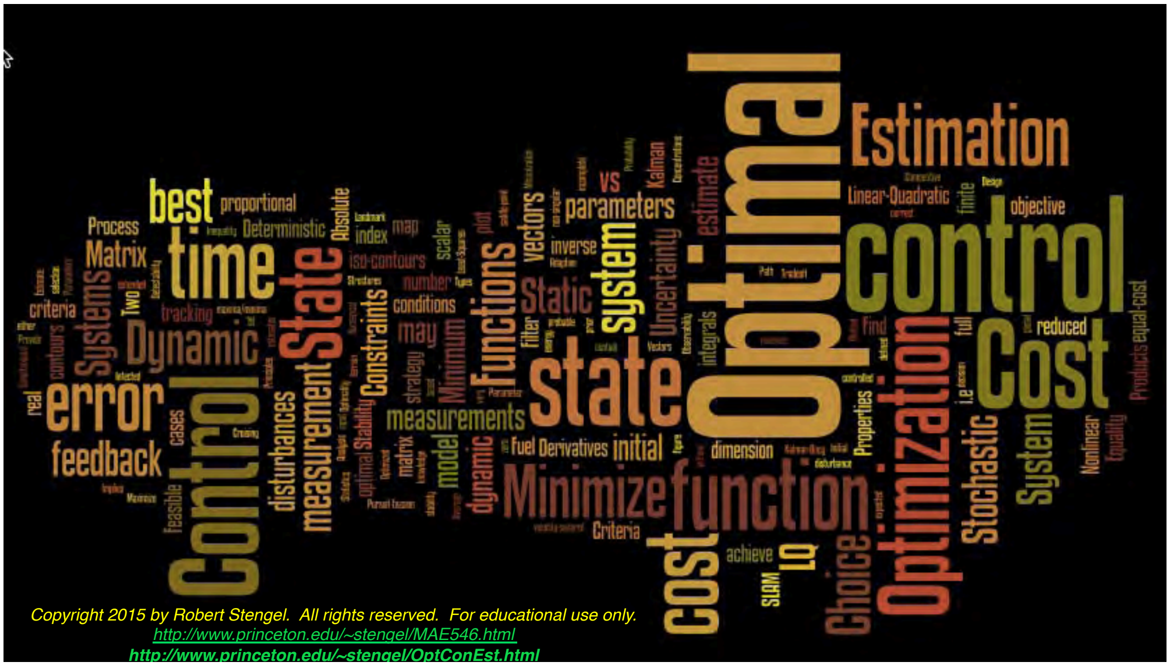
Optimal Control and Estimation

MAE 546, Princeton University
Robert Stengel, 2015



Preliminaries

Tuesday and Thursday, 3-4:20 pm
Room 305, Friend Center

- ***GRADING, tentative***
  - *Class participation: 50%*
  - *Final Project: 50%*

- ***Reference***
  - *R. Stengel, Optimal Control and Estimation, Dover, 1994*
  - *Various journal papers and book chapters*
- ***Resources***
  - *Blackboard*
    - *https://blackboard.princeton.edu/webapps/login*
  - *Course Home Page, Syllabus, and Links*
    - *www.princeton.edu/~stengel/MAE546.html*
  - *Engineering Library, including Databases and e-Journals*
    - *http://library.princeton.edu/catalogs/articles.php*
    - *http://libweb5.princeton.edu/databases/*

Syllabus - 1

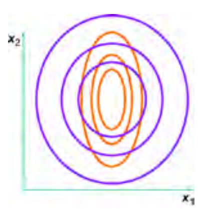
Week	Tuesday	Thursday
====	=======	========
1	Overview and Preliminaries Functions	Minimization of Static Cost
2	Principles for Optimal Control of Dynamic Systems	Principles for Optimal Control Part 2
3	Path Constraints and Numerical Optimization	Minimum-Time and -Fuel Optimization
4	Linear-Quadratic (LQ) Control	Dynamic System Stability
5	Linear-Quadratic Regulators	Cost Functions and Controller Structures
6	LQ Control System Design	Modal Properties of LQ Systems
	MID-TERM BREAK	

Syllabus - 2

Week	Tuesday	Thursday
====	=======	========
7	Spectral Properties of LQ Systems	Singular-Value Analysis
8	Probability and Statistics	Least-Squares Estimation for Static Systems
9	Propagation of Uncertainty in Dynamic Systems	Kalman Filter
10	Kalman-Bucy Filter	Nonlinear State Estimation
11	Nonlinear State Estimation	Adaptive State Estimation
12	Stochastic Optimal Control Control	Linear-Quadratic-Gaussian
	READING PERIOD	[Final Paper due on "Dean's Date"]

Seminar Format

- *Introduction*
- *Interactive Presentation and Discussion of Syllabus Topic*
  - *Course Slides*
  - *Assigned reading*
  - *Additional visuals*
- *Classes will adhere to Syllabus Schedule*
  - *Finish early, class is over*
  - *Don't finish, remainder of topics covered at your discretion reading*
- *After first few weeks, discussion led by pairs of students*
- *Discussion of recent journal papers*
- *Optional, ungraded homework assignments*

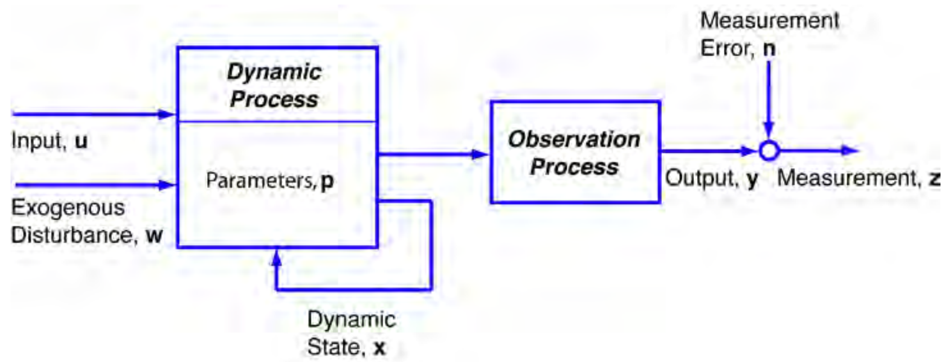


Typical Optimization Problems

- **Minimize the probable error in an estimate of the dynamic state of a system**
- **Maximize the probability of making a correct decision**
- **Minimize the time or energy required to achieve an objective**
- **Minimize the regulation error in a controlled system**

- *Estimation*
- *Control*

Dynamic Systems



Dynamic Process: Current state depends on prior state

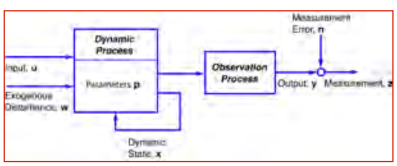
x	= dynamic state
u	= input
w	= exogenous disturbance
p	= parameter
t or *k*	= time or event index

Observation Process: Measurement may contain error or be incomplete

y	= output (error-free)
z	= measurement
n	= measurement error

- All of these quantities are **vectors**

Mathematical Models of Dynamic Systems



Dynamic Process: Current state depends on prior state

x	= dynamic state
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t	= time index

Observation Process: Measurement may contain error or be incomplete

y	= output (error-free)
z	= measurement
n	= measurement error

Continuous-time dynamic process: Vector Ordinary Differential Equation

$$\boxed{\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t]}$$

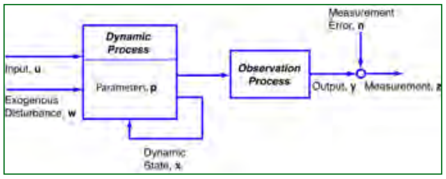
$$\boxed{t = \text{ time, s}}$$

Output Transformation

$$\boxed{\mathbf{y}(t) = \mathbf{h}[\mathbf{x}(t),\mathbf{u}(t)]}$$

Measurement with Error

$$\boxed{\mathbf{z}(t) = \mathbf{y}(t) + \mathbf{n}(t)}$$



Lateral Automobile Dynamics Example

Dynamic Process

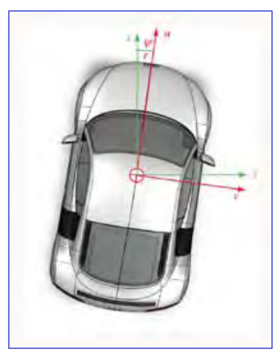
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{v}(t) \\ \dot{r}(t) \\ \dot{y}(t) \\ \dot{\psi}(t) \end{bmatrix} = \begin{bmatrix} \dfrac{Y(\mathbf{x},\mathbf{u},\mathbf{w})}{m} \\ \dfrac{N(\mathbf{x},\mathbf{u},\mathbf{w})}{I_{yy}} \\ u\sin\psi + v\cos\psi \\ r \end{bmatrix}$$

Observation Process

$$\mathbf{y} = \begin{bmatrix} y \\ \psi \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 + n_1 \\ y_2 + n_2 \end{bmatrix}$$

Example: Lateral Automobile Dynamics



Constant forward (axial) velocity, *u*

No rigid-body rolling motion

State Vector

$$\mathbf{x} = \begin{bmatrix} v \\ r \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} \text{Side velocity, m/s} \\ \text{Yaw angle rate, rad/s} \\ \text{Lateral position, m} \\ \text{Yaw angle, rad} \end{bmatrix}$$

Parameter Vector

$$\mathbf{p} = \begin{bmatrix} m \\ I_{zz} \\ \partial Y/\partial v \\ \partial Y/\partial \psi_{steer} \\ \ldots \\ \partial N/\partial v \\ \partial N/\partial \psi_{steer} \\ \ldots \end{bmatrix} = \begin{bmatrix} \text{mass, kg} \\ \text{Lateral moment of inertia, N-m} \\ \text{Side force sensitivity to side velocity, N/(m/s)} \\ \text{Side force sensitivity to steering angle, N/rad} \\ \ldots \\ \text{Yawing moment sensitivity to side velocity, N-m/(m/s)} \\ \text{Yawing moment sensitivity to steering angle, N-m/rad} \\ \ldots \end{bmatrix}$$

Control and Disturbance Vectors

$$\mathbf{u} = \psi_{steer} = \text{Steering angle, rad}$$

$$\mathbf{w} = \begin{bmatrix} v_{wind} \\ f_{road} \end{bmatrix} = \begin{bmatrix} \text{Crosswind, m/s} \\ \text{Side force on front wheel, N} \end{bmatrix}$$

Output and Measurement Vectors

$$\mathbf{y} = \begin{bmatrix} y \\ \psi \end{bmatrix} = \begin{bmatrix} \text{Lateral position, m} \\ \text{Yaw angle, rad} \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} y_{measured} \\ \psi_{measured} \end{bmatrix} = \begin{bmatrix} y + error \\ \psi + error \end{bmatrix}$$

Discrete-Time Models of Dynamic Systems

Dynamic Process: Current state depends on prior state

x	= dynamic state
u	= input
w	= exogenous disturbance
p	= parameter
t	= time index

Observation Process: Measurement may contain error or be incomplete

y	= output (error-free)
z	= measurement
n	= measurement error

Discrete-time dynamic process:
Vector Ordinary Difference Equation

$$\mathbf{x}_{k+1} == \mathbf{f}_k[\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k, \mathbf{p}_k, k]$$

$$k = \text{ time index, -}$$
$$\left(t_{k+1} - t_k\right) = \text{ time interval, s}$$

Output Transformation

$$\mathbf{y}_k = \mathbf{h}_k[\mathbf{x}_k, \mathbf{u}_k]$$

Measurement with Error

$$\mathbf{z}_k = \mathbf{y}_k + \mathbf{n}_k$$

Approximate Discrete-Time Lateral Automobile Dynamics Example

Approximate Dynamic Process (Rectangular Integration)

$$\mathbf{x}_{k+1} = \begin{bmatrix} v_{k+1} \\ r_{k+1} \\ y_{k+1} \\ \psi_{k+1} \end{bmatrix} \approx \begin{bmatrix} \dfrac{Y\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right)}{m} \\ \dfrac{N\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right)}{I_{yy}} \\ u_k \sin\psi_k + v_k \cos\psi_k \\ r_k \end{bmatrix} \left(t_{k+1} - t\ \right)$$

Observation Process

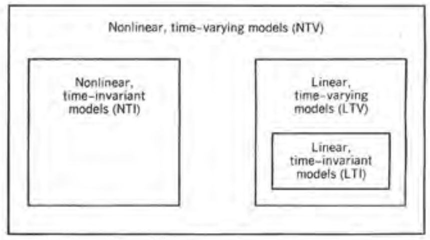
$$\mathbf{y}_k = \begin{bmatrix} y_k \\ \psi_k \end{bmatrix} = \begin{bmatrix} y_{1_k} \\ y_{2_k} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1_k} \\ x_{2_k} \\ x_{3_k} \\ x_{4_k} \end{bmatrix}$$

$$\mathbf{z}_k = \begin{bmatrix} z_{1_k} \\ z_{2_k} \end{bmatrix}$$

$$= \begin{bmatrix} y_{1_k} + n_{1_k} \\ y_{2_k} + n_{2_k} \end{bmatrix}$$

Dynamic System Model Types



- NTV
- NTI

$$\boxed{\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t\right]}$$

$$\boxed{\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t)\right]}$$

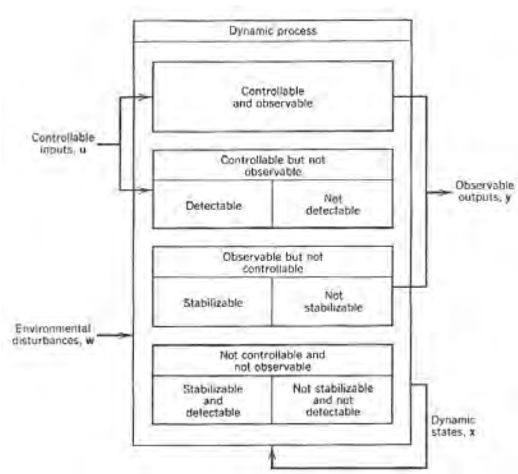
- LTV
- LTI

$$\boxed{\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t)}$$

$$\boxed{\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t) + \mathbf{L}\,\mathbf{w}(t)}$$

Controllability and Observability are Fundamental Characteristics of a Dynamic System

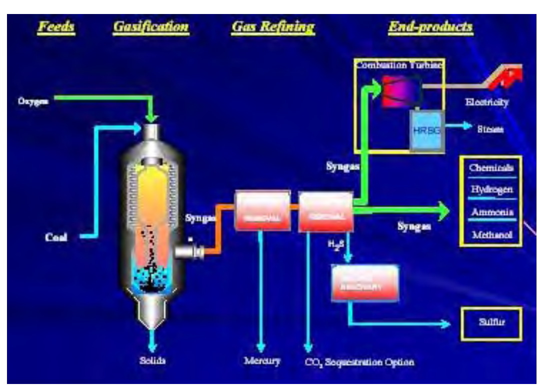
- **Controllability: State can be brought from an arbitrary initial condition to zero in finite time by the use of control**
- **Observability: Initial state can be derived from measurements over a finite time interval**
- **Subsets of the system may be either, both, or neither**
- **Effects of Stability**
  - **Stabilizability**
  - **Detectability**
- **Blocks subject to feedback control?**



Introduction to Optimization

Optimization Implies Choice

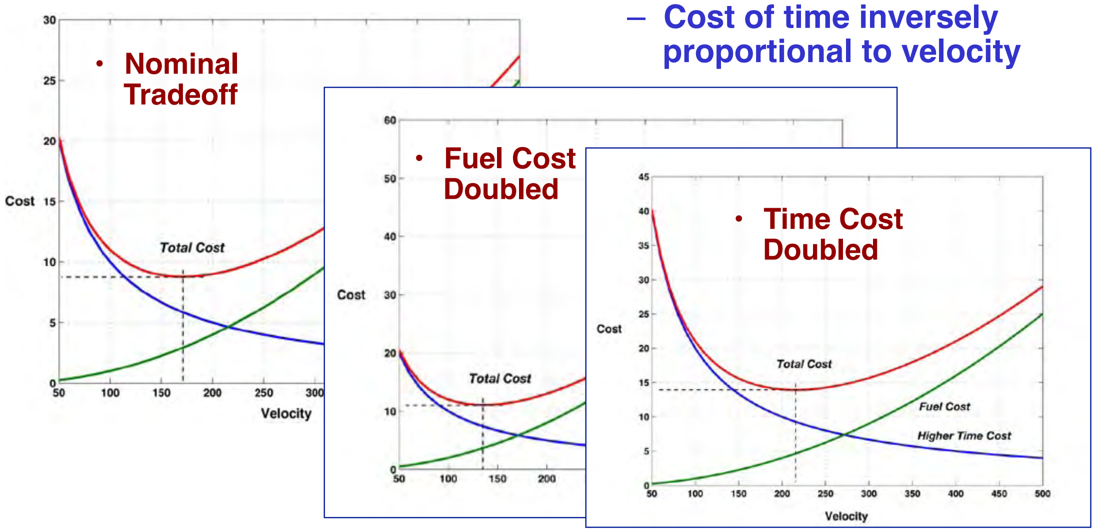
- Choice of best strategy
- Choice of best design parameters
- Choice of best control history
- Choice of best estimate
- Optimization is provided by selection of best control variable(s)



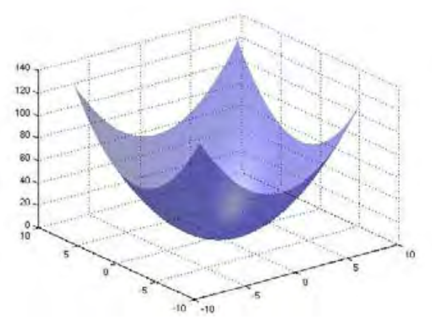
Tradeoff Between Two Cost Factors

Minimum-Cost Cruising Speed

– Fuel cost proportional to velocity-squared
– Cost of time inversely proportional to velocity



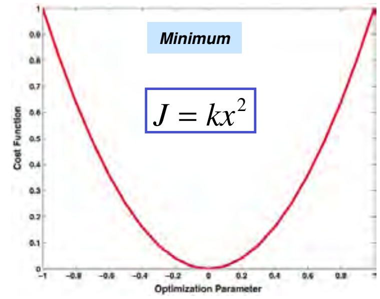
Desirable Characteristics of a Cost Function, *J*



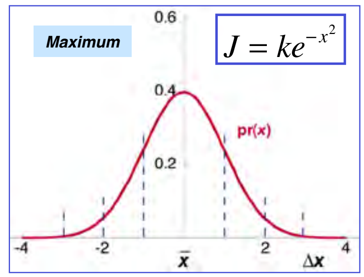
- Scalar
- Clearly defined (preferably unique) maximum or minimum
  – Local
  – Global
- Preferably positive-definite (i.e., always a positive number)

Criteria for Optimization

- Names for criteria
  - Figure of merit
  - Performance index
  - Utility function
  - Value function
  - <u>Cost function</u>, J
    - Optimal cost function = J^*
    - Optimal control = u^*

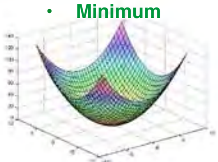


- Different criteria lead to different optimal solutions



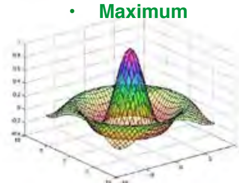
- Types of Optimality Criteria
  - Absolute
  - Regulatory
  - Feasible

- Minimum

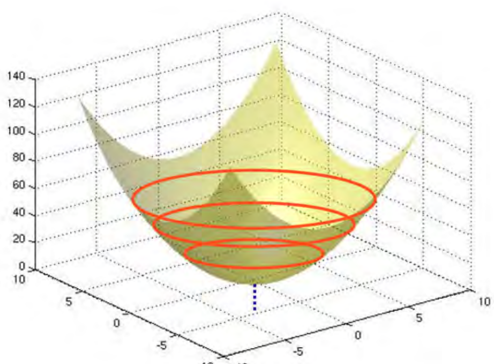


Cost Functions with Two Control Parameters

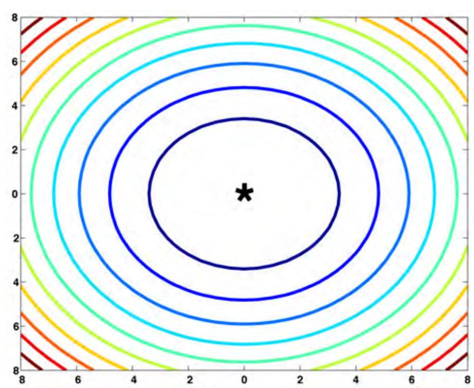
- Maximum

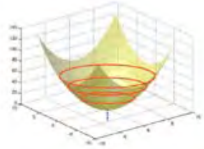


- 3-D plot of equal-cost contours (iso-contours)

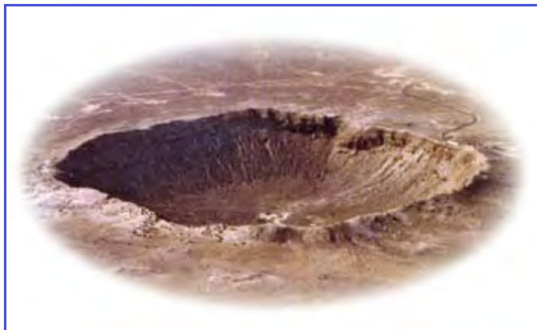


- 2-D plot of equal-cost contours (iso-contours)

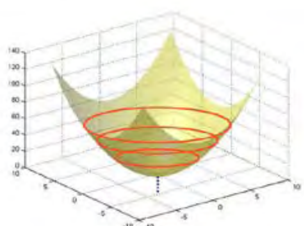




Real-World Topography

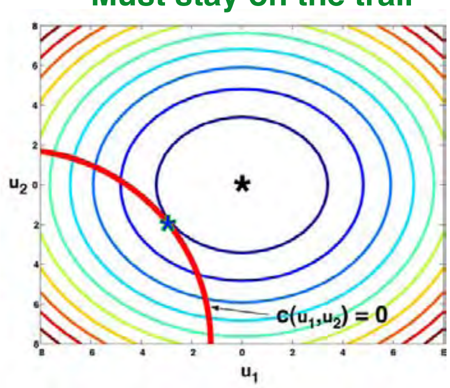


Local vs. global maxima/minima

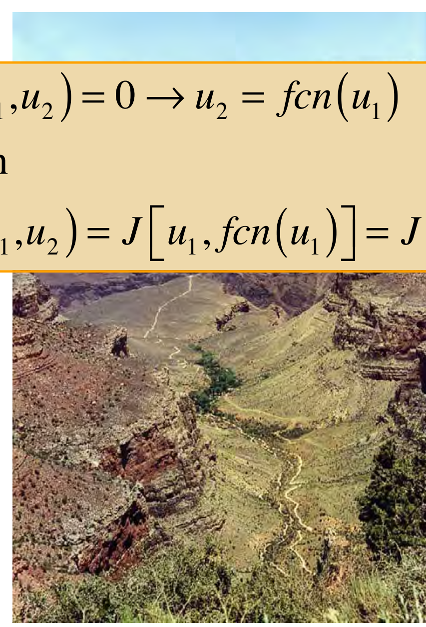


Cost Functions with Equality Constraints

- **Must stay on the trail**



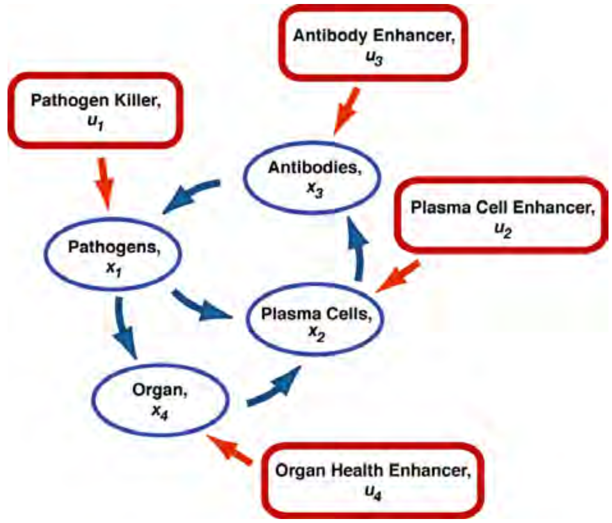
- **Equality constraint may allow control dimension to be reduced**



$$c(u_1,u_2)=0 \rightarrow u_2 = fcn(u_1)$$
then
$$J(u_1,u_2)=J\left[u_1, fcn(u_1)\right]=J'(u_1)$$

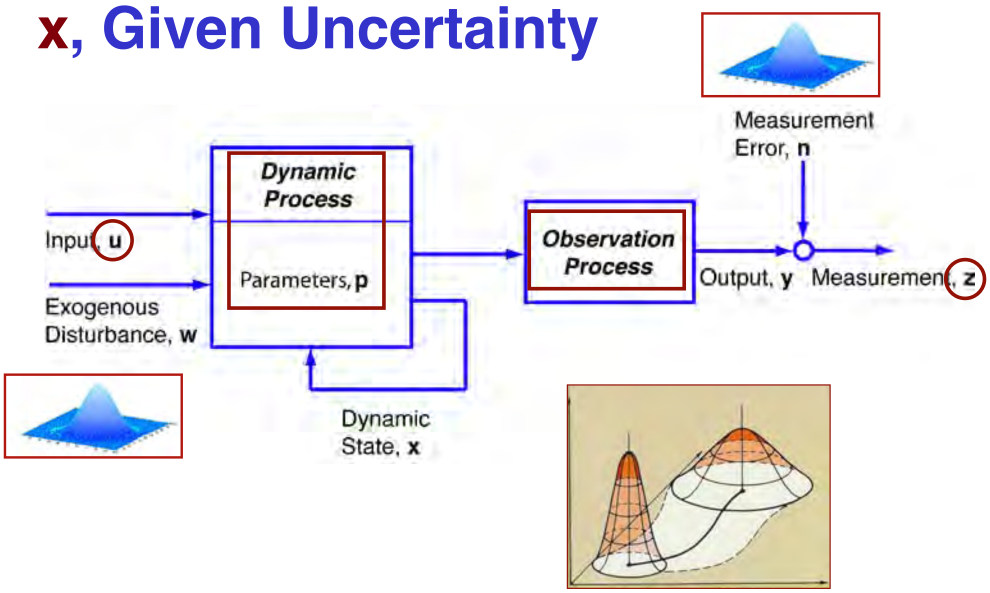
Example: Minimize Concentrations of Bacteria, Infected Cells, and Drug Usage

- x_1 = Concentration of a pathogen, which displays antigen
- x_2 = Concentration of plasma cells, which are carriers and producers of antibodies
- x_3 = Concentration of antibodies, which recognize antigen and kill pathogen
- x_4 = Relative characteristic of a damaged organ [0 = healthy, 1 = dead]

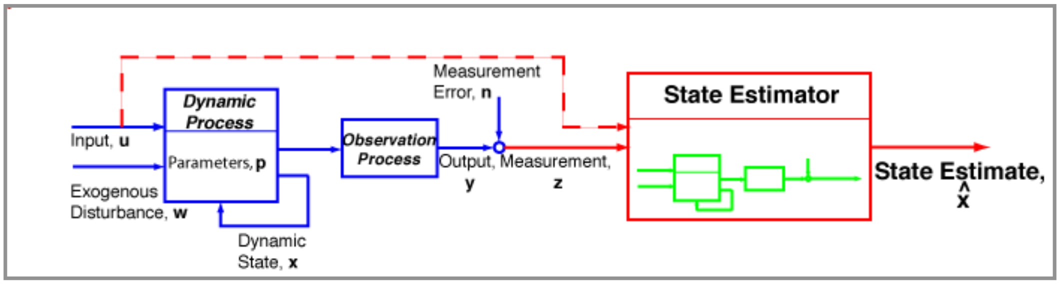


What is a reasonable cost function to minimize?

Optimal Estimate of the State, x, Given Uncertainty



Optimal State Estimation



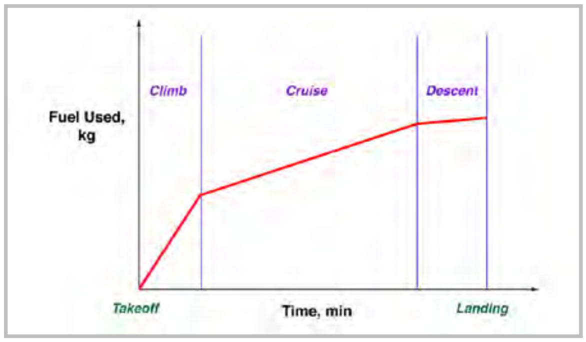
- **Goals**
  - **<u>Minimize</u> effects of measurement error on knowledge of the state**
  - **<u>Reconstruct</u> full state from reduced measurement set ($r \le n$)**
  - **<u>Average</u> redundant measurements ($r \ge n$) to estimate the full state**
- **Method**
  - **Provide optimal balance between measurements and estimates based on the dynamic model alone**

Typical Problems in Optimal Control and Estimation

Minimize an Absolute Criterion

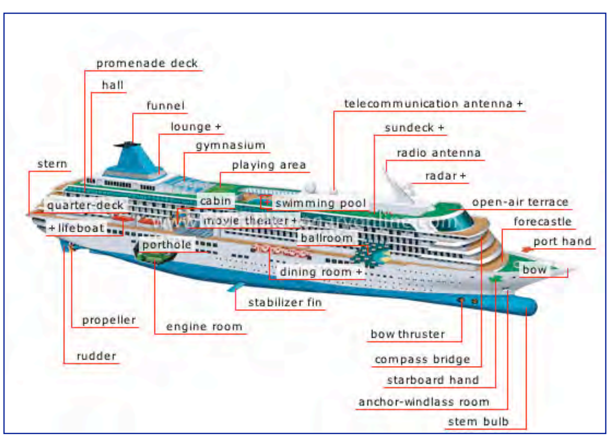
- Achieve a specific objective
  - Minimum time
  - Minimum fuel
  - Minimum financial cost
- to achieve a goal

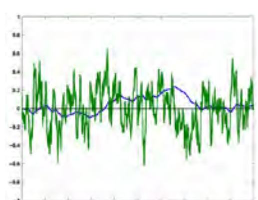
- What is the control variable?



Optimal System Regulation

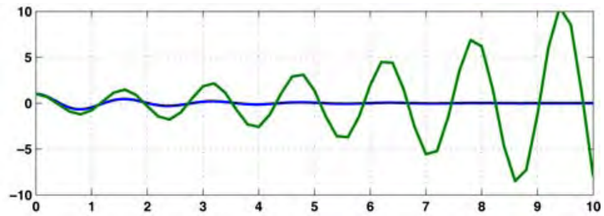
- Find feedback control gains that minimize tracking error in presence of random disturbances



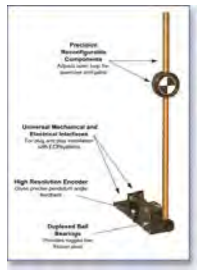


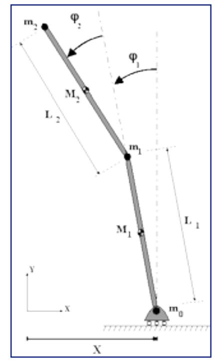
https://www.youtube.com/watch?v=bSmeQgUs6cg

Feasible Control Logic

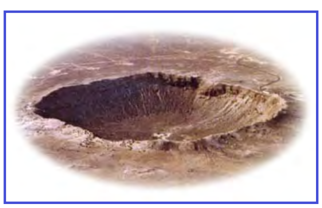


- **Find feedback control structure that guarantees stability (i.e., that keeps Δx from diverging)**



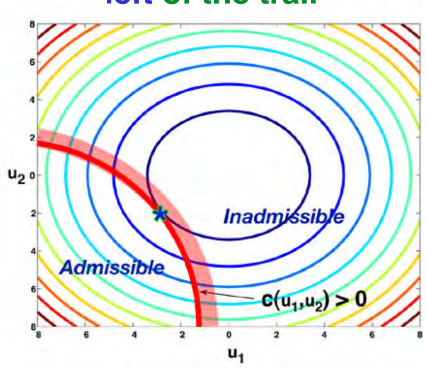


http://www.youtube.com/watch?v=8HDDzKxNMEY

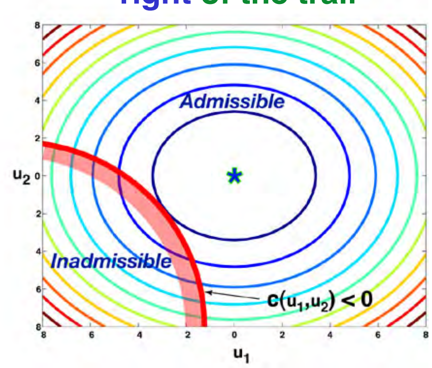


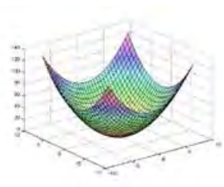
Cost Functions with Inequality Constraints

- **Must stay to the left of the trail**



- **Must stay to the right of the trail**

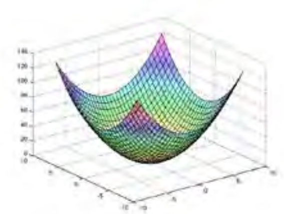




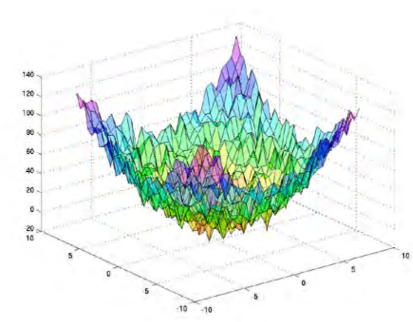
Static vs. Dynamic Optimization

- **Static**
  - **Optimal state, x*, and control, u*, are fixed, i.e., they do not change over time**
    - ***J** = J*(x*, u*)**
    - **Functional minimization (or maximization)**
    - **Parameter optimization**
- **Dynamic**
  - **Optimal state and control vary over time**
    - ***J** = J*[x*(*t*), u*(*t*)]**
    - **Optimal trajectory**
    - **Optimal feedback strategy**
- **Optimized cost function, *J**, is a scalar, real number in both cases**

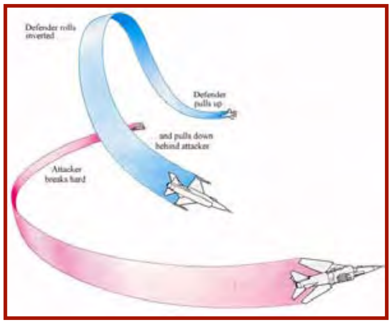
Deterministic vs. Stochastic Optimization



- **Deterministic**
  - **System model, parameters, initial conditions, and disturbances are known without error**
  - **Optimal control operates on the system with certainty**
    - ***J** = J*(x*, u*)**
- **Stochastic**
  - **Uncertainty in**
    - **system model**
    - **parameters**
    - **initial conditions**
    - **disturbances**
    - **resulting cost function**
  - **Optimal control minimizes the expected value of the cost:**
    - ***Optimal cost = E*{*J*[x*, u*]}**



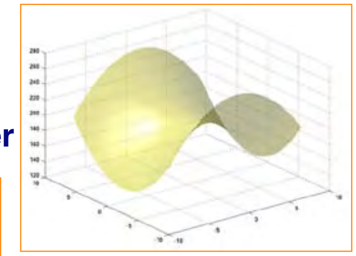
- **Cost function is a scalar, real number in both cases**



Example: Pursuit-Evasion: Competitive Optimization Problem

- Pursuer's goal: minimize final miss distance
- Evader's goal: maximize final miss distance

- "Minimax" (saddle-point) cost function

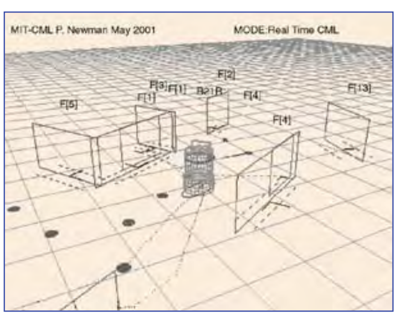


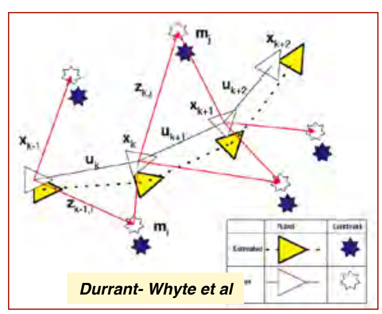
- Optimal control laws for pursuer and evader

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}_P(t) \\ \mathbf{u}_E(t) \end{bmatrix} = -\begin{bmatrix} \mathbf{C}_P(t) & \mathbf{C}_{PE}(t) \\ \mathbf{C}_{EP}(t) & \mathbf{C}_E(t) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_P(t) \\ \hat{\mathbf{x}}_E(t) \end{bmatrix}$$

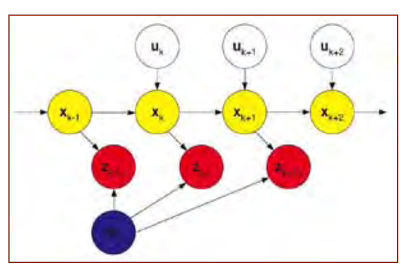
Example of a *differential game*, Isaacs (1965), Bryson & Ho (1969)

Example: Simultaneous Location and Mapping (SLAM)





- Build or update a local map within an unknown environment
  - Stochastic map, defined by mean and covariance
  - SLAM Algorithm = State estimation with extended Kalman filter
  - Landmark and terrain tracking

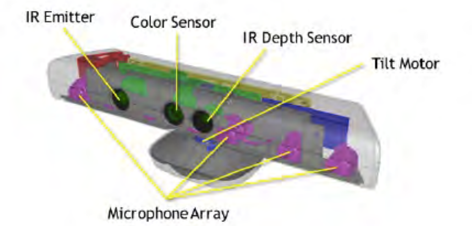


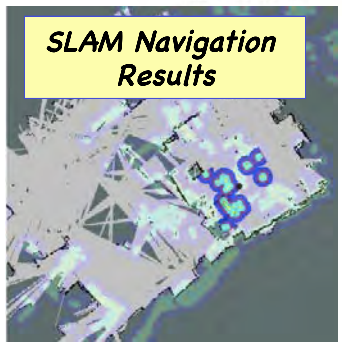
SLAM Senior Thesis Research

Quan Zhou, ‘15, David Fridovich-Keil, ‘15, and Michael Yuan,’15

Turtlebot

Kinect 1





Next Time: Minimization of Static Cost Functions

Reading: Optimal Control and Estimation (OCE): Chapter 1, Section 2.1

Supplemental Material

Seminar Grading Rubric

(from CMU, 2000)

Component	Sophisticated	Competent	Not Yet Competent	Unacceptable
Conduct	Student shows respect for members of the class, both in speech and manner, and for the method of shared inquiry and peer discussion. Does not dominate discussion. Student challenges ideas respectfully, encourages and supports others to do the same.	Student shows respect for members of the class and for the method of shared inquiry and peer discussion. Participates regularly in the discussion but occasionally has difficulty accepting challenges to his/her ideas or maintaining respectful attitude when challenging others' ideas.	Student shows little respect for the class or the process as evidenced by speech and manner. Sometimes resorts to ad hominem attacks when in disagreement with others.	Student shows a lack of respect for members of the group and the discussion process. Often dominates the discussion or disengages from the process. When contributing, can be argumentative or dismissive of others' ideas, or resorts to ad hominem attacks.
Ownership/ Leadership	Takes responsibility for maintaining the flow and quality of the discussion whenever needed. Helps to redirect or refocus discussion when it becomes sidetracked or unproductive. Makes efforts to engage reluctant participants. Provides constructive feedback and support to others.	Will take on responsibility for maintaining flow and quality of discussion, and encouraging others to participate but either is not always effective or is effective but does not regularly take on the responsibility.	Rarely takes an active role in maintaining the flow or direction of the discussion. When put in a leadership role, often acts as a guard rather than a facilitator: constrains or biases the content and flow of the discussion.	Does not play an active role in maintaining the flow of discussion or undermines the efforts of others who are trying to facilitate discussion.

Seminar Grading Rubric
(from CMU, 2000)

Reasoning	Arguments or positions are reasonable and supported with evidence from the readings. Often deepens the conversation by going beyond the text, recognizing implications and extensions of the text. Provides analysis of complex ideas that help deepen the inquiry and further the conversation.	Arguments or positions are reasonable and mostly supported by evidence from the readings. In general, the comments and ideas contribute to the group's understanding of the material and concepts.	Contributions to the discussion are more often based on opinion or unclear views than on reasoned arguments or positions based on the readings. Comments or questions suggest a difficulty in following complex lines of argument or student's arguments are convoluted and difficult to follow.	Comments are frequently so illogical or without substantiation that others are unable to critique or even follow them. Rather than critique the text the student may resort to ad hominem attacks on the author instead.
Listening	Always actively attends to what others say as evidenced by regularly building on, clarifying, or responding to their comments. Often reminds group of comments made by someone earlier that are pertinent.	Usually listens well and takes steps to check comprehension by asking clarifying and probing questions, and making connections to earlier comments. Responds to ideas and questions offered by other participants.	Does not regularly listen well as indicated by the repetition of comments or questions presented earlier, or frequent non sequiturs.	Behavior frequently reflects a failure to listen or attend to the discussion as indicated by repetition of comments and questions, non sequiturs, off-task activities.
Reading	Student has carefully read and understood the readings as evidenced by oral contributions; familiarity with main ideas, supporting evidence and secondary points. Comes to class prepared with questions and critiques of the readings.	Student has read and understood the readings as evidenced by oral contributions. The work demonstrates a grasp of the main ideas and evidence but sometimes interpretations are questionable. Comes prepared with questions.	Student has read the material, but comments often indicate that he/she didn't read or think carefully about it, or misunderstood or forgot many points. Class conduct suggests inconsistent commitment to preparation.	Student either is unable to adequately understand and interpret the material or has frequently come to class unprepared, as indicated by serious errors or an inability to answer basic questions or contribute to discussion.

Math Review

- *Scalars and Vectors*
- *Matrices, Transpose, and Trace*
- *Sums and Multiplication*
- *Vector Products*
- *Matrix Products*
- *Derivatives, Integrals, and Identity Matrix*
- *Matrix Inverse*

Scalars and Vectors

- **Scalar**: usually lower case: *a, b, c, …, x, y, z*
- **Vector**: usually bold or with underbar: **x** or <u>x</u>
  - Ordered set
  - Column of scalars
  - Dimension = *n* x 1

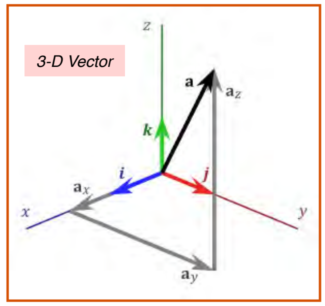
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad ; \quad \mathbf{y} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

3 x 1 *4 x 1*

- **Transpose**: interchange rows and columns

$$\mathbf{x}^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

1 x 3



Matrices and Transpose

- **Matrix**:
  - Usually bold capital or capital: **F** or F
  - Dimension = (*m* x *n*)
- **Transpose**:
  - Interchange rows and columns

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \\ l & m & n \end{bmatrix}$$

4 x 3

$$\mathbf{A}^T = \begin{bmatrix} a & d & g & l \\ b & e & h & m \\ c & f & k & n \end{bmatrix}$$

3 x 4

Trace of a Square Matrix

$$\text{Trace of } \mathbf{A} = \sum_{i=1}^{n} a_{ii}$$

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}; \quad Tr(\mathbf{A}) = a + e + i$$

Sums and Multiplication by a Scalar

- **Operands must be conformable**
- **Conformable vectors and matrices are added term by term**

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \;;\; \mathbf{z} = \begin{bmatrix} c \\ d \end{bmatrix} \;;\; \mathbf{x} + \mathbf{z} = \begin{bmatrix} (a+c) \\ (b+d) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \;;\; \mathbf{B} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \;;\; \mathbf{A} + \mathbf{B} = \begin{bmatrix} (a_1+b_1) & (a_2+b_2) \\ (a_3+b_3) & (a_4+b_4) \end{bmatrix}$$

- **Multiplication of vector by scalar is**
  - **associative**
  - **commutative**
  - **distributive**

$$a\mathbf{x} = \mathbf{x}a = \begin{bmatrix} ax_1 \\ ax_2 \\ ax_3 \end{bmatrix}$$

$$a\mathbf{x}^T = \begin{bmatrix} ax_1 & ax_2 & ax_3 \end{bmatrix}$$

Vector Products

- **Inner (dot) product of vectors produces a scalar**

$$\mathbf{x}^T\mathbf{x} = \mathbf{x}\bullet\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1^2 + x_2^2 + x_3^2)$$

$(1\times m)(m\times 1) = (1\times 1)$

- **Outer product of vectors produces a matrix**

$$\mathbf{x}\mathbf{x}^T = \mathbf{x}\otimes\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 \end{bmatrix}$$

$(m\times 1)(1\times m) = (m\times m)$

Matrix Products

- **Matrix-vector product transforms one vector into another**

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \\ l & m & n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 + cx_3 \\ dx_1 + ex_2 + fx_3 \\ gx_1 + hx_2 + kx_3 \\ lx_1 + mx_2 + nx_3 \end{bmatrix}$$

$(n\times 1) = (n\times m)(m\times 1)$

- **Matrix-matrix product produces a new matrix**

$$\boxed{\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \; ; \; \mathbf{B} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \; ; \; \mathbf{AB} = \begin{bmatrix} (a_1b_1 + a_2b_3) & (a_1b_2 + a_2b_4) \\ (a_3b_1 + a_4b_3) & (a_3b_2 + a_4b_4) \end{bmatrix}}$$

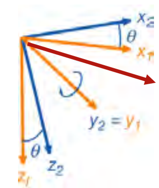
$(n\times m) = (n\times l)(l\times m)$

Examples

- **Inner product**

$$\mathbf{x}^T\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (1+4+9) = 14 = \left(\mathit{length}\right)^2$$

- **Rotation of expression for velocity vector through pitch angle**



$$\mathbf{y} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_2 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_1 = \begin{bmatrix} v_{x_1}\cos\theta + v_{z_1}\sin\theta \\ v_{y_1} \\ -v_{x_1}\cos\theta + v_{z_1}\sin\theta \end{bmatrix}$$

- **Matrix product**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$$

Vector Transformation Example

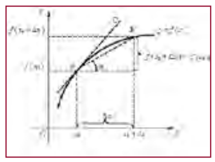
$$\boxed{\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 2 & 4 & 6 \\ 3 & -5 & 7 \\ 4 & 1 & 8 \\ -9 & -6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \boxed{= \begin{bmatrix} (2x_1 + 4x_2 + 6x_3) \\ (3x_1 - 5x_2 + 7x_3) \\ (4x_1 + x_2 + 8x_3) \\ (-9x_1 - 6x_2 - 3x_3) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}$$

$$(n \times 1) = (n \times m)(m \times 1)$$

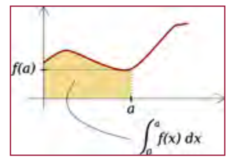
Derivatives and Integrals of Vectors

- **Derivatives and integrals of vectors are vectors of derivatives and integrals**

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{bmatrix}$$



$$\int \mathbf{x}\, dt = \begin{bmatrix} \int x_1\, dt \\ \int x_2\, dt \\ \int x_3\, dt \end{bmatrix}$$



Matrix Identity and Inverse

- **Identity matrix: no change when it multiplies a conformable vector or matrix**

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\mathbf{y} = \mathbf{I}\mathbf{y}}$$

- **A non-singular square matrix multiplied by its inverse forms an identity matrix**

$$\boxed{\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}}$$

$$\begin{aligned}
\mathbf{A}\mathbf{A}^{-1} &= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Matrix Inverse

- A non-singular square matrix multiplied by its inverse forms an identity matrix

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

- The inverse allows a reverse transformation of vectors of equal dimension

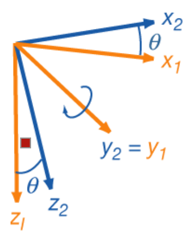
$$\mathbf{y} = \mathbf{A}\mathbf{x}; \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{y}; \quad \dim(\mathbf{x}) = \dim(\mathbf{y}) = (n \times 1); \quad \dim(\mathbf{A}) = (n \times n)$$

$$[\mathbf{A}]^{-1} = \frac{\mathrm{Adj}(\mathbf{A})}{|\mathbf{A}|} = \frac{\mathrm{Adj}(\mathbf{A})}{\det \mathbf{A}} \quad \frac{(n \times n)}{(1 \times 1)}$$

$$= \frac{\mathbf{C}^T}{\det \mathbf{A}}; \quad \mathbf{C} = \textit{matrix of cofactors}$$

Cofactors are signed minors of ***A***

*ij^{th} minor of **A** is the determinant of **A** with the i^{th} row and j^{th} column removed*



Matrix Inverse Example

Transformation

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_1$$

Inverse Transformation

$$\mathbf{x}_1 = \mathbf{A}^{-1}\mathbf{x}_2$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_1 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_2$$