Advanced Problems of Lateral-Directional Dynamics

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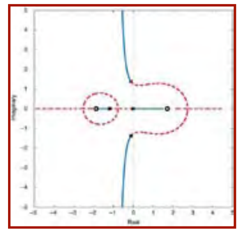
MAE 331, 2014

- Fourth-order dynamics
  - Steady-state response to control
  - Transfer functions
  - Frequency response
  - Root locus analysis of parameter variations
- Residualization
- Roll-spiral oscillation



Flight Dynamics
595-627

http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

Stability-Axis Lateral-Directional Equations

- With idealized aileron and rudder effects (i.e., $N_{\delta A} = L_{\delta R} = 0$)

$$\begin{bmatrix} \Delta\dot{r}(t) \\ \Delta\dot{\beta}(t) \\ \Delta\dot{p}(t) \\ \Delta\dot{\phi}(t) \end{bmatrix} = \left[\begin{array}{cc|cc} N_r & N_\beta & N_p & 0 \\ -1 & \dfrac{Y_\beta}{V_N} & 0 & \dfrac{g}{V_N} \\ \hline L_r & L_\beta & L_p & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} \Delta r(t) \\ \Delta\beta(t) \\ \Delta p(t) \\ \Delta\phi(t) \end{bmatrix} + \left[\begin{array}{c|c} \sim 0 & N_{\delta R} \\ 0 & 0 \\ \hline L_{\delta A} & \sim 0 \\ 0 & 0 \end{array}\right] \begin{bmatrix} \Delta\delta A \\ \Delta\delta R \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} = \begin{bmatrix} \Delta r \\ \Delta\beta \\ \Delta p \\ \Delta\phi \end{bmatrix} = \begin{bmatrix} \textit{Yaw Rate Perturbation} \\ \textit{Sideslip Angle Perturbation} \\ \hline \textit{Roll Rate Perturbation} \\ \textit{Roll Angle Perturbation} \end{bmatrix}$$

$$\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} \Delta\delta A \\ \Delta\delta R \end{bmatrix} = \begin{bmatrix} \textit{Aileron Perturbation} \\ \textit{Rudder Perturbation} \end{bmatrix}$$

Lateral-Directional Characteristic Equation

$$
\begin{aligned}
\Delta_{LD}(s) = {} & s^4 + \left(L_p + N_r + Y_\beta / V_N\right) s^3 \\
& + \left[N_\beta - L_r N_p + L_p Y_\beta / V_N + N_r \left(Y_\beta / V_N + L_p\right)\right] s^2 \\
& + \left[Y_\beta / V_N \left(L_r N_p - L_p N_r\right) + L_\beta \left(N_p - g / V_N\right)\right] s \\
& + g / V_N \left(L_\beta N_r - L_r N_\beta\right) \\
& = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0
\end{aligned}
$$

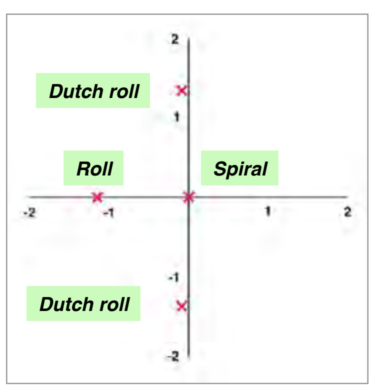
Typically factors into real spiral and roll roots and an oscillatory pair of Dutch roll roots

$$
\Delta_{LD}(s) = \left(s - \lambda_S\right)\left(s - \lambda_R\right)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{DR}
$$

Business Jet Example of Lateral-Directional Characteristic Equation

$$
\Delta_{LD}(s) = \left(s - 0.00883\right)\left(s + 1.2\right)\left[s^2 + 2\left(0.08\right)\left(1.39\right)s + 1.39^2\right]
$$

Slightly unstable Spiral | *Stable Roll* | *Lightly damped Dutch roll*



Steady-State Response

$$\boxed{\Delta \mathbf{x}_S = -\mathbf{F}^{-1}\mathbf{G}\,\Delta \mathbf{u}_S}$$

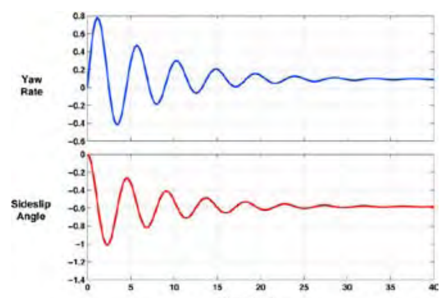
Equilibrium Response of 2nd-Order Dutch Roll Model

- Equilibrium response to constant rudder

$$\begin{bmatrix} \Delta r_{SS} \\ \Delta \beta_{SS} \end{bmatrix} = -\frac{\begin{bmatrix} \dfrac{Y_\beta}{V_N} & -N_\beta \\ 1 & N_r \end{bmatrix}}{\left(\dfrac{Y_\beta}{V_N} N_r + N_\beta\right)} \begin{bmatrix} N_{\delta R} \\ 0 \end{bmatrix} \Delta \delta R_{SS}$$

$$\Delta r_S = -\frac{\left(\dfrac{Y_\beta}{V_N} N_{\delta R}\right)}{\left(\dfrac{Y_\beta}{V_N} N_r + N_\beta\right)} \Delta \delta R_S$$

$$\Delta \beta_S = -\frac{N_{\delta R}}{\left(\dfrac{Y_\beta}{V_N} N_r + N_\beta\right)} \Delta \delta R_S$$



Steady yaw rate and sideslip angle are not zero

Equilibrium Response of Roll-Spiral Model

- **Equilibrium state with constant aileron**

$$\begin{bmatrix} \Delta p_{SS} \\ \Delta \phi_{SS} \end{bmatrix} = -\begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix} \Delta \delta A_{SS}$$

but

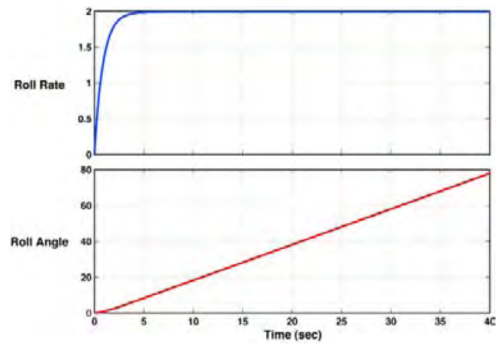
$$\begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 0 & 0 \\ -1 & L_p \end{bmatrix}}{0}$$

taken alone

$$\Delta p_S = -L_p^{-1} L_{\delta A} \Delta \delta A_S$$

$$\Delta p_S = -\frac{L_{\delta A}}{L_p} \Delta \delta A_S$$

$$\Delta \phi(t)_S = -\int_0^t \frac{L_{\delta A}}{L_p} \Delta \delta A_S \, dt$$



- **Steady roll rate proportional to aileron**
- **Roll angle, integral of roll rate, continually increases**

Equilibrium Response of 4th-Order Model

Equilibrium state with constant aileron and rudder deflection

$$\begin{bmatrix} \Delta r_S \\ \Delta \beta_S \\ \Delta p_S \\ \Delta \phi_S \end{bmatrix} = -\left[\begin{array}{cc|cc} N_r & N_\beta & N_p & 0 \\ -1 & \dfrac{Y_\beta}{V_N} & 0 & \dfrac{g}{V_N} \\ \hline L_r & L_\beta & L_p & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]^{-1} \begin{bmatrix} \sim 0 & N_{\delta R} \\ 0 & 0 \\ L_{\delta A} & \sim 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \delta A_S \\ \Delta \delta R_S \end{bmatrix}$$

Equilibrium Response of the 4th-Order Lateral-Directional Model

$$\boxed{\Delta\mathbf{y}_S = \mathbf{H_x}\Delta\mathbf{x}_S = -\mathbf{H_x}\mathbf{F}^{-1}\mathbf{G}\,\Delta\mathbf{u}_S}$$

With H_x = Identity matrix

$$\begin{bmatrix} \Delta r_S \\ \Delta\beta_S \\ \Delta p_S \\ \Delta\phi_S \end{bmatrix} = \frac{\left[\begin{array}{c|c} \frac{g}{V_N} L_{\delta A} N_\beta & -\frac{g}{V_N} L_\beta N_{\delta R} \\ \frac{g}{V_N} L_{\delta A} N_r & \frac{g}{V_N} L_r N_{\delta R} \\ 0 & 0 \\ \left(N_\beta + N_r \frac{Y_\beta}{V_N}\right) L_{\delta A} & -\left(L_\beta + L_r \frac{Y_\beta}{V_N}\right) N_{\delta R} \end{array}\right]}{\frac{g}{V_N}\left(L_\beta N_r - L_r N_\beta\right)} \begin{bmatrix} \Delta\delta A_S \\ \Delta\delta R_S \end{bmatrix}$$

Steady-state roll rate is zero

Aileron and rudder commands produce steady-state yaw rate, sideslip angle, and roll angle

Stability and Transient Response

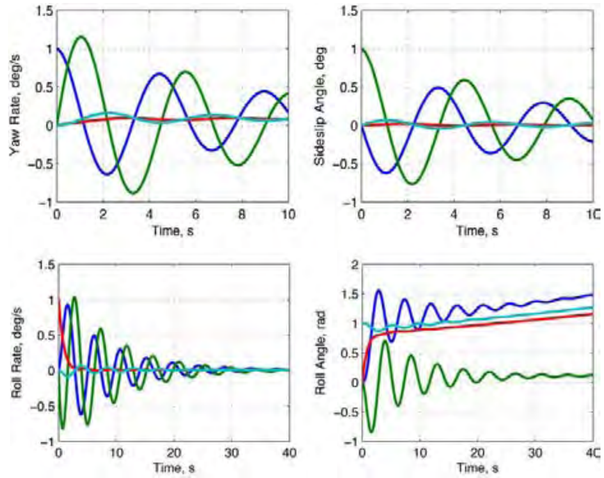
4^{th}-Order Initial-Condition Responses of Business Jet

Initial yaw rate

Initial sideslip angle

Initial roll rate

Initial roll angle



- **Initial roll angle and rate have little effect on yaw rate and sideslip angle responses**
- **Initial yaw rate and sideslip angle have large effect on roll rate and roll angle responses**

Effects of Variation in Primary Stability Derivatives

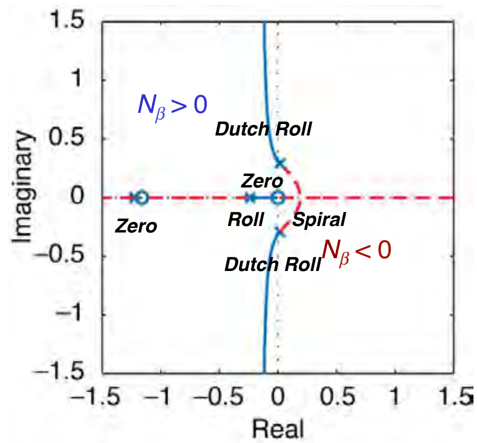
N_β Effect on 4th-Order Roots

- Group $\Delta(s)$ terms multiplied by N_β to form numerator
- Denominator formed from remaining terms of $\Delta(s)$

Root Locus Gain = Directional Stability

$$\Delta_{LD}(s) = d(s) + N_\beta n(s) = 0$$

$$k\frac{n(s)}{d(s)} = -1 = \frac{N_\beta (s - z_1)(s - z_2)}{(s-\lambda_1)(s-\lambda_2)(s^2 + 2\zeta\omega_n s + \omega_n{}^2)}$$



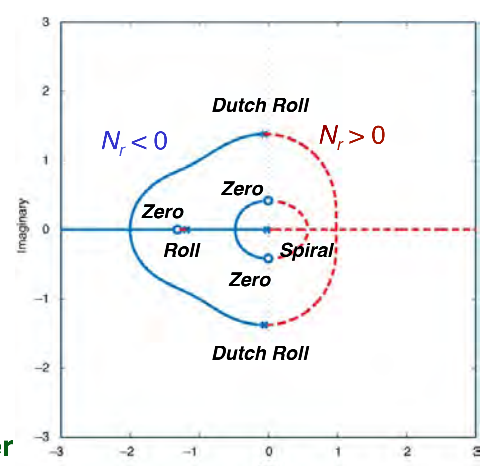
- Positive N_β
  - Increases Dutch roll natural frequency
  - Damping ratio decreases but remains stable
  - Spiral mode drawn toward origin
  - Roll mode unchanged
- Negative N_β destabilizes Dutch roll mode

N_r Effect on 4th-Order Roots

Root Locus Gain = Yaw Damping

$$\Delta_{LD}(s) = d(s) + N_r n(s) = 0$$

$$k\frac{n(s)}{d(s)} = -1 = \frac{N_r (s - z_1)(s^2 + 2\mu\nu_n s + \nu_n{}^2)}{(s-\lambda_1)(s-\lambda_2)(s^2 + 2\zeta\omega_n s + \omega_n{}^2)}$$



- Negative N_r
  - Increases Dutch roll damping
  - Draws spiral and roll modes together drawn toward origin
- Positive N_r destabilizes Dutch roll mode

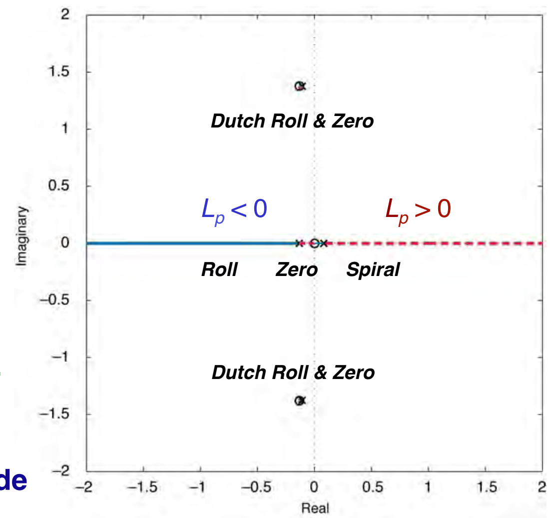
L_p Effect on 4th-Order Roots

Root Locus Gain = Roll Damping

$$\Delta_{LD}(s) = d(s) + L_p n(s) = 0$$

$$k\frac{n(s)}{d(s)} = -1 = \frac{L_p s\left(s^2 + 2\mu\nu_n s + \nu_n^2\right)}{(s-\lambda_1)(s-\lambda_2)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- Negative L_p
  - Decreases roll mode time constant
  - Draws spiral and roll modes together drawn toward origin
- Positive L_p destabilizes roll mode
- L_p has negligible effect on spiral mode
- Normally negative; however, can become positive at high angle of attack



Coupling Stability Derivatives and Their Effects

L_β Effect on 4th-Order Roots

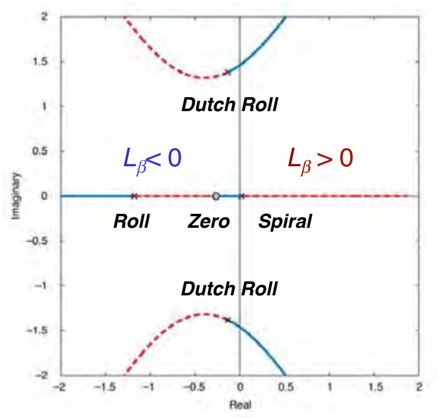
Root Locus Gain = Dihedral Effect

$$\Delta_{LD}(s) = d(s) + L_\beta\left(g/V_N - N_p\right)n(s) = 0$$

$$k\frac{n(s)}{d(s)} = -1 = \frac{L_\beta\left(g/V_N - N_p\right)\left(s - z_1\right)}{\left(s-\lambda_S\right)\left(s-\lambda_R\right)\left(s^2 + 2\zeta\omega_{n_{DR}}s + \omega_{n_{DR}}^2\right)}$$

$$\Delta_{LD}(s) = \left(s - 0.00883\right)\left(s+1.2\right)\left[s^2 + 2\left(0.08\right)\left(1.39\right)s + 1.39^2\right]$$

Bizjet Example

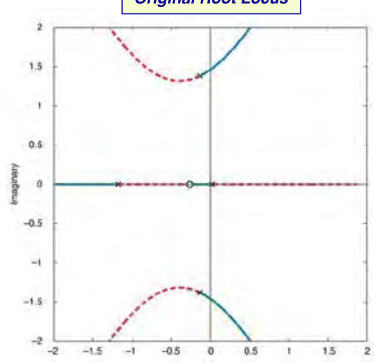


- **Negative L_β**
  - **Stabilizes spiral and roll modes but ...**
  - **Destabilizes Dutch roll mode**
- **Positive L_β does the opposite**

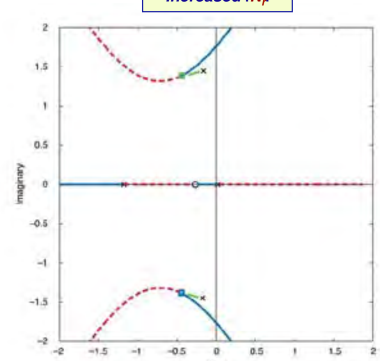
Stabilizing Lateral-Directional Motions

- **Provide sufficient L_β (–) to stabilize the spiral mode**
- **Provide sufficient N_r (–) to damp the Dutch roll mode**

Original Root Locus



Increased $|N_r|$

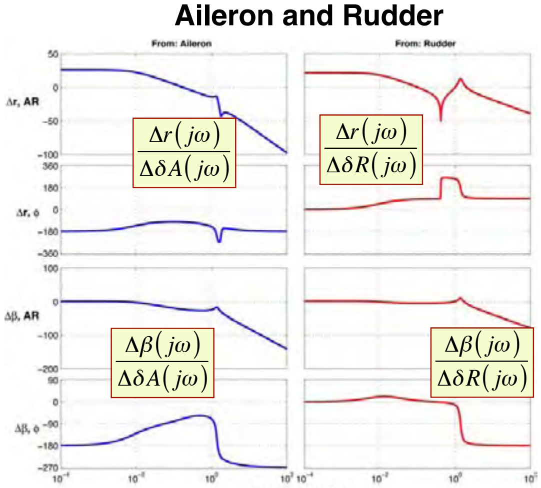


How can L_β and N_r be adjusted "artificially", i.e., by closed-loop control?

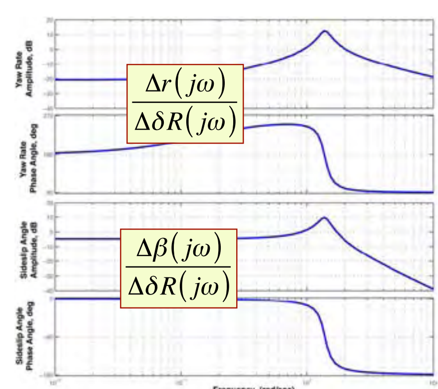
Fourth-Order Frequency Response

Yaw Rate and Sideslip Angle Frequency Responses of Business Jet

4th-Order Response to Aileron and Rudder



2nd-Order Response to Rudder



Yawing response to aileron is not negligible

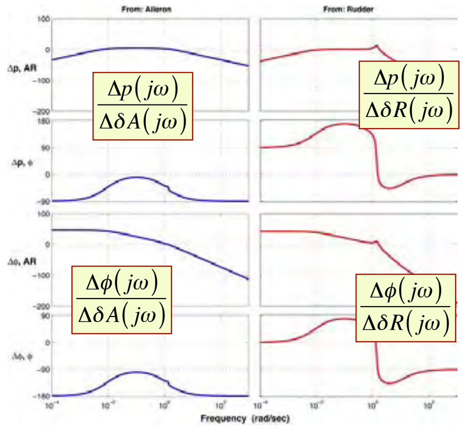
Yaw rate response is poorly characterized by the 2nd-order model below the Dutch roll natural frequency

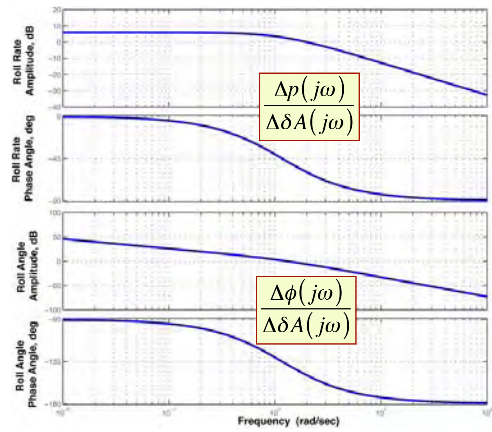
Sideslip angle response is adequately characterized by the 2nd-order model

Roll Rate and Roll Angle Frequency Responses of Business Jet

4th-Order Response to Aileron and Rudder

2nd-Order Response to Aileron



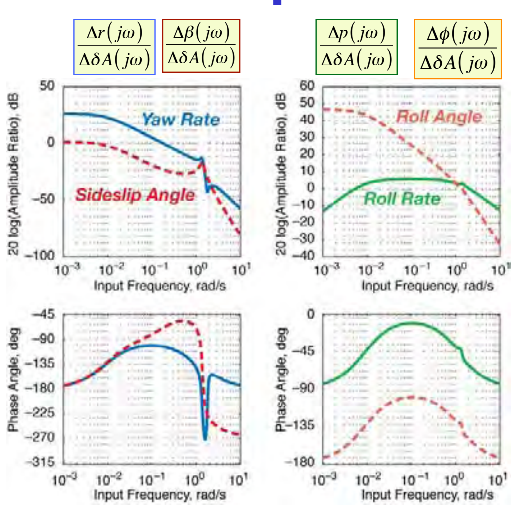


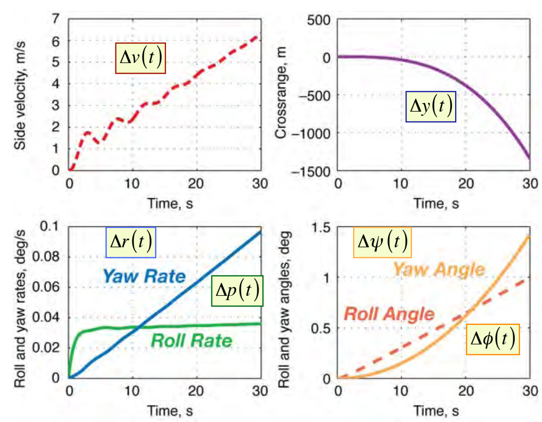
Roll response to rudder is not negligible

Roll rate response is marginally well characterized by the 2nd-order model

Roll angle response is poorly characterized at low frequency by the 2nd-order model

Frequency and Step Responses to Aileron Input



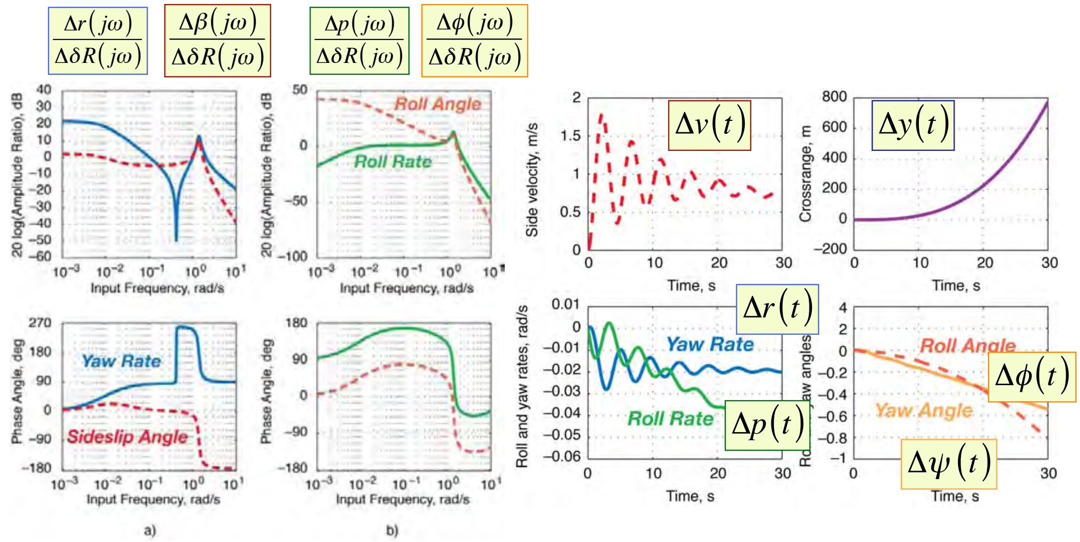


Yaw/sideslip sensitivity in the vicinity of the Dutch roll natural frequency

Roll rate response is relatively benign

Ratio of roll angle to sideslip response is important to the pilot

Frequency and Step Responses to Rudder Input



Yaw response variability near and below the Dutch roll natural frequency

Significant roll rate response near the Dutch roll natural frequency

Lightly damped yaw/sideslip response would be hard to control precisely

Reduction of Model Order by Residualization

Approximate Low-Order Response

- Dynamic model order can be reduced when
  - One mode is stable and well-damped, and it and is faster than the other
  - The two modes are coupled

$$\begin{bmatrix} \Delta\dot{\mathbf{x}}_{fast} \\ \Delta\dot{\mathbf{x}}_{slow} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{fast} & \mathbf{F}_{slow}^{fast} \\ \mathbf{F}_{fast}^{slow} & \mathbf{F}_{slow} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_{fast} \\ \Delta\mathbf{x}_{slow} \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{fast} \\ \mathbf{G}_{slow} \end{bmatrix} \Delta\mathbf{u}$$

Express as 2 separate equations

$$\begin{aligned} \Delta\dot{\mathbf{x}}_f &= \mathbf{F}_f \Delta\mathbf{x}_f + \mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}_f \Delta\mathbf{u} \\ \Delta\dot{\mathbf{x}}_s &= \mathbf{F}_f^s \Delta\mathbf{x}_f + \mathbf{F}_s \Delta\mathbf{x}_s + \mathbf{G}_s \Delta\mathbf{u} \end{aligned}$$

Approximation for Fast-Mode Response

Assume that fast mode reaches steady state very quickly compared to slow-mode response

$$\begin{aligned} \Delta\dot{\mathbf{x}}_f &\approx \mathbf{0} \approx \mathbf{F}_f \Delta\mathbf{x}_f + \mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}_f \Delta\mathbf{u} \\ \Delta\dot{\mathbf{x}}_s &= \mathbf{F}_f^s \Delta\mathbf{x}_f + \mathbf{F}_s \Delta\mathbf{x}_s + \mathbf{G}_s \Delta\mathbf{u} \end{aligned}$$

Steady-state solution for $\Delta\mathbf{x}_{fast}$

$$\begin{aligned} \mathbf{0} &\approx \mathbf{F}_f \Delta\mathbf{x}_f + \mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}_f \Delta\mathbf{u} \\ \mathbf{F}_f \Delta\mathbf{x}_f &= -\mathbf{F}_s^f \Delta\mathbf{x}_s - \mathbf{G}_f \Delta\mathbf{u} \end{aligned}$$

$$\boxed{\Delta\mathbf{x}_f = -\mathbf{F}_f^{-1}\left(\mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}_f \Delta\mathbf{u}\right)}$$

Adjust Slow-Mode Equation for Fast-Mode Steady State

Substitute quasi-steady $\Delta \mathbf{x}_{fast}$ in differential equation for $\Delta \mathbf{x}_{slow}$

$$\begin{aligned}\Delta \dot{\mathbf{x}}_s &= -\mathbf{F}_f^s \left[\mathbf{F}_f^{-1} \left(\mathbf{F}_s^f \Delta \mathbf{x}_s + \mathbf{G}_f \Delta \mathbf{u} \right) \right] + \mathbf{F}_s \Delta \mathbf{x}_s + \mathbf{G}_s \Delta \mathbf{u} \\ &= \left[\mathbf{F}_s - \mathbf{F}_f^s \mathbf{F}_f^{-1} \mathbf{F}_s^f \right] \Delta \mathbf{x}_s + \left[\mathbf{G}_s - \mathbf{F}_f^s \mathbf{F}_f^{-1-1} \mathbf{G}_f \right] \Delta \mathbf{u}\end{aligned}$$

Residualized differential equation for $\Delta \mathbf{x}_{slow}$

$$\Delta \dot{\mathbf{x}}_s = \mathbf{F}'_s \Delta \mathbf{x}_s + \mathbf{G}'_s \Delta \mathbf{u}$$

where

$$\begin{aligned}\mathbf{F}'_s &= \left[\mathbf{F}_s - \mathbf{F}_f^s \mathbf{F}_f^{-1} \mathbf{F}_s^f \right] \\ \mathbf{G}'_s &= \left[\mathbf{G}_s - \mathbf{F}_f^s \mathbf{F}_f^{-1} \mathbf{G}_f \right]\end{aligned}$$

Model of the Residualized Roll-Spiral Mode

Yawing motion is assumed to be instantaneous compared to rolling motions

Residualized roll/spiral equation

$$\begin{bmatrix} \Delta \dot{p} \\ \Delta \dot{\phi} \end{bmatrix} = \left[\begin{array}{c|c} \left[L_p - \dfrac{N_p \left(L_r \dfrac{Y_\beta}{V_N} + L_\beta \right)}{\left(N_\beta + N_r \dfrac{Y_\beta}{V_N} \right)} \right] & \left[\dfrac{\dfrac{g}{V_N} \left(L_r N_\beta - L_\beta N_r \right)}{\left(N_\beta + N_r \dfrac{Y_\beta}{V_N} \right)} \right] \\ \hline 1 & 0 \end{array} \right] \begin{bmatrix} \Delta p \\ \Delta \phi \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} f_{11} & f_{12} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta \phi \end{bmatrix} + \cdots$$

Roots of the Residualized Roll-Spiral Mode

$$\left|s\mathbf{I}-\mathbf{F}'_{RS}\right| = \left| s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} f_{11} & f_{12} \\ 1 & 0 \end{bmatrix} \right| = \Delta_{RS_{res}}$$

$$= s^2 - \left[L_p - N_p \left(\frac{L_\beta + L_r Y_\beta / V_N}{N_\beta + N_r Y_\beta / V_N} \right) \right] s + \frac{g}{V_N} \left(\frac{L_\beta N_r - L_r N_\beta}{N_\beta + N_r Y_\beta / V_N} \right)$$

$$= (s-\lambda_S)(s-\lambda_R) \quad or \quad \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{RS} = 0$$

For the business jet model

$$\Delta_{RS_{res}} = s^2 + 1.0894s - 0.0108 = 0$$

$$= \boxed{(s-0.0098)(s+1.1)} = (s-\lambda_S)(s-\lambda_R)$$

Slightly unstable spiral mode

Similar to 4th-order roll-spiral results

$$\Delta_{LD}(s) = \boxed{(s-0.00883)(s+1.2)}\left[s^2 + 2(0.08)(1.39)s + 1.39^2\right]$$

Oscillatory Roll-Spiral Mode

$$\Delta_{RS_{res}} = (s-\lambda_S)(s-\lambda_R) \quad or \quad \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{RS}$$

The characteristic equation factors into real or complex roots

Real roots are <u>roll mode</u> and <u>spiral mode</u> when

$$L_\beta N_r < L_r N_\beta$$

Complex roots produce <u>roll-spiral oscillation</u> or "lateral phugoid mode" when

$$L_\beta N_r > L_r N_\beta \quad \textbf{and}$$

$$N_p\left[\left(L_\beta + L_r Y_\beta / V_N\right) \Big/ 2\sqrt{\frac{g}{V_N}\left(L_\beta N_r - L_r N_\beta\right)} \right] < 1$$

Roll-Spiral Oscillation of the M2-F2 Lifting Body Test Vehicle

Return to Flying Qualities Criteria

Flight Dynamics
624-629

Supplemental Material

Equilibrium Response of 4th-Order Model

- Equilibrium state with constant aileron and spiral wind perturbations

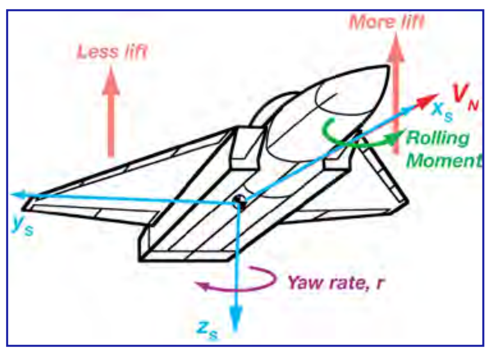
$$\begin{bmatrix} \Delta r_{SS} \\ \Delta\beta_{SS} \\ \Delta p_{SS} \\ \Delta\phi_{SS} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \\ 0 & 0 \\ e & f \end{bmatrix} \begin{bmatrix} \Delta\delta A_{SS} \\ \Delta\delta R_{SS} \end{bmatrix}$$

- **Observations**
  - Aileron command
  - Rudder command
  - Steady-state roll rate is zero
  - Steady-state roll angle is bounded

Effects of Variation in Secondary Stability Derivatives

Roll Acceleration Due to Yaw Rate, L_r

$$L_r \approx C_{l_r}\left(\frac{\rho V_N^2}{2I_{xx}}\right)Sb$$
$$= C_{l_{\hat{r}}}\left(\frac{b}{2V_N}\right)\left(\frac{\rho V_N^2}{2I_{xx}}\right)Sb = C_{l_{\hat{r}}}\left(\frac{\rho V_N}{4I_{xx}}\right)Sb^2$$



- **Wing is the principal contributor**
  - **Differential lift induced by yaw rate**

$$\left(C_{l_{\hat{r}}}\right)_{Wing} = \frac{\partial(\Delta C_l)_{Wing}}{\partial \hat{r}} = -\frac{C_{L_\alpha}}{12}\left(\frac{1+3\lambda}{1+\lambda}\right)\left(\frac{M^2\cos^2\Lambda - 2}{M^2\cos^2\Lambda - 1}\right)$$

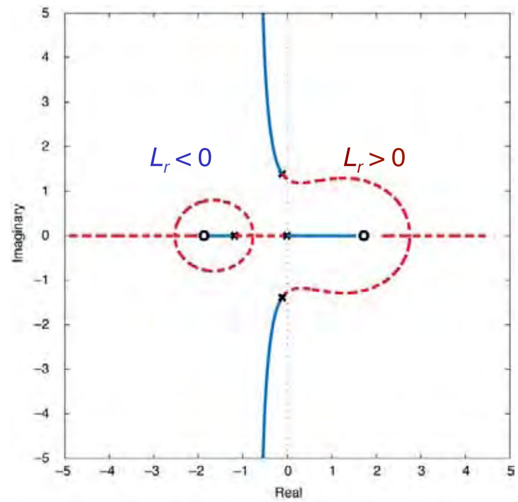
- **Thin triangular wing**

$$\left(C_{l_{\hat{r}}}\right)_{Wing} = \frac{\pi\alpha_N}{9AR}$$

- **Vertical tail**

$$\left(C_{l_{\hat{r}}}\right)_{Vertical\ Tail} = \frac{z_{vt}}{l_{vt}}\left(C_{n_{\hat{r}}}\right)_{Vertical\ Tail}$$

L_r Effect on 4th-Order Roots

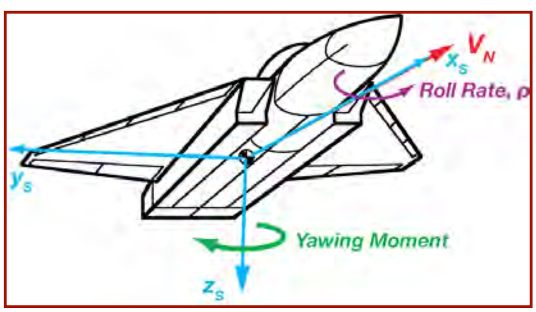


Root Locus Gain = Roll Due to Yaw Rate

$$\Delta_{LD}(s) = d(s) + L_r N_p n(s) = 0$$

$$\frac{kn(s)}{d(s)} = -1 = \frac{L_r N_p (s - z_1)(s - z_2)}{(s - \lambda_1)(s - \lambda_2)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- **Effect depends on the sign of N_p (negative here)**
- **Similar to N_β effect on the Dutch roll, but opposite to its effect on the spiral mode**



Yaw Acceleration Due to Roll Rate, N_p

$$N_p \approx C_{n_p}\left(\frac{\rho V_N^2}{2I_{zz}}\right)Sb$$
$$= C_{n_{\hat{p}}}\left(\frac{b}{2V_N}\right)\left(\frac{\rho V_N^2}{2I_{zz}}\right)Sb = C_{n_{\hat{p}}}\left(\frac{\rho V_N}{4I_{xx}}\right)Sb^2$$

Wing is the principal contributor

Differential yaw moment induced by roll rate

$$\left(C_{n_{\hat{p}}}\right)_{Wing} = \frac{\partial\left(\Delta C_n\right)_{Wing}}{\partial \hat{p}} = \frac{1}{12}\left(\frac{1+3\lambda}{1+\lambda}\right)\left(\frac{\partial C_{D_{Parasite,Wing}}}{\partial \alpha} \pm C_L\right)$$

(–): Subsonic
(+): Supersonic

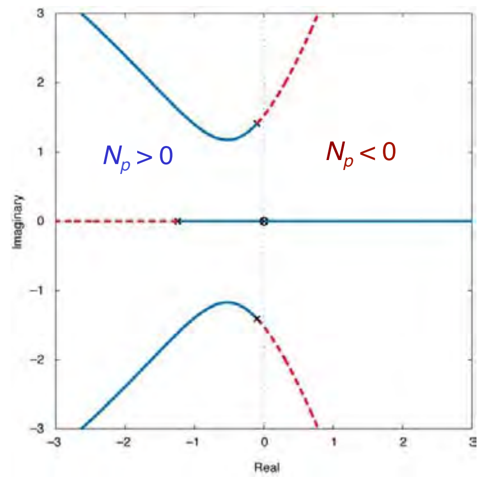
Thin triangular wing

$$\left(C_{n_{\hat{p}}}\right)_{Wing} = -\frac{\pi \alpha_N}{9AR}$$

Vertical tail

$$\left(C_{n_{\hat{p}}}\right)_{Vertical\ Tail} = -2\alpha_N\left(\frac{l_{vt}}{b}\right)\left(C_{n_\beta}\right)_{Vertical\ Tail}$$

N_p Effect on 4th-Order Roots

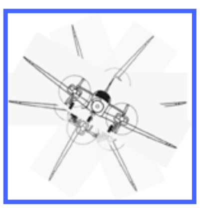


Root Locus Gain = Yaw due to Roll Rate

$$\Delta_{LD}(s) = d(s) + N_p n(s) = 0$$

$$\frac{kn(s)}{d(s)} = -1 = \frac{N_p s(s - z_1)}{(s-\lambda_1)(s-\lambda_2)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- **Tends to have opposite signs in sub- and supersonic flight**
- **Effect is analogous to L_β effect**



Approximate Roll and Spiral Modes

- **Roll rate is damped by L_p**
- **Roll angle is a pure integral of roll rate**

$$\begin{bmatrix} \Delta\dot{p} \\ \Delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta\phi \end{bmatrix} + \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix} \Delta\delta A$$

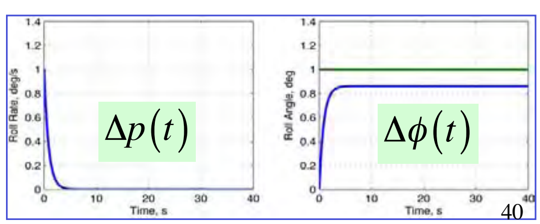
Characteristic polynomial has real roots

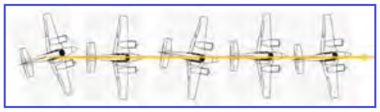
$$\Delta_{RS}(s) = s(s - L_p)$$

$$\lambda_S = 0 \quad \textit{Neutral stability}$$

$$\lambda_R = L_p \quad \textit{Generally < 0}$$

Initial condition response





Approximate Dutch Roll Mode

$$\begin{bmatrix} \Delta\dot{r} \\ \Delta\dot{\beta} \end{bmatrix} = \begin{bmatrix} N_r & N_\beta \\ \left(\frac{Y_r}{V_N} - 1\right) & \frac{Y_\beta}{V_N} \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta\beta \end{bmatrix} + \begin{bmatrix} N_{\delta R} \\ \frac{Y_{\delta R}}{V_N} \end{bmatrix} \Delta\delta R$$

- **Characteristic polynomial, natural frequency, and damping ratio**

$$\Delta_{DR}(s) = s^2 - \left(N_r + {Y_\beta}/{V_N}\right)s + \left[N_\beta\left(1 - {Y_r}/{V_N}\right) + N_r\, {Y_\beta}/{V_N}\right]$$

$$\omega_{n_{DR}} = \sqrt{N_\beta\left(1 - {Y_r}/{V_N}\right) + N_r\, {Y_\beta}/{V_N}}$$

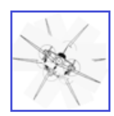
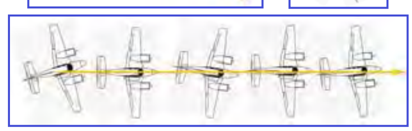
$$\zeta_{DR} = -\left(N_r + {Y_\beta}/{V_N}\right) \Big/ 2\sqrt{N_\beta\left(1 - {Y_r}/{V_N}\right) + N_r\, {Y_\beta}/{V_N}}$$

- **With negligible side-force sensitivity to yaw rate, Y_r**

$$\omega_{n_{DR}} = \sqrt{N_\beta + N_r\, {Y_\beta}/{V_N}}$$

$$\zeta_{DR} = -\left(N_r + {Y_\beta}/{V_N}\right) \Big/ 2\sqrt{N_\beta + N_r\, {Y_\beta}/{V_N}}$$

Bizjet Fourth- and Second-Order Models and Eigenvalues

Fourth-Order Model

F =				G =		Eigenvalue		Damping	Freq. (rad/s)
-0.1079	1.9011	0.0566	0	0	-1.1196	0.00883	*Unstable*		
-1	-0.1567	0	0.0958	0	0	-1.2			
0.2501	-2.408	-1.1616	0	2.3106	0	-1.16e-01 + 1.39e+00j		8.32E-02	1.39E+00
0	0	1	0	0	0	-1.16e-01 - 1.39e+00j		8.32E-02	1.39E+00

Dutch Roll Approximation

F =		G =	Eigenvalue	Damping	Freq. (rad/s)
-0.1079	1.9011	-1.1196	-1.32e-01 + 1.38e+00j	9.55E-02	1.38E+00
-1	-0.1567	0	-1.32e-01 - 1.38e+00j	9.55E-02	1.38E+00

Roll-Spiral Approximation

F =		G =	Eigenvalue	Damping	Freq. (rad/s)
-1.1616	0	2.3106	0		
1	0	0	-1.16		

- **2nd-order-model eigenvalues are close to those of the 4th-order model**
- **Eigenvalue magnitudes of Dutch roll and roll roots are similar**

Residualized Roll-Spiral Mode

- Assume that the Dutch roll mode is stable and faster than the roll mode
- Calculate effect of the quasi-steady Dutch roll on the roll and spiral modes

$$\begin{bmatrix} \Delta\dot{\mathbf{x}}_{DR} \\ \Delta\dot{\mathbf{x}}_{RS} \end{bmatrix} \approx$$

$$\begin{bmatrix} \mathbf{0} \\ \Delta\dot{\mathbf{x}}_{RS} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{DR} & \mathbf{F}_{RS}^{DR} \\ \mathbf{F}_{DR}^{RS} & \mathbf{F}_{RS} \end{bmatrix} \begin{bmatrix} \Delta x_{DR} \\ \Delta x_{RS} \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{DR} \\ \mathbf{G}_{RS} \end{bmatrix} \left[\begin{bmatrix} \Delta\delta A \\ \Delta\delta R \end{bmatrix} \right]$$

Residualized Roll-Spiral Mode

- Assume that the Dutch roll mode is stable and faster than the roll mode
- Calculate effect of the quasi-steady Dutch roll on the roll and spiral modes

$$\Delta\mathbf{x}_{DR} = -\mathbf{F}_{DR}^{-1}\left\{ \mathbf{F}_{RS}^{DR}\Delta\mathbf{x}_{RS} + \mathbf{G}_{DR}\begin{bmatrix} \Delta\delta A \\ \Delta\delta R \end{bmatrix} \right\}$$

$$\Delta\dot{\mathbf{x}}_{RS} = \mathbf{F}_{RS}\Delta\mathbf{x}_{RS} - \mathbf{F}_{DR}^{RS}\mathbf{F}_{DR}^{-1}\left\{ \mathbf{F}_{RS}^{DR}\Delta\mathbf{x}_{RS} + \mathbf{G}_{DR}\begin{bmatrix} \Delta\delta A \\ \Delta\delta R \end{bmatrix} \right\} + \mathbf{G}_{RS}\begin{bmatrix} \Delta\delta A \\ \Delta\delta R \end{bmatrix}$$

$$= \mathbf{F}'_{RS}\Delta\mathbf{x}_{RS} + \mathbf{G}'_{RS}\begin{bmatrix} \Delta\delta A \\ \Delta\delta R \end{bmatrix}$$

"Dihedral Effect": Roll Acceleration Sensitivity to Sideslip Angle, L_β

$$L_\beta \approx C_{l_\beta}\left(\frac{\rho V^2}{2I_{xx}}\right)Sb$$

Typically < 0 for stability

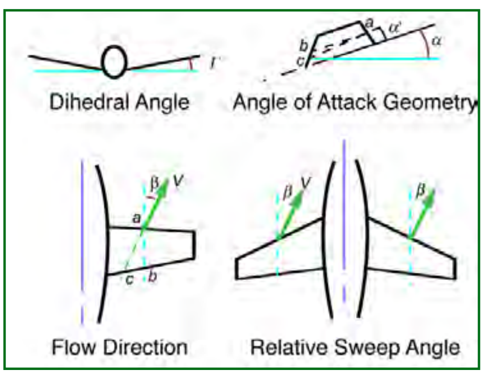
- Wing, wing-fuselage interference, and vertical tail are principal contributors

$$C_{l_\beta} \approx \left(C_{l_\beta}\right)_{Wing} + \left(C_{l_\beta}\right)_{Wing-Fuselage} + \left(C_{l_\beta}\right)_{Vertical\ Tail}$$

"Dihedral Effect": Roll Acceleration Sensitivity to Sideslip Angle, L_β

$$L_\beta \approx C_{l_\beta}\left(\frac{\rho V^2}{2I_{xx}}\right)Sb$$

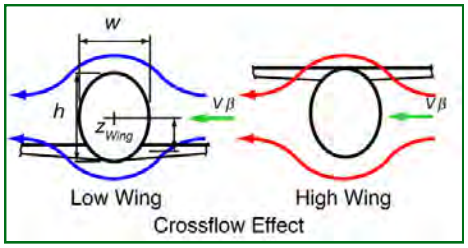
- Dihedral and sweep effect

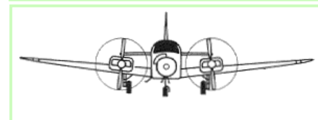


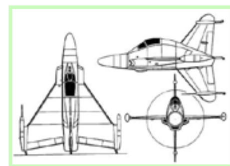
- Tapered, trapezoidal, swept wing

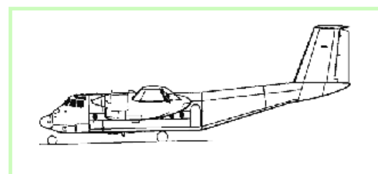
$$\left(C_{l_\beta}\right)_{Wing} = \frac{1+2\lambda}{6\left(1+\lambda\right)}\left(\Gamma C_{L_{\alpha_{wing}}} + \frac{C_L \tan\Lambda}{1-M^2\cos^2\Lambda}\right)$$

Wing and Tail Location Effects on L_β









High/low wing effect

$$\left(C_{l_\beta}\right)_{Wing-Fuselage} = 1.2\sqrt{AR}\,\frac{z_{Wing}\left(h+w\right)}{b^2}$$

Vertical tail effect

$$\left(C_{l_\beta}\right)_{Vertical\ Tail} \approx \frac{z_{vt}}{b}\left(C_{Y_\beta}\right)_{Vertical\ Tail}$$