Modal Properties of Linear-Quadratic Regulators

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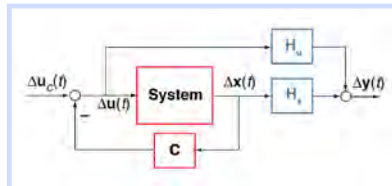
- Frequency domain models of optimally regulated systems
- Determinant identities
- Transmission zeros
- Root locus analysis of optimally regulated systems
- Eigenvectors of linearly regulated systems



http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Linear-Quadratic Control

Quadratic cost function for infinite final time

$$J = \frac{1}{2}\int_{t_o}^{\infty}\begin{bmatrix} \Delta\mathbf{x}^T(t) & \Delta\mathbf{u}^T(t) \end{bmatrix}\begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t) \end{bmatrix} dt = \frac{1}{2}\int_{t_o}^{\infty}\left[\Delta\mathbf{x}^T(t)\mathbf{Q}\Delta\mathbf{x}(t) + \Delta\mathbf{u}(t)\mathbf{R}\Delta\mathbf{u}^T(t)\right]dt$$

- Linear, time-invariant dynamic system

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)$$

- Adjoint equation derives from Euler-Lagrange equations ("adjoint system")

$$\Delta\dot{\boldsymbol{\lambda}}(t) = -\mathbf{Q}\Delta\mathbf{x}(t) - \mathbf{F}^T\Delta\boldsymbol{\lambda}(t)$$

- Optimal control is proportional to adjoint vector

$$\Delta\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{G}^T\Delta\boldsymbol{\lambda}(t)$$

hence

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) - \mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T\Delta\boldsymbol{\lambda}(t)$$

Coupling of State and Adjoint System Dynamics Due to Optimal Control

State and adjoint equations are coupled by optimal control

$$\begin{bmatrix} \Delta\dot{\mathbf{x}}(t) \\ \Delta\dot{\boldsymbol{\lambda}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F} & -\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T \\ -\mathbf{Q} & -\mathbf{F}^T \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\boldsymbol{\lambda}(t) \end{bmatrix}$$

or

$$\Delta\dot{\boldsymbol{\chi}}(t) = \mathbf{F}'\Delta\boldsymbol{\chi}(t)$$
$$\dim(\Delta\boldsymbol{\chi}) = 2n \times 1$$

State Dynamics	***Adjoint Effect on State***

$$\mathbf{F}' = \begin{bmatrix} \mathbf{F} & -\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T \\ -\mathbf{Q} & -\mathbf{F}^T \end{bmatrix}$$

State Effect on Adjoint	***Adjoint System Dynamics***

Frequency Domain Model of State-Adjoint System

Laplace transform of state and adjoint equations, neglecting initial/final conditions

$$\begin{bmatrix} s\Delta\mathbf{x}(s) \\ s\Delta\boldsymbol{\lambda}(s) \end{bmatrix} = \begin{bmatrix} \mathbf{F} & -\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T \\ -\mathbf{Q} & -\mathbf{F}^T \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}(s) \\ \Delta\boldsymbol{\lambda}(s) \end{bmatrix} \triangleq \mathbf{F}' \begin{bmatrix} \Delta\mathbf{x}(s) \\ \Delta\boldsymbol{\lambda}(s) \end{bmatrix}$$

$\left(s\mathbf{I}_{2n} - \mathbf{F}'\right)$ is the **characteristic matrix** of the coupled system

Characteristic polynomial of the control-coupled system

$$\left|\left(s\mathbf{I}_{2n} - \mathbf{F}'\right)\right| = \begin{vmatrix} \left(s\mathbf{I}_n - \mathbf{F}\right) & \mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T \\ \mathbf{Q} & \left(s\mathbf{I}_n + \mathbf{F}^T\right) \end{vmatrix} = \mathbf{0}$$

$$\Delta_{\text{coupled}}(s) = s^{2n} + a_{2n-1}s^{2n-1} + \cdots + a_1 s + a_o$$
$$= (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_{2n}) = 0$$

Uncoupled State and Adjoint Dynamics

Uncoupled System

With no coupling due to control

$$\begin{bmatrix} s\Delta\mathbf{x}(s) \\ s\Delta\boldsymbol{\lambda}(s) \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ -\mathbf{Q} & -\mathbf{F}^T \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}(s) \\ \Delta\boldsymbol{\lambda}(s) \end{bmatrix}$$

Determinants of the uncoupled system's characteristic matrix

$$\begin{vmatrix} \left(s\mathbf{I}_n - \mathbf{F}\right) & \mathbf{0} \\ \mathbf{Q} & \left(s\mathbf{I}_n + \mathbf{F}^T\right) \end{vmatrix} = \left|\left(s\mathbf{I}_n - \mathbf{F}\right)\left(s\mathbf{I}_n + \mathbf{F}^T\right)\right| = \left|s\mathbf{I}_n - \mathbf{F}\right|\left|s\mathbf{I}_n + \mathbf{F}^T\right| = \mathbf{0}$$

Eigenvalues of a block triangular matrix are eigenvalues of the diagonal block matrices

Two Determinant Identities

1) Determinant is a scalar

$$\left|\mathbf{A}^T\right| = \left|\mathbf{A}\right|$$

2) Square matrices, scalar determinant

$$\left|\mathbf{AB}\right| = \left|\mathbf{BA}\right| = \left|\mathbf{A}\right|\left|\mathbf{B}\right|$$

Characteristic Equation of the Uncoupled System

$$\left|s\mathbf{I}_{2n} - \mathbf{F}'_{\text{uncoupled}}\right| = \left|s\mathbf{I}_n - \mathbf{F}\right|\left|s\mathbf{I}_n + \mathbf{F}^T\right| = \mathbf{0}$$

Dynamic system

$$\left|s\mathbf{I}_n - \mathbf{F}\right| = \Delta_{\text{state}}(s) \triangleq \Delta_{\text{OL}}(s) = \mathbf{0}$$

Adjoint system

$$\begin{aligned}\left|s\mathbf{I}_n + \mathbf{F}^T\right| &= \Delta_{\text{adjoint}}(s) = (-1)^n\left|-s\mathbf{I}_n - \mathbf{F}^T\right| \\ &= (-1)^n \Delta_{\text{state}}(-s) \triangleq (-1)^n \Delta_{\text{OL}}(-s) = \mathbf{0}\end{aligned}$$

Eigenvalue Symmetry

Uncoupled system polynomials: <u>same</u> polynomial of (s) and ($-s$)

$$\boxed{\Delta_{\text{uncoupled}}(s) \triangleq (-1)^n \Delta_{OL}(s)\Delta_{OL}(-s)}$$

Characteristic polynomial has 2n roots
With no control coupling, ...

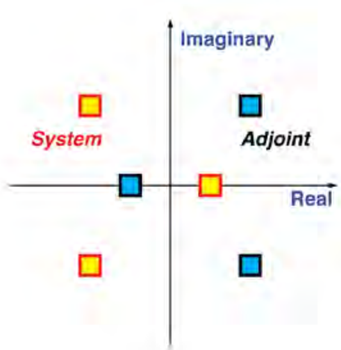
$$\begin{aligned}\Delta_{\text{uncoupled}}(s) &= s^{2n} + a_{2n-1}s^{2n-1} + \cdots + a_1 s + a_o = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_{2n}) \\ &= \left[(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)\right]_{\text{state}}\left[(s-\lambda_{n+1})(s-\lambda_{n+2})\cdots(s-\lambda_{2n})\right]_{\text{adjoint}}\end{aligned}$$

$$\begin{aligned}\Delta_{\text{uncoupled}}(s) &= \left|s\mathbf{I}_n - \mathbf{F}\right|\left|s\mathbf{I}_n + \mathbf{F}^T\right| = \left|s\mathbf{I}_n - \mathbf{F}\right|\left|s\mathbf{I}_n + \mathbf{F}\right| \\ &= (-1)^n \Delta_{OL}(s)\Delta_{OL}(-s) = \mathbf{0}\end{aligned}$$

Eigenvalue Symmetry

Eigenvalues are <u>mirrored about the imaginary axis</u>

$$\begin{aligned}\Delta_{\text{uncoupled}}(s) &= \left[(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)\right]_{\text{state}}\left[(-s-\lambda_1)(-s-\lambda_2)\cdots(-s-\lambda_n)\right]_{\text{adjoint}} \\ &= (-1)^n\left[(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)\right]_{\text{state}}\left[(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n)\right]_{\text{adjoint}} \\ &= 0\end{aligned}$$



Coupled State and Adjoint Dynamics

More Determinant Identities

3) from Laplace expansion

$$\begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{I}_m \end{vmatrix} = \begin{vmatrix} \mathbf{A} & 0 \\ \mathbf{B}^T & \mathbf{I}_m \end{vmatrix} = |\mathbf{A}|$$

4) From (2) and (3)

$$\begin{vmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{A}_3 & \mathbf{A}_4 \end{vmatrix} = \left| \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{A}_3 & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_4 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{A}_3 & \mathbf{I}_m \end{vmatrix} \begin{vmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_4 \end{vmatrix} = |\mathbf{A}_1| |\mathbf{A}_4|$$

Schur's Formula

5. a) Non-singular A_1

$$\begin{vmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{vmatrix} = \begin{vmatrix} \mathbf{I}_m & \mathbf{0} \\ -\mathbf{A}_3\mathbf{A}_1^{-1} & \mathbf{I}_n \end{vmatrix} \begin{vmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{vmatrix} = \begin{vmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{0} & \left(\mathbf{A}_4 - \mathbf{A}_3\mathbf{A}_1^{-1}\mathbf{A}_2\right) \end{vmatrix}$$

$$= \left|\mathbf{A}_1\right| \left|\mathbf{A}_4 - \mathbf{A}_3\mathbf{A}_1^{-1}\mathbf{A}_2\right|$$

5. b) Non-singular A_4

$$\begin{vmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{vmatrix} = \begin{vmatrix} \mathbf{I}_m & -\mathbf{A}_2\mathbf{A}_4^{-1} \\ \mathbf{0} & \mathbf{I}_n \end{vmatrix} \begin{vmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{vmatrix} = \begin{vmatrix} \left(\mathbf{A}_2 - \mathbf{A}_2\mathbf{A}_4^{-1}\right) & \mathbf{0} \\ \mathbf{A}_3 & \mathbf{A}_4 \end{vmatrix}$$

$$= \left|\mathbf{A}_4\right| \left|\mathbf{A}_1 - \mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{A}_3\right|$$

Application of Schur's Formula to System Polynomial

With coupling due to optimal control, ...

$$\begin{vmatrix} \left(s\mathbf{I}_n - \mathbf{F}\right) & \mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T \\ \mathbf{Q} & \left(s\mathbf{I}_n - \mathbf{F}^T\right) \end{vmatrix} \triangleq \begin{vmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{vmatrix}$$

$$\begin{vmatrix} \left(s\mathbf{I}_n - \mathbf{F}\right) & \mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T \\ \mathbf{Q} & \left(s\mathbf{I}_n + \mathbf{F}^T\right) \end{vmatrix} \triangleq \begin{vmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{vmatrix} = \left|\mathbf{A}_1\right| \left|\mathbf{A}_4 - \mathbf{A}_3\mathbf{A}_1^{-1}\mathbf{A}_2\right|$$

$$= \left|s\mathbf{I}_n - \mathbf{F}\right| \left|\left(s\mathbf{I}_n + \mathbf{F}^T\right) - \mathbf{Q}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T\right|$$

Manipulation of Characteristic Determinant

From prior determinant identities, [5. a and 2]

$$\begin{vmatrix} (s\mathbf{I}_n - \mathbf{F}) & \mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T \\ \mathbf{Q} & (s\mathbf{I}_n + \mathbf{F}^T) \end{vmatrix} =$$

$$= \left|s\mathbf{I}_n - \mathbf{F}\right| \left|(s\mathbf{I}_n + \mathbf{F}^T) - \mathbf{Q}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T\right|$$

$$= \left|s\mathbf{I}_n - \mathbf{F}\right| \left|s\mathbf{I}_n + \mathbf{F}^T\right| \left|\mathbf{I}_n - \mathbf{Q}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T(s\mathbf{I}_n + \mathbf{F}^T)^{-1}\right|$$

Application of Schur's Formula

In Schur's formula, [5. a) and 5. b)], choose

$$\mathbf{A}_1 = \mathbf{I}_m; \quad \mathbf{A}_4 = \mathbf{I}_n$$

Then

$$\begin{vmatrix} \mathbf{I}_m & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{I}_n \end{vmatrix} = \left|\mathbf{I}_m\right| \left|\mathbf{I}_n - \mathbf{A}_3\mathbf{I}_m^{-1}\mathbf{A}_2\right| = \left|\mathbf{I}_n - \mathbf{A}_3\mathbf{A}_2\right|$$

$$= \left|\mathbf{I}_n\right| \left|\mathbf{I}_m - \mathbf{A}_2\mathbf{I}_n^{-1}\mathbf{A}_3\right| = \left|\mathbf{I}_m - \mathbf{A}_2\mathbf{A}_3\right|$$

$$\left|\mathbf{I}_n - \mathbf{A}_3\mathbf{A}_2\right| = \left|\mathbf{I}_m - \mathbf{A}_2\mathbf{A}_3\right|$$

Application to 3rd term in determinant, ...

$$\left|\mathbf{I}_n - \mathbf{Q}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T(s\mathbf{I}_n + \mathbf{F}^T)^{-1}\right| =$$

$$= \left|\mathbf{I}_m + \mathbf{R}^{-1}\mathbf{G}^T(-s\mathbf{I}_n - \mathbf{F}^T)^{-1}\mathbf{Q}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\right|$$

Coupled-System Characteristic Equation Retains Mirror Symmetry

... which leads to

$$\begin{vmatrix} (s\mathbf{I}_n - \mathbf{F}) & \mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T \\ \mathbf{Q} & (s\mathbf{I}_n + \mathbf{F}^T) \end{vmatrix} \triangleq \Delta_{\mathbf{CL}}(s)\Delta_{\mathbf{CL}}(-s)$$

$$= \left|s\mathbf{I}_n - \mathbf{F}\right|\left|s\mathbf{I}_n + \mathbf{F}^T\right|\left|\mathbf{I}_n - \mathbf{Q}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T(s\mathbf{I}_n + \mathbf{F}^T)^{-1}\right|$$

$$= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \mathbf{R}^{-1}\mathbf{G}^T(-s\mathbf{I}_n - \mathbf{F}^T)^{-1}\mathbf{Q}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\right|$$

Output Matrix Derived from State Weighting Matrix

Define

$$\mathbf{Q} = \mathbf{H}^T\mathbf{H}$$

and

$$\Delta\mathbf{y}(t) = \mathbf{H}\Delta\mathbf{x}(t)$$
$$\Delta\mathbf{y}(s) = \mathbf{H}\Delta\mathbf{x}(s)$$

$$J = \frac{1}{2}\int_{t_o}^{\infty}\left[\Delta\mathbf{x}^T(t)\mathbf{Q}\Delta\mathbf{x}(t) + \Delta\mathbf{u}(t)\mathbf{R}\Delta\mathbf{u}^T(t)\right]dt$$

$$= \frac{1}{2}\int_{t_o}^{\infty}\left[\Delta\mathbf{x}^T(t)\mathbf{H}^T\mathbf{H}\Delta\mathbf{x}(t) + \Delta\mathbf{u}(t)\mathbf{R}\Delta\mathbf{u}^T(t)\right]dt$$

$$= \frac{1}{2}\int_{t_o}^{\infty}\left[\Delta\mathbf{y}^T(t)\Delta\mathbf{y}(t) + \Delta\mathbf{u}(t)\mathbf{R}\Delta\mathbf{u}^T(t)\right]dt$$

Cost Function Transfer Matrix

$$\mathbf{Y}_1(s) = \mathbf{H}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G} = \frac{\mathbf{V}_1(s)}{\Delta_{\mathbf{OL}}(s)} \qquad \frac{(r \times m)}{(1 \times 1)}$$

- **Analogous to regular transfer function matrix**
  - **"Output" is "square root" of state integrand term in cost function**
  - **Numerator reflects relative weighting of state elements**
  - **Denominator is the uncoupled characteristic polynomial**

Expression of Characteristic Polynomial Using Cost Function Transfer Matrix

$$\mathbf{Y}_1(s) = \mathbf{H}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G} = \frac{\mathbf{V}_1(s)}{\Delta_{\mathbf{OL}}(s)} \qquad \frac{(r \times m)}{(1 \times 1)}$$

Cost function transfer matrix affects state and adjoint characteristic polynomials

$$\begin{aligned}
&\Delta_{\mathbf{CL}}(s)\Delta_{\mathbf{CL}}(-s) = \\
&= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \mathbf{R}^{-1}\mathbf{G}^T(-s\mathbf{I}_n - \mathbf{F}^T)^{-1}\mathbf{H}^T\mathbf{H}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\right| \\
&\triangleq (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \mathbf{R}^{-1}\mathbf{Y}_1^T(-s)\mathbf{Y}_1(s)\right|
\end{aligned}$$

Determinant of $\mathbf{Y}_1(s)$

H need not be square, but cost observability/detectability criterion must be satisfied to guarantee closed-loop stability

$$\dim(\mathbf{H}) = r \times n \qquad (\mathbf{F},\mathbf{H})\ \textbf{observable}$$
$$\text{Rank}(\mathbf{H}) = \text{Rank}(\mathbf{Q}) \le \textbf{min}(r,n)$$

$$\text{If } \dim(\Delta\mathbf{u}) = m \times 1, \quad \text{and } r = m$$
$$\text{Then } \mathbf{Y}_1(s) \text{ is } \textbf{square} \quad [r \times m]$$

If $\mathbf{Y}_1(s)$ is square, it possesses a determinant

$$\left|\mathbf{Y}_1(s)\right| = \left|\mathbf{H}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\right| = \left|\frac{\mathbf{V}_1(s)}{\Delta_{\mathbf{OL}}(s)}\right|$$
$$= \left|\frac{\mathbf{V}_1(s)}{\left|s\mathbf{I}_n - \mathbf{F}\right|}\right| = \frac{\left|\mathbf{V}_1(s)\right|}{\left|s\mathbf{I}_n - \mathbf{F}\right|}$$
$$\triangleq \frac{\psi_1(s)}{\Delta_{\mathbf{OL}}(s)}\ (\textbf{scalar})$$

Transmission Zeros of Cost Function Transfer Matrix

Transmission Zeros of $|\mathbf{Y}_1(s)|$

$$
\left|\mathbf{Y}_1(s)\right| = \left|\mathbf{H}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}\right| = \frac{\left|\mathbf{V}_1(s)\right|}{\left|s\mathbf{I}_n - \mathbf{F}\right|}
$$

$$
\triangleq \frac{\psi_1(s)}{\Delta_{\mathbf{OL}}(s)}
$$

Roots of the numerator are the <u>transmission zeros</u> of $|\mathbf{Y}_1(s)|$

$$
\psi_1(s) = a_q s^q + a_{q-1}s^{q-1} + \cdots + a_1 s + a_0
$$

$$
= a_q\left(s - z_1\right)\left(s - z_2\right)\cdots\left(s - z_q\right)
$$

Number of zeros $= q = (n - m - d)$

$d =$ **Rank deficiency** of $\mathbf{HG}$

2^{nd}-Order System Examples of Cost Transfer Function

- **Example 6.4-1 a)**

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

$$
\mathbf{Y}_1(s) = \mathbf{H}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}
$$

$$
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} (s-a) & 0 \\ 0 & (s-d) \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
= \frac{\begin{bmatrix} (s-d) & 0 \\ 0 & (s-a) \end{bmatrix}}{(s-a)(s-d)} = \begin{bmatrix} \frac{1}{(s-a)} & 0 \\ 0 & \frac{1}{(s-d)} \end{bmatrix}
$$

$$
\left|\mathbf{Y}_1(s)\right| = \begin{vmatrix} \frac{1}{(s-a)} & 0 \\ 0 & \frac{1}{(s-d)} \end{vmatrix} = \frac{1}{(s-a)(s-d)} = \frac{\psi_1(s)}{\Delta_{\mathbf{OL}}(s)}
$$

$\therefore$ **No transmission zeros in system**

2^{nd}-Order System Examples of Cost Transfer Function

Example 6.4-1 (d)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} u; \quad y = \begin{bmatrix} g & h \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{Y}_1(s) = \frac{(eg+fh)s + \left[e(hc-gd) + f(gb-ha)\right]}{\Delta_{\mathbf{OL}}(s)}$$

$$= \frac{(eg+fh)\left\{s + \left[\dfrac{e(hc-gd)+f(gb-ha)}{(eg+fh)}\right]\right\}}{\Delta_{\mathbf{OL}}(s)}$$

$$\textbf{Zero at } s = \frac{-\left[e(hc-gd)+f(gb-ha)\right]}{(eg+fh)}$$

Scalar Cost Function Multiplier

Use $|\mathbf{Y}_1(s)|$ in the characteristic polynomial

$$\Delta_{\mathbf{CL}}(s)\Delta_{\mathbf{CL}}(-s) = (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \mathbf{R}^{-1}\mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{H}^T\mathbf{H}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}\right|$$

$$= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \mathbf{R}^{-1}\mathbf{Y}_1^T(-s)\mathbf{Y}_1(s)\right|$$

Scalar multiplier for control weighting matrix

$$\textbf{Define} \quad \mathbf{R} = \rho^2\mathbf{R}_o$$

Closed-loop characteristic polynomial

$$\Delta_{\mathbf{CL}}(s)\Delta_{\mathbf{CL}}(-s) =$$

$$= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \left(\rho^2\mathbf{R}_o\right)^{-1}\mathbf{Y}_1^T(-s)\mathbf{Y}_1(s)\right|$$

$$= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \frac{1}{\rho^2}\mathbf{R}_o^{-1}\mathbf{Y}_1^T(-s)\mathbf{Y}_1(s)\right|$$

Weighted Cost Function Transfer Matrix, $\mathbf{Y}_2(s)$

Square root of control weight inverse

$$\boxed{\text{Define} \quad \mathbf{U}^T\mathbf{U} = \mathbf{R}_o^{-1}}$$

Incorporate U in definition of $|\mathbf{Y}_2(s)|$

$$\begin{aligned} \mathbf{Y}_2(s) &\triangleq \mathbf{Y}_1(s)\mathbf{U} \\ &= \mathbf{H}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\mathbf{U} = \frac{\mathbf{V}_2(s)}{\Delta_{\mathbf{OL}}(s)} \quad (r \times m) \end{aligned}$$

Closed-loop characteristic polynomial

$$\begin{aligned} \Delta_{\text{coupled}}(s) &= \Delta_{\mathbf{CL}}(s)\Delta_{\mathbf{CL}}(-s) = \\ &= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \frac{1}{\rho^2}\mathbf{U}^T\mathbf{Y}_1^T(-s)\mathbf{Y}_1(s)\mathbf{U}\right| \\ &\triangleq (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \frac{1}{\rho^2}\mathbf{Y}_2^T(-s)\mathbf{Y}_2(s)\right| \end{aligned}$$

Effects of Cost Function Weights on Root Locations

Variations in Cost Function Transfer Matrix

- Open-loop poles are independent of **Q** and **R**
- **Q** and **R**$_o$ specify locations of transmission zeros

$$\mathbf{Q} = \mathbf{H}^T\mathbf{H}; \quad \mathbf{R}_o^{-1} = \mathbf{U}^T\mathbf{U}$$

$$\mathbf{Y}_2(s) = \mathbf{H}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\mathbf{U} = \frac{\mathbf{V}_2(s)}{\Delta_{\mathbf{OL}}(s)}$$

$$|\mathbf{Y}_2(s)| = \frac{|\mathbf{V}_2(s)|}{|s\mathbf{I}_n - \mathbf{F}|} \triangleq \frac{\psi_2(s)}{\Delta_{\mathbf{OL}}(s)} \quad (\textbf{scalar})$$

- Roots of characteristic equation vary with $1/\rho^2$

$$\Delta_{\textbf{coupled}}(s) = \Delta_{\mathbf{CL}}(s)\Delta_{\mathbf{CL}}(-s)$$
$$= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \frac{1}{\rho^2}\mathbf{U}^T\mathbf{Y}_1^T(-s)\mathbf{Y}_1(s)\mathbf{U}\right|$$
$$\triangleq (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \frac{1}{\rho^2}\mathbf{Y}_2^T(-s)\mathbf{Y}_2(s)\right|$$

Coupled-System Eigenvalues

Roots of characteristic equation vary with $1/\rho^2$

$$(-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \frac{1}{\rho^2}\mathbf{Y}_2^T(-s)\mathbf{Y}_2(s)\right| \xrightarrow[\rho^2 \to \infty]{} (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)|\mathbf{I}_m|$$
$$= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)$$

Open-loop system roots may be stable or unstable

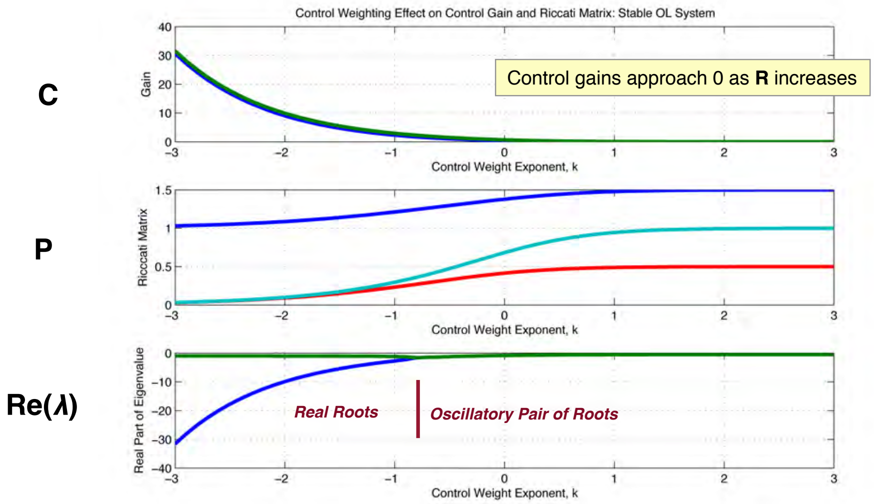
$$\Delta_{\textbf{coupled}}(s) = s^{2n} + a_{2n-1}s^{2n-1} + \cdots + a_1 s + a_o$$
$$= (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_{2n})$$
$$= (-1)^n\left[(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)\right]_{\mathbf{OL}}\left[(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n)\right]_{\mathbf{OL}} = 0$$

But we know that the all closed-loop system roots must be stable

Control Weighting Effect for <u>Stable</u> 2nd-Order System

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

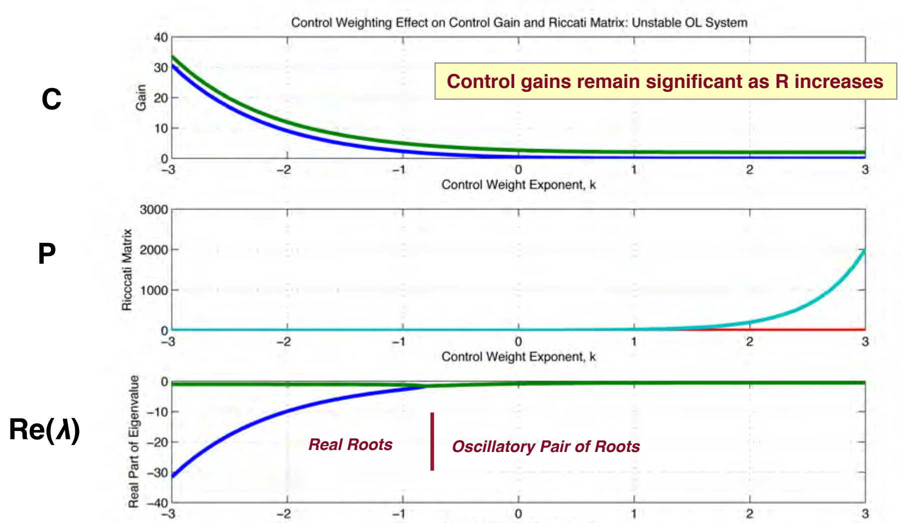
$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{R} = r \text{ in } \left(10^{-3}, 10^{3}\right)$$



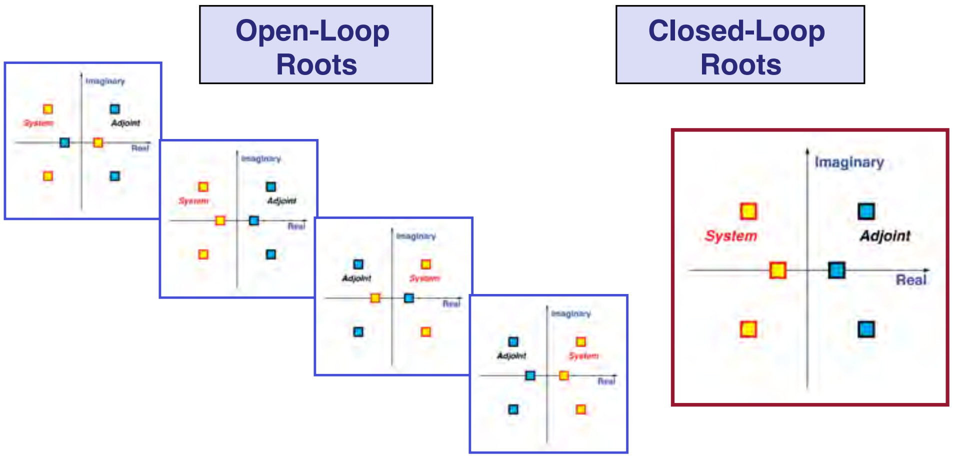
Control Weighting Effect for <u>Unstable</u> 2nd-Order System

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{R} = r \text{ in } \left(10^{-3}, 10^{3}\right)$$



All Closed-Loop System State Roots Go To Stable Images as $\rho^2 \rightarrow \infty$



All closed-loop **adjoint-system roots** are in right half plane

What Happens to the Roots when $\rho^2 \rightarrow \infty$?

Closed-loop characteristic polynomial

$$\Delta_{\text{coupled}}(s) = (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|\mathbf{I}_m + \frac{1}{\rho^2}\mathbf{Y}_2^T(-s)\mathbf{Y}_2(s)\right|$$

$$\dim\left[\mathbf{Y}_2(s)\right] = r \times m$$

Consider case in which $r = m$, *i.e.*, $\mathbf{Y}_2(s)$ is square

$|\mathbf{Y}_2(s)|$ possesses a determinant

$$\left|\mathbf{Y}_2(s)\right| = \frac{\psi_2(s)}{\Delta_{\mathbf{OL}}(s)}$$

Characteristic Polynomial of Scalar Coupled System

Let $r = m = 1$

$$\left|\mathbf{Y}_2(s)\right| = \frac{\left|\mathbf{V}_2(s)\right|}{\Delta_{\mathbf{OL}}(s)} = Y_2(s) = \frac{\psi_2(s)}{\Delta_{\mathbf{OL}}(s)}$$

Closed-loop characteristic polynomial

$$\Delta_{\text{coupled}}(s) =$$
$$= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left|1 + \frac{1}{\rho^2}Y_2(-s)Y_2(s)\right|$$
$$= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left[1 + \frac{1}{\rho^2}Y_2(-s)Y_2(s)\right]$$

Characteristic Polynomial of Scalar Coupled System

Closed-loop characteristic polynomial

$$\Delta_{\text{coupled}}(s) =$$
$$= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)\left[1 + \left(\frac{1}{\rho^2}\right)\frac{\psi_2(-s)\psi_2(s)}{(-1)^n \Delta_{\mathbf{OL}}(-s)\Delta_{\mathbf{OL}}(s)}\right]$$
$$= (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s) + \left(\frac{1}{\rho^2}\right)\psi_2(-s)\psi_2(s)$$

Numerator polynomial multiplied by $(-1)^q$

$$\psi_2(s) = a_q(s - z_1)(s - z_2)\cdots(s - z_q)$$
$$\psi_2(-s) = (-1)^q a_q(s + z_1)(s + z_2)\cdots(s + z_q)$$

Significance of Powers of (–1)

Denominator polynomial multiplied by $(-1)^n$

$$(-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s) =$$
$$= (-1)^n \left[(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)\right]\left[(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n)\right]$$

Closed-loop characteristic polynomial

$$\Delta_{\mathbf{coupled}}(s) = (-1)^n \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s)$$
$$+(-1)^q \left(\frac{a_q^{\ 2}}{\rho^2}\right)\left[(s+z_1)(s+z_2)\cdots(s+z_q)\right]\left[(s-z_1)(s-z_2)\cdots(s-z_q)\right]$$

Mirror-image poles and zeros

Characteristic Polynomial of the Coupled System Can Be Put In Root-Locus Form

Because characteristic polynomial equals zero, $(-1)^n$ multiplier can be discarded in the solution for roots

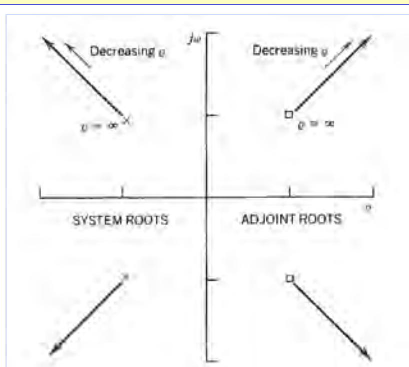
$$(-1)^{-n} \Delta_{\mathbf{coupled}}(s) = 0 = \Delta_{\mathbf{coupled}}(s)$$
$$= \Delta_{\mathbf{OL}}(s)\Delta_{\mathbf{OL}}(-s) + (-1)^{(q-n)} \left(\frac{a_q^{\ 2}}{\rho^2}\right)\left[(s+z_1)(s+z_2)\cdots(s+z_q)\right]\left[(s-z_1)(s-z_2)\cdots(s-z_q)\right]$$
$$\triangleq D(s) + KN(s) \quad \left[\textbf{Root locus form}\right]$$

Poles and zeros of the coupled system's root locus are symmetric about the imaginary axis

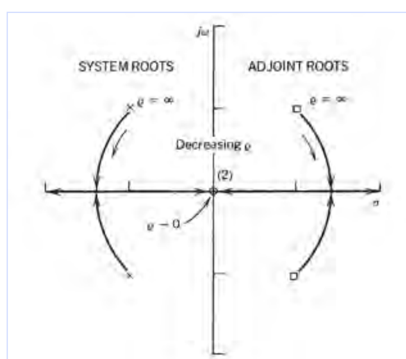
Root Locus Analysis of the Coupled State/Adjoint System

Optimal Closed-Loop Root Loci of Two 2nd-Order Systems

Complex Roots, No Transmission Zero

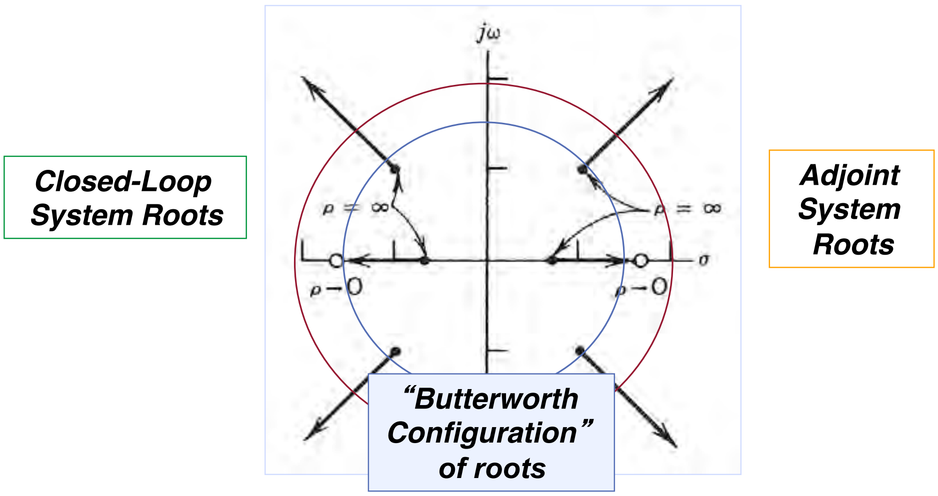


Complex Roots, One Transmission Zero



"Center of gravity" is always at the origin for state/adjoint root locus
Roots originating in the left/right half plane remain in the left/right half plane
ROOT LOCUS IS SYMMETRIC ABOUT THE IMAGINARY AXIS
Closed-loop system is stable for all values of ρ

Optimal Closed-Loop Root Locus of 3[rd]-Order System with One Transmission Zero



Closed-loop system is stable for all values of ρ

Eigenvectors

Eigenvectors

Eigenvalues of square matrix, F

$$\left| s\mathbf{I} - \mathbf{F} \right| = \Delta(s) =$$
$$= (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n) = 0$$

Eigenvectors of F

Defined within an arbitrary constant

Constant may be real or complex

$$(\lambda_i \mathbf{I} - \mathbf{F})\mathbf{e}_i = 0, \quad i = 1, n$$
$$(\lambda_i \mathbf{I} - \mathbf{F})\alpha \mathbf{e}_i = 0$$

$$\dim(\mathbf{e}_i) = n \times 1$$

Computation of Eigenvectors and the Modal Matrix

Calculation of each eigenvector typically produces *n* linearly dependent columns related by multiplicative constants

$$Adj(\lambda_i \mathbf{I} - \mathbf{F}) = \begin{bmatrix} \beta_1 \mathbf{e}_i & \beta_2 \mathbf{e}_i & \cdots & \beta_n \mathbf{e}_i \end{bmatrix}, \quad i = 1, n$$

E: Modal Matrix of F

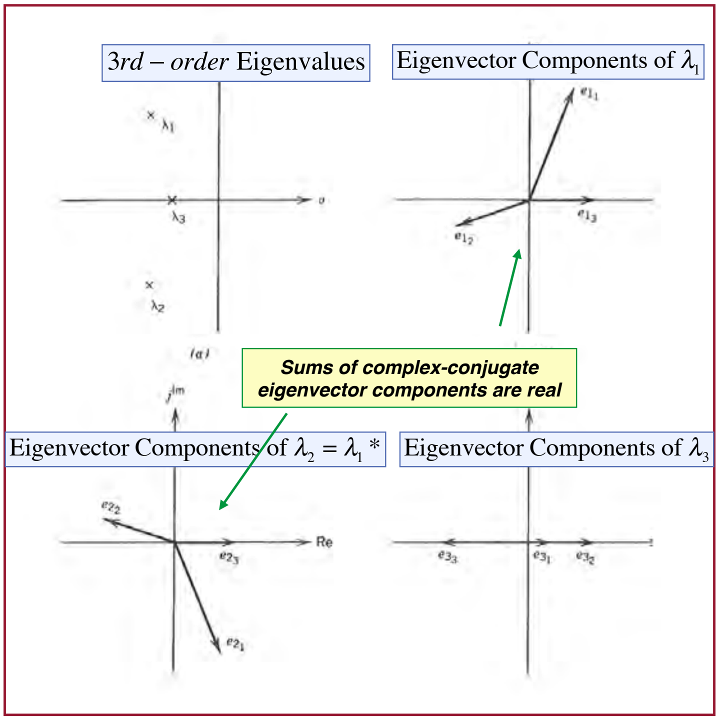
One eigenvector for each eigenvalue forms a column of E

E is normalized appropriately, e.g., $|\mathbf{e}_i| = 1$ or $|\mathbf{E}| = 1$

$$\mathbf{E} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}; \quad \dim(\mathbf{E}) = n \times n$$

Complex Eigenvectors

- If eigenvalues are <u>complex</u>, then corresponding eigenvectors are <u>complex</u>
- If eigenvalues are <u>complex conjugates</u>, then corresponding eigenvectors are <u>complex conjugates</u>



Property of the Modal Matrix, E

Each eigenvector satisfies the following

$$\begin{bmatrix} \mathbf{F}\mathbf{e}_1 = \lambda_1 \mathbf{e}_1 \\ \mathbf{F}\mathbf{e}_2 = \lambda_2 \mathbf{e}_2 \\ \cdots \\ \mathbf{F}\mathbf{e}_n = \lambda_n \mathbf{e}_n \end{bmatrix} \Rightarrow \mathbf{FE} = \mathbf{E}\boldsymbol{\Lambda}$$

Diagonal matrix of eigenvalues

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Modal Matrix, E, Can Be Used as a Similarity Transformation to "Diagonalize" F

If F is real and symmetric

Eigenvectors are real

E is an orthonormal transformation

Transpose = Inverse

$$\boxed{\mathbf{\Lambda} = \mathbf{E}^T \mathbf{F} \mathbf{E}}$$

$$\boxed{\mathbf{F} = \mathbf{E} \mathbf{\Lambda} \mathbf{E}^T}$$

2nd-Order Example

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}; \quad [s\mathbf{I} - \mathbf{F}] = \begin{bmatrix} s & -1 \\ \omega_n^2 & (s + 2\zeta\omega_n) \end{bmatrix}$$

$$\Delta(s) = |s\mathbf{I} - \mathbf{F}| = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2}$$

For $\zeta > 1$, roots are real

$$\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2}$$
$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

For $\zeta < 1$, roots are complex

$$\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2}$$
$$= -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}, \quad j = \sqrt{-1}$$

= Complex conjugates, $\lambda_1, \lambda_1{}^*$

Eigenvectors for 2nd-Order Example

$$
\begin{aligned}
Adj\left[\lambda_i \mathbf{I}-\mathbf{F}\right] &= Adj\begin{bmatrix} \lambda_i & -1 \\ \omega_n{}^2 & \left(\lambda_i + 2\zeta\omega_n\right) \end{bmatrix} \\
&= \left[\begin{array}{c|c} \left(\lambda_i + 2\zeta\omega_n\right) & 1 \\ -\omega_n{}^2 & \lambda_i \end{array}\right] \\
&\triangleq \begin{bmatrix} \beta_1 \mathbf{e}_i & \beta_2 \mathbf{e}_i \end{bmatrix}, \quad i = 1,2
\end{aligned}
$$

Each column is a representation of the eigenvector

2nd-Order Example with Real Roots

$$
Adj\left[\lambda_i \mathbf{I}-\mathbf{F}\right] = \begin{bmatrix} \left(-\zeta\omega_n\left(1 \mp \sqrt{1-\frac{1}{\zeta^2}}\right) + 2\zeta\omega_n\right) & 1 \\ -\omega_n^2 & -\zeta\omega_n\left(1 \mp \sqrt{1-\frac{1}{\zeta^2}}\right) \end{bmatrix}
$$

Modal Matrix

$$
\mathbf{E} = \left[\begin{array}{c|c} 1 & 1 \\ -\zeta\omega_n\left(1 - \sqrt{1-\frac{1}{\zeta^2}}\right) & -\zeta\omega_n\left(1 + \sqrt{1-\frac{1}{\zeta^2}}\right) \end{array}\right]
$$

Eigenvalues

$$
\mathbf{\Lambda} = \begin{bmatrix} -\zeta\omega_n\left(1 - \sqrt{1-\frac{1}{\zeta^2}}\right) & 0 \\ 0 & -\zeta\omega_n\left(1 + \sqrt{1-\frac{1}{\zeta^2}}\right) \end{bmatrix}
$$

Eigenvectors of Regulated Systems

Example with Single Control Variable

$$\left| s\mathbf{I}_n - (\mathbf{F} - \mathbf{GC}) \right| = 0$$

n eigenvalues

$1 \times n$ elements of $\mathbf{C}$

- **State equation and control law**

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ 0 & f_4 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \Delta u_1$$

$$\Delta u_1 = -\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

- **Closed-loop system dynamics**

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ 0 & f_4 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

$$= \begin{bmatrix} (f_1 - g_1 c_1) & -g_1 c_2 \\ -g_2 c_1 & (f_4 - g_2 c_2) \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Relationship between eigenvalues and elements of C is unique

2 parameters, 2 unknowns

$$\left\{ s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} (f_1 - g_1 c_1) & -g_1 c_2 \\ -g_2 c_1 & (f_4 - g_2 c_2) \end{bmatrix} \right\} = \begin{bmatrix} s-(f_1 - g_1 c_1) & g_1 c_2 \\ g_2 c_1 & s-(f_4 - g_2 c_2) \end{bmatrix}$$
$$= \left[s-(f_1 - g_1 c_1) \right]\left[s-(f_4 - g_2 c_2) \right] + g_1 c_2 g_2 c_1$$
$$= s^2 - \left[(f_1 - g_1 c_1) + (f_4 - g_2 c_2) \right] s + \left[g_1 c_2 g_2 c_1 + (f_1 - g_1 c_1)(f_4 - g_2 c_2) \right]$$
$$= (s - \lambda_1)(s - \lambda_2) = s^2 - (\lambda_1 + \lambda_2)s + \lambda_1 \lambda_2 = 0$$

Relationship between eigenvector components and elements of C is unique

$$Adj\left[\lambda_i \mathbf{I} - (\mathbf{F} - \mathbf{GC}) \right] = Adj\begin{bmatrix} \left[\lambda_i - (f_1 - g_1 c_1) \right] & g_1 c_2 \\ g_2 c_1 & \lambda_i - (f_4 - g_2 c_2) \end{bmatrix}$$
$$\triangleq \begin{bmatrix} \beta_1 \mathbf{e}_i & \beta_2 \mathbf{e}_i \end{bmatrix}$$

$$\left| s\mathbf{I}_n - (\mathbf{F} - \mathbf{GC}) \right| = 0$$

n eigenvalues

$2 \times n$ elements of **C**

Example with Two Control Variables

- **State equation and control law**

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ 0 & f_4 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix}$$
$$\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = -\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

- **Closed-loop system dynamics**

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ 0 & f_4 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} - \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$
$$= \begin{bmatrix} (f_1 - c_1) & c_2 \\ c_3 & (f_4 - c_4) \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Relationship between eigenvalues and elements of C is <u>not unique</u>

2 parameters, 4 unknowns

$$\left\{ s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} (f_1 - c_1) & c_2 \\ c_3 & (f_4 - c_4) \end{bmatrix} \right\} = \begin{bmatrix} s-(f_1 - c_1) & -c_2 \\ -c_3 & s-(f_4 - c_4) \end{bmatrix}$$

$$= \left[s-(f_1 - c_1)\right]\left[s-(f_4 - c_4)\right] + c_2 c_3 = 0 = s^2 - \left[(f_1 - c_1)+(f_4 - c_4)\right]s + \left[c_2 c_3 + (f_1 - c_1)(f_4 - c_4)\right]$$

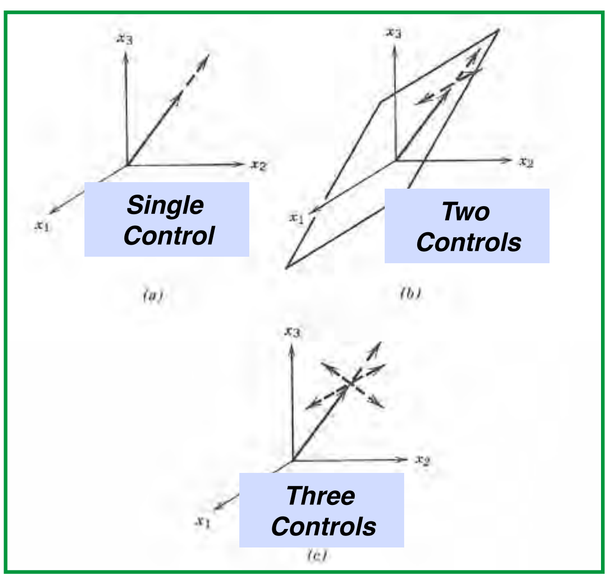
$$= (s-\lambda_1)(s-\lambda_2) = s^2 - (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2 = 0$$

Relationship between eigenvectors and elements of C is <u>not unique</u>

$$Adj\left[\lambda_i \mathbf{I} - (\mathbf{F} - \mathbf{GC})\right] = Adj\begin{bmatrix} \lambda_i - (f_1 - c_1) & -c_2 \\ -c_3 & \lambda_i - (f_4 - c_4) \end{bmatrix}$$

$$\triangleq \begin{bmatrix} \beta_1 \mathbf{e}_i & \beta_2 \mathbf{e}_i \end{bmatrix}$$

Eigenvectors Can Be Placed within an *m*-Space, $m \le n$



Modal Control Vector

$$\boxed{\Delta\mathbf{u}_i = -\mathbf{C}\Delta\mathbf{x}_i, \quad i = 1,n}$$

State eigenvector

$$\boxed{\left[s_i\mathbf{I}_n - (\mathbf{F} - \mathbf{G}\mathbf{C})\right]\Delta\mathbf{x}_i = \mathbf{0}, \quad i = 1,n}$$

Modal control vector of an eigenvalue's state eigenvector

$$\boxed{(s_i\mathbf{I}_n - \mathbf{F})\Delta\mathbf{x}_i = \mathbf{G}\Delta\mathbf{u}_i = -\mathbf{G}\mathbf{C}\Delta\mathbf{x}_i = \mathbf{0}, \quad i = 1,n}$$

Eigenvector satisfies the following equation

$$\Delta\mathbf{x}_i = -(s_i\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\mathbf{C}\Delta\mathbf{x}_i$$

$$\left[\mathbf{I}_n + (s_i\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\mathbf{C}\right]\Delta\mathbf{x}_i = \mathbf{0}$$

Modal Control Vector

Modal control vector satisfies a similar equation

$$\Delta\mathbf{u}_i = -\mathbf{C}\Delta\mathbf{x}_i = -\mathbf{C}(s_i\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\Delta\mathbf{u}_i$$

$$\left[\mathbf{I}_m + \mathbf{C}(s_i\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\right]\Delta\mathbf{u}_i = 0$$

Eigenvector and modal control vector equations have the same eigenvalues

Because $\left|\mathbf{I}_n - \mathbf{A}_3\mathbf{A}_2\right| = \left|\mathbf{I}_m - \mathbf{A}_2\mathbf{A}_3\right|$

$$\left|\mathbf{I}_n + (s_i\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\mathbf{C}\right| = \left|\mathbf{I}_m + \mathbf{C}(s_i\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}\right|$$

State eigenvector can be adjusted in a space with a dimension of Rank (C)

$$\boxed{\mathrm{Rank}(\mathbf{C}) \le \min(m,n)}$$

Next Time: Spectral Analysis of Linear-Quadratic Regulators

Supplemental Material

2^{nd}-Order System Examples of Cost Transfer Function

- **Example 6.4-1 b)**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{Y}_1(s) = \frac{\begin{bmatrix} (s-d) & b \\ c & (s-a) \end{bmatrix}}{\left[s^2 - (a+d)s + (ad-bc)\right]}$$

$$\left|\mathbf{Y}_1(s)\right| = \frac{\left[s^2 - (a+d)s + (ad-bc)\right]}{\left[s^2 - (a+d)s + (ad-bc)\right]^2} = \frac{1}{\left[s^2 - (a+d)s + (ad-bc)\right]}$$

$\therefore$ **No transmission zeros in system**

2^{nd}-Order System Examples of Cost Transfer Function

- **Example 6.4-1 c)**

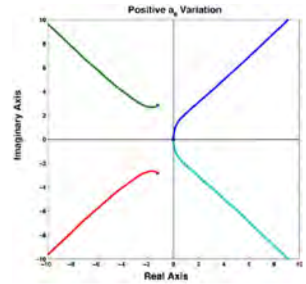
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{Y}_1(s) = \frac{\begin{bmatrix} (s-d)+c & (s-a) \\ c & (s-a) \end{bmatrix}}{(s-a)(s-d)}$$

$$\left|\mathbf{Y}_1(s)\right| = \frac{\left[s^2 - (a+d)s + ad\right]}{\left[(s-a)(s-d)\right]^2} = \frac{1}{\left[s^2 - (a+d)s + (ad)\right]}$$

$\therefore$ **No transmission zeros in system**

Root Locus Construction Rules

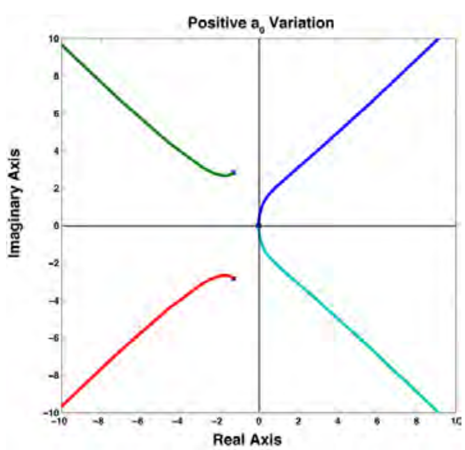


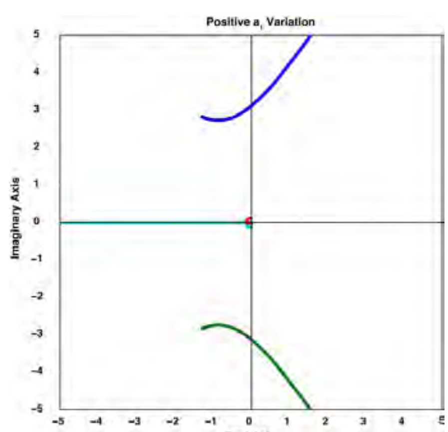
- All points on the locus of roots must satisfy the equation *K[N(s)/D(s)]* = –1
  - Phase angle of transfer function = –180 deg
- Number of roots (or *poles*) of the denominator = *n*
- Number of zeros of the numerator = *q*

Origins of Roots (for *K* = 0)

- Origins of the roots are the Poles of *D(s)*

$$\boxed{\Delta(s) = D(s) + KN(s) \xrightarrow[K \to 0]{} D(s)}$$





Destinations of Roots (for K -> ±∞)

q roots go to the zeros of $N(s)$

$$\boxed{\frac{D(s)+KN(s)}{K}=\frac{D(s)}{K}+N(s)\xrightarrow[K\to\infty]{}N(s)}$$

($n-q$) roots go to infinite radius from the origin

$$\boxed{\frac{D(s)+KN(s)}{N(s)}=\frac{D(s)}{N(s)}+K\xrightarrow[K\to\infty]{}s^{(n-q)}\pm R\to\pm\infty}$$

$$s^{(n-q)}=(R)e^{-j180^\circ}\to\infty \quad or \quad (R)e^{-j360^\circ}\to-\infty$$

which leads to

$$s=(R)e^{-j180^\circ/(n-q)}\to\infty \quad or \quad (R)e^{-j360^\circ/(n-q)}\to-\infty$$

Three More Root Locus Construction Rules

Angles of asymptotes for the ($n-q$) roots going to ∞

$$K\to+\infty:\quad \theta(rad)=\frac{\pi+2k\pi}{n-q},\quad k=0,1,\ldots,(n-q)-1$$

$$K\to-\infty:\quad \theta(rad)=\frac{2k\pi}{n-q},\quad k=0,1,\ldots,(n-q)-1$$

Origin of asymptotes = "center of gravity"

$$"c.g."=\frac{\sum_{i=1}^{n}\sigma_{\lambda_i}-\sum_{j=1}^{q}\sigma_{z_j}}{n-q}$$

Locus on real axis

$K>0$: Any segment with odd number of poles and zeros to the right

$K<0$: Any segment with even number of poles and zeros to the right

Characteristic Polynomial of the Coupled System, Root-Locus Form

When entire characteristic polynomial is multiplied by $(-1)^n$, Numerator multiplier is $(-1)^{(n-q)}$

$$\Delta_{\text{coupled}}(s) =$$
$$= (-1)^n \left\{ \Delta_{\text{OL}}(s)\Delta_{\text{OL}}(-s) + (-1)^{(q-n)}\left(\frac{a_q^{\;2}}{\rho^2}\right)\left[(s+z_1)(s+z_2)\cdots(s+z_q)\right]\left[(s-z_1)(s-z_2)\cdots(s-z_q)\right]\right\}$$
$$= 0$$

K, $D(s)$, and $N(s)$ of the Root Locus Equation

Sign of root locus gain depends on $(-1)^{(q-n)}$

$$(-1)^{-n}\,\Delta_{\text{coupled}}(s) = 0$$
$$= \left[(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)\right]\left[(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n)\right]$$
$$+(-1)^{(q-n)}\left(\frac{a_q^{\;2}}{\rho^2}\right)\left[(s+z_1)(s+z_2)\cdots(s+z_q)\right]\left[(s-z_1)(s-z_2)\cdots(s-z_q)\right]$$
$$\triangleq D(s) + KN(s)$$

where $$K = (-1)^{(q-n)}\left(\frac{a_q^{\;2}}{\rho^2}\right)$$

$$D(s) = \left[(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)\right]\left[(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n)\right]$$
$$N(s) = \left[(s+z_1)(s+z_2)\cdots(s+z_q)\right]\left[(s-z_1)(s-z_2)\cdots(s-z_q)\right]$$