Aircraft Equations of Motion - 2

Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2014

Learning Objectives

- How is a rotating reference frame described in an inertial reference frame?
- Is the transformation singular?
- What adjustments must be made to expressions for forces and moments in a non-inertial frame?
- How are the 6-DOF equations implemented in a computer?
- Damping effects

Reading:
Flight Dynamics
161-180

```
%   FLIGHT --  6-DOF Trim, Linear Model, and Flight Path Simulation
%   October 15, 2008
%   ===============================================================
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    clear
    global GEAR CONTROL SPOIL u x V parhis

%   This is the SCRIPT FILE.  It contains the Main Program, which:
%       Defines initial conditions
%       Calculates longitudinal trim condition
%       Calculates stability-and-control derivatives
%       Simulates flight path using nonlinear equations of motion

%   Functions used by FLIGHT:
%       AeroModel.m     Aerodynamic coefficients of the aircraft, thrust model,
%                       and geometric and inertial properties
%       Atmos.m         Air density, sound speed
%       ControlSystem.m Control law
%       DCM.m           Direction-cosine matrix
%       EoM.m           Equations of motion for integration
%       LinModel.m      Equations of motion for linear model definition
%       TrimCost.m      Cost function for trim solution
%       WindField.m     Wind velocity components

%   DEFINITION OF THE STATE VECTOR
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%       x(6) =      Negative of c.m. altitude WRT Earth, ze = -h, m
%       x(7) =      Body-axis roll rate, pr, rad/s
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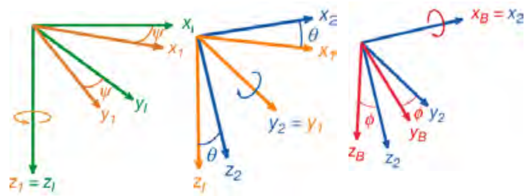
http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

Euler Angle Rates



Euler-Angle Rates and Body-Axis Rates

Body-axis angular rate vector (orthogonal)

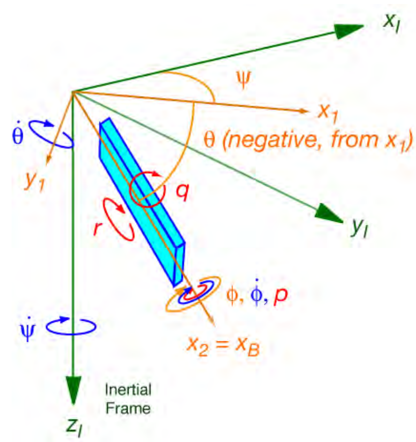
$$\boldsymbol{\omega}_B = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Form a non-orthogonal vector of Euler angles

$$\boldsymbol{\Theta} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

Euler-angle rate vector

$$\dot{\boldsymbol{\Theta}} = \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} \neq \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_I$$



Relationship Between Euler-Angle Rates and Body-Axis Rates

- $\dot\psi$ is measured in the Inertial Frame
- $\dot\theta$ is measured in Intermediate Frame #1
- $\dot\phi$ is measured in Intermediate Frame #2

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{I}_3 \begin{bmatrix} \dot\phi \\ 0 \\ 0 \end{bmatrix} + \mathbf{H}_2^B \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix} + \mathbf{H}_2^B \mathbf{H}_1^2 \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix}$$

- **... which is**

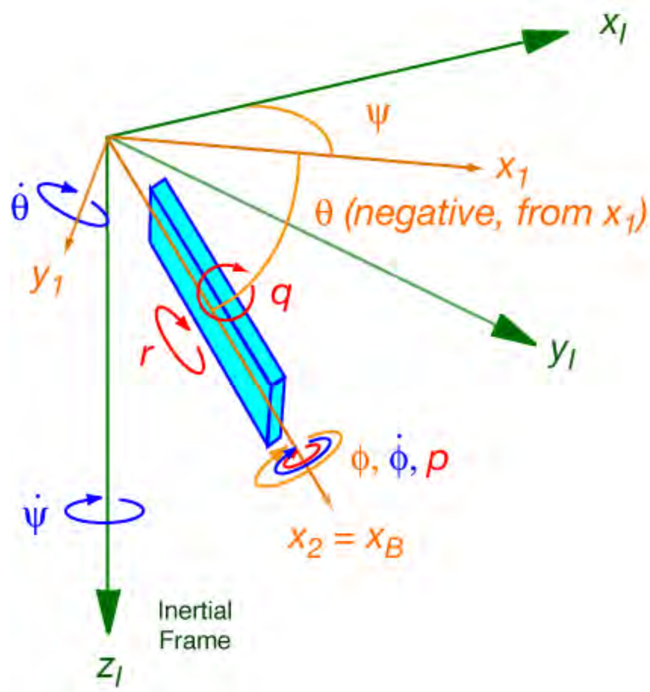
$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \mathbf{L}_I^B \dot{\boldsymbol{\Theta}}$$

Can the inversion become singular?
What does this mean?

- **Inverse transformation $[(.)^{-1} \neq (.)^T]$**

$$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{L}_B^I \boldsymbol{\omega}_B$$

Euler-Angle Rates and Body-Axis Rates



Avoiding the Singularity at $\theta = \pm 90°$

- **Don't use Euler angles as primary definition of angular attitude**
- **Alternatives to Euler angles**
  - **Direction cosine (rotation) matrix**
  - **Quaternions**
- **<u>Propagation of rotation matrix</u> (9 parameters)**
  - **From previous lecture**

$$\dot{\mathbf{H}}_B^I \mathbf{h}_B = \tilde{\boldsymbol{\omega}}_I \mathbf{H}_B^I \mathbf{h}_B$$

Consequently

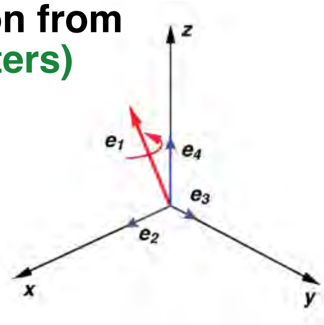
$$\boxed{\dot{\mathbf{H}}_I^B(t) = -\tilde{\boldsymbol{\omega}}_B(t)\mathbf{H}_I^B(t) = -\begin{bmatrix} 0 & -r(t) & q(t) \\ r(t) & 0 & -p(t) \\ -q(t) & p(t) & 0(t) \end{bmatrix}_B \mathbf{H}_I^B(t)}$$

$$\boxed{\mathbf{H}_I^B(0) = \mathbf{H}_I^B(\phi_0, \theta_0, \psi_0)}$$

Avoiding the Singularity at $\theta = \pm 90°$

- **Quaternion vector: single rotation from inertial to body frame (4 parameters)**

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} \text{Rotation angle, rad} \\ x\text{-component of rotation axis} \\ y\text{-component of rotation axis} \\ z\text{-component of rotation axis} \end{bmatrix}$$



- **Propagation of quaternion vector**
  - see *Flight Dynamics* for details

$$\dot{\mathbf{e}}(t) = \begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \\ \dot{e}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & -r(t) & -q(t) & -p(t) \\ r(t) & 0 & -p(t) & q(t) \\ q(t) & p(t) & 0 & -r(t) \\ p(t) & -q(t) & r(t) & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{bmatrix} = \mathbf{Q}(t)\mathbf{e}(t); \quad \mathbf{e}(0) = \mathbf{e}(\phi_0, \theta_0, \psi_0)$$

Rigid-Body Equations of Motion

Point-Mass Dynamics

- Inertial rate of change of translational position

$$\mathbf{v}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\boxed{\dot{\mathbf{r}}_I = \mathbf{v}_I = \mathbf{H}_B^I \mathbf{v}_B}$$

- Body-axis rate of change of translational velocity
  - Identical to angular-momentum transformation

$$\dot{\mathbf{v}}_I = \frac{1}{m}\mathbf{F}_I$$

$$\dot{\mathbf{v}}_B = \mathbf{H}_I^B \dot{\mathbf{v}}_I - \tilde{\boldsymbol{\omega}}_B \mathbf{v}_B = \frac{1}{m}\mathbf{H}_I^B \mathbf{F}_I - \tilde{\boldsymbol{\omega}}_B \mathbf{v}_B$$

$$= \frac{1}{m}\mathbf{F}_B - \tilde{\boldsymbol{\omega}}_B \mathbf{v}_B$$

$$\mathbf{F}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_B = \begin{bmatrix} C_X \bar{q} S \\ C_Y \bar{q} S \\ C_Z \bar{q} S \end{bmatrix}$$

Rigid-Body Equations of Motion (Euler Angles)

- Rate of change of Translational Position

$$\boxed{\dot{\mathbf{r}}_I(t) = \mathbf{H}_B^I(t)\mathbf{v}_B(t)}$$

- Rate of change of Angular Position

$$\boxed{\dot{\boldsymbol{\Theta}}_I(t) = \mathbf{L}_B^I(t)\boldsymbol{\omega}_B(t)}$$

- Rate of change of Translational Velocity

$$\boxed{\dot{\mathbf{v}}_B(t) = \frac{1}{m(t)}\mathbf{F}_B(t) + \mathbf{H}_I^B(t)\mathbf{g}_I - \tilde{\boldsymbol{\omega}}_B(t)\mathbf{v}_B(t)}$$

- Rate of change of Angular Velocity

$$\boxed{\dot{\boldsymbol{\omega}}_B(t) = \boldsymbol{I}_B^{-1}(t)\left[\mathbf{M}_B(t) - \tilde{\boldsymbol{\omega}}_B(t)\boldsymbol{I}_B(t)\boldsymbol{\omega}_B(t)\right]}$$

- Translational Position

$$\mathbf{r}_I = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_I$$

- Angular Position

$$\boldsymbol{\Theta}_I = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}_I$$

- Translational Velocity

$$\mathbf{v}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}_B$$

- Angular Velocity

$$\boldsymbol{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_B$$

Aircraft Characteristics Expressed in Body Frame of Reference

Aerodynamic and thrust force

$$\mathbf{F}_B = \begin{bmatrix} X_{aero} + X_{thrust} \\ Y_{aero} + Y_{thrust} \\ Z_{aero} + Z_{thrust} \end{bmatrix}_B = \begin{bmatrix} C_{X_{aero}} + C_{X_{thrust}} \\ C_{Y_{aero}} + C_{Y_{thrust}} \\ C_{Z_{aero}} + C_{Z_{thrust}} \end{bmatrix}_B \frac{1}{2}\rho V^2 S = \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix}_B \bar{q} S$$

Aerodynamic and thrust moment

$$\mathbf{M}_B = \begin{bmatrix} L_{aero} + L_{thrust} \\ M_{aero} + M_{thrust} \\ N_{aero} + N_{thrust} \end{bmatrix}_B = \begin{bmatrix} \left(C_{l_{aero}} + C_{l_{thrust}}\right) b \\ \left(C_{m_{aero}} + C_{m_{thrust}}\right) \bar{c} \\ \left(C_{n_{aero}} + C_{n_{thrust}}\right) b \end{bmatrix}_B \frac{1}{2}\rho V^2 S = \begin{bmatrix} C_l b \\ C_m \bar{c} \\ C_n b \end{bmatrix}_B \bar{q} S$$

Inertia matrix

$$\boldsymbol{I}_B = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}_B$$

Reference Lengths

b = wing span

$\bar{c}$ = mean aerodynamic chord

Rigid-Body Equations of Motion: Position

- **Rate of change of Translational Position**

$$\dot{x}_I = \left(\cos\theta\cos\psi\right)u + \left(-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi\right)v + \left(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi\right)w$$
$$\dot{y}_I = \left(\cos\theta\sin\psi\right)u + \left(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi\right)v + \left(-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi\right)w$$
$$\dot{z}_I = \left(-\sin\theta\right)u + \left(\sin\phi\cos\theta\right)v + \left(\cos\phi\cos\theta\right)w$$

- **Rate of change of Angular Position**

$$\dot{\phi} = p + \left(q\sin\phi + r\cos\phi\right)\tan\theta$$
$$\dot{\theta} = q\cos\phi - r\sin\phi$$
$$\dot{\psi} = \left(q\sin\phi + r\cos\phi\right)\sec\theta$$

Rigid-Body Equations of Motion: Rate

- **Rate of change of Translational Velocity**

$$
\begin{aligned}
\dot{u} &= X/m - g\sin\theta + rv - qw \\
\dot{v} &= Y/m + g\sin\phi\cos\theta - ru + pw \\
\dot{w} &= Z/m + g\cos\phi\cos\theta + qu - pv
\end{aligned}
$$

- **Rate of change of Angular Velocity**

$$
\begin{aligned}
\dot{p} &= \left(I_{zz}L + I_{xz}N - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)p + \left[I_{xz}^2 + I_{zz}\left(I_{zz} - I_{yy}\right)\right]r\right\}q\right) \Big/ \left(I_{xx}I_{zz} - I_{xz}^2\right) \\
\dot{q} &= \frac{1}{I_{yy}}\left[M - \left(I_{xx} - I_{zz}\right)pr - I_{xz}\left(p^2 - r^2\right)\right] \\
\dot{r} &= \left(I_{xz}L + I_{xx}N - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)r + \left[I_{xz}^2 + I_{xx}\left(I_{xx} - I_{yy}\right)\right]p\right\}q\right) \Big/ \left(I_{xx}I_{zz} - I_{xz}^2\right)
\end{aligned}
$$

Mirror symmetry, $I_{xz} \neq 0$

FLIGHT - Computer Program to Solve the 6-DOF Equations of Motion

FLIGHT - MATLAB Program

```
%   FLIGHT --  6-DOF Trim, Linear Model, and Flight Path Simulation
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```

http://www.princeton.edu/~stengel/FlightDynamics.html

FLIGHT - MATLAB Program

```
%   DEFINITION OF THE CONTROL VECTOR
%       u(1) =      Elevator, dEr, rad
%       u(2) =      Aileron, dAr, rad
%       u(3) =      Rudder, dRr, rad
%       u(4) =      Throttle, dT, %
%       u(5) =      Asymmetric Spoiler, dASr, rad
%       u(6) =      Flap, dFr, rad
%       u(7) =      Stabilator, dSr, rad

%   BEGINNING of MAIN PROGRAM
%   =========================

    'FLIGHT'
    date

%   FLIGHT Flags (1 = ON, 0 = OFF)
    TRIM =      1;      % Trim flag (= 1 to calculate trim)
    LINEAR =    1;      % Linear model flag (= 1 to calculate F and G)
    SIMUL =     1;      % Flight path flag (= 1 for nonlinear simulation)
    GEAR =      0;      % Landing gear DOWN (= 1) or UP (= 0)
    SPOIL =     0;      % Symmetric Spoiler DEPLOYED (= 1) or CLOSED (= 0)
    CONTROL =   0;      % Feedback control ON (= 1) or OFF (= 0)
    dF =        0;      % Flap setting, deg
```

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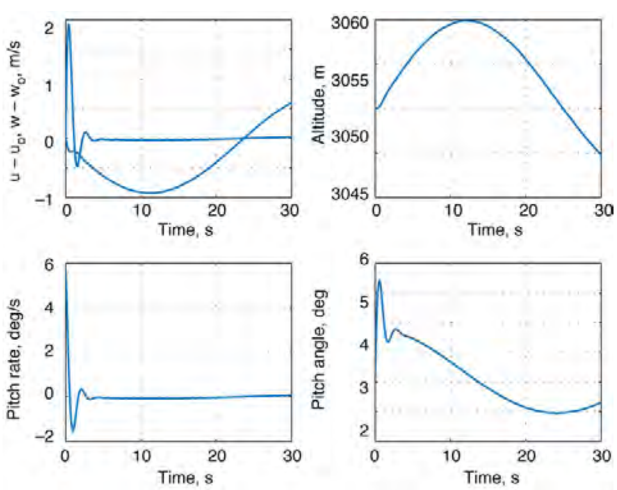
Examples from FLIGHT

Longitudinal Transient Response to Initial Pitch Rate

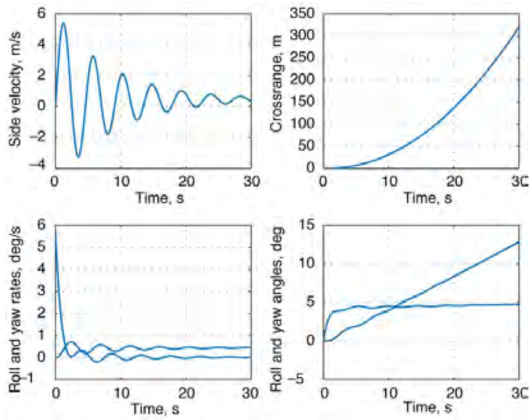


Bizjet, M = 0.3, Altitude = 3,052 m

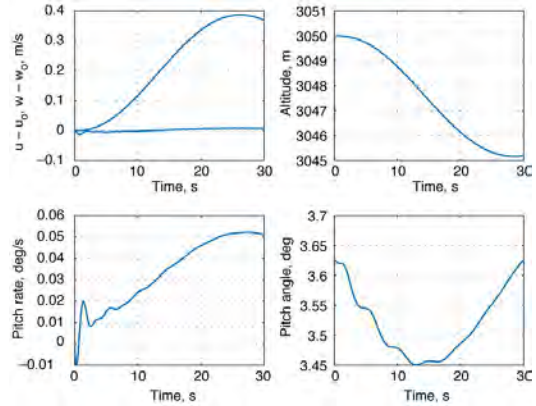
- **For a symmetric aircraft, longitudinal perturbations <u>do not</u> induce lateral-directional motions**

Transient Response to Initial Roll Rate

Lateral-Directional Response



Longitudinal Response

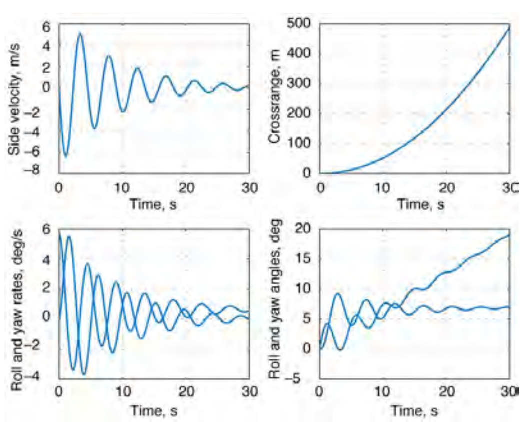


Bizjet, M = 0.3, Altitude = 3,052 m

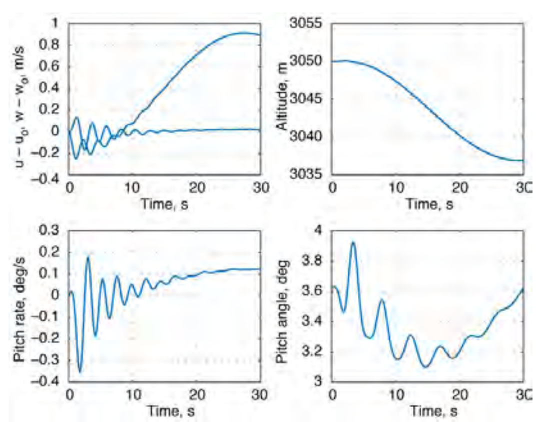
- **For a symmetric aircraft, lateral-directional perturbations do induce longitudinal motions**

Transient Response to Initial Yaw Rate

Lateral-Directional Response



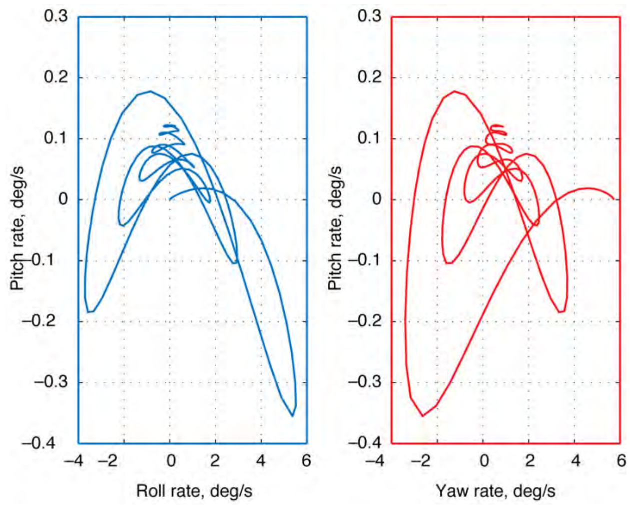
Longitudinal Response



Bizjet, M = 0.3, Altitude = 3,052 m

Crossplot of Transient Response to Initial Yaw Rate

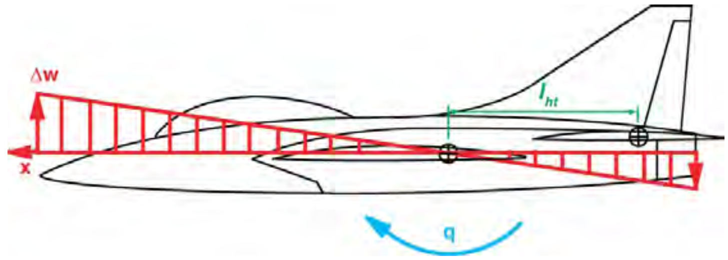
Longitudinal-Lateral-Directional Coupling



Bizjet, M = 0.3, Altitude = 3,052 m

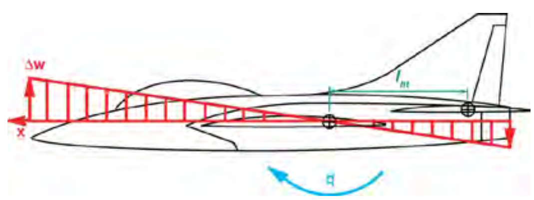
Aerodynamic Damping

Pitching Moment due to Pitch Rate



$$M_B = C_m \bar{q} S \bar{c} \approx \left(C_{m_o} + C_{m_q} q + C_{m_\alpha} \alpha \right) \bar{q} S \bar{c}$$
$$\approx \left(C_{m_o} + \frac{\partial C_m}{\partial q} q + C_{m_\alpha} \alpha \right) \bar{q} S \bar{c}$$

Angle of Attack Distribution Due to Pitch Rate



- **Aircraft pitching at a constant rate, *q* rad/s, produces a normal velocity distribution along *x***

$$\Delta w = -q \Delta x$$

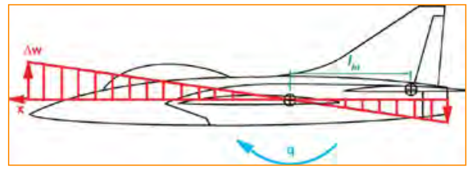
- **Corresponding angle of attack distribution**

$$\Delta \alpha = \frac{\Delta w}{V} = \frac{-q \Delta x}{V}$$

- **Angle of attack perturbation at tail center of pressure**

$$\boxed{\Delta \alpha_{ht} = \frac{q l_{ht}}{V}} \quad \boxed{l_{ht} = \text{horizontal tail distance from c.m.}}$$

Horizontal Tail Lift Due to Pitch Rate



- **Incremental tail lift due to pitch rate, referenced to tail area, S_{ht}**

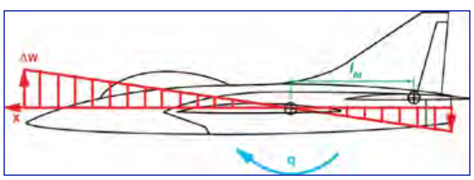
$$\Delta L_{ht} = \left(\Delta C_{L_{ht}}\right)_{ht} \frac{1}{2}\rho V^2 S_{ht}$$

- **Incremental tail lift coefficient due to pitch rate, referenced to wing area, S**

$$\left(\Delta C_{L_{ht}}\right)_{aircraft} = \left(\Delta C_{L_{ht}}\right)_{ht}\left(\frac{S_{ht}}{S}\right) = \left[\left(\frac{\partial C_{L_{ht}}}{\partial \alpha}\right)_{aircraft} \Delta\alpha\right] = \left(\frac{\partial C_{L_{ht}}}{\partial \alpha}\right)_{aircraft}\left(\frac{q l_{ht}}{V}\right)$$

- **Lift coefficient sensitivity to pitch rate referenced to wing area**

$$C_{L_{q_{ht}}} \equiv \frac{\partial\left(\Delta C_{L_{ht}}\right)_{aircraft}}{\partial q} = \left(\frac{\partial C_{L_{ht}}}{\partial \alpha}\right)_{aircraft}\left(\frac{l_{ht}}{V}\right)$$



Moment Coefficient Sensitivity to Pitch Rate of the Horizontal Tail

- **Differential pitch moment due to pitch rate**

$$\frac{\partial \Delta M_{ht}}{\partial q} = C_{m_{q_{ht}}} \frac{1}{2}\rho V^2 S\bar{c} = -C_{L_{q_{ht}}}\left(\frac{l_{ht}}{V}\right)\frac{1}{2}\rho V^2 S\bar{c}$$
$$= -\left[\left(\frac{\partial C_{L_{ht}}}{\partial \alpha}\right)_{aircraft}\left(\frac{l_{ht}}{V}\right)\right]\left(\frac{l_{ht}}{\bar{c}}\right)\frac{1}{2}\rho V^2 S\bar{c}$$

- **Coefficient derivative with respect to pitch rate**

$$C_{m_{q_{ht}}} = -\frac{\partial C_{L_{ht}}}{\partial \alpha}\left(\frac{l_{ht}}{V}\right)\left(\frac{l_{ht}}{\bar{c}}\right) = -\frac{\partial C_{L_{ht}}}{\partial \alpha}\left(\frac{l_{ht}}{\bar{c}}\right)^2\left(\frac{\bar{c}}{V}\right)$$

- **Coefficient derivative <u>with respect to normalized pitch rate</u> is insensitive to velocity**

$$C_{m_{\hat{q}_{ht}}} = \frac{\partial C_{m_{ht}}}{\partial \hat{q}} = \frac{\partial C_{m_{ht}}}{\partial\left(q\bar{c}/2V\right)} = -2\frac{\partial C_{L_{ht}}}{\partial \alpha}\left(\frac{l_{ht}}{\bar{c}}\right)^2$$

Pitch-Rate Derivative Definitions

- **Pitch-rate derivatives are often expressed in terms of a normalized pitch rate**

$$\hat{q} = \frac{q\bar{c}}{2V}$$

- **Then**

$$C_{m_{\hat{q}}} = \frac{\partial C_m}{\partial \hat{q}} = \frac{\partial C_m}{\partial \left(q\bar{c} \big/ 2V \right)} = \left(\frac{2V}{\bar{c}} \right) C_{m_q}$$

Pitching moment sensitivity to pitch rate

$$C_{m_q} = \frac{\partial C_m}{\partial q} = \left(\frac{\bar{c}}{2V} \right) C_{m_{\hat{q}}}$$

But dynamic equations require *$\partial C_m / \partial q$*

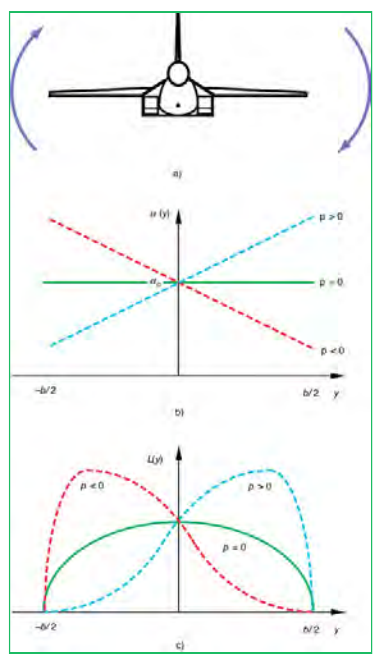
$$\frac{\partial M}{\partial q} = C_{m_q} \left(\rho V^2 / 2 \right) S\bar{c} = C_{m_{\hat{q}}} \left(\frac{\bar{c}}{2V} \right) \left(\frac{\rho V^2}{2} \right) S\bar{c} = C_{m_{\hat{q}}} \left(\frac{\rho V S \bar{c}^2}{4} \right)$$

Roll Damping Due to Roll Rate

$$C_{l_p} \left(\frac{\rho V^2}{2} \right) Sb = C_{l_{\hat{p}}} \left(\frac{b}{2V} \right) \left(\frac{\rho V^2}{2} \right) Sb$$
$$= C_{l_{\hat{p}}} \left(\frac{\rho V}{4} \right) Sb^2$$

< 0 for stability

$$\hat{p} = \frac{pb}{2V}$$



- **Vertical tail, horizontal tail, and wing are principal contributors**

$$C_{l_{\hat{p}}} \approx \left(C_{l_{\hat{p}}} \right)_{Vertical\ Tail} + \left(C_{l_{\hat{p}}} \right)_{Horizontal\ Tail} + \left(C_{l_{\hat{p}}} \right)_{Wing}$$

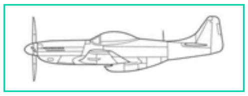
- **Wing with taper**

$$\left(C_{l_{\hat{p}}} \right)_{Wing} = \frac{\partial \left(\Delta C_l \right)_{Wing}}{\partial \hat{p}} = -\frac{C_{L_\alpha}}{12} \left(\frac{1 + 3\lambda}{1 + \lambda} \right)$$

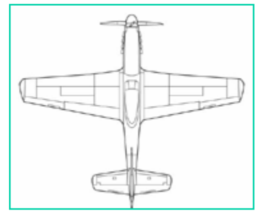
- **Thin triangular wing**

$$\left(C_{l_{\hat{p}}} \right)_{Wing} = -\frac{\pi AR}{32}$$

NACA-TR-1098, 1952
NACA-TR-1052, 1951



Roll Damping Due to Roll Rate



- **Tapered vertical tail**

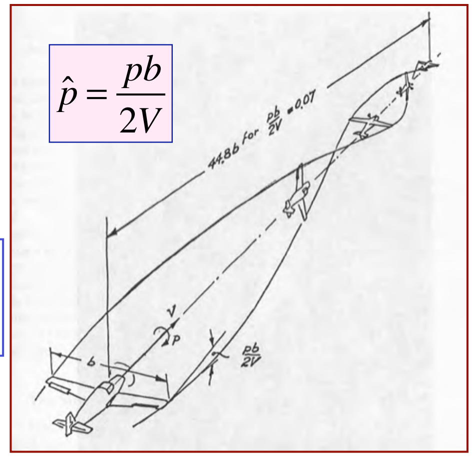
$$\left(C_{l_{\hat{p}}}\right)_{vt} = \frac{\partial\left(\Delta C_l\right)_{vt}}{\partial \hat{p}} = -\frac{C_{Y_{\beta_{vt}}}}{12}\left(\frac{S_{vt}}{S}\right)\left(\frac{1+3\lambda}{1+\lambda}\right)$$

- **Tapered horizontal tail**

$$\left(C_{l_{\hat{p}}}\right)_{ht} = \frac{\partial\left(\Delta C_l\right)_{ht}}{\partial \hat{p}} = -\frac{C_{L_{\alpha_{ht}}}}{12}\left(\frac{S_{ht}}{S}\right)\left(\frac{1+3\lambda}{1+\lambda}\right)$$

- ***pb/2V*** **describes helix angle for a steady roll**

$$\hat{p} = \frac{pb}{2V}$$



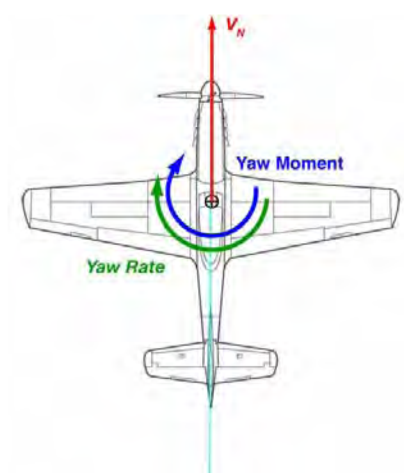
Yaw Damping Due to Yaw Rate

$$C_{n_r}\left(\frac{\rho V^2}{2}\right)Sb = C_{n_{\hat{r}}}\left(\frac{b}{2V}\right)\left(\frac{\rho V^2}{2}\right)Sb$$

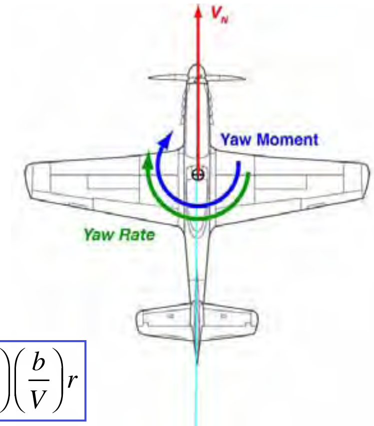
$$= C_{n_{\hat{r}}}\left(\frac{\rho V}{4}\right)Sb^2$$

< 0 for stability

$$\hat{r} = \frac{rb}{2V}$$



Yaw Damping Due to Yaw Rate



$$C_{n_{\hat{r}}} \approx \left(C_{n_{\hat{r}}}\right)_{Vertical\ Tail} + \left(C_{n_{\hat{r}}}\right)_{Wing}$$

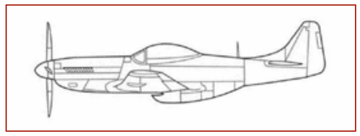
$$\hat{r} = \frac{rb}{2V}$$

- **Vertical tail contribution**

$$\Delta\left(C_n\right)_{Vertical\ Tail} = -\left(C_{n_\beta}\right)_{Vertical\ Tail}\left(\frac{rl_{vt}}{V}\right) = -\left(C_{n_\beta}\right)_{Vertical\ Tail}\left(\frac{l_{vt}}{b}\right)\left(\frac{b}{V}\right)r$$

$$\left(C_{n_{\hat{r}}}\right)_{vt} = \frac{\partial\Delta\left(C_n\right)_{Vertical\ Tail}}{\partial\left(\frac{rb}{2V}\right)} = \frac{\partial\Delta\left(C_n\right)_{Vertical\ Tail}}{\partial\hat{r}} = -2\left(C_{n_\beta}\right)_{Vertical\ Tail}\left(\frac{l_{vt}}{b}\right)$$

- **Wing contribution**



$$\left(C_{n_{\hat{r}}}\right)_{Wing} = k_0 C_L^2 + k_1 C_{D_{Parasite,\,Wing}}$$

k_0 and k_1 are functions of aspect ratio and sweep angle

NACA-TR-1098, 1952
NACA-TR-1052, 1951

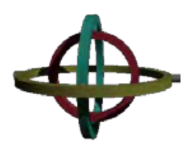
Next Time: Aircraft Control Devices and Systems

Reading:
Flight Dynamics
214-234

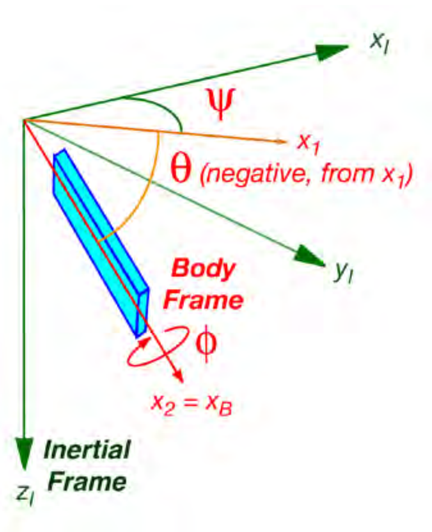
Supplemental Material

Airplane Angular Attitude (Position)

$$\boxed{(\psi,\theta,\phi)}$$



- **Euler angles**
  - 3 angles that relate one Cartesian coordinate frame to another
  - defined by sequence of 3 rotations about individual axes
  - intuitive description of angular attitude
  - Euler angle rates have a nonlinear relationship to body-axis angular rate vector
  - Transformation of rates is singular at 2 orientations, ±90°



Airplane Angular Attitude (Position)

$$\mathbf{H}_I^B(\phi,\theta,\psi) = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}_I^B$$

$$\mathbf{H}_I^B(\phi,\theta,\psi) = \mathbf{H}_2^B(\phi)\mathbf{H}_1^2(\theta)\mathbf{H}_I^1(\psi)$$

$$\left[\mathbf{H}_I^B(\phi,\theta,\psi)\right]^{-1} = \left[\mathbf{H}_I^B(\phi,\theta,\psi)\right]^T = \mathbf{H}_B^I(\psi,\theta,\phi)$$

- **Rotation matrix**
  - orthonormal transformation
  - inverse = transpose
  - linear propagation from one attitude to another, based on body-axis rate vector
  - 9 parameters, 9 equations to solve
  - solution for Euler angles from parameters is intricate

Airplane Angular Attitude (Position)

Rotation Matrix

$$\mathbf{H}_I^B(\phi,\theta,\psi) = \mathbf{H}_2^B(\phi)\mathbf{H}_1^2(\theta)\mathbf{H}_I^1(\psi)$$

$$\mathbf{H}_I^B(\phi,\theta,\psi) =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_I^B(\phi,\theta,\psi) =$$

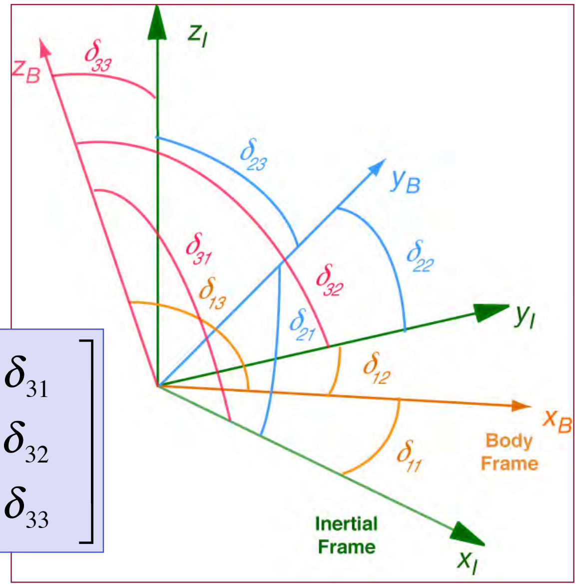
$$\begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix}$$

$\mathbf{H}_I^B\mathbf{H}_B^I = \mathbf{I}$ for all (ϕ,θ,ψ), i.e., No Singularities

Airplane Angular Attitude (Position)

Rotation Matrix = Direction Cosine Matrix

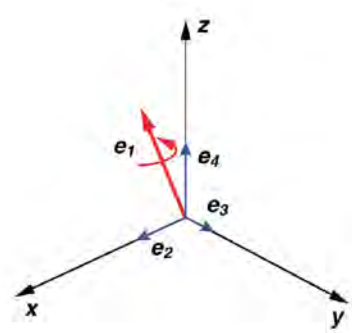
Angles between each *I* axis and each *B* axis



$$
\mathbf{H}_I^B = \begin{bmatrix} \cos\delta_{11} & \cos\delta_{21} & \cos\delta_{31} \\ \cos\delta_{12} & \cos\delta_{22} & \cos\delta_{32} \\ \cos\delta_{13} & \cos\delta_{23} & \cos\delta_{33} \end{bmatrix}
$$

Airplane Angular Attitude (Position)

$$
\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} \text{Rotation angle, rad} \\ x\text{-component of rotation axis} \\ y\text{-component of rotation axis} \\ z\text{-component of rotation axis} \end{bmatrix}
$$



- **Quaternion vector**
  - single rotation from inertial to body frame
  - non-singular, linear propagation of attitude based on body-axis rate vector
  - 4 parameters; more compact than the rotation matrix
  - solution for rotation matrix and Euler angles from parameters is intricate

Airplane Angular Attitude (Position)

Rotation Matrix from Quaternion Vector

$$\mathbf{H}_I^B = \begin{bmatrix} e_1^2 - e_2^2 - e_3^2 + e_4^2 & 2(e_1e_2 + e_3e_4) & 2(e_2e_4 - e_1e_3) \\ 2(e_3e_4 - e_1e_2) & e_1^2 - e_2^2 + e_3^2 + e_4^2 & 2(e_2e_3 + e_1e_4) \\ 2(e_1e_3 + e_2e_4) & 2(e_2e_3 - e_1e_4) & e_1^2 + e_2^2 - e_3^2 + e_4^2 \end{bmatrix}$$

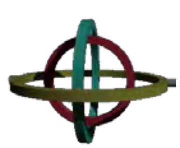
Euler Angles from Quaternion: see p. 186, Flight Dynamics

Euler Angle Dynamics

$$\dot{\mathbf{\Theta}} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{L}_B^I \boldsymbol{\omega}_B$$

$\mathbf{L}_B^I$ is not orthonormal

$\mathbf{L}_B^I$ is singular when $\theta = \pm 90°$



Rigid-Body Equations of Motion (Euler Angles)

$$\boxed{\dot{\mathbf{r}}_I(t) = \mathbf{H}_B^I(t)\mathbf{v}_B(t)}$$

$$\boxed{\dot{\boldsymbol{\Theta}}_I(t) = \mathbf{L}_B^I(t)\boldsymbol{\omega}_B(t)}$$

$\mathbf{H}_B^I, \mathbf{H}_I^B$ are functions of $\boldsymbol{\Theta}$

$$\boxed{\dot{\mathbf{v}}_B(t) = \frac{1}{m(t)}\mathbf{F}_B(t) + \mathbf{H}_I^B(t)\mathbf{g}_I - \tilde{\boldsymbol{\omega}}_B(t)\mathbf{v}_B(t)}$$

$$\boxed{\dot{\boldsymbol{\omega}}_B(t) = \boldsymbol{I}_B^{-1}(t)\left[\mathbf{M}_B(t) - \tilde{\boldsymbol{\omega}}_B(t)\boldsymbol{I}_B(t)\boldsymbol{\omega}_B(t)\right]}$$

Rotation Matrix Dynamics

$$\boxed{\dot{\mathbf{H}}_B^I\mathbf{h}_B = \tilde{\boldsymbol{\omega}}_I\mathbf{h}_I = \tilde{\boldsymbol{\omega}}_I\mathbf{H}_B^I\mathbf{h}_B}$$

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$\boxed{\dot{\mathbf{H}}_B^I = \tilde{\boldsymbol{\omega}}_I\mathbf{H}_B^I}$$

$$\boxed{\dot{\mathbf{H}}_I^B = -\tilde{\boldsymbol{\omega}}_B\mathbf{H}_I^B}$$

Rigid-Body Equations of Motion (Attitude from 9-Element Rotation Matrix)

$$\boxed{\dot{\mathbf{r}}_I = \mathbf{H}_B^I \mathbf{v}_B}$$

$$\boxed{\dot{\mathbf{H}}_I^B = -\tilde{\boldsymbol{\omega}}_B \mathbf{H}_I^B}$$

$$\boxed{\dot{\mathbf{v}}_B = \frac{1}{m}\mathbf{F}_B + \mathbf{H}_I^B \mathbf{g}_I - \tilde{\boldsymbol{\omega}}_B \mathbf{v}_B}$$

$$\boxed{\dot{\boldsymbol{\omega}}_B = \boldsymbol{I}_B^{-1}\left(\mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B\right)}$$

No need for Euler angles to solve the dynamic equations

Quaternion Vector Dynamics

$$\dot{\mathbf{e}}(t) = \begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \\ \dot{e}_4(t) \end{bmatrix} = \mathbf{Q}(t)\mathbf{e}(t)$$

$$= \begin{bmatrix} 0 & -r(t) & -q(t) & -p(t) \\ r(t) & 0 & -p(t) & q(t) \\ q(t) & p(t) & 0 & -r(t) \\ p(t) & -q(t) & r(t) & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{bmatrix}$$

Rigid-Body Equations of Motion
(Attitude from 4-Element Quaternion Vector)

$$\boxed{\dot{\mathbf{r}}_I = \mathbf{H}_B^I \mathbf{v}_B}$$

$$\boxed{\dot{\mathbf{e}} = \mathbf{Q}\mathbf{e}}$$

$\mathbf{H}_B^I, \mathbf{H}_I^B$ are functions of $\mathbf{e}$

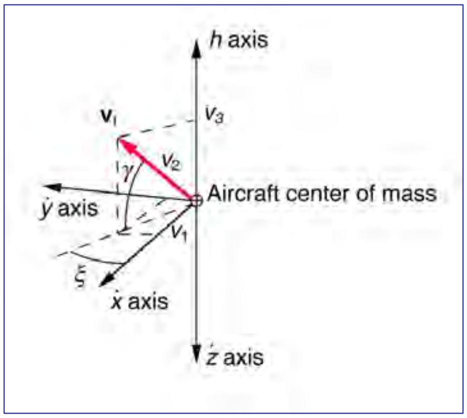
$$\boxed{\dot{\mathbf{v}}_B = \frac{1}{m}\mathbf{F}_B + \mathbf{H}_I^B \mathbf{g}_I - \tilde{\boldsymbol{\omega}}_B \mathbf{v}_B}$$

$$\boxed{\dot{\boldsymbol{\omega}}_B = \boldsymbol{I}_B^{-1}\left(\mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B\right)}$$

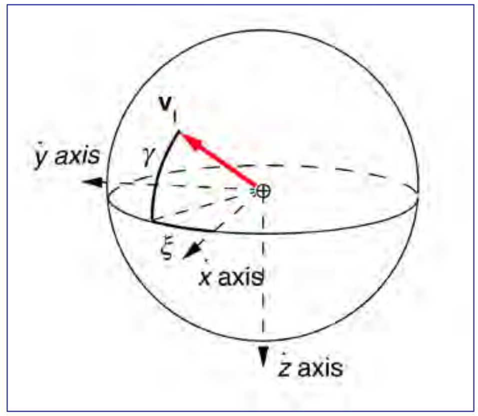
Alternative Reference Frames

Velocity Orientation in an Inertial Frame of Reference

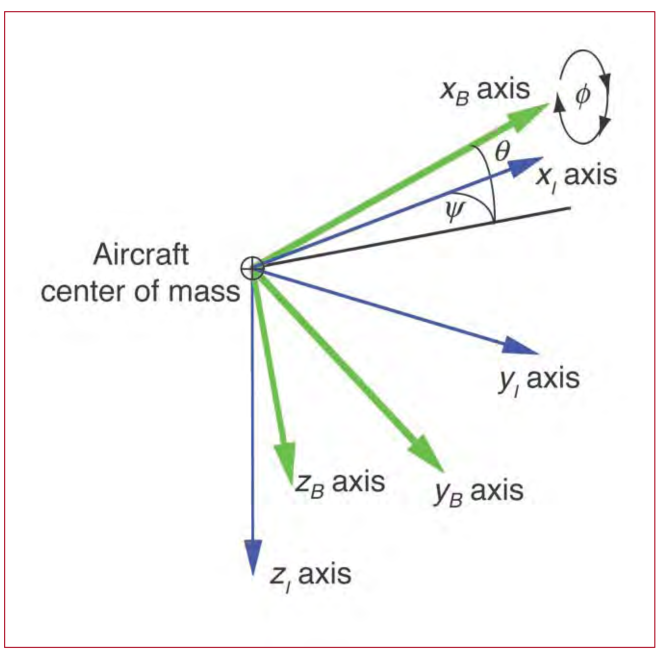
Polar Coordinates



Projected on a Sphere

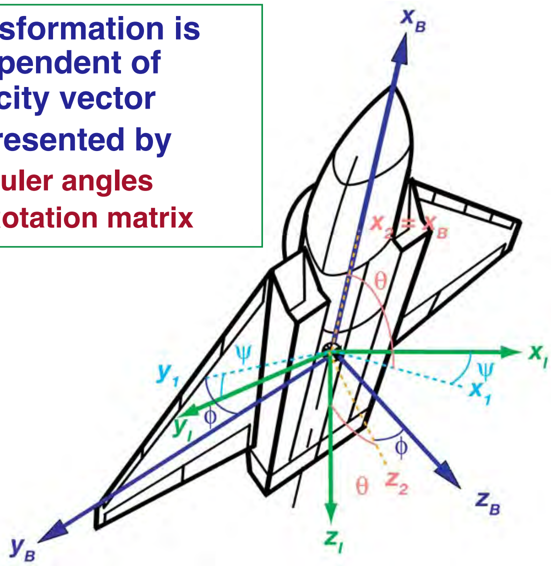


Body Orientation with Respect to an Inertial Frame



Relationship of Inertial Axes to Body Axes

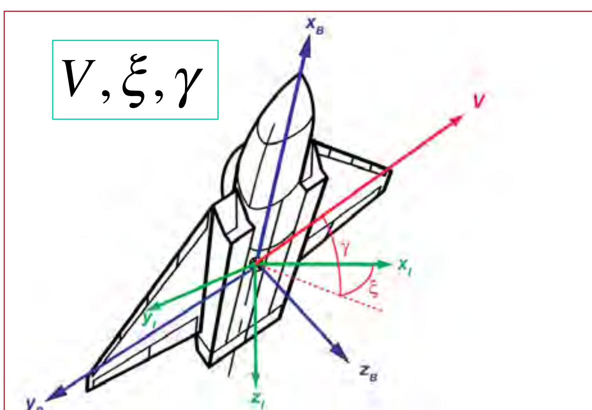
- Transformation is independent of velocity vector
- Represented by
  - Euler angles
  - Rotation matrix

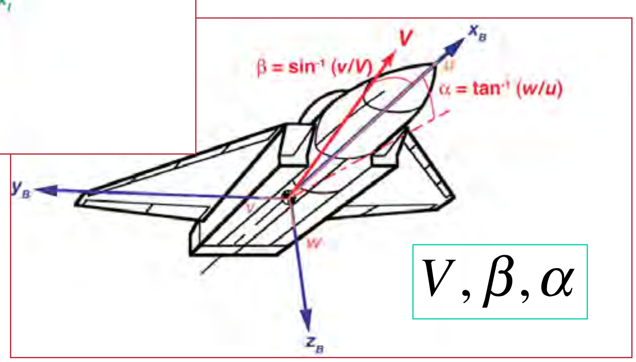


$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{H}_I^B \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \mathbf{H}_B^I \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Velocity-Vector Components of an Aircraft

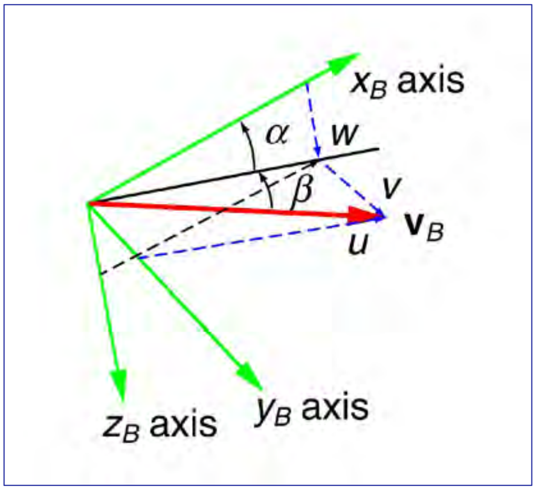


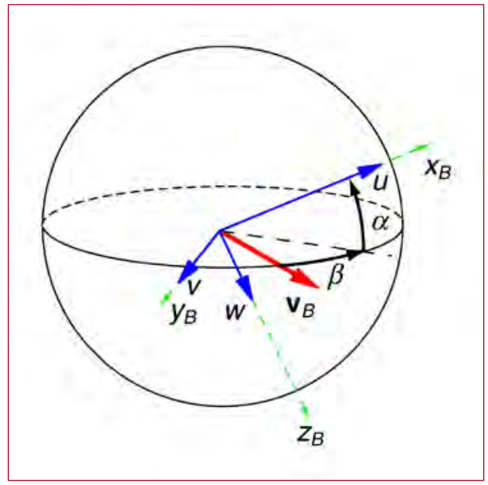


Velocity Orientation with Respect to the Body Frame

Polar Coordinates

Projected on a Sphere

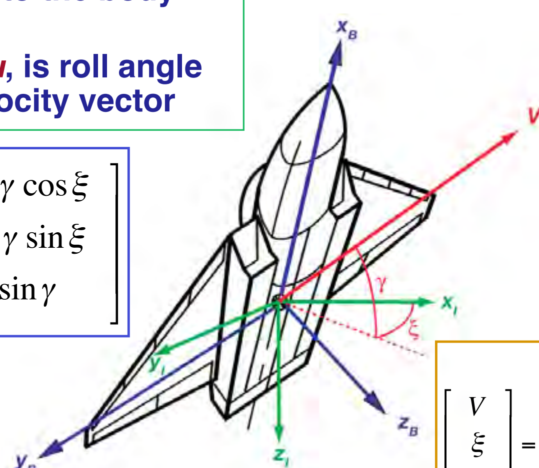




Relationship of Inertial Axes to Velocity Axes

- No reference to the body frame
- Bank angle, μ, is roll angle about the velocity vector

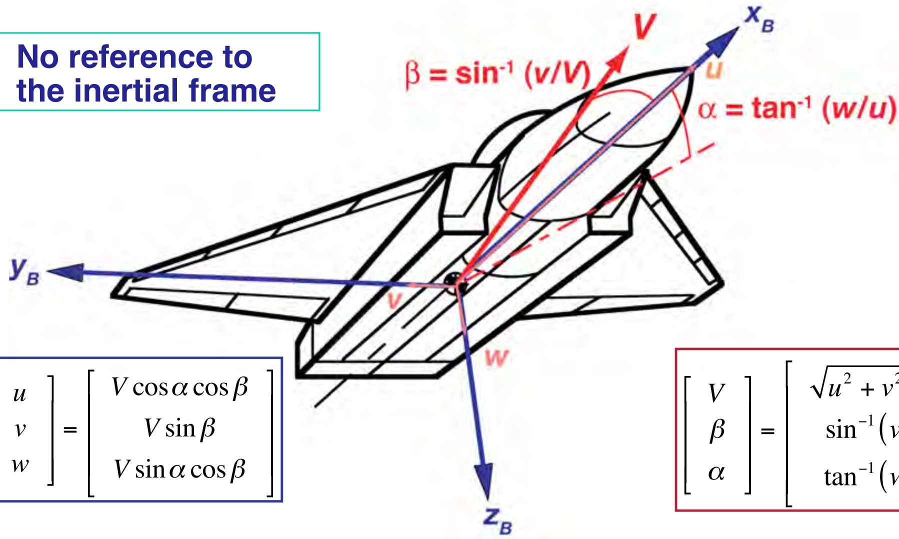
$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_I = \begin{bmatrix} V\cos\gamma\cos\xi \\ V\cos\gamma\sin\xi \\ -V\sin\gamma \end{bmatrix}$$



$$\begin{bmatrix} V \\ \xi \\ \gamma \end{bmatrix} = \begin{bmatrix} \sqrt{v_x^2 + v_y^2 + v_z^2} \\ \sin^{-1}\left[v_y / \left(v_x^2 + v_y^2\right)^{1/2}\right] \\ \sin^{-1}\left(-v_z / V\right) \end{bmatrix}$$

Relationship of Body Axes to Wind Axes

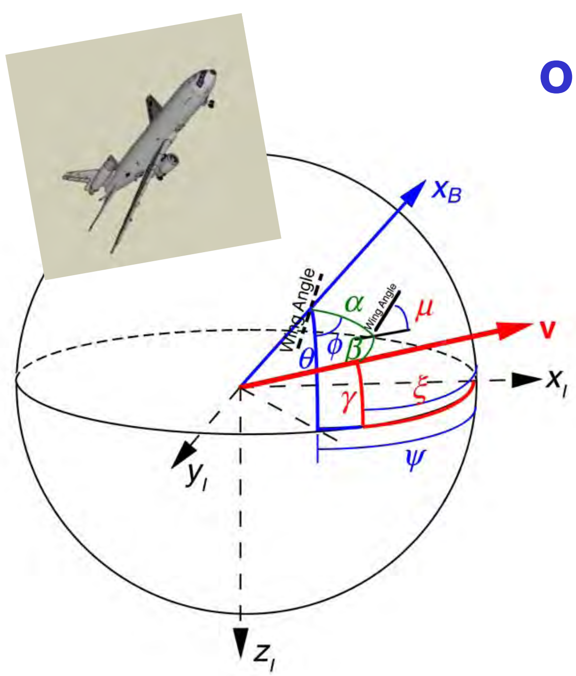
- No reference to the inertial frame



$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} V\cos\alpha\cos\beta \\ V\sin\beta \\ V\sin\alpha\cos\beta \end{bmatrix}$$

$$\begin{bmatrix} V \\ \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \sqrt{u^2+v^2+w^2} \\ \sin^{-1}(v/V) \\ \tan^{-1}(w/u) \end{bmatrix}$$

Angles Projected on the Unit Sphere



- Origin is airplane's center of mass

α : *angle of attack*
β : *sideslip angle*
γ : *vertical flight path angle*
ξ : *horizontal flight path angle*
ψ : *yaw angle*
θ : *pitch angle*
ϕ : *roll angle (about body x – axis)*
μ : *bank angle (about velocity vector)*

Alternative Frames of Reference

- Orthonormal transformations connect all reference frames