# Spacecraft Guidance


## Space System Design, MAE 342, Princeton University
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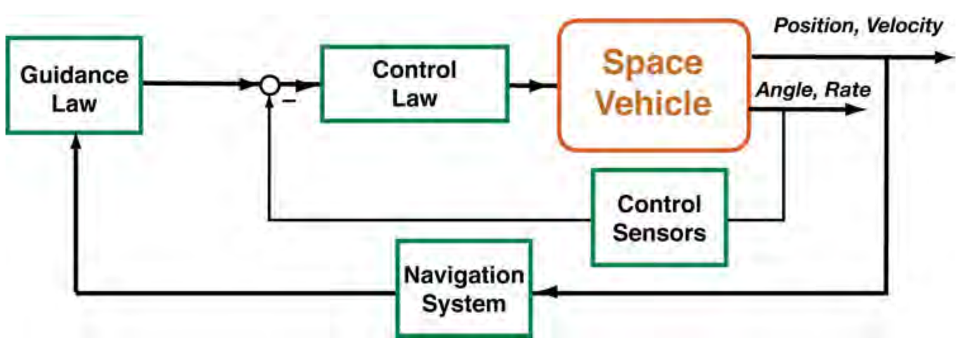
- Oberth's "Synergy Curve"
- Explicit ascent guidance
- Impulsive $\Delta V$ maneuvers
- Hohmann transfer between circular orbits
- Sphere of gravitational influence
- Synodic periods and launch windows
- Hyperbolic orbits and escape trajectories
- Battin's universal formulas
- Lambert's time-of-flight theorem (hyperbolic orbit)
- Fly-by (swingby) trajectories for gravity assist





# Guidance, Navigation, and Control




- **Navigation:** Where are we?
- **Guidance:** How do we get to our destination?
- **Control:** What do we tell our vehicle to do?


# Energy Gained from Propellant

**Specific energy = energy per unit weight**

$$\mathbb{E} = h + \frac{V^2}{2g}$$

$h$ : height; $V$ : velocity

**Rate of change of specific energy per unit of expended propellant mass**

$$\frac{d\mathbb{E}}{dm} = \frac{dh}{dm} + \frac{V}{g}\frac{dV}{dm} = \frac{1}{(dm/dt)}\left(\frac{dh}{dt} + \frac{V}{g}\frac{dV}{dt}\right)$$

$$= \frac{1}{(dm/dt)}\left(\frac{dh}{dt} + \frac{1}{g}\mathbf{v}^T\frac{d\mathbf{v}}{dt}\right) = \frac{1}{(dm/dt)}\left(V\sin\gamma + \frac{1}{g}\mathbf{v}^T(\mathbf{T} - m\mathbf{g})\right)$$

$$= \frac{1}{(dm/dt)}\left(V\sin\gamma + \frac{VT}{mg}\cos\alpha - V\sin\gamma\right)$$



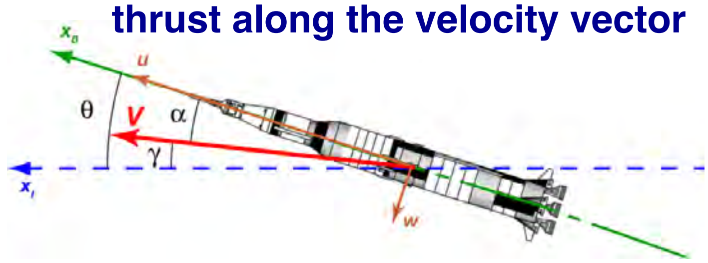
# Oberth's Synergy Curve

$\gamma$ : Flight Path Angle

$\theta$: Pitch Angle

$\alpha$: Angle of Attack

***dE/dm*** **maximized when** ***α*** **= 0, or** ***θ*** **=** ***γ, i.e.,*** **thrust along the velocity vector**




**Approximate round-earth equations of motion**

$$\frac{dV}{dt} = \frac{T}{m}\cos\alpha - \frac{Drag}{m} - g\sin\gamma$$

$$\frac{d\gamma}{dt} = \frac{T}{mV}\sin\alpha + \left(\frac{V}{r} - \frac{g}{V}\right)\cos\gamma$$

# Gravity-Turn Pitch Program

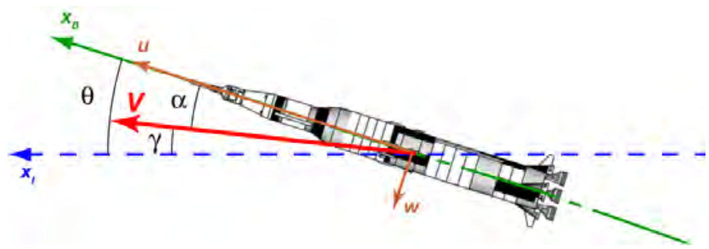
**With angle of attack, *α* = 0**

$$\frac{d\gamma}{dt} = \frac{d\theta}{dt} = \left(\frac{V}{r} - \frac{g}{V}\right)\cos\gamma$$

**Locally optimal flight path**

**Minimizes aerodynamic loads**

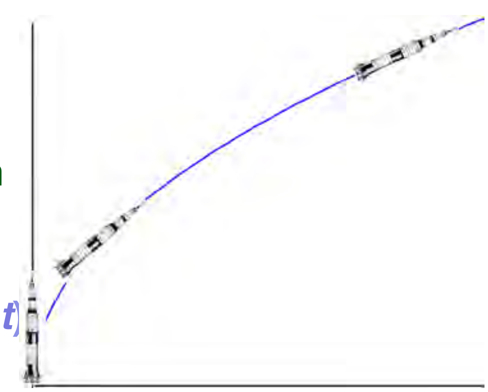
**Feedback controller minimizes *α* or load factor**






# Gravity-Turn Flight Path

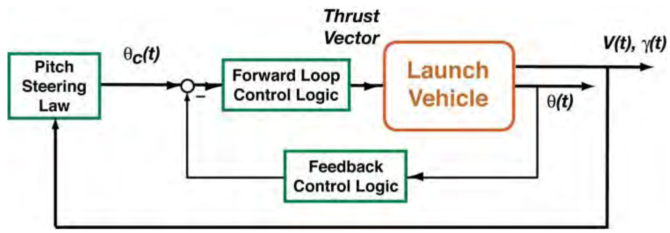
- **Gravity-turn flight path is function of 3 variables**
  - Initial pitchover angle (from vertical launch)
  - Velocity at pitchover
  - Acceleration profile, *T*(*t*)/*m*(*t*)



**Gravity-turn program closely approximated by tangent steering laws** *(see Supplemental Material)*

# Feedback Control Law

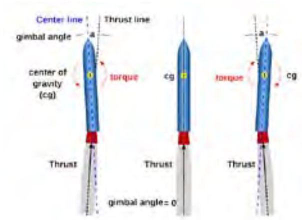
## Errors due to disturbances and modeling errors corrected by feedback control




$$\text{Motor Gimbal Angle}(t) \triangleq \delta_G(t) = c_\theta\left[\theta_{des}(t) - \theta(t)\right] - c_q q(t)$$

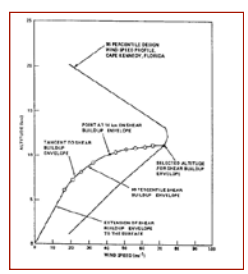
$$\theta_{des} = \text{ Desired pitch angle; } q = \frac{d\theta}{dt} = \text{pitch rate}$$

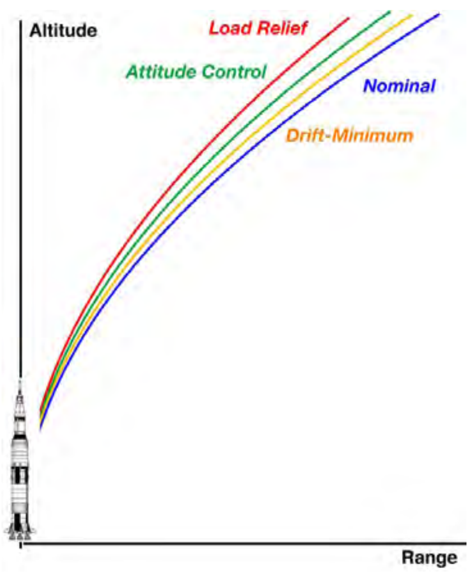
$$c_\theta, c_q : \text{Feedback control law gains}$$





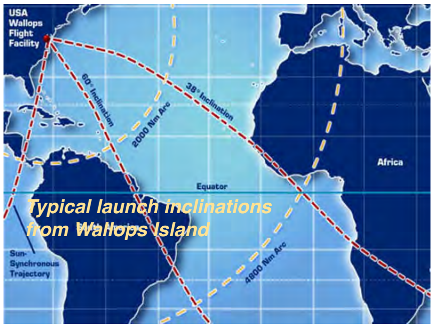

# Thrust Vector Control During Launch Through Wind Profile

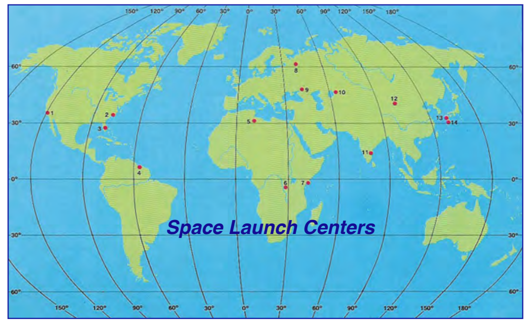





- **Attitude control**
  - **Attitude and rate feedback**
- **Drift-minimum control**
  - **Attitude and accelerometer feedback**
  - **Increased loads**
- **Load relief control**
  - **Rate and accelerometer feedback**
  - **Increased drift**

# Effect of Launch Latitude on Orbital Parameters



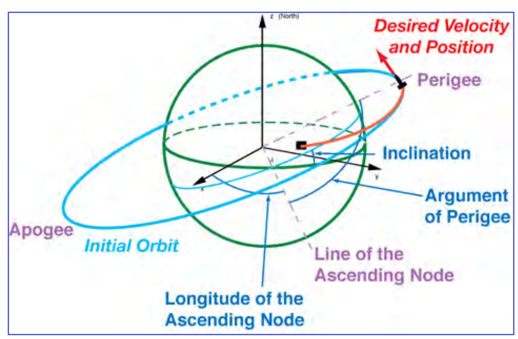

- Launch latitude establishes <u>minimum</u> orbital inclination (without "dogleg" maneuver)
- Time of launch establishes line of nodes
- Argument of perigee established by
  - Launch trajectory
  - On-orbit adjustment



# Guidance Law for Launch to Orbit

(Brand, Brown, Higgins, and Pu, CSDL, 1972)

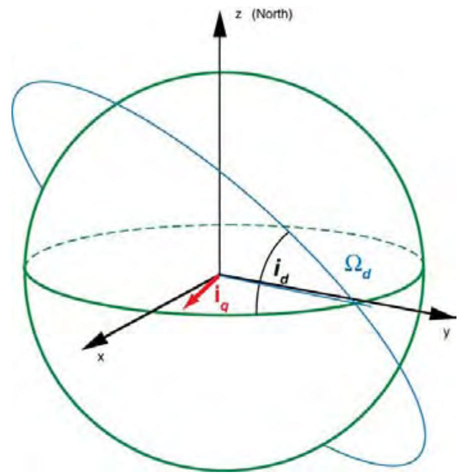
- Initial conditions
  - End of pitch program, outside atmosphere
- Final condition
  - Insertion in desired orbit
- Initial inputs
  - Desired radius
  - Desired velocity magnitude
  - Desired flight path angle
  - Desired inclination angle
  - Desired longitude of the ascending/descending node
- Continuing outputs
  - Unit vector describing desired thrust direction
  - Throttle setting, % of maximum thrust



# Guidance Program Initialization

- Thrust acceleration estimate
- Mass/mass flow rate
- Acceleration limit (~ 3g)
- Effective exhaust velocity
- Various coefficients
- Unit vector normal to desired orbital plane, $\mathbf{i}_q$

$$\mathbf{i}_q = \begin{bmatrix} \sin i_d \sin \Omega_d \\ \sin i_d \cos \Omega_d \\ \cos i_d \end{bmatrix}$$




$i_d$ : Desired inclination angle of final orbit

$\Omega_d$ : Desired longitude of descending node



# Guidance Program Operation: Position and Velocity

- Obtain thrust acceleration estimate, $a_T$, from guidance system
- Compute corresponding mass, mass flow rate, and throttle setting, $\delta T$

$\mathbf{i}_r = \dfrac{\mathbf{r}}{r}$ : Unit vector aligned with local vertical

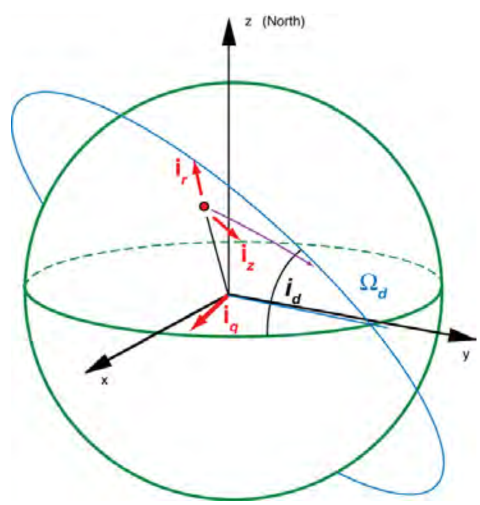
$\mathbf{i}_z = \mathbf{i}_r \times \mathbf{i}_q$ : Downrange direction

**Position**

$$\begin{bmatrix} r \\ y \\ z \end{bmatrix} = \begin{bmatrix} |\mathbf{r}| \\ r \sin^{-1}\left(\mathbf{i}_r \bullet \mathbf{i}_q\right) \\ open \end{bmatrix}$$

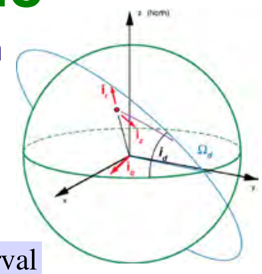
**Velocity**

$$\begin{bmatrix} \dot{r} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{IMU} \bullet \mathbf{i}_r \\ \mathbf{v}_{IMU} \bullet \mathbf{i}_q \\ \mathbf{v}_{IMU} \bullet \mathbf{i}_z \end{bmatrix}$$




$\mathbf{v}_{IMU}$ : Velocity estimate in IMU frame

# Guidance Program: Velocity and Time to Go



### Effective gravitational acceleration

$$g_{eff} = -\frac{\mu}{r^2} + \frac{|\mathbf{r} \times \mathbf{v}|^2}{r^3}$$

### Time to go (to motor burnout)

$$t_{go_{new}} = t_{go_{old}} - \Delta t$$

$\Delta t$ : Guidance command interval

### Velocity to be gained

$$\mathbf{v}_{go} = \begin{bmatrix} (\dot{r}_d - \dot{r}) - g_{eff} t_{go}/2 \\ -\dot{y} \\ \dot{z}_d - \dot{z} \end{bmatrix}$$

### Time to go prediction (prior to acceleration limiting)

$$t_{go} = \frac{m}{\dot{m}}\left(1 - e^{-v_{go}/c_{eff}}\right)$$

$c_{eff}$ : Effective exhaust velocity



# Guidance Program Commands

**Guidance law:** required radial and cross-range accelerations

$$a_{T_r} = a_T\left[A + B(t - t_o)\right] - g_{eff}$$
$$a_{T_y} = a_T\left[C + D(t - t_o)\right]$$

$a_T$ = Net available acceleration

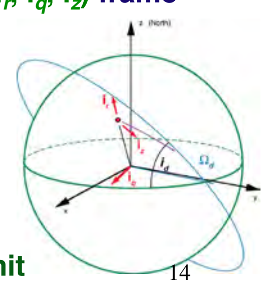
Guidance coefficients, ***A***, ***B***, ***C***, and ***D*** are functions of

$$(r_d, r, \dot{r}, t_{go})$$
$$(y, \dot{y}, t_{go})$$

plus $c_{eff}$, $m/\dot{m}$, Acceleration limit

Required thrust direction, $\mathbf{i}_T$ (i.e., vehicle orientation in ($\mathbf{i}_r$, $\mathbf{i}_q$, $\mathbf{i}_z$) frame

$$\mathbf{a}_T = \begin{bmatrix} a_{T_r} \\ a_{T_y} \\ \text{what's left over} \end{bmatrix}; \quad i_T = \frac{\mathbf{a}_T}{|\mathbf{a}_T|}$$

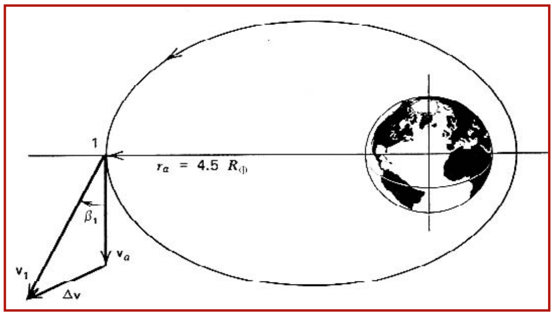


Throttle command is a function of $a_T$ (i.e., acceleration magnitude) and acceleration limit

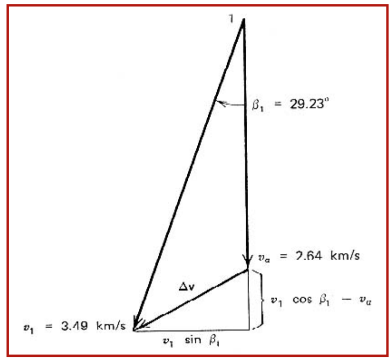
# Impulsive $\Delta V$ Orbital Maneuver

- If rocket burn time is short compared to orbital period (e.g., seconds compared to hours), impulsive $\Delta V$ approximation can be made
  - Change in position during burn is ~ zero
  - Change in velocity is ~ instantaneous

Velocity impulse at apogee



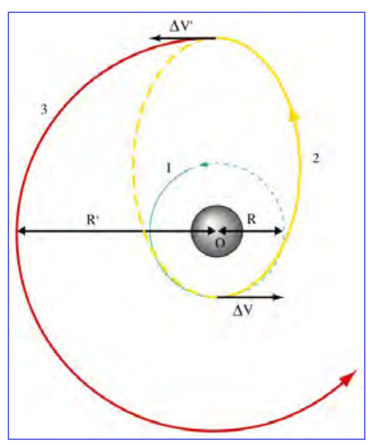

Vector diagram of velocity change



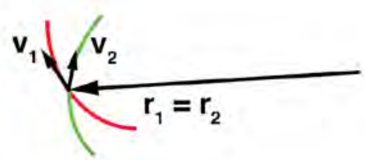



# Orbit Change due to Impulsive $\Delta V$

- Maximum energy change accrues when $\Delta V$ is aligned with the instantaneous orbital velocity vector
  - Energy change -> Semi-major axis change
  - Maneuver at perigee raises or lowers apogee
  - Maneuver at apogee raises or lowers perigee
- Optimal transfer from one circular orbit to another involves two impulses *[Hohmann transfer]*
- Other maneuvers
  - In-plane parameter change
  - Orbital plane change



# Assumptions for Impulsive Maneuver




**Instantaneous change in velocity vector**

$$\boxed{\mathbf{v}_2 = \mathbf{v}_1 + \Delta\mathbf{v}_{rocket}}$$

**Negligible change in radius vector**

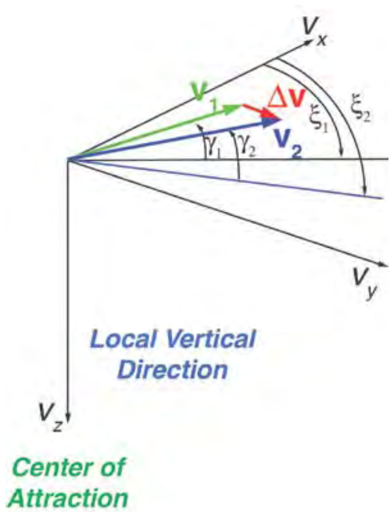
$$\boxed{\mathbf{r}_2 = \mathbf{r}_1}$$

**Therefore, new orbit intersects old orbit**
**Velocities different at the intersection**



# Geometry of Impulsive Maneuver

**Change in velocity magnitude, |v|, vertical flight path angle, $\gamma$, and horizontal flight path angle, $\xi$**

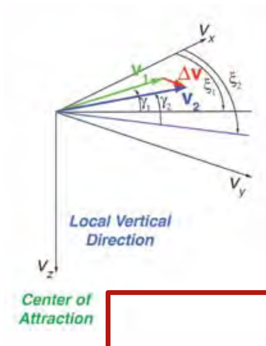



$$\mathbf{v}_1 = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_1 = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_1 = \begin{bmatrix} V\cos\gamma\cos\xi \\ V\cos\gamma\sin\xi \\ -V\sin\gamma \end{bmatrix}_1$$

$$\mathbf{v}_2 = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_2 = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_2 = \begin{bmatrix} V\cos\gamma\cos\xi \\ V\cos\gamma\sin\xi \\ -V\sin\gamma \end{bmatrix}_2$$

$$\Delta\mathbf{v} = \begin{bmatrix} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{bmatrix} = \begin{bmatrix} \left(v_{x_2} - v_{x_1}\right) \\ \left(v_{y_2} - v_{y_1}\right) \\ \left(v_{z_2} - v_{z_1}\right) \end{bmatrix}$$

# Required $\Delta$v for Impulsive Maneuver




$$\Delta\mathbf{v} = \begin{bmatrix} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{bmatrix} = \begin{bmatrix} \left(v_{x_2} - v_{x_1}\right) \\ \left(v_{y_2} - v_{y_1}\right) \\ \left(v_{z_2} - v_{z_1}\right) \end{bmatrix}$$

$$\Delta\mathbf{v}_{rocket} = \begin{bmatrix} \Delta V_{rocket} \\ \xi_{rocket} \\ \gamma_{rocket} \end{bmatrix} = \begin{bmatrix} \left(\Delta v_x^2 + \Delta v_y^2 + \Delta v_z^2\right)^{1/2}_{rocket} \\ \sin^{-1}\left(\dfrac{\Delta v_y}{\left(\Delta v_x^2 + \Delta v_y^2\right)^{1/2}}\right)_{rocket} \\ \sin^{-1}\left(\dfrac{\Delta v_z}{\Delta V}\right)_{rocket} \end{bmatrix}$$



# Single Impulse Orbit Adjustment

## Coplanar (i.e., in-plane) maneuvers

- Change energy
- Change angular momentum
- Change eccentricity

$$\mathbb{E} = \frac{1}{2}v^2 - \mu/r = \left(e^2 - 1\right)\mu^2/h^2$$

$$h = \sqrt{\frac{\mu^2\left(e^2 - 1\right)}{\mathbb{E}}} = \sqrt{\frac{\mu^2\left(e^2 - 1\right)}{v^2/2 - \mu/r}}$$

$$e = \sqrt{1 + 2\,\mathbb{E}h^2/\mu^2}$$

- Required velocity increment

$$v_{new} \triangleq v_{old} + \Delta v_{rocket} = \sqrt{2\left(\mathbb{E}_{new} + \mu/r\right)}$$

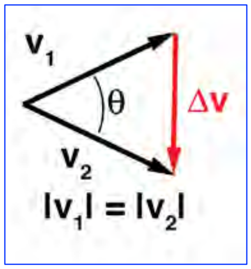
$$= \sqrt{2\left[\left(e_{new}^2 - 1\right)\mu^2/h_{new}^2 + \mu/r\right]}$$

$$\Delta v_{rocket} = v_{new} - v_{old}$$

# Single Impulse Orbit Adjustment
## Coplanar (i.e., in-plane) maneuvers

- Change semi-major axis
  - magnitude
  - orientation (i.e., argument of perigee); in-plane isosceles triangle

$$a_{new} = \frac{h_{new}^2/\mu}{1-e_{new}^2}$$




- Change apogee or perigee
  - radius
  - velocity

$$r_{perigee} = a(1-e)$$

$$r_{apogee} = a(1+e)$$

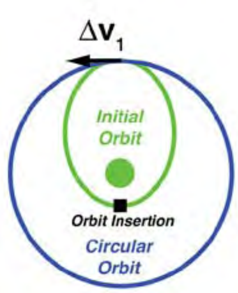
$$v_{perigee} = \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)}$$

$$v_{apogee} = \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)}$$



# In-Plane Orbit Circularization

**Initial orbit is elliptical, with apogee radius equal to desired circular orbit radius**




Initial Orbit

$$a = \left(r_{cir(target)} + r_{insertion}\right)/2$$

$$e = \left(r_{cir(target)} - r_{insertion}\right)/2a$$

$$v_{apogee} = \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)}$$

**Velocity in circular orbit is a function of the radius**

"Vis viva" equation:

$$v_{cir} = \sqrt{\mu\left(\frac{2}{r_{cir}} - \frac{1}{a_{cir}}\right)} = \sqrt{\mu\left(\frac{2}{r_{cir}} - \frac{1}{r_{cir}}\right)} = \sqrt{\frac{\mu}{r_{cir}}}$$

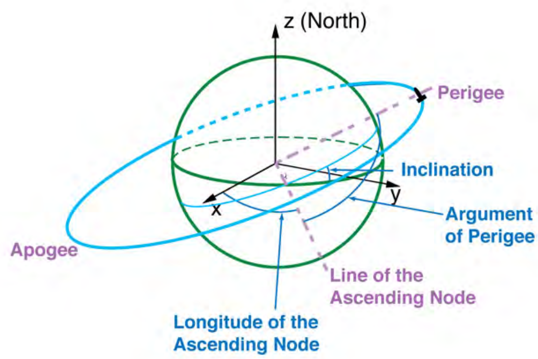
Rocket must provide the difference

$$\Delta v_{rocket} = v_{cir} - v_{apogee}$$

# Single Impulse Orbit Adjustment

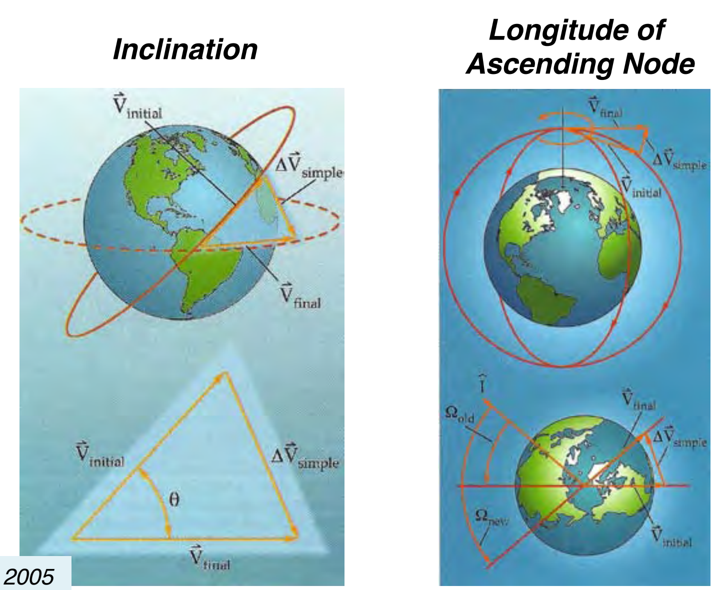
## Out-of-plane maneuvers

- Change orbital inclination
- Change longitude of the ascending node
- $v_1$, $\Delta v$, and $v_2$ form isosceles triangle perpendicular to the orbital plane to leave in-plane parameters unchanged





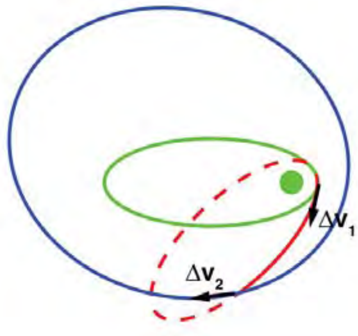

# Change in Inclination and Longitude of Ascending Node




Sellers, 2005

# Two Impulse Maneuvers

**Transfer to Non-Intersecting Orbit**

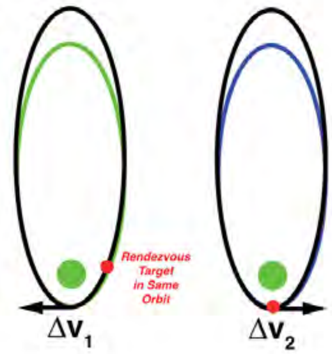



**1st Δv produces target orbit intersection**

**2nd Δv matches target orbit**

**Minimize ($|\Delta v_1| + |\Delta v_2|$) to minimize propellant use**

**Phasing Orbit**




**Rendezvous with trailing spacecraft in same orbit**

**At perigee, increase speed to increase orbital period**

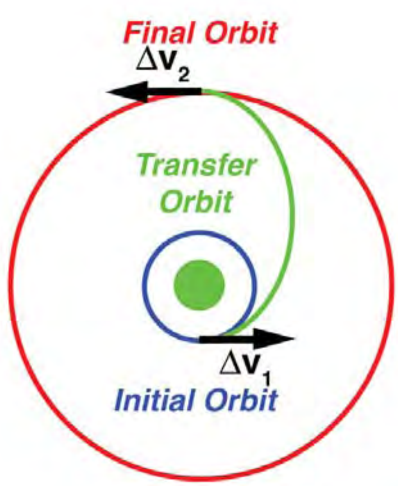
**At future perigee, decrease speed to resume original orbit**



# Hohmann Transfer between Coplanar Circular Orbits

## *(Outward transfer example)*

**Thrust in direction of motion at transfer perigee and apogee**




$$v_{cir_1} = \sqrt{\frac{\mu}{r_{cir_1}}}$$

$$v_{cir_2} = \sqrt{\frac{\mu}{r_{cir_2}}}$$

**Transfer Orbit**

$$a = \left(r_{cir_1} + r_{cir_2}\right)/2$$

$$e = \left(r_{cir_2} - r_{cir_1}\right)/2a$$

$$v_{p_{transfer}} = \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)}$$

$$v_{a_{transfer}} = \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)}$$

# Outward Transfer Orbit Velocity Requirements

**Δ*v* at 1st Burn**

$$\Delta v_1 = v_{p_{transfer}} - v_{cir_1} = v_{cir_1}\left(\sqrt{\frac{2r_{cir_2}}{r_{cir_1}+r_{cir_2}}} - 1\right)$$

**Δ*v* at 2nd Burn**

$$\Delta v_2 = v_{cir_1} - v_{a_{transfer}} = v_{cir_2}\left(1 - \sqrt{\frac{2r_{cir_1}}{r_{cir_1}+r_{cir_2}}}\right)$$

$$v_{cir_2} = v_{cir_1}\sqrt{\frac{r_{cir_1}}{r_{cir_2}}}$$

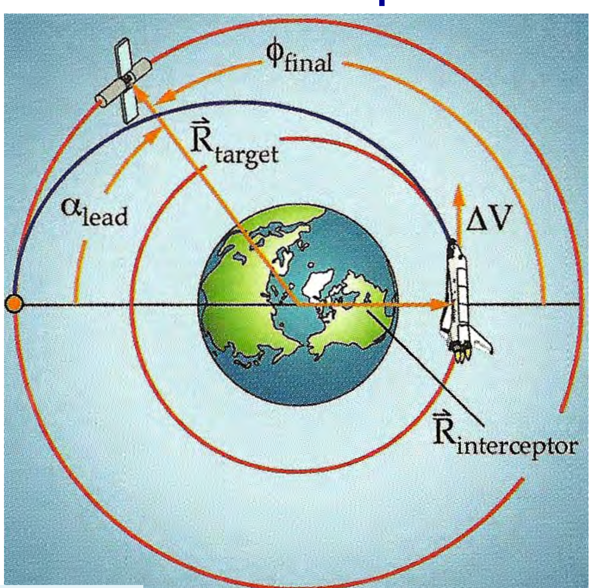
**Hohmann Transfer is energy-optimal for 2-impulse transfer between circular orbits and $r_2/r_1 < 11.94$**

$$\Delta v_{total} = v_{cir_1}\left[\sqrt{\frac{2r_{cir_2}}{r_{cir_1}+r_{cir_2}}}\left(1 - \frac{r_{cir_1}}{r_{cir_2}}\right) + \sqrt{\frac{r_{cir_1}}{r_{cir_2}}} - 1\right]$$



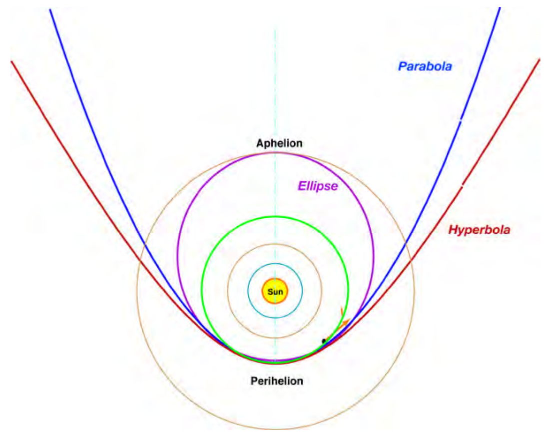
# Rendezvous Requires Phasing of the Maneuver

## Transfer orbit time equals target's time to reach rendezvous point




Sellers, 2005

# Solar Orbits



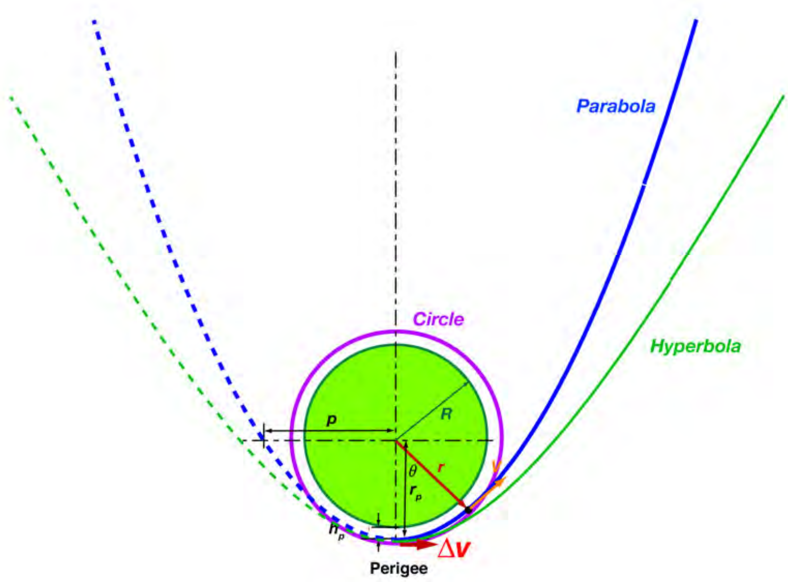

- **Same equations used for Earth-referenced orbits**
  - **Dimensions of the orbit**
  - **Position and velocity of the spacecraft**
  - **Period of elliptical orbits**
  - **Different gravitational constant**

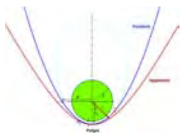
$$\boxed{\mu_{Sun} = 1.3327 \times 10^{11} \mathrm{km}^3/\mathrm{s}^2}$$



# Escape from a Circular Orbit

## Minimum escape trajectory shape is a parabola





# In-plane Parameters of Earth Escape Trajectories

## Dimensions of the orbit

$$p = h^2 / \mu = \text{"The parameter" or semi-latus rectum}$$

$$h = \text{Angular momentum about center of mass}$$

$$e = \sqrt{1 + 2\mathbb{E}p / \mu} = \text{Eccentricity} \geq 1$$

$$\mathbb{E} = \text{Specific energy}, \ \geq 0$$

$$a = \frac{p}{1-e^2} = \text{Semi-major axis}, < 0$$

$$r_{perigee} = a(1-e) = \text{Perigee radius}$$



# In-plane Parameters of Earth Escape Trajectories

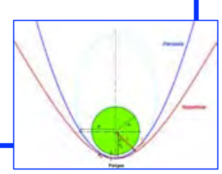
## Position and velocity of the spacecraft

$$r = \frac{p}{1 + e\cos\theta} = \text{Radius of the spacecraft}$$

$$\theta = \text{True anomaly}$$

$$V = \sqrt{2\mu\left(\frac{1}{r} - \frac{1}{2a}\right)} = \text{Velocity of the spacecraft}$$

$$V_{perigee} \geq \sqrt{2\mu / r_{perigee}}$$



# Escape from Circular Orbit

**Velocity in circular orbit**

$$V_c = \sqrt{\mu\left(\frac{2}{r_c} - \frac{1}{r_c}\right)} = \sqrt{\frac{\mu}{r_c}}$$

**Velocity at perigee of parabolic orbit**

$$V_{perigee} = \sqrt{\mu\left(\frac{2}{r_c} - \frac{1}{(a \to \infty)}\right)} = \sqrt{\frac{2\mu}{r_c}}$$

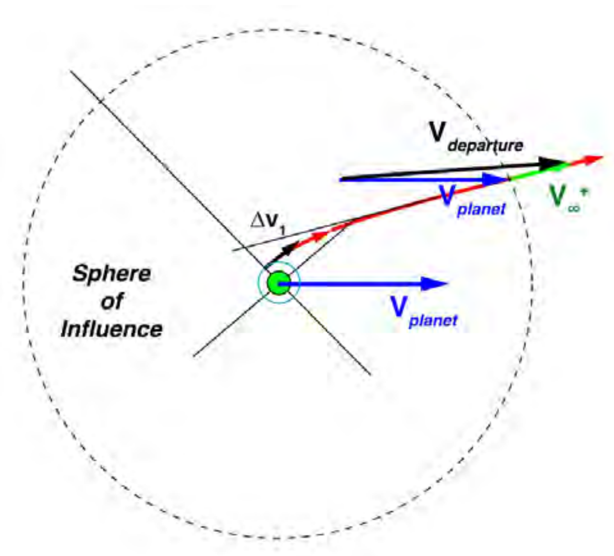
**Velocity increment required for escape**

$$\Delta V_{escape} = V_{perigee_{parabola}} - V_c == \sqrt{\frac{2\mu}{r_c}} - \sqrt{\frac{\mu}{r_c}} \approx 0.414 V_c$$



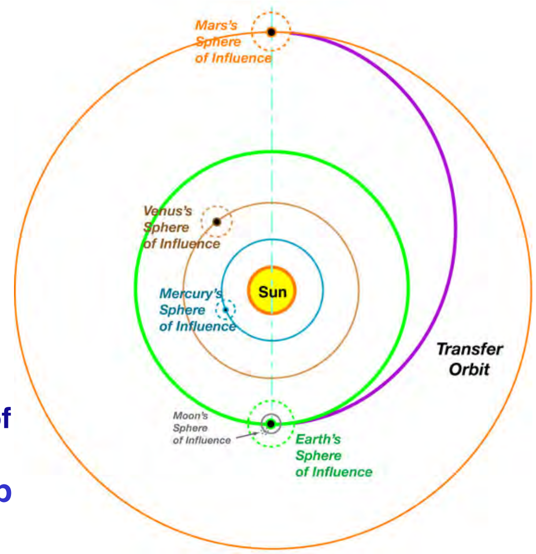
# Earth Escape Trajectory

**$\Delta v_1$ to increase speed to escape velocity**
**Velocity required for transfer at sphere of influence**



# Transfer Orbits and Spheres of Influence

- Sphere of Influence (Laplace):
  - Radius within which gravitational effects of planet are more significant than those of the Sun
- Patched-conic section approximation
  - Sequence of 2-body orbits
  - Outside of planet's sphere of influence, Sun is the center of attraction
  - Within planet's sphere of influence, planet is the center of attraction
- Fly-by (swingby) trajectories dip into intermediate object's sphere of influence for gravity assist



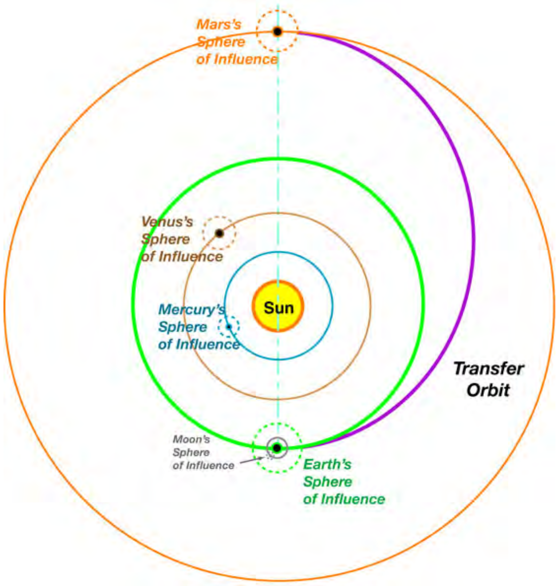



# Solar System Spheres of Influence

$$\text{for } \frac{m_{Planet}}{m_{Sun}} << 1, \quad r_{SI} \simeq r_{Planet-Sun}\left(\frac{m_{Planet}}{m_{Sun}}\right)^{2/5}$$

| Planet | Sphere of Influence, km |
|---|---|
| Mercury | 112,000 |
| Venus | 616,000 |
| Earth | 929,000 |
| Mars | 578,000 |
| Jupiter | 48,200,000 |
| Saturn | 54,500,000 |
| Uranus | 51,800,000 |
| Neptune | 86,800,000 |
| Pluto | 27,000,000-45,000,000 |

# Interplanetary Mission Planning




- **Example: Direct Hohmann Transfer from Earth Orbit to Mars Orbit (No fly-bys)**

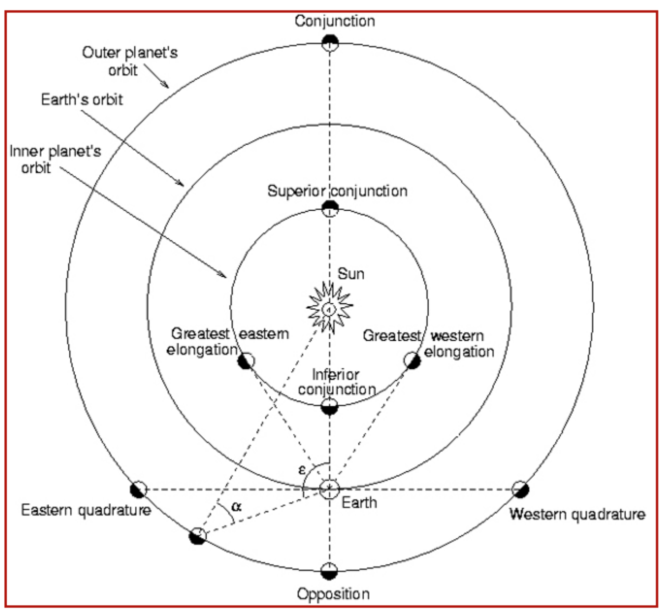
1) **Calculate required perigee velocity for transfer orbit - Sun as center of attraction: Elliptical orbit**
2) **Calculate $\Delta v$ required to reach Earth's sphere of influence with velocity required for transfer – Earth as center of attraction: Hyperbolic orbit**
3) **Calculate $\Delta v$ required to enter circular orbit about Mars, given transfer apogee velocity – Mars as center of attraction: Hyperbolic orbit**



# Launch Opportunities for Fixed Transit Time: The Synodic Period

- **Synodic Period, $S_n$: The time between conjunctions**
  - $P_A$: Period of Planet $A$
  - $P_B$: Period of Planet B
- **Conjunction: Two planets, $A$ and $B$, in a line or at some fixed angle**

$$\boxed{S_n = \frac{P_A P_B}{P_A - P_B}}$$

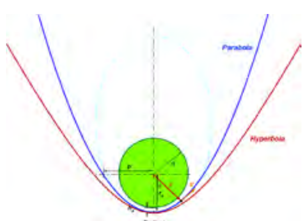


# Launch Opportunities for Fixed Transit Time: The Synodic Period

| Planet | Synodic Period with respect to Earth, days | Period |
|---|---|---|
| Mercury | 116 | 88 days |
| Venus | 584 | 225 days |
| Earth | - | 365 days |
| Mars | 780 | 687 days |
| Jupiter | 399 | 11.9 yr |
| Saturn | 378 | 29.5 yr |
| Uranus | 370 | 84 yr |
| Neptune | 367 | 165 yr |
| Pluto | 367 | 248 yr |



# Hyperbolic Orbits



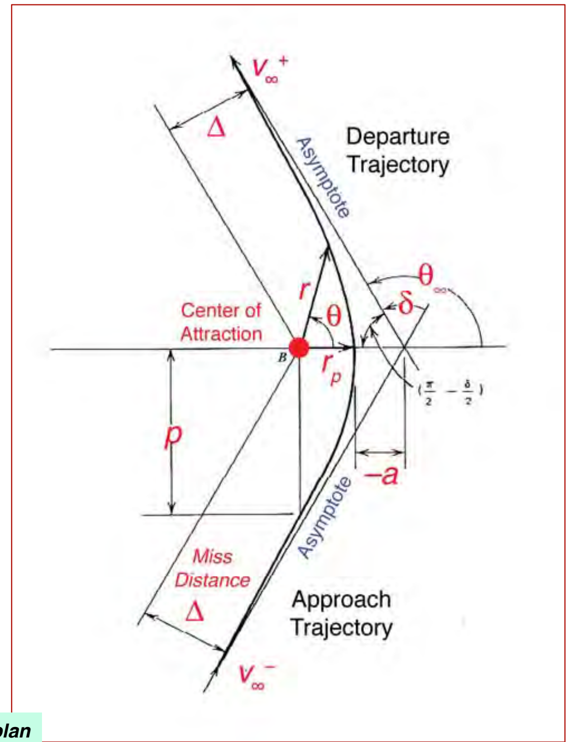
| Orbit Shape | Eccentricity, e | Energy, E |
|---|---|---|
| Circle | 0 | <0 |
| Ellipse | 0 < e <1 | <0 |
| Parabola | 1 | 0 |
| Hyperbola | >1 | >0 |

$$\mathbb{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}, \quad \therefore a < 0$$

**Velocity remains positive as radius approaches ∞**

$$v \xrightarrow[r \to \infty]{} v_\infty$$

$$\therefore \mathbb{E}_\infty = \frac{v_\infty^2}{2}, \quad \text{and } v_\infty = \sqrt{-\frac{\mu}{a}} \quad \text{or} \quad a = -\frac{\mu}{v_\infty^2}$$




Kaplan

# Hyperbolic Encounter with a Planet

- Trajectory is deflected by target planet's gravitational field
  - In-plane
  - Out-of-plane
- Velocity w.r.t. Sun is increased or decreased

$\Delta$ : Miss Distance, km

$\delta$ : Deflection Angle, deg or rad



# Hyperbolic Orbits

***Asymptotic Value of True Anomaly***

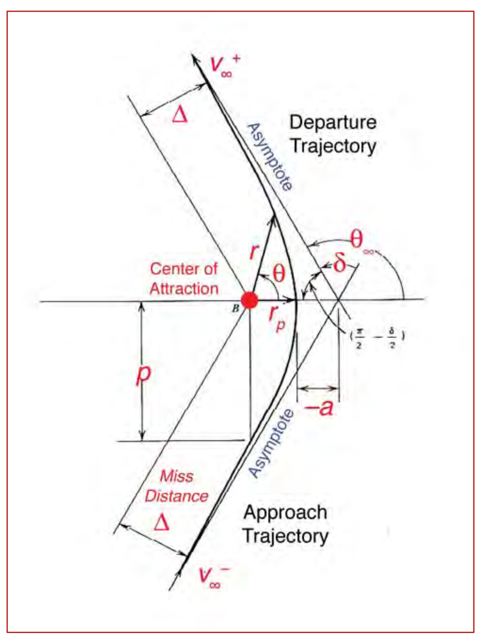
**Polar Equation for a Conic Section**

$$r = \frac{p}{1 + e\cos\theta} = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta}$$

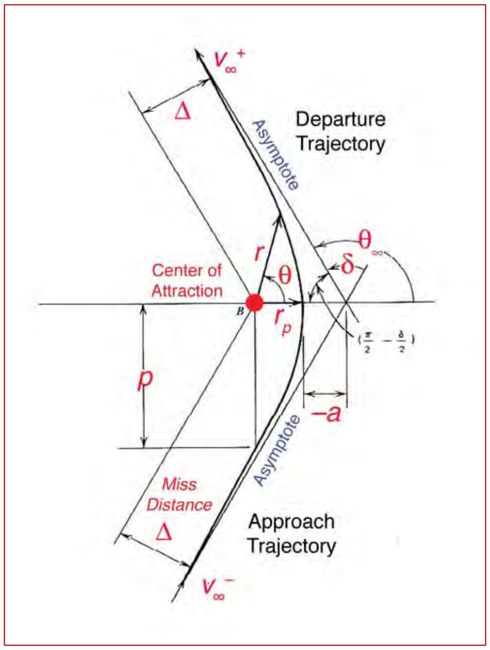
$$\cos\theta = \frac{1}{e}\left[\frac{a\left(1 - e^2\right)}{r} - 1\right]$$

$$\theta \xrightarrow[r \leftrightarrow \infty]{} \theta_\infty$$

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right)$$



# Hyperbolic Orbits




***Angular Momentum***

$$h = \text{Constant} = v_\infty \Delta$$
$$= \sqrt{\mu p} = \sqrt{\mu a\left(1-e^2\right)} = \sqrt{\frac{\mu^2\left(e^2-1\right)}{v_\infty^2}}$$

***Eccentricity***

$$e = \sqrt{1+\frac{2h^2\mathbb{E}}{\mu^2}} = \sqrt{1+\frac{v_\infty^4\Delta^2}{\mu^2}}$$

***Perigee Radius***

$$r_p = a(1-e) = \frac{\mu}{v_\infty^2}(e-1)$$

***Eccentricity***

$$e = \left[1+\frac{r_p v_\infty^2}{\mu}\right]$$



# Hyperbolic Mean and Eccentric Anomalies

$H$ :   Hyperbolic Eccentric Anomaly

$$M = e\sinh H - H$$

***Newton's method of successive approximation***
to find *H* from *M*, similar to solution for *E* (Lecture 2)

$$\theta(t) = 2\tan^{-1}\left[\sqrt{\frac{e+1}{e-1}}\tanh\frac{H(t)}{2}\right]$$

$$r = a\left(1 - e\cosh H\right)$$

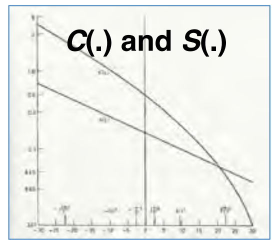
**see Ch. 7, *Kaplan, 1976***

# Battin's Universal Formulas for Conic Section Position and Velocity as Functions of Time

$$\mathbf{r}(t_2)=\left[1-\frac{\chi^2}{r(t_1)}C\left(\frac{\chi^2}{a}\right)\right]\mathbf{r}(t_1)+\left[t_2-\frac{\chi^3}{\sqrt{\mu}}S\left(\frac{\chi^2}{a}\right)\right]\mathbf{v}(t_1)$$

$$\mathbf{v}(t_2)=\frac{\sqrt{\mu}}{r(t_1)r(t_2)}\left[\frac{\chi^3}{a}S\left(\frac{\chi^2}{a}\right)-\chi\right]\mathbf{r}(t_1)+\left[1-\frac{\chi^2}{r(t_2)}C\left(\frac{\chi^2}{a}\right)\right]\mathbf{v}(t_1)$$

$$\chi=\begin{cases}\sqrt{a}\left[E(t_2)-E(t_1)\right], & \text{Ellipse}\\ \sqrt{-a}\left[H(t_2)-H(t_1)\right], & \text{Hyperbola}\\ \sqrt{p}\left[\tan\dfrac{\theta(t_2)}{2}-\tan\dfrac{\theta(t_1)}{2}\right], & \text{Parabola}\end{cases}$$




see Ch. 7, ***Kaplan, 1976; also Battin, 1964***

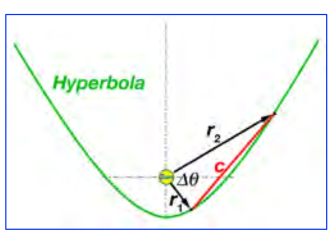


# Lambert's Time-of-Flight Theorem *(Hyperbolic Orbit)*

$$(t_2-t_1)=\sqrt{\frac{-a^3}{\mu}}\left[(\sinh\gamma-\gamma)+(\sinh\delta-\delta)\right]$$

***where***

$$\gamma\triangleq 2\sinh^{-1}\sqrt{\frac{r_1+r_2+c}{-4a}};\quad \delta\triangleq 2\sinh^{-1}\sqrt{\frac{r_1+r_2-c}{-4a}}$$

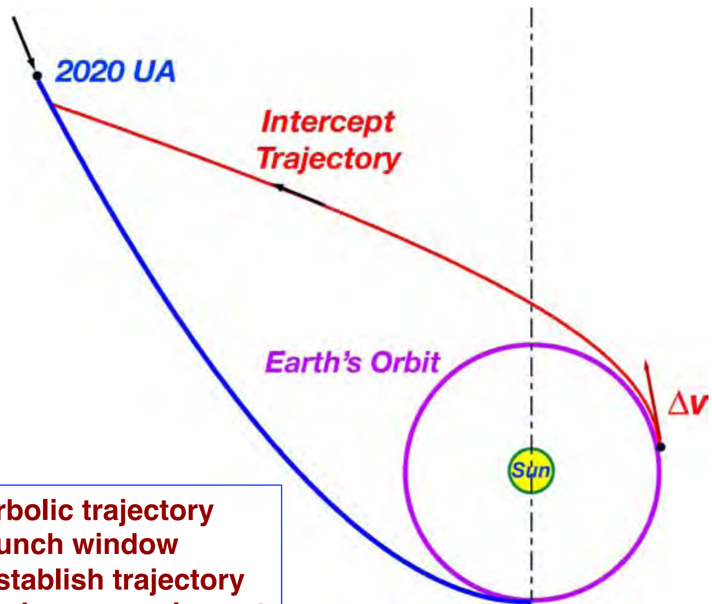



see Ch. 7, ***Kaplan, 1976; also Battin, 1964***

*http://www.mathworks.com/matlabcentral/fileexchange/39530-lambert-s-problem*

*http://www.mathworks.com/matlabcentral/fileexchange/26348-robust-solver-for-lambert-s-orbital-boundary-value-problem*

# Asteroid Encounter

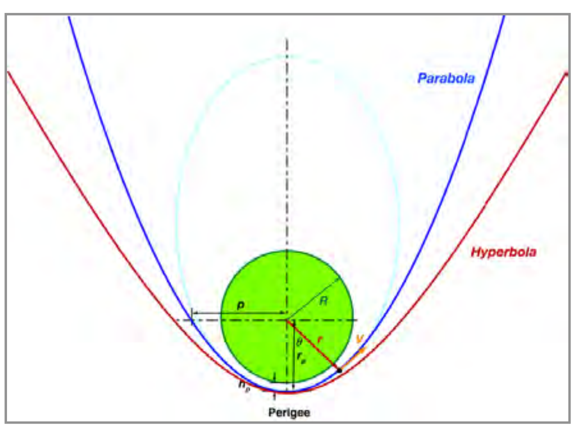



Hyperbolic trajectory
Launch window
$\Delta V$ to establish trajectory
$\Delta V$ for rendezvous or impact



# Swing-By/Fly-By Trajectories

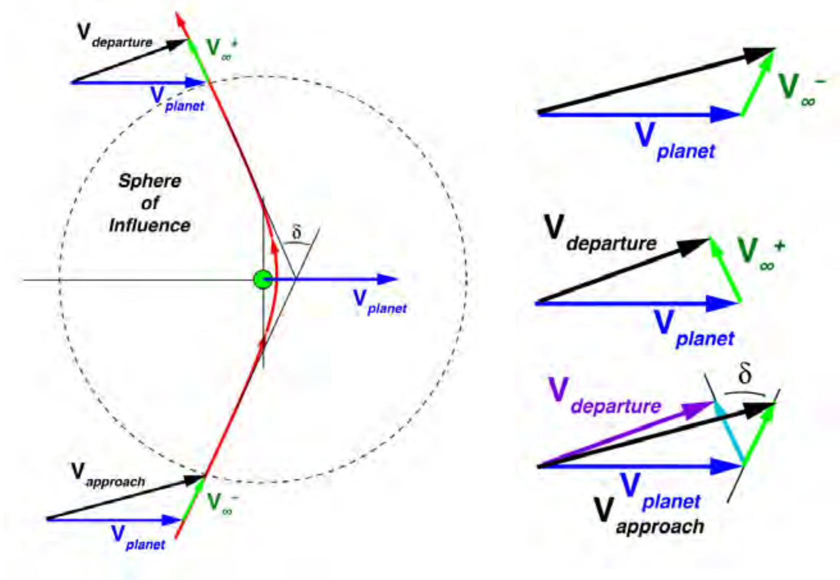
- Hyperbolic encounters with planets and the moon provide gravity assist
  - Shape, energy, and duration of transfer orbit altered
  - Potentially large reduction in rocket $\Delta V$ required to accomplish mission




Why does gravity assist work?

# Effect of Target Planet's Gravity on Probe's Sun-Relative Velocity

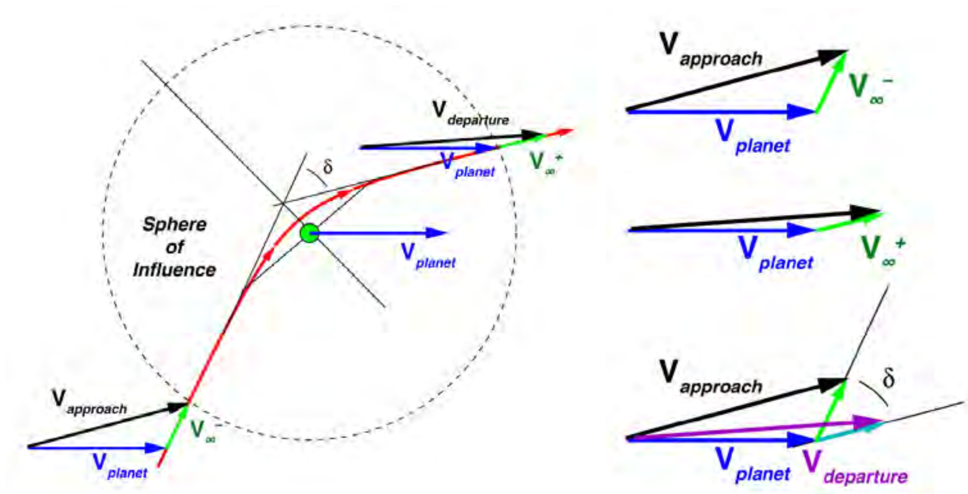
***Deflection – Velocity Reduction***





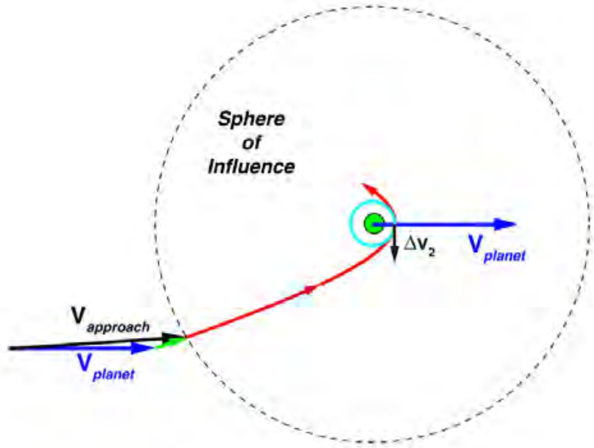

# Effect of Target Planet's Gravity on Probe's Sun-Relative Velocity

***Deflection – Velocity Addition***



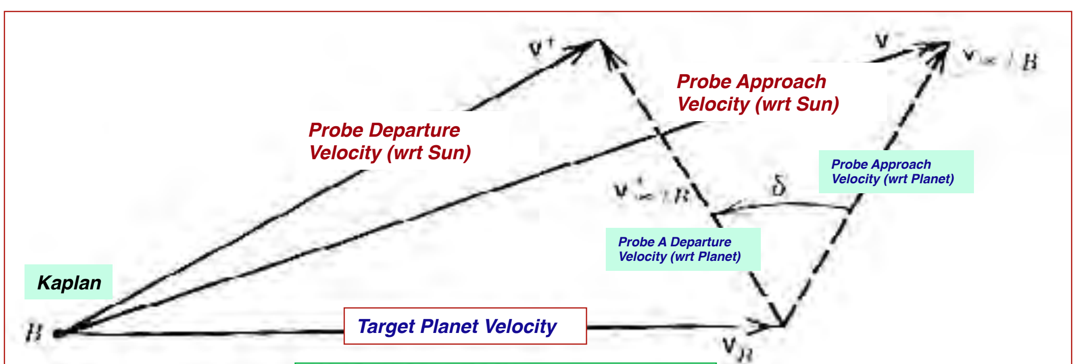
# Planet Capture Trajectory

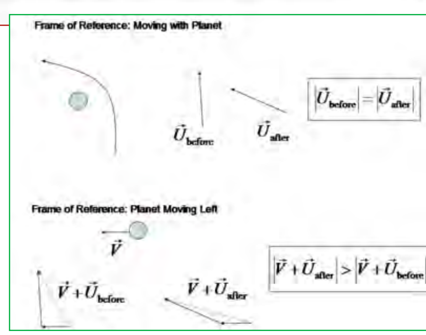
## Hyperbolic approach to planet's sphere of influence
## Δv to decrease speed to circular velocity






# Effect of Target Planet's Gravity on Probe's Velocity

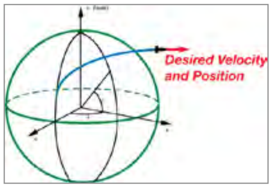


# Next Time:
# Spacecraft Environment

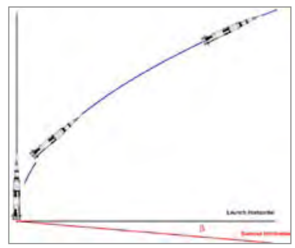


# Supplemental Material




# Phases of Ascent Guidance

- **Vertical liftoff**
- **Roll to launch azimuth**
- **Pitch program to atmospheric "exit"**
  - Jet stream penetration
  - Booster cutoff and staging
- **Explicit guidance to desired orbit**
  - Booster separation
  - Acceleration limiting
  - Orbital insertion

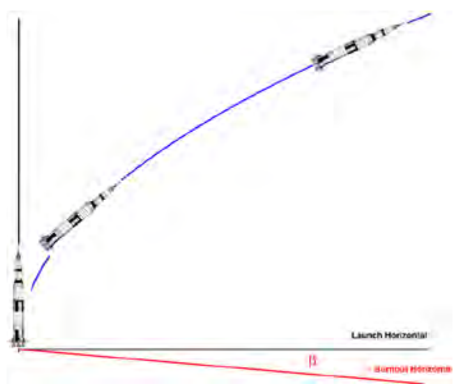


# Tangent Steering Laws

- **Neglecting surface curvature**

$$\tan\theta(t) = \tan\theta_o\left(1 - \frac{t}{t_{BO}}\right)$$

- **"Open-loop" command, i.e., no feedback of vehicle state**

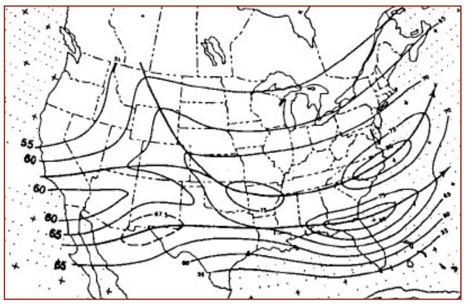


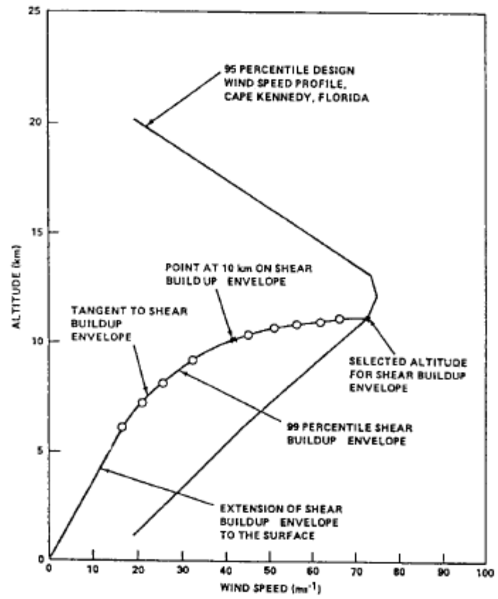

- **Accounting for effect of Earth surface curvature on burnout flight path angle**

$$\tan\theta(t) = \tan\theta_o\left[1 - \frac{t}{t_{BO}} - \tan\beta\left(\frac{t}{t_{BO}}\right)\right]$$

# Jet Stream Profiles

- Launch vehicle must able to fly through strong wind profiles
- Design profiles assume 95th-99th-percentile worst winds and wind shear








# Longitudinal (2-D) Equations of Motion for Re-Entry

**Differential equations for velocity ($x_1 = V$), flight path angle ($x_2 = \gamma$), altitude ($x_3 = h$), and range ($x_4$)**

**Angle of attack ($\alpha$) is optimization control variable**

$$\dot{x}_1 = -D(x_1, x_3, \alpha)/m - g\cos x_2$$

$$\dot{x}_2 = \left[g/x_1 - x_1/(R + x_3)\right]\sin x_2 - L(x_1, x_3, \alpha)/mx_1$$

$$\dot{x}_3 = x_1 \cos x_2$$

$$\dot{x}_4 = x_1 \sin x_2 / (1 + x_3/R)$$

*D* = drag
*L* = lift
*M* = pitch moment
*R* = Earth radius

**Equations of motion define the dynamic constraint**

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t), \alpha(t)\right]$$

# A Different Approach to Guidance: Optimizing a Cost Function

- **Minimize a scalar function, *J*, of terminal and integral costs**

$$J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t), \mathbf{u}(t)\right] dt$$

***L*[.]: Lagrangian**

**with respect to the control, u(*t*), in (*t*$_o$,*t*$_f$), subject to a dynamic constraint**

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)], \quad \mathbf{x}(t_o) \ given$$

dim(**x**) = *n* x 1
dim(**f**) = *n* x 1
dim(**u**) = *m* x 1



# Guidance Cost Function

$$\phi\left[\mathbf{x}(t_f)\right]$$
$$L\left[\mathbf{x}(t), \mathbf{u}(t)\right]$$

- **Terminal cost, e.g., in final position and velocity**
- **Integral cost, e.g., tradeoff between control usage and trajectory error**

- **Minimization of cost function determines the optimal state and control, x* and u*, over the flight path duration**

$$\min_u J = \min_u \left\{ \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t), \mathbf{u}(t)\right] dt \right\}$$
$$= \phi\left[\mathbf{x}^*(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}^*(t), \mathbf{u}^*(t)\right] dt \rightarrow \left[\mathbf{x}^*(t), \mathbf{u}^*(t)\right]$$

# Example of Re-Entry Flight Path Cost Function

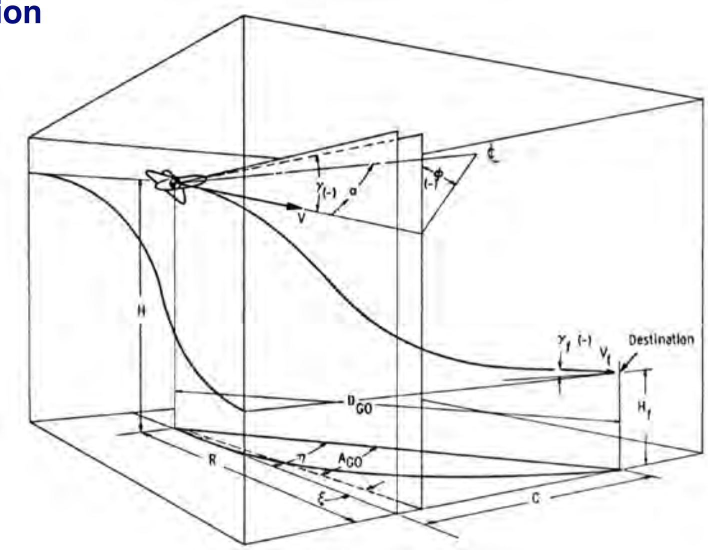
$$J = a\left[V(t_f) - V_d\right]^2 + b\left[r(t_f) - r_d\right]^2 + \int_{t_o}^{t_f} c\left[\alpha(t)\right]^2 dt$$

- **Cost function includes**
  - **Terminal range and velocity**
  - **Penalty on control use**
  - ***a*, *b*, and *c* tradeoff importance of each factor**
- **Minimization of this cost function**
  - **Defines the optimal path, x*($t$), from $t_o$ to $t_f$**
  - **Defines the optimal control, $\alpha$*($t$), from $t_o$ to $t_f$**



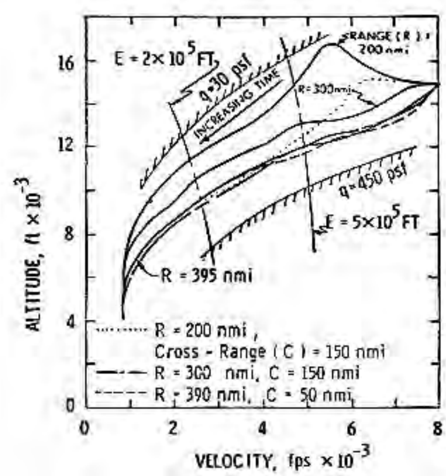
# Extension to Three Dimensions

- **Add roll angle as a control; add crossrange as a state**
- **For the guidance law, replace range and crossrange from the starting point by distance to go and azimuth to go to the destination**

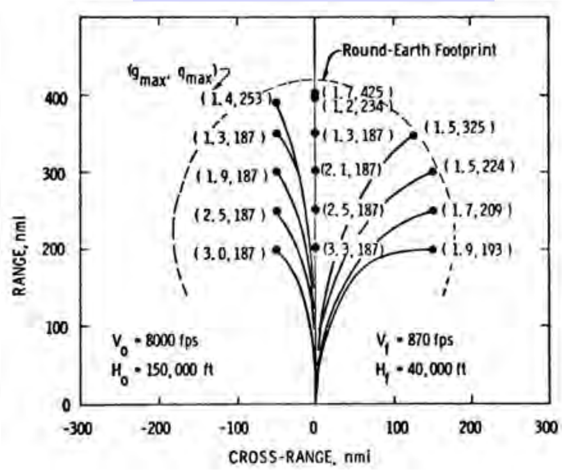


# Optimal Trajectories for Space Shuttle Transition

**Altitude vs. Velocity**



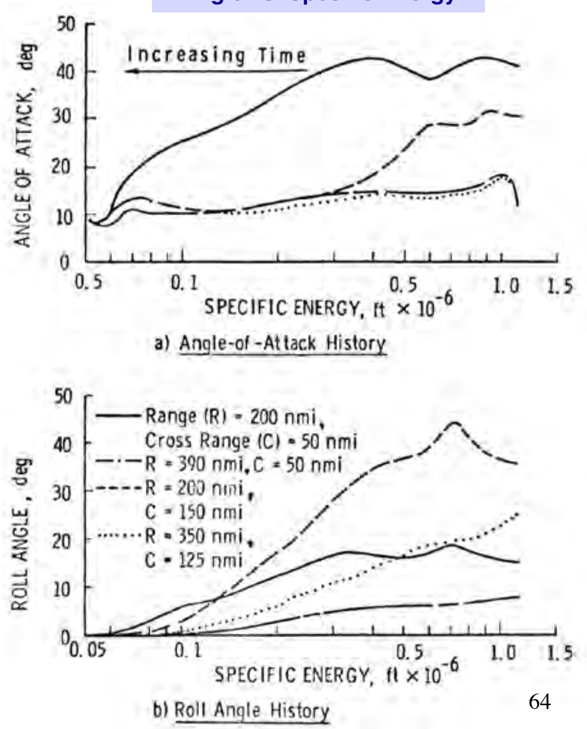

**Range vs. Cross-Range ("footprint")**






# Optimal Controls for Space Shuttle Transition

**Angle of Attack and Roll Angle vs. Specific Energy**



# Optimal Guidance System Derived from Optimal Trajectories

**Angle of Attack and Roll Angle vs. Specific Energy**

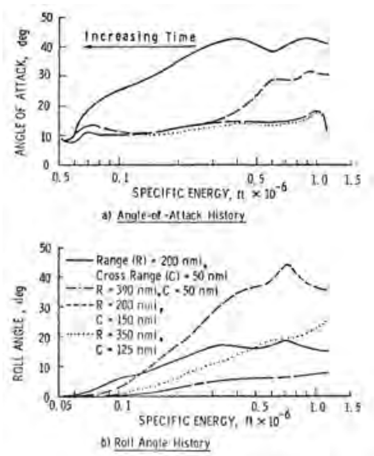
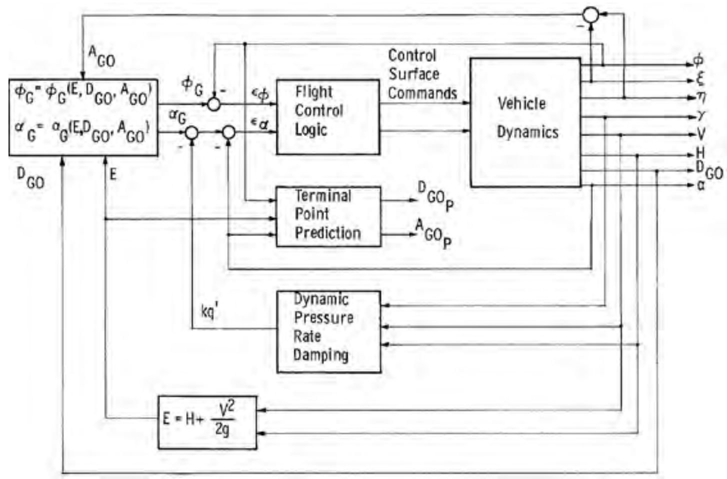



**Diagram of Energy-Guidance Law**

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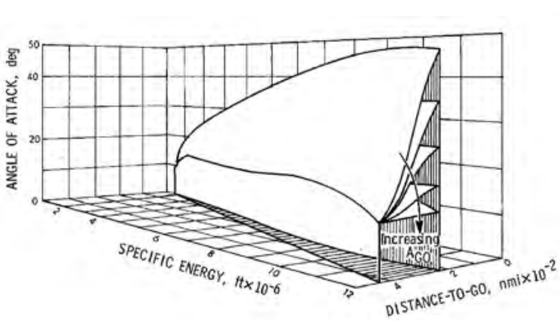
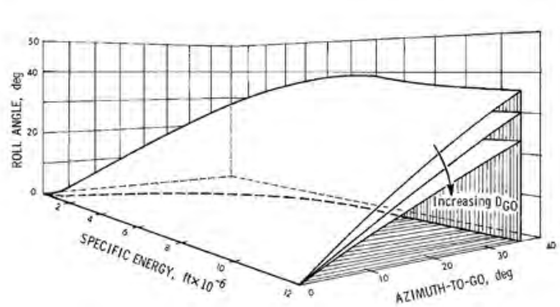
# Guidance Functions for Space Shuttle Transition

**Angle of Attack Guidance Function**




**Roll Angle Guidance Function**

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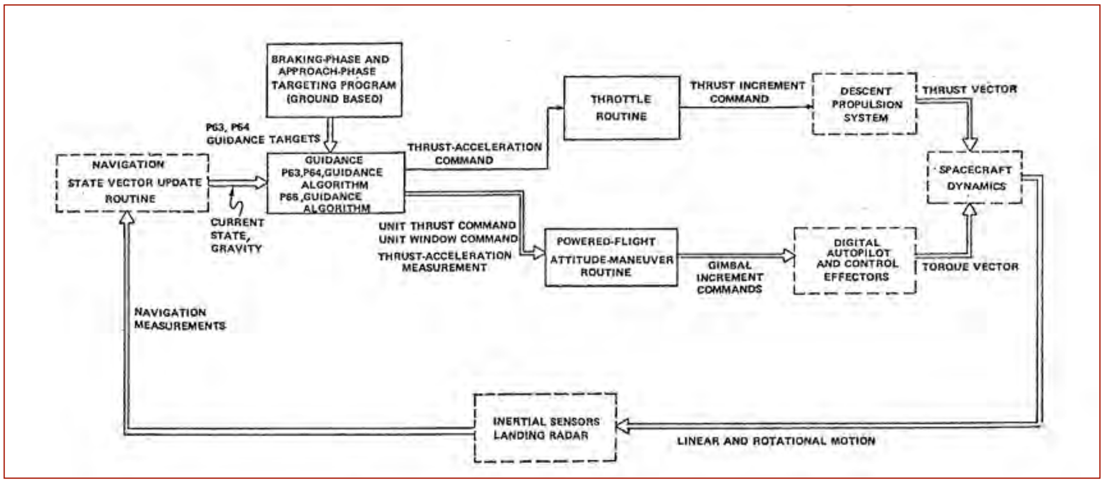
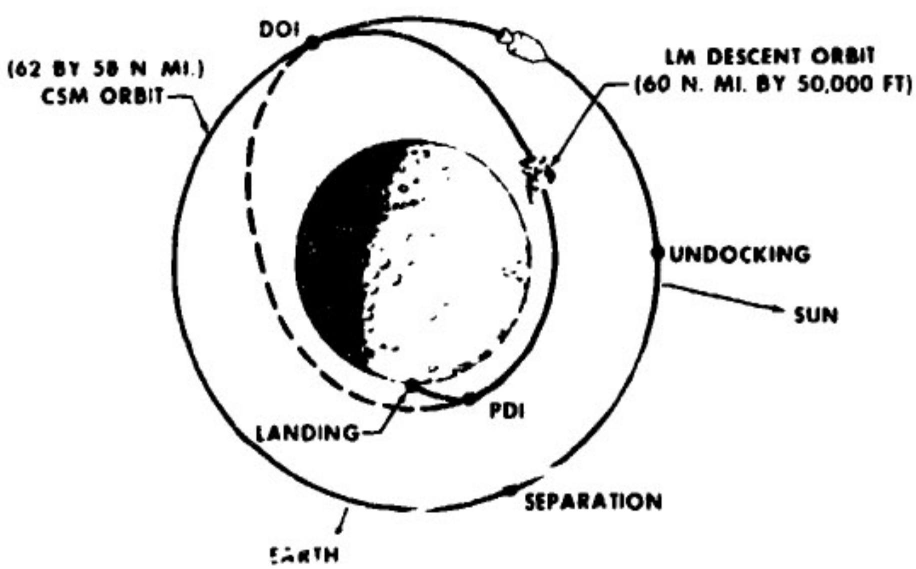
# Lunar Module Navigation, Guidance, and Control Configuration

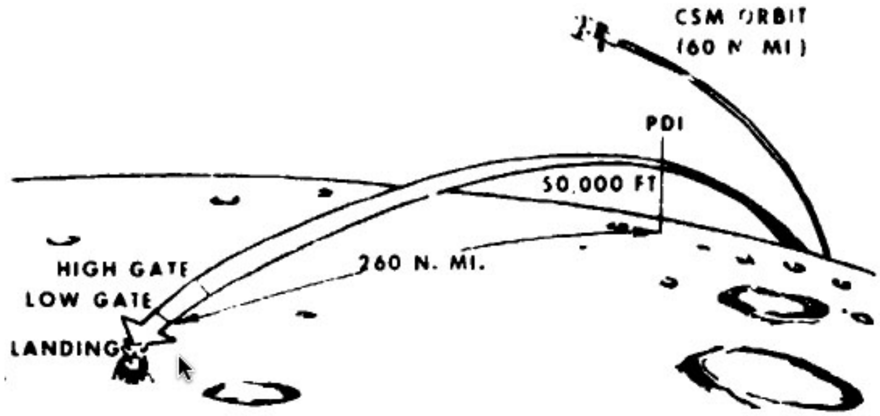





# Lunar Module Transfer Ellipse to Powered Descent Initiation



# Lunar Module Powered Descent




| PHASE | INITIAL EVENT | DESIGN CRITERIA |
|---|---|---|
| BRAKING | PDI | MINIMIZE PROPELLANT USAGE |
| APPROACH | HIGH GATE | CREW VISIBILITY |
| LANDING | LOW GATE | MANUAL CONTROL |

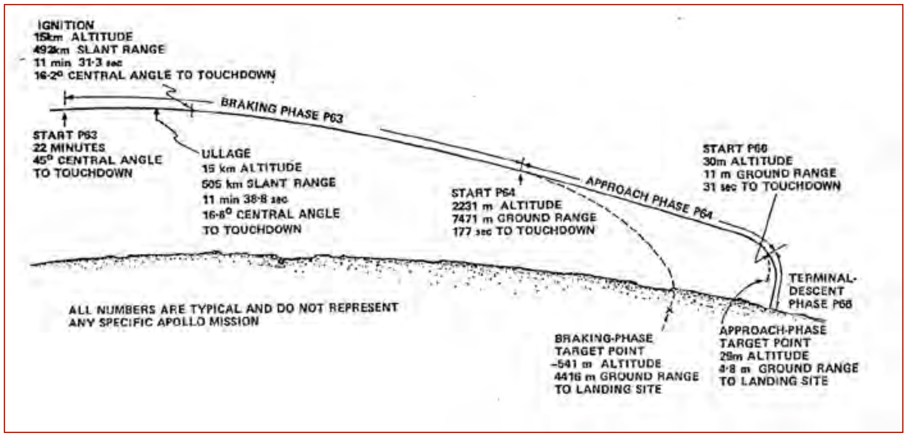


# Lunar Module Descent Events

| Event | TFI,[a] min:sec | Inertial velocity, fps | Altitude rate, fps | Altitude, ft | ΔV, fps |
|---|---|---|---|---|---|
| A Ullage | -00:07 | | | | |
| B Powered descent initiation | 00:00 | 5560 | -4 | 48 814 | 0 |
| C Throttle to maximum thrust | 00:26 | 5529 | -3 | 48 725 | 31 |
| D Rotate to windows-up position | 02:56 | 4000 | -50 | 44 934 | 1572 |
| E LR altitude update | 04:18 | 3065 | -89 | 39 201 | 2536 |
| F Throttle recovery | 06:24 | 1456 | -106 | 24 639 | 4239 |
| G LR velocity update | 06:42 | 1315 | -127 | 22 644 | 4399 |
| H High gate | 08:26 | 506 | -145 | 7 515 | 5375 |
| I Low gate | 10:06 | 55 ([b]68) | -16 | 512 | 6176 |
| J Touchdown (probe contact) | 11:54 | -15 ([b]0) | -3 | 12 | 6775 |

[a]Time from ignition of the DPS.
[b]Horizontal velocity relative to surface.

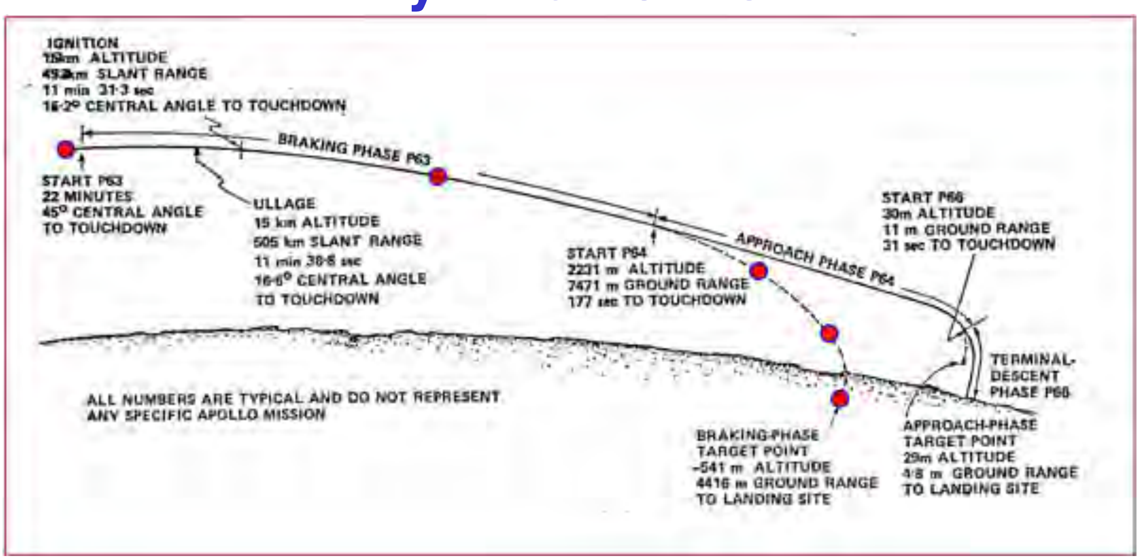
# Lunar Module Descent Targeting Sequence




**Braking Phase (P63)**
**Approach Phase (P64)**
**Terminal Descent Phase (P66)**



# Characterize Braking Phase By Five Points



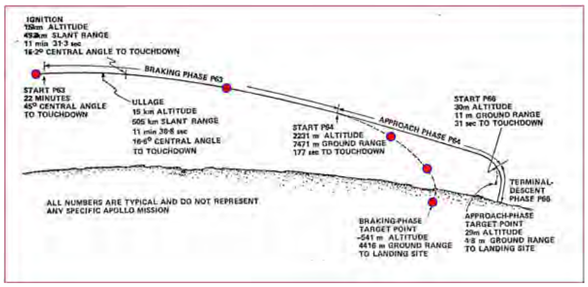
# Lunar Module Descent Guidance Logic

## (Klumpp, *Automatica*, 1974)

- Reference (nominal) trajectory, $\mathbf{r}_r(t)$, from target position back to starting point (Braking Phase example)
  - Three 4th-degree polynomials in time
  - 5 points needed to specify each polynomial

$$\mathbf{r}_r(t) = \mathbf{r}_t + \mathbf{v}_t t + \mathbf{a}_t \frac{t^2}{2} + \mathbf{j}_t \frac{t^3}{6} + \mathbf{s}_t \frac{t^4}{24}$$

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$






# Coefficients of the Polynomials

$$\mathbf{r}_r(t) = \mathbf{r}_t + \mathbf{v}_t t + \mathbf{a}_t \frac{t^2}{2} + \mathbf{j}_t \frac{t^3}{6} + \mathbf{s}_t \frac{t^4}{24}$$

- **r** = position vector
- **v** = velocity vector
- **a** = acceleration vector
- **j** = jerk vector (time derivative of acceleration)
- **s** = snap vector (time derivative of jerk)

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} j_x \\ j_y \\ j_z \end{bmatrix} \qquad \mathbf{s} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}$$

# Corresponding Reference Velocity and Acceleration Vectors

$$\boxed{\mathbf{v}_r(t) = \mathbf{v}_t + \mathbf{a}_t t + \mathbf{j}_t \frac{t^2}{2} + \mathbf{s}_t \frac{t^3}{6}}$$

$$\boxed{\mathbf{a}_r(t) = \mathbf{a}_t + \mathbf{j}_t t + \mathbf{s}_t \frac{t^2}{2}}$$

- **$\mathbf{a}_r(t)$ is the reference control vector**
  - **Descent engine thrust / mass = total acceleration**
  - **Vector components controlled by orienting yaw and pitch angles of the Lunar Module**



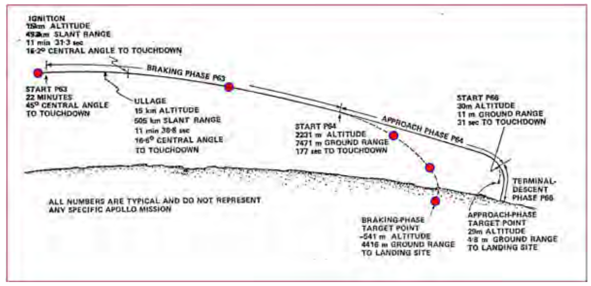

# Guidance Logic Defines Desired Acceleration Vector

- **If initial conditions, dynamic model, and thrust control were perfect, $\mathbf{a}_r(t)$ would produce $\mathbf{r}_r(t)$**

$$\boxed{\mathbf{a}_r(t) = \mathbf{a}_t + \mathbf{j}_t t + \mathbf{s}_t \frac{t^2}{2} \Rightarrow \mathbf{r}_r(t) = \mathbf{r}_t + \mathbf{v}_t t + \mathbf{a}_t \frac{t^2}{2} + \mathbf{j}_t \frac{t^3}{6} + \mathbf{s}_t \frac{t^4}{24}}$$

- **... but they are not**
- **Therefore, feedback control is required to follow the reference trajectory**

# Guidance Law for the Lunar Module Descent




## Linear feedback guidance law

$$\boxed{\mathbf{a}_{command}(t) = \mathbf{a}_r(t) + \mathbf{K}_V\left[\mathbf{v}_{measured}(t) - \mathbf{v}_r(t)\right] + \mathbf{K}_R\left[\mathbf{r}_{measured}(t) - \mathbf{r}_r(t)\right]}$$

$\mathbf{K}_V$ : velocity error gain
$\mathbf{K}_R$ : position error gain

**Nominal acceleration profile is corrected for measured differences between actual and reference flight paths**

**Considerable modifications made in actual LM implementation (see Klumpp's original paper on *Blackboard*)**