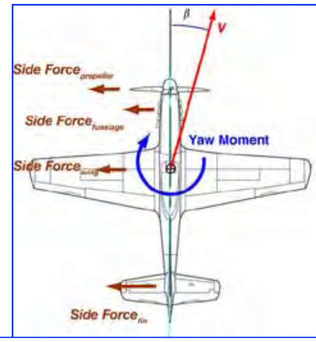
Linearized Lateral-Directional Equations of Motion

Robert Stengel, Aircraft Flight Dynamics MAE 331, 2014



Learning Objectives

- 6^{th}-order -> 4^{th}-order -> hybrid equations
- Dynamic stability derivatives
- Dutch roll mode
- Roll and spiral modes

Reading:
Flight Dynamics
574-591

http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

Lateral-Directional LTI Dynamics "Wordle"

6-Component Lateral-Directional Equations of Motion

$$\dot{v} = Y_B / m + g\sin\phi\cos\theta - ru + pw$$
$$\dot{y}_I = (\cos\theta\sin\psi)u + (\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)v + (-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi)w$$
$$\dot{p} = \left(I_{zz}L_B + I_{xz}N_B - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)p + \left[I_{xz}^2 + I_{zz}\left(I_{zz} - I_{yy}\right)\right]r\right\}q\right) \div \left(I_{xx}I_{zz} - I_{xz}^2\right)$$
$$\dot{r} = \left(I_{xz}L_B + I_{xx}N_B - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)r + \left[I_{xz}^2 + I_{xx}\left(I_{xx} - I_{yy}\right)\right]p\right\}q\right) \div \left(I_{xx}I_{zz} - I_{xz}^2\right)$$
$$\dot{\phi} = p + (q\sin\phi + r\cos\phi)\tan\theta$$
$$\dot{\psi} = (q\sin\phi + r\cos\phi)\sec\theta$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \mathbf{x}_{LD_6} \qquad \begin{bmatrix} v \\ y \\ p \\ r \\ \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \text{Side Velocity} \\ \text{Crossrange} \\ \text{Body-Axis Roll Rate} \\ \text{Body-Axis Yaw Rate} \\ \text{Roll Angle about Body } x \text{ Axis} \\ \text{Yaw Angle about Inertial } x \text{ Axis} \end{bmatrix}$$

4- Component Lateral-Directional Equations of Motion

Nonlinear Dynamic Equations, neglecting crossrange and yaw angle

$$\dot{v} = Y_B / m + g\sin\phi\cos\theta - ru + pw$$
$$\dot{p} = \left(I_{zz}L_B + I_{xz}N_B - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)p + \left[I_{xz}^2 + I_{zz}\left(I_{zz} - I_{yy}\right)\right]r\right\}q\right) \div \left(I_{xx}I_{zz} - I_{xz}^2\right)$$
$$\dot{r} = \left(I_{xz}L_B + I_{xx}N_B - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)r + \left[I_{xz}^2 + I_{xx}\left(I_{xx} - I_{yy}\right)\right]p\right\}q\right) \div \left(I_{xx}I_{zz} - I_{xz}^2\right)$$
$$\dot{\phi} = p + (q\sin\phi + r\cos\phi)\tan\theta$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{x}_{LD_4} \qquad \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} = \begin{bmatrix} \text{Side Velocity} \\ \text{Body-Axis Roll Rate} \\ \text{Body-Axis Yaw Rate} \\ \text{Roll Angle about Body } x \text{ Axis} \end{bmatrix}$$

Lateral-Directional Equations of Motion

Assuming Steady, Level Longitudinal Flight

Longitudinal variables are constant

$$\dot{v} = Y_B / m + g\sin\phi\cos\theta_N - ru_N + pw_N$$

$$\dot{p} = \left(I_{zz}L_B + I_{xz}N_B\right) / \left(I_{xx}I_{zz} - I_{xz}^2\right)$$

$$\dot{r} = \left(I_{xz}L_B + I_{xx}N_B\right) / \left(I_{xx}I_{zz} - I_{xz}^2\right)$$

$$\dot{\phi} = p + \left(r\cos\phi\right)\tan\theta_N$$

$$q_N = 0$$
$$\gamma_N = 0$$
$$\theta_N = \alpha_N$$

Lateral-Directional Force and Moments in Steady, Level Flight

Dynamic pressure is constant

$$Y_B = C_{Y_B}\frac{1}{2}\rho_N V_N^2 S$$

$$L_B = C_{l_B}\frac{1}{2}\rho_N V_N^2 Sb$$

$$N_B = C_{n_B}\frac{1}{2}\rho_N V_N^2 Sb$$

Body-Axis Side Force
Body-Axis Rolling Moment
Body-Axis Yawing Moment

Linearized Lateral-Directional Equations of Motion in Steady, Level Flight

Body-Axis Perturbation Equations of Motion

$$
\begin{bmatrix} \Delta\dot{v}(t) \\ \Delta\dot{p}(t) \\ \Delta\dot{r}(t) \\ \Delta\dot{\phi}(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial p} & \dfrac{\partial f_1}{\partial r} & \dfrac{\partial f_1}{\partial \phi} \\ \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial p} & \dfrac{\partial f_2}{\partial r} & \dfrac{\partial f_2}{\partial \phi} \\ \dfrac{\partial f_3}{\partial v} & \dfrac{\partial f_3}{\partial p} & \dfrac{\partial f_3}{\partial r} & \dfrac{\partial f_3}{\partial \phi} \\ \dfrac{\partial f_4}{\partial v} & \dfrac{\partial f_4}{\partial p} & \dfrac{\partial f_4}{\partial r} & \dfrac{\partial f_4}{\partial \phi} \end{bmatrix} \begin{bmatrix} \Delta v(t) \\ \Delta p(t) \\ \Delta r(t) \\ \Delta\phi(t) \end{bmatrix}
$$

$$
+\left[Control\right]+\left[Disturbance\right]
$$

Body-Axis Perturbation Variables

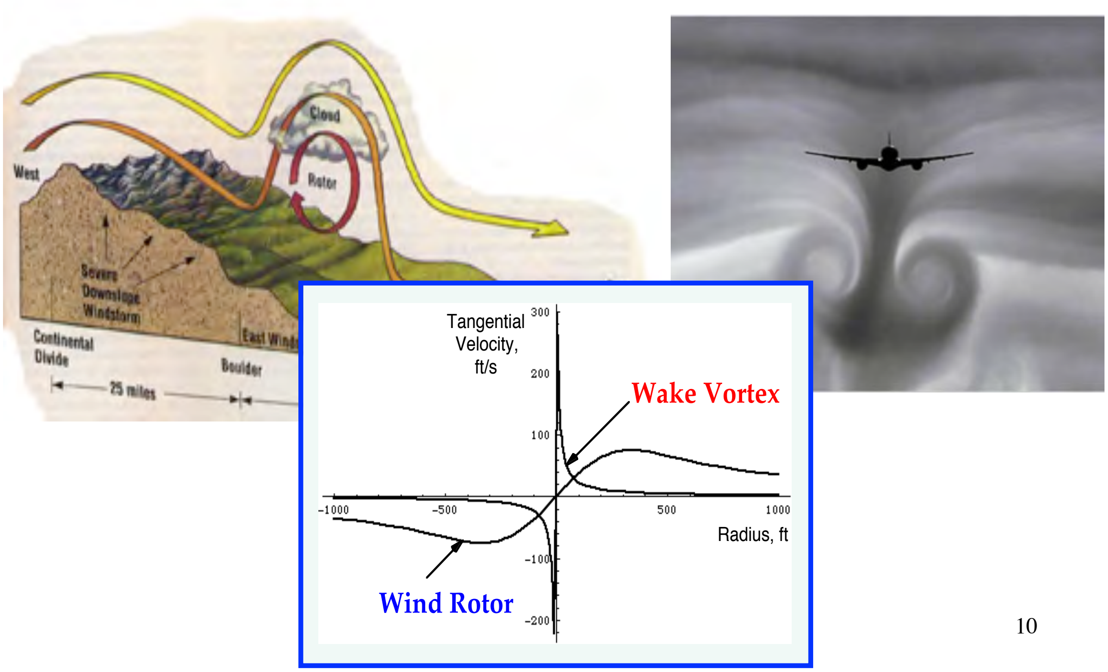
$$\begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{bmatrix} = \begin{bmatrix} \text{Side Velocity Perturbation} \\ \text{Body-Axis Roll Rate Perturbation} \\ \text{Body-Axis Yaw Rate Perturbation} \\ \text{Roll Angle about Body } x \text{ Axis Perturbation} \end{bmatrix}$$

$$\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} \Delta \delta A \\ \Delta \delta R \end{bmatrix} = \begin{bmatrix} \text{Aileron Perturbation} \\ \text{Rudder Perturbation} \end{bmatrix}$$

$$\begin{bmatrix} \Delta w_1 \\ \Delta w_2 \end{bmatrix} = \begin{bmatrix} \Delta v_{wind} \\ \Delta p_{wind} \end{bmatrix} = \begin{bmatrix} \text{Side Wind Perturbation} \\ \text{Vortical Wind Perturbation} \end{bmatrix}$$

Vortical Wind Perturbations

Wind Rotor ***Wake Vortex***



Side Velocity Dynamics

Nonlinear equation

$$\dot{v} = Y_B / m + g \sin\phi \cos\theta_N - r u_N + p w_N$$

First row of linearized dynamic equation

$$\Delta\dot{v}(t) = \left[\frac{\partial f_1}{\partial v}\Delta v(t) + \frac{\partial f_1}{\partial p}\Delta p(t) + \frac{\partial f_1}{\partial r}\Delta r(t) + \frac{\partial f_1}{\partial \phi}\Delta\phi(t)\right]$$
$$+\left[\frac{\partial f_1}{\partial \delta A}\Delta\delta A(t) + \frac{\partial f_1}{\partial \delta R}\Delta\delta R(t)\right]$$
$$+\left[\frac{\partial f_1}{\partial v_{wind}}\Delta v_{wind} + \frac{\partial f_1}{\partial p_{wind}}\Delta p_{wind}\right]$$

Side Velocity Sensitivity to State Perturbations

$$\dot{v} = Y_B / m + g \sin\phi \cos\theta_N - r u_N + p w_N$$

Coefficients in first row of F

$$\frac{\partial f_1}{\partial v} = \frac{1}{m}\left(C_{Y_v}\frac{\rho_N V_N^2}{2}S\right) \triangleq Y_v$$

$$\frac{\partial f_1}{\partial p} = \frac{1}{m}\left(C_{Y_p}\frac{\rho_N V_N^2}{2}S\right) + w_N \triangleq Y_p + w_N$$

$$\frac{\partial f_1}{\partial r} = \frac{1}{m}\left(C_{Y_r}\frac{\rho_N V_N^2}{2}S\right) - u_N \triangleq Y_r - u_N$$

$$\frac{\partial f_1}{\partial \phi} = g\cos\phi\cos\alpha_N \approx g$$

Side Velocity Sensitivity to Control and Disturbance Perturbations

Coefficients in first rows of G and L

$$\frac{\partial f_1}{\partial \delta A} = \frac{1}{m}\left[C_{Y_{\delta A}} \frac{\rho_N V_N^{\,2}}{2} S \right] \triangleq Y_{\delta A}$$

$$\frac{\partial f_1}{\partial \delta R} = \frac{1}{m}\left[C_{Y_{\delta R}} \frac{\rho_N V_N^{\,2}}{2} S \right] \triangleq Y_{\delta R}$$

$$\frac{\partial f_1}{\partial v_{wind}} = -\frac{\partial f_1}{\partial v}$$

$$\frac{\partial f_1}{\partial p_{wind}} = -\frac{\partial f_1}{\partial p}$$

Roll Rate Dynamics

Nonlinear equation

$$\dot{p} = \left(I_{zz} L_B + I_{xz} N_B \right) \Big/ \left(I_{xx} I_{zz} - I_{xz}^2 \right)$$

Second row of linearized dynamic equation

$$\begin{aligned}
\Delta \dot{p}(t) = & \left[\frac{\partial f_2}{\partial v} \Delta v(t) + \frac{\partial f_2}{\partial p} \Delta p(t) + \frac{\partial f_2}{\partial r} \Delta r(t) + \frac{\partial f_2}{\partial \phi} \Delta \phi(t) \right] \\
& + \left[\frac{\partial f_2}{\partial \delta A} \Delta \delta A(t) + \frac{\partial f_2}{\partial \delta R} \Delta \delta R(t) \right] \\
& + \left[\frac{\partial f_2}{\partial v_{wind}} \Delta v_{wind} + \frac{\partial f_2}{\partial p_{wind}} \Delta p_{wind} \right]
\end{aligned}$$

Roll Rate Sensitivity to State Perturbations

$$\dot{p} = \left(I_{zz}L_B + I_{xz}N_B\right) / \left(I_{xx}I_{zz} - I_{xz}^2\right)$$

Coefficients in second row of F

$$\frac{\partial f_2}{\partial v} = \left(I_{zz}\frac{\partial L_B}{\partial v} + I_{xz}\frac{\partial N_B}{\partial v} \right) / \left(I_{xx}I_{zz} - I_{xz}^2\right) \triangleq L_v$$

$$\frac{\partial f_2}{\partial p} = \left(I_{zz}\frac{\partial L_B}{\partial p} + I_{xz}\frac{\partial N_B}{\partial p} \right) / \left(I_{xx}I_{zz} - I_{xz}^2\right) \triangleq L_p$$

$$\frac{\partial f_2}{\partial r} = \left(I_{zz}\frac{\partial L_B}{\partial r} + I_{xz}\frac{\partial N_B}{\partial r} \right) / \left(I_{xx}I_{zz} - I_{xz}^2\right) \triangleq L_r$$

$$\frac{\partial f_2}{\partial \phi} = 0$$

Yaw Rate Dynamics

Nonlinear equation

$$\dot{r} = \left(I_{xz}L_B + I_{xx}N_B\right) / \left(I_{xx}I_{zz} - I_{xz}^2\right)$$

Third row of linearized dynamic equation

$$\begin{aligned}\Delta\dot{r}(t) = &\left[\frac{\partial f_3}{\partial v}\Delta v(t) + \frac{\partial f_3}{\partial p}\Delta p(t) + \frac{\partial f_3}{\partial r}\Delta r(t) + \frac{\partial f_3}{\partial \phi}\Delta \phi(t) \right] \\ &+ \left[\frac{\partial f_3}{\partial \delta A}\Delta \delta A(t) + \frac{\partial f_3}{\partial \delta R}\Delta \delta R(t) \right] \\ &+ \left[\frac{\partial f_3}{\partial v_{wind}}\Delta v_{wind} + \frac{\partial f_3}{\partial p_{wind}}\Delta p_{wind} \right]\end{aligned}$$

Yaw Rate Sensitivity to State Perturbations

$$\dot{r} = \left(I_{xz} L_B + I_{xx} N_B\right) \Big/ \left(I_{xx} I_{zz} - I_{xz}^2\right)$$

Coefficients in third row of F

$$\frac{\partial f_3}{\partial v} = \left(I_{xz} \frac{\partial L_B}{\partial v} + I_{xx} \frac{\partial N_B}{\partial v} \right) \Big/ \left(I_{xx} I_{zz} - I_{xz}^2 \right) \triangleq N_v$$

$$\frac{\partial f_3}{\partial p} = \left(I_{xz} \frac{\partial L_B}{\partial p} + I_{xx} \frac{\partial N_B}{\partial p} \right) \Big/ \left(I_{xx} I_{zz} - I_{xz}^2 \right) \triangleq N_p$$

$$\frac{\partial f_3}{\partial r} = \left(I_{xz} \frac{\partial L_B}{\partial r} + I_{xx} \frac{\partial N_B}{\partial r} \right) \Big/ \left(I_{xx} I_{zz} - I_{xz}^2 \right) \triangleq N_r$$

$$\frac{\partial f_3}{\partial \phi} = 0$$

Roll Angle Dynamics

Nonlinear equation

$$\dot{\phi} = p + \left(r \cos\phi \right) \tan\theta_N$$

Fourth row of linearized dynamic equation

$$\begin{aligned}
\Delta\dot{\phi}(t) = & \left[\frac{\partial f_4}{\partial v} \Delta v(t) + \frac{\partial f_4}{\partial p} \Delta p(t) + \frac{\partial f_4}{\partial r} \Delta r(t) + \frac{\partial f_4}{\partial \phi} \Delta\phi(t) \right] \\
& + \left[\frac{\partial f_4}{\partial \delta A} \Delta\delta A(t) + \frac{\partial f_4}{\partial \delta R} \Delta\delta R(t) \right] \\
& + \left[\frac{\partial f_4}{\partial v_{wind}} \Delta v_{wind} + \frac{\partial f_4}{\partial p_{wind}} \Delta p_{wind} \right]
\end{aligned}$$

Roll Angle Sensitivity to State Perturbations

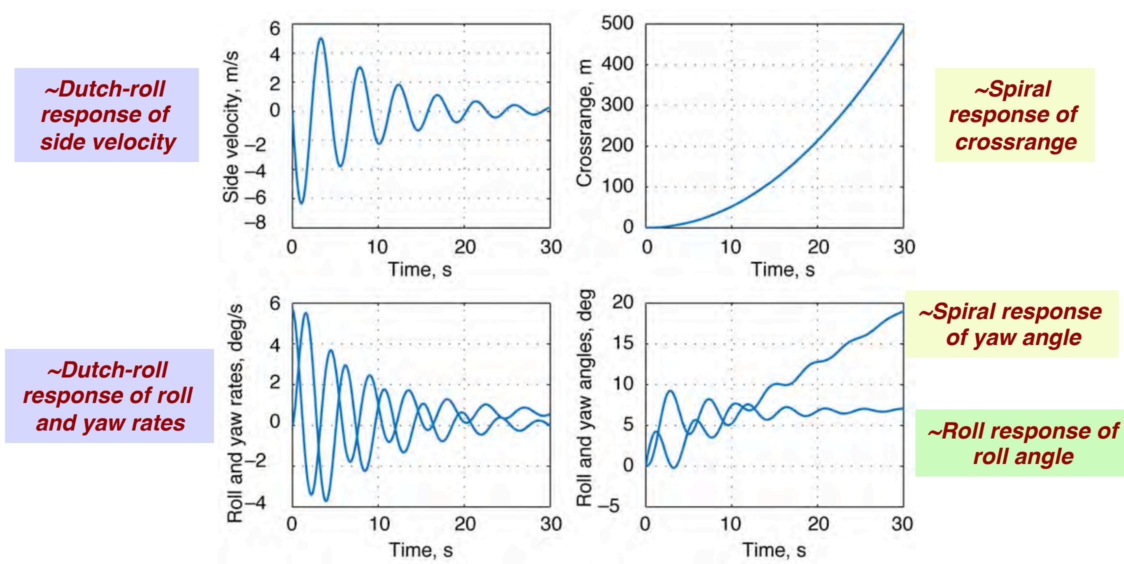
$$\boxed{\dot{\phi} = p + \left(r\cos\phi\right)\tan\theta_N}$$

Coefficients in fourth row of F

$$\boxed{\frac{\partial f_4}{\partial v} = 0} \qquad \boxed{\frac{\partial f_4}{\partial r} = \cos\phi_N \tan\theta_N \approx \tan\theta_N}$$

$$\boxed{\frac{\partial f_4}{\partial p} = 1} \qquad \boxed{\frac{\partial f_4}{\partial \phi} = r_N \sin\phi_N \tan\theta_N = 0}$$

Linearized Lateral-Directional Response to Initial Yaw Rate



Dimensional Stability Derivatives

Stability Matrix

$$\begin{bmatrix} \partial f_1/\partial v & \partial f_1/\partial p & \partial f_1/\partial r & \partial f_1/\partial \phi \\ \partial f_2/\partial v & \partial f_2/\partial p & \partial f_2/\partial r & \partial f_2/\partial \phi \\ \partial f_3/\partial v & \partial f_3/\partial p & \partial f_3/\partial r & \partial f_3/\partial \phi \\ \partial f_4/\partial v & \partial f_4/\partial p & & \end{bmatrix}$$

$$= \begin{bmatrix} Y_v & \left(Y_p + w_N\right) & \left(Y_r - u_N\right) & g\cos\theta_N \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & \tan\theta_N & 0 \end{bmatrix}$$

Dimensional Control- and Disturbance-Effect Derivatives

Control Effect Matrix

$$\begin{bmatrix} \partial f_1/\partial\delta A & \partial f_1/\partial\delta R \\ \partial f_2/\partial\delta A & \partial f_2/\partial\delta R \\ \partial f_3/\partial\delta A & \partial f_3/\partial\delta R \\ \partial f_4/\partial\delta A & \partial f_4/\partial\delta R \end{bmatrix} = \begin{bmatrix} Y_{\delta A} & Y_{\delta R} \\ L_{\delta A} & L_{\delta R} \\ N_{\delta A} & N_{\delta R} \\ 0 & 0 \end{bmatrix}$$

Disturbance Effect Matrix

$$\begin{bmatrix} \partial f_1/\partial v_{wind} & \partial f_1/\partial p_{wind} \\ \partial f_2/\partial v_{wind} & \partial f_2/\partial p_{wind} \\ \partial f_3/\partial v_{wind} & \partial f_3/\partial p_{wind} \\ \partial f_4/\partial v_{wind} & \partial f_4/\partial p_{wind} \end{bmatrix} = \begin{bmatrix} Y_v & Y_p \\ L_v & L_p \\ N_v & N_p \\ 0 & 0 \end{bmatrix}$$

LTI Body-Axis Perturbation Equations of Motion

Rolling and yawing motions

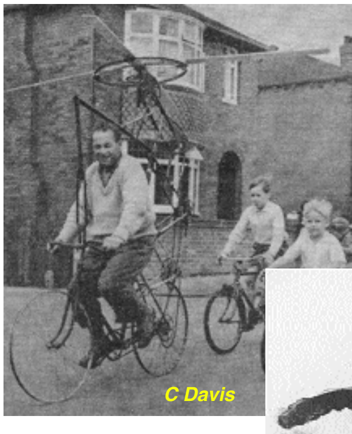
$$
\begin{bmatrix} \Delta\dot{v}(t) \\ \Delta\dot{p}(t) \\ \Delta\dot{r}(t) \\ \Delta\dot{\phi}(t) \end{bmatrix} = \begin{bmatrix} Y_v & (Y_p + w_N) & (Y_r - u_N) & g\cos\theta_N \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & \tan\theta_N & 0 \end{bmatrix} \begin{bmatrix} \Delta v(t) \\ \Delta p(t) \\ \Delta r(t) \\ \Delta\phi(t) \end{bmatrix}
$$

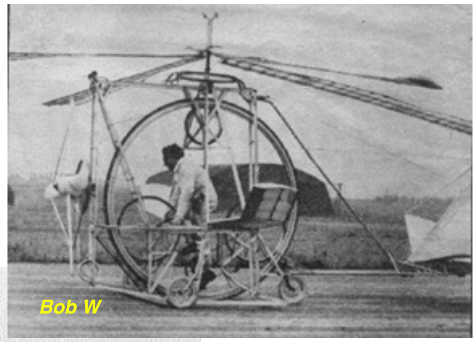
$$
+ \begin{bmatrix} Y_{\delta A} & Y_{\delta R} \\ L_{\delta A} & L_{\delta R} \\ N_{\delta A} & N_{\delta R} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta A(t) \\ \Delta\delta R(t) \end{bmatrix} + \begin{bmatrix} Y_v & Y_p \\ L_v & L_p \\ N_v & N_p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta v_{wind} \\ \Delta p_{wind} \end{bmatrix}
$$

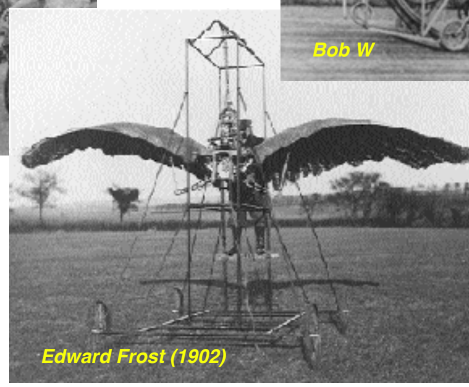
Mythical/Historical Factoids

Daedalus and Icarus, father and son, Attempt to escape from Crete *(<630 BC)*

Other human-powered airplanes that didn't work

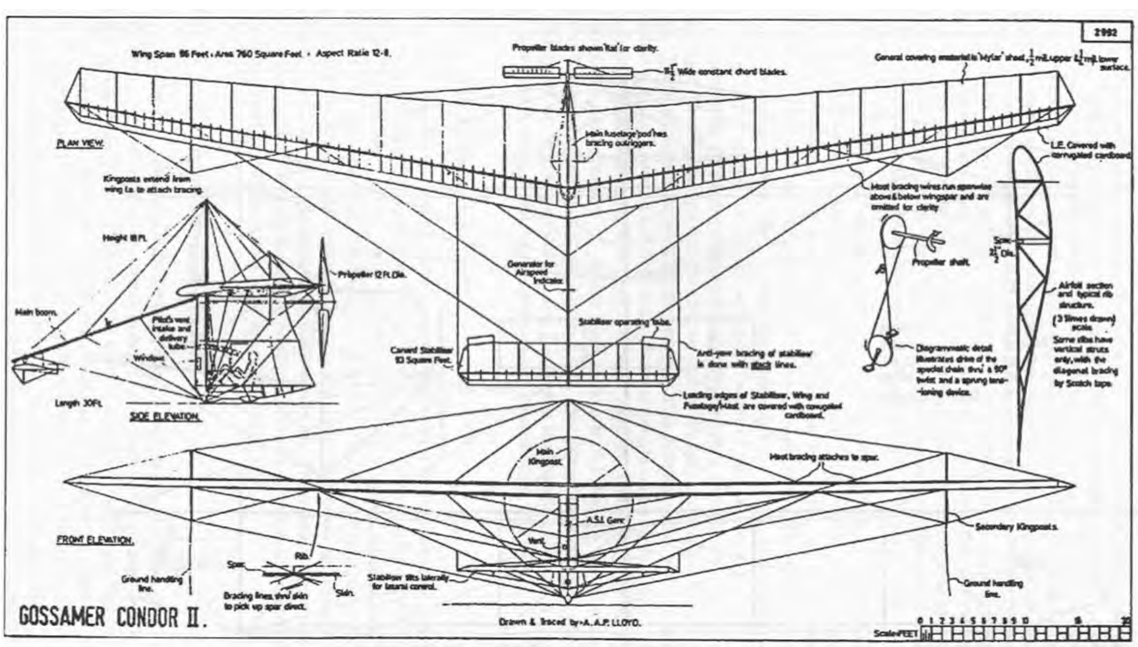






AeroVironment Gossamer Condor

Winner of the 1st Kremer Prize: *Figure 8* around pylons half-mile apart (1977)



AeroVironment Gossamer Albatross and MIT Monarch

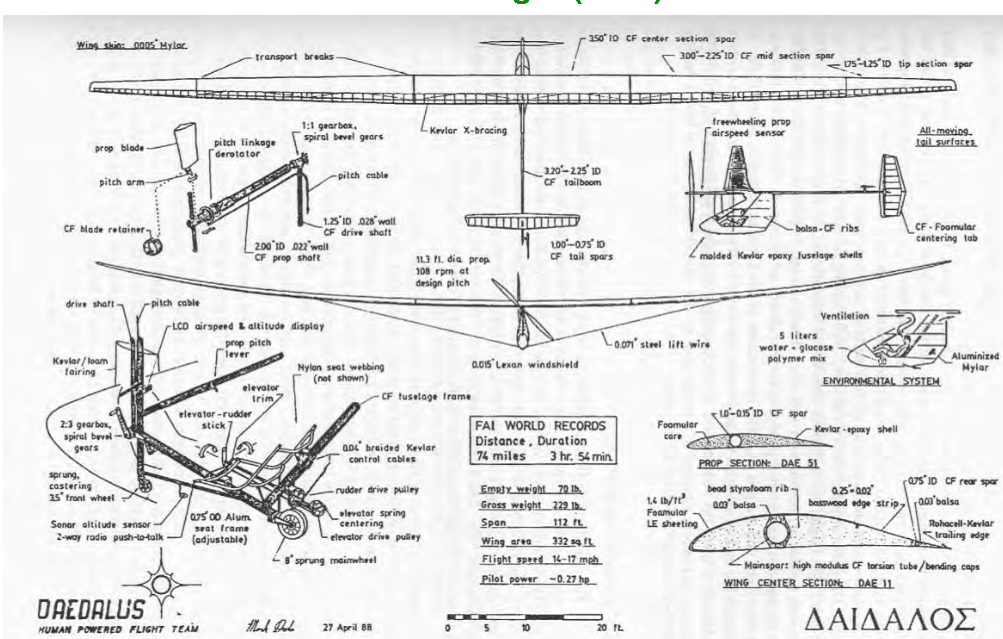
2nd Kremer Prize: crossing the English Channel (1979)

3rd Kremer Prize: 1500 m on closed course in < 3 min (1984)

Gene Larrabee, 'Mr. Propeller' of human-powered flight, dies at 82 (1/11/2003)
"The Albatross' pilot could stay aloft only 10 minutes at first. With (Larrabee's) propeller, he stayed up for over an hour on his first flight …. There was no way the Albatross could cross the Channel, which took almost three hours, without (Larrabee's) propeller." *(D. Wilson)*

MIT Daedalus

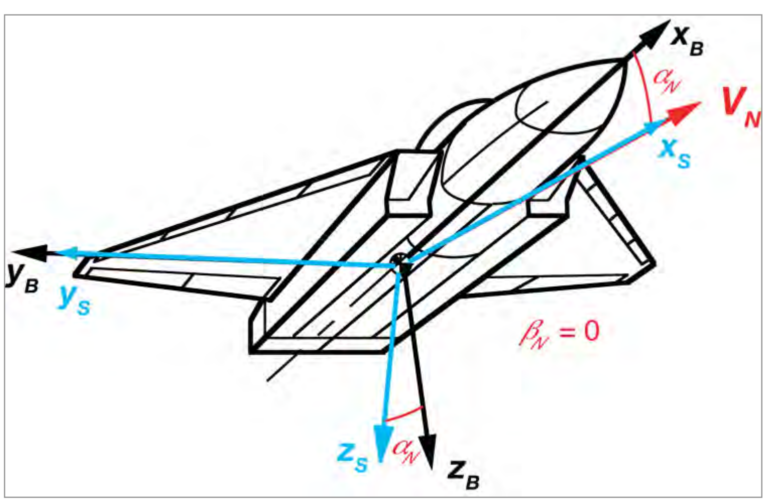
Flew 74 miles across the Aegean Sea, completing Daedalus' s intended flight (1988)



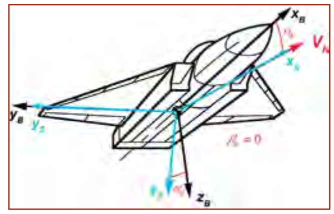
Stability Axis Representation of Dynamics

Stability Axes

- Stability axes are an alternative set of body axes
- Nominal *x* axis is offset from the body centerline by the nominal angle of attack, α_N



Transformation from Original Body Axes to Stability Axes



$$\mathbf{H}_B^S = \begin{bmatrix} \cos\alpha_N & 0 & \sin\alpha_N \\ 0 & 1 & 0 \\ -\sin\alpha_N & 0 & \cos\alpha_N \end{bmatrix}$$

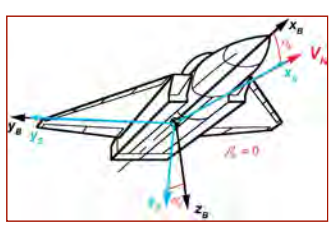
$$\begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{bmatrix}_S = H_B^S \begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{bmatrix}_B \qquad \begin{bmatrix} \Delta p \\ \Delta q \\ \Delta r \end{bmatrix}_S = H_B^S \begin{bmatrix} \Delta p \\ \Delta q \\ \Delta r \end{bmatrix}_B$$

Side velocity (Δv) and pitch rate (Δq) are unchanged by the transformation

Stability-Axis State Vector

Rotate body axes to stability axes

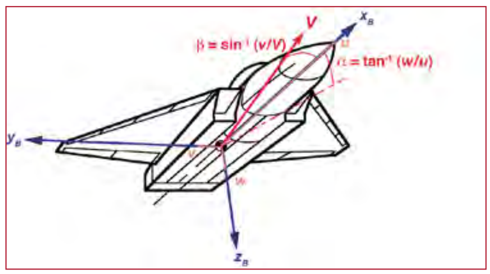
$$\begin{bmatrix} \Delta v(t) \\ \Delta p(t) \\ \Delta r(t) \\ \Delta \phi(t) \end{bmatrix}_{Body-Axis} \Rightarrow \measuredangle\alpha_N \Rightarrow \begin{bmatrix} \Delta v(t) \\ \Delta p(t) \\ \Delta r(t) \\ \Delta \phi(t) \end{bmatrix}_{Stability-Axis}$$



Stability-Axis State Vector

Replace side velocity by sideslip angle

$$\begin{bmatrix} \Delta v(t) \\ \Delta p(t) \\ \Delta r(t) \\ \Delta \phi(t) \end{bmatrix}_{Stability-Axis} \Rightarrow \Delta\beta \approx \frac{\Delta v}{V_N} \Rightarrow \begin{bmatrix} \Delta \beta(t) \\ \Delta p(t) \\ \Delta r(t) \\ \Delta \phi(t) \end{bmatrix}_{Stability-Axis}$$



Stability-Axis State Vector

Revise state order

$$\begin{bmatrix} \Delta \beta(t) \\ \Delta p(t) \\ \Delta r(t) \\ \Delta \phi(t) \end{bmatrix}_{Stability-Axis} \Rightarrow \begin{bmatrix} \Delta r(t) \\ \Delta \beta(t) \\ \Delta p(t) \\ \Delta \phi(t) \end{bmatrix}_{Stability-Axis}$$

$$= \begin{bmatrix} \text{Stability-Axis Yaw Rate Perturbation} \\ \text{Sideslip Angle Perturbation} \\ \text{Stability-Axis Roll Rate Perturbation} \\ \text{Stability-Axis Roll Angle Perturbation} \end{bmatrix}$$

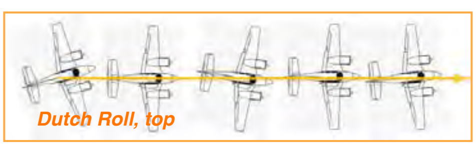
Stability-Axis Lateral-Directional Equations

$$
\begin{bmatrix} \Delta\dot{r}(t) \\ \Delta\dot{\beta}(t) \\ \Delta\dot{p}(t) \\ \Delta\dot{\phi}(t) \end{bmatrix}_S =
\begin{bmatrix} N_r & N_\beta & N_p & 0 \\ \left(\dfrac{Y_r}{V_N}-1\right) & \dfrac{Y_\beta}{V_N} & \dfrac{Y_p}{V_N} & \dfrac{g}{V_N} \\ L_r & L_\beta & L_p & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_S
\begin{bmatrix} \Delta r(t) \\ \Delta\beta(t) \\ \Delta p(t) \\ \Delta\phi(t) \end{bmatrix}_S
$$

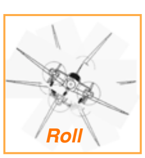
$$
+\begin{bmatrix} N_{\delta A} & N_{\delta R} \\ \dfrac{Y_{\delta A}}{V_N} & \dfrac{Y_{\delta R}}{V_N} \\ L_{\delta A} & L_{\delta R} \\ 0 & 0 \end{bmatrix}_S
\begin{bmatrix} \Delta\delta A(t) \\ \Delta\delta R(t) \end{bmatrix}
+\begin{bmatrix} N_\beta & N_p \\ \dfrac{Y_\beta}{V_N} & \dfrac{Y_p}{V_N} \\ L_\beta & L_p \\ 0 & 0 \end{bmatrix}_S
\begin{bmatrix} \Delta\beta_{wind} \\ \Delta p_{wind} \end{bmatrix}
$$

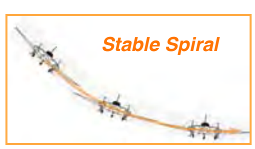
Why Modify the Equations?

Dutch-roll motion is primarily described by stability-axis yaw rate and sideslip angle



Roll and spiral motions are primarily described by stability-axis roll rate and roll angle





Why Modify the Equations?

Effects of **Dutch roll** perturbations on **Dutch roll** motion

Effects of **roll-spiral** perturbations on **Dutch roll** motion

$$\mathbf{F}_{LD} = \begin{bmatrix} \mathbf{F}_{DR} & \mathbf{F}_{RS}^{DR} \\ \mathbf{F}_{DR}^{RS} & \mathbf{F}_{RS} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{DR} & small \\ small & \mathbf{F}_{RS} \end{bmatrix} \approx \begin{bmatrix} \mathbf{F}_{DR} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{RS} \end{bmatrix}$$

Effects of **Dutch roll** perturbations on **roll-spiral** motion

Effects of **roll-spiral** perturbations on **roll-spiral** motion

... but are the off-diagonal blocks really small?

Stability, Control, and Disturbance Matrices

$$\mathbf{F}_{LD} = \begin{bmatrix} \mathbf{F}_{DR} & \mathbf{F}_{RS}^{DR} \\ \mathbf{F}_{DR}^{RS} & \mathbf{F}_{RS} \end{bmatrix} = \left[\begin{array}{cc|cc} N_r & N_\beta & N_p & 0 \\ \left(\frac{Y_r}{V_N} - 1\right) & \frac{Y_\beta}{V_N} & \frac{Y_p}{V_N} & \frac{g}{V_N} \\ \hline L_r & L_\beta & L_p & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} = \begin{bmatrix} \Delta r \\ \Delta \beta \\ \Delta p \\ \Delta \phi \end{bmatrix}$$

$$\mathbf{G}_{LD} = \left[\begin{array}{c|c} N_{\delta A} & N_{\delta R} \\ \frac{Y_{\delta A}}{V_N} & \frac{Y_{\delta R}}{V_N} \\ \hline L_{\delta A} & L_{\delta R} \\ 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} \Delta \delta A \\ \Delta \delta R \end{bmatrix}$$

$$\mathbf{L}_{LD} = \left[\begin{array}{c|c} N_\beta & N_p \\ \frac{Y_\beta}{V_N} & \frac{Y_p}{V_N} \\ \hline L_\beta & L_p \\ 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} \Delta w_1 \\ \Delta w_2 \end{bmatrix} = \begin{bmatrix} \Delta \delta A \\ \Delta \delta R \end{bmatrix}$$

2^{nd}-Order Approximate Modes of Lateral-Directional Motion

2^{nd}-Order Approximations in System Matrices

$$\mathbf{F}_{LD} = \begin{bmatrix} \mathbf{F}_{DR} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{RS} \end{bmatrix} = \left[\begin{array}{cc|cc} N_r & N_\beta & 0 & 0 \\ \left(\dfrac{Y_r}{V_N} - 1\right) & \dfrac{Y_\beta}{V_N} & 0 & 0 \\ \hline 0 & 0 & L_p & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\mathbf{G}_{LD} = \left[\begin{array}{c|c} N_{\delta R} & 0 \\ \dfrac{Y_{\delta R}}{V_N} & 0 \\ \hline 0 & L_{\delta A} \\ 0 & 0 \end{array}\right] \qquad \mathbf{L}_{LD} = \left[\begin{array}{c|c} N_\beta & 0 \\ \dfrac{Y_\beta}{V_N} & 0 \\ \hline 0 & L_p \\ 0 & 0 \end{array}\right]$$

2^{nd}-Order Models of Lateral-Directional Motion

Approximate Dutch Roll Equation

$$\Delta\dot{\mathbf{x}}_{DR} = \begin{bmatrix} \Delta\dot{r} \\ \Delta\dot{\beta} \end{bmatrix} \approx \begin{bmatrix} N_r & N_\beta \\ \left(\frac{Y_r}{V_N} - 1\right) & \frac{Y_\beta}{V_N} \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta\beta \end{bmatrix} + \begin{bmatrix} N_{\delta R} \\ \frac{Y_{\delta R}}{V_N} \end{bmatrix} \Delta\delta R + \begin{bmatrix} N_\beta \\ \frac{Y_\beta}{V_N} \end{bmatrix} \Delta\beta_{wind}$$

Approximate Spiral-Roll Equation

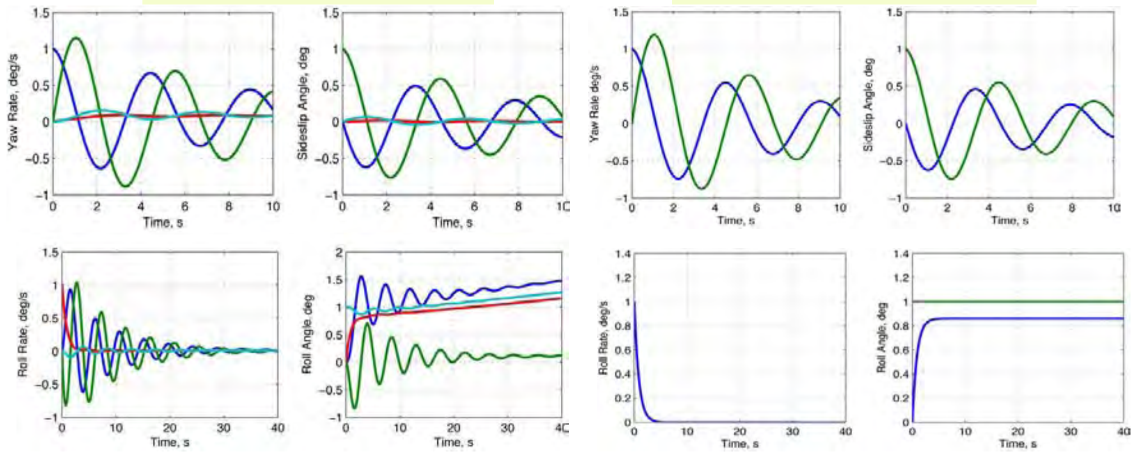
$$\Delta\dot{\mathbf{x}}_{RS} = \begin{bmatrix} \Delta\dot{p} \\ \Delta\dot{\phi} \end{bmatrix} \approx \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta\phi \end{bmatrix} + \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix} \Delta\delta A + \begin{bmatrix} L_p \\ 0 \end{bmatrix} \Delta p_{wind}$$

Comparison of 4^{th}- and 2^{nd}-Order Dynamic Models

Comparison of 2nd- and 4th-Order Initial-Condition Responses of Business Jet

4 initial conditions [$r(0)$, $\beta(0)$, $p(0)$, $\phi(0)$]

Fourth-Order Response **_Second-Order Response_**



- Speed and damping of responses is adequately portrayed by 2nd-order models
- Roll-spiral modes have little effect on yaw rate and sideslip angle responses
- BUT Dutch roll mode has large effect on roll rate and roll angle responses

Next Time: Analysis of Time Response

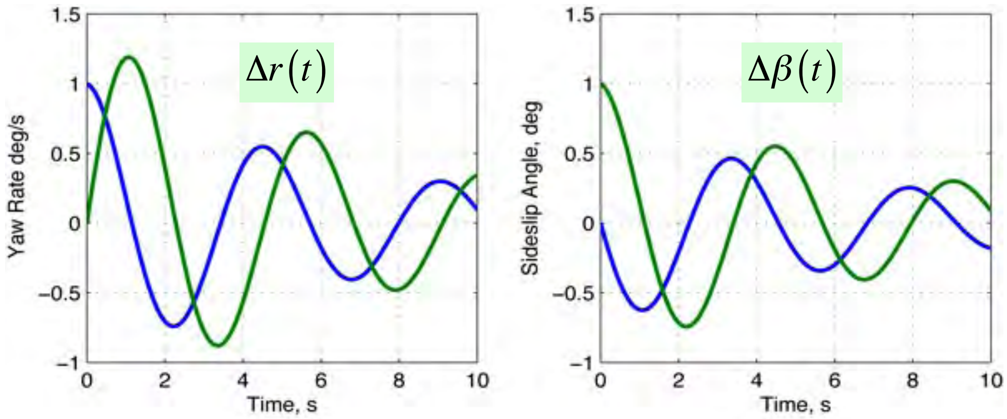
Reading:
Flight Dynamics
298-313, 338-342

Supplemental Material

Primary Lateral-Directional Control Derivatives

$$\boxed{\begin{aligned} L_{\delta A} &= C_{l_{\delta A}}\left(\frac{\rho_N V_N^2}{2I_{xx}}\right)Sb \\ N_{\delta R} &= C_{n_{\delta R}}\left(\frac{\rho_N V_N^2}{2I_{zz}}\right)Sb \end{aligned}}$$

Initial Condition Response of Approximate Dutch Roll Mode



Dutch roll Video

Calspan Variable-Stability Learjet Dutch roll Oscillation
http://www.youtube.com/watch?v=1_TW9oz99NQ