Linear-Quadratic-Gaussian Controllers

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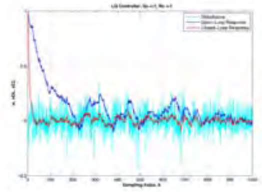
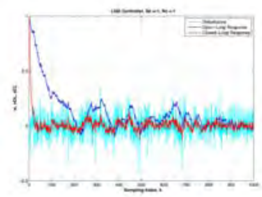
Optimal Control and Estimation MAE 546
Princeton University, 2015

- LTI dynamic system
- Asymptotic stability of the constant-gain LQG regulator
- Coupling due to parameter uncertainty
- Robustness (loop transfer) recovery
- Stochastic robustness analysis and design





http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

The Problem: Control to Minimize Cost, Subject to Dynamic Constraint, Uncertain Disturbances, and Measurement Error

Dynamic System

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{L}\mathbf{w}(t), \quad \mathbf{x}(0) = \mathbf{x}_o$$
$$\mathbf{z}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{n}(t)$$

Cost Function

$$\min_{\mathbf{u}} V(t_o) = \min_{\mathbf{u}} J(t_f)$$

$$= \frac{1}{2} \min_{\mathbf{u}} E\left\{ E\left[\mathbf{x}^T(t_f)\mathbf{S}(t_f)\mathbf{x}(t_f) \mid \Im_D\right] + E\left\{ \int_0^{t_f} \left[\begin{array}{cc} \mathbf{x}^T(t) & \mathbf{u}^T(t) \end{array} \right] \left[\begin{array}{cc} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{array} \right] \left[\begin{array}{c} \mathbf{x}(t) \\ \mathbf{u}(t) \end{array} \right] dt \,\middle|\, \Im_D \right\} \right\}$$

$$\Im_D(t) = \left\{ \hat{\mathbf{x}}(t), \mathbf{P}(t), \mathbf{u}(t) \right\}$$

Initial Conditions and Dimensions

$$E\left[\mathbf{x}(0)\right]=\hat{\mathbf{x}}_o;\quad E\left\{\left[\mathbf{x}(0)-\hat{\mathbf{x}}(0)\right]\left[\mathbf{x}(0)-\hat{\mathbf{x}}(0)\right]^T\right\}=\mathbf{P}(0)$$

$$E\left[\mathbf{w}(t)\right]=\mathbf{0};\quad E\left[\mathbf{w}(t)\mathbf{w}^T(\tau)\right]=\mathbf{W}\delta(t-\tau)$$

$$E\left[\mathbf{n}(t)\right]=\mathbf{0};\quad E\left[\mathbf{n}(t)\mathbf{n}^T(\tau)\right]=\mathbf{N}\delta(t-\tau)$$

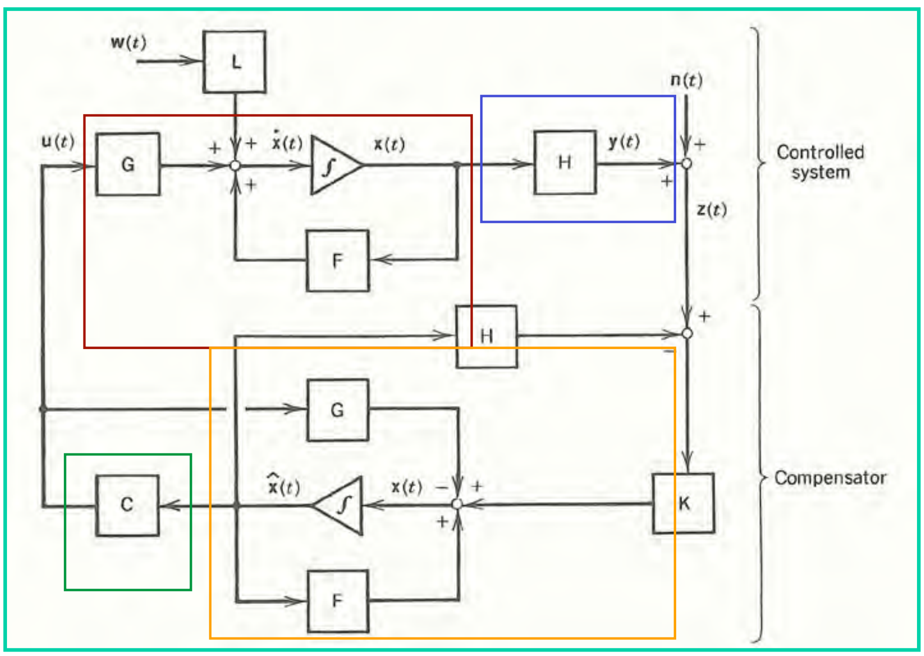
$$E\left[\mathbf{w}(t)\mathbf{n}^T(\tau)\right]=0$$

Statistics

$$\dim\left[\mathbf{x}(t)\right]=n\times 1$$
$$\dim\left[\mathbf{u}(t)\right]=m\times 1$$
$$\dim\left[\mathbf{w}(t)\right]=s\times 1$$
$$\dim\left[\mathbf{z}(t)\right]=\dim\left[\mathbf{n}(t)\right]=r\times 1$$

Dimensions

Linear-Quadratic-Gaussian Control



The Equations
(Continuous-Time Model)

System State and Measurement

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t)$$
$$\mathbf{z}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{n}(t)$$

Control Law

$$\mathbf{u}(t) = -\mathbf{C}(t)\hat{\mathbf{x}}(t) + \mathbf{C}_F(t)\mathbf{y}_C(t)$$

State Estimate

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{K}(t)\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right] \\ &= \left[\mathbf{F}(t) - \mathbf{G}(t)\mathbf{C}(t) - \mathbf{K}(t)\mathbf{H}\right]\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{C}_F(t)\mathbf{y}_C(t) + \mathbf{K}(t)\mathbf{z}(t)\end{aligned}$$

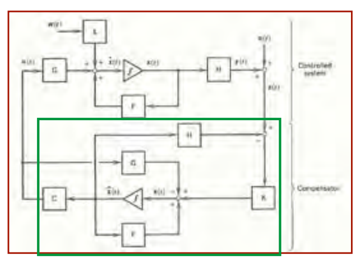
Estimator Gain and State Covariance Estimate

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^T\mathbf{N}^{-1}(t)$$
$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{L}(t)\mathbf{W}(t)\mathbf{L}^T(t) - \mathbf{P}(t)\mathbf{H}^T\mathbf{N}^{-1}(t)\mathbf{H}\mathbf{P}(t)$$

Control Gain and Adjoint Covariance Estimate

$$\mathbf{C}(t) = \mathbf{R}^{-1}(t)\mathbf{G}^T(t)\mathbf{S}(t)$$
$$\dot{\mathbf{S}}(t) = -\mathbf{Q}(t) - \mathbf{F}(t)^T\mathbf{S}(t) - \mathbf{S}(t)\mathbf{F}(t) + \mathbf{S}(t)\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{G}^T(t)\mathbf{S}(t)$$

Estimator in the Feedback Loop



Linear-Gaussian (LG) state estimator adds dynamics to the feedback signal

$$\mathbf{u}(t) = -\mathbf{C}(t)\hat{\mathbf{x}}(t)$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{K}(t)\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right]$$

Thus, state estimator can be viewed as a "compensator"

Bandwidth of the compensation is dependent on the multivariable signal/noise ratio, $\mathbf{PH}^T\mathbf{N}^{-1}$

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^T\mathbf{N}^{-1}$$
$$\dot{\mathbf{P}}(t) = \mathbf{F}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T + \mathbf{L}\mathbf{W}\mathbf{L}^T - \mathbf{P}(t)\mathbf{H}^T\mathbf{N}^{-1}\mathbf{H}\mathbf{P}(t)$$

Scalar LTI Example of Estimator Compensation

Dynamic system and measurement

$$\dot{x} = x + w; \quad z = Hx + n$$

Estimator differential equation

$$\dot{\hat{x}} = \hat{x} + K(z - H\hat{x}) = (1 - KH)\hat{x} + Kz$$

Laplace transform of estimator

$$[s - (1 - KH)]\hat{x}(s) = Kz(s)$$

Estimator transfer function

Low-pass filter

$$\hat{x}(s) = \frac{K}{[s - (1 - KH)]} z(s)$$

$$\frac{\hat{x}(s)}{z(s)} = \frac{K}{[s - (1 - KH)]}$$

Steady-State Scalar Filter Gain

$$Signal \text{ "Power"} = State\ Estimate\ Variance = P$$

$$Noise \text{ "Power"} = Measurement\ Error\ Variance = N$$

$$H = Projection\ from\ Noise\ Space\ to\ Signal\ Space$$

Constant, scalar filter gain

$$K = \frac{PH}{N}$$

Algebraic Riccati equation

$$0 = 2P + W - \frac{P^2 H^2}{N}; \quad P^2 - \frac{2N}{H^2}P - \frac{WN}{H^2} = 0$$

$$P = \frac{N}{H^2} \pm \sqrt{\left(\frac{N}{H^2}\right)^2 + \frac{WN}{H^2}} = \frac{N}{H^2}\left[1 \pm \sqrt{1 + \frac{WH^2}{N}}\right]$$

Steady-State Filter Gain

$$K = \frac{\left\{\frac{N}{H^2}\left[1+\sqrt{1+\frac{WH^2}{N}}\right]\right\}H}{N} = \left\{\frac{1}{H}\left[1+\sqrt{1+\frac{WH^2}{N}}\right]\right\}$$

$$K \xrightarrow[W \gg N]{} \sqrt{\frac{W}{N}}$$

Dynamic Constraint on the Certainty-Equivalent Cost

P(t) is independent of u(t); therefore

$$\min_{\mathbf{u}} J = \min_{\mathbf{u}} J_{CE} + J_S$$

J_{CE} is

Identical in form to the deterministic cost function

Minimized subject to dynamic constraint based on the state estimate

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{K}(t)\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right]$$

Kalman-Bucy Filter Provides Estimate of the State Mean Value

Filter residual is a Gaussian process

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{F}\hat{\mathbf{x}}(t)+\mathbf{G}\mathbf{u}(t)+\mathbf{K}(t)\left[\underline{\mathbf{z}(t)-\mathbf{H}\hat{\mathbf{x}}(t)}\right]\\ &\triangleq \mathbf{F}\hat{\mathbf{x}}(t)+\mathbf{G}\mathbf{u}(t)+\underline{\mathbf{K}(t)\boldsymbol{\varepsilon}(t)}\end{aligned}$$

Filter equation is analogous to <u>deterministic</u> dynamic constraint on deterministic cost function

$$\dot{\mathbf{x}}(t)=\mathbf{F}\mathbf{x}(t)+\mathbf{G}\mathbf{u}(t)+\underline{\mathbf{L}\mathbf{w}(t)}$$

Control That Minimizes the Certainty-Equivalent Cost

Optimizing control history is generated by a time-varying feedback control law

$$\mathbf{u}(t)=-\mathbf{C}(t)\hat{\mathbf{x}}(t)$$

The control gain is the same as the deterministic gain

$$\mathbf{C}(t)=\mathbf{R}^{-1}\mathbf{G}^T\mathbf{S}(t)$$

$$\dot{\mathbf{S}}(t)=-\mathbf{Q}-\mathbf{F}^T\mathbf{S}(t)-\mathbf{S}(t)\mathbf{F}+\mathbf{S}(t)\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T\mathbf{S}(t)$$

$$\mathbf{S}(t_f)\ \text{given}$$

Optimal Cost for the Continuous-Time LQG Controller

Certainty-equivalent cost

$$J_{CE} = \frac{1}{2}\mathrm{Tr}\left[\mathbf{S}(0)E\left[\hat{\mathbf{x}}(0)\hat{\mathbf{x}}^T(0)\right] + \int_0^{t_f}\mathbf{S}(t)\mathbf{K}(t)\mathbf{N}\mathbf{K}^T(t)\,dt\right]$$

Total cost

$$J = J_{CE} + J_S$$

$$= \frac{1}{2}\mathrm{Tr}\left\{\mathbf{S}(0)E\left[\hat{\mathbf{x}}(0)\hat{\mathbf{x}}^T(0)\right] + \int_0^{t_f}\mathbf{S}(t)\mathbf{K}(t)\mathbf{N}\mathbf{K}^T(t)\,dt\right\}$$

$$+\frac{1}{2}\mathrm{Tr}\left[\mathbf{S}(t_f)\mathbf{P}(t_f) + \int_0^{t_f}\mathbf{Q}\mathbf{P}(t)\,dt\right]$$

Discrete-Time LQG Controller

Kalman filter produces state estimate

$$\hat{\mathbf{x}}_k(-) = \mathbf{\Phi}\hat{\mathbf{x}}_{k-1}(+) + \mathbf{\Gamma}\mathbf{C}_{k-1}\hat{\mathbf{x}}_{k-1}(+)$$

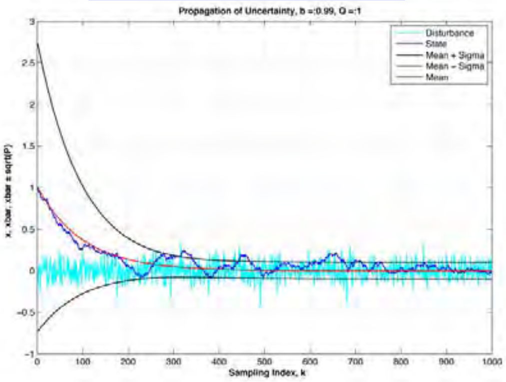
$$\hat{\mathbf{x}}_k(+) = \hat{\mathbf{x}}_k(-) + \mathbf{K}_k\left[\mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_k(-)\right]$$

Closed-loop system uses state estimate for feedback control

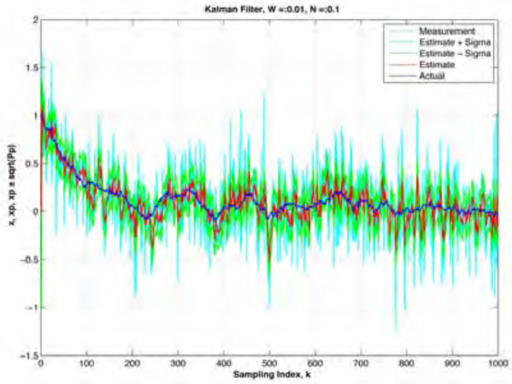
$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k - \mathbf{\Gamma}\mathbf{C}_k\hat{\mathbf{x}}_k(+)$$

Response of Discrete-Time 1st-Order System to Disturbance, and Kalman Filter Estimate from Noisy Measurement

Propagation of Uncertainty

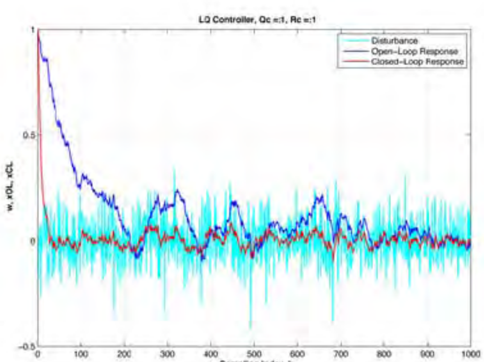


Kalman Filter, Uncontrolled System

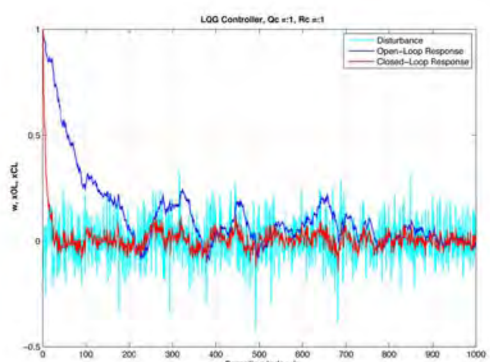


Comparison of 1st-Order Discrete-Time LQ and LQG Control Response

Linear-Quadratic Control with Noise-free Measurement



Linear-Quadratic-Gaussian Control with Noisy Measurement



Asymptotic Stability of the LQG Regulator

System Equations with LQG Control

With perfect knowledge of the system

$$\boxed{\begin{gathered}\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{L}\mathbf{w}(t) \\ \dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{K}(t)\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right]\end{gathered}}$$

State estimate error

$$\boxed{\boldsymbol{\varepsilon}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)}$$

State estimate error dynamics

$$\boxed{\dot{\boldsymbol{\varepsilon}}(t) = (\mathbf{F} - \mathbf{K}\mathbf{H})\boldsymbol{\varepsilon}(t) + \mathbf{L}\mathbf{w}(t) - \mathbf{K}\mathbf{n}(t)}$$

Control-Loop and Estimator Eigenvalues are Uncoupled

$$\boxed{\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\boldsymbol{\varepsilon}}(t) \end{bmatrix} = \begin{bmatrix} (\mathbf{F}-\mathbf{GC}) & \mathbf{GC} \\ \mathbf{0} & (\mathbf{F}-\mathbf{KH}) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\varepsilon}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{L} & -\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{n}(t) \end{bmatrix}}$$

Upper-block-triangular stability matrix

LQG system is stable because

(F – GC) is stable

(F – KH) is stable

Estimate error affects state response

$$\boxed{\dot{\mathbf{x}}(t) = (\mathbf{F}-\mathbf{GC})\mathbf{x}(t) + \mathbf{GC}\boldsymbol{\varepsilon}(t) + \mathbf{Lw}(t)}$$

Actual state does not affect error response

Disturbance affects both equally

Parameter Uncertainty Introduces Coupling

Coupling Due To Parameter Uncertainty

Actual System: $\{\mathbf{F}_A, \mathbf{G}_A, \mathbf{H}_A\}$

Assumed System: $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$

$$\dot{\mathbf{x}}(t) = \mathbf{F}_A\mathbf{x}(t) + \mathbf{G}_A\mathbf{u}(t) + \mathbf{L}\mathbf{w}(t)$$
$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{K}(t)\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right]$$

$$\mathbf{z}(t) = \mathbf{H}_A\mathbf{x}(t) + \mathbf{n}(t)$$
$$\mathbf{u}(t) = -\mathbf{C}(t)\hat{\mathbf{x}}(t)$$

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\boldsymbol{\varepsilon}}(t) \end{bmatrix} = \begin{bmatrix} (\mathbf{F}_A - \mathbf{G}_A\mathbf{C}) & \mathbf{G}_A\mathbf{C} \\ \left[(\mathbf{F}_A - \mathbf{F}) - (\mathbf{G}_A - \mathbf{G})\mathbf{C} - \mathbf{K}(\mathbf{H}_A - \mathbf{H})\right] & \left[\mathbf{F} + (\mathbf{G}_A - \mathbf{G})\mathbf{C} - \mathbf{K}\mathbf{H}\right] \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\varepsilon}(t) \end{bmatrix} + \cdots$$

Closed-loop control and estimator responses are coupled

Effects of Parameter Uncertainty on Closed-Loop Stability

$$\left|s\mathbf{I}_{2n} - \mathbf{F}_{CL}\right| = \begin{vmatrix} \left[s\mathbf{I}_n - (\mathbf{F}_A - \mathbf{G}_A\mathbf{C})\right] & -\mathbf{G}_A\mathbf{C} \\ -\left[(\mathbf{F}_A - \mathbf{F}) - (\mathbf{G}_A - \mathbf{G})\mathbf{C} - \mathbf{K}(\mathbf{H}_A - \mathbf{H})\right] & \left\{s\mathbf{I}_n - \left[\mathbf{F} + (\mathbf{G}_A - \mathbf{G})\mathbf{C} - \mathbf{K}\mathbf{H}\right]\right\} \end{vmatrix}$$
$$= \Delta_{CL}(s) = 0$$

- **Uncertain parameters affect closed-loop eigenvalues**
- **Coupling can lead to instability for numerous reasons**
  - **Improper control gain**
  - **Control effect on estimator block**
  - **Redistribution of damping**

Doyle's Counter-Example of LQG Robustness (1978)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w$$

Unstable Plant

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n$$

Design Matrices

$$\mathbf{Q} = Q \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad \mathbf{R} = 1; \quad \mathbf{W} = W \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad \mathbf{N} = 1$$

Control and Estimator Gains

$$\mathbf{C} = \left(2 + \sqrt{4+Q}\right) \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} c & c \end{bmatrix}$$

$$\mathbf{K} = \left(2 + \sqrt{4+W}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}$$

Uncertainty in the Control Effect

System Matrices

$$\mathbf{F}_A = \mathbf{F}; \quad \mathbf{G}_A = \begin{bmatrix} 0 \\ \mu \end{bmatrix}; \quad \mathbf{H}_A = \mathbf{H}$$

Characteristic Equation

$$\begin{vmatrix} (s-1) & -1 & 0 & 0 \\ 0 & (s-1) & \mu c & \mu c \\ -k & 0 & (s-1+k) & -1 \\ -k & 0 & (c+k) & (s-1+c) \end{vmatrix} = 0$$

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = \Delta_{CL}(s) = 0$$

Stability Effect of Parameter Variation

Routh's Stability Criterion (necessary condition)

- All coefficients of $\Delta(s)$ must be positive for stability

- μ is nominally equal to 1
- μ can force a_0 and a_1 to change sign
- Result is dependent on magnitude of ck

$$a_1 = k + c - 4 + 2(\mu - 1)ck$$
$$a_0 = 1 + (1 - \mu)ck$$

- Arbitrarily small uncertainty, $\mu = 1 + \varepsilon$, could cause unstability
- Not surprising: **uncertainty is in the control effect**

The Counter-Example Raises a Flag

Solution

Choose *Q* and *W* to be small, increasing allowable range of μ

- **However, ... The counter-example is irrelevant because it does not satisfy the requirements for LQ and LG stability**
  - **The open-loop system is unstable, so it requires feedback control to restore stability**
  - **To guarantee stability, Q and W must be positive definite, but**

$$\mathbf{Q} = Q\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad hence, |\mathbf{Q}| = 0$$

$$\mathbf{W} = W\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad hence, |\mathbf{W}| = 0$$

Robustness
(Loop Transfer Recovery)

Loop-Transfer Recovery

(Doyle and Stein, 1979)

- **Proposition: LQG and LQ robustness would be the same if the control vector had the same effect on the state and its estimate**

$$\boxed{\begin{gathered}\mathbf{Cx}(t) \text{ and } \mathbf{C\hat{x}}(t) \text{ produce } \textbf{same expected value of control}, \quad E\left[\mathbf{u}(t)\right] \\ \text{but not the same} \\ E\left\{\left[\mathbf{u}_{LQ}(t)-\mathbf{u}_{LQG}(t)\right]\left[\mathbf{u}_{LQ}(t)-\mathbf{u}_{LQG}(t)\right]^T\right\} \\ \text{as } \hat{\mathbf{x}}(t) \text{ contains measurement errors but } \mathbf{x}(t) \text{ does not}\end{gathered}}$$

- **Therefore, restoring the correct mean value from $\mathbf{z}(t)$ restores closed-loop robustness**
- **Solution: Increase the assumed "process noise" for estimator design as follows** (see text for details)

$$\boxed{\mathbf{W} = \mathbf{W}_o + k^2\mathbf{G}\mathbf{G}^T}$$

Stochastic Robustness Analysis and Design

Expression of Uncertainty in the System Model

System uncertainty may be expressed as

- Elements of $\mathbf{F}$
- Coefficients of $\Delta(s)$
- Eigenvalues, λ
- Frequency response/singular values/time response, $A(j\omega)$, $\sigma(j\omega)$, $\mathbf{x}(t)$

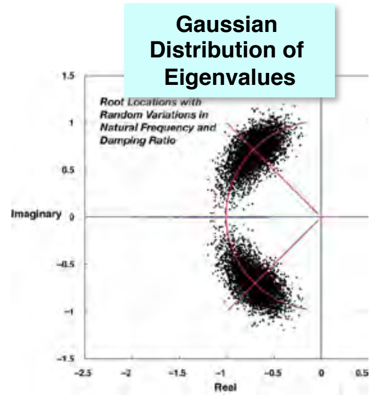
- **Variation may be**
  - **Deterministic, e.g.,**
    - **Upper/lower bounds ("worst-case")**
  - **Probabilistic, e.g.,**
    - **Gaussian distribution**
- **Bounded variation is equivalent to probabilistic variation with uniform distribution**

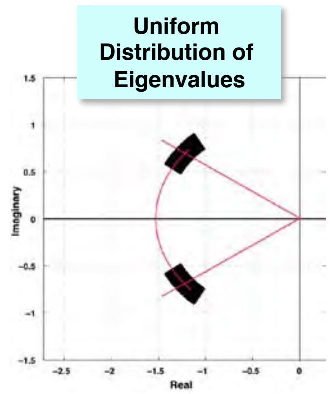
Stochastic Root Locus: Uncertain Damping Ratio and Natural Frequency

Laplace transform of dynamic model

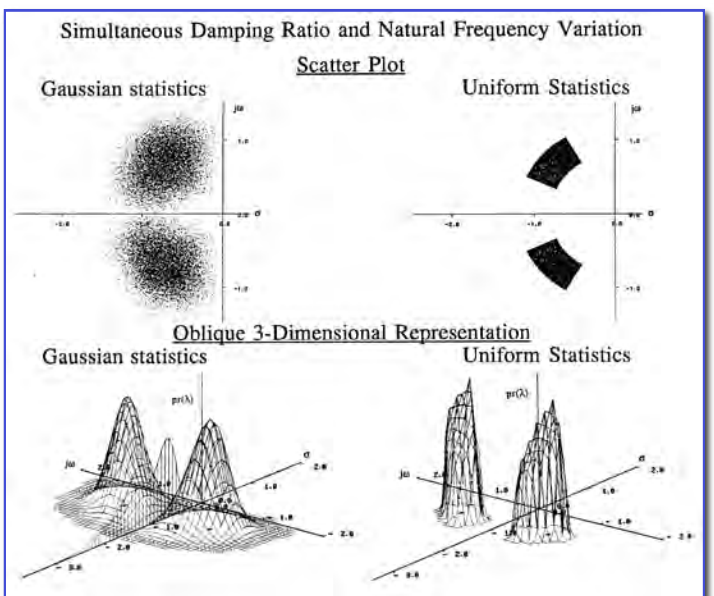
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Gaussian statistics	Uniform Statistics
$E(\zeta) = \bar{\zeta} = 0.707 \quad E[(\zeta - \bar{\zeta})^2] = 0.2^2$	$\zeta = [\,0.507, 0.907]$
$E(\omega_n) = \bar{\omega}_n = 1.0 \quad E[(\omega_n - \bar{\omega}_n)^2] = 0.2^2$	$\omega_n = [0.8, 1.2]$





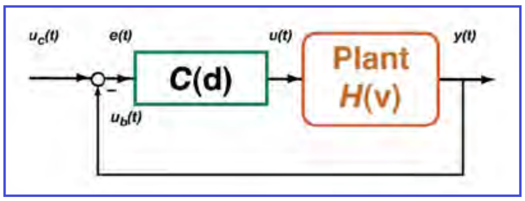
Probability of Instability



- Nonlinear mapping from probability density functions (*pdf*) of uncertain parameters to pdf of roots
- Finite probability of instability with Gaussian (unbounded) distribution
- Zero probability of instability for some uniform distributions

Probabilistic Control Design

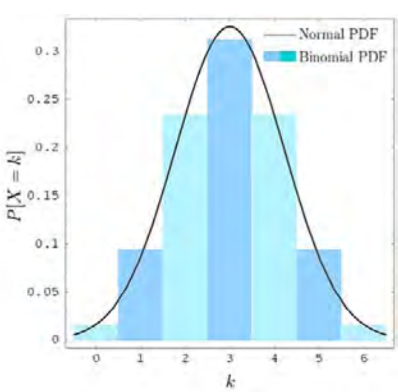
- Design constant-parameter controller (CPC) for satisfactory stability and performance in an uncertain environment
- Monte Carlo Evaluation of simulated system response with
  – competing CPC designs [Design parameters = d]
  – given statistical model of uncertainty in the plant [Uncertain plant parameters = v]

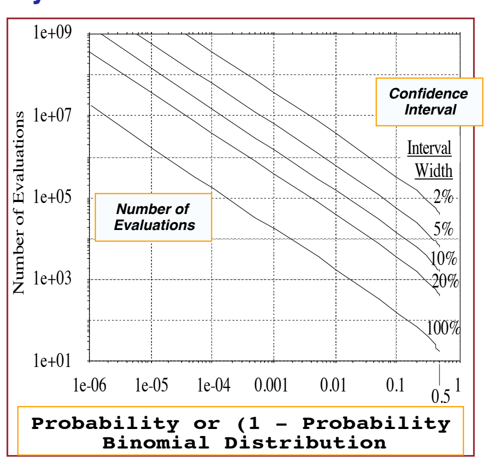


- Search for best CPC
  – Exhaustive search
  – Random search
  – Multivariate line search
  – Genetic algorithm
  – Simulated annealing

Design Outcome Follows Binomial Distribution

- Binomial distribution: Satisfactory/Unsatisfactory
- Confidence intervals of probability estimate are functions of
  – Actual probability
  – Number of trials





Maximum Information Entropy when Pr = 0.5

Example: Probability of Stable Control of an Unstable Plant

Longitudinal dynamics for a Forward-Swept-Wing Airplane

$$\mathbf{F} = \begin{bmatrix} -2gf_{11}/V & \rho V^2 f_{12}/2 & \rho V f_{13} & -g \\ -45/V^2 & \rho V f_{22}/2 & 1 & 0 \\ 0 & \rho V^2 f_{32}/2 & \rho V f_{33} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} g_{11} & g_{12} \\ 0 & 0 \\ g_{31} & g_{32} \\ 0 & 0 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} V\text{, Airspeed} \\ \alpha\text{, Angle of attack} \\ q\text{, Pitch rate} \\ \theta\text{, Pitch angle} \end{bmatrix}$$

Example: Probability of Stable Control of an Unstable Plant

Nominal eigenvalues (one unstable)

$$\lambda_{1-4} = -0.1 \pm 0.057j, \quad -5.15, \quad 3.35$$

Air density and airspeed, ρ and V, have uniform distributions($\pm 30\%$)

10 coefficients have Gaussian distributions ($\sigma = 30\%$)

$$\mathbf{p} = \begin{bmatrix} \rho & V & f_{11} & f_{12} & f_{13} & f_{22} & f_{32} & f_{33} & g_{11} & g_{12} & g_{31} & g_{32} \end{bmatrix}^T$$

Environment ***Uncontrolled Dynamics*** ***Control Effect***

LQ Regulators for the Example

Three stabilizing feedback control laws

Case a) LQR with low control weighting

$$\mathbf{Q} = diag(1,1,1,0); \quad \mathbf{R} = (1,1); \quad \lambda_{1-4_{nominal}} = -35, -5.1, -3.3, -.02$$

$$\mathbf{C} = \begin{bmatrix} 0.17 & 130 & 33 & 0.36 \\ 0.98 & -11 & -3 & -1.1 \end{bmatrix}$$

Case b) LQR with high control weighting

$$\mathbf{Q} = diag(1,1,1,0); \quad \mathbf{R} = (1000,1000); \quad \lambda_{1-4_{nominal}} = -5.2, -3.4, -1.1, -.02$$

$$\mathbf{C} = \begin{bmatrix} 0.03 & 83 & 21 & -0.06 \\ 0.01 & -63 & -16 & -1.9 \end{bmatrix}$$

Case c) Case b with gains multiplied by 5 for bandwidth (loop-transfer) recovery

$$\lambda_{1-4_{nominal}} = -32, -5.2, -3.4, -0.01$$

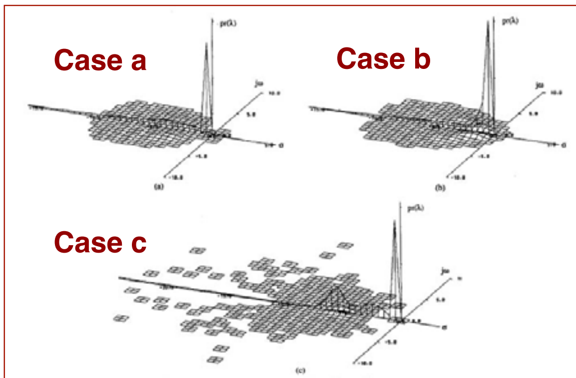
$$\mathbf{C} = \begin{bmatrix} 0.13 & 413 & 105 & -0.32 \\ 0.05 & -313 & -81 & -1.1-9.5 \end{bmatrix}$$

Stochastic Robustness

(Ray, Stengel, 1991)

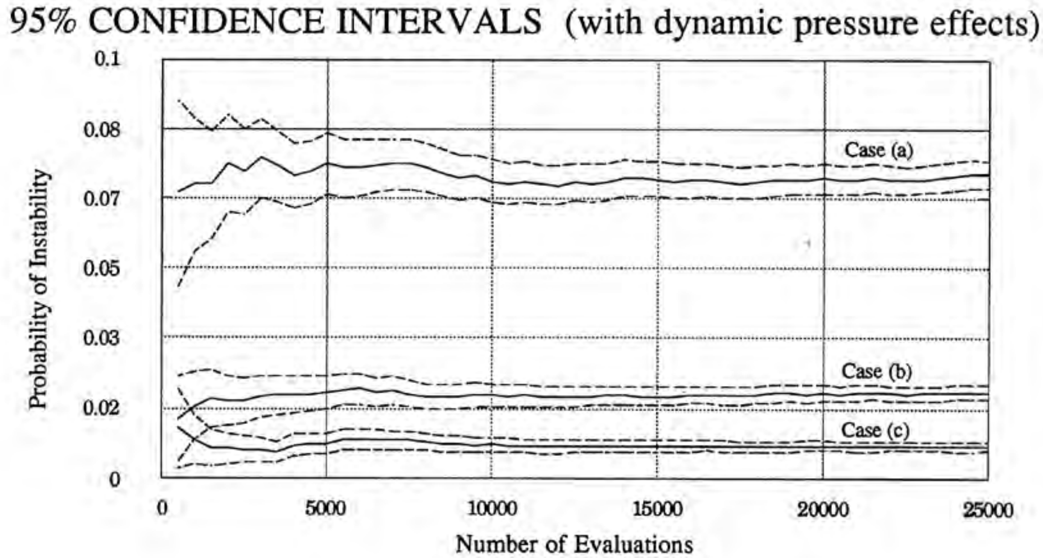
- Distribution of closed-loop roots with
  - Gaussian uncertainty in 10 parameters
  - Uniform uncertainty in velocity and air density
- 25,000 Monte Carlo evaluations

- Probability of instability
- a) Pr = 0.072
- b) Pr = 0.021
- c) Pr = 0.0076

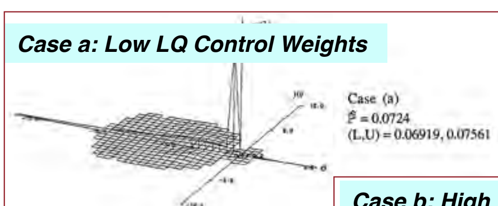


Stochastic Root Locus

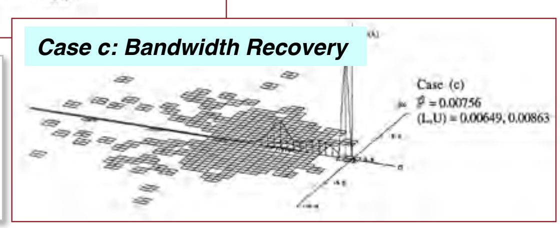
Probabilities of Instability for the Three Cases



Stochastic Root Loci for the Three Cases



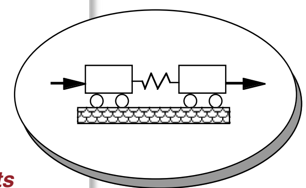
with Gaussian Aerodynamic Uncertainty



- Probabilities of instability with 30% uniform aerodynamic uncertainty
  - Case a: 3.4×10^{-4}
  - Case b: 0
  - Case c: 0

ACC Benchmark Control Problem, 1991

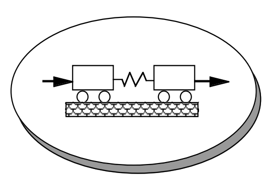
- Parameters of 4^{th}-order mass-spring system
  - Uniform probability density functions for
    - $0.5 < m_1, m_2 < 1.5$
    - $0.5 < k < 2$
- Probability of Instability, P_i
  - $m_i = 1$ (unstable) or 0 (stable)
- Probability of Settling Time Exceedance, P_{ts}
  - $m_{ts} = 1$ (exceeded) or 0 (not exceeded)
- Probability of Control Limit Exceedance, P_u
  - $m_u = 1$ (exceeded) or 0 (not exceeded)
- Design Cost Function
- 10 controllers designed for the competition



$$J = aP_i^2 + bP_{ts}^2 + cP_u^2$$

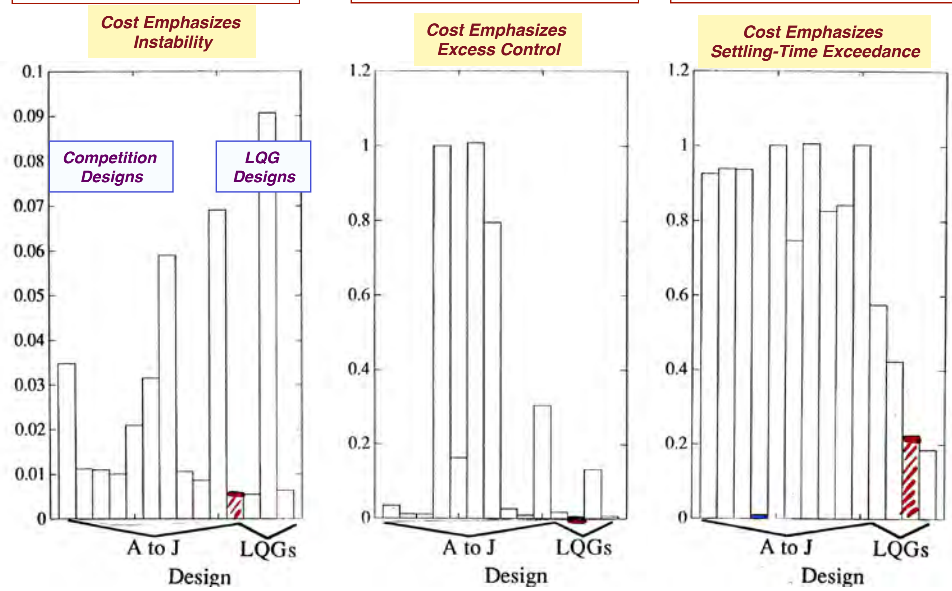
Stochastic LQG Design for Benchmark Control Problem

- SISO Linear-Quadratic-Gaussian Regulators (Marrison)
  - Implicit model following with control-rate weighting and scalar output (5^{th} order)
  - Kalman filter with single measurement (4^{th} order)
  - Design parameters
    - Control cost function weights
    - Springs and masses in ideal model
    - Estimator weights
  - Search
    - Multivariate line search
    - Genetic algorithm



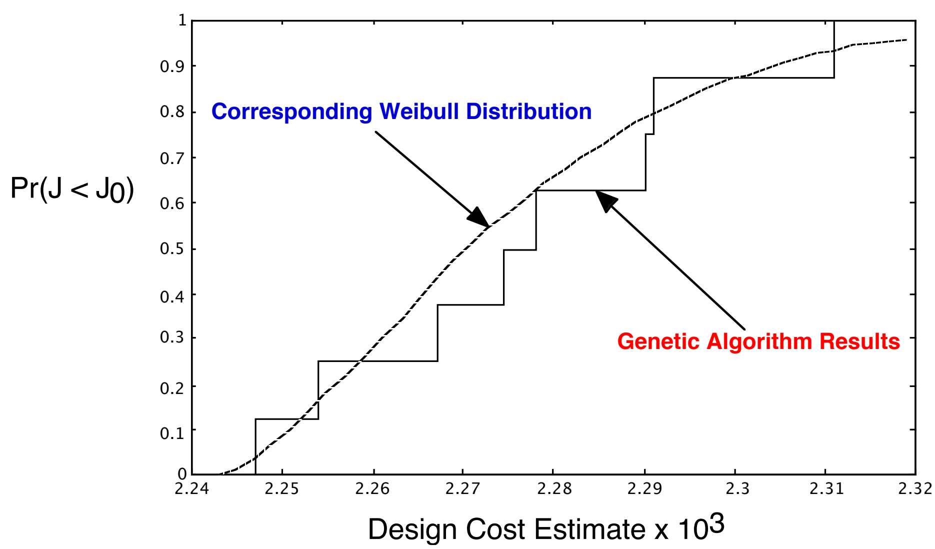
Comparison of Design Costs for Benchmark Control Problem

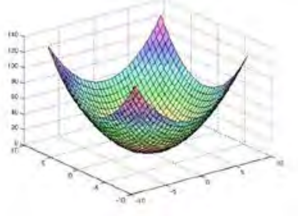
$J = P_i^2 + 0.01P_{ts}^2 + 0.01P_u^2$ | $J = 0.01P_i^2 + 0.01P_{ts}^2 + P_u^2$ | $J = 0.01P_i^2 + P_{ts}^2 + 0.01P_u^2$

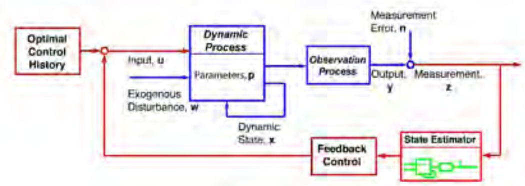


Stochastic LQG controller more robust in 39 of 40 benchmark controller comparisons

Estimation of Minimum Design Cost Using Jackknife/Bootstrapping Evaluation






Optimal Control History
Dynamic Process
Input, u
Parameters, p
Exogenous Disturbance, w
Dynamic State, x
Observation Process
Measurement Error, n
Output, y
Measurement, z
State Estimator
Feedback Control

Thank you!