Linear-Quadratic Control System Design

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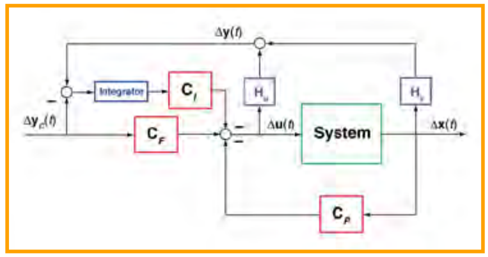
- Control system configurations
  - Proportional-integral
  - Proportional-integral-filtering
  - Model following
- Root locus analysis



System Equilibrium at Desired Output

Recall

$$\begin{aligned}\mathbf{0} &= \mathbf{F}\Delta\mathbf{x}^* + \mathbf{G}\Delta\mathbf{u}^* + \mathbf{L}\Delta\mathbf{w}^* \\ \Delta\mathbf{y}^* &= \mathbf{H}_\mathbf{x}\Delta\mathbf{x}^* + \mathbf{H}_\mathbf{u}\Delta\mathbf{u}^* + \mathbf{H}_\mathbf{w}\Delta\mathbf{w}^*\end{aligned}$$

$$\begin{bmatrix}\mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{bmatrix} = \begin{bmatrix}\mathbf{F} & \mathbf{G} \\ \mathbf{H}_\mathbf{x} & \mathbf{H}_\mathbf{u}\end{bmatrix}^{-1}$$

Equilibrium solution

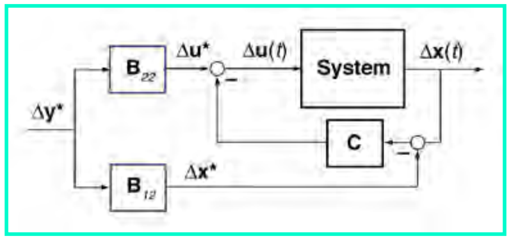
$$\begin{aligned}\Delta\mathbf{x}^* &= \mathbf{B}_{12}\Delta\mathbf{y}_C - \left(\mathbf{B}_{11}\mathbf{L} + \mathbf{B}_{12}\mathbf{H}_\mathbf{w}\right)\Delta\mathbf{w}^* \\ \Delta\mathbf{u}^* &= \mathbf{B}_{22}\Delta\mathbf{y}_C - \left(\mathbf{B}_{21}\mathbf{L} + \mathbf{B}_{22}\mathbf{H}_\mathbf{w}\right)\Delta\mathbf{w}^*\end{aligned}$$

where

$$\begin{aligned}\mathbf{B}_{11} &= \mathbf{F}^{-1}\left(-\mathbf{G}\mathbf{B}_{21} + \mathbf{I}_n\right) \\ \mathbf{B}_{12} &= -\mathbf{F}^{-1}\mathbf{G}\mathbf{B}_{22} \\ \mathbf{B}_{21} &= -\mathbf{B}_{22}\mathbf{H}_\mathbf{x}\mathbf{F}^{-1} \\ \mathbf{B}_{22} &= \left(-\mathbf{H}_\mathbf{x}\mathbf{F}^{-1}\mathbf{G} + \mathbf{H}_\mathbf{u}\right)^{-1}\end{aligned}$$

Non-Zero Steady-State Regulation with Proportional LQ Regulator

Command input provides equilibrium state and control values



Control law with command input

$$
\begin{aligned}
\Delta\mathbf{u}(t) &= \Delta\mathbf{u}^*(t) - \mathbf{C}\left[\Delta\mathbf{x}(t) - \Delta\mathbf{x}^*(t)\right] \\
&= \mathbf{B}_{22}\Delta\mathbf{y}^* - \mathbf{C}\left[\Delta\mathbf{x}(t) - \mathbf{B}_{12}\Delta\mathbf{y}^*\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\mathbf{B}_{22} + \mathbf{C}\mathbf{B}_{12}\right)\Delta\mathbf{y}^* - \mathbf{C}\Delta\mathbf{x}(t) \\
&\triangleq \mathbf{C}_F\Delta\mathbf{y}^* - \mathbf{C}_B\Delta\mathbf{x}(t)
\end{aligned}
$$

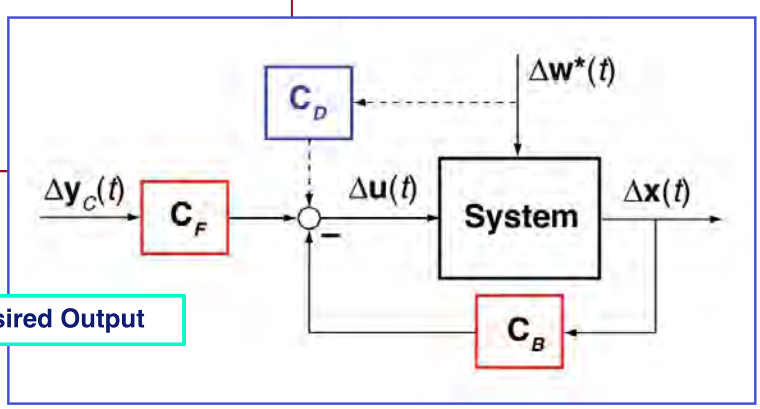
LQ Regulator with Forward Gain Matrix

$$
\begin{aligned}
\Delta\mathbf{u}(t) &= \Delta\mathbf{u}^*(t) - \mathbf{C}\left[\Delta\mathbf{x}(t) - \Delta\mathbf{x}^*(t)\right] \\
&= \mathbf{C}_F\Delta\mathbf{y}^* - \mathbf{C}_B\Delta\mathbf{x}(t)
\end{aligned}
$$

where

$$\mathbf{C}_F \triangleq \mathbf{B}_{22} + \mathbf{C}\mathbf{B}_{12}$$

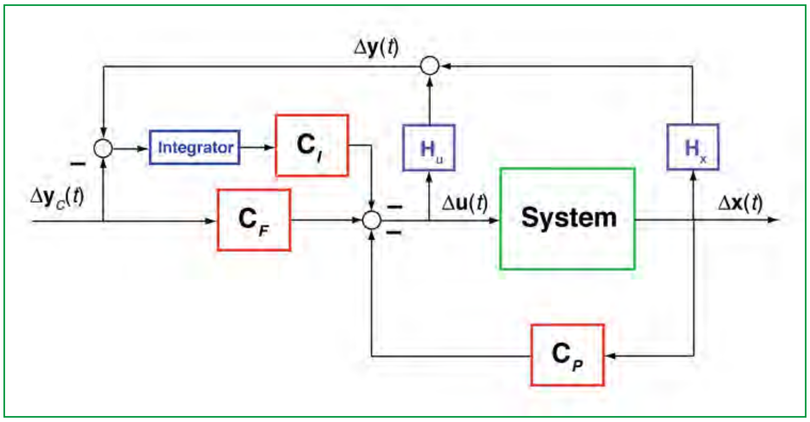
$$\mathbf{C}_B \triangleq \mathbf{C}$$



Input = Desired Output

LQ PI Command Response Block Diagram

Integrate error in desired (commanded) response



Formulating Proportional-Integral Control as a Linear-Quadratic Problem

LTI system with command input

$$\begin{aligned}\Delta\dot{\mathbf{x}}(t) &= \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t) \\ \Delta\mathbf{y}_C &= \mathbf{H_x}\Delta\mathbf{x}^* + \mathbf{H_u}\Delta\mathbf{u}^*\end{aligned}$$

Desired steady-state response to command

$$\Delta\mathbf{x}^* = \mathbf{B}_{12}\Delta\mathbf{y}_C \qquad \Delta\mathbf{u}^* = \mathbf{B}_{22}\Delta\mathbf{y}_C$$

Perturbations from desired response

$$\begin{aligned}\Delta\tilde{\mathbf{x}}(t) &= \Delta\mathbf{x}(t) - \Delta\mathbf{x}^* \\ \Delta\tilde{\mathbf{u}}(t) &= \Delta\mathbf{u}(t) - \Delta\mathbf{u}^* \\ \Delta\tilde{\mathbf{y}}(t) &= \Delta\mathbf{y}(t) - \Delta\mathbf{y}_C\end{aligned}$$

LQ Proportional-Integral (*PI*) Control with Command Input

Integral state

$$\Delta\tilde{\boldsymbol{\xi}}(t) = \int_0^t \Delta\tilde{\mathbf{y}}(t)\,dt = \int_0^t \left[\mathbf{H_x}\Delta\tilde{\mathbf{x}}(t) + \mathbf{H_u}\Delta\tilde{\mathbf{u}}(t)\right]dt$$

$$\Delta\tilde{\boldsymbol{\chi}}(t) \triangleq \begin{bmatrix} \Delta\tilde{\mathbf{x}}(t) \\ \Delta\tilde{\boldsymbol{\xi}}(t) \end{bmatrix}$$

Augmented dynamic system, referenced to desired steady state

$$\Delta\dot{\tilde{\mathbf{x}}}(t) = \mathbf{F}\Delta\tilde{\mathbf{x}}(t) + \mathbf{G}\Delta\tilde{\mathbf{u}}(t)$$
$$\Delta\dot{\tilde{\boldsymbol{\xi}}}(t) = \mathbf{H_x}\Delta\tilde{\mathbf{x}}(t) + \mathbf{H_u}\Delta\tilde{\mathbf{u}}(t)$$

$$\begin{bmatrix} \Delta\dot{\tilde{\mathbf{x}}}(t) \\ \Delta\dot{\tilde{\boldsymbol{\xi}}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{H_x} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{x}}(t) \\ \Delta\tilde{\boldsymbol{\xi}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{G} \\ \mathbf{H_u} \end{bmatrix} \Delta\tilde{\mathbf{u}}(t)$$

$$\Delta\dot{\tilde{\boldsymbol{\chi}}}(t) = \mathbf{F}_\chi\Delta\tilde{\boldsymbol{\chi}}(t) + \mathbf{G}_\chi\Delta\tilde{\mathbf{u}}(t)$$

Augmented Cost Function

$$J = \frac{1}{2}\int_0^\infty \left[\Delta\tilde{\mathbf{x}}^T(t)\mathbf{Q_x}\Delta\tilde{\mathbf{x}}(t) + \Delta\tilde{\boldsymbol{\xi}}^T(t)\mathbf{Q}_\xi\Delta\tilde{\boldsymbol{\xi}}(t) + \Delta\tilde{\mathbf{u}}^T(t)\mathbf{R}\Delta\tilde{\mathbf{u}}(t)\right]dt$$

$$= \frac{1}{2}\int_0^\infty \left[\Delta\tilde{\boldsymbol{\chi}}^T(t)\begin{bmatrix} \mathbf{Q_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_\xi \end{bmatrix}\Delta\tilde{\boldsymbol{\chi}}(t) + \Delta\tilde{\mathbf{u}}^T(t)\mathbf{R}\Delta\tilde{\mathbf{u}}(t)\right]dt$$

subject to

$$\Delta\dot{\tilde{\boldsymbol{\chi}}}(t) = \mathbf{F}_\chi\Delta\tilde{\boldsymbol{\chi}}(t) + \mathbf{G}_\chi\Delta\tilde{\mathbf{u}}(t)$$

$$\Delta\tilde{\boldsymbol{\chi}}(t) \triangleq \begin{bmatrix} \Delta\tilde{\mathbf{x}}(t) \\ \Delta\tilde{\boldsymbol{\xi}}(t) \end{bmatrix}$$

LQ Proportional-Integral (*PI*) Control with Command Input

The cost function is minimized by

$$\boxed{\Delta\tilde{\mathbf{u}}(t) = -\mathbf{C}_\chi \Delta\tilde{\boldsymbol{\chi}}(t)}$$

The control signal includes the error between the commanded and actual response

$$\begin{aligned}\Delta\mathbf{u}(t) - \Delta\mathbf{u}^* &= -\mathbf{C}_\chi\left[\Delta\boldsymbol{\chi}(t) - \Delta\boldsymbol{\chi}^*\right]\\ &= -\mathbf{C}_P\left[\Delta\mathbf{x}(t) - \Delta\mathbf{x}^*\right] - \mathbf{C}_I\left\{\int_0^t \left[\Delta\mathbf{y}(t) - \Delta\mathbf{y}_C\right]dt\right\}\end{aligned}$$

$$\begin{aligned}\Delta\tilde{\mathbf{x}}(t) &= \Delta\mathbf{x}(t) - \Delta\mathbf{x}^*\\ \Delta\tilde{\mathbf{u}}(t) &= \Delta\mathbf{u}(t) - \Delta\mathbf{u}^*\\ \Delta\tilde{\mathbf{y}}(t) &= \Delta\mathbf{y}(t) - \Delta\mathbf{y}_C\end{aligned}$$

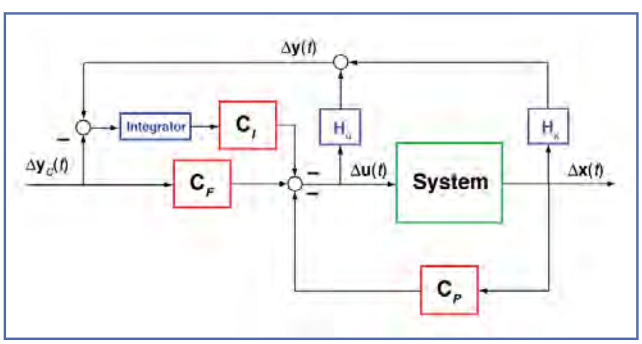
$$= -\mathbf{C}_P\left[\Delta\mathbf{x}(t) - \Delta\mathbf{x}^*\right] - \mathbf{C}_I\left\{\int_0^t \left[\left(\mathbf{H}_\mathbf{x}\Delta\mathbf{x} + \mathbf{H}_\mathbf{u}\Delta\mathbf{u}\right) - \Delta\mathbf{y}_C\right]dt\right\}$$

LQ Proportional-Integral (*PI*) Control with Command Input

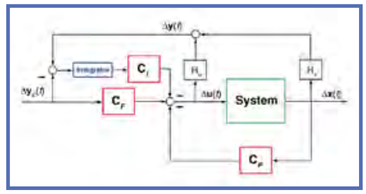
The cost function is minimized by a control law of the form

$$\Delta\mathbf{u}(t) = \left(\mathbf{B}_{22} + \mathbf{C}_P\mathbf{B}_{12}\right)\Delta\mathbf{y}_C - \mathbf{C}_P\Delta\mathbf{x}(t) + \mathbf{C}_I\int_0^t \left[\Delta\mathbf{y}_C - \Delta\mathbf{y}(t)\right]dt$$

$$= \mathbf{C}_F\Delta\mathbf{y}_C - \mathbf{C}_P\Delta\mathbf{x}(t) + \mathbf{C}_I\int_0^t \left[\Delta\mathbf{y}_C - \Delta\mathbf{y}(t)\right]dt$$



Integrating Action Sets Equilibrium Command Error to Zero



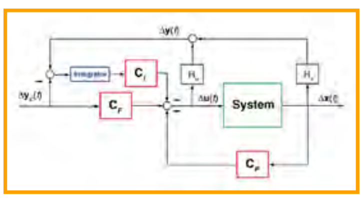
The closed-loop system

$$\begin{bmatrix} \Delta\dot{\tilde{\mathbf{x}}}(t) \\ \Delta\dot{\tilde{\boldsymbol{\xi}}}(t) \end{bmatrix} = \begin{bmatrix} \left(\mathbf{F}-\mathbf{G}\mathbf{C}_P\right) & -\mathbf{G}\mathbf{C}_I \\ \left(\mathbf{H}_\mathbf{x}-\mathbf{H}_\mathbf{u}\mathbf{C}_P\right) & -\mathbf{H}_\mathbf{u}\mathbf{C}_I \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{x}}(t) \\ \Delta\tilde{\boldsymbol{\xi}}(t) \end{bmatrix} \quad \text{is stable}$$

Therefore

$$\begin{aligned} \Delta\tilde{\mathbf{x}}(t) &= \left[\Delta\mathbf{x}(t)-\Delta\mathbf{x}^*\right] \xrightarrow[t\to\infty]{} \mathbf{0} \\ \Delta\tilde{\mathbf{u}}(t) &= \left[\Delta\mathbf{u}(t)-\Delta\mathbf{u}^*\right] \xrightarrow[t\to\infty]{} \mathbf{0} \\ \Delta\tilde{\mathbf{y}}(t) &= \left[\Delta\mathbf{y}(t)-\Delta\mathbf{y}_C\right] \xrightarrow[t\to\infty]{} \mathbf{0} \end{aligned}$$

$$\begin{aligned} \Delta\mathbf{x}(t) &\xrightarrow[t\to\infty]{} \Delta\mathbf{x}^* \\ \Delta\mathbf{u}(t) &\xrightarrow[t\to\infty]{} \Delta\mathbf{u}^* \\ \Delta\mathbf{y}(t) &\xrightarrow[t\to\infty]{} \Delta\mathbf{y}_C \end{aligned}$$



Equilibrium Error Due to Constant Disturbance is Zero

Equilibrium response to constant disturbance is constant

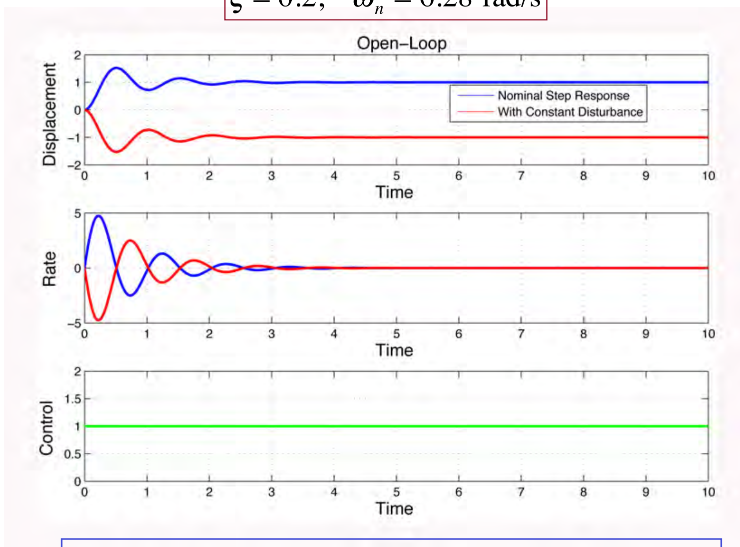
$$\begin{bmatrix} \Delta\tilde{\mathbf{x}}^*(t) \\ \Delta\tilde{\boldsymbol{\xi}}^*(t) \end{bmatrix} = -\begin{bmatrix} \left(\mathbf{F}-\mathbf{G}\mathbf{C}_P\right) & -\mathbf{G}\mathbf{C}_I \\ \left(\mathbf{H}_\mathbf{x}-\mathbf{H}_\mathbf{u}\mathbf{C}_P\right) & -\mathbf{H}_\mathbf{u}\mathbf{C}_I \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{L} \\ \mathbf{0} \end{bmatrix} \Delta\mathbf{w}^*$$

Therefore

$$\begin{aligned} \Delta\mathbf{x}(t) &\xrightarrow[t\to\infty]{} \Delta\mathbf{x}^* \\ \Delta\mathbf{u}(t) &\xrightarrow[t\to\infty]{} \Delta\mathbf{u}^* + \Delta\mathbf{u}_{\mathbf{w}^*} \\ \Delta\mathbf{y}(t) &\xrightarrow[t\to\infty]{} \Delta\mathbf{y}_C \end{aligned}$$

Example: Open-Loop Response of a 2nd-Order System, with and without Constant Disturbance

$$\zeta = 0.2, \quad \omega_n = 6.28 \text{ rad/s}$$

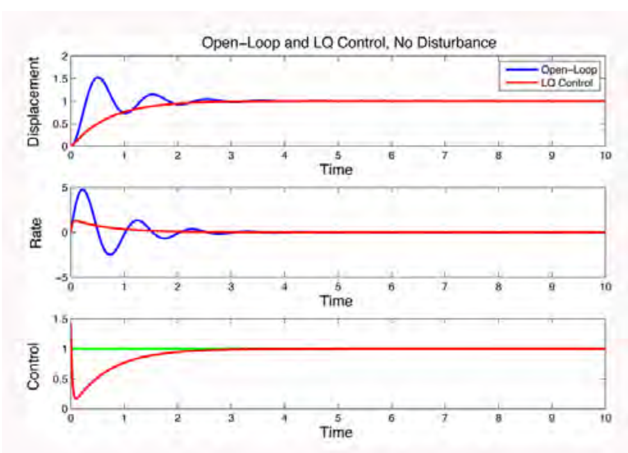


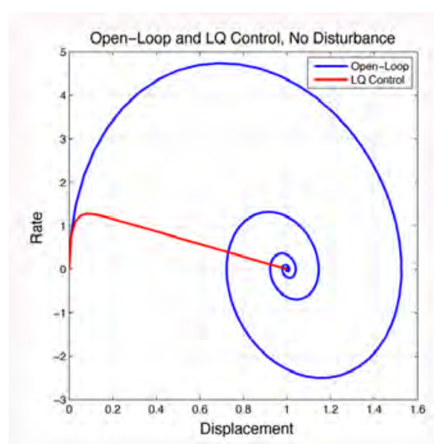
$$\Delta\dot{\mathbf{x}} = \mathbf{F}\Delta\mathbf{x} + \mathbf{G}\Delta\mathbf{y}_C, \quad \Delta\mathbf{y}_C = 1$$

$$\Delta\dot{\mathbf{x}} = \mathbf{F}\Delta\mathbf{x} + \mathbf{G}\Delta\mathbf{y}_C + \mathbf{G}\Delta\mathbf{w}^*, \quad \Delta\mathbf{y}_C = 1, \quad \Delta\mathbf{w}^* = -2$$

Example: Open-Loop and LQ Control of 2nd-Order System

Step input, with and without LQ Control, No Disturbance





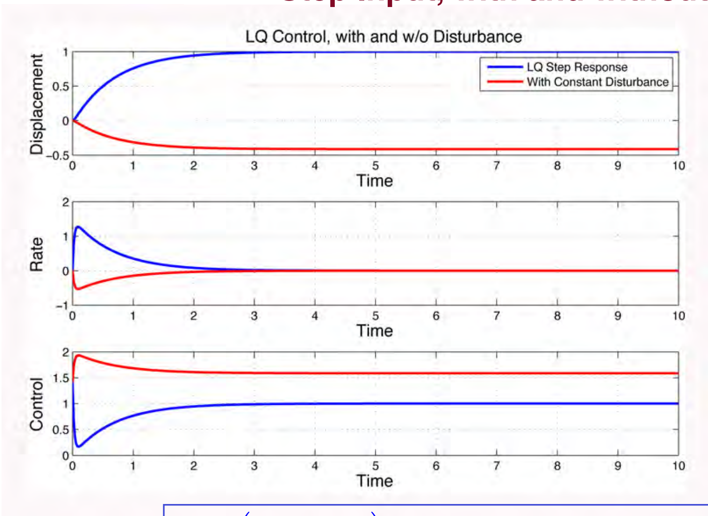
$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{R} = 1$$

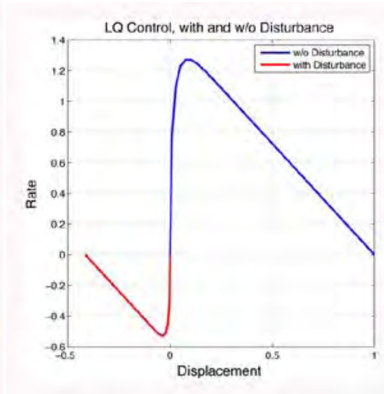
$$\Delta\dot{\mathbf{x}} = \mathbf{F}\Delta\mathbf{x} + \mathbf{G}\Delta\mathbf{y}_C, \quad \Delta\mathbf{y}_C = 1$$

$$\Delta\dot{\mathbf{x}} = \left(\mathbf{F} - \mathbf{G}\mathbf{C}_B\right)\Delta\mathbf{x} + \mathbf{G}\mathbf{C}_F\Delta\mathbf{y}_C, \quad \Delta\mathbf{y}_C = 1$$

Example: LQ Control, with and without Disturbance

Step Input, with and without Disturbance

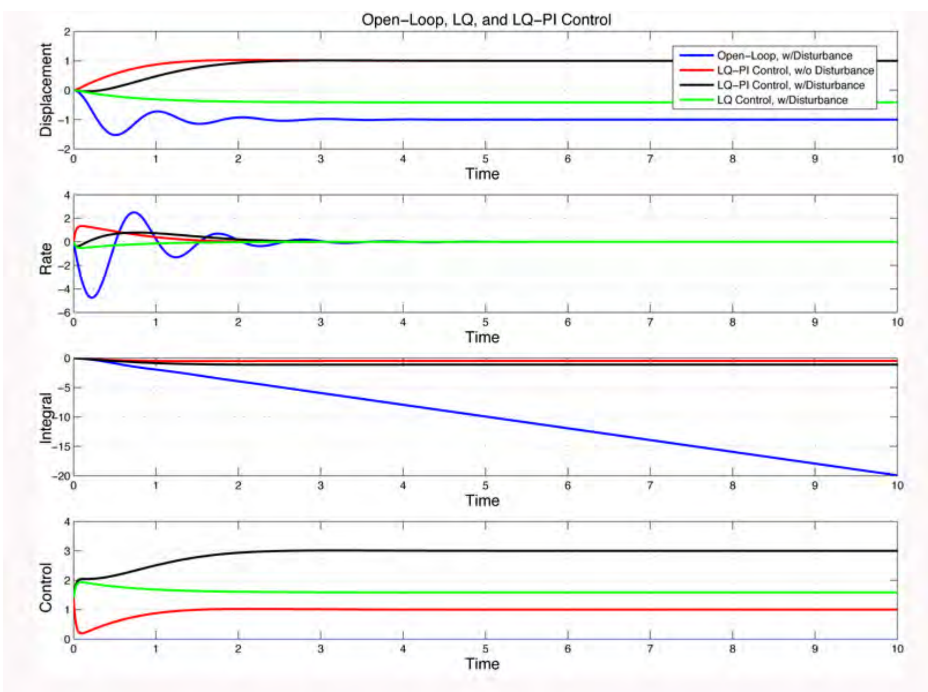




$$\Delta\dot{\mathbf{x}} = \left(\mathbf{F} - \mathbf{G}\mathbf{C}_B\right)\Delta\mathbf{x} + \mathbf{G}\mathbf{C}_F\Delta\mathbf{y}_C, \quad \Delta\mathbf{y}_C = 1$$

$$\Delta\dot{\mathbf{x}} = \left(\mathbf{F} - \mathbf{G}\mathbf{C}_B\right)\Delta\mathbf{x} + \mathbf{G}\mathbf{C}_F\Delta\mathbf{y}_C + \mathbf{G}\Delta\mathbf{w}, \quad \Delta\mathbf{y}_C = 1, \quad \Delta\mathbf{w} = -2$$

Example: Open-Loop, LQ, and LQ Proportional-Integral Control of 2nd-Order System



Example: Open-Loop, LQ, and LQ Proportional-Integral Control of 2nd-Order System

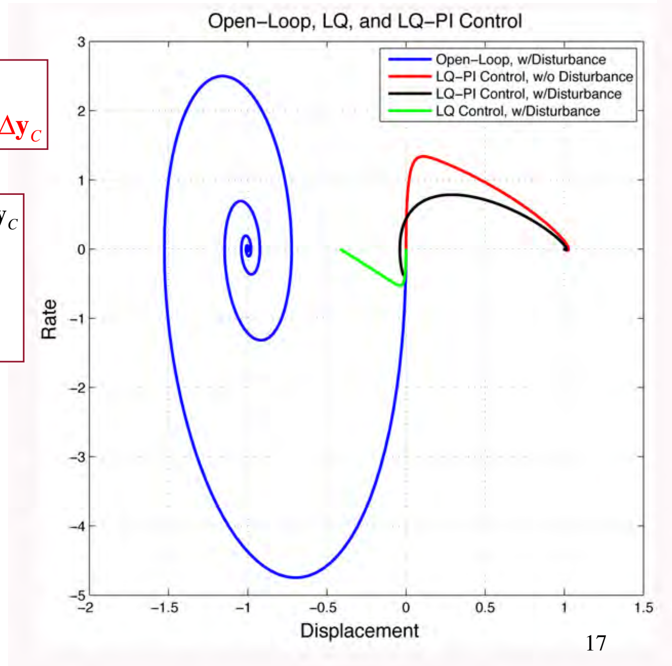
$$\Delta\dot{\mathbf{x}} = \mathbf{F}\Delta\mathbf{x} + \mathbf{G}\Delta\mathbf{y}_C + \mathbf{G}\Delta\mathbf{w}^*$$

$$\Delta\dot{\mathbf{x}} = \left(\mathbf{F} - \mathbf{G}\mathbf{C}_B\right)\Delta\mathbf{x} - \mathbf{G}\mathbf{C}_I\Delta\boldsymbol{\xi} + \mathbf{G}\mathbf{C}_F\Delta\mathbf{y}_C$$

$$\Delta\dot{\mathbf{x}} = \left(\mathbf{F} - \mathbf{G}\mathbf{C}_B\right)\Delta\mathbf{x} - \mathbf{G}\mathbf{C}_I\Delta\boldsymbol{\xi} + \mathbf{G}\mathbf{C}_F\Delta\mathbf{y}_C + \mathbf{G}\Delta\mathbf{w}^*$$

$$\Delta\dot{\mathbf{x}} = \left(\mathbf{F} - \mathbf{G}\mathbf{C}_B\right)\Delta\mathbf{x} + \mathbf{G}\mathbf{C}_F\Delta\mathbf{y}_C + \mathbf{G}\Delta\mathbf{w}^*$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix} \qquad \mathbf{R} = 1$$



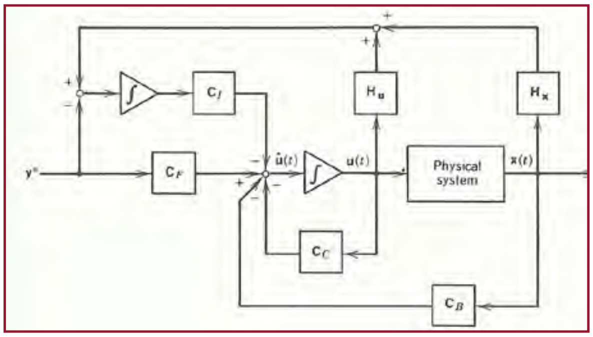
Proportional-Integral-Filter (*PIF*) Controller

- Introduce
  - Integration of command-response error
  - Low-pass filtering of actuator input

$$\begin{bmatrix} \Delta\dot{\tilde{\mathbf{x}}}(t) \\ \Delta\dot{\tilde{\mathbf{u}}}(t) \\ \Delta\dot{\tilde{\boldsymbol{\xi}}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{H}_\mathbf{x} & \mathbf{H}_\mathbf{u} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{x}}(t) \\ \Delta\tilde{\mathbf{u}}(t) \\ \Delta\tilde{\boldsymbol{\xi}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \Delta\mathbf{v}(t)$$

$$J = \frac{1}{2}\int_0^\infty \left[\begin{bmatrix} \Delta\tilde{\mathbf{x}}^T(t) & \Delta\tilde{\mathbf{u}}^T(t) & \Delta\tilde{\boldsymbol{\xi}}^T(t) \end{bmatrix} \begin{bmatrix} \mathbf{Q}_\mathbf{x} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_\mathbf{u} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_\xi \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{x}}(t) \\ \Delta\tilde{\mathbf{u}}(t) \\ \Delta\tilde{\boldsymbol{\xi}}(t) \end{bmatrix} + \Delta\mathbf{v}^T(t)\mathbf{R}_\mathbf{v}\Delta\mathbf{v}(t) \right] dt$$

Optimal *PIF* Control Law

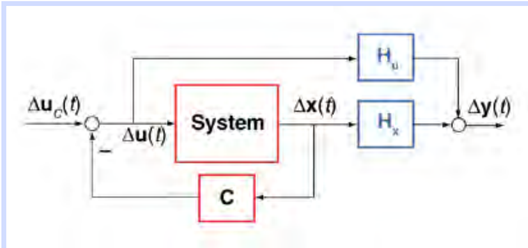


- Pure integration (high low-frequency gain)
- Low-pass filtering for smooth actuator command
- Lead (derivative) compensation
- Zero steady-state error
- Satisfies Bode criteria

$$\boxed{\Delta\mathbf{v}(t) = \mathbf{C}_F\Delta\tilde{\mathbf{y}}(t) - \mathbf{C}_B\Delta\tilde{\mathbf{x}}(t) - \mathbf{C}_I\Delta\tilde{\boldsymbol{\xi}}(t) - \mathbf{C}_C\Delta\tilde{\mathbf{u}}(t) = \Delta\dot{\tilde{\mathbf{u}}}_A(t)}$$

LQ Model-Following Control

Implicit Model-Following LQ Regulator



Actual and Ideal Models

$$\Delta\dot{\mathbf{x}}(t)=\mathbf{F}\Delta\mathbf{x}(t)+\mathbf{G}\Delta\mathbf{u}(t)$$
$$\Delta\dot{\mathbf{x}}_M(t)=\mathbf{F}_M\Delta\mathbf{x}_M(t)$$

$$J=\frac{1}{2}\int_0^\infty\left\{\left[\Delta\dot{\mathbf{x}}(t)-\Delta\dot{\mathbf{x}}_M(t)\right]^T\mathbf{Q}_M\left[\Delta\dot{\mathbf{x}}(t)-\Delta\dot{\mathbf{x}}_M(t)\right]\right\}dt\triangleq\frac{1}{2}\int_0^\infty\left\{\begin{bmatrix}\Delta\mathbf{x}^T(t) & \Delta\mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}\mathbf{Q} & \mathbf{M}\\ \mathbf{M}^T & \mathbf{R}\end{bmatrix}_{IMF}\begin{bmatrix}\Delta\mathbf{x}(t)\\ \Delta\mathbf{u}(t)\end{bmatrix}\right\}dt$$

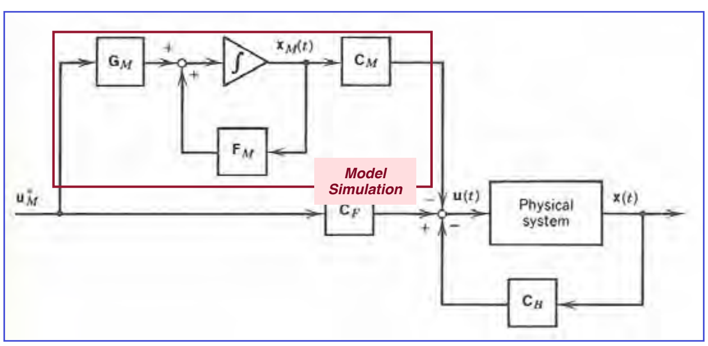
Cost-minimizing control law

$$\Delta\mathbf{u}(t)=\Delta\mathbf{u}_C(t)-\mathbf{C}_M\Delta\mathbf{x}(t)$$

$$\Delta\dot{\mathbf{x}}(t)=\mathbf{F}\Delta\mathbf{x}(t)+\mathbf{G}\left[\Delta\mathbf{u}_C(t)-\mathbf{C}_M\Delta\mathbf{x}(t)\right]$$
$$=\left[\mathbf{F}-\mathbf{G}\mathbf{C}_M\right]\Delta\mathbf{x}(t)+\mathbf{G}\Delta\mathbf{u}_C(t)$$

LQ control shifts closed-loop roots toward desired values

Explicit Model Following



- Model of the ideal system is <u>explicitly</u> included in the control law
  - Could have lower dimension than actual system
  - Here, we assume dimensions are the same

$$\begin{bmatrix}\Delta\dot{\tilde{\mathbf{x}}}(t)\\ \Delta\dot{\tilde{\mathbf{x}}}_M(t)\end{bmatrix}=\begin{bmatrix}\mathbf{F} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}_M\end{bmatrix}\begin{bmatrix}\Delta\tilde{\mathbf{x}}(t)\\ \Delta\tilde{\mathbf{x}}_M(t)\end{bmatrix}+\begin{bmatrix}\mathbf{G} & \mathbf{0}\\ \mathbf{0} & \mathbf{G}_M\end{bmatrix}\begin{bmatrix}\Delta\tilde{\mathbf{u}}(t)\\ \Delta\tilde{\mathbf{u}}_M(t)\end{bmatrix}$$

- Control law forces actual system to mimic the ideal system

Explicit Model Following

Output vector = error between actual and ideal state vectors

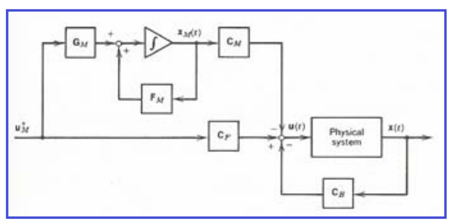
$$\boxed{\Delta\tilde{\mathbf{y}}(t) \triangleq \Delta\tilde{\mathbf{x}}(t) - \Delta\tilde{\mathbf{x}}_M(t) = \begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{x}}(t) \\ \Delta\tilde{\mathbf{x}}_M(t) \end{bmatrix}}$$

Output vector cost function

$$J = \frac{1}{2}\int_0^{\infty} \left[\Delta\tilde{\mathbf{y}}^T(t)\mathbf{Q}\Delta\tilde{\mathbf{y}}(t) + \Delta\tilde{\mathbf{u}}^T(t)\mathbf{R}\Delta\tilde{\mathbf{u}}(t) \right] dt$$

$$J = \frac{1}{2}\int_0^{\infty} \left[\begin{bmatrix} \Delta\tilde{\mathbf{x}}^T(t) & \Delta\tilde{\mathbf{x}}_M^T(t) \end{bmatrix} \begin{bmatrix} \mathbf{Q} & -\mathbf{Q} \\ -\mathbf{Q} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{x}}(t) \\ \Delta\tilde{\mathbf{x}}_M(t) \end{bmatrix} + \Delta\tilde{\mathbf{u}}^T(t)\mathbf{R}\Delta\tilde{\mathbf{u}}(t) \right] dt$$

Algebraic Riccati Equation



- **Algebraic Riccati equation**

$$\mathbf{0} = -\begin{bmatrix} \mathbf{F}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_M^T \end{bmatrix} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12} & \mathbf{P}_{22} \end{bmatrix} - \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12} & \mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_M \end{bmatrix} - \begin{bmatrix} \mathbf{Q} & -\mathbf{Q} \\ -\mathbf{Q} & \mathbf{Q} \end{bmatrix} + \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12} & \mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12} & \mathbf{P}_{22} \end{bmatrix}$$

- **Three equations**
  - First is the LQ Riccati equation for the actual system; it solves for $\mathbf{P}_{11}$

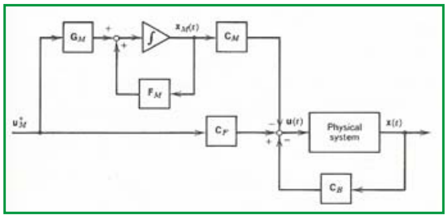
$$\mathbf{0} = -\mathbf{F}^T\mathbf{P}_{11} - \mathbf{P}_{11}\mathbf{F} - \mathbf{Q} + \mathbf{P}_{11}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T\mathbf{P}_{11}$$

  - Second solves for $\mathbf{P}_{12}$

$$\mathbf{0} = \left(-\mathbf{F}^T\mathbf{P}_{11} - \mathbf{P}_{11}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T\right)\mathbf{P}_{12} - \mathbf{F}_M + \mathbf{Q}$$

  - Third solves for $\mathbf{P}_{22}$

$$\mathbf{0} = -\mathbf{F}_M^T\mathbf{P}_{22} - \mathbf{P}_{22}\mathbf{F}_M - \mathbf{Q} + \mathbf{P}_{12}^T\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T\mathbf{P}_{12}$$

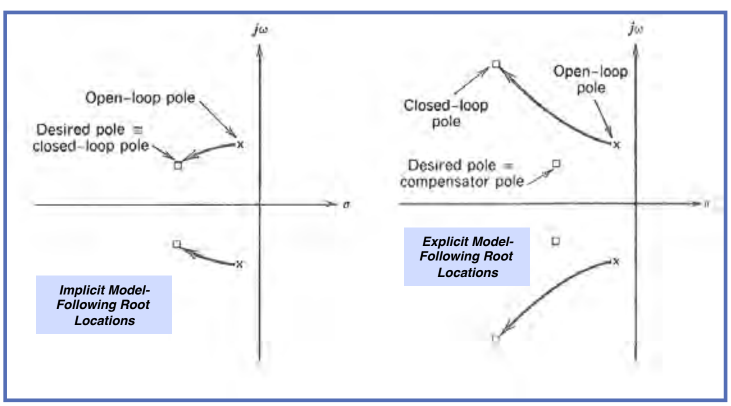


Explicit Model Following

$$\mathbf{C} = \mathbf{R}^{-1}\begin{pmatrix} \mathbf{G}^T & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^T & \mathbf{P}_{22} \end{pmatrix} = \mathbf{R}^{-1}\mathbf{G}^T\begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_B & \mathbf{C}_M \end{pmatrix}$$

- **Feedback gain is independent of the forward gains**
- **Therefore, it determines the stability and bandwidth of the actual system**
- **Forward gains, $\mathbf{C}_F$ and $\mathbf{C}_M$, act as a "pre-filter" that shapes the command input to have ideal system dynamics**

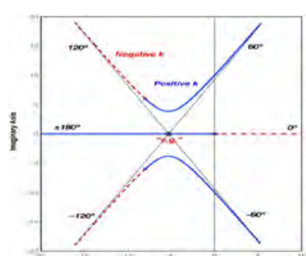
Closed-Loop Root Locations for Implicit and Explicit Model Following



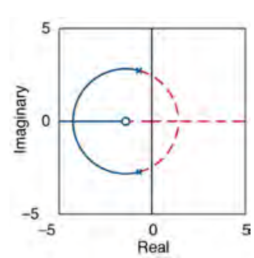
- **Implicit model-following system has *n* roots**
  - ***n* LQ closed-loop roots approach roots of ideal system**
  - **Relatively small feedback gains**
- **Explicit model-following system has (*n* + 1) to *2n* roots**
  - ***n* LQ closed-loop roots forced to large, fast values**
  - **1 to *n* ideal system roots specified as input to the LQ compensator**
  - **Relatively large feedback gains**

Root Locus Analysis

Root Locus Analysis of Control Effects on System Dynamics



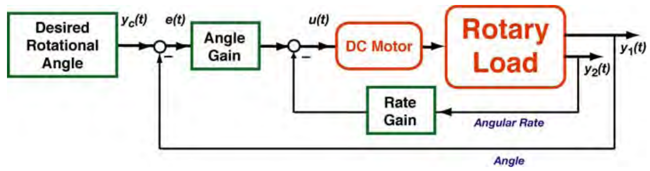
- **Graphical depiction of control effects on location of eigenvalues of F (or roots of the characteristic polynomial)**
- **Evan's rules for root locus construction**



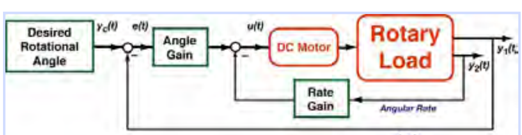
Locus: "the set of all points whose location is determined by stated conditions" (*Webster's Dictionary*)

Root Loci for Angle and Rate Feedback

- Variation of roots as a scalar gain, c_i, goes from 0 to ∞
- Example: DC motor control

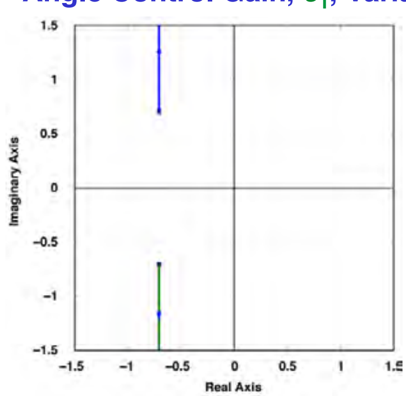



```
%   Root Locus of DC Motor Angle Control

    F   =  [0 1;-1 -1.414];
    G   =   [0;1];

    Hx1 =   [1 0];  %   Angle Output
    Hx2 =   [0 1];  %   Angular Rate Output

    Sys1    =   ss(F,G,Hx1,0);
    Sys2    =   ss(F,G,Hx2,0);

    rlocus(Sys1), grid
    figure
    rlocus(Sys2), grid
```

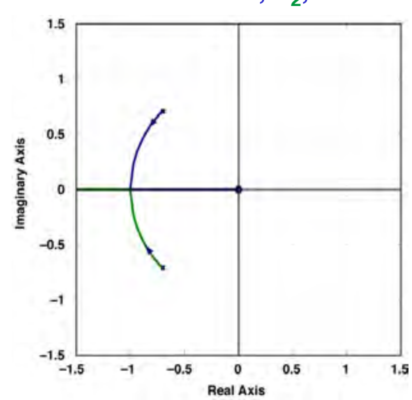



Root Loci for Angle and Rate Feedback

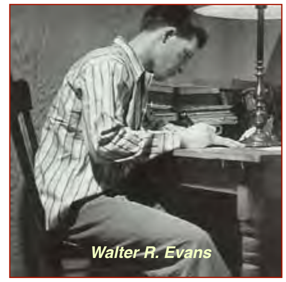
- Angle Control Gain, c_1, Variation



- Rate Control Gain, c_2, Variation



Effect of Parameter Variations on Root Location



Example: Characteristic equation of aircraft longitudinal motion

$$
\begin{aligned}
\Delta_{Lon}(s) &= s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \\
&= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) = (s-\lambda_1)(s-\lambda_1^*)(s-\lambda_3)(s-\lambda_3^*) \\
&= \left(s^2 + 2\zeta_P\omega_{n_P}s + \omega_{n_P}^2\right)\left(s^2 + 2\zeta_{SP}\omega_{n_{SP}}s + \omega_{n_{SP}}^2\right) = 0
\end{aligned}
$$

- **What effect would variations in a_i have on the locations (or locus) of roots?**
  - Let "root locus gain" = k = $c_i = a_i$ (just a notation change)
    - Option 1: Vary k and calculate roots for each new value
    - Option 2: Apply Evans's Rules of Root Locus Construction

Effect of a_0 Variation on Longitudinal Root Location

Example: $k = a_0$

$$
\begin{aligned}
\Delta_{Lon}(s) &= \left[s^4 + a_3 s^3 + a_2 s^2 + a_1 s\right] + [k] \equiv d(s) + kn(s) \\
&= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) = 0
\end{aligned}
$$

$d(s)$: Polynomial in s

$n(s)$: Polynomial in s

where

$$
\begin{aligned}
d(s) &= s^4 + a_3 s^3 + a_2 s^2 + a_1 s \\
&= (s-\lambda'_1)(s-\lambda'_2)(s-\lambda'_3)(s-\lambda'_4)
\end{aligned}
$$

$$n(s) = 1$$

Effect of a_1 Variation on Longitudinal Root Location

Example: $k = a_1$

$$\Delta_{Lon}(s) = s^4 + a_3 s^3 + a_2 s^2 + ks + a_0 \equiv d(s) + kn(s)$$
$$= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) = 0$$

where

$$d(s) = s^4 + a_3 s^3 + a_2 s^2 + a_0$$
$$= (s-\lambda'_1)(s-\lambda'_2)(s-\lambda'_3)(s-\lambda'_4)$$
$$n(s) = s$$

Three Equivalent Expressions for the Polynomial

$$d(s) + k\, n(s) = 0$$

$$1 + k\frac{n(s)}{d(s)} = 0$$

$$k\frac{n(s)}{d(s)} = -1 = (1)e^{-j\pi(rad)} = (1)e^{-j180(\deg)}$$

Example: Effect of a_0 Variation

Original 4th-order polynomial

$$\Delta_{Lon}(s) = s^4 + 2.57s^3 + 9.68s^2 + 0.202s + 0.145 = 0$$

Example: $k = a_0$

$$\begin{aligned}\Delta(s) &= s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \\ &= \left(s^4 + a_3 s^3 + a_2 s^2 + a_1 s\right) + k \\ &= s\left(s^3 + a_3 s^2 + a_2 s + a_1\right) + k \\ &= s\left(s + 0.21\right)\left[s^2 + 2.55s + 9.62\right] + k\end{aligned}$$

$$\frac{k}{s\left(s + 0.21\right)\left[s^2 + 2.55s + 9.62\right]} = -1$$

Example: Effect of a_1 Variation

Example: $k = a_1$

$$\begin{aligned}\Delta(s) &= s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \\ &= s^4 + a_3 s^3 + a_2 s^2 + ks + a_0 \\ &= \left(s^4 + a_3 s^3 + a_2 s^2 + a_0\right) + ks \\ &= \left[s^2 - 0.00041s + 0.015\right]\left[s^2 + 2.57s + 9.67\right] + ks\end{aligned}$$

$$\frac{ks}{\left[s^2 - 0.00041s + 0.015\right]\left[s^2 + 2.57s + 9.67\right]} = -1$$

The Root Locus Criterion

- All points on the locus of roots must satisfy the equation *k[n(s)/d(s)]* = –1
- Phase angle(–1) = ±180 deg

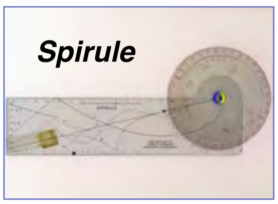
- Number of roots (or *poles*) of the denominator = *n*
- Number of zeros of the numerator = *q*

$$k = a_0: \quad k\frac{n(s)}{d(s)} = k\frac{1}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s} = -1$$

- Number of roots = 4
- Number of zeros = 0
- $(n - q) = 4$

$$k = a_1: \quad k\frac{n(s)}{d(s)} = k\frac{s}{s^4 + a_3 s^3 + a_2 s^2 + a_0} = -1$$

- Number of roots = 4
- Number of zeros = 1
- $(n - q) = 3$

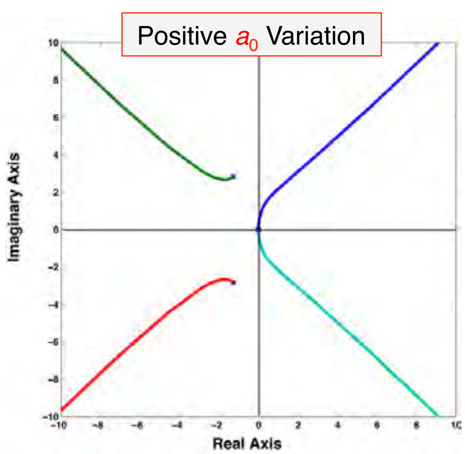


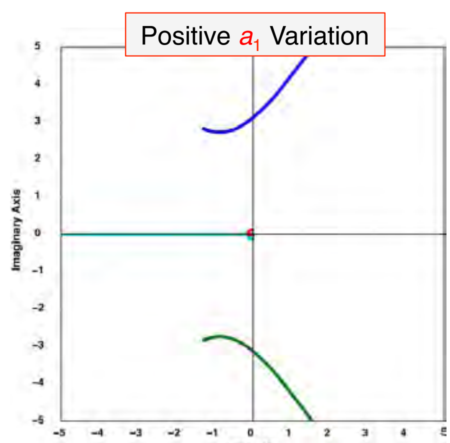
- Manual graphical construction of the root locus
- Invented by Walter Evans

Origins of Roots (for *k* = 0)

- Origins of the roots are the Poles of *d(s)*

$$\Delta(s) = d(s) + kn(s) \xrightarrow[k \to 0]{} d(s)$$



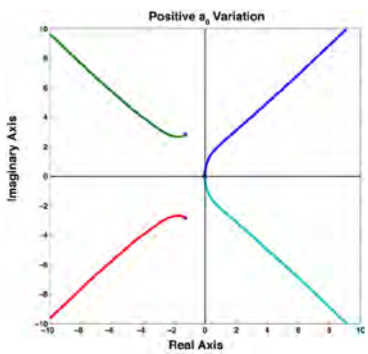


<u>Destinations</u> of Roots (for k -> ±∞)

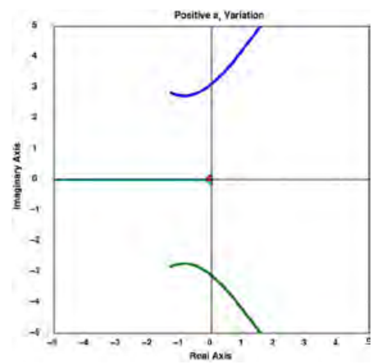
- ***q*** **roots go to the zeros of** ***n(s)***

$$\boxed{\frac{d(s)+kn(s)}{k}=\frac{d(s)}{k}+n(s)\xrightarrow[k\to\infty]{}n(s)}$$

No zeros when k = a_0



One zero at origin when k = a_1



<u>Destinations</u> of Roots (for k -> ±∞)

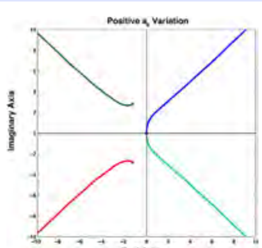
(*n* – *q*) roots go to infinite radius from the origin

$$\boxed{\frac{d(s)+kn(s)}{n(s)}=\left[\frac{d(s)}{n(s)}+k\right]\xrightarrow[k\to\pm\infty]{}\left[s^{(n-q)}\pm R\right]\to\pm\infty}$$

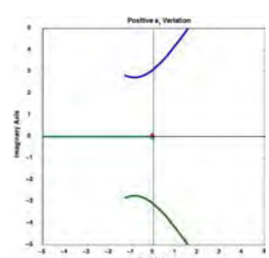
$$\boxed{s^{(n-q)}=Re^{-j180^\circ}\xrightarrow[k\to+\infty]{}\infty \quad or \quad Re^{-j360^\circ}\xrightarrow[k\to-\infty]{}-\infty}$$

$$\boxed{s=Re^{-j180^\circ/(n-q)}\xrightarrow[k\to+\infty]{}\infty \quad or \quad Re^{-j360^\circ/(n-q)}\xrightarrow[k\to-\infty]{}-\infty}$$

4 roots to infinite radius



3 roots to infinite radius



($n - q$) Roots Approach Asymptotes as $k \rightarrow \pm\infty$

Asymptote angles for positive k

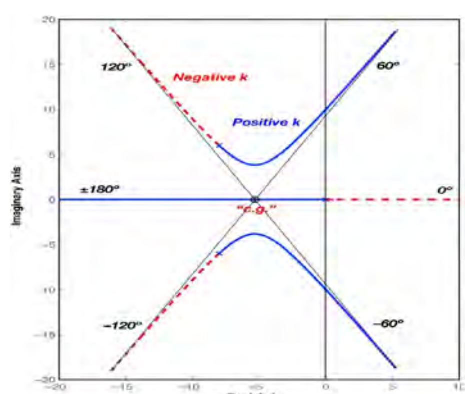
$$\theta(rad) = \frac{\pi + 2m\pi}{n - q}, \quad m = 0, 1, \ldots, (n - q) - 1$$

Asymptote angles for negative k

$$\theta(rad) = \frac{2m\pi}{n - q}, \quad m = 0, 1, \ldots, (n - q) - 1$$

Origin of Asymptotes = "Center of Gravity"

$$"c.g." = \frac{\sum_{i=1}^{n} \sigma_{\lambda_i} - \sum_{j=1}^{q} \sigma_{z_j}}{n - q}$$

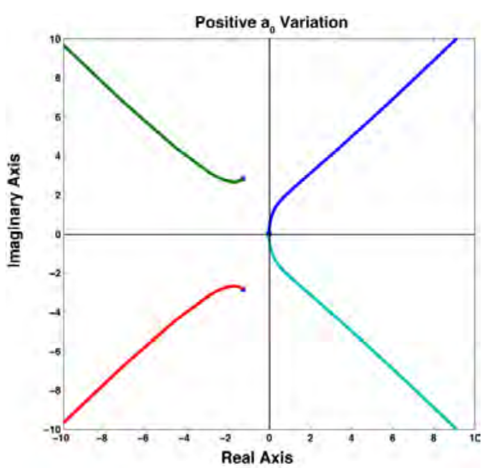


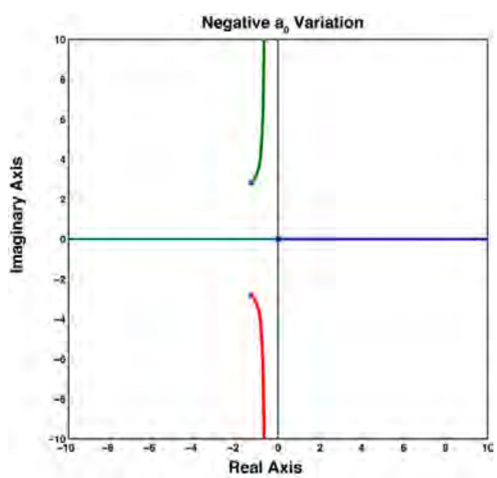
Root Locus on Real Axis

- **Locus on real axis**
  - **k > 0: Any segment with odd number of poles and zeros to the right**
  - **k < 0: Any segment with even number of poles and zeros to the right**

First Example: $k = a_0$

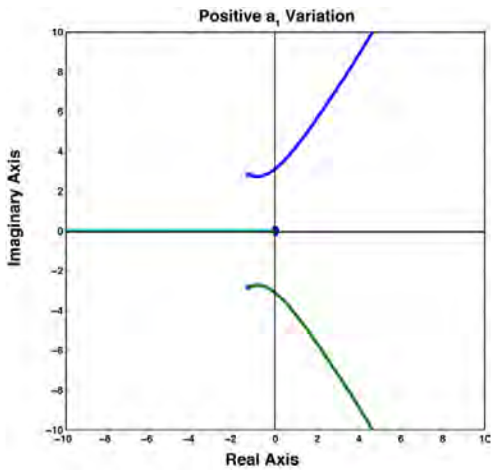
$$\boxed{\frac{k}{s(s+0.21)\left[s^2+2.55s+9.62\right]}=-1}$$

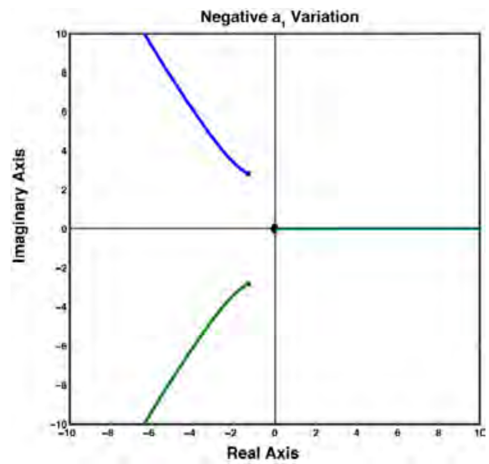




Second Example: $k = a_1$

$$\boxed{\frac{ks}{\left[s^2 - 0.00041s + 0.015\right]\left[s^2 + 2.57s + 9.67\right]} = -1}$$





Next Time: Modal Properties of LQ Regulators

Supplemental Material

Truncation and Residualization

Reduction of Dynamic Model Order

- Separation of high-order models into loosely coupled or decoupled lower order approximations
  - [Rigid body] + [Flexible modes]
  - Chemical/biological process with fast and slow reactions
  - Economic system with local and global components
  - Social networks with large and small clusters

$$\begin{bmatrix} \Delta\dot{\mathbf{x}}_{fast} \\ \Delta\dot{\mathbf{x}}_{slow} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{fast} & \mathbf{F}_{slow}^{fast} \\ \mathbf{F}_{fast}^{slow} & \mathbf{F}_{slow} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_{fast} \\ \Delta\mathbf{x}_{slow} \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{fast} & \mathbf{G}_{slow}^{fast} \\ \mathbf{G}_{fast}^{slow} & \mathbf{G}_{slow} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}_{fast} \\ \Delta\mathbf{u}_{slow} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{F}_f & small \\ small & \mathbf{F}_s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_f \\ \Delta\mathbf{x}_s \end{bmatrix} + \begin{bmatrix} \mathbf{G}_f & small \\ small & \mathbf{G}_s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}_f \\ \Delta\mathbf{u}_s \end{bmatrix}$$

Truncation of a Dynamic Model

- Dynamic model order reduction when
  - Two modes are only slightly coupled
  - Time scales of motions are far apart
  - Forcing terms are largely independent

$$\begin{bmatrix} \Delta\dot{\mathbf{x}}_f \\ \Delta\dot{\mathbf{x}}_s \end{bmatrix} = \begin{bmatrix} \mathbf{F}_f & \mathbf{F}_s^f \\ \mathbf{F}_f^s & \mathbf{F}_s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_f \\ \Delta\mathbf{x}_s \end{bmatrix} + \begin{bmatrix} \mathbf{G}_f & \mathbf{G}_s^f \\ \mathbf{G}_f^s & \mathbf{G}_s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}_f \\ \Delta\mathbf{u}_s \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{F}_f & small \\ small & \mathbf{F}_s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_f \\ \Delta\mathbf{x}_s \end{bmatrix} + \begin{bmatrix} \mathbf{G}_f & small \\ small & \mathbf{G}_s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}_f \\ \Delta\mathbf{u}_s \end{bmatrix}$$

$$\approx \boxed{\begin{bmatrix} \mathbf{F}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_s \end{bmatrix}} \begin{bmatrix} \Delta\mathbf{x}_f \\ \Delta\mathbf{x}_s \end{bmatrix} + \boxed{\begin{bmatrix} \mathbf{G}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_s \end{bmatrix}} \begin{bmatrix} \Delta\mathbf{u}_f \\ \Delta\mathbf{u}_s \end{bmatrix}$$

- Approximation: Modes can be analyzed and control systems can be designed separately

$$\begin{aligned} \Delta\dot{\mathbf{x}}_f &= \mathbf{F}_f\Delta\mathbf{x}_f + \mathbf{G}_f\Delta\mathbf{u}_f \\ \Delta\dot{\mathbf{x}}_s &= \mathbf{F}_s\Delta\mathbf{x}_s + \mathbf{G}_s\Delta\mathbf{u}_s \end{aligned}$$

Residualization of a Dynamic Model

- Dynamic model order reduction when
  - Two modes are coupled
  - Time scales of motions are separated
  - Fast mode is stable

$$\begin{bmatrix} \Delta\dot{\mathbf{x}}_f \\ \Delta\dot{\mathbf{x}}_s \end{bmatrix} = \begin{bmatrix} \mathbf{F}_f & \mathbf{F}_s^f \\ \mathbf{F}_f^s & \mathbf{F}_s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_f \\ \Delta\mathbf{x}_s \end{bmatrix} + \begin{bmatrix} \mathbf{G}_f & \mathbf{G}_s{}^f \\ \mathbf{G}_f{}^s & \mathbf{G}_s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}_f \\ \Delta\mathbf{u}_s \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{F}_f & small \\ small & \mathbf{F}_s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_f \\ \Delta\mathbf{x}_s \end{bmatrix} + \begin{bmatrix} \mathbf{G}_f & small \\ small & \mathbf{G}_s \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}_f \\ \Delta\mathbf{u}_s \end{bmatrix}$$

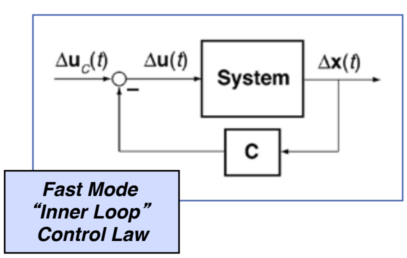
- Approximation: Motions can be analyzed separately using different "clocks"
  - Fast mode reaches steady state instantaneously on slow-mode time scale
  - Slow mode produces slowly changing bias disturbances on fast-mode time scale

Residualized Fast Mode

Fast mode dynamics

$$\Delta\dot{\mathbf{x}}_f = \mathbf{F}_f \Delta\mathbf{x}_f + \mathbf{G}_f \Delta\mathbf{u}_f$$
$$+ \left(\mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}^f{}_s \Delta\mathbf{u}_s \right)_{\sim Bias}$$

If fast mode is not stable, it could be stabilized by "inner loop" control



$$\Delta\dot{\mathbf{x}}_f = \mathbf{F}_f \Delta\mathbf{x}_f + \mathbf{G}_f \left(\Delta\mathbf{u}_c - \mathbf{C}_f \Delta\mathbf{x}_f \right)$$
$$+ \left(\mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}_s^f \Delta\mathbf{u}_s \right)_{\sim Bias}$$
$$= \left(\mathbf{F}_f - \mathbf{G}_f \mathbf{C}_f \right) \Delta\mathbf{x}_f + \mathbf{G}_f \Delta\mathbf{u}_{f_c}$$
$$+ \left(\mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}_s^f \Delta\mathbf{u}_s \right)_{\sim Bias}$$

Fast Mode in Quasi-Steady State

Assume that fast mode reaches steady state on a time scale that is short compared to the slow mode

$$\begin{aligned} 0 &\approx \mathbf{F}_f \Delta\mathbf{x}_f + \mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}_f \Delta\mathbf{u}_f + \mathbf{G}_s^f \Delta\mathbf{u}_s \\ \Delta\dot{\mathbf{x}}_s &= \mathbf{F}_f^s \Delta\mathbf{x}_f + \mathbf{F}_s \Delta\mathbf{x}_s + \mathbf{G}_s \Delta\mathbf{u}_s + \mathbf{G}_f^s \Delta\mathbf{u}_f \end{aligned}$$

Algebraic solution for fast variable

$$\begin{aligned} 0 &\approx \mathbf{F}_f \Delta\mathbf{x}_f + \mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}_f \Delta\mathbf{u}_f + \mathbf{G}_s^f \Delta\mathbf{u}_s \\ \mathbf{F}_f \Delta\mathbf{x}_f &= -\mathbf{F}_s^f \Delta\mathbf{x}_s - \mathbf{G}_f \Delta\mathbf{u}_f - \mathbf{G}_s^f \Delta\mathbf{u}_s \\ \Delta\mathbf{x}_f &= -\mathbf{F}_f^{-1}\left(\mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}_f \Delta\mathbf{u}_f + \mathbf{G}_s^f \Delta\mathbf{u}_s\right) \end{aligned}$$

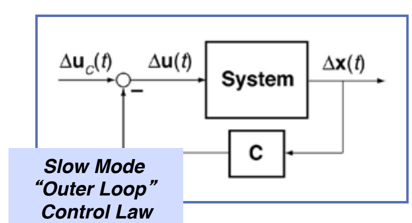
Residualized Slow Mode

Substitute quasi-steady fast variable in differential equation for slow variable

$$\begin{aligned} \Delta\dot{\mathbf{x}}_s &= -\mathbf{F}_f^s\left[\mathbf{F}_f^{-1}\left(\mathbf{F}_s^f \Delta\mathbf{x}_s + \mathbf{G}_f \Delta\mathbf{u}_f + \mathbf{G}^f{}_s \Delta\mathbf{u}_s\right)\right] + \mathbf{F}_s \Delta\mathbf{x}_s + \mathbf{G}_s \Delta\mathbf{u}_s + \mathbf{G}^s{}_f \Delta\mathbf{u}_f \\ &= \left[\mathbf{F}_s - \mathbf{F}_f^s \mathbf{F}_f^{-1} \mathbf{F}_s^f\right]\Delta\mathbf{x}_s + \left[\mathbf{G}_s - \mathbf{F}_f^s \mathbf{F}_f^{-1} \mathbf{G}_s^f\right]\Delta\mathbf{u}_s + \left[\mathbf{G}_f^s - \mathbf{F}_f^s \mathbf{F}_f^{-1} \mathbf{G}_f\right]\Delta\mathbf{u}_f \end{aligned}$$

Residualized equation for slow variable

$$\Delta\dot{\mathbf{x}}_s = \mathbf{F}_{s_{NEW}} \Delta\mathbf{x}_s + \mathbf{G}_{s_{NEW}} \begin{bmatrix} \Delta\mathbf{u}_f \\ \Delta\mathbf{u}_s \end{bmatrix}$$



Control law can be designed for reduced-order slow model, assuming inner loop has been stabilized separately