Time Response of Linear, Time-Invariant (LTI) Systems

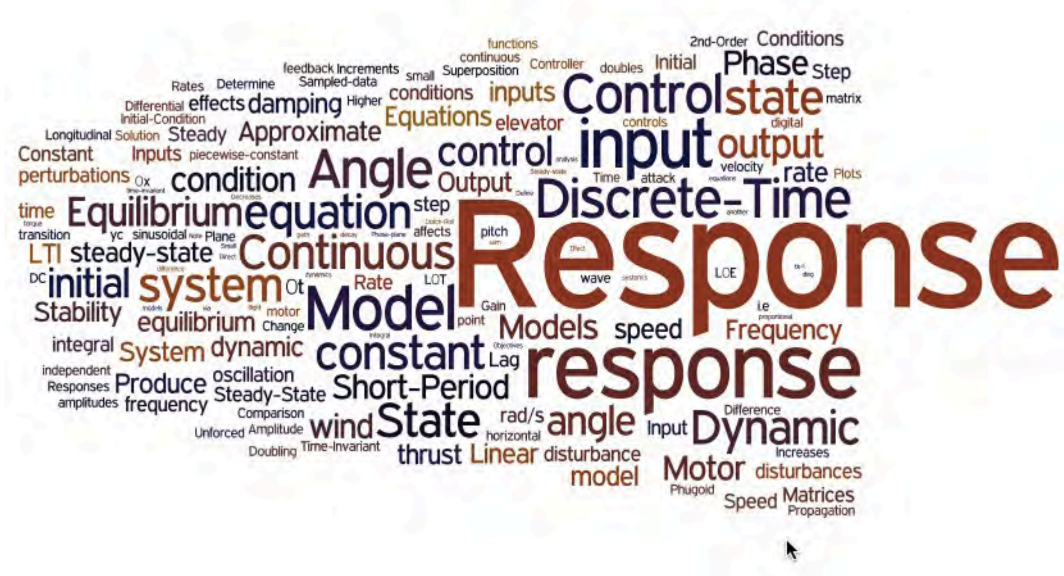
Robert Stengel, Aircraft Flight Dynamics
MAE 331, 2014

Learning Objectives

- **Methods of time-domain analysis**
  - **Continuous- and discrete-time models**
  - **Transient response to initial conditions and inputs**
  - **Steady-state (equilibrium) response**
  - **Phase-plane plots**
  - **Response to sinusoidal input**

Reading:
Flight Dynamics
298-313, 338-342
Airplane Stability and Control
Sections 11.1-11.12

Analysis of LTI System Time Response "Wordle"



2nd-Order Short-Period (LTI) Model

$$\begin{bmatrix} \Delta\dot{q}(t) \\ \Delta\dot{\alpha}(t) \end{bmatrix} = \begin{bmatrix} M_q & M_\alpha \\ \left(1 - L_q / V_N\right) & -L_\alpha / V_N \end{bmatrix} \begin{bmatrix} \Delta q(t) \\ \Delta\alpha(t) \end{bmatrix} + \begin{bmatrix} M_{\delta E} \\ -L_{\delta E} / V_N \end{bmatrix} \Delta\delta E(t) + \begin{bmatrix} M_\alpha \\ -L_\alpha / V_N \end{bmatrix} \Delta\alpha_{wind}(t)$$

- What can we do with it?
  - Integrate equations to obtain time histories of initial condition, control, and disturbance effects
  - Examine steady-state conditions
  - Identify effects of parameter variations
  - Determine modes of motion
  - Define frequency response

Gain insights about system dynamics

Linear, Time-Invariant System Model

Dynamic equation (ordinary differential equation)

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t), \quad \Delta\mathbf{x}(t_o)\ given$$

Output equation (algebraic transformation)

$$\Delta\mathbf{y}(t) = \mathbf{H}_\mathbf{x}\Delta\mathbf{x}(t) + \mathbf{H}_\mathbf{u}\Delta\mathbf{u}(t) + \mathbf{H}_\mathbf{w}\Delta\mathbf{w}(t)$$

State and output dimensions need not be the same

$$\dim\left[\Delta\mathbf{x}(t)\right] = (n \times 1)$$
$$\dim\left[\Delta\mathbf{y}(t)\right] = (r \times 1)$$

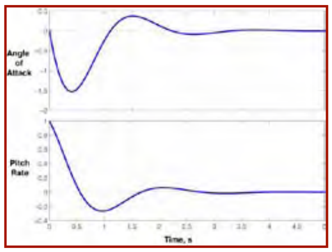
System Response to Inputs and Initial Conditions

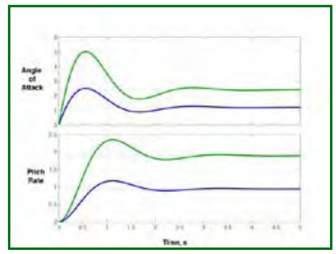
- **Solution of the linear, time-invariant (LTI) dynamic model**

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t), \quad \Delta\mathbf{x}(t_o)\ given$$

$$\Delta\mathbf{x}(t) = \Delta\mathbf{x}(t_o) + \int_{t_o}^{t}\left[\boxed{\mathbf{F}\Delta\mathbf{x}(\tau)} + \boxed{\mathbf{G}\Delta\mathbf{u}(\tau) + \mathbf{L}\Delta\mathbf{w}(\tau)}\right]d\tau$$

- **... has two parts**
  - **Unforced (homogeneous) response to initial conditions**
  - **Forced response to control and disturbance inputs**





Response to Initial Conditions

Unforced Response to Initial Conditions

Neglecting forcing functions

$$\Delta\mathbf{x}(t) = \Delta\mathbf{x}(t_o) + \int_{t_o}^{t} \left[\mathbf{F}\Delta\mathbf{x}(\tau)\right] d\tau = e^{\mathbf{F}(t-t_o)}\Delta\mathbf{x}(t_o) = \mathbf{\Phi}\left(t-t_o\right)\Delta\mathbf{x}(t_o)$$

The state transition matrix, Φ, propagates the state from t_o to t by a single multiplication

$$e^{\mathbf{F}(t-t_o)} = \textbf{Matrix Exponential}$$
$$= \mathbf{I} + \mathbf{F}\left(t-t_o\right) + \frac{1}{2!}\left[\mathbf{F}\left(t-t_o\right)\right]^2 + \frac{1}{3!}\left[\mathbf{F}\left(t-t_o\right)\right]^3 + \ldots$$
$$= \mathbf{\Phi}\left(t-t_o\right) = \textbf{State Transition Matrix}$$

Initial-Condition Response via State Transition

Incremental propagation of Δx

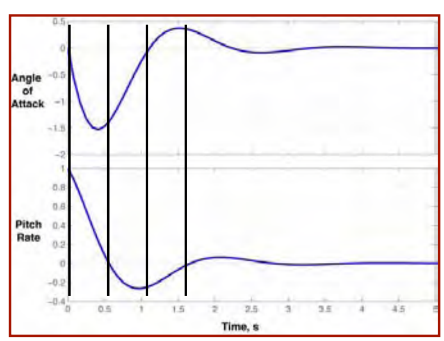
$$\Delta\mathbf{x}(t_1) = \mathbf{\Phi}\left(t_1-t_o\right)\Delta\mathbf{x}(t_o)$$
$$\Delta\mathbf{x}(t_2) = \mathbf{\Phi}\left(t_2-t_1\right)\Delta\mathbf{x}(t_1)$$
$$\Delta\mathbf{x}(t_3) = \mathbf{\Phi}\left(t_3-t_2\right)\Delta\mathbf{x}(t_2)$$
$$\ldots$$

$$\Delta\mathbf{x}(t_1) = \mathbf{\Phi}\left(\delta t\right)\Delta\mathbf{x}(t_o) = \mathbf{\Phi}\Delta\mathbf{x}(t_o)$$
$$\Delta\mathbf{x}(t_2) = \mathbf{\Phi}\Delta\mathbf{x}(t_1) = \mathbf{\Phi}^2\Delta\mathbf{x}(t_o)$$
$$\Delta\mathbf{x}(t_3) = \mathbf{\Phi}\Delta\mathbf{x}(t_2) = \mathbf{\Phi}^3\Delta\mathbf{x}(t_o)$$
$$\ldots$$

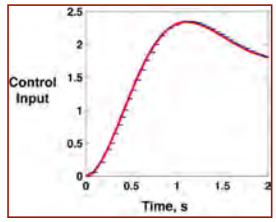
If $(t_{k+1} - t_k) = \delta t$ = constant, state transition matrix is constant

$$\mathbf{\Phi} = \mathbf{I} + \mathbf{F}\left(\delta t\right) + \frac{1}{2!}\left[\mathbf{F}\left(\delta t\right)\right]^2 + \frac{1}{3!}\left[\mathbf{F}\left(\delta t\right)\right]^3 + \ldots$$

Propagation is exact



Discrete-Time Dynamic Model



Response to continuous controls and disturbances

$$\Delta\mathbf{x}(t_{k+1}) = \Delta\mathbf{x}(t_k) + \int_{t_k}^{t_{k+1}} \left[\mathbf{F}\Delta\mathbf{x}(\tau) + \boxed{\mathbf{G}\Delta\mathbf{u}(\tau) + \mathbf{L}\Delta\mathbf{w}(\tau)}\right] d\tau$$

Response to piecewise-constant controls and disturbances

$$\Delta\mathbf{x}(t_{k+1}) = \mathbf{\Phi}(\delta t)\Delta\mathbf{x}(t_k) + \mathbf{\Phi}(\delta t)\int_{t_k}^{t_{k+1}} \left[e^{-\mathbf{F}(\tau - t_k)}\right] d\tau \boxed{\left[\mathbf{G}\Delta\mathbf{u}(t_k) + \mathbf{L}\Delta\mathbf{w}(t_k)\right]}$$

$$\boxed{= \mathbf{\Phi}\Delta\mathbf{x}(t_k) + \mathbf{\Gamma}\Delta\mathbf{u}(t_k) + \mathbf{\Lambda}\Delta\mathbf{w}(t_k)}$$

Ordinary Difference Equation

With piecewise-constant inputs, control and disturbance effects taken outside the integral

Discrete-time model of continuous system = Sampled-data model

Sampled-Data Control- and Disturbance-Effect Matrices

$$\Delta\mathbf{x}(t_k) = \mathbf{\Phi}\Delta\mathbf{x}(t_{k-1}) + \mathbf{\Gamma}\Delta\mathbf{u}(t_{k-1}) + \mathbf{\Lambda}\Delta\mathbf{w}(t_{k-1})$$

$$\mathbf{\Gamma} = \left(e^{\mathbf{F}\delta t} - \mathbf{I}\right)\mathbf{F}^{-1}\mathbf{G}$$
$$= \left(\mathbf{I} - \frac{1}{2!}\mathbf{F}\delta t + \frac{1}{3!}\mathbf{F}^2\delta t^2 - \frac{1}{4!}\mathbf{F}^3\delta t^3 + \ldots\right)\mathbf{G}\delta t$$

$$\mathbf{\Lambda} = \left(e^{\mathbf{F}\delta t} - \mathbf{I}\right)\mathbf{F}^{-1}\mathbf{L}$$
$$= \left(\mathbf{I} - \frac{1}{2!}\mathbf{F}\delta t + \frac{1}{3!}\mathbf{F}^2\delta t^2 - \frac{1}{4!}\mathbf{F}^3\delta t^3 + \ldots\right)\mathbf{L}\delta t$$

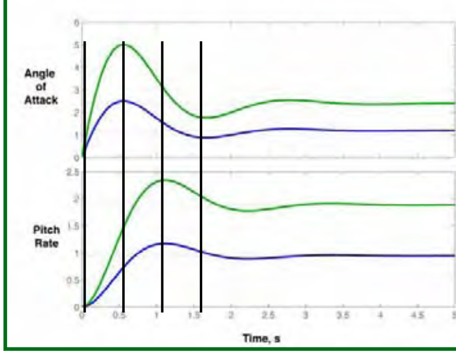
- As δt becomes very small

$$\mathbf{\Phi} \xrightarrow[\delta t \to 0]{} \left(\mathbf{I} + \mathbf{F}\delta t\right)$$
$$\mathbf{\Gamma} \xrightarrow[\delta t \to 0]{} \mathbf{G}\delta t$$
$$\mathbf{\Lambda} \xrightarrow[\delta t \to 0]{} \mathbf{L}\delta t$$

Discrete-Time Response to Inputs

Propagation of Δx, with constant Φ, Γ, and Λ

$$\begin{aligned}
\Delta\mathbf{x}(t_1) &= \mathbf{\Phi}\Delta\mathbf{x}(t_o)+\mathbf{\Gamma}\Delta\mathbf{u}(t_o)+\mathbf{\Lambda}\Delta\mathbf{w}(t_o)\\
\Delta\mathbf{x}(t_2) &= \mathbf{\Phi}\Delta\mathbf{x}(t_1)+\mathbf{\Gamma}\Delta\mathbf{u}(t_1)+\mathbf{\Lambda}\Delta\mathbf{w}(t_1)\\
\Delta\mathbf{x}(t_3) &= \mathbf{\Phi}\Delta\mathbf{x}(t_2)+\mathbf{\Gamma}\Delta\mathbf{u}(t_2)+\mathbf{\Lambda}\Delta\mathbf{w}(t_2)\\
&\ldots
\end{aligned}$$



$$\delta t = t_{k+1} - t_k$$

Continuous- and Discrete-Time Short-Period System Matrices

- **Continuous-time ("analog") system**

$$\mathbf{F} = \begin{bmatrix} -1.2794 & -7.9856 \\ 1 & -1.2709 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} -9.069 \\ 0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} -7.9856 \\ -1.2709 \end{bmatrix}$$

- **Sampled-data ("digital") system**

$$\delta t = t_{k+1} - t_k$$

- **δt = 0.01 s**

$$\mathbf{\Phi} = \begin{bmatrix} 0.987 & -0.079 \\ 0.01 & 0.987 \end{bmatrix} \quad \mathbf{\Gamma} = \begin{bmatrix} -0.09 \\ -0.0004 \end{bmatrix} \quad \mathbf{\Lambda} = \begin{bmatrix} -0.079 \\ -0.013 \end{bmatrix}$$

- **δt = 0.1 s**

$$\mathbf{\Phi} = \begin{bmatrix} 0.845 & -0.694 \\ 0.0869 & 0.846 \end{bmatrix} \quad \mathbf{\Gamma} = \begin{bmatrix} -0.84 \\ -0.0414 \end{bmatrix} \quad \mathbf{\Lambda} = \begin{bmatrix} -0.694 \\ -0.154 \end{bmatrix}$$

- **δt = 0.5 s**

$$\mathbf{\Phi} = \begin{bmatrix} 0.0823 & -1.475 \\ 0.185 & 0.0839 \end{bmatrix} \quad \mathbf{\Gamma} = \begin{bmatrix} -2.492 \\ -0.643 \end{bmatrix} \quad \mathbf{\Lambda} = \begin{bmatrix} -1.475 \\ -0.916 \end{bmatrix}$$

δt has a large effect on the "digital" model

Continuous- and Discrete-Time Short-Period Models

Learjet 23
$M_N = 0.3,\ h_N = 3{,}050$ m
$V_N = 98.4$ m/s

Differential Equations Produce State Rates of Change

$$\begin{bmatrix} \Delta\dot{q}(t) \\ \Delta\dot{\alpha}(t) \end{bmatrix} = \begin{bmatrix} -1.3 & -8 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} \Delta q(t) \\ \Delta\alpha(t) \end{bmatrix} + \begin{bmatrix} -9.1 \\ 0 \end{bmatrix} \Delta\delta E(t)$$

Difference Equations Produce State Increments

$\delta t = 0.1\sec$

$$\begin{bmatrix} \Delta q_{k+1} \\ \Delta\alpha_{k+1} \end{bmatrix} = \begin{bmatrix} 0.85 & -0.7 \\ 0.09 & 0.85 \end{bmatrix} \begin{bmatrix} \Delta q_k \\ \Delta\alpha_k \end{bmatrix} + \begin{bmatrix} -0.84 \\ -0.04 \end{bmatrix} \Delta\delta E_k$$

Note individual acceleration and difference sensitivities to state and control perturbations

Continuous- and Discrete-Time Roll-Spiral Models

Differential Equations Produce State Rates of Change

$$\begin{bmatrix} \Delta\dot{p}(t) \\ \Delta\dot{\phi}(t) \end{bmatrix} \approx \begin{bmatrix} -1.2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta p(t) \\ \Delta\phi(t) \end{bmatrix} + \begin{bmatrix} 2.3 \\ 0 \end{bmatrix} \Delta\delta A(t)$$

Difference Equations Produce State Increments

$\delta t = 0.1\sec$

$$\begin{bmatrix} \Delta p_{k+1} \\ \Delta\phi_{k+1} \end{bmatrix} \approx \begin{bmatrix} 0.89 & 0 \\ 0.09 & 1 \end{bmatrix} \begin{bmatrix} \Delta p_k \\ \Delta\phi_k \end{bmatrix} + \begin{bmatrix} 0.24 \\ -0.01 \end{bmatrix} \Delta\delta A_k$$

Continuous- and Discrete-Time Dutch-Roll Models

Differential Equations Produce State Rates of Change

$$\begin{bmatrix} \Delta\dot{r}(t) \\ \Delta\dot{\beta}(t) \end{bmatrix} \approx \begin{bmatrix} -0.11 & 1.9 \\ -1 & -0.16 \end{bmatrix} \begin{bmatrix} \Delta r(t) \\ \Delta\beta(t) \end{bmatrix} + \begin{bmatrix} -1.1 \\ 0 \end{bmatrix} \Delta\delta R(t)$$

Difference Equations Produce State Increments

$\delta t = 0.1\,\text{sec}$

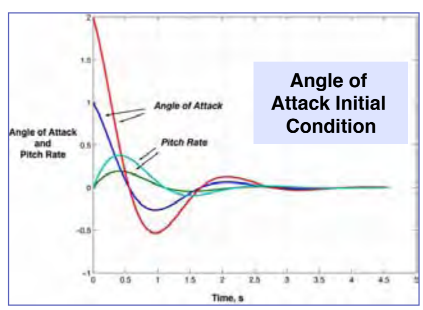
$$\begin{bmatrix} \Delta r_{k+1} \\ \Delta\beta_{k+1} \end{bmatrix} \approx \begin{bmatrix} 0.98 & 0.19 \\ -0.1 & 0.97 \end{bmatrix} \begin{bmatrix} \Delta r_k \\ \Delta\beta_k \end{bmatrix} + \begin{bmatrix} -0.11 \\ 0.01 \end{bmatrix} \Delta\delta R_k$$

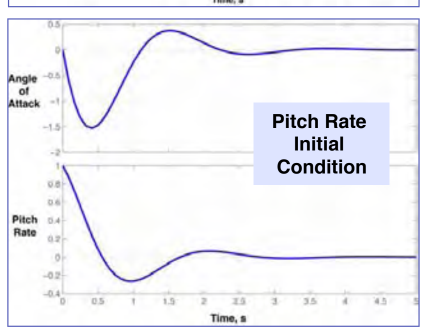
Initial-Condition Response

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1.2794 & -7.9856 \\ 1 & -1.2709 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} -9.069 \\ 0 \end{bmatrix} \Delta\delta E$$

$$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Delta\delta E$$

```
%    Short-Period Linear Model - Initial Condition

     F       =    [-1.2794 -7.9856;1. -1.2709];
     G       =    [-9.069;0];
     Hx      =    [1 0;0 1];
     sys     =    ss(F, G, Hx,0);

     xo          =    [1;0];
     [y1,t1,x1]    =   initial(sys, xo);

     xo          =    [2;0];
     [y2,t2,x2]    =   initial(sys, xo);
     plot(t1,y1,t2,y2), grid

     figure
     xo      =    [0;1];
     initial(sys, xo), grid
```





Doubling the initial condition doubles the output

Historical Factoids

Commercial Aircraft of the 1940s

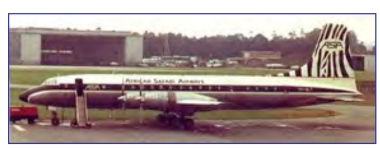
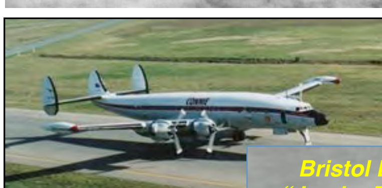
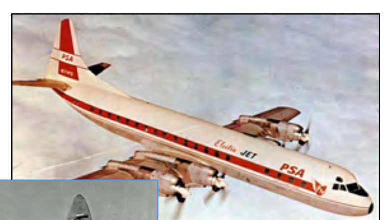
- **Pre-WWII designs**
- **Development enhanced by military transport and bomber versions**
  - ***Douglas DC-4*** **(adopted as** ***C-54)***
  - ***Boeing Stratoliner 377*** **(from** ***B-29, C-97*****)**
  - ***Lockheed Constellation 749*** **(from** ***C-69*****)**





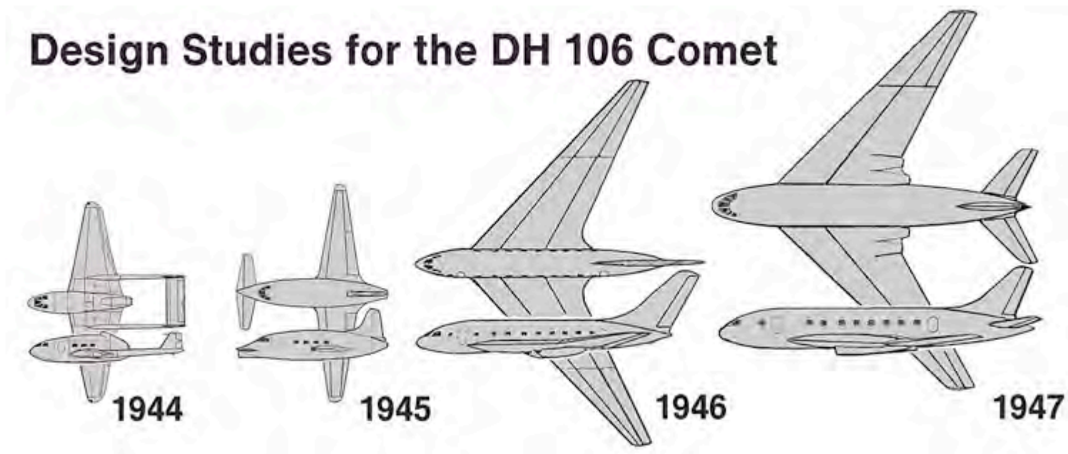

Commercial Propeller-Driven Aircraft of the 1950s

- **Reciprocating and turboprop engines**
- ***Douglas DC-6, DC-7, Lockheed Starliner 1649, Vickers Viscount, Bristol Britannia, Lockheed Electra 188***


Bristol Brabazon
“Jumbo Turboprop”

Brabazon Committee study for a post-WWII jet-powered mailplane with small passenger compartment



Commercial Jets of the 1950s

- **Low-bypass ratio turbojet engines**
- ***deHavilland DH 106 Comet* (1951)**
  - **1st commercial jet transport**
  - **engines buried in wings**
  - **early takeoff accidents and in-flight fatigue failures**
- ***Boeing 707* (1957)**
  - **derived from *USAF KC-135***
  - **engines on pylons below wings**
  - **largest aircraft of its time**
- ***Sud-Aviation Caravelle* (1959)**
  - **1st aircraft with twin aft-mounted engines**


Boeing 707


deHavilland Comet


Sud-Aviation Caravelle

Fatigue Failure of the deHavilland Comet

- 3 in-flight breakups in first 2 years of commercial operation
- Structural test revealed the cause
- Pressurization cycling produced fatigue failure at stress concentration points
- Re-designed *Comet* flew to 1997; *RAF Nimrod* operation to 2011


FIG. 6. AERIAL VIEW OF COMET G-ALYU IN TESTING TANK.

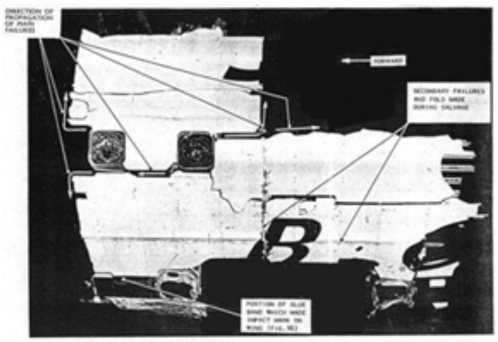


FIG. 12. PHOTOGRAPH OF WRECKAGE AROUND ADF AERIAL WINDOWS—G-ALYP.

Superposition of Linear Responses

Step Response

Step Input

$$\Delta\delta E(t) = \begin{cases} 0, & t < 0 \\ -1, & t \geq 0 \end{cases}$$

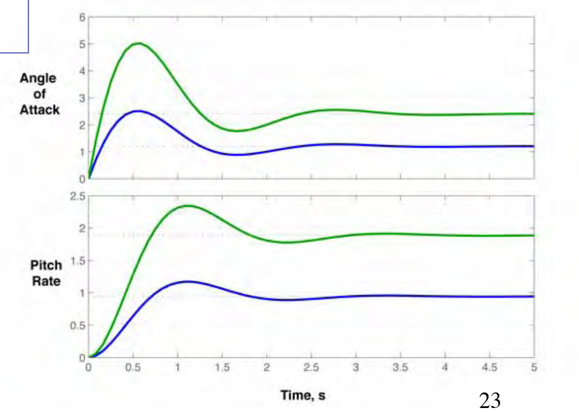
```
%   Short-Period Linear Model - Step

    F       =   [-1.2794 -7.9856;1. -1.2709];
    G       =   [-9.069;0];
    Hx      =   [1 0;0 1];
    sys     =   ss(F, -G, Hx,0); % (-1)*Step
    sys2    =   ss(F, -2*G, Hx,0); % (-1)*Step

%   Step response
    step(sys, sys2), grid
```

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1.2794 & -7.9856 \\ 1 & -1.2709 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} -9.069 \\ 0 \end{bmatrix} \Delta\delta E$$

$$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Delta\delta E$$

- Stability, speed of response, and damping are independent of the initial condition or input

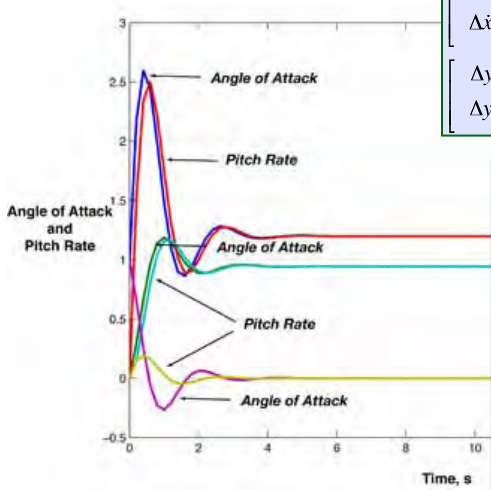
Doubling the input doubles the output



Superposition of Linear Responses

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1.2794 & -7.9856 \\ 1 & -1.2709 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} -9.069 \\ 0 \end{bmatrix} \Delta\delta E$$

$$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Delta\delta E$$




```
%   Short-Period Linear Model - Superposition

    F       =   [-1.2794 -7.9856;1. -1.2709];
    G       =   [-9.069;0];
    Hx      =   [1 0;0 1];
    sys     =   ss(F, -G, Hx,0); % (-1)*Step

    xo      =   [1; 0];
    t       =   [0:0.2:20];
    u       =   ones(1,length(t));

    [y1,t1,x1]  =   lsim(sys,u,t,xo);
    [y2,t2,x2]  =   lsim(sys,u,t);

    u       =   zeros(1,length(t));
    [y3,t3,x3]  =   lsim(sys,u,t,xo);

    plot(t1,y1,t2,y2,t3,y3), grid
```

Stability, speed of response, and damping are independent of the initial condition or input

2nd-Order Comparison: Continuous- and Discrete-Time LTI Longitudinal Models

Differential Equations Produce State Rates of Change

Short Period

$$\begin{bmatrix} \Delta\dot{q}(t) \\ \Delta\dot{\alpha}(t) \end{bmatrix} = \begin{bmatrix} -1.3 & -8 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} \Delta q(t) \\ \Delta\alpha(t) \end{bmatrix} + \begin{bmatrix} -9.1 \\ 0 \end{bmatrix} \Delta\delta E(t)$$

Phugoid

$$\begin{bmatrix} \Delta\dot{V}(t) \\ \Delta\dot{\gamma}(t) \end{bmatrix} \approx \begin{bmatrix} -0.02 & -9.8 \\ 0.02 & 0 \end{bmatrix} \begin{bmatrix} \Delta V(t) \\ \Delta\gamma(t) \end{bmatrix} + \begin{bmatrix} 4.7 \\ 0 \end{bmatrix} \Delta\delta T(t)$$

Difference Equations Produce State Increments

$\delta t = 0.1 \sec$

Short Period

$$\begin{bmatrix} \Delta q_{k+1} \\ \Delta\alpha_{k+1} \end{bmatrix} = \begin{bmatrix} 0.85 & -0.7 \\ 0.09 & 0.85 \end{bmatrix} \begin{bmatrix} \Delta q_k \\ \Delta\alpha_k \end{bmatrix} + \begin{bmatrix} -0.84 \\ -0.04 \end{bmatrix} \Delta\delta E_k$$

Phugoid

$$\begin{bmatrix} \Delta V_{k+1} \\ \Delta\gamma_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -0.98 \\ 0.002 & 1 \end{bmatrix} \begin{bmatrix} \Delta V_k \\ \Delta\gamma_k \end{bmatrix} + \begin{bmatrix} 0.47 \\ 0.0005 \end{bmatrix} \Delta\delta T_k$$

4th- Order Comparison: Continuous- and Discrete-Time Longitudinal Models

Phugoid and Short Period

Differential Equations Produce State Rates of Change

$$\begin{bmatrix} \Delta\dot{V}(t) \\ \Delta\dot{\gamma}(t) \\ \Delta\dot{q}(t) \\ \Delta\dot{\alpha}(t) \end{bmatrix} = \left[\begin{array}{cc|cc} -0.02 & -9.8 & 0 & 0 \\ 0.02 & 0 & 0 & 1.3 \\ \hline 0 & 0 & -1.3 & -8 \\ -0.02 & 0 & 1 & -1.3 \end{array}\right] \begin{bmatrix} \Delta V(t) \\ \Delta\gamma(t) \\ \Delta q(t) \\ \Delta\alpha(t) \end{bmatrix} + \left[\begin{array}{c|c} 4.7 & 0 \\ 0 & 0 \\ \hline 0 & -9.1 \\ 0 & 0 \end{array}\right] \begin{bmatrix} \Delta\delta T(t) \\ \Delta\delta E(t) \end{bmatrix}$$

Difference Equations Produce State Increments

$\delta t = 0.1 \sec$

$$\begin{bmatrix} \Delta V_{k+1} \\ \Delta\gamma_{k+1} \\ \Delta q_{k+1} \\ \Delta\alpha_{k+1} \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & -0.98 & -0.002 & -0.06 \\ 0.002 & 1 & 0.006 & 0.12 \\ \hline 0.0001 & 0 & 0.84 & -0.69 \\ -0.002 & 0.0001 & 0.09 & 0.84 \end{array}\right] \begin{bmatrix} \Delta V_k \\ \Delta\gamma_k \\ \Delta q_k \\ \Delta\alpha_k \end{bmatrix} + \left[\begin{array}{c|c} 0.47 & 0.0005 \\ 0.0005 & -0.002 \\ \hline 0 & -0.84 \\ 0 & -0.04 \end{array}\right] \begin{bmatrix} \Delta\delta T_k \\ \Delta\delta E_k \end{bmatrix}$$

Equilibrium Response

Equilibrium Response

Dynamic equation

$$\boxed{\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t)}$$

At equilibrium, the state is unchanging

$$\boxed{\mathbf{0} = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t)}$$

Constant values denoted by (.)*

$$\boxed{\Delta\mathbf{x}^* = -\mathbf{F}^{-1}\left(\mathbf{G}\Delta\mathbf{u}^* + \mathbf{L}\Delta\mathbf{w}^*\right)}$$

Steady-State Condition

- **If the system is also stable, an equilibrium point is a steady-state point, i.e.,**
  - **Small disturbances decay to the equilibrium condition**

2nd-order example

System Matrices

$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

Equilibrium Response with Constant Inputs

$$\begin{bmatrix} \Delta x_1 * \\ \Delta x_2 * \end{bmatrix} = -\frac{\begin{bmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{bmatrix}}{\left(f_{11}f_{22} - f_{12}f_{21}\right)} \left[\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \Delta u * + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \Delta w * \right]$$

Requirement for Stability

$$\left| s\mathbf{I} - \mathbf{F} \right| = \Delta(s) = s^2 + \left(f_{11} + f_{22}\right)s + \left(f_{11}f_{22} - f_{12}f_{21}\right)$$
$$= \left(s - \lambda_1\right)\left(s - \lambda_2\right) = 0$$
$$\mathrm{Re}\left(\lambda_i\right) < 0$$

Phase Plane Plots

A 2nd-Order Dynamic Model

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n{}^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix}$$

$\Delta x_1(t)$: Displacement (or Position)

$\Delta x_2(t)$: Rate of change of Position

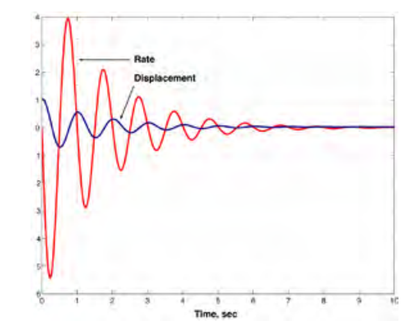
ω_n: Natural frequency, rad/s

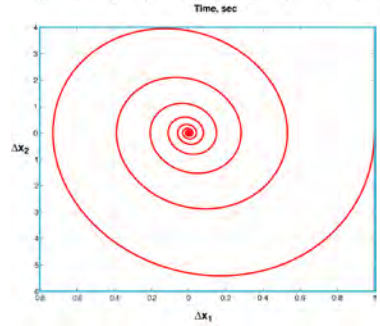
ζ: Damping ratio, -

State ("Phase") Plane Plots

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix}$$

```
%    2nd-Order Model - Initial Condition Response

     clear
     z            =    0.1;  % Damping ratio
     wn           =    6.28; % Natural frequency, rad/s
     F            =    [0 1;-wn^2 -2*z*wn];
     G            =    [1 -1;0 2];
     Hx           =    [1 0;0 1];
     sys          =    ss(F, G, Hx,0);
     t            =    [0:0.01:10];
     xo           =    [1;0];
     [y1,t1,x1]   =    initial(sys, xo, t);

     plot(t1,y1)
     grid on

     figure
     plot(y1(:,1),y1(:,2))
     grid on
```



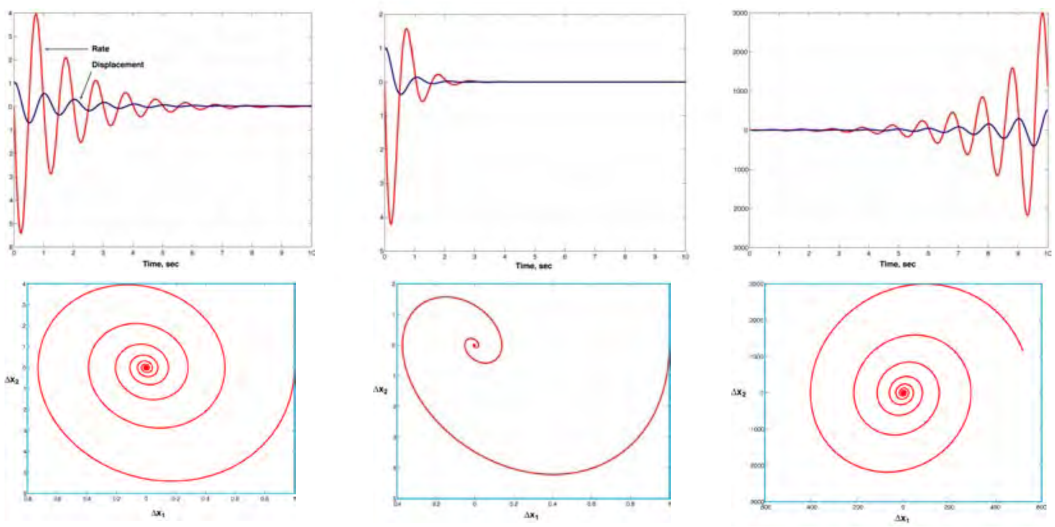


Cross-plot of one component against another

Time is not shown explicitly

Dynamic Stability Changes the State-Plane Spiral

Damping ratio = 0.1 | **Damping ratio = 0.3** | **Damping ratio = –0.1**



Equilibrium Response of Approximate Phugoid Model

Equilibrium state with constant thrust and wind perturbations

$$\boxed{\Delta \mathbf{x}_P^* = -\mathbf{F}_P^{-1}\left(\mathbf{G}_P \Delta \mathbf{u}_P^* + \mathbf{L}_P \Delta \mathbf{w}_P^*\right)}$$

$$\boxed{\begin{bmatrix} \Delta V^* \\ \Delta \gamma^* \end{bmatrix} = -\begin{bmatrix} 0 & \dfrac{V_N}{L_V} \\ \dfrac{-1}{g} & \dfrac{V_N D_V}{g L_V} \end{bmatrix} \left\{ \begin{bmatrix} T_{\delta T} \\ \dfrac{L_{\delta T}}{V_N} \end{bmatrix} \Delta \delta T^* + \begin{bmatrix} D_V \\ \dfrac{-L_V}{V_N} \end{bmatrix} \Delta V_W^* \right\}}$$

Equilibrium Response of Approximate Phugoid Model

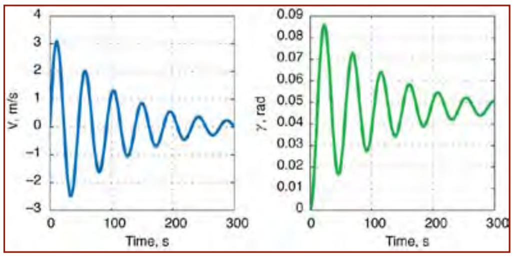
$$\Delta V^* = -\cancel{\frac{L_{\delta T}}{L_V}}\Delta\delta T^* + \Delta V_W^*$$

$$\Delta\gamma^* = \frac{1}{g}\left(T_{\delta T} + L_{\delta T}\cancel{\frac{D_V}{L_V}}\right)\Delta\delta T^*$$

Steady horizontal wind affects velocity but not flight path angle

- **With $L_{\delta T} \sim 0$, i.e., no lift produced directly by thrust, steady-state velocity depends only on the horizontal wind**
- **Constant thrust produces steady climb rate**

Corresponding dynamic response to thrust step, with $L_{\delta T} = 0$



Equilibrium Response of Approximate Short-Period Model

Equilibrium state with constant elevator and wind perturbations

$$\Delta\mathbf{x}_{SP}^* = -\mathbf{F}_{SP}^{-1}\left(\mathbf{G}_{SP}\Delta\mathbf{u}_{SP}^* + \mathbf{L}_{SP}\Delta\mathbf{w}_{SP}^*\right)$$

$$\begin{bmatrix}\Delta q^* \\ \Delta\alpha^*\end{bmatrix} = -\frac{\begin{bmatrix}\frac{L_\alpha}{V_N} & M_\alpha \\ 1 & -M_q\end{bmatrix}}{\left(\frac{L_\alpha}{V_N}M_q + M_\alpha\right)}\left\{\begin{bmatrix}M_{\delta E} \\ -\frac{L_{\delta E}}{V_N}\end{bmatrix}\Delta\delta E^* - \begin{bmatrix}M_\alpha \\ \frac{-L_\alpha}{V_N}\end{bmatrix}\Delta\alpha_W^*\right\}$$

Equilibrium Response of Approximate Short-Period Model

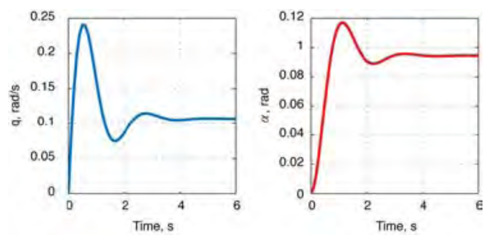
$$\Delta q^* = -\frac{\left(\frac{L_\alpha}{V_N} M_{\delta E}\right)}{\left(\frac{L_\alpha}{V_N} M_q + M_\alpha\right)} \Delta\delta E^*$$

$$\Delta\alpha^* = -\frac{\left(M_{\delta E}\right)}{\left(\frac{L_\alpha}{V_N} M_q + M_\alpha\right)} \Delta\delta E + \Delta\alpha_W^*$$

with $L_{\delta E} = 0$

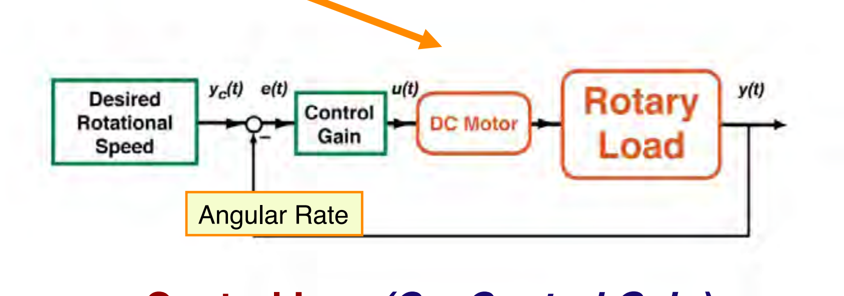
- **Steady pitch rate and angle of attack response to elevator are not zero**
- **Steady vertical wind affects steady-state angle of attack but not pitch rate**

Dynamic response to elevator step with $L_{\delta E} = 0$



Scalar Frequency Response

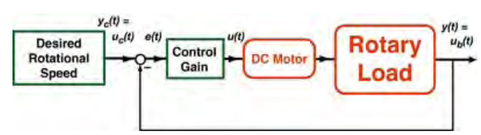
Speed Control of Direct-Current Motor



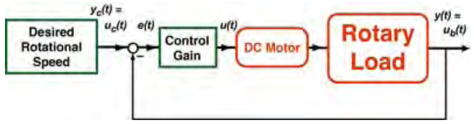
Control Law ***(C = Control Gain)***

$$u(t) = C\,e(t)$$
where
$$e(t) = y_c(t) - y(t)$$

Characteristics of the Motor



- **Simplified Dynamic Model**
  - Rotary inertia, ***J***, is the sum of motor and load inertias
  - Internal damping neglected
  - Output speed, ***y(t)***, rad/s, is an integral of the control input, ***u(t)***
  - Motor control torque is proportional to ***u(t)***
  - Desired speed, $y_c(t)$, rad/s, is constant
  - Control gain, ***C***, scales command-following error to motor input voltage



Model of Dynamics and Speed Control

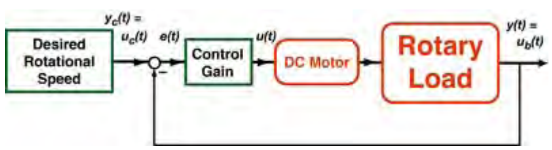
Dynamic equation

$$\frac{dy(t)}{dt} = \frac{u(t)}{J} = \frac{Ce(t)}{J} = \frac{C}{J}\left[y_c(t) - y(t)\right], \quad y(0) \text{ given}$$

Integral of the equation, with $y(0) = 0$

$$y(t) = \frac{1}{J}\int_0^t u(t)dt = \frac{C}{J}\int_0^t e(t)dt = \frac{C}{J}\int_0^t \left[y_c(t) - y(t)\right]dt$$

Direct integration of $y_c(t)$
Negative feedback of $y(t)$

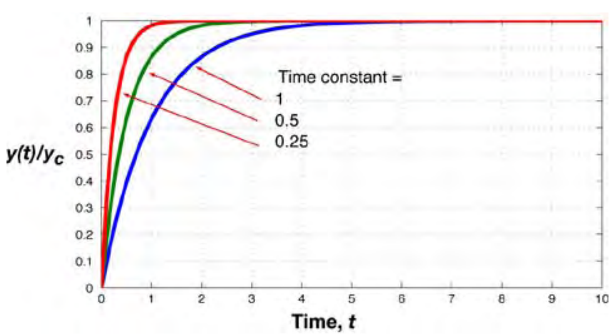


Step Response of Speed Controller

- Solution of the integral, with step command

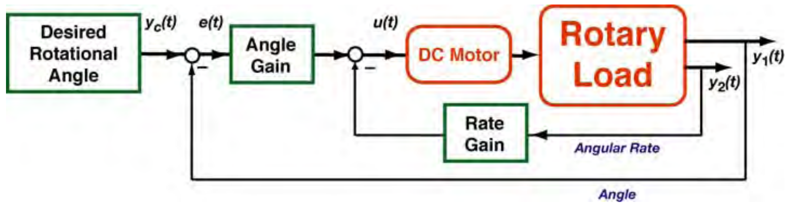
$$y_c(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases}$$

$$y(t) = y_c\left[1 - e^{-\left(\frac{C}{J}\right)t}\right] = y_c\left[1 - e^{\lambda t}\right] = y_c\left[1 - e^{-t/\tau}\right]$$



- where
  - $\lambda = -C/J$ = eigenvalue or root of the system (rad/s)
  - $\tau = J/C$ = time constant of the response (sec)

Angle Control of a DC Motor



Control law with angle and angular rate feedback

$$u(t) = c_1\left[y_c(t) - y_1(t)\right] - c_2 y_2(t)$$

Closed-loop dynamic equation, with y(*t*) = I$_2$ x(*t*)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c_1/J & -c_2/J \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ c_1/J \end{bmatrix} y_c$$

$$\omega_n = \sqrt{c_1/J}; \quad \zeta = (c_2/J)/2\omega_n$$

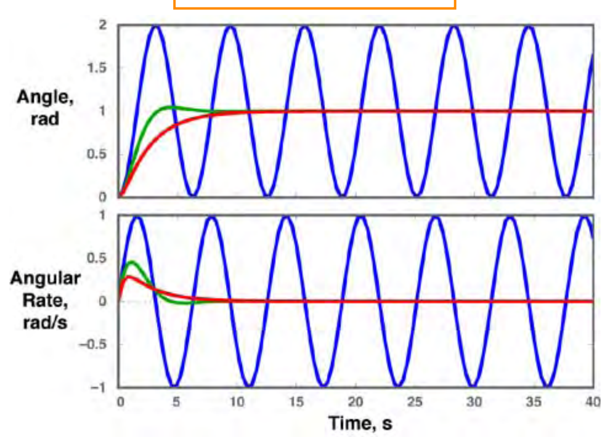
Step Response of Angle Controller, with Angle and Rate Feedback

- **Single natural frequency, three damping ratios**

$$\omega_n = \sqrt{c_1/J}; \quad \zeta = (c_2/J)/2\omega_n$$

$c_1/J = 1$
$c_2/J = 0, 1.414, 2.828$

```
%   Step Response of Damped
Angle Control

    F1  =   [0 1;-1 0];
    G1  =   [0;1];

    F1a =  [0 1;-1 -1.414];
    F1b =   [0 1;-1 -2.828];

    Hx  =   [1 0;0 1];

    Sys1    =   ss(F1,G1,Hx,0);
    Sys2    =   ss(F1a,G1,Hx,0);
    Sys3    =   ss(F1b,G1,Hx,0);

    step(Sys1,Sys2,Sys3)
```



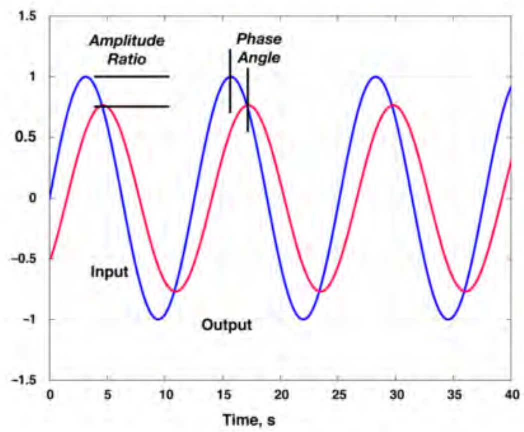
Angle Response to a Sinusoidal Angle Command

$$y_C(t) = y_{C_{peak}} \sin \omega t$$

- Output wave lags behind the input wave
- Input and output amplitudes different

$$\textbf{Amplitude Ratio}\ (AR) = \frac{y_{peak}}{y_{C_{peak}}}$$

$$\textbf{Phase Angle}(\phi) = -360 \frac{\Delta t_{peak}}{Period}, \text{deg}$$

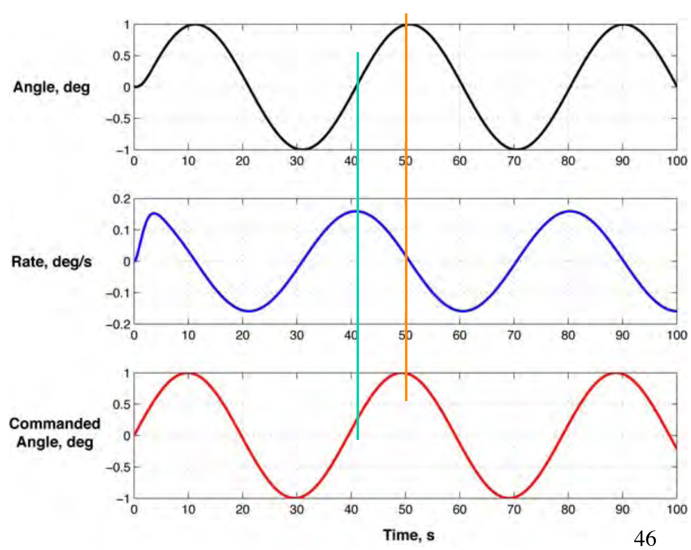


Effect of Input Frequency on Output Amplitude and Phase Angle

$$y_c(t) = \sin\left(t / 6.28\right), \text{deg}$$

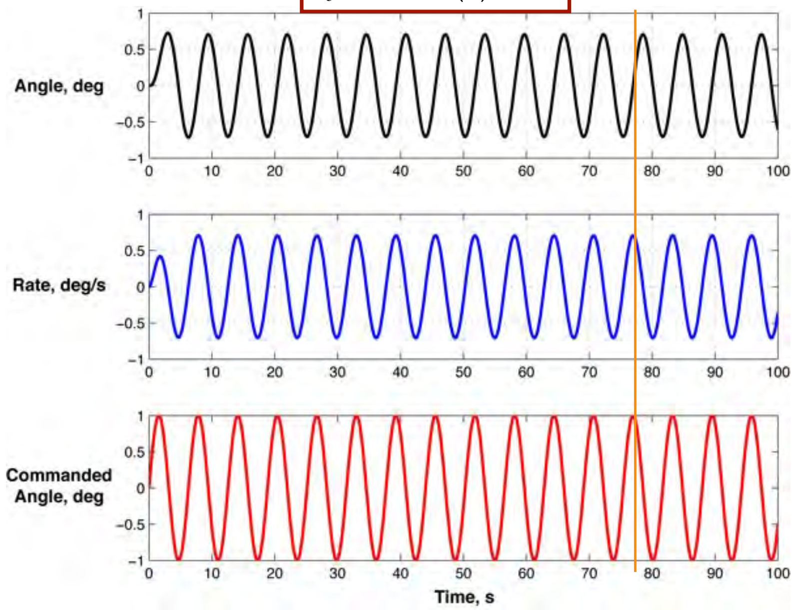
$$\omega_n = 1\ rad / s$$
$$\zeta = 0.707$$

- With low input frequency, input and output amplitudes are about the same
- Rate oscillation "leads" angle oscillation by ~90 deg
- Lag of angle output oscillation, compared to input, is small



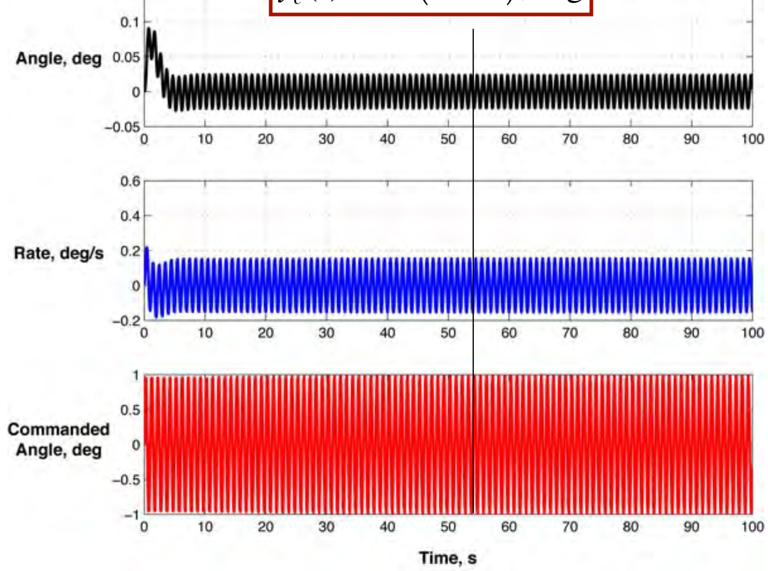
At Higher Input Frequency, Phase Angle Lag Increases

$$y_c(t) = \sin(t), \text{deg}$$



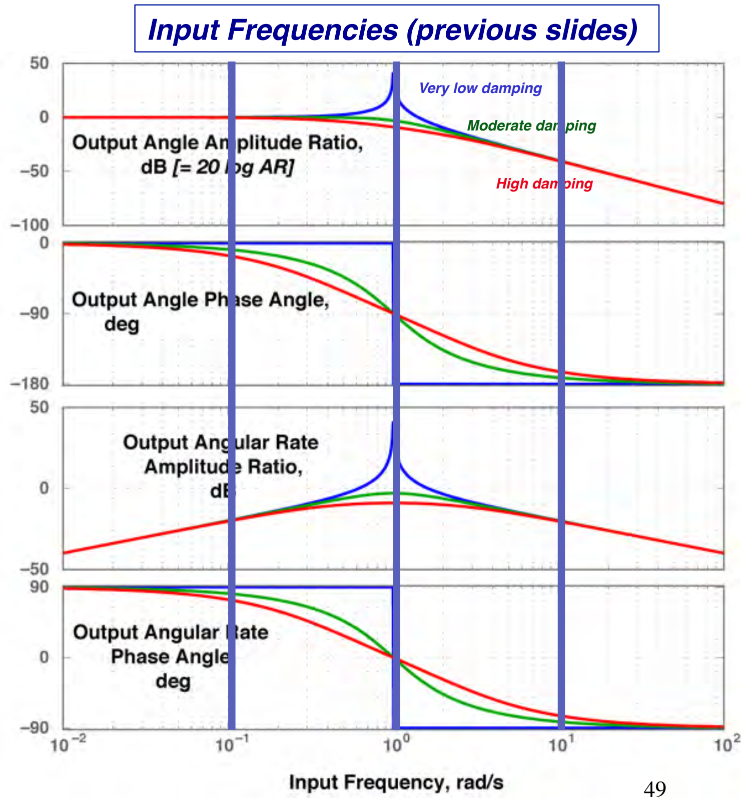
At Even Higher Frequency, Amplitude Ratio Decreases and Phase Lag Increases

$$y_c(t) = \sin(6.28t), \text{deg}$$



Angle and Rate Response of a DC Motor over Wide Input-Frequency Range

- **Long-term response of a dynamic system to sinusoidal inputs over a range of frequencies**
  - Determine **experimentally** from time response *or*
  - Compute the **Bode plot** of the system's **transfer functions** *(TBD)*



Next Time: Transfer Functions and Frequency Response

Reading:
Flight Dynamics
342-357

Supplemental Material

Example: Aerodynamic Angle, Linear Velocity, and Angular Rate Perturbations

Learjet 23
M_N = 0.3, h_N = 3,050 m
V_N = 98.4 m/s

Aerodynamic angle and linear velocity perturbations

$$\Delta\alpha \simeq \Delta w / V_N$$
$$\Delta\alpha = 1° \rightarrow \Delta w = 0.01745 \times 98.4 = 1.7\, m/s$$
$$\Delta\beta \simeq \Delta v / V_N$$
$$\Delta\beta = 1° \rightarrow \Delta v = 0.01745 \times 98.4 = 1.7\, m/s$$

Angular rate and linear velocity perturbations

$$\Delta p = 1° / s$$
$$\Delta w_{wingtip} = \Delta p \left[b / 2 \right] = 0.01745 \times 5.25 = 0.09\, m/s$$
$$\Delta q = 1° / s$$
$$\Delta w_{nose} = \Delta q \left[x_{nose} - x_{cm} \right] = 0.01745 \times 6.4 = 0.11\, m/s$$
$$\Delta r = 1° / s$$
$$\Delta v_{nose} = \Delta r \left[x_{nose} - x_{cm} \right] = 0.01745 \times 6.4 = 0.11\, m/s$$