Stochastic Control

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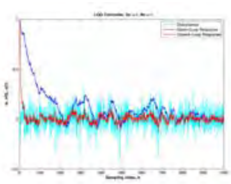
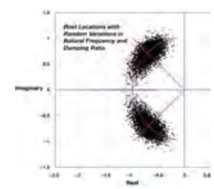
Robotics and Intelligent Systems, MAE 345, Princeton University, 2015

Learning Objectives

- **Overview of the Linear-Quadratic-Gaussian (LQG) Regulator**
- **Introduction to Stochastic Robust Control Laws**

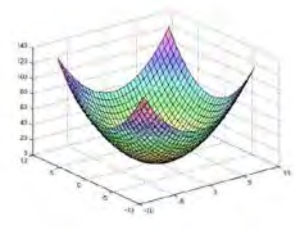




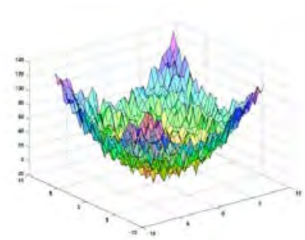
Stochastic Optimal Control

Deterministic vs. Stochastic Optimal Control

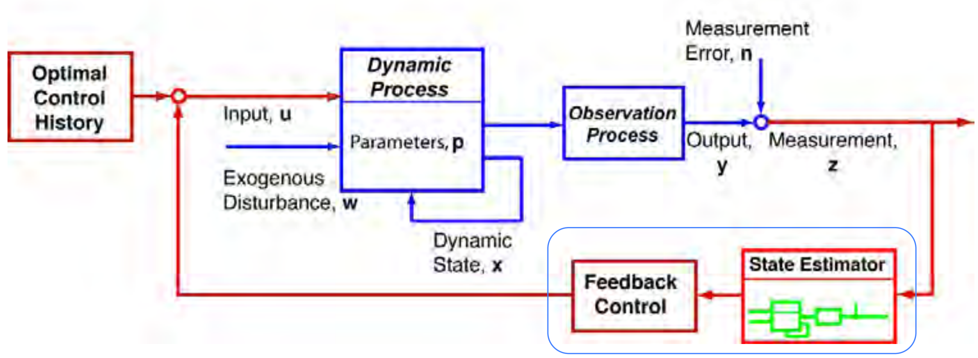
- **Deterministic control**
  - Known dynamic process
    - precise input
    - precise initial condition
    - precise measurement
  - Optimal control minimizes $J^* = J(\mathbf{x}^*, \mathbf{u}^*)$



- **Stochastic control**
  - Known dynamic process
    - unknown input
    - imprecise initial condition
    - imprecise or incomplete measurement
  - Optimal control minimizes $E\{J[\mathbf{x}^*, \mathbf{u}^*]\}$



Linear-Quadratic-Gaussian (LQG) Control of a Dynamic Process



$$\mathbf{z}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{n}(t)$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{K}(t)\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right]$$

$$\mathbf{u}(t) = -\mathbf{C}(t)\,\hat{\mathbf{x}}(t) + \mathbf{u}_{command}$$

Linear-Quadratic (LQ) Control Equations
(Continuous-Time Model)

Open-Loop System State Dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t)$$

Control Law

$$\mathbf{u}(t) = -\mathbf{C}\mathbf{x}(t) + \mathbf{u}_{command}$$

Closed-Loop System State Dynamics

$$\dot{\mathbf{x}}(t) = (\mathbf{F} - \mathbf{G}\mathbf{C})\mathbf{x}(t) + \mathbf{G}\mathbf{C}\mathbf{u}_{command}$$

Characteristic Equation

$$\left|s\mathbf{I} - (\mathbf{F} - \mathbf{G}\mathbf{C})\right| = 0$$

How many eigenvalues?

n

Stable or unstable?

Stable, with correct design criteria, F, and G

Linear-Quadratic-Gaussian (LQG) Control Equations
(Continuous-Time Model)

Open-Loop System State Dynamics and Measurement

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{L}\mathbf{w}(t)$$

$$\mathbf{z}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{n}(t)$$

State Estimate

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{K}\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right]$$

Control Law

$$\mathbf{u}(t) = -\mathbf{C}\hat{\mathbf{x}}(t) + \mathbf{u}_{command}$$

Linear-Quadratic-Gaussian (LQG) Control Equations
(Continuous-Time Model)

Closed-Loop System State and Estimate Dynamics
(neglect command)

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\left[-\mathbf{C}\hat{\mathbf{x}}(t)\right] + \mathbf{L}\mathbf{w}(t)$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\left[-\mathbf{C}\hat{\mathbf{x}}(t)\right] + \mathbf{K}\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right]$$

How many eigenvalues? ***2n***

Stable or unstable? ***TBD***

LQG Separation Property

Optimal estimation algorithm does not depend on the optimal control algorithm

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^T\mathbf{N}^{-1}(t)$$
$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{L}(t)\mathbf{W}(t)\mathbf{L}^T(t) - \mathbf{P}(t)\mathbf{H}^T\mathbf{N}^{-1}(t)\mathbf{H}\mathbf{P}(t)$$

Optimal control algorithm does not depend on the optimal estimation algorithm

$$\mathbf{C}(t) = \mathbf{R}^{-1}(t)\mathbf{G}^T(t)\mathbf{S}(t)$$
$$\dot{\mathbf{S}}(t) = -\mathbf{Q}(t) - \mathbf{F}(t)^T\mathbf{S}(t) - \mathbf{S}(t)\mathbf{F}(t) + \mathbf{S}(t)\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{G}^T(t)\mathbf{S}(t)$$

LQG Certainty Equivalence

Stochastic feedback control is computed from optimal estimate of the state

$$\boxed{\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{G}^T(t)\mathbf{S}(t)\hat{\mathbf{x}}(t) = -\mathbf{C}(t)\,\hat{\mathbf{x}}(t)}$$

Stochastic feedback control law is the same as the deterministic control law

$$\boxed{\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{G}^T(t)\mathbf{S}(t)\mathbf{x}(t) = -\mathbf{C}(t)\,\mathbf{x}(t)}$$

Asymptotic Stability of the LQG Regulator

(with no parameter uncertainty)

System Equations with LQG Control

With perfect knowledge of the system

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\left[-\mathbf{C}\hat{\mathbf{x}}(t)\right] + \mathbf{L}\mathbf{w}(t)}$$

$$\boxed{\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\left[-\mathbf{C}\hat{\mathbf{x}}(t)\right] + \mathbf{K}\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right]}$$

State estimate error

$$\boxed{\boldsymbol{\varepsilon}(t) \triangleq \mathbf{x}(t) - \hat{\mathbf{x}}(t)}$$

State estimate error dynamics

$$\boxed{\dot{\boldsymbol{\varepsilon}}(t) = (\mathbf{F} - \mathbf{K}\mathbf{H})\boldsymbol{\varepsilon}(t) + \mathbf{L}\mathbf{w}(t) - \mathbf{K}\mathbf{n}(t)}$$

Control-Loop and Estimator Eigenvalues are Uncoupled

$$\boxed{\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\boldsymbol{\varepsilon}}(t) \end{bmatrix} = \begin{bmatrix} (\mathbf{F} - \mathbf{G}\mathbf{C}) & \mathbf{G}\mathbf{C} \\ \mathbf{0} & (\mathbf{F} - \mathbf{K}\mathbf{H}) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\varepsilon}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{L} & -\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{n}(t) \end{bmatrix}}$$

Upper-block-triangular stability matrix

LQG system is stable because

- **(F – GC) is stable**
- **(F – KH) is stable**

Estimate error affects state response

$$\boxed{\dot{\mathbf{x}}(t) = (\mathbf{F} - \mathbf{G}\mathbf{C})\mathbf{x}(t) + \mathbf{G}\mathbf{C}\boldsymbol{\varepsilon}(t) + \mathbf{L}\mathbf{w}(t)}$$

Actual state does not affect error response

Disturbance affects both equally

Discrete-Time LQG Controller

Kalman filter produces state estimate

$$\boxed{\hat{\mathbf{x}}_k(-) = \mathbf{\Phi}\hat{\mathbf{x}}_{k-1}(+) - \mathbf{\Gamma}\mathbf{C}_{k-1}\hat{\mathbf{x}}_{k-1}(+)}$$

$$\boxed{\hat{\mathbf{x}}_k(+) = \hat{\mathbf{x}}_k(-) + \mathbf{K}_k\left[\mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_k(-)\right]}$$

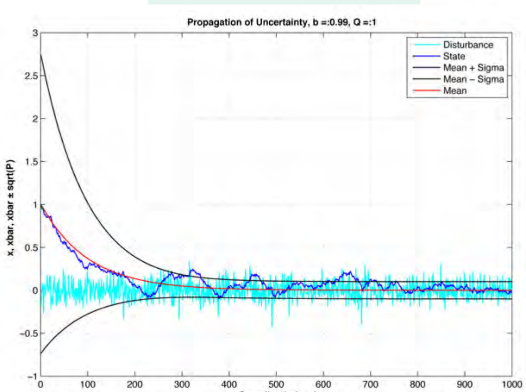
Closed-loop system uses state estimate for feedback control ($u_{command} = 0$)

$$\boxed{\mathbf{u}_k = -\mathbf{C}_k\hat{\mathbf{x}}_k(+)}$$

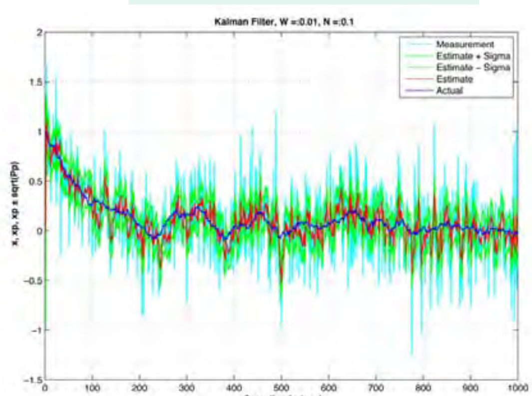
$$\boxed{\mathbf{x}_{k+1}(-) = \mathbf{\Phi}\mathbf{x}_k(-) - \mathbf{\Gamma}\mathbf{C}_k\hat{\mathbf{x}}_k(+)}$$

Response of Discrete-Time 1st-Order System to Disturbance, and Kalman Filter Estimate from Noisy Measurement

Propagation of Uncertainty

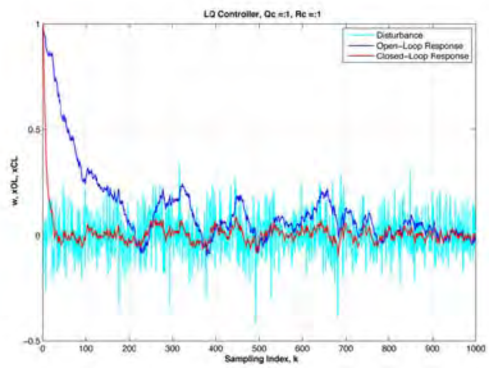


Kalman Filter, Uncontrolled System

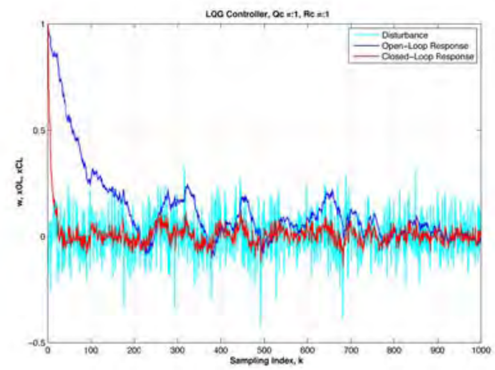


Comparison of 1st-Order Discrete-Time LQ and LQG Control Response

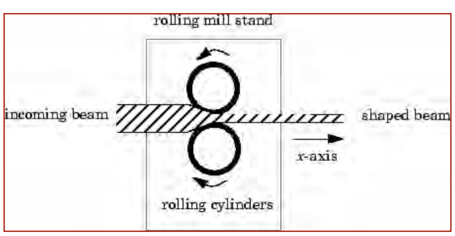
Linear-Quadratic Control with Noise-free Measurement



Linear-Quadratic-Gaussian Control with Noisy Measurement

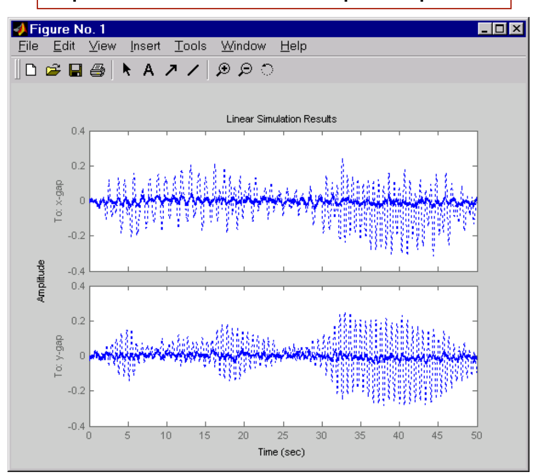


MATLAB Demo: LQG Rolling Mill Control System Design Example



- **Maintain desired thickness of shaped beam**
- **Account for random**
  - **variations in thickness/ hardness of incoming beam**
  - **eccentricity in rolling cylinders**
  - **measurement errors**

Open- and Closed-Loop Response



http://www.mathworks.com/help/control/ug/lqg-regulation-rolling-mill-example.html

Robust Stochastic Control

Stochastic, Robust, and Adaptive Control

- Stochastic controller
  - minimize response to random initial conditions, disturbances, and measurement errors
  - perfect knowledge of the plant
- Robust controller
  - fixed gains and structure
  - minimize likelihood of instability or unsatisfactory performance due to parameter uncertainty in the plant
- Adaptive controller
  - variable gains and/or structure
  - minimize likelihood of instability or unsatisfactory performance due to plant parameter uncertainty, disturbances, and measurement errors

Practical controller may have elements of all three

Robust Control System Design

- Make closed-loop response insensitive to plant parameter variations
- Robust controller
  - Fixed gains and structure
  - Minimize likelihood of instability
  - Minimize likelihood of unsatisfactory performance

Probabilistic Robust Control Design

- Design a fixed-parameter controller for stochastic robustness
- Monte Carlo Evaluation of competing designs
- Genetic Algorithm or Simulated Annealing search for best design

Representations of Uncertainty

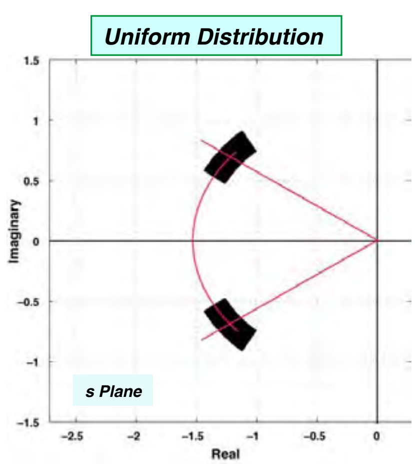
Characteristic equation of the uncontrolled system

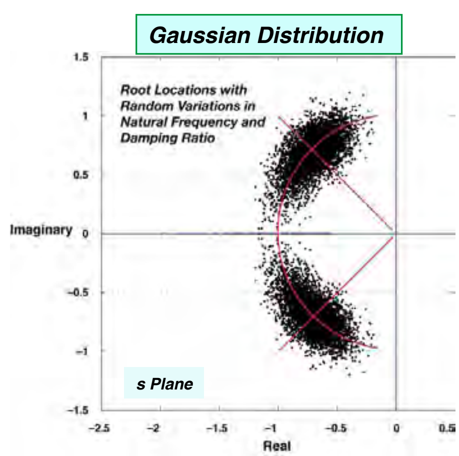
$$
\begin{aligned}
\left|s\mathbf{I}-\mathbf{F}\right| &= \det\left(s\mathbf{I}-\mathbf{F}\right) \triangleq \\
\Delta(s) &= s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \\
&= \left(s-\lambda_1\right)\left(s-\lambda_2\right)(\ldots)\left(s-\lambda_n\right) = 0
\end{aligned}
$$

- **Uncertainty can be expressed in**
  - **Elements of F**
  - **Coefficients of $\Delta(s)$**
  - **Eigenvalues of F**

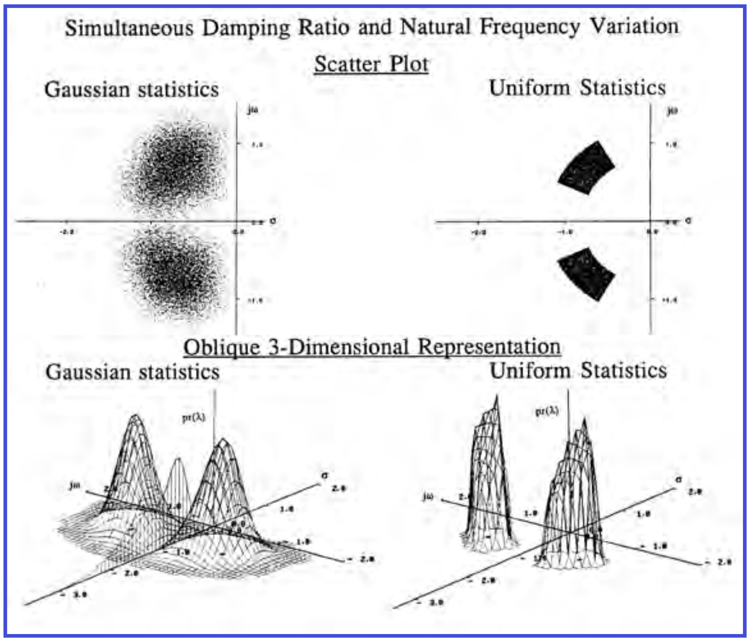
Root Locations for an Uncertain 2nd-Order System

- **Variation may be represented by**
  - **Worst-case, e.g., Upper/lower bounds of uniform distribution**
  - **Probability, e.g., Gaussian distribution**



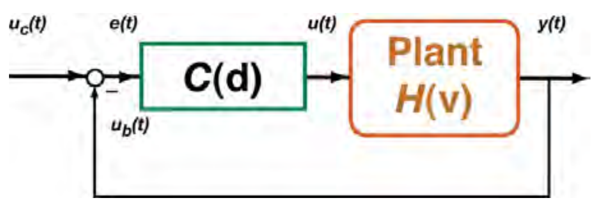


"3-D" Stochastic Root Loci for 2^{nd}-Order Example



- Root distributions are nonlinear functions of parameter distributions
- Unbounded parameter distributions always lead to non-zero probability of instability
- Bounded distributions may be guaranteed to be stable

Probability of Satisfying a Design Metric



$$Pr(\mathbf{d},\mathbf{v}) \approx \frac{1}{N}\sum_{i=1}^{N} e\left[C(\mathbf{d}),H(\mathbf{v})\right]$$

- Probability of satisfying a design metric
  - **d**: Control design parameter vector [e.g., SA, GA, …]
  - **v**: Uncertain plant parameter vector [e.g., RNG]
  - *e*: Binary indicator, e.g., 0: satisfactory 1: unsatisfactory
  - *H*(v): Plant
  - *C*(d): Controller (Compensator)

Design Control System to Minimize Probability of Instability

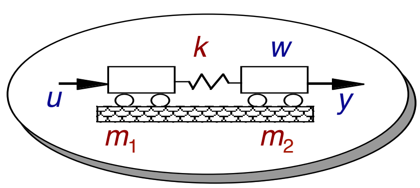
- Characteristic equation of the closed-loop system

$$\Delta_{closed-loop}(s) = \left| s\mathbf{I} - \left[\mathbf{F}(\mathbf{v}) - \mathbf{G}(\mathbf{v})\mathbf{C}(\mathbf{d}) \right] \right|$$
$$= \left[(s-\lambda_1)(s-\lambda_2)(\ldots)(s-\lambda_n) \right]_{closed-loop} = 0$$

- Monte Carlo evaluation of probability of instability with uncertain plant parameters
- Minimize probability of instability using numerical search of control parameters

$$\min_{\mathbf{d}} \left\{ \Pr\left[\mathrm{Re}\left(\lambda_i, i = 1, n \right) \right] > 0 \right\}$$

Control Design Example*



- **Challenge:** Design a feedback compensator for a 4^{th}-order spring-mass system ("the plant") whose parameters are bounded but unknown
  - Minimize the likelihood of instability
  - Satisfy a settling time requirement
  - Don't use too much control

** 1990 American Control Conference Robust Control Benchmark Problem*

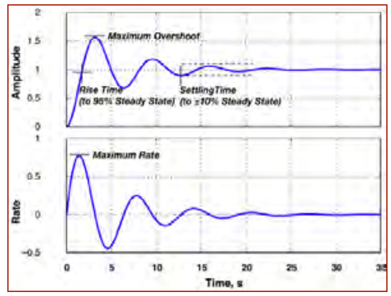
Design Cost Function

- Probability of Instability, Pr_i
  - e_i = 1 (unstable) or 0 (stable)
- Probability of Settling Time Exceedance, Pr_{ts}
  - e_{ts} = 1 (exceeded) or 0 (not exceeded)
- Probability of Control Limit Exceedance, Pr_u
  - e_u = 1 (exceeded) or 0 (not exceeded)
- Each metric has a binomial distribution

$$\mathrm{pr}(x) = \frac{n!}{k!(n-k)!} p(x)^k \left[1 - p(x)\right]^{n-k} \triangleq \begin{pmatrix} n \\ k \end{pmatrix} p(x)^k \left[1 - p(x)\right]^{n-k}$$

$=$ probability of exactly k successes in n trials, in $(0,1)$

$\sim$ normal distribution for large n

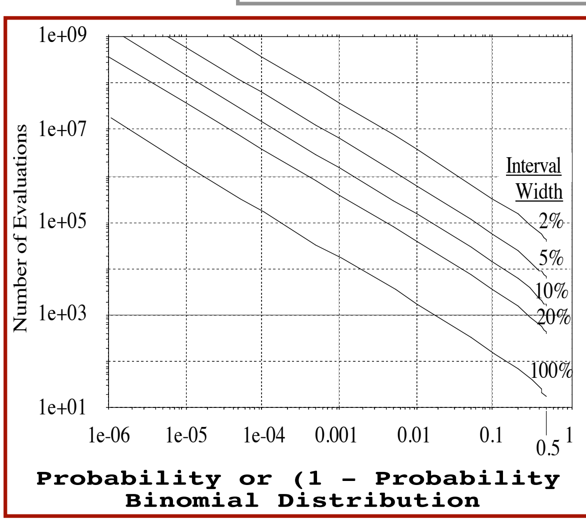


- Design Cost Function
  - *High probabilities weighted more than low probabilities*
  - $J = aPr_i^2 + bPr_{ts}^2 + c\,Pr_u^2$
  - $a = 1$
  - $b = c = 0.01$

Monte Carlo Evaluation of Probability of Satisfying a Design Metric

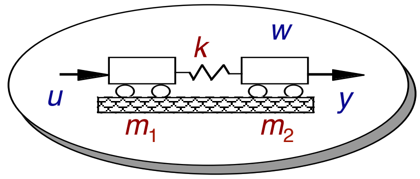
$$Pr_k(\mathbf{d},\mathbf{v}) \approx \frac{1}{N}\sum_{i=1}^{N} e_k\left[C(\mathbf{d}), H(\mathbf{v})\right], \quad k = 1,3$$

$$J = aPr_i^2(\mathbf{d},\mathbf{v}) + bPr_{ts}^2(\mathbf{d},\mathbf{v}) + cPr_u^2(\mathbf{d},\mathbf{v})$$



- Compute **v** using random number generators over N trials
  - Required number of trials depends on outcome probability and desired confidence interval
- Search for best **d** using a genetic algorithm to minimize J

Uncertain Plant*



$$\boxed{y = x_2 + n}$$

Plant dynamic equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} w$$

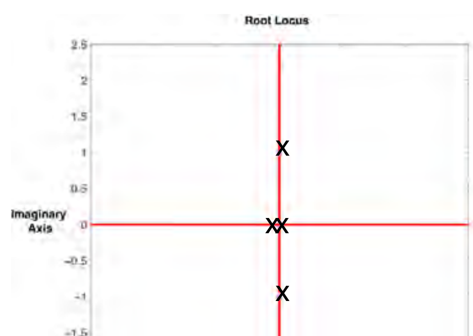
4th-Order Plant characteristic equation

$$\Delta(s) = s^2 \left[s^2 + k\frac{(m_1 + m_2)}{m_1 m_2} \right] = s^2 \left[s^2 - \omega_n^2 \right]$$

** 1990 American Control Conference Robust Control Benchmark Problem*

Parameter Uncertainties, Root Locus, and Control Law

- **Parameters of mass-spring system**
  - **Uniform probability density functions for**
    - **$0.5 < m_1, m_2 < 1.5$**
    - **$0.5 < -k < 2$**
- **Effects of parameters on root locations (right)**



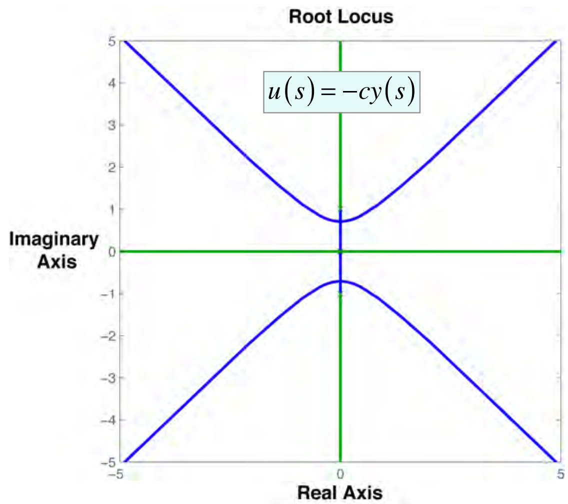
$$\Delta(s) = s^2 \left[s^2 + k\frac{(m_1 + m_2)}{m_1 m_2} \right] = s^2 \left[s^2 - \omega_n^2 \right]$$

- **Single-input/single-output feedback control law**

$$\boxed{u(s) = -C(s)y(s)}$$

Mass-Spring-Mass Stabilization Requires Compensation

- **Proportional feedback alone cannot stabilize the system**
- **Feedback of either sign drives at least one root into the right half plane**



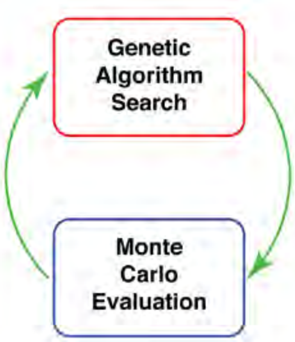
Search-and-Sweep Design of Family of Robust Feedback Compensators

Begin with lowest-order feedback compensator

$$C_{12}(s) = \frac{a_0 + a_1 s}{b_0 + b_1 s + b_2 s^2} \equiv C(\mathbf{d})$$

Arrange parameters as binary design vector

$$\mathbf{d} = \{a_0, a_1, b_0, b_1, b_2\}$$



$$\mathbf{d}^* = \{a_0^*, a_1^*, b_0^*, b_1^*, b_2^*\}$$

Search for design vector, *d*, that minimizes *J*

$$m_1 = rand(1) + 0.5$$
$$m_2 = rand(1) + 0.5$$
$$k = -1.5 * rand(1) + 0.5$$

Monte Carlo evaluation with uncertain parameters, v

Search-and-Sweep Design of Family of Robust Feedback Compensators

1) **Define next higher-order compensator**

$$C_{22}(s)=\frac{a_0+a_1 s+a_2 s^2}{b_0+b_1 s+b_2 s^2}$$

2) **Optimize over all parameters, including optimal coefficients in starting population**

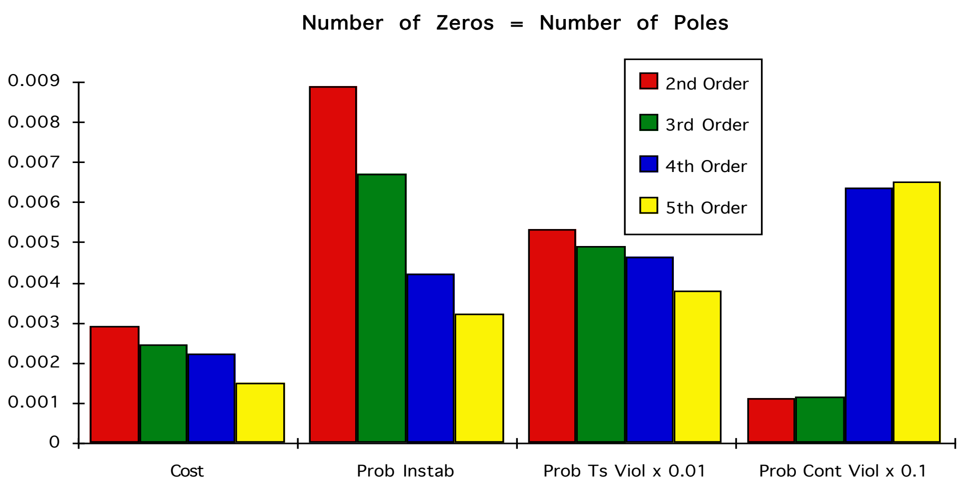
$$\mathbf{d}=\left\{a_0^*,a_1^*,a_2,b_0^*,b_1^*,b_2^*\right\}\Rightarrow \mathbf{d}^{**}=\left\{a_0^{**},a_1^{**},a_2^{**},b_0^{**},b_1^{**},b_2^{**}\right\}$$

3) **Sweep to satisfactory design or no further improvement**

$$C_{23}(s)=\frac{a_0+a_1 s+a_2 s^2}{b_0+b_1 s+b_2 s^2+b_3 s^3}\qquad C_{33}(s)=\frac{a_0+a_1 s+a_2 s^2+a_3 s^3}{b_0+b_1 s+b_2 s^2+b_3 s^3}$$

$$C_{34}(s)=\frac{a_0+a_1 s+a_2 s^2+a_3 s^3}{b_0+b_1 s+b_2 s^2+b_3 s^3+b_4 s^4}\quad \ldots$$

Design Cost and Probabilities for Optimal 2nd- to 5th-Order Compensators



Next Time:
Parameter Estimation and Adaptive Control

Supplemental Material

Linear-Quadratic Gaussian Optimal Control Law

- **Minimize expected value of cost, subject to uncertainty***

$$\boxed{\min_u E(J) \approx E(J^*)}$$

- **Stochastic optimal feedback control law combines the linear-optimal control law with a linear-optimal state estimate**

$$\boxed{\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{G}^T(t)\mathbf{S}(t)\hat{\mathbf{x}}(t) = -\mathbf{C}(t)\,\hat{\mathbf{x}}(t)}$$

  – where $\hat{\mathbf{x}}(t)$ is an optimal estimate of the state perturbation

* *See reading for details*

Example: Probability of Stable Control of an Unstable Plant

- **Longitudinal dynamics for a Forward-Swept-Wing Demonstrator**

$$\mathbf{F} = \begin{bmatrix} -2gf_{11}/V & \rho V^2 f_{12}/2 & \rho V f_{13} & -g \\ -45/V^2 & \rho V f_{22}/2 & 1 & 0 \\ 0 & \rho V^2 f_{32}/2 & \rho V f_{33} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} g_{11} & g_{12} \\ 0 & 0 \\ g_{31} & g_{32} \\ 0 & 0 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} V \\ \alpha \\ q \\ \theta \end{bmatrix}$$

- **Nominal eigenvalues (one unstable)**

$$\lambda_{1-4} = -0.1 \pm 0.057j, \quad -5.15, \quad 3.35$$

Air density and airspeed, ρ and V, have uniform distributions($\pm 30\%$)

10 coefficients have Gaussian distributions ($\sigma = 30\%$)

$$\mathbf{p} = \begin{bmatrix} \rho & V & f_{11} & f_{12} & f_{13} & f_{22} & f_{32} & f_{33} & g_{11} & g_{12} & g_{31} & g_{32} \end{bmatrix}^T$$

Environment | *Uncontrolled Dynamics* | *Control Effect*

LQ Regulators for the Example

- **Three stabilizing feedback control laws**
- **Case a) LQR with low control weighting**

$$\mathbf{Q} = diag(1,1,1,0); \quad \mathbf{R} = (1,1); \quad \lambda_{1-4_{nominal}} = \mathbf{-35,-5.1,-3.3,-.02}$$
$$\mathbf{C} = \begin{bmatrix} 0.17 & 130 & 33 & 0.36 \\ 0.98 & -11 & -3 & -1.1 \end{bmatrix}$$

- **Case b) LQR with high control weighting**

$$\mathbf{Q} = diag(1,1,1,0); \quad \mathbf{R} = (1000,1000); \quad \lambda_{1-4_{nominal}} = -5.2,-3.4,-1.1,-.02$$
$$\mathbf{C} = \begin{bmatrix} 0.03 & 83 & 21 & -0.06 \\ 0.01 & -63 & -16 & -1.9 \end{bmatrix}$$

- **Case c) Case b with gains multiplied by 5 for bandwidth (loop-transfer) recovery**

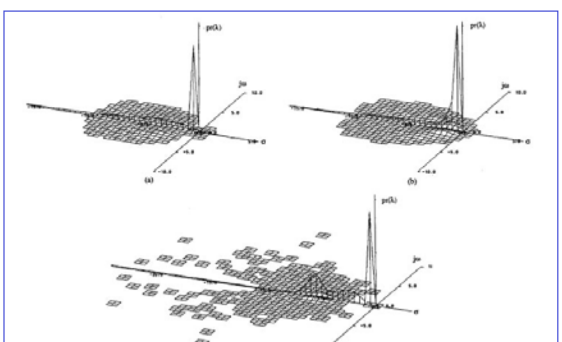
$$\lambda_{1-4_{nominal}} = \mathbf{-32,-5.2,-3.4,-0.01}$$
$$\mathbf{C} = \begin{bmatrix} 0.13 & 413 & 105 & -0.32 \\ 0.05 & -313 & -81 & -1.1-9.5 \end{bmatrix}$$

Stochastic Robustness

(Ray, Stengel, 1991)

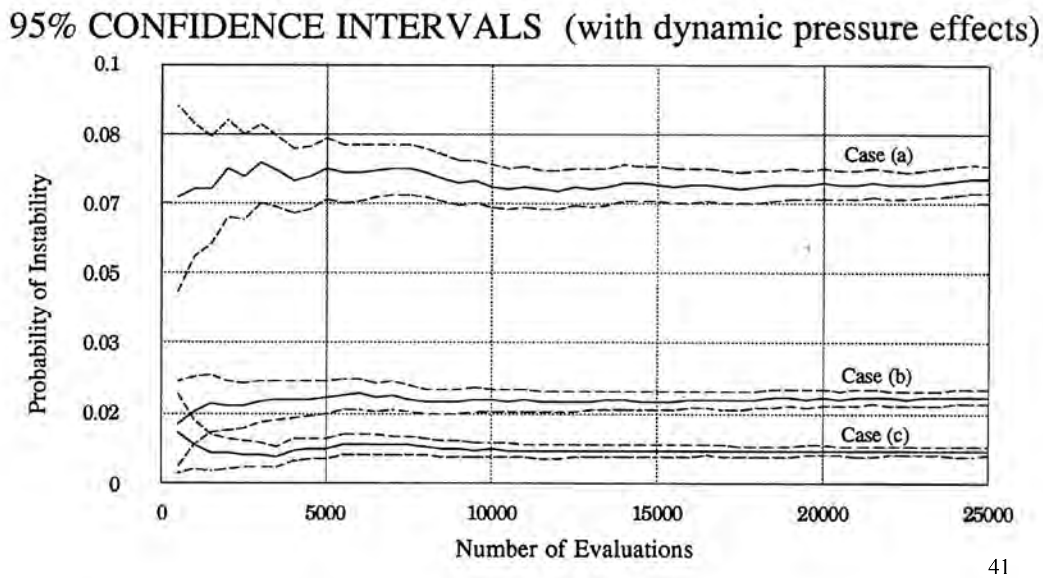
- **Distribution of closed-loop roots with**
  - **Gaussian uncertainty in 10 parameters**
  - **Uniform uncertainty in velocity and air density**
- **25,000 Monte Carlo evaluations**

- **Probability of instability**
- **a) Pr = 0.072**
- **b) Pr = 0.021**
- **c) Pr = 0.0076**

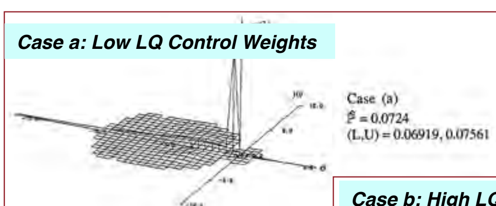


Stochastic Root Locus

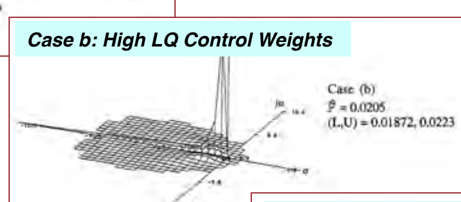
Probabilities of Instability for the Three Cases

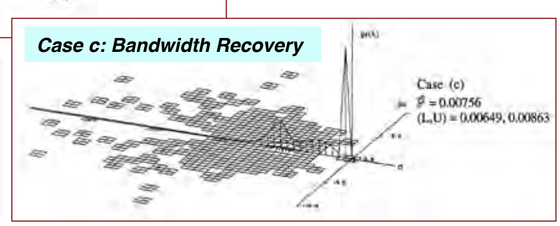


Stochastic Root Loci for the Three Cases



with Gaussian Aerodynamic Uncertainty





- Probabilities of instability with 30% uniform aerodynamic uncertainty
  - Case a: 3.4×10^{-4}
  - Case b: 0
  - Case c: 0