0,\ a < 0$
$2pa < 0$ and $q > 0$
$\therefore$ system is stable

$$\mathbf{PF} + \mathbf{F}^T\mathbf{P} = -\mathbf{Q}$$
with $p > 0,\ a > 0$
$2pa < 0$ and $q < 0$
$\therefore$ system is unstable

Lyapunov Stability and the HJB Equation

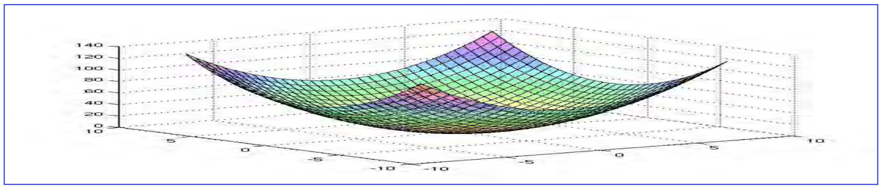
$$\boxed{V\left[\mathbf{x}(t)\right] = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)}$$

Lyapunov stability

$$\boxed{\frac{dV}{dt} < 0}$$

Dynamic programming optimality

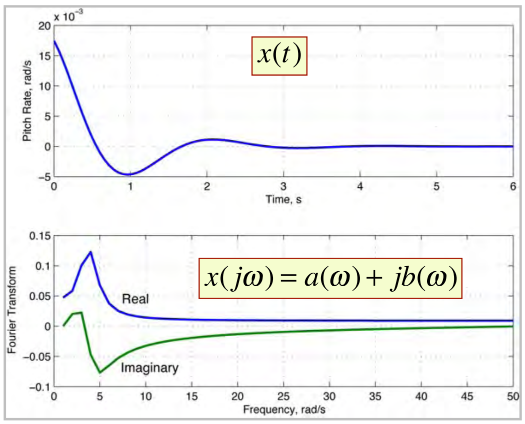
$$\boxed{\frac{\partial V^*}{\partial t} = -\min_{\mathbf{u}(t)} H}$$



Laplace Transforms and Linear System Stability

Fourier Transform of a Scalar Variable

$$F\left[x(t)\right] = x(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \quad \omega = frequency, rad / s$$



$$\begin{aligned} x(t): &\quad \text{real variable} \\ x(j\omega): &\quad \text{complex variable} \\ &= a(\omega) + jb(\omega) \\ &= A(\omega)e^{j\varphi(\omega)} \end{aligned}$$

$$\begin{aligned} A: &\quad amplitude \\ \varphi: &\quad phase\ angle \end{aligned}$$

Laplace Transforms of Scalar Variables

Laplace transform of a scalar variable is a complex number

s **is the Laplace operator, a complex variable**

$$\mathbf{L}\left[x(t)\right] = x(s) = \int_{0}^{\infty} x(t)e^{-st}\,dt, \quad s = \sigma + j\omega, \quad (j = i = \sqrt{-1})$$

Laplace transformation is a linear operation

Multiplication by a constant

$$\mathbf{L}\left[a\,x(t)\right] = a\,x(s)$$

Sum of Laplace transforms

$$\mathbf{L}\left[x_1(t) + x_2(t)\right] = x_1(s) + x_2(s)$$

$$\begin{aligned} x(t): &\quad \text{real variable} \\ x(s): &\quad \text{complex variable} \\ &= a(\omega) + jb(\omega) \\ &= A(\omega)e^{j\varphi(\omega)} \end{aligned}$$

Laplace Transforms of Vectors and Matrices

Laplace transform of a vector variable

$$L\left[\mathbf{x}(t)\right]=\mathbf{x}(s)=\begin{bmatrix} x_1(s) \\ x_2(s) \\ \ldots \end{bmatrix}$$

Laplace transform of a matrix variable

$$L\left[\mathbf{A}(t)\right]=\mathbf{A}(s)=\begin{bmatrix} a_{11}(s) & a_{12}(s) & \ldots \\ a_{21}(s) & a_{22}(s) & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

Laplace transform of a time-derivative

$$\boxed{L\left[\dot{\mathbf{x}}(t)\right]=s\mathbf{x}(s)-\mathbf{x}(0)}$$

Transformation of the System Equations

Time-Domain System Equations

$$\dot{\mathbf{x}}(t)=\mathbf{F}\,\mathbf{x}(t)+\mathbf{G}\,\mathbf{u}(t)$$

Dynamic Equation

$$\mathbf{y}(t)=\mathbf{H}_{\mathbf{x}}\mathbf{x}(t)+\mathbf{H}_{\mathbf{u}}\mathbf{u}(t)$$

Output Equation

Laplace Transforms of System Equations

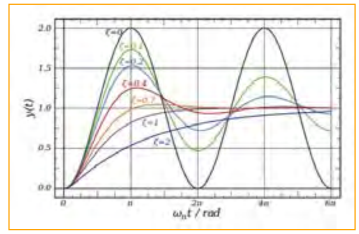
$$s\mathbf{x}(s)-\mathbf{x}(0)=\mathbf{F}\,\mathbf{x}(s)+\mathbf{G}\,\mathbf{u}(s)$$

Dynamic Equation

$$\mathbf{y}(s)=\mathbf{H}_{\mathbf{x}}\mathbf{x}(s)+\mathbf{H}_{\mathbf{u}}\mathbf{u}(s)$$

Output Equation

Second-Order Oscillator



Differential Equations for 2nd-Order System

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u(t)$$

Dynamic Equation

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Output Equation

Laplace Transforms of 2nd-Order System

$$\begin{bmatrix} sx_1(s) - x_1(0) \\ sx_2(s) - x_2(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u(s)$$

Dynamic Equation

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix}$$

Output Equation

Laplace Transform of the State Vector Response to Initial Condition and Control

Rearrange Laplace Transform of Dynamic Equation

$$s\mathbf{x}(s) - \mathbf{F}\,\mathbf{x}(s) = \mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)$$

$$[s\mathbf{I} - \mathbf{F}]\mathbf{x}(s) = \mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)$$

$$\mathbf{x}(s) = [s\mathbf{I} - \mathbf{F}]^{-1}[\mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)]$$

The matrix inverse is

$$[s\mathbf{I} - \mathbf{F}]^{-1} = \frac{Adj(s\mathbf{I} - \mathbf{F})}{|s\mathbf{I} - \mathbf{F}|} \quad (n\ x\ n)$$

$$Adj(s\mathbf{I} - \mathbf{F}): \quad Adjoint\ matrix \quad (n \times n) \quad Transpose\ of\ matrix\ of\ cofactors$$

$$|s\mathbf{I} - \mathbf{F}| = \det(s\mathbf{I} - \mathbf{F}): \quad Determinant \quad (1 \times 1)$$

Characteristic Polynomial of a Dynamic System

Matrix Inverse

$$\left[s\mathbf{I}-\mathbf{F}\right]^{-1}=\frac{Adj\left(s\mathbf{I}-\mathbf{F}\right)}{\left|s\mathbf{I}-\mathbf{F}\right|}\quad (n\ x\ n)$$

Characteristic matrix of the system

$$\left(s\mathbf{I}-\mathbf{F}\right)=\begin{pmatrix}\left(s-f_{11}\right) & -f_{12} & \ldots & -f_{1n}\\ -f_{21} & \left(s-f_{22}\right) & \ldots & -f_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ -f_{n1} & -f_{n2} & \ldots & \left(s-f_{nn}\right)\end{pmatrix}\quad (n\ x\ n)$$

Characteristic polynomial of the system

$$\begin{gathered}\left|s\mathbf{I}-\mathbf{F}\right|=\det\left(s\mathbf{I}-\mathbf{F}\right)\\ \equiv\Delta(s)=s^{n}+a_{n-1}s^{n-1}+\ldots+a_{1}s+a_{0}\end{gathered}$$

Eigenvalues

Eigenvalues of the System

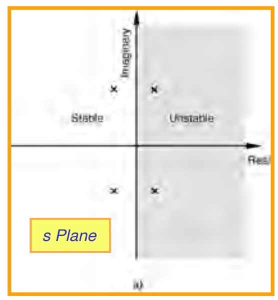
Characteristic equation of the system

$$\Delta(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$
$$= (s-\lambda_1)(s-\lambda_2)(\ldots)(s-\lambda_n) = 0$$

Eigenvalues, λ_i, are solutions (roots) of the polynomial, $\Delta(s) = 0$

$$\lambda_i = \sigma_i + j\omega_i$$

$$\lambda^*_i = \sigma_i - j\omega_i$$



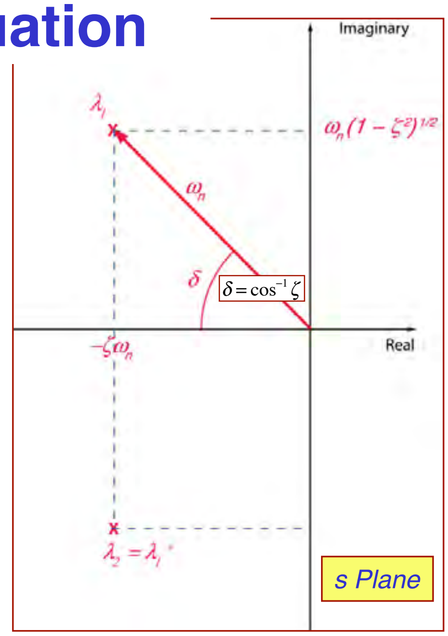
Factors of a 2nd-Degree Characteristic Equation

$$|s\mathbf{I} - \mathbf{F}| = \begin{vmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{vmatrix} \triangleq \Delta(s)$$
$$= s^2 - (f_{12} + f_{21})s + (f_{11}f_{22} + f_{12}f_{21})$$
$$= (s-\lambda_1)(s-\lambda_2) = 0 \quad \text{[real or complex roots]}$$
$$= s^2 + 2\zeta\omega_n s + \omega_n^2 \quad \text{with complex-conjugate roots}$$

$$\lambda_1 = \sigma_1, \quad \lambda_2 = \sigma_2$$

$$\lambda_1 = \sigma_1 + j\omega_1$$
$$\lambda_2 = \sigma_1 - j\omega_1$$

ω_n : natural frequency, rad/s
ζ : damping ratio, dimensionless



z Transforms and Discrete-Time Systems

Application of Dirac Delta Function to Sampling Process

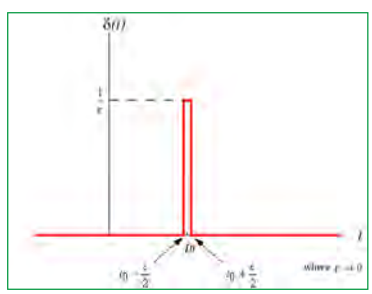
- **Periodic sequence of numbers**

$$\boxed{\Delta x_k = \Delta x(t_k) = \Delta x(k\Delta t)}$$

- **Dirac delta function**

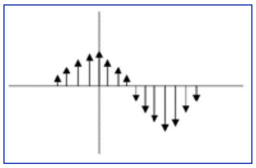
$$\delta(t_0 - k\Delta t) = \begin{cases} \infty, & (t_0 - k\Delta t) = 0 \\ 0, & (t_0 - k\Delta t) \neq 0 \end{cases}$$

$$\int_{(t_0 - k\Delta t)-\varepsilon}^{(t_0 - k\Delta t)+\varepsilon} \delta(t_0 - k\Delta t)\,dt = 1$$



- **Periodic sequence of scaled delta functions**

$$\boxed{\Delta x(k\Delta t)\,\delta(t_0 - k\Delta t)}$$



Laplace Transform of a Periodic Scalar Sequence

- **Periodic sequence of numbers** $\Delta x_k = \Delta x(t_k) = \Delta x(k\Delta t)$
- **Periodic sequence of scaled delta functions** $\Delta x(k\Delta t)\delta(t - k\Delta t)$
- **Laplace transform of the delta function sequence**

$$L\left[\Delta x(k\Delta t)\delta(t-k\Delta t)\right] = \Delta x(z) = \int_0^\infty \Delta x(k\Delta t)\delta(t-k\Delta t)e^{-s\Delta t}dt$$
$$= \sum_{k=0}^{\infty}\Delta x(k\Delta t)e^{-sk\Delta t} \triangleq \sum_{k=0}^{\infty}\Delta x(k\Delta t)z^{-k}$$

z Transform of the Periodic Sequence

z transform is the Laplace transform of the delta function sequence

$$L\left[\Delta x(k\Delta t)\delta(t-k\Delta t)\right] = \sum_{k=0}^{\infty}\Delta x(k\Delta t)e^{-sk\Delta t} \triangleq \sum_{k=0}^{\infty}\Delta x(k\Delta t)z^{-k}$$

z Transform (time-shift) Operator

$$z \triangleq e^{s\Delta t} \quad \text{[advance by one sampling interval]}$$
$$z^{-1} \triangleq e^{-s\Delta t} \quad \text{[delay by one sampling interval]}$$

z Transform of a Discrete-Time Dynamic System

System equation in sampled time domain

$$\Delta\mathbf{x}_{k+1} = \mathbf{\Phi}\,\Delta\mathbf{x}_k + \mathbf{\Gamma}\,\Delta\mathbf{u}_k + \mathbf{\Lambda}\Delta\mathbf{w}_k$$

Laplace transform of sampled-data system equation ("*z* Transform")

$$z\Delta\mathbf{x}(z) - \Delta\mathbf{x}(0) = \mathbf{\Phi}\,\Delta\mathbf{x}(z) + \mathbf{\Gamma}\Delta\,\mathbf{u}(z) + \mathbf{\Lambda}\Delta\mathbf{w}(z)$$

z Transform of a Discrete-Time Dynamic System

Rearrange

$$z\Delta\mathbf{x}(z) - \mathbf{\Phi}\,\Delta\mathbf{x}(z) = \Delta\mathbf{x}(0) + \mathbf{\Gamma}\Delta\,\mathbf{u}(z) + \mathbf{\Lambda}\Delta\mathbf{w}(z)$$

Collect terms

$$\left(z\mathbf{I} - \mathbf{\Phi}\right)\Delta\mathbf{x}(z) = \Delta\mathbf{x}(0) + \mathbf{\Gamma}\Delta\,\mathbf{u}(z) + \mathbf{\Lambda}\Delta\mathbf{w}(z)$$

Pre-multiply by inverse

$$\Delta\mathbf{x}(z) = \left(z\mathbf{I} - \mathbf{\Phi}\right)^{-1}\left[\Delta\mathbf{x}(0) + \mathbf{\Gamma}\Delta\,\mathbf{u}(z) + \mathbf{\Lambda}\Delta\mathbf{w}(z)\right]$$

Characteristic Matrix and Determinant of Discrete-Time System

$$\boxed{\Delta\mathbf{x}(z) = (z\mathbf{I} - \mathbf{\Phi})^{-1}\left[\Delta\mathbf{x}(0) + \mathbf{\Gamma}\Delta\,\mathbf{u}(z) + \mathbf{\Lambda}\Delta\mathbf{w}(z)\right]}$$

Inverse matrix

$$\boxed{(z\mathbf{I} - \mathbf{\Phi})^{-1} = \frac{Adj(z\mathbf{I} - \mathbf{\Phi})}{\left|z\mathbf{I} - \mathbf{\Phi}\right|} \quad (n\ x\ n)}$$

Characteristic polynomial of the discrete-time model

$$\left|z\mathbf{I} - \mathbf{\Phi}\right| = \det(z\mathbf{I} - \mathbf{\Phi}) \equiv \Delta(z)$$
$$= z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

Eigenvalues (or Roots) of the Discrete-Time System

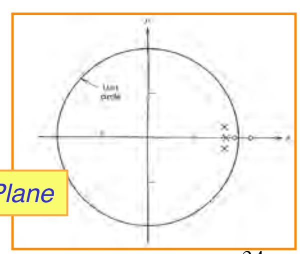
Characteristic equation of the system

$$\Delta(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$
$$= (z - \lambda_1)(z - \lambda_2)(\ldots)(z - \lambda_n) = 0$$

Eigenvalues, λ_i, of the state transition matrix, $\mathbf{\Phi}$, are solutions (roots) of the polynomial, $\Delta(z) = 0$

Eigenvalues are complex numbers that can be plotted in the *z* plane

$$\lambda_i = \sigma_i + j\omega_i \qquad \lambda_i^* = \sigma_i - j\omega_i$$



Laplace Transforms of Continuous- and Discrete-Time State-Space Models

Initial condition and disturbance effect neglected

$$
\boxed{\begin{aligned}
\Delta\mathbf{x}(s) &= \left(s\mathbf{I}-\mathbf{F}\right)^{-1}\mathbf{G}\Delta\mathbf{u}(s)\\
\Delta\mathbf{y}(s) &= \mathbf{H}\left(s\mathbf{I}-\mathbf{F}\right)^{-1}\mathbf{G}\Delta\mathbf{u}(s)
\end{aligned}}
$$

Equivalent discrete-time model

$$
\boxed{\begin{aligned}
\Delta\mathbf{x}(z) &= \left(z\mathbf{I}-\mathbf{\Phi}\right)^{-1}\mathbf{\Gamma}\Delta\mathbf{u}(z)\\
\Delta\mathbf{y}(z) &= \mathbf{H}\left(z\mathbf{I}-\mathbf{\Phi}\right)^{-1}\mathbf{\Gamma}\Delta\mathbf{u}(z)
\end{aligned}}
$$

Scalar Transfer Functions of Continuous- and Discrete-Time Systems

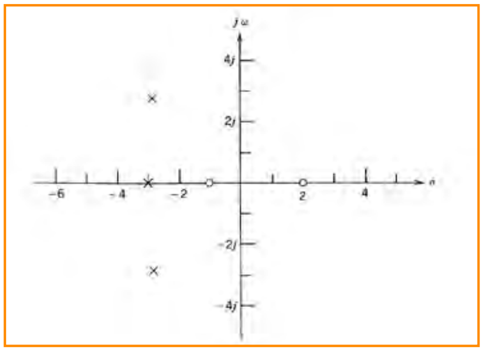
$$
\boxed{\begin{aligned}
\dim(\mathbf{H}) &= 1\times n\\
\dim(\mathbf{G}) &= n\times 1
\end{aligned}}
$$

$$
\boxed{\frac{\Delta y(s)}{\Delta u(s)} = \mathbf{H}\left(s\mathbf{I}-\mathbf{F}\right)^{-1}\mathbf{G} = \frac{\mathbf{H}Adj\left(s\mathbf{I}-\mathbf{F}\right)\mathbf{G}}{\left|s\mathbf{I}-\mathbf{F}\right|} = Y(s)}
$$

$$
\boxed{\frac{\Delta y(z)}{\Delta u(z)} = \mathbf{H}\left(z\mathbf{I}-\mathbf{\Phi}\right)^{-1}\mathbf{\Gamma} = \frac{\mathbf{H}Adj\left(s\mathbf{I}-\mathbf{\Phi}\right)\mathbf{\Gamma}}{\left|s\mathbf{I}-\mathbf{\Phi}\right|} = Y(z)}
$$

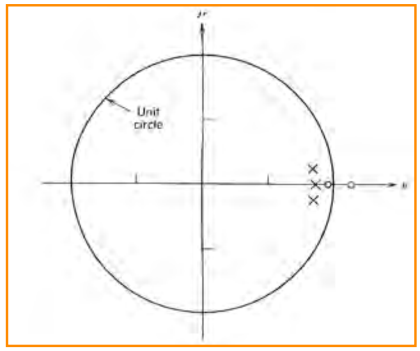
Comparison of *s*-Plane and *z*-Plane Plots of Poles and Zeros

- ***s*-Plane Plot of Poles and Zeros**
  - **Poles in left-half-plane are *stable***
  - **Zeros in left-half-plane are *minimum phase***



Note correspondence of configurations

- ***z*-Plane Plot of Poles and Zeros**
  - **Poles within unit circle are *stable***
  - **Zeros within unit circle are *minimum phase***



Increasing sampling rate moves poles and zeros toward the (1,0) point

Next Time: Time-Invariant Linear-Quadratic Regulators

Supplementary Material

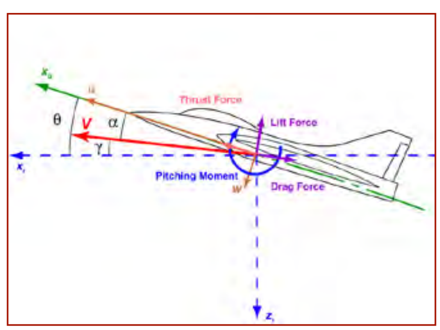
Small Perturbations from Steady, Level Flight

$$\boxed{\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t)}$$

$$\Delta\mathbf{x}(t) = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} = \begin{bmatrix} \Delta V \\ \Delta\gamma \\ \Delta q \\ \Delta\alpha \end{bmatrix} \begin{array}{c} \text{velocity, m/s} \\ \text{flight path angle, rad} \\ \text{pitch rate, rad/s} \\ \text{angle of attack, rad} \end{array}$$

$$\Delta\mathbf{u}(t) = \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} \Delta\delta E \\ \Delta\delta T \end{bmatrix} \begin{array}{c} \text{elevator angle, rad} \\ \text{throttle setting, \%} \end{array}$$

$$\Delta\mathbf{w}(t) = \begin{bmatrix} \Delta w_1 \\ \Delta w_2 \end{bmatrix} = \begin{bmatrix} \Delta V_w \\ \Delta\alpha_w \end{bmatrix} \begin{array}{c} \text{\textasciitilde horizontal wind, m/s} \\ \text{\textasciitilde vertical wind/V}_{\text{nom}}\text{, rad} \end{array}$$



Eigenvalues of Aircraft Longitudinal Modes of Motion

$$
\begin{aligned}
|s\mathbf{I}-\mathbf{F}| = \det(s\mathbf{I}-\mathbf{F}) \equiv \Delta(s) &= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4)\\
&= (s-\lambda_P)(s-\lambda^*_P)(s-\lambda_{SP})(s-\lambda^*_{SP})\\
&= \left(s^2+2\zeta_P\omega_{n_P}s+\omega_{n_P}^2\right)\left(s^2+2\zeta_{SP}\omega_{n_{SP}}s+\omega_{n_{SP}}^2\right)=0
\end{aligned}
$$

Eigenvalues determine the damping and natural frequencies of the linear system's modes of motion

$\left(\zeta_P,\omega_{n_P}\right)$: phugoid (long-period) mode

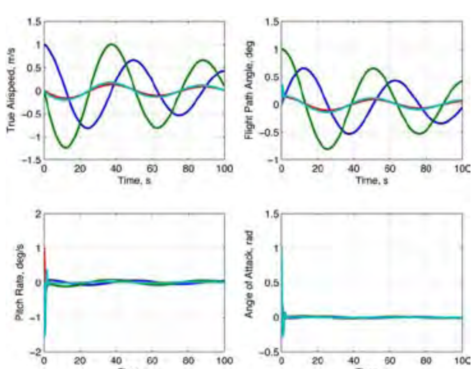
$\left(\zeta_{SP},\omega_{n_{SP}}\right)$: short-period mode

Initial-Condition Response of Business Jet at TwoTime Scales

$$
\Delta\mathbf{x}(t)=\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\\ \Delta x_4\end{bmatrix}=\begin{bmatrix}\Delta V\\ \Delta\gamma\\ \Delta q\\ \Delta\alpha\end{bmatrix}\begin{matrix}\text{velocity, m/s}\\ \text{flight path angle, rad}\\ \text{pitch rate, rad/s}\\ \text{angle of attack, rad}\end{matrix}
$$

Same 4th-order responses viewed over different periods of time

- **0 - 100 sec**
- **Reveals Long-Period Mode**



- **0 - 6 sec**
- **Reveals Short-Period Mode**