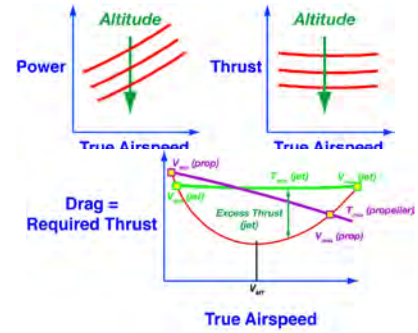
Cruising Flight Performance

Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2014



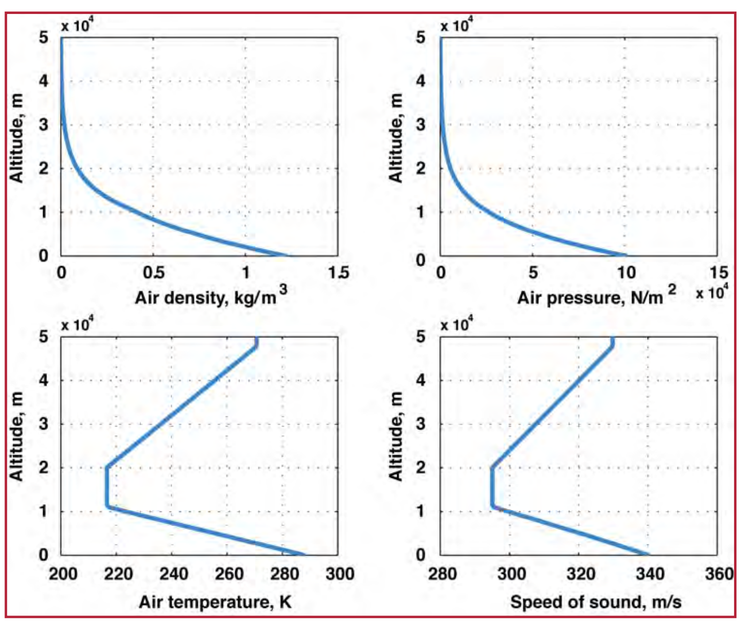
Learning Objectives

- *Definitions of airspeed*
- *Performance parameters*
- *Steady cruising flight conditions*
- *Breguet range equations*
- *Optimize cruising flight for minimum thrust and power*
- *Flight envelope*

Reading:
Flight Dynamics
Aerodynamic Coefficients, 118-130
Airplane Stability and Control
Chapter 6

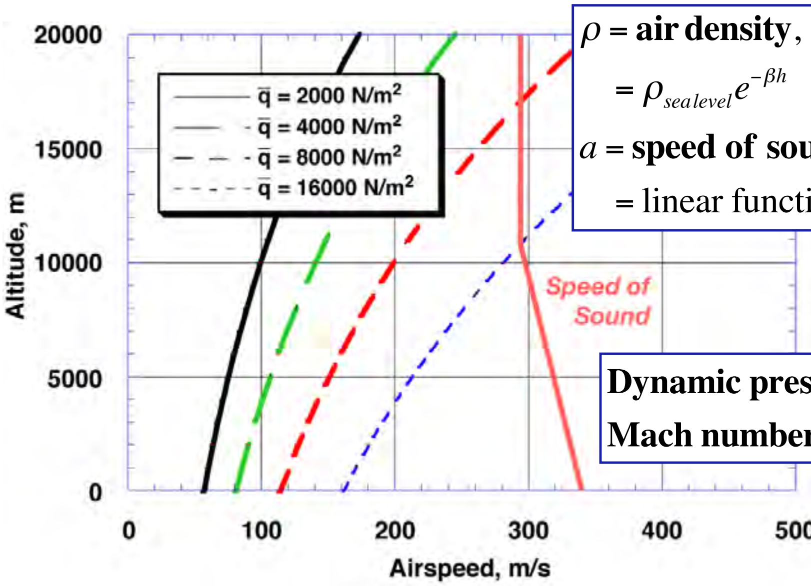
http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

U.S. Standard Atmosphere, 1976



http://en.wikipedia.org/wiki/U.S._Standard_Atmosphere

Dynamic Pressure and Mach Number



$\rho =$ **air density**, function of height

$= \rho_{sealevel} e^{-\beta h}$

$a =$ **speed of sound**

$=$ linear function of height

Dynamic pressure $= \bar{q} \triangleq \rho V^2 / 2$

Mach number $= V/a$

Definitions of Airspeed

- **Airspeed is speed of aircraft measured with respect to air mass**
  - **Airspeed = Inertial speed if wind speed = 0**

- **Indicated Airspeed (IAS)**

$$IAS = \sqrt{2\left(p_{stagnation} - p_{ambient}\right)/\rho_{SL}} = \sqrt{\frac{2\left(p_{total} - p_{static}\right)}{\rho_{SL}}}$$

$$= \sqrt{\frac{2q_c}{\rho_{SL}}}, \text{ with } q_c \triangleq \textbf{impact pressure}$$

- **Calibrated Airspeed (CAS)***

$$CAS = IAS \ corrected \ for \ instrument \ and \ position \ errors$$

$$= \sqrt{\frac{2\left(q_c\right)_{corr-1}}{\rho_{SL}}}$$

*** Kayton & Fried, 1969; NASA TN-D-822, 1961***

Definitions of Airspeed

- **Airspeed is speed of aircraft measured with respect to air mass**
  - **Airspeed = Inertial speed if wind speed = 0**

- **Equivalent Airspeed (EAS)***

$$\boxed{EAS = CAS \ corrected \ for \ compressibility \ effects = \sqrt{\frac{2\left(q_c\right)_{corr-2}}{\rho_{SL}}}}$$

- **True Airspeed (TAS)***

$$\boxed{V \triangleq TAS = EAS\sqrt{\frac{\rho_{SL}}{\rho(z)}} = IAS_{corrected}\sqrt{\frac{\rho_{SL}}{\rho(z)}}}$$

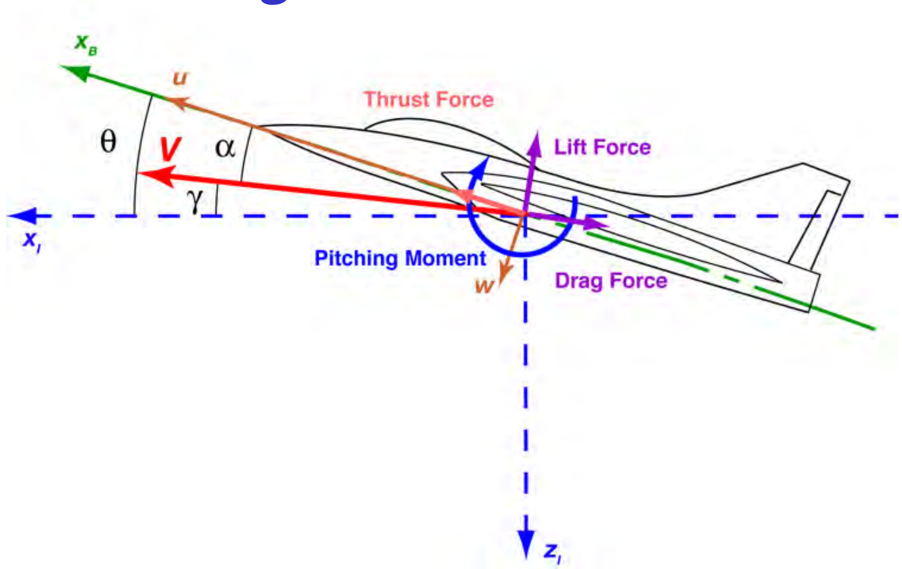
- **Mach number**

$$\boxed{M = \frac{TAS}{a}}$$

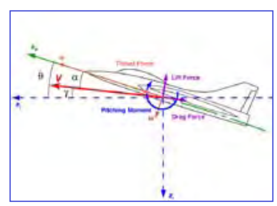
*** Kayton & Fried, 1969; NASA TN-D-822, 1961***

Flight in the Vertical Plane

Longitudinal Variables



Longitudinal Point-Mass Equations of Motion



- **Assume thrust is aligned with the velocity vector (small-angle approximation for α)**
- **Mass = constant**

$$\dot{V} = \frac{\left(C_T\cos\alpha - C_D\right)\frac{1}{2}\rho V^2 S - mg\sin\gamma}{m} \approx \frac{\left(C_T - C_D\right)\frac{1}{2}\rho V^2 S - mg\sin\gamma}{m}$$

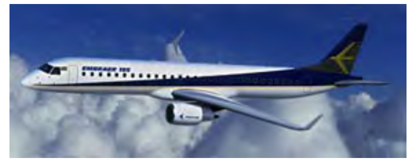
$$\dot{\gamma} = \frac{\left(C_T\sin\alpha + C_L\right)\frac{1}{2}\rho V^2 S - mg\cos\gamma}{mV} \approx \frac{C_L\frac{1}{2}\rho V^2 S - mg\cos\gamma}{mV}$$

$$\dot{h} = -\dot{z} = -v_z = V\sin\gamma$$

$$\dot{r} = \dot{x} = v_x = V\cos\gamma$$

$V = velocity =$ Earth-relative airspeed
$=$ True airspeed with zero wind
$\gamma = flight\ path\ angle$
$h = height\ (altitude)$
$r = range$

Conditions for Steady, Level Flight



- Flight path angle = 0
- Altitude = constant
- Airspeed = constant
- Dynamic pressure = constant

$$0 = \frac{(C_T - C_D)\frac{1}{2}\rho V^2 S}{m}$$

- Thrust = Drag

$$0 = \frac{C_L \frac{1}{2}\rho V^2 S - mg}{mV}$$

- Lift = Weight

$$\dot{h} = 0$$

$$\dot{r} = V$$

Power and Thrust

- Propeller

$$Power = P = T \times V = C_T \frac{1}{2}\rho V^3 S \approx independent\ of\ airspeed$$

- Turbojet

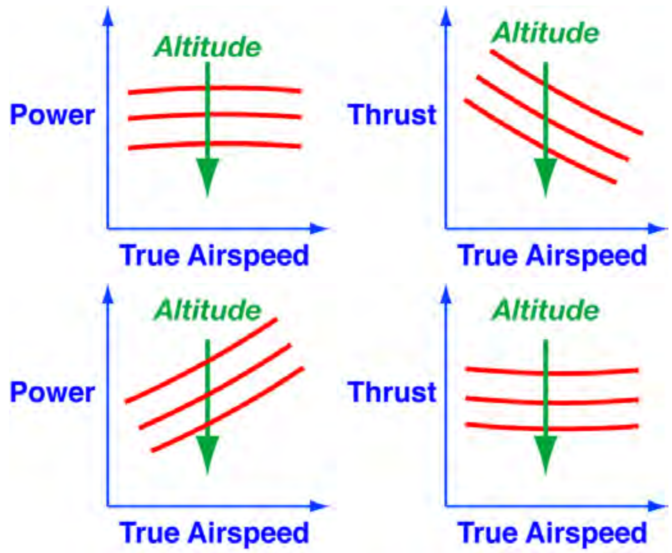
$$Thrust = T = C_T \frac{1}{2}\rho V^2 S \approx independent\ of\ airspeed$$

- Throttle Effect

$$T = T_{\max}\delta T = C_{T_{\max}}\delta T \bar{q} S, \quad 0 \le \delta T \le 1$$

Typical Effects of Altitude and Velocity on Power and Thrust

- **Propeller**
- **Turbojet**



Models for Altitude Effect on Turbofan Thrust

From *Flight Dynamics*, pp.117-118

$$Thrust = C_T\left(V,\delta T\right)\frac{1}{2}\rho\left(h\right)V^2 S$$
$$= \left(k_o + k_1 V^\eta\right)\frac{1}{2}\rho\left(h\right)V^2 S\delta T, \text{ N}$$

$k_o =$ Static thrust coefficient at sea level

$k_1 =$ Velocity sensitivity of thrust coefficient

$\eta =$ Exponent of velocity sensitivity

$= -2$ for turbojet

$\delta T =$ Throttle setting, $(0,1)$

$$\rho\left(h\right) = \rho_{SL} e^{-\beta h}, \quad \rho_{SL} = 1.225\, kg/m^3, \quad \beta = \left(1/9{,}042\right)m^{-1}$$

Models for Altitude Effect on Turbofan Thrust

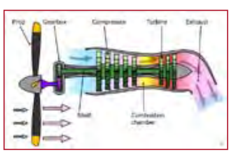
From *AeroModelMach.m* in *FLIGHT.m, Flight Dynamics,*
http://www.princeton.edu/~stengel/AeroModelMach.m

```
[airDens,airPres,temp,soundSpeed] = Atmos(-x(6));
Thrust =  u(4) * StaticThrust * (airDens / 1.225)^0.7 * (1 - exp((-x(6) – 17000)/2000));
```

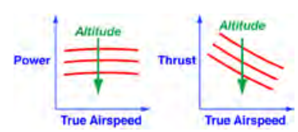
$$
\begin{aligned}
\text{Atmos(-x(6))}&: \text{ 1976 U.S. Standard Atmosphere function} \\
\text{-x(6)} = h &= \text{ Altitude, m} \\
\text{airDens } = \rho &= \text{ Air density at altitude } h, \text{kg/m}^3 \\
\text{u(4)} = \delta T &= \text{ Throttle setting, } (0,1)
\end{aligned}
$$

Empirical fit to match known characteristics of powerplant for generic business jet

```
(airDens / 1.225)^0.7 * (1 - exp((-x(6) – 17000)/2000))
```

Thrust of a Propeller-Driven Aircraft



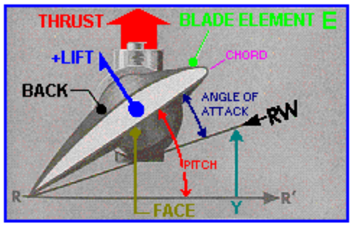
- **With constant *rpm*, variable-pitch propeller**

$$
T = \eta_P \eta_I \frac{P_{engine}}{V} = \eta_{net} \frac{P_{engine}}{V}
$$

$$
\begin{aligned}
&where \\
&\quad \eta_P = propeller\ efficiency \\
&\quad \eta_I = ideal\ \ propulsive\ efficiency \\
&\quad \eta_{net_{\max}} \approx 0.85 - 0.9
\end{aligned}
$$

- **Efficiencies decrease with airspeed**
- **Engine power decreases with altitude**
  - **Proportional to air density, w/o supercharger**

Propeller Efficiency, η_P, and Advance Ratio, J



- **Advance Ratio**

$$J = \frac{V}{nD}$$

$$\begin{aligned} &where \\ &V = airspeed, m/s \\ &n = rotation\ rate, revolutions/s \\ &D = propeller\ diameter, m \end{aligned}$$

Effect of propeller-blade pitch angle

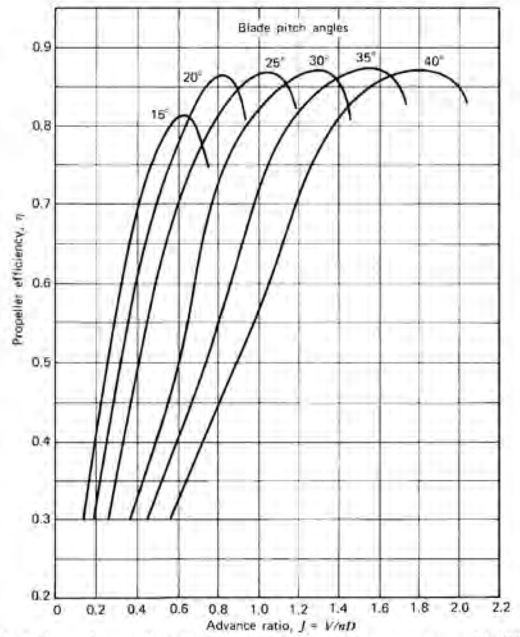
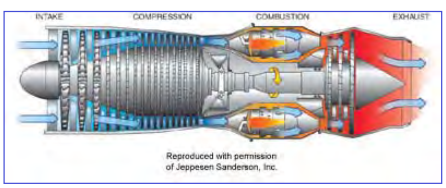
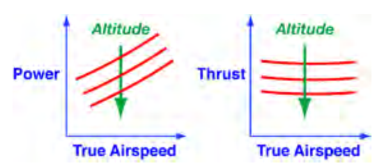


Figure 6.19 Estimated propeller efficiency for the Piper Cherokee Arrow PA-28R.

from McCormick



Thrust of a Turbojet Engine



$$T = \dot{m}V\left\{\left[\left(\frac{\theta_o}{\theta_o - 1}\right)\left(\frac{\theta_t}{\theta_t - 1}\right)(\tau_c - 1) + \frac{\theta_t}{\theta_o \tau_c}\right]^{1/2} - 1\right\}$$

$$\dot{m} = \dot{m}_{air} + \dot{m}_{fuel}$$

$$\theta_o = \left(p_{stag} / p_{ambient}\right)^{(\gamma-1)/\gamma}; \quad \gamma = ratio\ of\ specific\ heats \approx 1.4$$

$$\theta_t = (turbine\ inlet\ temp./freestream\ ambient\ temp.)$$

$$\tau_c = (compressor\ outlet\ temp./compressor\ inlet\ temp.)$$

from Kerrebrock

- **Little change in thrust with airspeed below M_{crit}**
- **Decrease with increasing altitude**

Performance Parameters

- **Lift-to-Drag Ratio** $\boxed{L/D = C_L/C_D}$
- **Load Factor** $\boxed{n = L/W = L/mg, \text{"}g\text{"}s}$
- **Thrust-to-Weight Ratio** $\boxed{T/W = T/mg, \text{"}g\text{"}s}$
- **Wing Loading** $\boxed{W/S, \quad N/m^2 \; or \; lb/ft^2}$

Steady, Level Flight

Trimmed Lift Coefficient, C_L

- **Trimmed lift coefficient, C_L**
  - Proportional to weight and wing loading factor
  - Decreases with V^2
  - At constant true airspeed, increases with altitude

$$\boxed{W = C_{L_{trim}}\left(\frac{1}{2}\rho V^2\right)S = C_{L_{trim}}\bar{q}S}$$

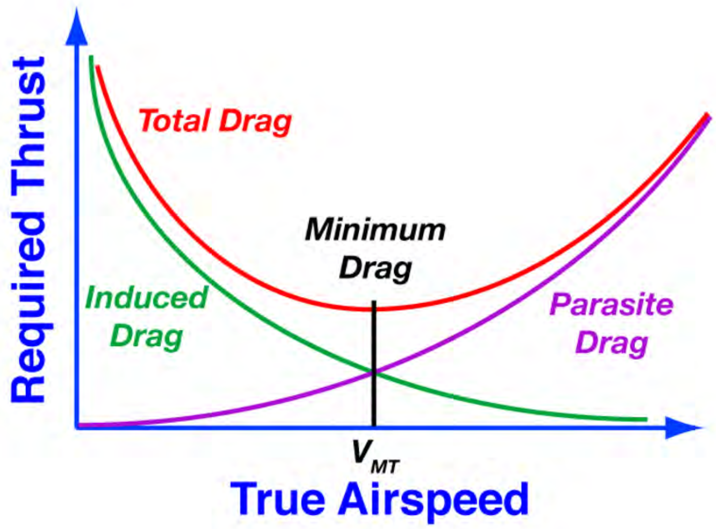
$$\boxed{C_{L_{trim}} = \frac{1}{\bar{q}}(W/S) = \frac{2}{\rho V^2}(W/S) = \left(\frac{2\,e^{\beta h}}{\rho_0 V^2}\right)(W/S)}$$

Trimmed Angle of Attack, α

- **Trimmed angle of attack, α**
  - Constant if dynamic pressure and weight are constant
  - If dynamic pressure decreases, angle of attack must increase

$$\boxed{\alpha_{trim} = \frac{2W/\rho V^2 S - C_{L_o}}{C_{L_\alpha}} = \frac{\frac{1}{\bar{q}}(W/S) - C_{L_o}}{C_{L_\alpha}}}$$

Thrust Required for Steady, Level Flight



Thrust Required for Steady, Level Flight

- **Trimmed thrust**

Parasitic Drag — Induced Drag

$$T_{trim} = D_{cruise} = C_{D_o}\left(\frac{1}{2}\rho V^2 S\right) + \varepsilon \frac{2W^2}{\rho V^2 S}$$

- **Minimum required thrust conditions**

Necessary Condition = Zero Slope

$$\frac{\partial T_{trim}}{\partial V} = C_{D_o}\left(\rho V S\right) - \frac{4\varepsilon W^2}{\rho V^3 S} = 0$$

Necessary and Sufficient Conditions for Minimum Required Thrust

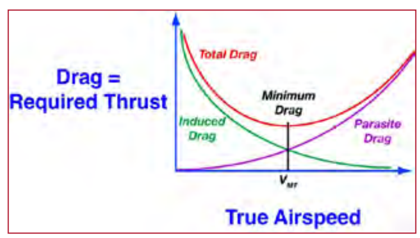
Necessary Condition = Zero Slope

$$C_{D_o}(\rho V S) = \frac{4\varepsilon W^2}{\rho V^3 S}$$

Sufficient Condition for a Minimum = Positive Curvature when slope = 0

$$\frac{\partial^2 T_{trim}}{\partial V^2} = \underset{(+)}{C_{D_o}(\rho S)} + \underset{(+)}{\frac{12\varepsilon W^2}{\rho V^4 S}} > 0$$

Airspeed for Minimum Thrust in Steady, Level Flight



- **Satisfy necessary condition**

$$V^4 = \left(\frac{4\varepsilon}{C_{D_o}\rho^2}\right)(W/S)^2$$

- **Fourth-order equation for velocity**
  - **Choose the positive root**

$$V_{MT} = \sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)\sqrt{\frac{\varepsilon}{C_{D_o}}}}$$

Lift, Drag, and Thrust Coefficients in Minimum-Thrust Cruising Flight

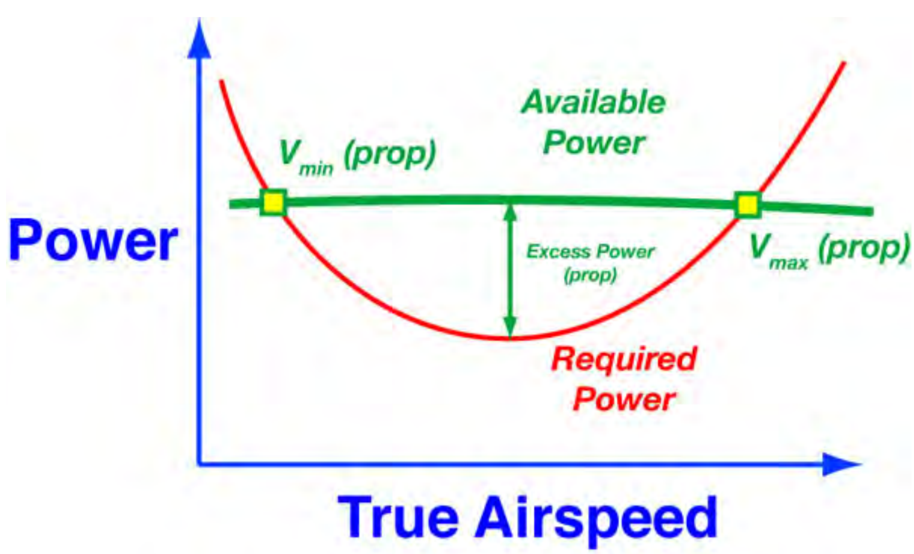
Lift coefficient

$$\boxed{\begin{aligned} C_{L_{MT}} &= \frac{2}{\rho V_{MT}^2}\left(\frac{W}{S}\right) \\ &= \sqrt{\frac{C_{D_o}}{\varepsilon}} = \left(C_L\right)_{(L/D)_{\max}} \end{aligned}}$$

Drag and thrust coefficients

$$\boxed{\begin{aligned} C_{D_{MT}} &= C_{D_o} + \varepsilon C_{L_{MT}}^2 = C_{D_o} + \varepsilon \frac{C_{D_o}}{\varepsilon} \\ &= 2C_{D_o} \equiv C_{T_{MT}} \end{aligned}}$$

Power Required for Steady, Level Flight



Power Required for Steady, Level Flight

- **Trimmed power**

Parasitic Drag — Induced Drag

$$P_{trim} = T_{trim}V = D_{cruise}V = \left[C_{D_o}\left(\frac{1}{2}\rho V^2 S\right) + \frac{2\varepsilon W^2}{\rho V^2 S}\right]V$$

- **Minimum required power conditions**

$$\frac{\partial P_{trim}}{\partial V} = C_{D_o}\frac{3}{2}\left(\rho V^2 S\right) - \frac{2\varepsilon W^2}{\rho V^2 S} = 0$$

Airspeed for Minimum Power in Steady, Level Flight

- **Satisfy necessary condition**

$$C_{D_o}\frac{3}{2}\left(\rho V^2 S\right) = \frac{2\varepsilon W^2}{\rho V^2 S}$$

- **Fourth-order equation for velocity**
  - Choose the positive root

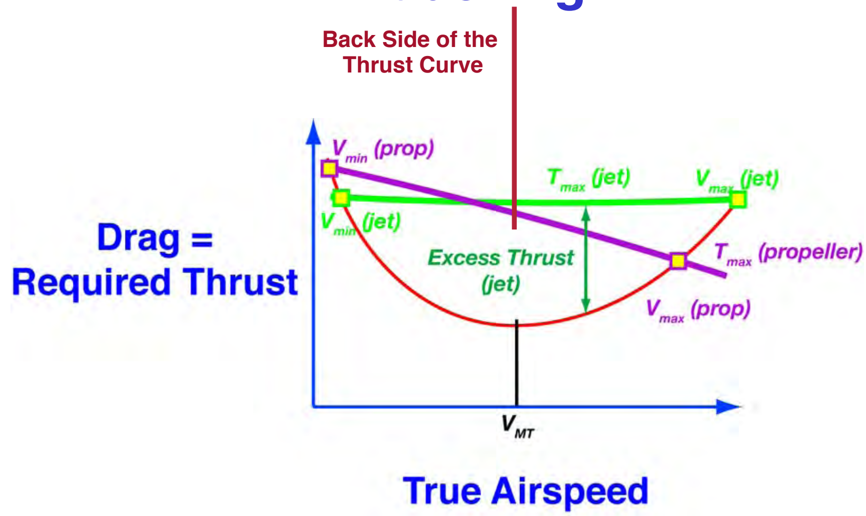
$$V_{MP} = \sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)\sqrt{\frac{\varepsilon}{3C_{D_o}}}}$$

- **Corresponding lift and drag coefficients**

$$C_{L_{MP}} = \sqrt{\frac{3C_{D_o}}{\varepsilon}}$$

$$C_{D_{MP}} = 4C_{D_o}$$

Achievable Airspeeds in Constant-Altitude Flight



- Two equilibrium airspeeds for a given thrust or power setting
  - Low speed, high C_L, high α
  - High speed, low C_L, low α
- Achievable airspeeds between minimum and maximum values with maximum thrust or power

Achievable Airspeeds for Jet in Cruising Flight

Thrust = constant

$$T_{avail} = C_D \bar{q} S = C_{D_o}\left(\frac{1}{2}\rho V^2 S\right) + \frac{2\varepsilon W^2}{\rho V^2 S}$$

$$C_{D_o}\left(\frac{1}{2}\rho V^4 S\right) - T_{avail} V^2 + \frac{2\varepsilon W^2}{\rho S} = 0$$

$$V^4 - \frac{T_{avail}}{C_{D_o}\rho S} V^2 + \frac{4\varepsilon W^2}{C_{D_o}(\rho S)^2} = 0$$

4th-order algebraic equation for V

Achievable Airspeeds for Jet in Cruising Flight

Solutions for V^2 can be put in quadratic form and solved easily

$$V^2 \triangleq x; \quad V = \pm\sqrt{x}$$

$$V^4 - \frac{T_{avail}}{C_{D_o}\rho S}V^2 + \frac{4\varepsilon W^2}{C_{D_o}(\rho S)^2} = 0$$

$$x^2 + bx + c = 0$$

$$x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c} = V^2$$

Thrust Required and Thrust Available for a Typical Bizjet

- Available thrust decreases with altitude, and range of achievable airspeeds decreases
- Stall limitation at low speed
- Mach number effect on lift and drag increases thrust required at high speed

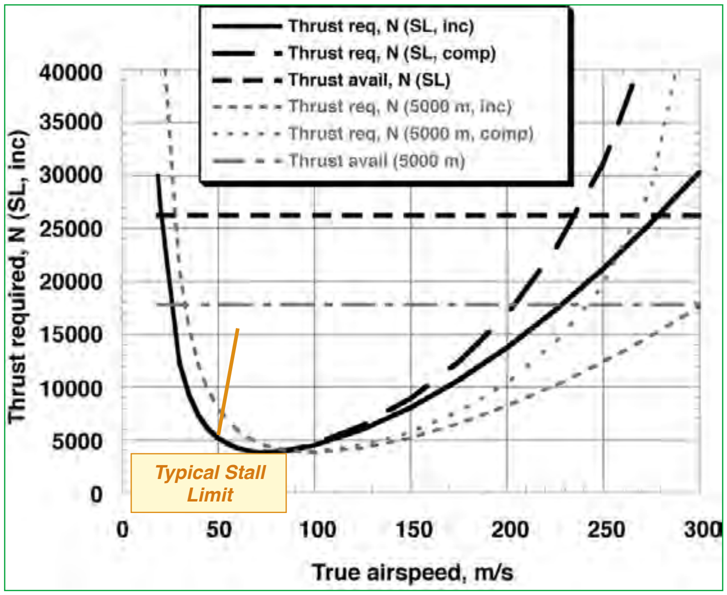
Typical Simplified Jet Thrust Model

$$T_{\max}(h) = T_{\max}(SL)\left[\frac{\rho(SL)e^{-\beta h}}{\rho(SL)}\right]^x = T_{\max}(SL)\left[e^{-\beta h}\right]^x = T_{\max}(SL)e^{-x\beta h}$$

- With empirical correction to force thrust to zero at a given altitude, h_{max}. c is a convergence factor.

$$T_{\max}(h) = T_{\max}(SL)e^{-x\beta h}\left[1 - e^{-(h-h_{\max})/c}\right]$$

Thrust Required and Thrust Available for a Typical Bizjet



Historical Factoid

- **Aircraft Flight Distance Records**

http://en.wikipedia.org/wiki/Flight_distance_record

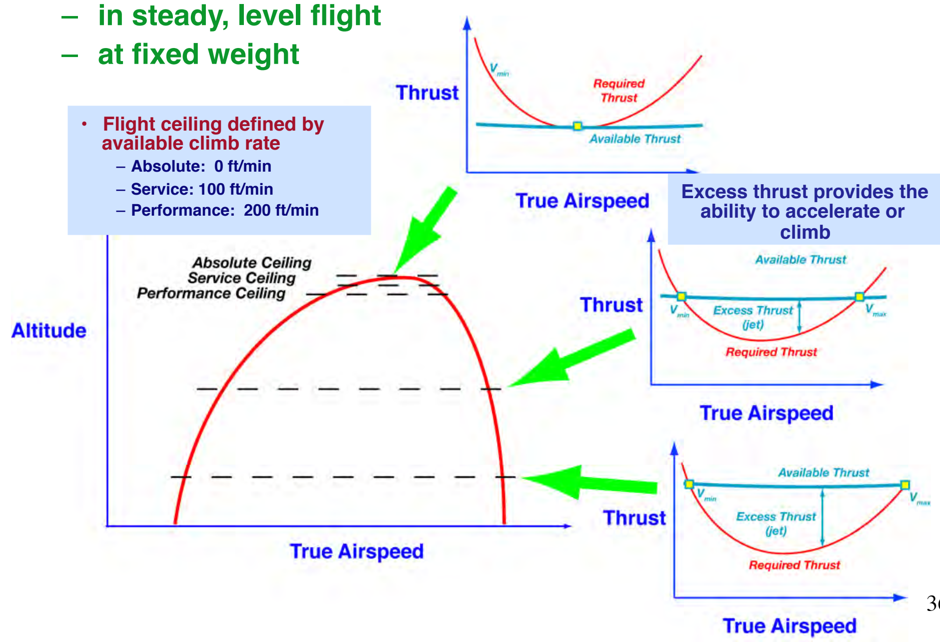
- **Aircraft Flight Endurance Records**

http://en.wikipedia.org/wiki/Flight_endurance_record

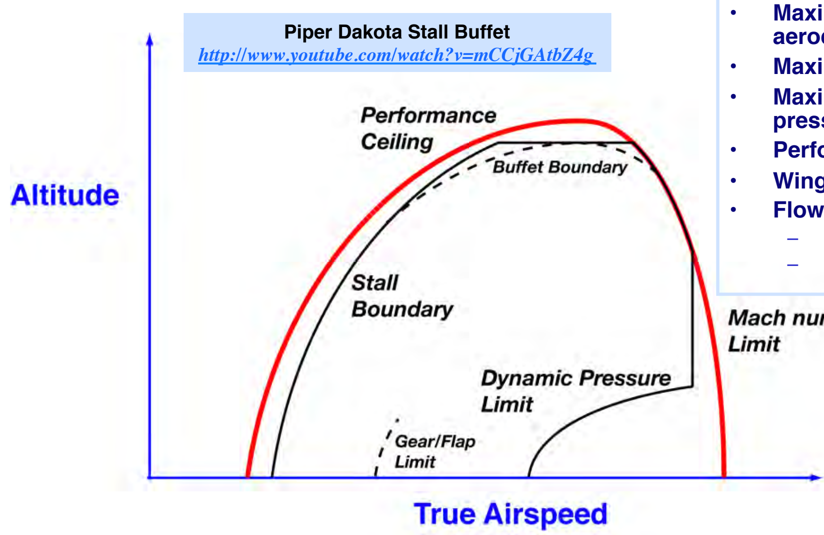
The Flight Envelope

Flight Envelope Determined by Available Thrust

- All altitudes and airspeeds at which an aircraft can fly
  - in steady, level flight
  - at fixed weight

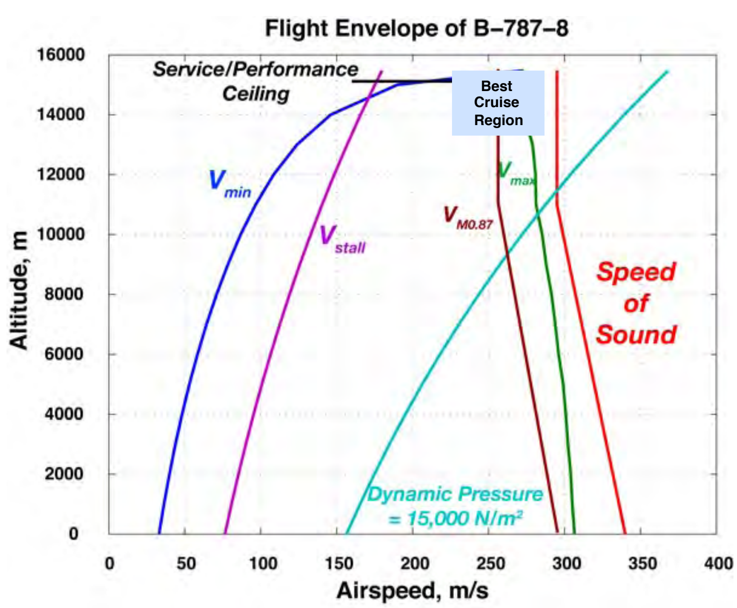


Additional Factors Define the Flight Envelope



- Maximum Mach number
- Maximum allowable aerodynamic heating
- Maximum thrust
- Maximum dynamic pressure
- Performance ceiling
- Wing stall
- Flow-separation buffet
  - Angle of attack
  - Local shock waves

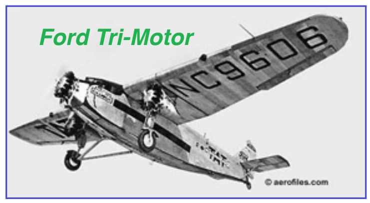
Boeing 787 Flight Envelope (HW #5, 2008)



Historical Factoids

Air Commerce Act of 1926

- Airlines formed to carry mail and passengers:
  - Northwest (1926)
  - Eastern (1927), bankruptcy
  - Pan Am (1927), bankruptcy
  - Boeing Air Transport (1927), became United (1931)
  - Delta (1928), consolidated with Northwest, 2010
  - American (1930)
  - TWA (1930), acquired by American
  - Continental (1934), consolidated with United, 2010



http://www.youtube.com/watch?v=3a8G87qnZz4

Commercial Aircraft of the 1930s

- Streamlining, engine cowlings
- *Douglas DC-1, DC-2, DC-3*

- *Lockheed 14 Super Electra, Boeing 247*, exterior and interior

Comfort and Elegance by the End of the Decade

- *Boeing 307*, 1st pressurized cabin (1936), flight engineer, *B-17* pre-cursor, large dorsal fin (exterior and interior)

- Sleeping bunks on transcontinental planes (e.g., *DC-3*)
- Full-size dining rooms on flying boats

41

Seaplanes Became the First TransOceanic Air Transports

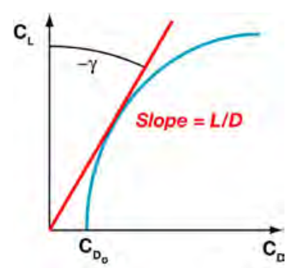
- PanAm led the way
  - 1st scheduled TransPacific flights(1935)
  - 1st scheduled TransAtlantic flights(1938)
  - 1st scheduled non-stop Trans-Atlantic flights (*VS-44*, 1939)
- *Boeing B-314, Vought-Sikorsky VS-44, Shorts Solent*
- Superseded by more efficient landplanes (lighter, less drag)

http://www.youtube.com/watch?v=x8SkeE1h_-A

42

Optimal Cruising Flight

Maximum Lift-to-Drag Ratio



- Lift-to-drag ratio

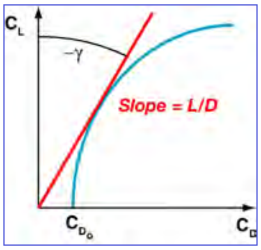
$$L/D = C_L/C_D = \frac{C_L}{C_{D_o} + \varepsilon C_L^2}$$

- Satisfy necessary condition for a maximum

$$\frac{\partial\left(C_L/C_D\right)}{\partial C_L} = \frac{1}{C_{D_o} + \varepsilon C_L^2} - \frac{2\varepsilon C_L^2}{\left(C_{D_o} + \varepsilon C_L^2\right)^2} = 0$$

- Lift coefficient for maximum *L/D* and minimum thrust are the same

$$\boxed{\left(C_L\right)_{L/D_{\max}} = \sqrt{\frac{C_{D_o}}{\varepsilon}} = C_{L_{MT}}}$$



Airspeed, Drag Coefficient, and Lift-to-Drag Ratio for L/D_{max}

Airspeed

$$\boxed{V_{L/D_{max}} = V_{MT} = \sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)\sqrt{\frac{\varepsilon}{C_{D_o}}}}}$$

Drag Coefficient

$$\boxed{\left(C_D\right)_{L/D_{max}} = C_{D_o} + C_{D_o} = 2C_{D_o}}$$

Maximum L/D

$$\boxed{\left(L/D\right)_{max} = \frac{\sqrt{C_{D_o}/\varepsilon}}{2C_{D_o}} = \frac{1}{2\sqrt{\varepsilon C_{D_o}}}}$$

- Maximum *L/D* depends only on induced drag factor and zero-α drag coefficient
- Induced drag factor and zero-α drag coefficient are functions of Mach number

Cruising Range and Specific Fuel Consumption

- Thrust = Drag
- Lift = Weight

$$0 = \left(C_T - C_D\right)\frac{1}{2}\rho V^2 S \Big/ m$$
$$0 = \left(C_L \frac{1}{2}\rho V^2 S - mg\right) \Big/ mV$$

- Level flight

$$\dot{h} = 0$$
$$\dot{r} = V$$

- Thrust specific fuel consumption, *TSFC* = c_T
  - Fuel mass burned per sec per unit of thrust

$$c_T : \frac{kg/s}{kN} \qquad \dot{m}_f = -c_T T$$

- Power specific fuel consumption, *PSFC* = c_P
  - Fuel mass burned per sec per unit of power

$$c_P : \frac{kg/s}{kW} \qquad \dot{m}_f = -c_P P$$

Historical Factoid

- **Louis Breguet (1880-1955), aviation pioneer**
  - **Gyroplane (1905), flew vertically in 1907**
  - **Breguet Type 1 (1909), fixed-wing aircraft**
  - **Formed Compagnie des messageries aériennes (1919), predecessor of *Air France***
- **Breguet Aviation manufactured numerous military and commericial aircraft until after World War II; teamed with BAC in SEPECAT**
- **Merged with Dassault in 1971**

Breguet Range Equation for Jet Aircraft

Rate of change of range with respect to weight of fuel burned

$$\frac{dr}{dm} = \frac{dr/dt}{dm/dt} = \frac{\dot{r}}{\dot{m}} = \frac{V}{\left(-c_T T\right)} = -\frac{V}{c_T D} = -\left(\frac{L}{D}\right)\frac{V}{c_T mg}$$

$$dr = -\left(\frac{L}{D}\right)\frac{V}{c_T mg}dm$$

Range traveled

$$Range = R = \int_0^R dr = -\int_{W_i}^{W_f}\left(\frac{L}{D}\right)\left(\frac{V}{c_T g}\right)\frac{dm}{m}$$

Maximum Range of a Jet Aircraft Flying at Constant Altitude

- At constant altitude $V_{cruise}(t) = \sqrt{\dfrac{2W(t)}{C_L \rho(h_{fixed}) S}}$

$$Range = -\int_{W_i}^{W_f} \left(\frac{C_L}{C_D}\right)\left(\frac{1}{c_T g}\right)\sqrt{\frac{2}{C_L \rho S}}\frac{dm}{m^{1/2}}$$
$$= \left(\frac{\sqrt{C_L}}{C_D}\right)\left(\frac{2}{c_T g}\right)\sqrt{\frac{2}{\rho S}}\left(m_i^{1/2} - m_f^{1/2}\right)$$

- Range is maximized when

$$\left(\frac{\sqrt{C_L}}{C_D}\right) = maximum \quad and \begin{cases} \rho = minimum \\ h = maximum \end{cases}$$

Breguet Range Equation for Jet Aircraft

For constant true airspeed, $V = V_{cruise}$

$$R = -\left(\frac{L}{D}\right)\left(\frac{V_{cruise}}{c_T g}\right)\ln(m)\Big|_{m_i}^{m_f}$$
$$= \left(\frac{L}{D}\right)\left(\frac{V_{cruise}}{c_T g}\right)\ln\left(\frac{m_i}{m_f}\right)$$

$$R = \left(V_{cruise}\frac{C_L}{C_D}\right)\left(\frac{1}{c_T g}\right)\ln\left(\frac{m_i}{m_f}\right)$$

- V_{cruise} as fast as possible
- Respect M_{crit}
- ρ as small as possible
- h as high as possible

Maximize Jet Aircraft Range Using Optimal Cruise-Climb

$$\frac{\partial R}{\partial C_L} \propto \frac{\partial\left(V_{cruise} C_L / C_D\right)}{\partial C_L} = \frac{\partial\left[V_{cruise} C_L / \left(C_{D_o} + \varepsilon C_L^2\right)\right]}{\partial C_L} = 0$$

$$V_{cruise} = \sqrt{2W / C_L \rho S}$$

Assume $\sqrt{2W(t)/\rho(h)S} = \text{constant}$

i.e., airplane **climbs at constant *TAS*** as fuel is burned

Maximize Jet Aircraft Range Using Optimal Cruise-Climb

$$\frac{\partial\left[V_{cruise} C_L / \left(C_{D_o} + \varepsilon C_L^2\right)\right]}{\partial C_L} = \sqrt{\frac{2W}{\rho S}} \frac{\partial\left[C_L^{1/2} / \left(C_{D_o} + \varepsilon C_L^2\right)\right]}{\partial C_L} = 0$$

Optimal values: (see Supplemental Material)

$C_{L_{MR}} = \sqrt{\frac{C_{D_o}}{3\varepsilon}}$: **Lift Coefficient for Maximum Range**

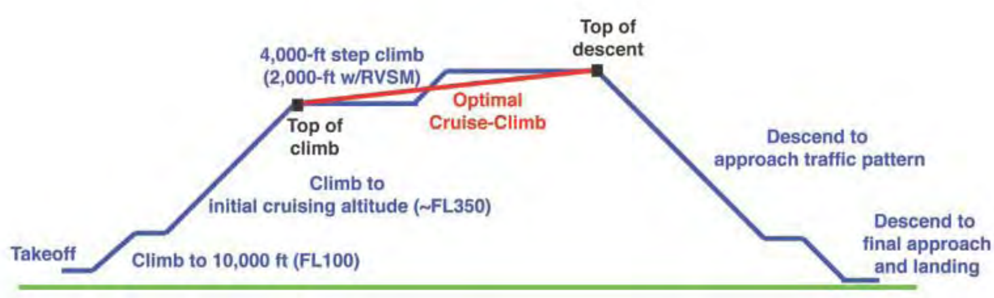
$$C_{D_{MR}} = C_{D_o} + \frac{C_{D_o}}{3} = \frac{4}{3} C_{D_o}$$

$$V_{cruise-climb} = \sqrt{2W(t)/C_{L_{MR}}\rho(h)S} = a(h) M_{cruise\text{-}climb}$$

$a(h)$: Speed of sound; $M_{cruise\text{-}climb}$: Mach number

Step-Climb Approximates Optimal Cruise-Climb

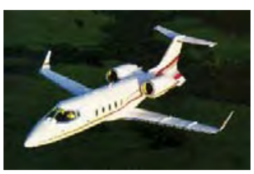
- Cruise-climb usually violates air traffic control rules
- Constant-altitude cruise does not
- Compromise: Step climb from one allowed altitude to the next as fuel is burned



Next Time:
Gliding, Climbing, and Turning Flight

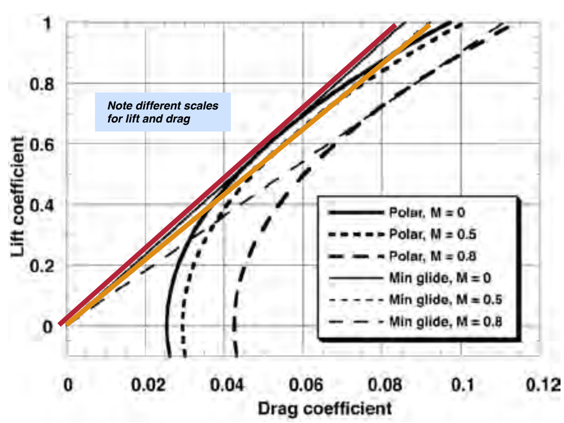
Reading:
Flight Dynamics
Aerodynamic Coefficients, 130-141, 147-155

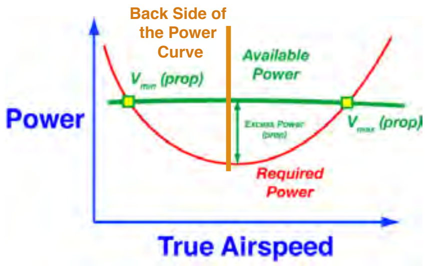
Supplemental Material



Lift-Drag Polar for a Typical Bizjet

- *L/D* equals slope of line drawn from the origin
  - Single maximum for a given polar
  - Two solutions for lower *L/D* (high and low airspeed)
  - Available *L/D* decreases with Mach number
- Intercept for L/D_{max} depends only on ε and zero-lift drag





Achievable Airspeeds in Propeller-Driven Cruising Flight

- **Power = constant**

$$P_{avail} = T_{avail} V$$
$$V^4 - \frac{P_{avail} V}{C_{D_o} \rho S} + \frac{4 \varepsilon W^2}{C_{D_o} (\rho S)^2} = 0$$

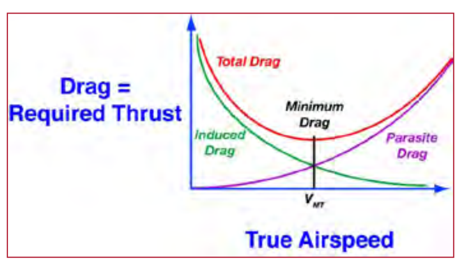
- **Solutions for *V* cannot be put in quadratic form; solution is more difficult, e.g., Ferrari' s method**

$$aV^4 + (0)V^3 + (0)V^2 + dV + e = 0$$

- **Best bet: *roots* in MATLAB**

P-51 Mustang Minimum-Thrust Example

$$\begin{aligned} &Wing\ Span = 37\ ft\ (9.83\ m) \\ &Wing\ Area = 235\ ft^2\ (21.83\ m^2) \\ &Loaded\ Weight = 9{,}200\ lb\ (3{,}465\ kg) \\ &C_{D_o} = 0.0163 \\ &\varepsilon = 0.0576 \\ &W/S = 39.3\ lb/ft^2\ (1555.7\ N/m^2) \end{aligned}$$



Airspeed for minimum thrust

$$V_{MT} = \sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}\sqrt{\frac{\varepsilon}{C_{D_o}}} = \sqrt{\frac{2}{\rho}(1555.7)}\sqrt{\frac{0.947}{0.0163}} = \frac{76.49}{\sqrt{\rho}}\ m/s$$

Altitude, m	Air Density, kg/m^3	VMT, m/s
0	1.23	69.11
2,500	0.96	78.20
5,000	0.74	89.15
10,000	0.41	118.87

P-51 Mustang Maximum L/D Example

$$\left(C_D\right)_{L/D_{\max}} = 2C_{D_o} = 0.0326$$

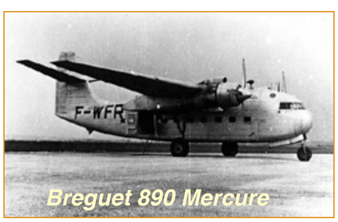
$$\left(C_L\right)_{L/D_{\max}} = \sqrt{\frac{C_{D_o}}{\varepsilon}} = C_{L_{MT}} = 0.531$$

$$Wing\ Span = 37\ ft\ (9.83\ m)$$
$$Wing\ Area = 235\ ft\ (21.83\ m^2)$$
$$Loaded\ Weight = 9{,}200\ lb\ (3{,}465\ kg)$$
$$C_{D_o} = 0.0163$$
$$\varepsilon = 0.0576$$
$$W/S = 1555.7\ N/m^2$$

$$\left(L/D\right)_{\max} = \frac{1}{2\sqrt{\varepsilon C_{D_o}}} = 16.31$$

$$V_{L/D_{\max}} = V_{MT} = \frac{76.49}{\sqrt{\rho}}\ m/s$$

Altitude, m	Air Density, kg/m^3	VMT, m/s
0	1.23	69.11
2,500	0.96	78.20
5,000	0.74	89.15
10,000	0.41	118.87



Breguet 890 Mercure

Breguet Range Equation for Propeller-Driven Aircraft

- **Rate of change of range with respect to weight of fuel burned**

$$\frac{dr}{dw} = \frac{\dot{r}}{\dot{w}} = \frac{V}{\left(-c_P P\right)} = -\frac{V}{c_P TV} = -\frac{V}{c_P DV} = -\left(\frac{L}{D}\right)\frac{1}{c_P W}$$

- **Range traveled**

$$Range = R = \int_0^R dr = -\int_{W_i}^{W_f} \left(\frac{L}{D}\right)\left(\frac{1}{c_P}\right)\frac{dw}{w}$$

Breguet Range Equation for Propeller-Driven Aircraft

- For constant true airspeed, $V = V_{cruise}$

$$R = -\left(\frac{L}{D}\right)\left(\frac{1}{c_P}\right)\ln\left(w\right)\Big|_{W_i}^{W_f}$$
$$= \left(\frac{C_L}{C_D}\right)\left(\frac{1}{c_P}\right)\ln\left(\frac{W_i}{W_f}\right)$$

- Range is maximized when

$$\left(\frac{C_L}{C_D}\right) = maximum = \left(L/D\right)_{\max}$$

P-51 Mustang Maximum Range (Internal Tanks only)

$$W = C_{L_{trim}}\bar{q}S$$
$$C_{L_{trim}} = \frac{1}{\bar{q}}(W/S) = \frac{2}{\rho V^2}(W/S) = \left(\frac{2\,e^{\beta h}}{\rho_0 V^2}\right)(W/S)$$

$$R = \left(\frac{C_L}{C_D}\right)_{\max}\left(\frac{1}{c_P}\right)\ln\left(\frac{W_i}{W_f}\right)$$
$$= (16.31)\left(\frac{1}{0.0017}\right)\ln\left(\frac{3{,}465+600}{3{,}465}\right)$$
$$= 1{,}530\ km\ ((825\ nm)$$

Maximize Jet Aircraft Range Using Optimal Cruise-Climb

$$\frac{\partial\left[V_{cruise}\, C_L \big/ \left(C_{D_o} + \varepsilon C_L^2\right)\right]}{\partial C_L} = \sqrt{\frac{2w}{\rho S}}\, \frac{\partial\left[C_L^{1/2} \big/ \left(C_{D_o} + \varepsilon C_L^2\right)\right]}{\partial C_L} = 0$$

$$\sqrt{\frac{2w}{\rho S}} = \text{Constant}; \;\; \text{let } C_L^{1/2} = x, \quad C_L = x^2$$

$$\frac{\partial}{\partial x}\left[\frac{x}{\left(C_{D_o} + \varepsilon x^4\right)}\right] = \frac{\left(C_{D_o} + \varepsilon x^4\right) - x\left(4\varepsilon x^3\right)}{\left(C_{D_o} + \varepsilon x^4\right)^2} = \frac{\left(C_{D_o} - 3\varepsilon x^4\right)}{\left(C_{D_o} + \varepsilon x^4\right)^2}$$

Optimal values:

$$C_{L_{MR}} = \sqrt{\frac{C_{D_o}}{3\varepsilon}} : \;\; C_{D_{MR}} = C_{D_o} + \frac{C_{D_o}}{3} = \frac{4}{3} C_{D_o}$$