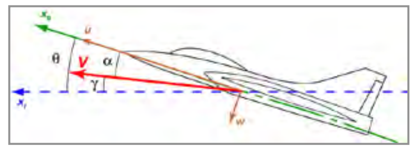
Linearized Longitudinal Equations of Motion

Robert Stengel, Aircraft Flight Dynamics
MAE 331, 2014

Learning Objectives

- 6^{th}-order -> 4^{th}-order -> hybrid equations
- Dynamic stability derivatives
- Long-period (phugoid) mode
- Short-period mode



Reading:
Flight Dynamics
452-464, 482-486
Airplane Stability and Control
Chapter 7

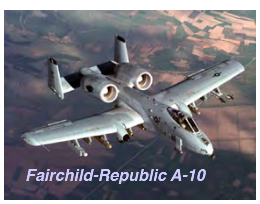
http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

The Jets at an Awkward Age

Chapter 7, *Airplane Stability and Control*, Abzug and Larrabee

- What are the principal subject and scope of the chapter?
- What technical ideas are needed to understand the chapter?
- During what time period did the events covered in the chapter take place?
- What are the three main "takeaway" points or conclusions from the reading?
- What are the three most surprising or remarkable facts that you found in the reading?

Longitudinal LTI Dynamics "Wordle"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6^{th}-Order Longitudinal Equations of Motion

- Symmetric aircraft
- Motions in the vertical plane
- Flat earth

Nonlinear Dynamic Equations

$$
\begin{aligned}
\dot{u} &= X/m - g\sin\theta - qw \\
\dot{w} &= Z/m + g\cos\theta + qu \\
\dot{x}_I &= (\cos\theta)u + (\sin\theta)w \\
\dot{z}_I &= (-\sin\theta)u + (\cos\theta)w \\
\dot{q} &= M / I_{yy} \\
\dot{\theta} &= q
\end{aligned}
$$

State Vector, 6 components

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \mathbf{x}_{Lon_6}
$$

$$
\begin{bmatrix} u \\ w \\ x \\ z \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} Axial\ Velocity \\ Vertical\ Velocity \\ Range \\ Altitude(-) \\ Pitch\ Rate \\ Pitch\ Angle \end{bmatrix}
$$

4^{th}-Order Longitudinal Equations of Motion

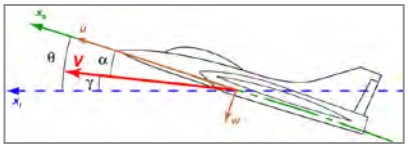
Nonlinear Dynamic Equations, neglecting range and altitude

$$
\begin{aligned}
\dot{u} &= f_1 = X/m - g\sin\theta - qw \\
\dot{w} &= f_2 = Z/m + g\cos\theta + qu \\
\dot{q} &= f_3 = M/I_{yy} \\
\dot{\theta} &= f_4 = q
\end{aligned}
$$

State Vector, 4 components

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{x}_{Lon_4}
\qquad
\begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} \textit{Axial Velocity}, \text{ m/s} \\ \textit{Vertical Velocity}, \text{ m/s} \\ \textit{Pitch Rate}, \text{ rad/s} \\ \textit{Pitch Angle}, \text{ rad} \end{bmatrix}
$$

Fourth-Order Hybrid Equations of Motion



Transform Longitudinal Velocity Components

Replace Cartesian body components of velocity by polar inertial components

Replace *X* and *Z* by *T*, *D*, and *L*

$$
\begin{aligned}
\dot{u} &= f_1 = X/m - g\sin\theta - qw \\
\dot{w} &= f_2 = Z/m + g\cos\theta + qu \\
\dot{q} &= f_3 = M/I_{yy} \\
\dot{\theta} &= f_4 = q
\end{aligned}
$$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} Axial\ Velocity \\ Vertical\ Velocity \\ Pitch\ Rate \\ Pitch\ Angle \end{bmatrix}
$$

7

Transform Longitudinal Velocity Components

Replace Cartesian body components of velocity by polar inertial components

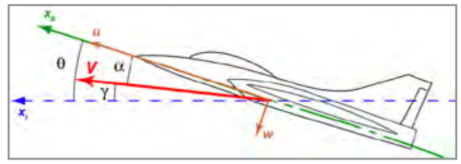
Replace *X* and *Z* by *T*, *D*, and *L*

$$
\begin{aligned}
\dot{V} &= f_1 = \left[T\cos(\alpha+i) - D - mg\sin\gamma\right]/m \\
\dot{\gamma} &= f_2 = \left[T\sin(\alpha+i) + L - mg\cos\gamma\right]/mV \\
\dot{q} &= f_3 = M/I_{yy} \\
\dot{\theta} &= f_4 = q
\end{aligned}
$$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} V \\ \gamma \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} Velocity \\ Flight\ Path\ Angle \\ Pitch\ Rate \\ Pitch\ Angle \end{bmatrix}
$$

i = Incidence angle of the thrust vector with respect to the centerline

Hybrid Longitudinal Equations of Motion



- **Replace pitch angle by angle of attack** $\boxed{\alpha = \theta - \gamma}$

$$
\begin{aligned}
\dot{V} &= f_1 = \left[T\cos(\alpha + i) - D - mg\sin\gamma\right]/m \\
\dot{\gamma} &= f_2 = \left[T\sin(\alpha + i) + L - mg\cos\gamma\right]/mV \\
\dot{q} &= f_3 = M / I_{yy} \\
\dot{\theta} &= f_4 = q
\end{aligned}
$$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} V \\ \gamma \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} \textit{Velocity} \\ \textit{Flight Path Angle} \\ \textit{Pitch Rate} \\ \textit{Pitch Angle} \end{bmatrix}
$$

Hybrid Longitudinal Equations of Motion

- **Replace pitch angle by angle of attack** $\boxed{\alpha = \theta - \gamma}$

$$
\begin{aligned}
\dot{V} &= f_1 = \left[T\cos(\alpha + i) - D - mg\sin\gamma\right]/m \\
\dot{\gamma} &= f_2 = \left[T\sin(\alpha + i) + L - mg\cos\gamma\right]/mV \\
\dot{q} &= f_3 = M / I_{yy} \\
\dot{\alpha} &= \dot{\theta} - \dot{\gamma} = f_4 = q - \frac{1}{mV}\left[T\sin(\alpha + i) + L - mg\cos\gamma\right]
\end{aligned}
$$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} V \\ \gamma \\ q \\ \alpha \end{bmatrix} = \begin{bmatrix} \textit{Velocity} \\ \textit{Flight Path Angle} \\ \textit{Pitch Rate} \\ \textit{Angle of Attack} \end{bmatrix}
$$

$$\boxed{\theta = \alpha + \gamma}$$

Why Transform Equations and State Vector?

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} V \\ \gamma \\ \hline q \\ \alpha \end{bmatrix} = \begin{bmatrix} Velocity \\ Flight\ Path\ Angle \\ \hline Pitch\ Rate \\ Angle\ of\ Attack \end{bmatrix}$$

- **Velocity and flight path angle typically have slow variations**
- **Pitch rate and angle of attack typically have quicker variations**

Small Perturbations from Steady Path Approximated by Linear Equations

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t)$$
$$\approx \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t] + \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t) + \mathbf{L}(t)\Delta\mathbf{w}(t)$$

Steady, Level Flight

$$\dot{\mathbf{x}}(t) = \mathbf{0} + \Delta\dot{\mathbf{x}}(t)$$
$$\approx \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t] + \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t)$$

Rates of change are “small”

Nominal Equations of Motion in Equilibrium (Trimmed Condition)

$$\boxed{\dot{\mathbf{x}}_N(t) = \mathbf{0} = \mathbf{f}[\mathbf{x}_N(t),\mathbf{u}_N(t),\mathbf{w}_N(t),t]}$$

$$\boxed{\mathbf{x}_N^T = \begin{bmatrix} V_N & \gamma_N & 0 & \alpha_N \end{bmatrix}^T = \textbf{constant}}$$

***T*, *D*, *L*, and *M* contain state, control, and disturbance effects**

$$
\begin{aligned}
\dot{V}_N &= 0 = f_1 = \left[T\cos(\alpha_N + i) - D - mg\sin\gamma_N\right]/m \\
\dot{\gamma}_N &= 0 = f_2 = \left[T\sin(\alpha_N + i) + L - mg\cos\gamma_N\right]/mV_N \\
\dot{q}_N &= 0 = f_3 = M/I_{yy} \\
\dot{\alpha}_N &= 0 = f_4 = (0) - \frac{1}{mV_N}\left[T\sin(\alpha_N + i) + L - mg\cos\gamma_N\right]
\end{aligned}
$$

(See Supplemental Material for trimmed solution)

Small Perturbations from Steady Path Approximated by Linear Equations

Linearized Equations of Motion

$$\Delta\dot{\mathbf{x}}_{Lon} = \begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\gamma} \\ \Delta\dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} = \mathbf{F}_{Lon}\begin{bmatrix} \Delta V \\ \Delta\gamma \\ \Delta q \\ \Delta\alpha \end{bmatrix} + \mathbf{G}_{Lon}\begin{bmatrix} \Delta\delta T \\ \Delta\delta E \\ \cdots \end{bmatrix} + \cdots$$

Linearized Equations of Motion

Phugoid (Long-Period) Motion

Short-Period Motion

Approximate Decoupling of Fast and Slow Modes of Motion

Hybrid linearized equations allow the two modes to be examined separately

Effects of **phugoid** perturbations on **phugoid** motion	Effects of **short-period** perturbations on **phugoid** motion

$$\mathbf{F}_{Lon} = \left[\begin{array}{c|c} \mathbf{F}_{Ph} & \mathbf{F}_{SP}^{Ph} \\ \hline \mathbf{F}_{Ph}^{SP} & \mathbf{F}_{SP} \end{array}\right]$$

Effects of **phugoid** perturbations on **short-period** motion	Effects of **short-period** perturbations on **short-period** motion

$$= \left[\begin{array}{c|c} \mathbf{F}_{Ph} & small \\ \hline small & \mathbf{F}_{SP} \end{array}\right] \approx \left[\begin{array}{c|c} \mathbf{F}_{Ph} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{F}_{SP} \end{array}\right]$$

Sensitivity Matrices for Longitudinal LTI Model

$$\boxed{\Delta\dot{\mathbf{x}}_{Lon}(t) = \mathbf{F}_{Lon}\Delta\mathbf{x}_{Lon}(t) + \mathbf{G}_{Lon}\Delta\mathbf{u}_{Lon}(t) + \mathbf{L}_{Lon}\Delta\mathbf{w}_{Lon}(t)}$$

$$\mathbf{F}_{Lon} = \begin{bmatrix} \frac{\partial f_1}{\partial V} & \frac{\partial f_1}{\partial \gamma} & \frac{\partial f_1}{\partial q} & \frac{\partial f_1}{\partial \alpha} \\ \frac{\partial f_2}{\partial V} & \frac{\partial f_2}{\partial \gamma} & \frac{\partial f_2}{\partial q} & \\ \frac{\partial f_3}{\partial V} & \frac{\partial f_3}{\partial \gamma} & \frac{\partial f_3}{\partial q} & \\ \frac{\partial f_4}{\partial V} & \frac{\partial f_4}{\partial \gamma} & \frac{\partial f_4}{\partial q} & \end{bmatrix}$$

$$\mathbf{G}_{Lon} = \begin{bmatrix} \frac{\partial f_1}{\partial \delta E} & \frac{\partial f_1}{\partial \delta T} & \frac{\partial f_1}{\partial \delta F} \\ \frac{\partial f_2}{\partial \delta E} & \frac{\partial f_2}{\partial \delta T} & \\ \frac{\partial f_3}{\partial \delta E} & \frac{\partial f_3}{\partial \delta T} & \\ \frac{\partial f_4}{\partial \delta E} & \frac{\partial f_4}{\partial \delta T} & \end{bmatrix}$$

$$\mathbf{L}_{Lon} = \begin{bmatrix} \frac{\partial f_1}{\partial V_{wind}} & \frac{\partial f_1}{\partial \alpha_{wind}} \\ \frac{\partial f_2}{\partial V_{wind}} & \frac{\partial f_2}{\partial \alpha_{wind}} \\ \frac{\partial f_3}{\partial V_{wind}} & \frac{\partial f_3}{\partial \alpha_{wind}} \\ \frac{\partial f_4}{\partial V_{wind}} & \frac{\partial f_4}{\partial \alpha_{wind}} \end{bmatrix}$$

Velocity Dynamics

Nonlinear equation

$$\dot{V} = f_1 = \frac{1}{m}\left[T\cos\alpha - D - mg\sin\gamma\right]$$

$$= \frac{1}{m}\left[C_T\cos\alpha\frac{\rho V^2}{2}S - C_D\frac{\rho V^2}{2}S - mg\sin\gamma\right]$$

Thrust along x_B

First row of linearized dynamic equation

$$\Delta\dot{V}(t) = \left[\frac{\partial f_1}{\partial V}\Delta V(t) + \frac{\partial f_1}{\partial \gamma}\Delta\gamma(t) + \frac{\partial f_1}{\partial q}\Delta q(t) + \frac{\partial f_1}{\partial \alpha}\Delta\alpha(t)\right]$$
$$+\left[\frac{\partial f_1}{\partial \delta E}\Delta\delta E(t) + \frac{\partial f_1}{\partial \delta T}\Delta\delta T(t) + \frac{\partial f_1}{\partial \delta F}\Delta\delta F(t)\right]$$
$$+\left[\frac{\partial f_1}{\partial V_{wind}}\Delta V_{wind} + \frac{\partial f_1}{\partial \alpha_{wind}}\Delta\alpha_{wind}\right]$$

Sensitivity of Velocity Dynamics to State Perturbations

$$\dot{V} = \left[(C_T \cos\alpha - C_D) \frac{\rho V^2}{2} S - mg \sin\gamma \right] \Big/ m$$

Coefficients in first row of F

$$\frac{\partial f_1}{\partial V} = \frac{1}{m}\left[\left(C_{T_V} \cos\alpha_N - C_{D_V} \right) \frac{\rho_N V_N^{\,2}}{2} S + \left(C_{T_N} \cos\alpha_N - C_{D_N} \right) \rho_N V_N S \right]$$

$$\frac{\partial f_1}{\partial \gamma} = \frac{-1}{m}\left[mg \cos\gamma_N \right] = -g \cos\gamma_N$$

$$\frac{\partial f_1}{\partial q} = \frac{-1}{m}\left[C_{D_q} \frac{\rho_N V_N^{\,2}}{2} S \right]$$

$$\frac{\partial f_1}{\partial \alpha} = \frac{-1}{m}\left[\left(C_{T_N} \sin\alpha_N + C_{D_\alpha} \right) \frac{\rho_N V_N^{\,2}}{2} S \right]$$

$$C_{T_V} \equiv \frac{\partial C_T}{\partial V} \qquad C_{D_V} \equiv \frac{\partial C_D}{\partial V} \qquad C_{D_q} \equiv \frac{\partial C_D}{\partial q} \qquad C_{D_\alpha} \equiv \frac{\partial C_D}{\partial \alpha}$$

Sensitivity of Velocity Dynamics to Control and Disturbance Perturbations

Coefficients in first rows of G and L

$$\frac{\partial f_1}{\partial \delta E} = \frac{-1}{m}\left[C_{D_{\delta E}} \frac{\rho_N V_N^{\,2}}{2} S \right]$$

$$\frac{\partial f_1}{\partial \delta T} = \frac{1}{m}\left[C_{T_{\delta T}} \cos\alpha_N \frac{\rho_N V_N^{\,2}}{2} S \right]$$

$$\frac{\partial f_1}{\partial \delta F} = \frac{-1}{m}\left[C_{D_{\delta F}} \frac{\rho_N V_N^{\,2}}{2} S \right]$$

$$C_{T_{\delta T}} \equiv \frac{\partial C_T}{\partial \delta T} \qquad C_{D_{\delta E}} \equiv \frac{\partial C_D}{\partial \delta E} \qquad C_{D_{\delta F}} \equiv \frac{\partial C_D}{\partial \delta F}$$

$$\frac{\partial f_1}{\partial V_{wind}} = -\frac{\partial f_1}{\partial V}$$

$$\frac{\partial f_1}{\partial \alpha_{wind}} = -\frac{\partial f_1}{\partial \alpha}$$

Flight Path Angle Dynamics

Nonlinear equation

$$\dot{\gamma} = f_2 = \frac{1}{mV}\left[T\sin\alpha + L - mg\cos\gamma\right]$$
$$= \frac{1}{mV}\left[C_T \sin\alpha \frac{\rho V^2}{2} S + C_L \frac{\rho V^2}{2} S - mg\cos\gamma\right]$$

Second row of linearized equation

$$\Delta\dot{\gamma}(t) = \left[\frac{\partial f_2}{\partial V}\Delta V(t) + \frac{\partial f_2}{\partial \gamma}\Delta\gamma(t) + \frac{\partial f_2}{\partial q}\Delta q(t) + \frac{\partial f_2}{\partial \alpha}\Delta\alpha(t)\right]$$
$$+\left[\frac{\partial f_2}{\partial \delta E}\Delta\delta E(t) + \frac{\partial f_2}{\partial \delta T}\Delta\delta T(t) + \frac{\partial f_2}{\partial \delta F}\Delta\delta F(t)\right]$$
$$+\left[\frac{\partial f_2}{\partial V_{wind}}\Delta V_{wind} + \frac{\partial f_2}{\partial \alpha_{wind}}\Delta\alpha_{wind}\right]$$

Sensitivity of Flight Path Angle Dynamics to State Perturbations

$$\dot{\gamma} = \left[(C_T \sin\alpha + C_L)\frac{\rho V^2}{2} S - mg\cos\gamma\right] \Big/ mV$$

Coefficients in second row of F

$$\frac{\partial f_2}{\partial V} = \frac{1}{mV_N}\left[\left(C_{T_V}\sin\alpha_N + C_{L_V}\right)\frac{\rho_N {V_N}^2}{2} S + \left(C_{T_N}\sin\alpha_N + C_{L_N}\right)\rho_N V_N S\right]$$
$$-\frac{1}{m{V_N}^2}\left[\left(C_{T_N}\sin\alpha_N + C_{L_N}\right)\frac{\rho_N {V_N}^2}{2} S - mg\cos\gamma_N\right]$$

$$\frac{\partial f_2}{\partial \gamma} = \frac{1}{mV_N}\left[mg\sin\gamma_N\right] = g\sin\gamma_N / V_N$$

$$\frac{\partial f_2}{\partial q} = \frac{1}{mV_N}\left[C_{L_q}\frac{\rho_N {V_N}^2}{2} S\right]$$

$$\frac{\partial f_2}{\partial \alpha} = \frac{1}{mV_N}\left[\left(C_{T_N}\cos\alpha_N + C_{L_\alpha}\right)\frac{\rho_N {V_N}^2}{2} S\right]$$

$$C_{T_V} \equiv \frac{\partial C_T}{\partial V}$$
$$C_{L_V} \equiv \frac{\partial C_L}{\partial V}$$
$$C_{L_q} \equiv \frac{\partial C_L}{\partial q}$$
$$C_{L_\alpha} \equiv \frac{\partial C_L}{\partial \alpha}$$

Pitch Rate Dynamics

Nonlinear equation

$$\dot{q} = f_3 = \frac{M}{I_{yy}} = \frac{C_m\left(\rho V^2/2\right)S\bar{c}}{I_{yy}}$$

Third row of linearized equation

$$\begin{aligned}\Delta\dot{q}(t) = &\left[\frac{\partial f_3}{\partial V}\Delta V(t) + \frac{\partial f_3}{\partial \gamma}\Delta\gamma(t) + \frac{\partial f_3}{\partial q}\Delta q(t) + \frac{\partial f_3}{\partial \alpha}\Delta\alpha(t)\right] \\ &+\left[\frac{\partial f_3}{\partial \delta E}\Delta\delta E(t) + \frac{\partial f_3}{\partial \delta T}\Delta\delta T(t) + \frac{\partial f_3}{\partial \delta F}\Delta\delta F(t)\right] \\ &+\left[\frac{\partial f_3}{\partial V_{wind}}\Delta V_{wind} + \frac{\partial f_3}{\partial \alpha_{wind}}\Delta\alpha_{wind}\right]\end{aligned}$$

Sensitivity of Pitch Rate Dynamics to State Perturbations

$$\dot{q} = C_m\left(\rho V^2/2\right)S\bar{c}/I_{yy}$$

Coefficients in third row of F

$$\frac{\partial f_3}{\partial V} = \frac{1}{I_{yy}}\left[C_{m_V}\frac{\rho_N V_N^{\ 2}}{2}S\bar{c} + C_{m_N}\rho_N V_N S\bar{c}\right]$$

$$\frac{\partial f_3}{\partial \gamma} = 0$$

$$\frac{\partial f_3}{\partial q} = \frac{1}{I_{yy}}\left[C_{m_q}\frac{\rho_N V_N^{\ 2}}{2}S\bar{c}\right]$$

$$\frac{\partial f_3}{\partial \alpha} = \frac{1}{I_{yy}}\left[C_{m_\alpha}\frac{\rho_N V_N^{\ 2}}{2}S\bar{c}\right]$$

$$C_{m_V} \equiv \frac{\partial C_m}{\partial V}$$
$$C_{m_q} \equiv \frac{\partial C_m}{\partial q}$$
$$C_{m_\alpha} \equiv \frac{\partial C_m}{\partial \alpha}$$

Angle of Attack Dynamics

Nonlinear equation

$$\boxed{\dot{\alpha} = f_4 = \dot{\theta} - \dot{\gamma} = q - \frac{1}{mV}\left[T \sin\alpha + L - mg\cos\gamma\right]}$$

Fourth row of linearized equation

$$\begin{aligned}
\Delta\dot{\alpha}(t) = &\left[\frac{\partial f_4}{\partial V}\Delta V(t) + \frac{\partial f_4}{\partial \gamma}\Delta\gamma(t) + \frac{\partial f_4}{\partial q}\Delta q(t) + \frac{\partial f_4}{\partial \alpha}\Delta\alpha(t)\right] \\
&+ \left[\frac{\partial f_4}{\partial \delta E}\Delta\delta E(t) + \frac{\partial f_4}{\partial \delta T}\Delta\delta T(t) + \frac{\partial f_4}{\partial \delta F}\Delta\delta F(t)\right] \\
&+ \left[\frac{\partial f_4}{\partial V_{wind}}\Delta V_{wind} + \frac{\partial f_4}{\partial \alpha_{wind}}\Delta\alpha_{wind}\right]
\end{aligned}$$

Sensitivity of Angle of Attack Dynamics to State Perturbations

$$\boxed{\dot{\alpha} = \dot{\theta} - \dot{\gamma} = q - \dot{\gamma}}$$

Coefficients in fourth row of F

$$\boxed{\frac{\partial f_4}{\partial V} = -\frac{\partial f_2}{\partial V}} \qquad \boxed{\frac{\partial f_4}{\partial q} = 1 - \frac{\partial f_2}{\partial q}}$$

$$\boxed{\frac{\partial f_4}{\partial \gamma} = -\frac{\partial f_2}{\partial \gamma}} \qquad \boxed{\frac{\partial f_4}{\partial \alpha} = -\frac{\partial f_2}{\partial \alpha}}$$

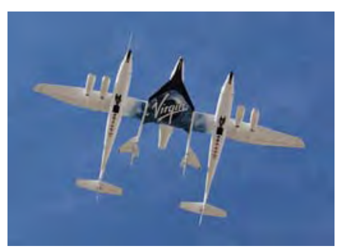
Alternative Approach:
Numerical Calculation of the Sensitivity Matrices *("1st Differences")*

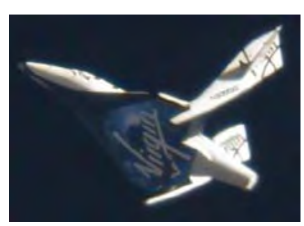
$$\frac{\partial f_1}{\partial V}(t) \approx \frac{ f_1\begin{bmatrix} (V+\Delta V) \\ \gamma \\ q \\ \alpha \end{bmatrix} - f_1\begin{bmatrix} (V-\Delta V) \\ \gamma \\ q \\ \alpha \end{bmatrix} \Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t) \\ \mathbf{w}=\mathbf{w}_N(t)}} }{2\Delta V}; \quad \frac{\partial f_1}{\partial \gamma}(t) \approx \frac{ f_1\begin{bmatrix} V \\ (\gamma+\Delta\gamma) \\ q \\ \alpha \end{bmatrix} - f_1\begin{bmatrix} V \\ (\gamma-\Delta\gamma) \\ q \\ \alpha \end{bmatrix} \Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t) \\ \mathbf{w}=\mathbf{w}_N(t)}} }{2\Delta\gamma}$$

$$\frac{\partial f_2}{\partial V}(t) \approx \frac{ f_2\begin{bmatrix} (V+\Delta V) \\ \gamma \\ q \\ \alpha \end{bmatrix} - f_2\begin{bmatrix} (V-\Delta V) \\ \gamma \\ q \\ \alpha \end{bmatrix} \Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t) \\ \mathbf{w}=\mathbf{w}_N(t)}} }{2\Delta V}; \quad \frac{\partial f_2}{\partial \gamma}(t) \approx \frac{ f_2\begin{bmatrix} V \\ (\gamma+\Delta\gamma) \\ q \\ \alpha \end{bmatrix} - f_2\begin{bmatrix} V \\ (\gamma-\Delta\gamma) \\ q \\ \alpha \end{bmatrix} \Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t) \\ \mathbf{w}=\mathbf{w}_N(t)}} }{2\Delta\gamma}$$

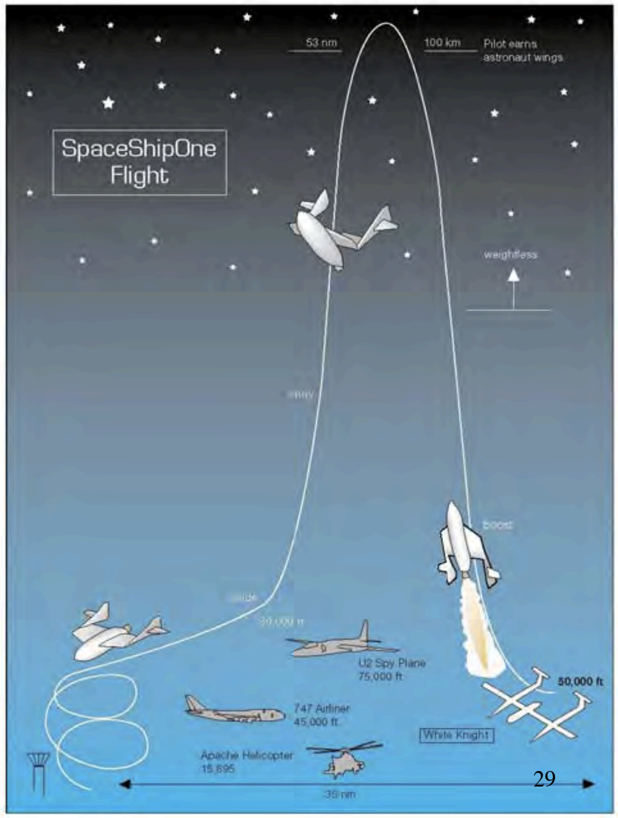
Remaining elements of F(*t*), G(*t*), and L(*t*) calculated accordingly

Current Events
SpaceShipTwo Accident
October 31, 2014

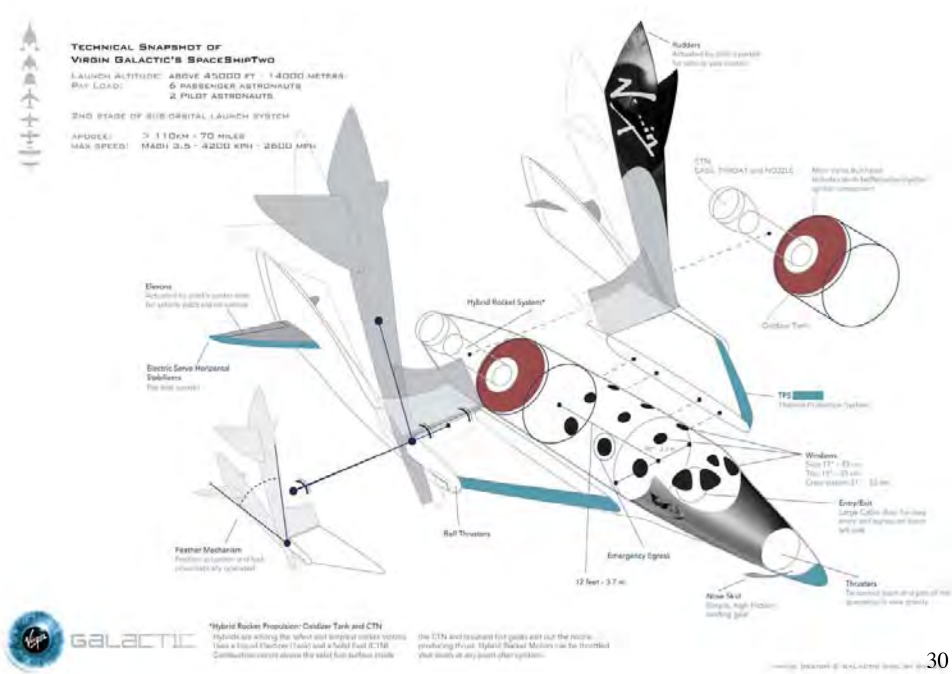



Flight Profile of *SpaceShipOne (Precursor to SpaceShipTwo)*



SpaceShipTwo Cutaway Diagram



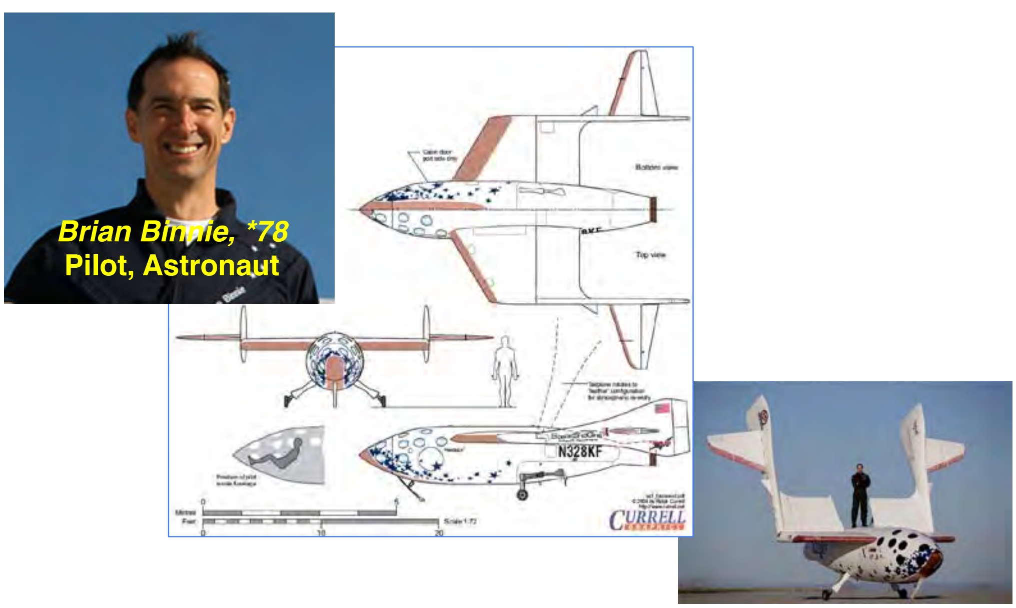
Current Events

SpaceShipTwo Accident
October 31, 2014

Probable Cause, as of 11/3/14
Premature Feathering at M = 1

Chris Hart, ‘68, *70
Acting Chairman, NTSB

SpaceShipOne
Ansari X Prize, December 17, 2003



Brian Binnie, *78
Pilot, Astronaut

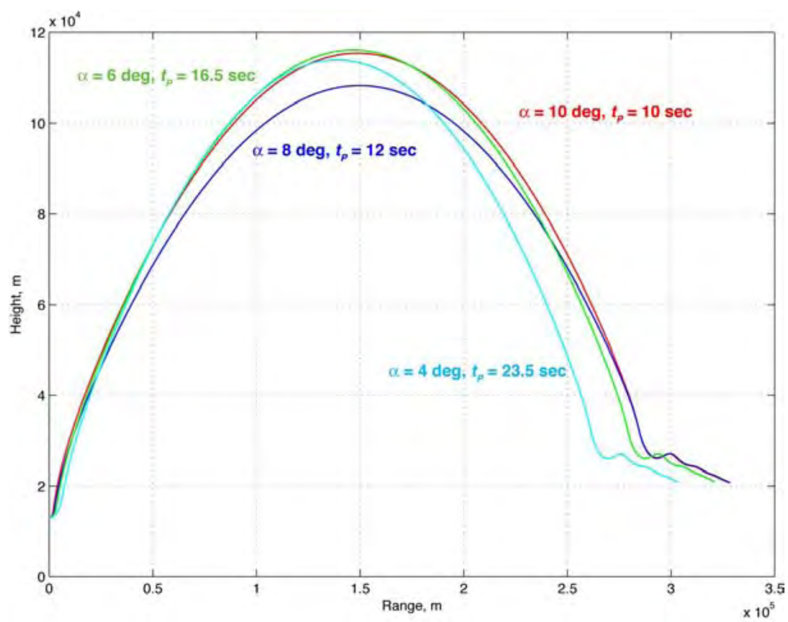
MAE 331
AIRCRAFT FLIGHT DYNAMICS
Assignment #4
due: October 21, 2010

Interest in space tourism is growing, as several companies compete to offer sub-orbital rides into space for paying customers. The door was opened on October 4, 2004 when Mojave Aerospace Ventures *SpaceShipOne* (Fig. 1) flew higher than 100 km for the second time in less than three weeks, winning the Ansari X-Prize (http://en.wikipedia.org/wiki/SpaceShipOne). Princeton alumnus, Brian Binnie, was at the controls for the award-winning flight.

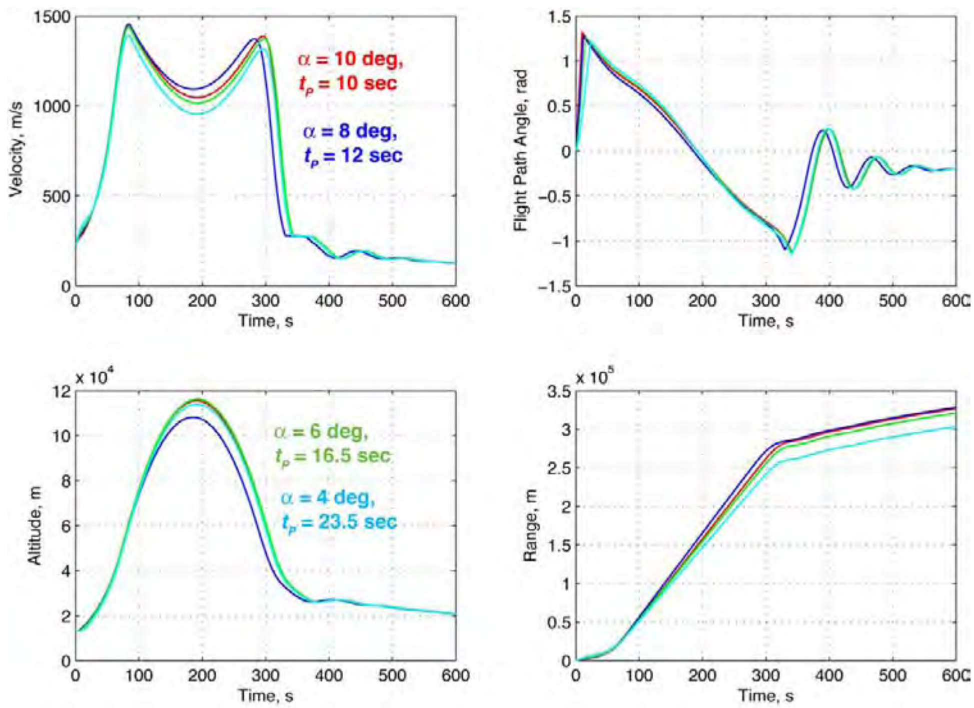
Figure 1. *SpaceShipOne*, in flight, and with astronaut/test pilot Brian Binnie, MAE *78.

This week's assignment is to simulate *SpaceShipOne's* flight. There are two major parts to the assignment. First, you will develop a longitudinal aerodynamic, inertial, and thrust model for the aircraft. Then, you will calculate the flight trajectory (Fig. 2) using point-mass longitudinal equations of motion. The dynamic equations are similar to those of Assignment #2, but are modified to include thrust, to portray the vehicle in conventional and "feathered" re-entry configuration over a range of angles of attack and Mach numbers, and to account for altitude-dependent variations in gravitational acceleration and atmospheric properties.

SpaceShipOne Altitude vs. Range MAE 331 Assignment #4, 2010



SpaceShipOne State Histories



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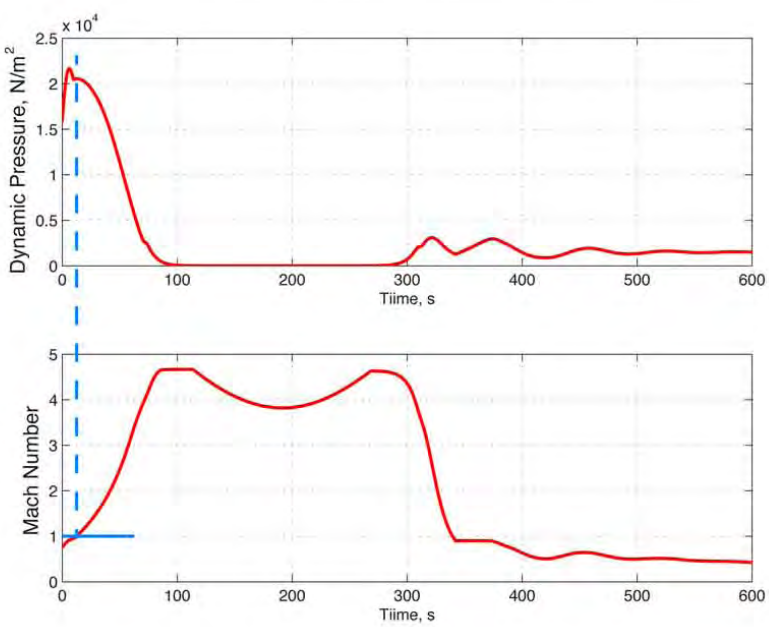
SpaceShipOne Dynamic Pressure and Mach Number Histories



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Dimensional Stability and Control Derivatives

Dimensional Stability-Derivative Notation

- **Redefine force and moment symbols as acceleration symbols**
- **Dimensional stability derivatives portray acceleration sensitivities to state perturbations**

$$\frac{Drag}{mass\ (m)} \Rightarrow D \propto \dot{V}$$

$$\frac{Lift}{mass} \Rightarrow L \propto V\dot{\gamma}$$

$$\frac{Moment}{moment\ of\ inertia\ (I_{yy})} \Rightarrow M \propto \dot{q}$$

<u>Dimensional</u> Stability-Derivative Notation

$$\boxed{\frac{\partial f_1}{\partial V} \equiv -D_V \triangleq \frac{1}{m}\left[\left(C_{T_V}\cos\alpha_N - C_{D_V}\right)\frac{\rho_N V_N{}^2}{2}S + \left(C_{T_N}\cos\alpha_N - C_{D_N}\right)\rho_N V_N S\right]}$$

Thrust and drag effects are combined and represented by one symbol

$$\boxed{\frac{\partial f_2}{\partial \alpha} \equiv L_\alpha / V_N \triangleq \frac{1}{mV_N}\left[\left(C_{T_N}\cos\alpha_N + C_{L_\alpha}\right)\frac{\rho_N V_N{}^2}{2}S\right]}$$

Thrust and lift effects are combined and represented by one symbol

$$\boxed{\frac{\partial f_3}{\partial \alpha} \equiv M_\alpha \triangleq \frac{1}{I_{yy}}\left[C_{m_\alpha}\frac{\rho_N V_N{}^2}{2}S\bar{c}\right]}$$

Longitudinal Stability Matrix

Effects of phugoid perturbations on phugoid motion

Effects of short-period perturbations on phugoid motion

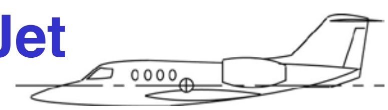
$$\boxed{\mathbf{F}_{Lon} = \begin{bmatrix} \mathbf{F}_{Ph} & \mathbf{F}_{SP}^{Ph} \\ \mathbf{F}_{Ph}^{SP} & \mathbf{F}_{SP} \end{bmatrix} = \left[\begin{array}{cc|cc} -D_V & -g\cos\gamma_N & -D_q & -D_\alpha \\ L_V/V_N & \dfrac{g}{V_N}\sin\gamma_N & L_q/V_N & L_\alpha/V_N \\ \hline M_V & 0 & M_q & M_\alpha \\ -L_V/V_N & -\dfrac{g}{V_N}\sin\gamma_N & \left(1 - L_q/V_N\right) & -L_\alpha/V_N \end{array}\right]}$$

Effects of phugoid perturbations on short-period motion

Effects of short-period perturbations on short-period motion

Comparison of Fourth- and Second-Order Dynamic Models

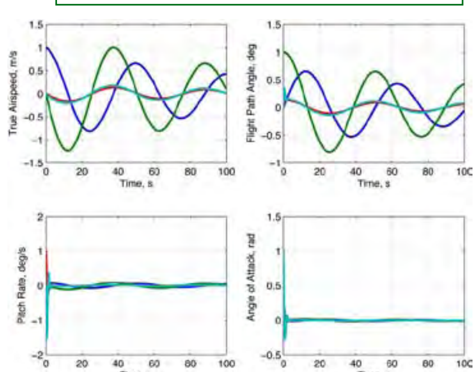
4^{th}-Order Initial-Condition Responses of Business Jet at Two Time Scales



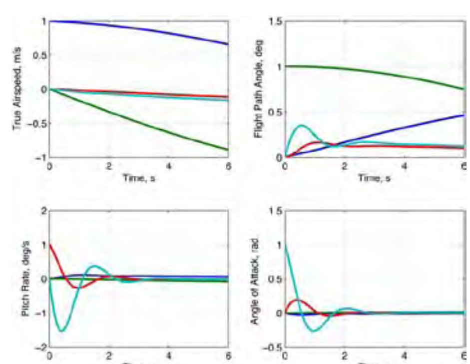
Plotted over different periods of time

4 initial conditions [*V*(0), *γ*(0), *q*(0), *α*(0)]

0 - 100 sec
- **Reveals Phugoid Mode**



0 - 6 sec
- **Reveals Short-Period Mode**



2nd-Order Models of Longitudinal Motion

Assume off-diagonal blocks of (4 x 4) stability matrix are negligible

$$\mathbf{F}_{Lon} = \begin{bmatrix} \mathbf{F}_{Ph} & \sim 0 \\ \sim 0 & \mathbf{F}_{SP} \end{bmatrix}$$

Approximate Phugoid Equation

$$\Delta\dot{\mathbf{x}}_{Ph} =$$

$$\begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\gamma} \end{bmatrix} \approx \begin{bmatrix} -D_V & -g\cos\gamma_N \\ L_V/V_N & \dfrac{g}{V_N}\sin\gamma_N \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix}$$

$$+ \begin{bmatrix} T_{\delta T} \\ L_{\delta T}/V_N \end{bmatrix} \Delta\delta T + \begin{bmatrix} -D_V \\ L_V/V_N \end{bmatrix} \Delta V_{wind}$$

2nd-Order Models of Longitudinal Motion

$$\mathbf{F}_{Lon} = \begin{bmatrix} \mathbf{F}_{Ph} & \sim 0 \\ \sim 0 & \mathbf{F}_{SP} \end{bmatrix}$$

Approximate Short-Period Equation

$$\Delta\dot{\mathbf{x}}_{SP} =$$

$$\begin{bmatrix} \Delta\dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} \approx \begin{bmatrix} M_q & M_\alpha \\ \left(1 - L_q/V_N\right) & -L_\alpha/V_N \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta\alpha \end{bmatrix}$$

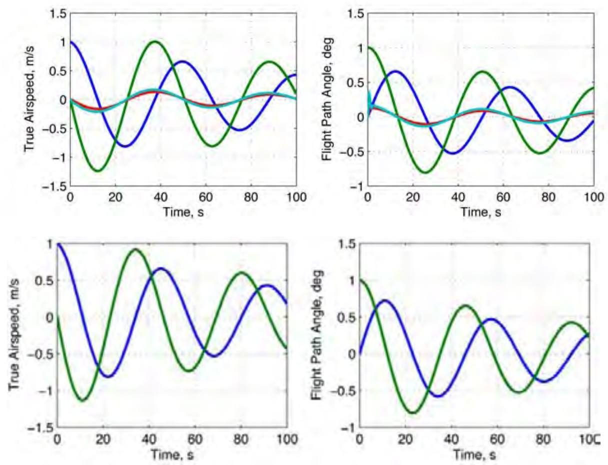
$$+ \begin{bmatrix} M_{\delta E} \\ -L_{\delta E}/V_N \end{bmatrix} \Delta\delta E + \begin{bmatrix} M_\alpha \\ -L_\alpha/V_N \end{bmatrix} \Delta\alpha_{wind}$$

Comparison of Bizjet 4th- and 2nd-Order Model Responses

Phugoid Time Scale, ~100 s

4th Order,
4 initial conditions
[$V(0)$, $\gamma(0)$, $q(0)$, $\alpha(0)$]

2nd Order,
2 initial conditions
[$V(0)$, $\gamma(0)$]



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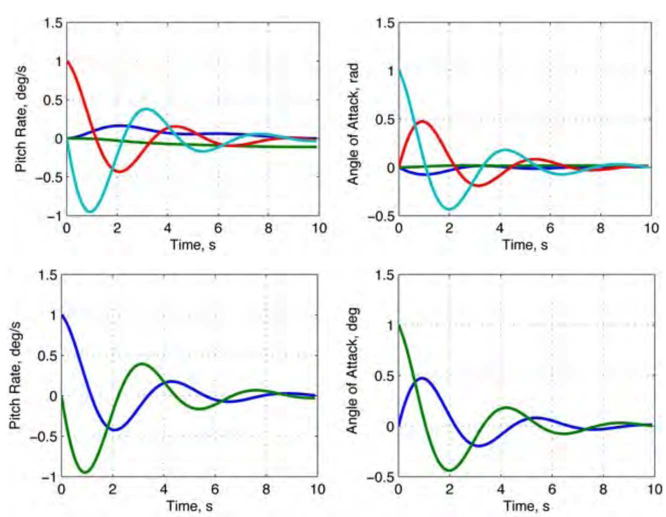
Comparison of Bizjet 4th- and 2nd-Order Model Responses

Short-Period Time Scale, ~10 s

4th Order,
4 initial conditions
[$V(0)$, $\gamma(0)$, $q(0)$, $\alpha(0)$]

2nd Order,
2 initial conditions
[$q(0)$, $\alpha(0)$]



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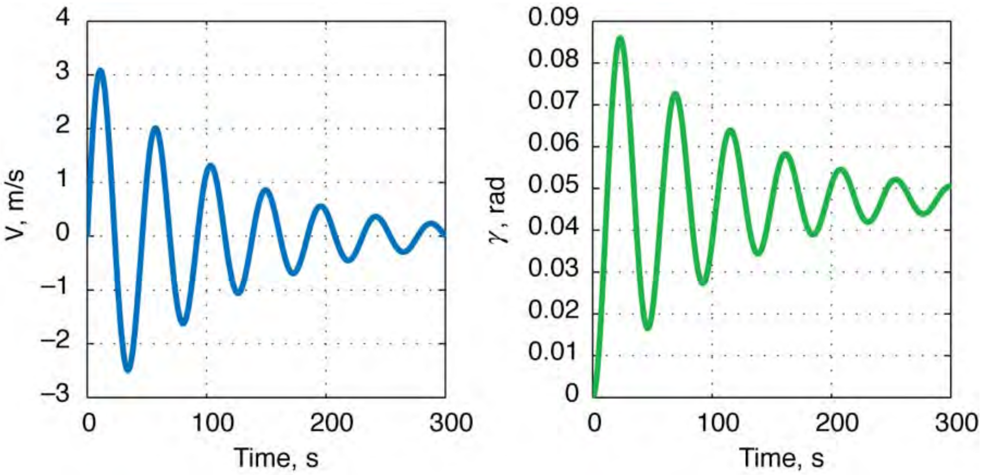
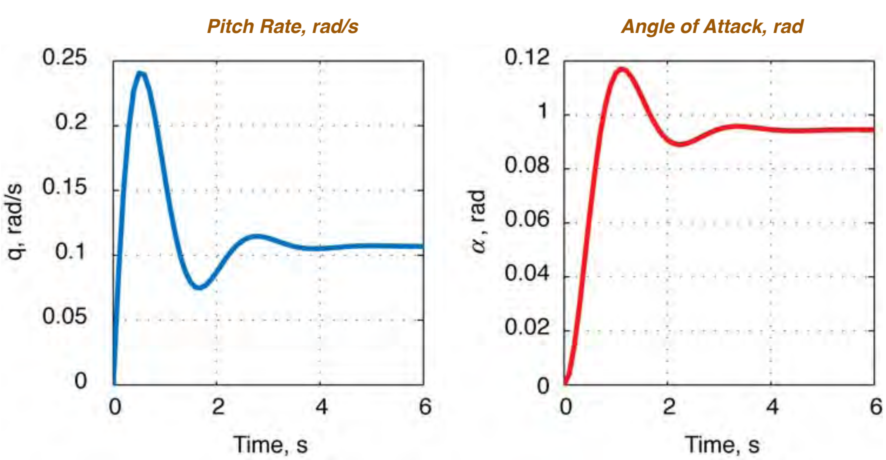
Approximate Phugoid Response to a 10% Thrust Increase



What is the steady-state response?

Approximate Short-Period Response to a 0.1-Rad Pitch Control Step Input



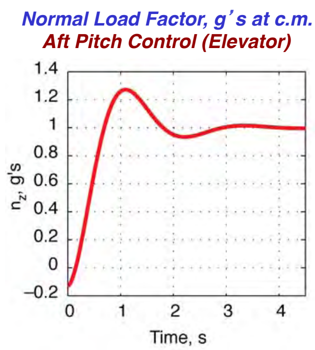
What is the steady-state response?

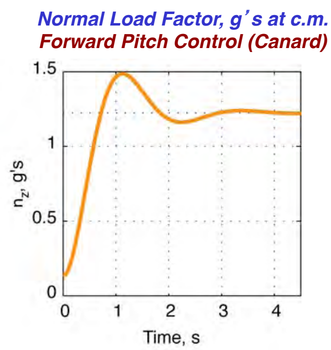
Normal Load Factor Response to a 0.1-Rad Pitch Control Step Input

- **Normal load factor at the center of mass**

$$n_z = \frac{V_N}{g}\left(\Delta\dot{\alpha} - \Delta q\right) = \frac{V_N}{g}\left(\frac{L_\alpha}{V_N}\Delta\alpha + \frac{L_{\delta E}}{V_N}\Delta\delta E\right)$$

- **Pilot focuses on normal load factor during rapid maneuvering**





Next Time: *Lateral-Directional Dynamics*

Reading:
Flight Dynamics
574-591

Supplemental Material

Trimmed Solution of the Equations of Motion

Flight Conditions for Steady, Level Flight

Nonlinear longitudinal model

$$
\begin{aligned}
\dot{V} &= f_1 = \frac{1}{m}\left[T\cos(\alpha+i) - D - mg\sin\gamma\right] \\
\dot{\gamma} &= f_2 = \frac{1}{mV}\left[T\sin(\alpha+i) + L - mg\cos\gamma\right] \\
\dot{q} &= f_3 = M / I_{yy} \\
\dot{\alpha} &= f_4 = \dot{\theta} - \dot{\gamma} = q - \frac{1}{mV}\left[T\sin(\alpha+i) + L - mg\cos\gamma\right]
\end{aligned}
$$

Nonlinear longitudinal model in equilibrium

$$
\begin{aligned}
\mathbf{0} &= f_1 = \frac{1}{m}\left[T\cos(\alpha+i) - D - mg\sin\gamma\right] \\
\mathbf{0} &= f_2 = \frac{1}{mV}\left[T\sin(\alpha+i) + L - mg\cos\gamma\right] \\
\mathbf{0} &= f_3 = M / I_{yy} \\
\mathbf{0} &= f_4 = \dot{\theta} - \dot{\gamma} = q - \frac{1}{mV}\left[T\sin(\alpha+i) + L - mg\cos\gamma\right]
\end{aligned}
$$

Numerical Solution for Level Flight Trimmed Condition

- **Specify desired altitude and airspeed, h_N and V_N**
- **Guess starting values for the trim parameters, δT_0, δE_0, and θ_0**
- **Calculate starting values of f_1, f_2, and f_3**

$$
\begin{aligned}
f_1 &= \frac{1}{m}\left[T(\delta T, \delta E, \theta, h, V)\cos(\alpha+i) - D(\delta T, \delta E, \theta, h, V)\right] \\
f_2 &= \frac{1}{mV_N}\left[T(\delta T, \delta E, \theta, h, V)\sin(\alpha+i) + L(\delta T, \delta E, \theta, h, V) - mg\right] \\
f_3 &= M(\delta T, \delta E, \theta, h, V) / I_{yy}
\end{aligned}
$$

- **f_1, f_2, and $f_3 = 0$ in equilibrium, but not for arbitrary δT_0, δE_0, and θ_0**
- **Define a scalar, positive-definite trim error cost function, e.g.,**

$$
J(\delta T, \delta E, \theta) = a\left(f_1^2\right) + b\left(f_2^2\right) + c\left(f_3^2\right)
$$

Minimize the Cost Function with Respect to the Trim Parameters

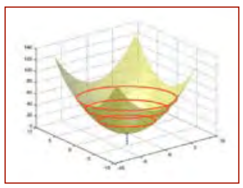
Error cost is "bowl-shaped"

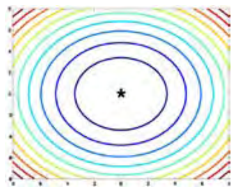
$$\boxed{J\left(\delta T,\delta E,\theta\right)=a\left(f_1^2\right)+b\left(f_2^2\right)+c\left(f_3^2\right)}$$

Cost is minimized at bottom of bowl, i.e., when

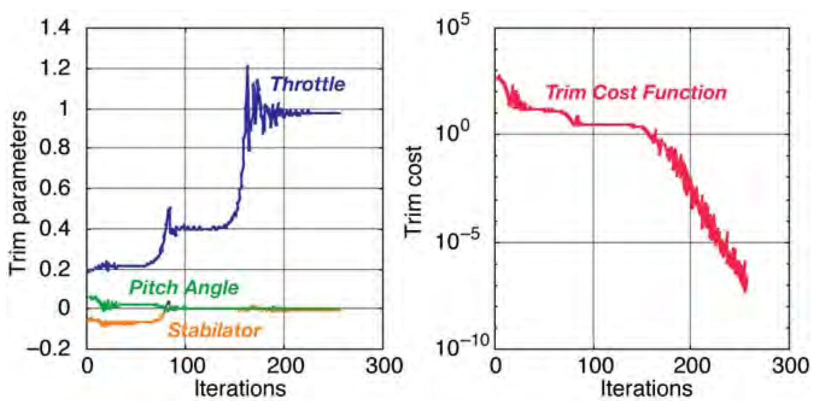
$$\boxed{\left[\begin{array}{ccc}\dfrac{\partial J}{\partial \delta T} & \dfrac{\partial J}{\partial \delta E} & \dfrac{\partial J}{\partial \theta}\end{array}\right]=\mathbf{0}}$$

Search to find the minimum value of *J*





Example of Search for Trimmed Condition (Fig. 3.6-9, *Flight Dynamics*)



In MATLAB, use *fminsearch* to find trim settings

$$\boxed{\left(\delta T^*,\delta E^*,\theta^*\right)=\text{fminsearch}\left[J,\left(\delta T,\delta E,\theta\right)\right]}$$

Elements of the Stability Matrix

Stability derivatives portray acceleration sensitivities to state perturbations

$$\frac{\partial f_1}{\partial V} \equiv -D_V; \quad \frac{\partial f_1}{\partial \gamma} = -g\cos\gamma_N; \quad \frac{\partial f_1}{\partial q} \equiv -D_q; \quad \frac{\partial f_1}{\partial \alpha} \equiv -D_\alpha$$

$$\frac{\partial f_2}{\partial V} \equiv L_V / V_N; \quad \frac{\partial f_2}{\partial \gamma} = \frac{g}{V_N}\sin\gamma_N; \quad \frac{\partial f_2}{\partial q} \equiv L_q / V_N; \quad \frac{\partial f_2}{\partial \alpha} \equiv L_\alpha / V_N$$

$$\frac{\partial f_3}{\partial V} \equiv M_V; \quad \frac{\partial f_3}{\partial \gamma} = 0; \quad \frac{\partial f_3}{\partial q} \equiv M_q; \quad \frac{\partial f_3}{\partial \alpha} \equiv M_\alpha$$

$$\frac{\partial f_4}{\partial V} \equiv -L_V / V_N; \quad \frac{\partial f_4}{\partial \gamma} = -\frac{g}{V_N}\sin\gamma_N; \quad \frac{\partial f_4}{\partial q} \equiv 1 - L_q / V_N; \quad \frac{\partial f_4}{\partial \alpha} \equiv -L_\alpha / V_N$$

Control and Disturbance Sensitivities in Flight Path Angle, Pitch Rate, and Angle-of-Attack Dynamics

$$\frac{\partial f_2}{\partial \delta E} = \frac{1}{mV_N}\left[C_{L_{\delta E}}\frac{\rho V_N^2}{2}S\right]$$
$$\frac{\partial f_2}{\partial \delta T} = \frac{1}{mV_N}\left[C_{T_{\delta T}}\sin\alpha_N\frac{\rho V_N^2}{2}S\right]$$
$$\frac{\partial f_2}{\partial \delta F} = \frac{1}{mV_N}\left[C_{L_{\delta F}}\frac{\rho V_N^2}{2}S\right]$$

$$\frac{\partial f_3}{\partial \delta E} = \frac{1}{I_{yy}}\left[C_{m_{\delta E}}\frac{\rho V_N^2}{2}S\bar{c}\right]$$
$$\frac{\partial f_3}{\partial \delta T} = \frac{1}{I_{yy}}\left[C_{m_{\delta T}}\frac{\rho V_N^2}{2}S\bar{c}\right]$$
$$\frac{\partial f_3}{\partial \delta F} = \frac{1}{I_{yy}}\left[C_{m_{\delta F}}\frac{\rho V_N^2}{2}S\bar{c}\right]$$

$$\frac{\partial f_4}{\partial \delta E} = -\frac{\partial f_2}{\partial \delta E}; \quad \frac{\partial f_4}{\partial \delta T} = -\frac{\partial f_2}{\partial \delta T}; \quad \frac{\partial f_4}{\partial \delta F} = -\frac{\partial f_2}{\partial \delta F}$$

$$\frac{\partial f_2}{\partial V_{wind}} = -\frac{\partial f_2}{\partial V}; \quad \frac{\partial f_2}{\partial \alpha_{wind}} = -\frac{\partial f_2}{\partial \alpha}$$

$$\frac{\partial f_3}{\partial V_{wind}} = -\frac{\partial f_3}{\partial V}; \quad \frac{\partial f_3}{\partial \alpha_{wind}} = -\frac{\partial f_3}{\partial \alpha}$$

$$\frac{\partial f_4}{\partial V_{wind}} = \frac{\partial f_2}{\partial V}; \quad \frac{\partial f_3}{\partial \alpha_{wind}} = \frac{\partial f_2}{\partial \alpha}$$

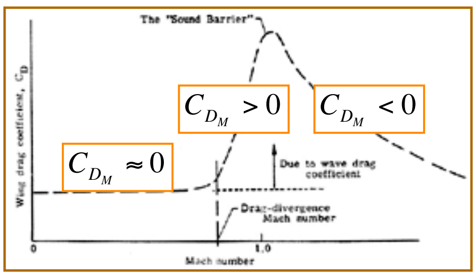
Velocity-Dependent Derivative Definitions

Air compressibility effects are a principal source of velocity dependence

$$C_{D_M} \equiv \frac{\partial C_D}{\partial M} = \frac{\partial C_D}{\partial (V/a)} = a\frac{\partial C_D}{\partial V}$$

a = *Speed of Sound*

M = *Mach number* $= V/a$



$$C_{D_V} \equiv \frac{\partial C_D}{\partial V} = \left(\frac{1}{a}\right)C_{D_M}$$

$$C_{L_V} \equiv \frac{\partial C_L}{\partial V} = \left(\frac{1}{a}\right)C_{L_M}$$

$$C_{m_V} \equiv \frac{\partial C_m}{\partial V} = \left(\frac{1}{a}\right)C_{m_M}$$

Wing Lift and Moment Coefficient Sensitivity to Pitch Rate

Straight-wing incompressible flow estimate (*Etkin*)

$$C_{L_{\hat{q}_{wing}}} = -2C_{L_{\alpha_{wing}}}\left(h_{cm} - 0.75\right)$$

$$C_{m_{\hat{q}_{wing}}} = -2C_{L_{\alpha_{wing}}}\left(h_{cm} - 0.5\right)^2$$

Straight-wing supersonic flow estimate (*Etkin*)

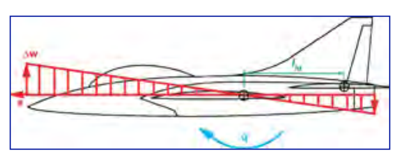
$$C_{L_{\hat{q}_{wing}}} = -2C_{L_{\alpha_{wing}}}\left(h_{cm} - 0.5\right)$$

$$C_{m_{\hat{q}_{wing}}} = -\frac{2}{3\sqrt{M^2-1}} - 2C_{L_{\alpha_{wing}}}\left(h_{cm} - 0.5\right)^2$$

Triangular-wing estimate (*Bryson, Nielsen*)

$$C_{L_{\hat{q}_{wing}}} = -\frac{2\pi}{3}C_{L_{\alpha_{wing}}}$$

$$C_{m_{\hat{q}_{wing}}} = -\frac{\pi}{3AR}$$



Control- and Disturbance-Effect Matrices

- **Control-effect derivatives portray acceleration sensitivities to control input perturbations**

$$\mathbf{G}_{Lon} = \left[\begin{array}{c|c|c} -D_{\delta E} & T_{\delta T} & -D_{\delta F} \\ L_{\delta E}/V_N & L_{\delta T}/V_N & L_{\delta F}/V_N \\ \hline M_{\delta E} & M_{\delta T} & M_{\delta F} \\ -L_{\delta E}/V_N & -L_{\delta T}/V_N & -L_{\delta F}/V_N \end{array}\right]$$

- **Disturbance-effect derivatives portray acceleration sensitivities to disturbance input perturbations**

$$\mathbf{L}_{Lon} = \left[\begin{array}{c|c} -D_{V_{wind}} & -D_{\alpha_{wind}} \\ L_{V_{wind}}/V_N & L_{\alpha_{wind}}/V_N \\ \hline M_{V_{wind}} & M_{\alpha_{wind}} \\ -L_{V_{wind}}/V_N & -L_{\alpha_{wind}}/V_N \end{array}\right]$$

Primary Longitudinal Stability Derivatives

$$\boxed{D_V \triangleq \frac{-1}{m}\left[\left(C_{T_V} - C_{D_V}\right)\frac{\rho V_N^2}{2}S + \left(C_{T_N} - C_{D_N}\right)\rho V_N S\right]}$$

$$\boxed{L_V \big/ V_N \simeq \frac{1}{mV_N}\left[C_{L_V}\frac{\rho V_N^2}{2}S + C_{L_N}\rho V_N S\right] - \frac{1}{mV_N^2}\left[C_{L_N}\frac{\rho V_N^2}{2}S - mg\right]}$$

$$\boxed{M_q = \frac{1}{I_{yy}}\left[C_{m_q}\frac{\rho V_N^2}{2}S\overline{c}\right]} \qquad \boxed{M_\alpha = \frac{1}{I_{yy}}\left[C_{m_\alpha}\frac{\rho V_N^2}{2}S\overline{c}\right]}$$

$$\boxed{L_\alpha \big/ V_N \simeq \frac{1}{mV_N}\left[\left(C_{T_N} + C_{L_\alpha}\right)\frac{\rho V_N^2}{2}S\right]}$$

Small angle assumptions

Primary Phugoid Control Derivatives

$$D_{\delta T} \simeq \frac{-1}{m}\left[C_{T_{\delta T}} \frac{\rho V_N{}^2}{2} S\right]$$

$$L_{\delta F}/V_N \simeq \frac{1}{mV_N}\left[C_{L_{\delta F}} \frac{\rho V_N{}^2}{2} S\right]$$

Primary Short-Period Control Derivatives

$$M_{\delta E} = C_{m_{\delta E}}\left(\frac{\rho_N V_N^2}{2 I_{yy}}\right) S\overline{c}$$

$$L_{\delta E}/V = C_{L_{\delta E}}\left(\frac{\rho_N V_N^2}{2m}\right) S$$

Flight Motions

Simulator Demonstration of
Short-Period Response to Elevator Deflection
http://www.youtube.com/watch?v=1O7ZqBS0_B8

Simulator Demonstration of Phugoid Response
http://www.youtube.com/watch?v=DEOGM_9NGTI

Dornier Do-128 Short-Period Demonstration
http://www.youtube.com/watch?v=3hdLXE0rc9Q

Dornier Do-128 Phugoid Demonstration
http://www.youtube.com/watch?v=jzxtpQ30nLg&feature=related