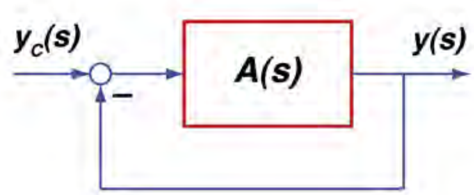
Singular Value Analysis of Linear-Quadratic Systems

Robert Stengel
Optimal Control and Estimation MAE 546
Princeton University, 2015

- Multivariable Nyquist Stability Criterion
- Matrix Norms and Singular Value Analysis
- Frequency domain measures of robustness
  - Stability Margins of Multivariable Linear-Quadratic Regulators

http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Scalar Transfer Function and Return Difference Function



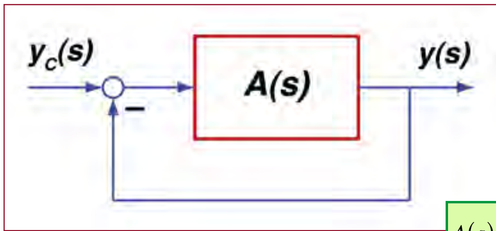
- Unit feedback control law

- Block diagram algebra

$$y(s) = A(s)\left[y_C(s) - y(s)\right]$$
$$\left[1 + A(s)\right]y(s) = A(s)y_C(s)$$
$$\frac{y(s)}{y_C(s)} = \frac{A(s)}{1 + A(s)} = A(s)\left[1 + A(s)\right]^{-1}$$

$A(s)$: **Transfer Function**

$\left[1 + A(s)\right]$: **Return Difference Function**



$$A(s)=\frac{kn(s)}{\Delta_{OL}(s)}$$

Relationship Between SISO Open- and Closed-Loop Characteristic Polynomials

$$\frac{y(s)}{y_C(s)}=\frac{kn(s)/\Delta_{OL}(s)}{\left[1+kn(s)/\Delta_{OL}(s)\right]}=\frac{kn(s)}{\Delta_{OL}(s)\left[1+kn(s)/\Delta_{OL}(s)\right]}$$
$$=\frac{kn(s)}{\left[\Delta_{OL}(s)+kn(s)\right]}=\frac{kn(s)}{\Delta_{CL}(s)}$$

- Closed-loop polynomial is open-loop polynomial multiplied by return difference function

$$\Delta_{CL}(s)=\Delta_{OL}(s)\left[1+A(s)\right]$$

Return Difference Function Matrix for the Multivariable LQ Regulator

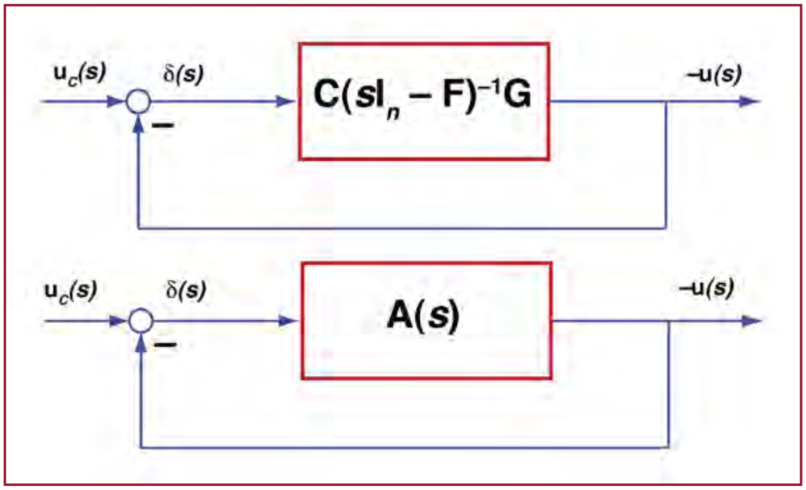
Open-loop system

$$s\Delta\mathbf{x}(s)=\mathbf{F}\Delta\mathbf{x}(s)+\mathbf{G}\Delta\mathbf{u}(s)$$
$$\left(s\mathbf{I}-\mathbf{F}\right)\Delta\mathbf{x}(s)=\mathbf{G}\Delta\mathbf{u}(s)$$
$$\Delta\mathbf{x}(s)=\left(s\mathbf{I}-\mathbf{F}\right)^{-1}\mathbf{G}\Delta\mathbf{u}(s)$$

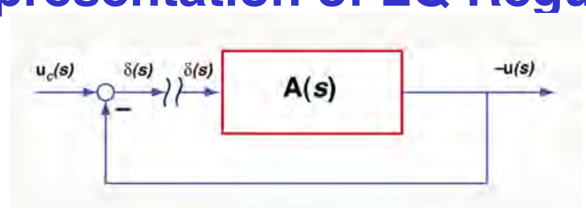
Linear-quadratic feedback control law

$$\Delta\mathbf{u}(s)=-\mathbf{R}^{-1}\mathbf{G}^T\mathbf{P}\Delta\mathbf{x}(s)\triangleq-\mathbf{C}\Delta\mathbf{x}(s)$$
$$=-\mathbf{C}\left(s\mathbf{I}-\mathbf{F}\right)^{-1}\mathbf{G}\Delta\mathbf{u}(s)\triangleq-\mathbf{A}(s)\Delta\mathbf{u}(s)$$

Multivariable LQ Regulator Portrayed as a Unit-Feedback System



Broken-Loop Analysis of Unit-Feedback Representation of LQ Regulator



Cut the loop as shown

Analyze signal flow from $\delta(s)$ to $\delta(s)$

$$\boldsymbol{\delta}(s) = \left[\mathbf{u}_C(s) - \mathbf{A}(s)\boldsymbol{\delta}(s)\right]$$

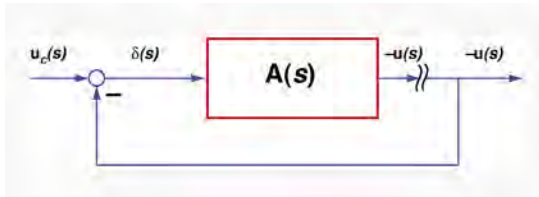
$$\left[\mathbf{I}_m + \mathbf{A}(s)\right]\boldsymbol{\delta}(s) = \mathbf{u}_C(s)$$

$$\boldsymbol{\delta}(s) = \left[\mathbf{I}_m + \mathbf{A}(s)\right]^{-1}\mathbf{u}_C(s)$$

$$-\mathbf{u}(s) = \mathbf{A}(s)\boldsymbol{\delta}(s) = \mathbf{A}(s)\left[\mathbf{I}_m + \mathbf{A}(s)\right]^{-1}\mathbf{u}_C(s)$$

Analogy to SISO closed-loop transfer function

Broken-Loop Analysis of Unit-Feedback Representation of LQ Regulator



Cut the loop as shown

Analyze signal flow from $-u(s)$ to $-u(s)$

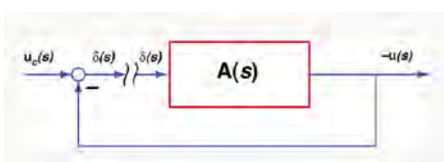
$$\mathbf{A}(s) = \mathbf{C}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}$$

$$-\mathbf{u}(s) = \mathbf{A}(s)\boldsymbol{\delta}(s) = \mathbf{A}(s)\left[\mathbf{u}_C(s) + \mathbf{u}(s)\right]$$

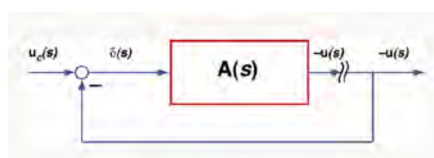
$$-\left[\mathbf{I}_m + \mathbf{A}(s)\right]\mathbf{u}(s) = \mathbf{A}(s)\mathbf{u}_C(s)$$

$$-\mathbf{u}(s) = \left[\mathbf{I}_m + \mathbf{A}(s)\right]^{-1}\mathbf{A}(s)\mathbf{u}_C(s)$$

Closed-Loop Transfer Function Matrix is Commutative



$$-\mathbf{u}(s) = \mathbf{A}(s)\left[\mathbf{I}_m + \mathbf{A}(s)\right]^{-1}\mathbf{u}_C(s)$$



$$-\mathbf{u}(s) = \left[\mathbf{I}_m + \mathbf{A}(s)\right]^{-1}\mathbf{A}(s)\mathbf{u}_C(s)$$

2nd-order example

$$\mathbf{A}\left[\mathbf{I}+\mathbf{A}\right]^{-1} = \left[\mathbf{I}+\mathbf{A}\right]^{-1}\mathbf{A}$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} a_1+1 & a_2 \\ a_3 & a_4+1 \end{bmatrix}^{-1} = \begin{bmatrix} a_1+1 & a_2 \\ a_3 & a_4+1 \end{bmatrix}^{-1}\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

$$= \begin{bmatrix} (a_1 a_4 - a_2 a_3 + a_1) & 1 \\ a_3 & (a_1 a_4 - a_2 a_3 + a_4) \end{bmatrix} \Big/ \det\left(\mathbf{I}_2 + \mathbf{A}\right)$$

Relationship Between Multi-Input/Multi-Output (MIMO) Open- and Closed-Loop Characteristic Polynomials

$$\left|\mathbf{I}_m + \mathbf{A}(s)\right| = \left|\mathbf{I}_m + \mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}\right|$$
$$= \left|\mathbf{I}_m + \frac{\mathbf{C}Adj\left(s\mathbf{I}_n - \mathbf{F}\right)\mathbf{G}}{\Delta_{\mathbf{OL}}(s)}\right|$$

$$\Delta_{\mathbf{OL}}(s)\left|\mathbf{I}_m + \frac{\mathbf{C}Adj\left(s\mathbf{I}_n - \mathbf{F}\right)\mathbf{G}}{\Delta_{\mathbf{OL}}(s)}\right| = \Delta_{\mathbf{OL}}(s)\left|\mathbf{I}_m + \mathbf{A}(s)\right| = \Delta_{\mathbf{CL}}(s) = 0$$

Closed-loop polynomial is open-loop polynomial multiplied by determinant of return difference function matrix

Multivariable Nyquist Stability Criterion

Ratio of Closed- to Open-Loop Characteristic Polynomials Tested in Nyquist Stability Criterion

Scalar Control

$$\frac{\Delta_{CL}(s)}{\Delta_{OL}(s)} = \left[1 + A(s)\right] = \left[1 + \frac{\mathbf{C}Adj(s\mathbf{I}_n - \mathbf{F})\mathbf{G}}{\Delta_{\mathbf{OL}}(s)}\right]$$
$$= a(s) + jb(s) \quad \textit{Scalar}$$

Multivariate Control

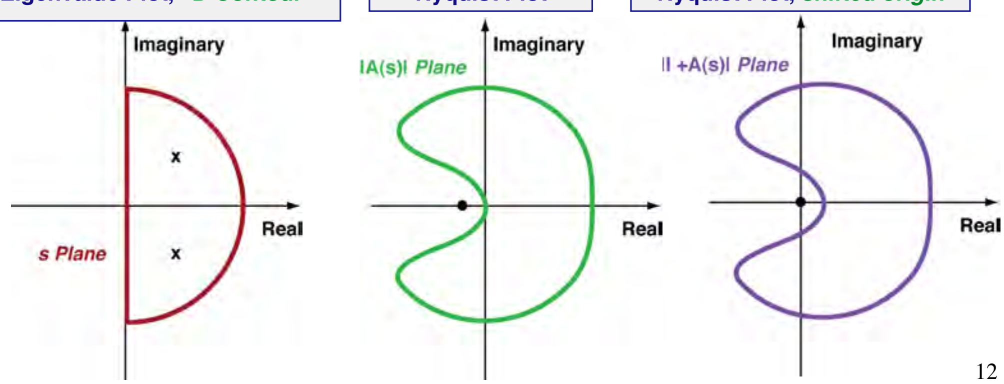
$$\frac{\Delta_{\mathbf{CL}}(s)}{\Delta_{\mathbf{OL}}(s)} = \left|\mathbf{I}_m + \mathbf{A}(s)\right| = \left|\mathbf{I}_m + \frac{\mathbf{C}Adj(s\mathbf{I}_n - \mathbf{F})\mathbf{G}}{\Delta_{\mathbf{OL}}(s)}\right|$$
$$= a(s) + jb(s) \quad \textit{Scalar}$$

Multivariable Nyquist Stability Criterion

$$\frac{\Delta_{\mathbf{CL}}(s)}{\Delta_{\mathbf{OL}}(s)} = \left|\mathbf{I}_m + \mathbf{A}(s)\right| \triangleq a(s) + jb(s) \quad \textit{Scalar}$$

Same stability criteria for encirclements of –1 point apply for scalar and vector control

Eigenvalue Plot, *“D Contour”* | Nyquist Plot | Nyquist Plot, *shifted origin*



Limits of Multivariable Nyquist Stability Criterion

$$\frac{\Delta_{CL}(s)}{\Delta_{OL}(s)} = \left|\mathbf{I}_m + \mathbf{A}(s)\right| \triangleq a(s) + jb(s) \quad \textit{Scalar}$$

- **Multivariable Nyquist Stability Criterion**
  - **Indicates stability of the nominal system**
  - **In the $|\mathbf{I} + \mathbf{A}(s)|$ plane, Nyquist plot depicts the ratio of closed-to-open-loop characteristic polynomials**
- **However, determinant is not a good indicator for the "size" of a matrix**
  - **Little can be said about robustness**
  - **Therefore, analogies to gain and phase margins are not readily identified**

Determinant is Not a Reliable Measure of Matrix "Size"

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \quad \left|\mathbf{A}_1\right| = 2$$

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 100 \\ 0 & 2 \end{bmatrix}; \quad \left|\mathbf{A}_2\right| = 2$$

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 100 \\ 0.02 & 2 \end{bmatrix}; \quad \left|\mathbf{A}_3\right| = 0$$

- **Qualitatively,**
  - **A_1 and A_2 have the same determinant**
  - **A_2 and A_3 are about the same size**

Matrix Norms and Singular Value Analysis

Vector Norms

Euclidean norm

$$\boxed{\|\mathbf{x}\| = \left(\mathbf{x}^T\mathbf{x}\right)^{1/2}}$$

Weighted Euclidean norm

$$\boxed{\|\mathbf{D}\mathbf{x}\| = \left(\mathbf{x}^T\mathbf{D}^T\mathbf{D}\mathbf{x}\right)^{1/2}}$$

For fixed value of ||x||,

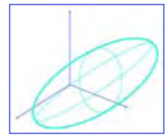
||Dx|| provides a measure of the "size" of D

Spectral Norm (or Matrix Norm)

Spectral norm has more than one "size"

$$\|\mathbf{D}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{Dx}\| \text{ is real-valued}$$

$$\dim(\mathbf{x}) = \dim(\mathbf{Dx}) = n \times 1; \quad \dim(\mathbf{D}) = n \times n$$



Also called Induced Euclidean norm

If D and x are complex

$$\|\mathbf{x}\| = \left(\mathbf{x}^H \mathbf{x}\right)^{1/2}$$

$$\|\mathbf{Dx}\| = \left(\mathbf{x}^H \mathbf{D}^H \mathbf{Dx}\right)^{1/2}$$

where

$$\mathbf{x}^H \triangleq \text{complex conjugate transpose of } \mathbf{x}$$

$$= \text{Hermitian transpose of } \mathbf{x}$$

Spectral Norm (or Matrix Norm)

Spectral norm of D

$$\boxed{\|\mathbf{D}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{Dx}\|}$$

D^TD or D^HD has *n* eigenvalues

Eigenvalues are all real, as D^TD is symmetric and D^HD is Hermitian

Square roots of eigenvalues are called singular values

Singular Values of D

Singular values of D

$$\sigma_i(\mathbf{D}) = \sqrt{\lambda_i\left(\mathbf{D}^T\mathbf{D}\right)}, \quad i = 1, n$$

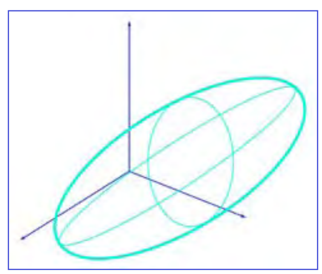
Maximum singular value of D

$$\sigma_{\max}(\mathbf{D}) \triangleq \bar{\sigma}(\mathbf{D}) \triangleq \|\mathbf{D}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{Dx}\|$$

Minimum singular value of D

$$\sigma_{\min}(\mathbf{D}) \triangleq \underline{\sigma}(\mathbf{D}) = 1\Big/\left\|\mathbf{D}^{-1}\right\| = \min_{\|\mathbf{x}\|=1} \|\mathbf{Dx}\|$$

Comparison of Determinants and Singular Values



$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \quad |\mathbf{A}_1| = 2$$

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 100 \\ 0 & 2 \end{bmatrix}; \quad |\mathbf{A}_2| = 2$$

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 100 \\ 0.02 & 2 \end{bmatrix}; \quad |\mathbf{A}_3| = 0$$

- Singular values provide a better portrayal of matrix "size", but ...
- "Size" is multi-dimensional
- Singular values describe magnitude along axes of a multi-dimensional ellipsoid

$e.g.,$

$$\frac{x^2}{\bar{\sigma}^2} + \frac{y^2}{\underline{\sigma}^2} = 1$$

$$\mathbf{A}_1: \quad \bar{\sigma}(\mathbf{A}_1) = 2; \quad \underline{\sigma}(\mathbf{A}_1) = 1$$

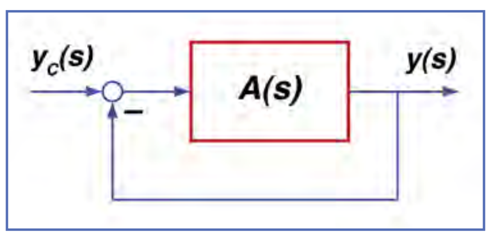
$$\mathbf{A}_2: \quad \bar{\sigma}(\mathbf{A}_2) = 100.025; \quad \underline{\sigma}(\mathbf{A}_2) = 0.02$$

$$\mathbf{A}_3: \quad \bar{\sigma}(\mathbf{A}_3) = 100.025; \quad \underline{\sigma}(\mathbf{A}_3) = 0$$

Stability Margins of Multivariable LQ Regulators

Bode Gain Criterion and the Closed-Loop Transfer Function

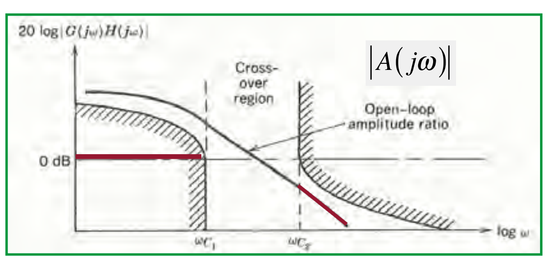
- Bode magnitude criterion for scalar open-loop transfer function
  - High gain at low input frequency
  - Low gain at high input frequency
- Behavior of unit-gain closed-loop transfer function with high and low open-loop amplitude ratio



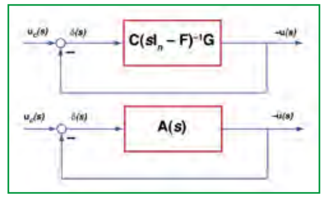
$$\left|\frac{y(j\omega)}{y_C(j\omega)}\right| = \frac{|A(j\omega)|}{|1+A(j\omega)|}$$

$$\xrightarrow[|A(j\omega)|\to 0]{} |A(j\omega)|$$

$$\xrightarrow[|A(j\omega)|\to \infty]{} 1$$



Additive Variations in A(s)



$$\mathbf{A}_o(s) = \mathbf{C}_o(s\mathbf{I}_n - \mathbf{F}_o)^{-1}\mathbf{G}_o$$

$$\mathbf{A}(s) = \mathbf{A}_o(s) + \Delta\mathbf{A}(s)$$

Connections to LQ open-loop transfer matrix

Gain Change

$$\Delta\mathbf{A}_C(s) = \Delta\mathbf{C}(s\mathbf{I}_n - \mathbf{F}_o)^{-1}\mathbf{G}_o$$

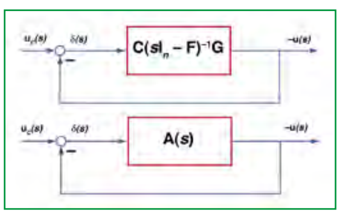
Control Effect Change

$$\Delta\mathbf{A}_G(s) = \mathbf{C}_o(s\mathbf{I}_n - \mathbf{F}_o)^{-1}\Delta\mathbf{G}$$

Stability Matrix Change

$$\Delta\mathbf{A}_F(s) = \mathbf{C}_o\left\{\left[s\mathbf{I}_n - (\mathbf{F}_o + \Delta\mathbf{F})\right]^{-1} - \left[s\mathbf{I}_n - (\mathbf{F}_o)\right]^{-1}\right\}\mathbf{G}_o$$

Conservative Bounds for Additive Variations in A(s)



Assume original system is stable

$$\mathbf{A}_o(s)\left[\mathbf{I}_m + \mathbf{A}_o(s)\right]^{-1}$$

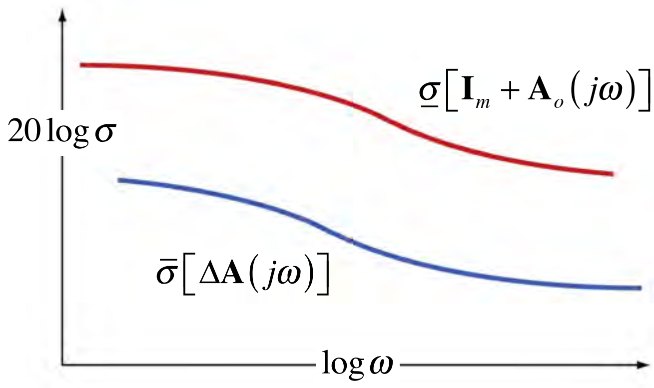
"Worst-case" additive variation does not de-stabilize if

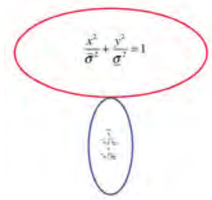
$$\bar{\sigma}\left[\Delta\mathbf{A}(j\omega)\right] < \underline{\sigma}\left[\mathbf{I}_m + \mathbf{A}_o(j\omega)\right], \quad 0 < \omega < \infty$$

Sandell, 1979

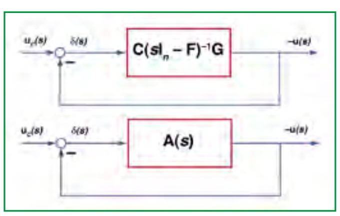
"Bode" Plot of Singular Values

Singular values have magnitude but not phase





Stability guaranteed for changing $\bar{\sigma}\left[\Delta\mathbf{A}(j\omega)\right]$
up to the point that it touches $\underline{\sigma}\left[\mathbf{I}_m + \mathbf{A}_o(j\omega)\right]$



Multiplicative Variations in A(*s*)

$$\mathbf{A}_o(s) = \mathbf{C}_o\left(s\mathbf{I}_n - \mathbf{F}_o\right)^{-1}\mathbf{G}_o$$

$$\mathbf{A}(s) = \mathbf{L}_{PRE}(s)\mathbf{A}_o(s) \text{ or } \mathbf{A}(s) = \mathbf{A}_o(s)\mathbf{L}_{POST}(s)$$

- **Very complex relationship to system equations; suppose**

$$\mathbf{L}(s) = \mathbf{I}_3 + \begin{bmatrix} l_{11}(s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{I}_3 + \Delta\mathbf{L}(s)$$

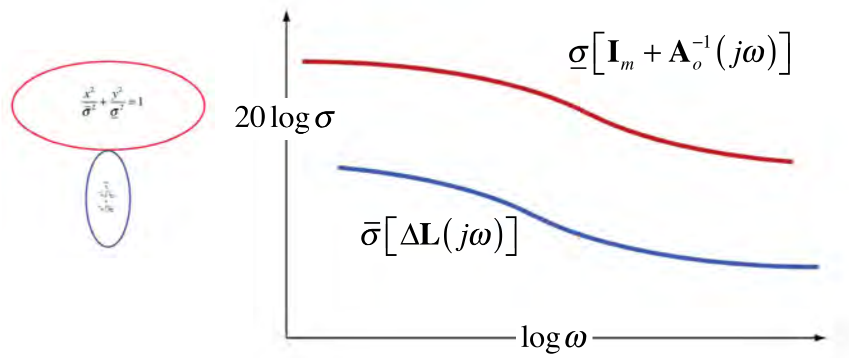
$\Delta\mathbf{L}(s)$ affects first row of $\mathbf{A}_o(s)$ for pre-multiplication
$\Delta\mathbf{L}(s)$ affects first column of $\mathbf{A}_o(s)$ for post-multiplication

Sandell, 1979

Bounds on Multiplicative Variations in A(s)

$$\mathbf{A}_o(s) = \mathbf{C}_o\left(s\mathbf{I}_n - \mathbf{F}_o\right)^{-1}\mathbf{G}_o$$

$$\bar{\sigma}\left[\Delta\mathbf{L}(j\omega)\right] < \underline{\sigma}\left[\mathbf{I}_m + \mathbf{A}_o^{-1}(j\omega)\right], \quad 0 < \omega < \infty$$



27

Desirable "Bode Gain Criterion" Attributes

At low frequency

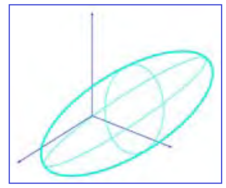
$$\underline{\sigma}\left[\mathbf{I}_m + \mathbf{A}(j\omega)\right] > \sigma_{\min}(\omega) > 1$$

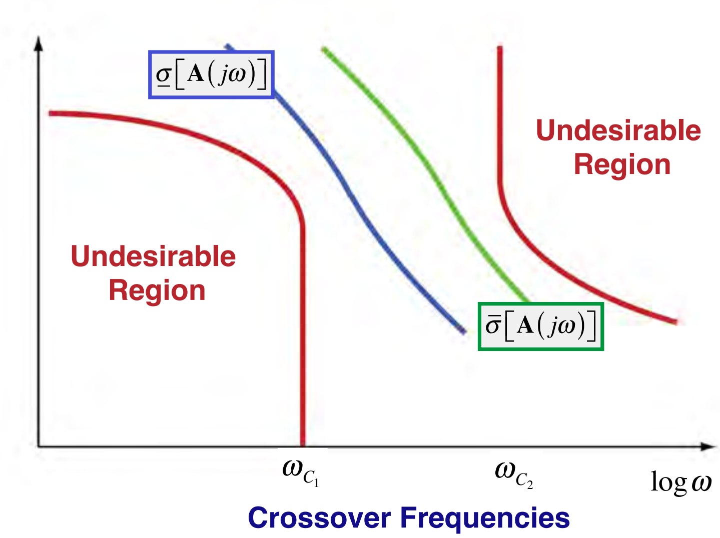
At high frequency

$$\bar{\sigma}\left\{\left[\mathbf{I}_m + \mathbf{A}^{-1}(j\omega)\right]^{-1}\right\} = \frac{1}{\underline{\sigma}\left[\mathbf{I}_m + \mathbf{A}^{-1}(j\omega)\right]} < \sigma_{\max}(\omega)$$

28

Desirable “Bode Gain Criterion” Attributes





Next Time: Probability and Statistics

Supplemental Material

Sensitivity and Complementary Sensitivity Matrices of A(s)

Sensitivity matrix

$$\boxed{\mathbf{S}(s) \triangleq \left[\mathbf{I}_m + \mathbf{A}(s)\right]^{-1}}$$

Inverse return difference matrix

$$\boxed{\left[\mathbf{I}_m + \mathbf{A}^{-1}(s)\right]}$$

Complementary sensitivity matrix

$$\boxed{\mathbf{T}(s) \triangleq \mathbf{A}(s)\left[\mathbf{I}_m + \mathbf{A}(s)\right]^{-1}}$$

Sensitivity and Complementary Sensitivity Matrices of A(s)

Small $\mathbf{S}(j\omega)$ implies low sensitivity to parameter variations as a function of frequency

$$\boxed{\mathbf{S}(j\omega) \triangleq \left[\mathbf{I}_m + \mathbf{A}(j\omega)\right]^{-1}}$$

Small $\mathbf{T}(j\omega)$ implies low noise response as a function of frequency

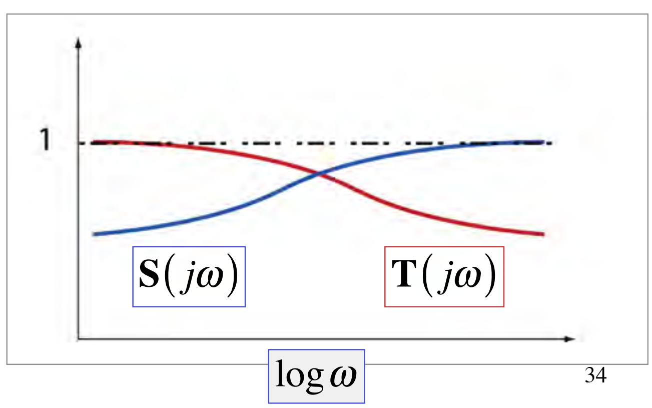
$$\boxed{\mathbf{T}(j\omega) \triangleq \mathbf{A}(j\omega)\left[\mathbf{I}_m + \mathbf{A}(j\omega)\right]^{-1}}$$

Sensitivity and Complementary Sensitivity Matrices of A(s)

- **But**

$$\boxed{\begin{aligned}\mathbf{S}(j\omega) + \mathbf{T}(j\omega) &\triangleq \left[\mathbf{I}_m + \mathbf{A}(j\omega)\right]^{-1} + \mathbf{A}(j\omega)\left[\mathbf{I}_m + \mathbf{A}(j\omega)\right]^{-1} \\ &= \left[\mathbf{I}_m + \mathbf{A}(j\omega)\right]\left[\mathbf{I}_m + \mathbf{A}(j\omega)\right]^{-1} = \mathbf{I}_m\end{aligned}}$$

- **Therefore, there is a tradeoff between robustness and noise suppression**



Alternative Criteria for Multiplicative Variations in A(*s*)

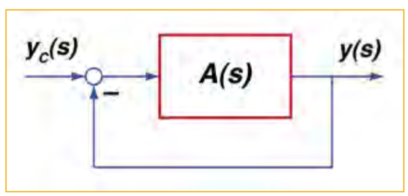
- **Definitions**

$\Delta_{OL}(s)$: Open-loop characteristic polynomial of original system
$\tilde{\Delta}_{OL}(s)$: Perturbed characteristic polynomial of original system
$\Delta_{CL}(s)$: Stable closed-loop characteristic polynomial of original system

$\left\{\tilde{\Delta}_{OL}(j\omega)=0\right\}$ implies that $\Delta_{OL}(j\omega)=0$
for any ω on Ω_R (i.e., vertical component of "D contour")
$\alpha=\underline{\sigma}\left[\mathbf{I}_m+\mathbf{A}_o(j\omega)\right]$ for any ω on Ω_R

Lehtomaki, Sandell, Athans, 1981

Alternative Criteria for Multiplicative Variations in A(*s*)

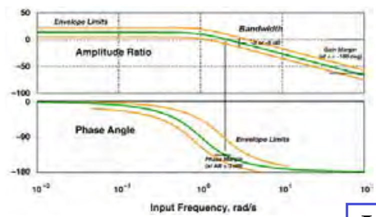


- **Perturbed closed-loop system is stable if**

$\bar{\sigma}\left[\mathbf{L}^{-1}(j\omega)-\mathbf{I}_m\right]<\alpha=\underline{\sigma}\left[\mathbf{I}_m+\mathbf{A}_o(j\omega)\right]$

And at least one of the following is satisfied:

- $\alpha<1$
- $\mathbf{L}^H(j\omega)+\mathbf{L}(j\omega)\geq 0$
- $4\left(\alpha^2-1\right)\underline{\sigma}^2\left[\mathbf{L}(j\omega)-\mathbf{I}_m\right]>\alpha^2\bar{\sigma}^2\left[\mathbf{L}(j\omega)+\mathbf{L}^H(j\omega)-2\mathbf{I}_m\right]$



Guaranteed Gain and Phase Margins

If

$\underline{\sigma}\left[\mathbf{I}_m + \mathbf{A}_o(j\omega)\right] > \alpha_o \leq 1$

- **Guaranteed Gain Margin**

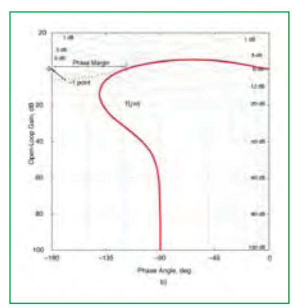
$$K = \frac{1}{1 \pm \alpha_o}$$

- **Guaranteed Phase Margin**

$$\varphi = \pm\cos\left(1 - \frac{\alpha_o^2}{2}\right)$$

In each of *m* control loops

Guaranteed Gain and Phase Margins



If

$\mathbf{L}^H(j\omega) + \mathbf{L}(j\omega) \geq 0$

and

$\mathbf{A}_o^H(j\omega) + \mathbf{A}_o(j\omega) \geq 0$

- **Guaranteed Gain Margin**

$$K = (0, \infty)$$

- **Guaranteed Phase Margin**

$$\varphi = \pm 90°$$

In each of *m* control loops

Control Design for Increased Gain Margin

- Obtain lowest possible LQ control gain matrix, C, by choosing large R
  - Gain margin is 1/2 of these gains
  - Speed of response (e.g., bandwidth) may be too slow
- Increase gains to restore desired bandwidth
- Control system is sub-optimal but has higher gain margin than LQ system designed for same bandwidth

Control Design for Increased Gain Margin

- High R, low-gain optimal controller

$$\boxed{\begin{aligned}&\mathbf{R} \triangleq \rho^2 \mathbf{R}_o\\ &\mathbf{F}^T\mathbf{P} + \mathbf{P}\mathbf{F} + \mathbf{Q} - \mathbf{P}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T\mathbf{P} = \mathbf{0}\\ &\mathbf{C}_{opt} = \mathbf{R}^{-1}\mathbf{G}^T\mathbf{P}\end{aligned}}$$

- Increased gain to restore bandwidth

$$\boxed{\mathbf{C}_{sub-opt} = \mathbf{R}_o^{-1}\mathbf{G}^T\mathbf{P} = \rho^2\mathbf{C}_{opt}}$$

- Increased gain margin for high-bandwidth controller

$$\boxed{K_{sub-opt} = \left(\frac{1}{2\rho^2}, \infty\right)}$$

Example: Control Design for Increased Robustness

(Ray, Stengel, 1991)

- **Open-loop longitudinal eigenvalues**

$$\boxed{\lambda_{1-4} = -0.1 \pm 0.057j, -5.15, 3.35}$$

- **Three controllers**
  - a) Q = diag(1 1 1 0) and R = 1
  - b) R = 1000
  - c) Case (b) with gains multiplied by 5

TABLE I
PARAMETERS FOR FORWARD-SWEPT-WING DEMONSTRATOR AIRCRAFT EXAMPLE

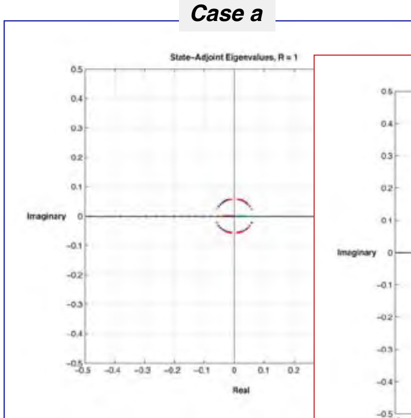
Case (a)	$C = \begin{bmatrix} 0.1714 & 130.26 & 33.165 & 0.364 \\ 0.984 & -11.387 & -2.968 & -1.133 \end{bmatrix}$	$Q = \mathrm{diag}(1, 1, 1, 0)$	$R = \mathrm{diag}(1, 1)$	$\lambda = -35.0, -5.14, -3.32, -0.0183$
Case (b)	$C = \begin{bmatrix} 0.0270 & 82.659 & 20.927 & -0.0638 \\ 0.0107 & -62.623 & -16.203 & -1.902 \end{bmatrix}$	$Q = \mathrm{diag}(1, 1, 1, 0)$	$R = 1000\,\mathrm{diag}(1, 1)$	$\lambda = -5.15, -3.36, -1.09, -0.0186$
Case (c)	$C = \begin{bmatrix} 0.1349 & 413.294 & 104.633 & -0.3191 \\ 0.0535 & -313.112 & -81.015 & -9.509 \end{bmatrix}$			$\lambda = -32.21, -5.15, -3.44, -0.01$

Root Loci for Three Cases

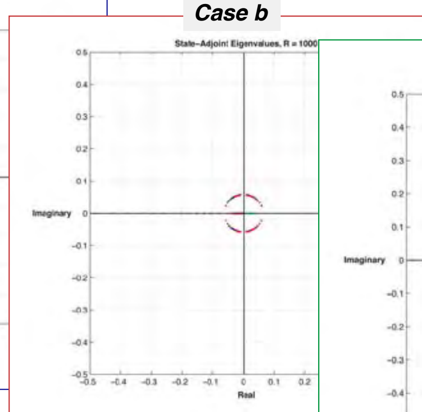
Transmission zeros

$$\boxed{z_{1,2} = \begin{bmatrix} 0 & -1.2 \end{bmatrix}}$$

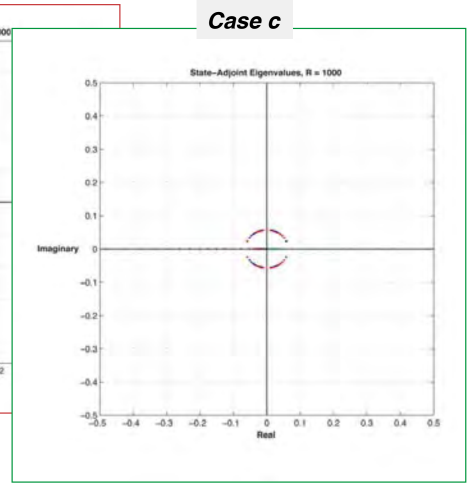
Case a



Case b



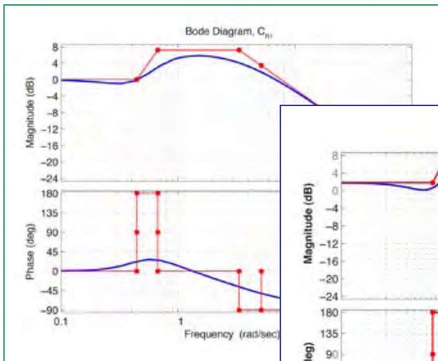
Case c

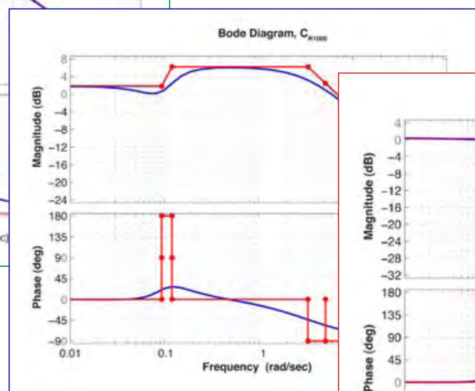


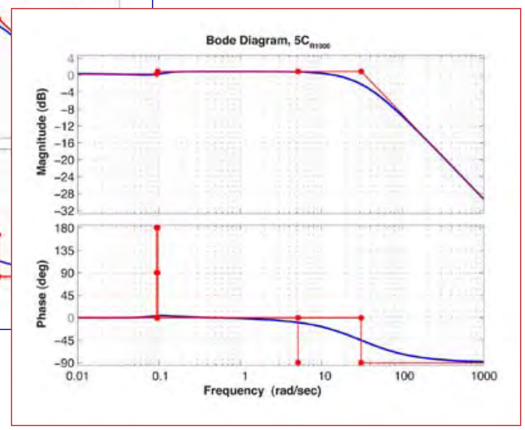
- **Closed-loop system roots**
  - Originate at stable images of open-loop poles
  - 2 roots to transmission zeros
  - 2 roots to $-\infty$, multiple Butterworth spacing

Loop Transfer Function Frequency Response with Elevator Control

$$H(j\omega) = \mathbf{C}(j\omega\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$$

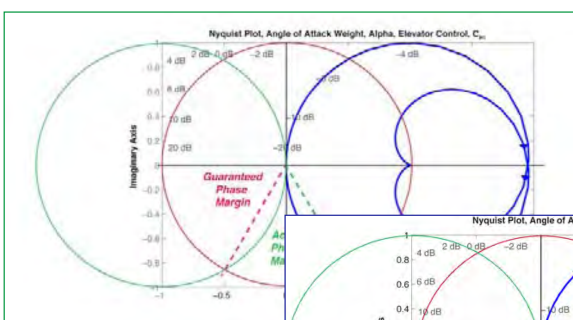


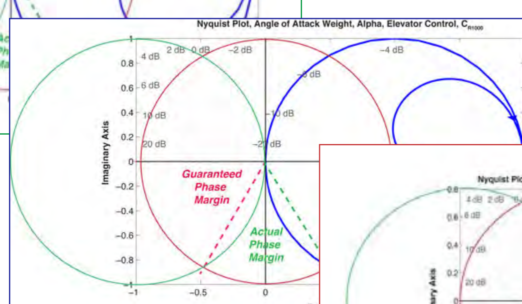


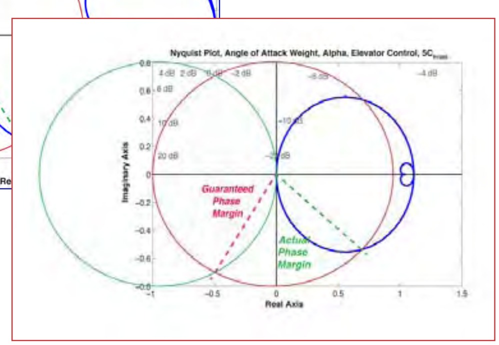


$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{R} = 1 \text{ or } 1000$$

Loop Transfer Function Nyquist Plots with Elevator Control



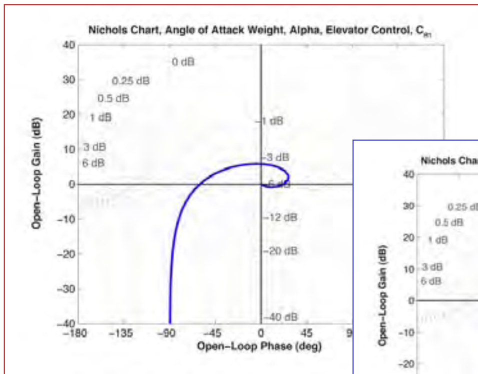


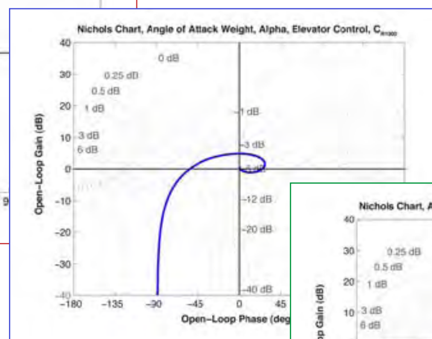


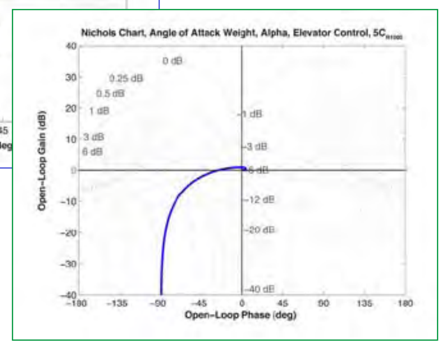
$$H(j\omega) = \mathbf{C}(j\omega\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$$

Loop Transfer Function Nichols Charts with Elevator Control

$$H(j\omega) = \mathbf{C}(j\omega\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$$





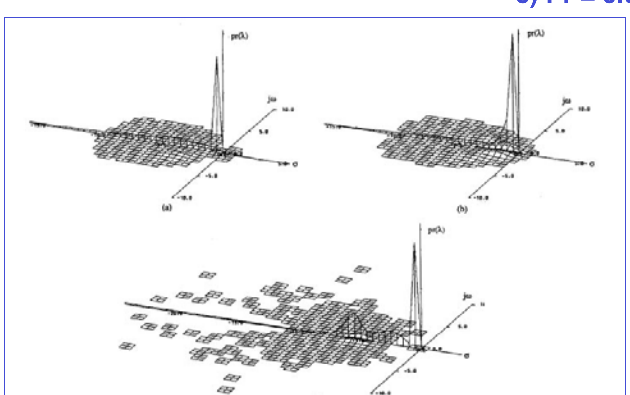


Probability of Instability Describes Robustness to Parameter Uncertainty

(Ray, Stengel, 1991)

- Distribution of closed-loop roots with
  - Gaussian uncertainty in 10 parameters
  - Uniform uncertainty in velocity and air density
- 25,000 Monte Carlo evaluations

- Probability of instability
- a) Pr = 0.072
- b) Pr = 0.021
- c) Pr = 0.0076



Stochastic Root Locus