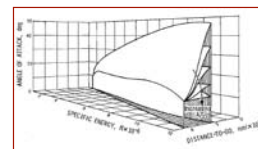
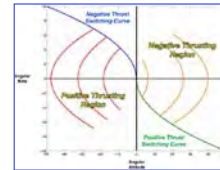


# Principles for Optimal Control, Part 2

Robert Stengel

Optimal Control and Estimation, MAE 546, Princeton University, 2015

- Minimum Principle
- Hamilton-Jacobi-Bellman Equation (Dynamic Programming)
- Terminal State Equality Constraint
- Linear, Time-Invariant, Minimum-Time Control Problem



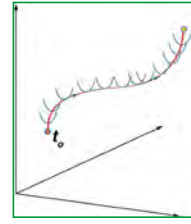
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<http://www.princeton.edu/~stengel/MAE546.html>  
<http://www.princeton.edu/~stengel/OptConEst.html>

1

## *The Minimum Principle*

2

# The Minimum Principle\*



Variational necessary and sufficient conditions  
imply that minimum **H** is optimal

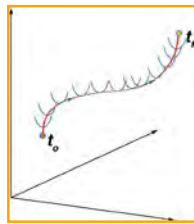
$$\frac{\partial H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t]}{\partial \mathbf{u}} = \mathbf{0} \quad \text{in } (t_0, t_f)$$

$$\frac{\partial^2 H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t]}{\partial \mathbf{u}^2} > 0 \quad \text{in } (t_0, t_f)$$

$$H^* = H[\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)] \leq H[\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}^*(t)]$$

\* After the "Maximum Principle" of Pontryagin, et al, 1950s (opposite convention for sign of Hamiltonian)

3



## Control Perturbation Can Only Increase Cost

Effect of control perturbation on optimal **H** and **J**\*

$$J[\mathbf{u}^*(t) + \Delta \mathbf{u}(t)] - J[\mathbf{u}^*(t)] = \phi[\mathbf{x}^*(t_f)] - \phi[\mathbf{x}^*(t_f)] + \int_{t_0}^{t_f} \left\{ H[\mathbf{x}^*(t), \mathbf{u}^*(t) + \Delta \mathbf{u}(t), \boldsymbol{\lambda}^*(t)] - \boldsymbol{\lambda}^{*T}(t) \dot{\mathbf{x}}^*(t) - \left\{ H[\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)] - \boldsymbol{\lambda}^{*T}(t) \dot{\mathbf{x}}^*(t) \right\} \right\} dt$$

Control perturbation has no effect on terminal cost or  $\boldsymbol{\lambda}^T \frac{\partial \mathbf{x}}{\partial t}$

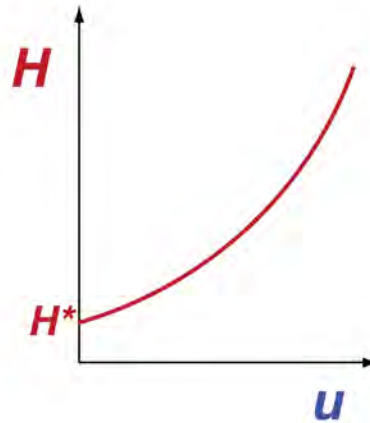
$$J[\mathbf{u}^*(t) + \Delta \mathbf{u}(t)] - J[\mathbf{u}^*(t)] = \int_{t_0}^{t_f} \left\{ H[\mathbf{x}^*(t), \mathbf{u}^*(t) + \Delta \mathbf{u}(t), \boldsymbol{\lambda}^*(t)] - \left\{ H[\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)] \right\} \right\} dt \geq 0$$

Assuming that  $\mathbf{x}^*(t)$  and  $\boldsymbol{\lambda}^*(t)$  are the optimal values

4

## Application of the Minimum Principle with Bounded Control

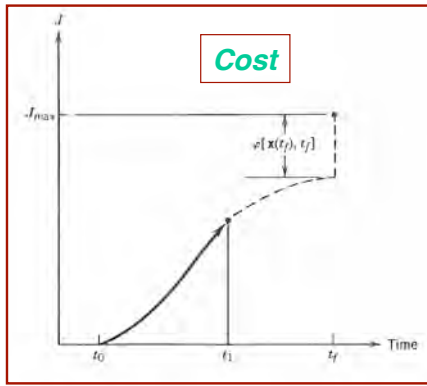
- Minimum principle applies
  - when control is limited such that  $\partial H / \partial u \neq 0$
  - in some cases of singular control, e.g. “bang-bang control” (TBD)



5

## *Dynamic Programming*

6



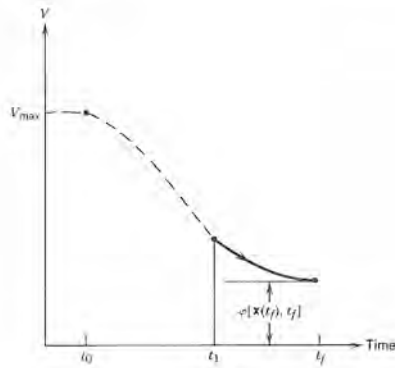
## Cost Function vs. Value Function

Optimal Cost Function (i.e., accrued cost) at  $t_1$

$$J^*(t_1) = \int_{t_0}^{t_1} L[\mathbf{x}(\tau), \mathbf{u}(\tau)] d\tau$$

Optimal Cost Function at  $t_f$

$$J^*(t_f) = \phi[\mathbf{x}(t_f)] + \int_{t_0}^{t_f} L[\mathbf{x}(\tau), \mathbf{u}(\tau)] d\tau \triangleq J^*_{\max}$$



7

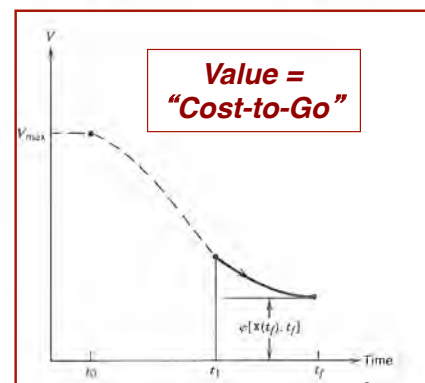
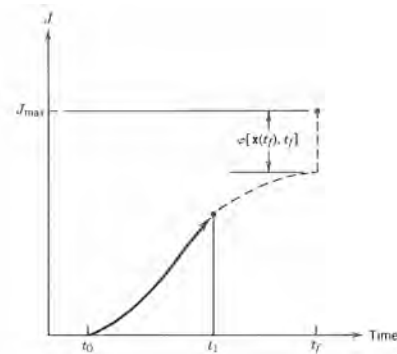
## Cost Function vs. Value Function

Optimal Value Function (i.e., remaining cost) at  $t_1$

$$\begin{aligned} V^*(x_1, t_1) &= \phi[\mathbf{x}^*(t_f)] + \int_{t_1}^{t_f} L[\mathbf{x}^*(\tau), \mathbf{u}^*(\tau)] d\tau \\ &= \phi[\mathbf{x}^*(t_f)] - \int_{t_f}^{t_1} L[\mathbf{x}^*(\tau), \mathbf{u}^*(\tau)] d\tau \\ &= \min_{\mathbf{u}} \left\{ \phi[\mathbf{x}^*(t_f)] - \int_{t_f}^{t_1} L[\mathbf{x}^*(\tau), \mathbf{u}(\tau)] d\tau \right\} \end{aligned}$$

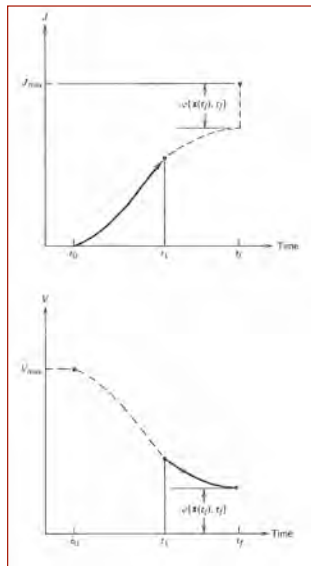
Optimal Value Function at  $t_0$

$$\begin{aligned} V^*(x_0, t_0) &= \phi[\mathbf{x}^*(t_f)] - \int_{t_f}^{t_0} L[\mathbf{x}^*(\tau), \mathbf{u}^*(\tau)] d\tau \\ &\triangleq V^*_{\max} = J^*_{\max} \end{aligned}$$



8

# Time Derivative of the Value Function



Optimal Value Function at  $t_1$

$$V^*(x_1, t_1) = \phi[\mathbf{x}^*(t_f)] - \int_{t_f}^{t_1} L[\mathbf{x}^*(\tau), \mathbf{u}^*(\tau)] d\tau$$

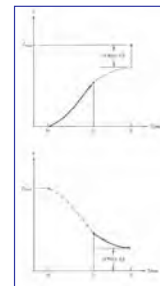
• Total time-derivative of  $V^*$

- Rate at which Value is spent
- Integrand of Value function

$$\begin{aligned} \left. \frac{dV^*}{dt} \right|_{t=t_1} &= -L[\mathbf{x}^*(t_1), \mathbf{u}^*(t_1)] \\ &= \left( \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial V^*}{\partial \mathbf{u}} \dot{\mathbf{u}} \right) \bigg|_{t=t_1} \end{aligned}$$

*= 0 on optimal trajectory* 9

## Dynamic Programming: Hamilton-Jacobi-Bellman Equation



Rearrange to solve for partial derivative wrt  $t$

$$\begin{aligned} \left. \frac{\partial V^*}{\partial t} \right|_{t=t_1} &= \left( \frac{dV^*}{dt} - \frac{\partial V^*}{\partial \mathbf{x}} \dot{\mathbf{x}} \right) \bigg|_{t=t_1} = \left( -L[\mathbf{x}^*, \mathbf{u}^*] - \frac{\partial V^*}{\partial \mathbf{x}} \dot{\mathbf{x}} \right) \bigg|_{t=t_1} \\ &= \left( -L[\mathbf{x}^*, \mathbf{u}^*] - \frac{\partial V^*}{\partial \mathbf{x}} \mathbf{f}[\mathbf{x}^*, \mathbf{u}^*] \right) \bigg|_{t=t_1} \end{aligned}$$

Define a **Hamiltonian** for the system

$$\begin{aligned} \left. \frac{\partial V^*}{\partial t} \right|_{t=t_1} &\triangleq -H \left\{ \mathbf{x}^*(t_1), \mathbf{u}^*(t_1), \frac{\partial V^*}{\partial \mathbf{x}}(t_1) \right\} \\ &\triangleq -\min_{\mathbf{u}} H \left\{ \mathbf{x}^*(t_1), \mathbf{u}(t_1), \frac{\partial V^*}{\partial \mathbf{x}}(t_1) \right\} \quad \text{in } [t_o, t_f] \end{aligned}$$

# Principle of Optimality (Bellman, 1957)

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

**HJB equation** is a partial differential equation

$$\left. \frac{\partial V^*}{\partial t} \right|_{t=t_1} \triangleq -H \left\{ \mathbf{x}^*(t_1), \mathbf{u}^*(t_1), \frac{\partial V^*}{\partial \mathbf{x}}(t_1) \right\}$$

$$\triangleq -\min_{\mathbf{u}} H \left\{ \mathbf{x}^*(t_1), \mathbf{u}(t_1), \frac{\partial V^*}{\partial \mathbf{x}}(t_1) \right\} \quad \text{in } [t_o, t_f]$$

**Boundary condition**

$$V^*[\mathbf{x}^*(t_f)] = \phi[\mathbf{x}^*(t_f)]$$

11

## Necessary and Sufficient Condition for Optimality

$$\left. \frac{\partial V^*}{\partial t} \right|_{t=t_1} = -\min_{\mathbf{u}(t)} H \left\{ \mathbf{x}^*(t_1), \mathbf{u}(t_1), \frac{\partial V^*}{\partial \mathbf{x}}(t_1) \right\}$$

Minimum of **H** w.r.t. **u(t)** requires  
**stationarity and convexity**

12

# $V^*[x(t),t]$ is a Hypersurface That Defines Minimum Cost Control

- $V^*[x(t),t]$  is the integral of the HJB equation

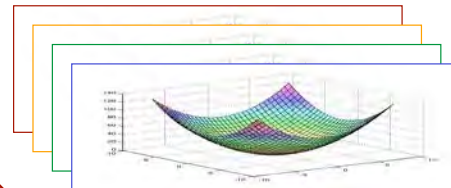
At the terminal time

- $V^*$  is a scalar function of the state

$$V^*[x^*(t_f)] = \phi[x^*(t_f)]$$

- Ideally, the time-varying hypersurface of  $V^*$  is bowl-shaped
- Minimum of hypersurface specifies optimal control policy

Time



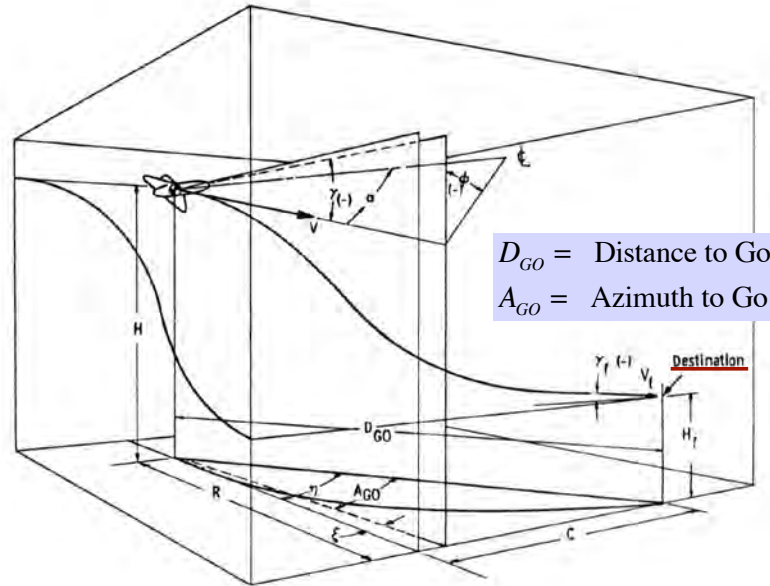
$$\mathbf{u}^*(t) = \mathbf{u}^* \left\{ V^*[x^*(t)] \right\}$$

13

## *Space Shuttle Reentry Example*

14

# Optimal Guidance for Space Shuttle Reentry

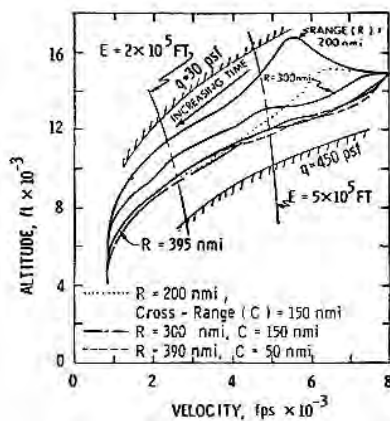


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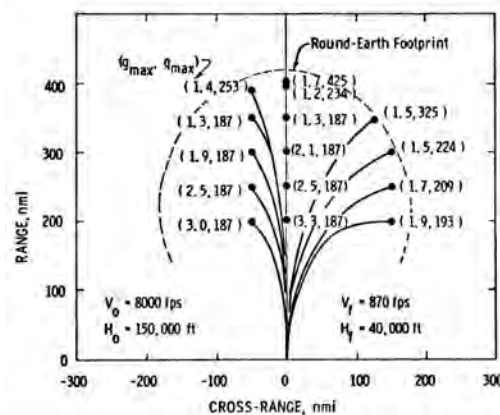


## Optimal Trajectories for Space Shuttle Reentry

Altitude vs. Velocity



Range vs. Cross-Range ("footprint")



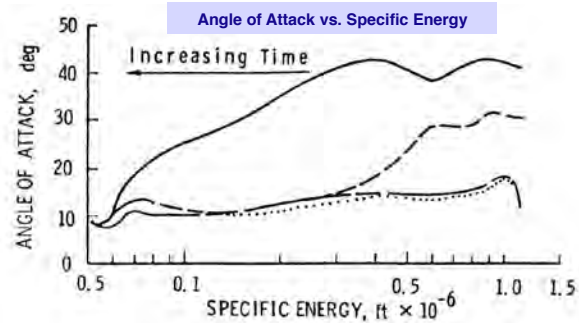
Numerical solutions using steepest-descent and conjugate-gradient algorithms

16

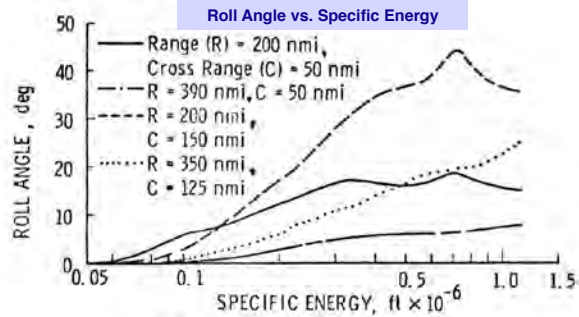


# Optimal Controls for Space Shuttle Reentry

- Independent variable is specific total energy rather than time
- On reentry, total energy decreases as time increases



a) Angle-of-Attack History



b) Roll Angle History

17

## Optimal Guidance System Derived from Optimal Trajectories

Angle of Attack and Roll Angle vs. Specific Energy

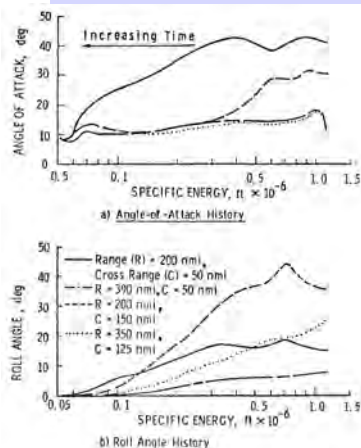
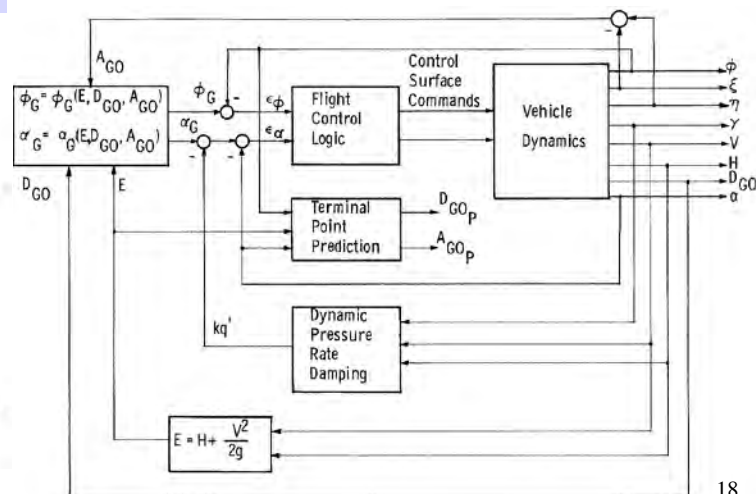


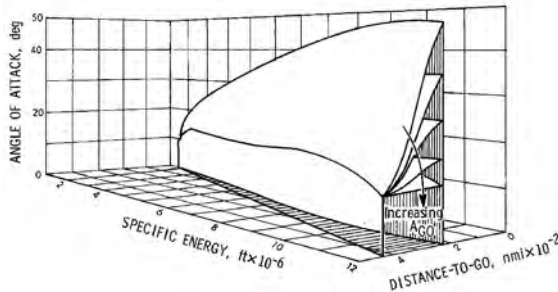
Diagram of Energy-Guidance Law



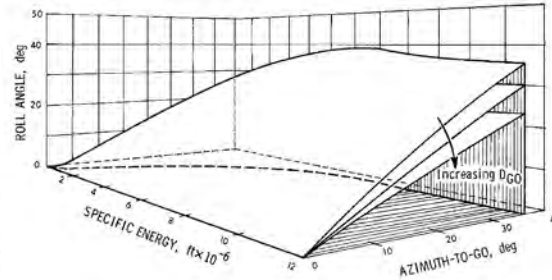
18

# Guidance Functions for Space Shuttle Reentry

Angle of Attack Guidance Function



Roll Angle Guidance Function



- Guidance surfaces can be implemented with
  - Table lookup
  - Computational neural networks

19

## Relationship of HJB Equation to Other Principles of Optimality

$$\left. \frac{\partial V^*}{\partial t} \right|_{t=t_1} \triangleq -H \left\{ \mathbf{x}^*(t_1), \mathbf{u}^*(t_1), \frac{\partial V^*}{\partial \mathbf{x}}(t_1) \right\}$$

$$\triangleq -\min_{\mathbf{u}} H \left\{ \mathbf{x}^*(t_1), \mathbf{u}(t_1), \frac{\partial V^*}{\partial \mathbf{x}}(t_1) \right\} \quad \text{in } [t_o, t_f]$$

Calculus of Variations  
(Euler-Lagrange Equations)

$$\frac{\partial V^*}{\partial \mathbf{x}}(t_1) = \boldsymbol{\lambda}^T(t_1)$$

Minimum Principle

$$\min_{\mathbf{u}} H \left\{ \mathbf{x}^*(t_1), \mathbf{u}(t_1), \frac{\partial V^*}{\partial \mathbf{x}}(t_1) \right\} \quad \text{in } [t_o, t_f] \quad \text{defines optimality}$$

20

# *Terminal State Equality Constraint*

21

**Minimize a Cost Function Subject to  
a Terminal State Equality Constraint**

$$\min_{\mathbf{u}(t)} J = \phi[\mathbf{x}(t_f)] + \int_{t_o}^{t_f} L[\mathbf{x}(t), \mathbf{u}(t)] dt$$

▪ subject to

**Dynamic Constraint**

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)], \quad \mathbf{x}(t_o) \text{ given}$$

**Terminal State Equality Constraint**

$$\psi[\mathbf{x}(t_f)] \equiv 0 \text{ (scalar)}$$

22

# Terminal State Equality Constraints

## Soft Constraint

$$\min_{\mathbf{u}(t)} \phi[\mathbf{x}(t_f)]$$

$$\phi[\mathbf{x}(t_f)] \approx 0 \text{ is OK}$$

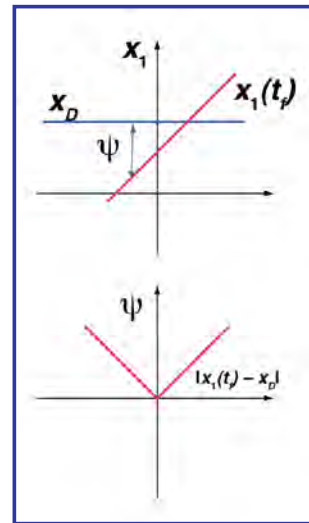
$$\psi[\mathbf{x}(t_f)] = x_{1_f} - x_D \equiv 0$$

## Hard Constraint

$$\psi[\mathbf{x}(t_f)] \equiv 0$$

$$\psi[\mathbf{x}(t_f)] = |x_{1_f} - x_D| \equiv 0$$

## Examples



23

# Cost Function Augmented by Terminal State Equality Constraint

$$J_{Constrained} = J_{Unconstrained} + \mu \psi[\mathbf{x}(t_f)]$$

$$\triangleq J_0 + \mu J_1$$

$\mu$  = constant scalar Lagrange multiplier for terminal constraint

- Separate solution into two parts
  - Optimize original cost function alone
  - Optimize for constraint alone

24

# Euler-Lagrange Equations and 1<sup>st</sup> Variation for Unconstrained Optimization

$$\lambda_0(t_f) = \left\{ \frac{\partial \phi[\mathbf{x}(t_f)]}{\partial \mathbf{x}} \right\}^T$$

$$\dot{\lambda}_0 = - \left\{ \frac{\partial H_0[\mathbf{x}, \mathbf{u}, \lambda_0, t]}{\partial \mathbf{x}} \right\}^T = - \left[ \frac{\partial L}{\partial \mathbf{x}} + \lambda_0^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]^T = - [L_{\mathbf{x}} + \mathbf{F}^T \lambda_0]$$

Assuming that these equations are satisfied, the first variation is

$$\Delta J_0 = \int_{t_o}^{t_f} \left( \frac{\partial H_0}{\partial \mathbf{u}} \Delta \mathbf{u} \right) dt = \int_{t_o}^{t_f} (L_{\mathbf{u}} + \lambda_0^T \mathbf{G}) \Delta \mathbf{u} dt$$

25

## Terminal Constraint “Cost” Augmented by Dynamic Constraint

$$\begin{aligned} J_1 &= \psi[\mathbf{x}(t_f)] + \int_{t_o}^{t_f} \{ \lambda_1^T(t) [\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)] - \dot{\mathbf{x}}(t)] \} dt \\ &= \psi[\mathbf{x}(t_f)] + \int_{t_o}^{t_f} \{ \lambda_1^T \mathbf{f}[\mathbf{x}, \mathbf{u}] - \lambda_1^T \dot{\mathbf{x}} \} dt \end{aligned}$$

$$\begin{aligned} J_1 &\triangleq \psi[\mathbf{x}(t_f)] + \int_{t_o}^{t_f} \{ H_1[\mathbf{x}, \mathbf{u}, \lambda_1^T] - \lambda_1^T \dot{\mathbf{x}} \} dt \\ &= \psi[\mathbf{x}(t_f)] + [\lambda_1^T(t_o) \mathbf{x}(t_o) - \lambda_1^T(t_f) \mathbf{x}(t_f)] \end{aligned}$$

$$H_1 \triangleq \lambda_1^T \mathbf{f}[\mathbf{x}, \mathbf{u}] + \int_{t_o}^{t_f} \{ H_1[\mathbf{x}, \mathbf{u}, \lambda_1] + \dot{\lambda}_1^T \mathbf{x} \} dt$$

26

# Euler-Lagrange Equations and 1<sup>st</sup> Variation for Terminal Constraint “Cost” Stationarity

$$\lambda_1(t_f) = \left\{ \frac{\partial \psi[\mathbf{x}(t_f)]}{\partial \mathbf{x}} \right\}^T$$

$$\dot{\lambda}_1 = - \left\{ \frac{\partial H_1[\mathbf{x}, \mathbf{u}, \lambda_1, t]}{\partial \mathbf{x}} \right\}^T = - \left[ \lambda_1^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]^T = - [\mathbf{F}^T \lambda_1]$$

Assuming that these equations are satisfied, the first variation is

$$\Delta J_1 = \int_{t_o}^{t_f} \left( \frac{\partial H_1}{\partial \mathbf{u}} \Delta \mathbf{u} \right) dt = \int_{t_o}^{t_f} (\lambda_1^T \mathbf{G} \Delta \mathbf{u}) dt$$

27

## First Variation of the Constrained Cost

$$\Delta J_C = \Delta J_0 + \mu \Delta J_1$$

= 0 for constrained stationarity

$$\Delta J_0 = \int_{t_o}^{t_f} \left( \frac{\partial H_0}{\partial \mathbf{u}} \Delta \mathbf{u} \right) dt$$

$$\Delta J_1 = \int_{t_o}^{t_f} \left( \frac{\partial H_1}{\partial \mathbf{u}} \Delta \mathbf{u} \right) dt$$

$$\Delta J_C = \Delta J_0 + \mu \Delta J_1$$

$$= \int_{t_o}^{t_f} \left( \frac{\partial H_0}{\partial \mathbf{u}} + \mu \frac{\partial H_1}{\partial \mathbf{u}} \right) \Delta \mathbf{u} dt = \int_{t_o}^{t_f} \left[ (L_u + \lambda_0^T \mathbf{G}) + \mu \lambda_1^T \mathbf{G} \right] \Delta \mathbf{u} dt$$

28

## First Variation of the Constrained Cost

$$\Delta J_C = \Delta J_0 + \mu \Delta J_1 = 0$$

$$= \int_{t_o}^{t_f} \left( \frac{\partial H_0}{\partial \mathbf{u}} + \mu \frac{\partial H_1}{\partial \mathbf{u}} \right) \Delta \mathbf{u} dt = \int_{t_o}^{t_f} \left[ (L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G}) + \mu \boldsymbol{\lambda}_1^T \mathbf{G} \right] \Delta \mathbf{u} dt$$

Control perturbation is arbitrary, so chose

$$\Delta \mathbf{u} = \varepsilon \left( \frac{\partial H_1}{\partial \mathbf{u}} \right)^T = \varepsilon (\boldsymbol{\lambda}_1^T \mathbf{G})^T, \quad \varepsilon = \text{arbitrary constant}$$

29

## First Variation of the Constrained Cost

$$\begin{aligned} \Delta J_C &= \int_{t_o}^{t_f} \left[ (L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G}) + \mu \boldsymbol{\lambda}_1^T \mathbf{G} \right] \varepsilon \mathbf{G}^T \boldsymbol{\lambda}_1 dt \\ &= \varepsilon \int_{t_o}^{t_f} \left[ (L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G}) \mathbf{G}^T \boldsymbol{\lambda}_1 + \mu \boldsymbol{\lambda}_1^T \mathbf{G} \mathbf{G}^T \boldsymbol{\lambda}_1 \right] dt \\ &= \varepsilon \left\{ \int_{t_o}^{t_f} \left[ (L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G}) \mathbf{G}^T \boldsymbol{\lambda}_1 \right] dt + \mu \int_{t_o}^{t_f} \left[ \boldsymbol{\lambda}_1^T \mathbf{G} \mathbf{G}^T \boldsymbol{\lambda}_1 \right] dt \right\} \triangleq \varepsilon (a + \mu b) \end{aligned}$$

Solution for terminal constraint Lagrange multiplier

$$\Delta J_C = 0 \quad \text{if} \quad \mu = -\frac{a}{b}$$

30

## Controllability Gramian

For control of the terminal constraint, the controllability gramian must not equal zero

$$b \triangleq \int_{t_o}^{t_f} [\lambda_1^T \mathbf{G} \mathbf{G}^T \lambda_1] dt \neq 0$$

A sufficient condition for optimality

31

## Optimizing Control for Terminal Constraint

Choose  $\mathbf{u}(t)$  such that

$$\begin{aligned} \frac{\partial H_c}{\partial \mathbf{u}} &= \left[ \frac{\partial H_0}{\partial \mathbf{u}} - \left( \frac{a}{b} \right) \frac{\partial H_1}{\partial \mathbf{u}} \right] \\ &= \left[ (L_{\mathbf{u}} + \lambda_0^T \mathbf{G}) - \left( \frac{a}{b} \right) \lambda_1^T \mathbf{G} \right] = 0 \end{aligned}$$

32



# *Linear, Time-Invariant Minimum-Time Problem*

33

## **Linear, Time-Invariant Minimum-Time Problem**

**Linear, time-invariant system, scalar control**

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}\mathbf{x}(t) + \mathbf{G}u(t), \quad \mathbf{x}(0) = \mathbf{x}_o$$

**Control constraint**

$$c(u) = |u| - 1 \leq 0$$

**Cost function**

$$J = \int_0^{t_f} dt$$

**Terminal constraint**

$$\psi[\mathbf{x}(t_f)] = 0$$

34

# Linear, Time-Invariant Minimum-Time Problem

Hamiltonian

$$H_C = 1 + \lambda^T (\mathbf{F}\mathbf{x} + \mathbf{G}u) + \mu\psi$$

Adjoint equation

$$\dot{\lambda} = -\left(\frac{\partial H_C}{\partial \mathbf{x}}\right)^T = -\mathbf{F}^T \lambda, \quad \lambda(t_f) = \left[\frac{\partial \psi}{\partial \mathbf{x}}(t_f)\right]$$

Open-end time problem

$$H_C^*(t_f) = 0$$

Time-invariant problem

$$H_C^*(t) = 0 \text{ on entire trajectory}$$

35

# Linear, Time-Invariant Minimum-Time Problem

Optimality conditions not satisfied

$$\frac{\partial H_C}{\partial u} = \lambda^T \mathbf{G}, \quad \therefore \frac{\partial^2 H_C}{\partial u^2} = 0 \Rightarrow \text{Singular problem (not convex)}$$

Minimum principle (smallest Hamiltonian) solves the problem

$$1 + \lambda^{*T} (\mathbf{F}\mathbf{x}^* + \mathbf{G}u^*) \leq 1 + \lambda^{*T} (\mathbf{F}\mathbf{x}^* + \mathbf{G}u)$$

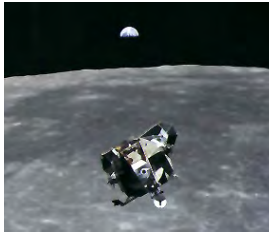
or

$$\lambda^{*T} (\mathbf{G}u^*) \leq \lambda^{*T} (\mathbf{G}u), \text{ most negative value}$$

Optimal control is a  
switching law

$$u^* = \begin{cases} +1, & \lambda^{*T} \mathbf{G} < 0 \\ -1, & \lambda^{*T} \mathbf{G} > 0 \end{cases}$$

36



## “Bang-Bang” Control of the Lunar Module

Second-order system with **ON/OFF** reaction control

$$\begin{bmatrix} \dot{\theta}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ g_A / I_{yy} \end{bmatrix} u(t)$$

Time evolution of the state while a thruster is on [ $u(t) = 1$ ]

**Angular rate**, deg/s:  $q(t) = (g_A / I_{yy})t + q(0)$

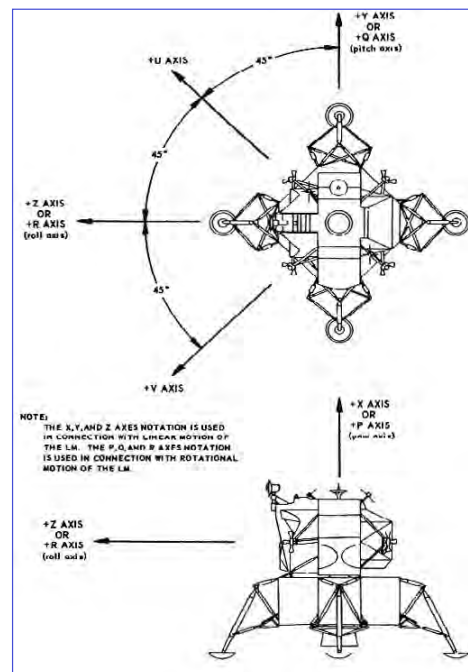
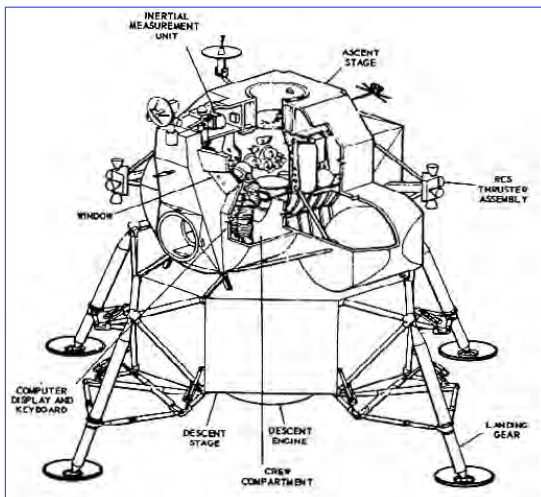
**Angle**, deg:  $\theta(t) = (g_A / I_{yy})t^2 / 2 + q(0)t + \theta(0)$

Neglecting initial conditions, what does the **phase-plane plot (pitch rate vs. pitch angle)** look like?

37

## Apollo Lunar Module Control

- **16 reaction control thrusters**
  - Control about 3 axes
  - Redundancy of thrusters
- **LM Digital Autopilot**



38

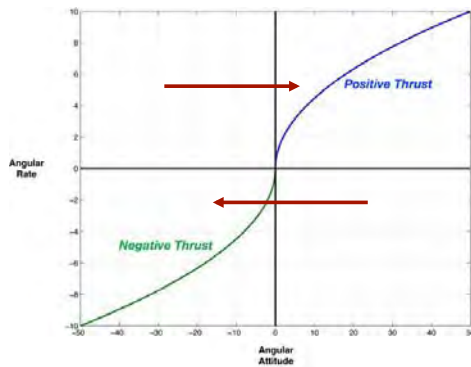
# Constant-Thrust (Acceleration) Trajectories



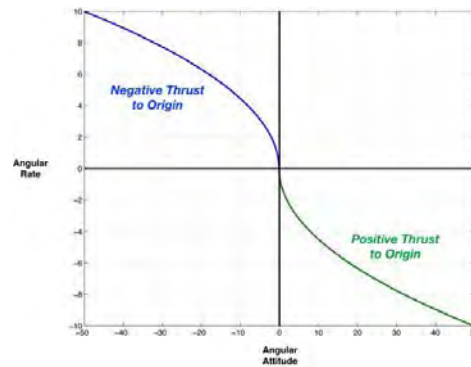
For  $u = 1$ ,  
Acceleration =  $g_A/l_{yy}$

For  $u = -1$ ,  
Acceleration =  $-g_A/l_{yy}$

Thrusting away from the origin



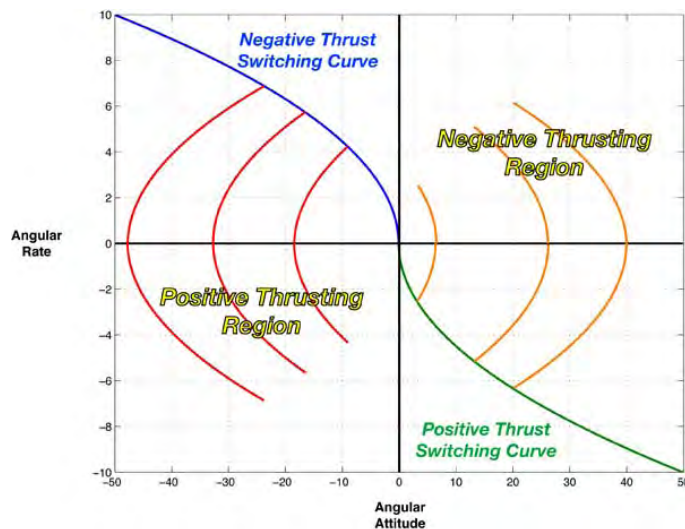
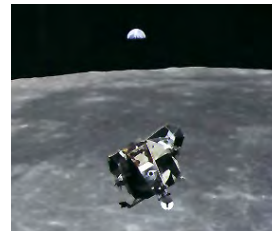
Thrusting to the origin



With zero thrust, what does the phase-plane plot look like?

39

# Switching-Curve Control Law for On-Off Thrusters



- Origin (i.e., zero rate and attitude error) can be reached from any point in the state space
- Control logic:
  - Thrust in one direction until switching curve is reached
  - Then reverse thrust
  - Switch thrust off when errors are zero

40

***Next Time:  
Constraints and  
Numerical Optimization***

***Reading  
OCE: Section 3.5, 3.6***

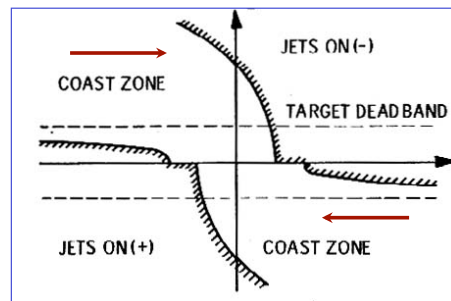
41

***Supplementary  
Material***



42

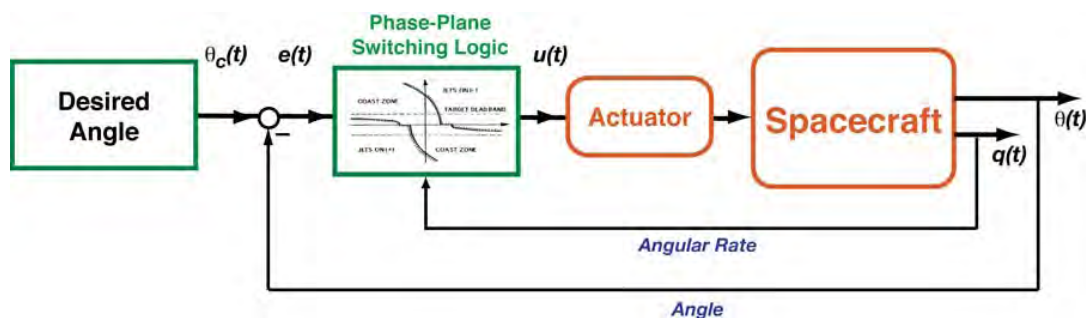
# Apollo Lunar Module Phase-Plane Control Logic



- Coast zones conserve RCS propellant by limiting angular rate
- With no coast zone, thrusters would chatter on and off at origin, wasting propellant
- State limit cycles about target attitude
- Switching curve shapes modified to provide robustness against modeling errors
  - RCS thrust level
  - Moment of inertia

43

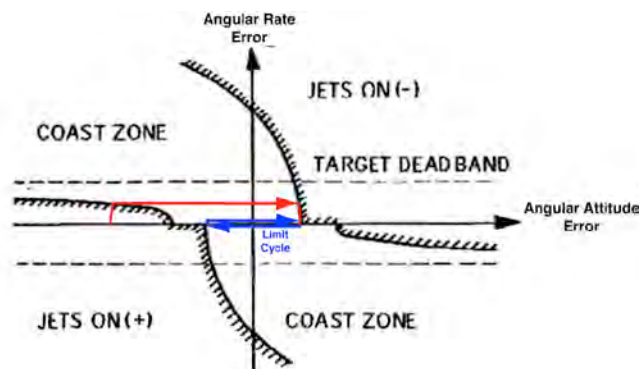
# Apollo Lunar Module Phase-Plane Control Law



- Switching logic implemented in the Apollo Guidance & Control Computer
- More efficient than a linear control law for on-off actuators

44

# Typical Phase-Plane Trajectory



- With angle error, RCS turned on until reaching OFF switching curve
- Phase point drifts until reaching ON switching curve
- RCS turned off when rate is 0-
- Limit cycle maintained with minimum-impulse RCS firings
  - Amplitude =  $\pm 1$  deg (coarse),  $\pm 0.1$  deg (fine)