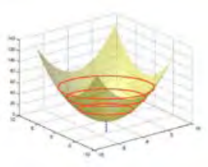
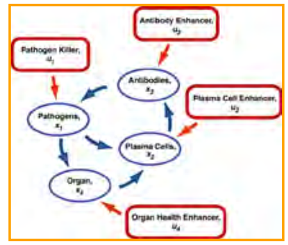


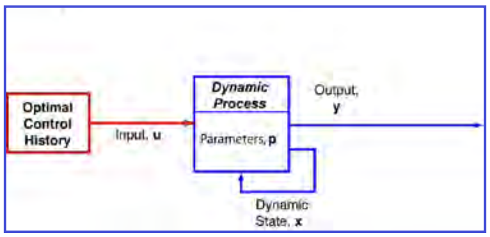
Dynamic Optimal Control

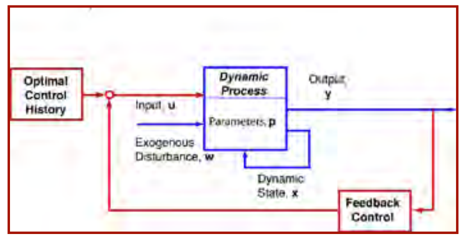
Robert Stengel
Robotics and Intelligent Systems MAE 345, Princeton University, 2015

Learning Objectives

- Examples of cost functions
- Necessary conditions for optimality
- Calculation of optimal trajectories
- Design of optimal feedback control laws







Integrated Effect can be a Scalar "Cost"

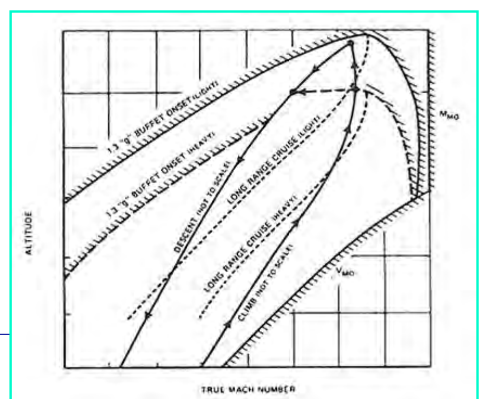
Time

$$J = \int_0^{\text{final time}} (1)\,dt$$

Fuel

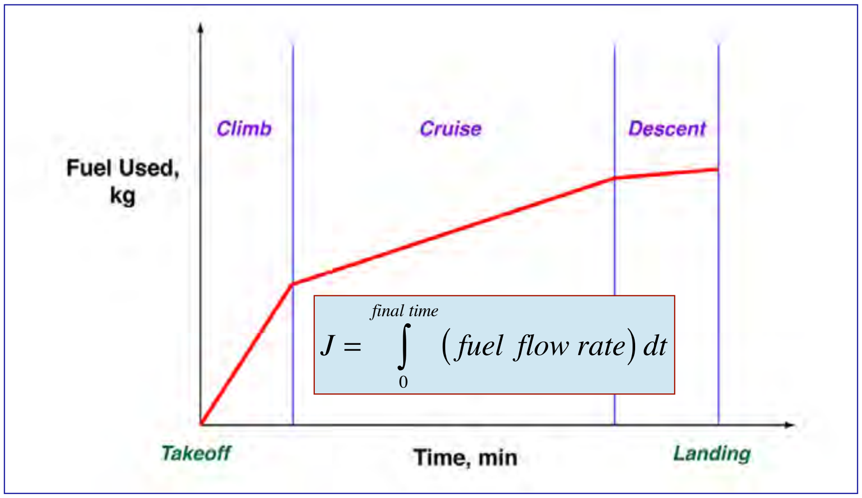
$$J = \int_0^{\text{final range}} (\text{fuel use per kilometer})\,dR$$

Financial cost of time and fuel



$$J = \int_0^{\text{final time}} (\text{cost per hour})\,dt$$
$$= \int_0^{\text{final time}} \left[\left(\frac{\text{cost}}{\text{hour}}\right) + \left(\frac{\text{cost}}{\text{liter}}\right)\left(\frac{\text{liter}}{\text{kilometer}}\right)\frac{dR}{dt} \right] dt$$

Cost Accumulates from Start to Finish



Optimal System Regulation

Cost functions that penalize state deviations over a time interval:

Quadratic scalar variation

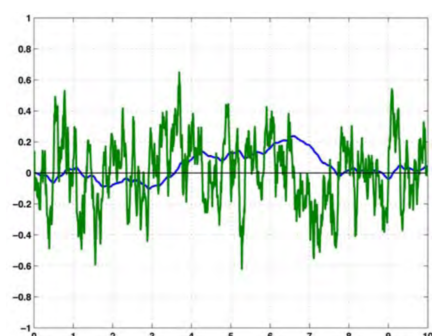
$$J = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\Delta x^2 \right) dt < \infty$$

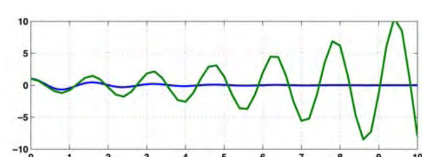
Vector variation

$$J = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\Delta \mathbf{x}^T \Delta \mathbf{x} \right) dt < \infty$$

Weighted vector variation

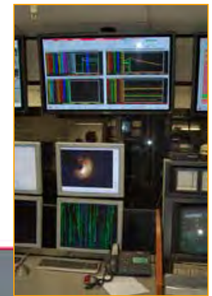
$$J = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\Delta \mathbf{x}^T \mathbf{Q} \Delta \mathbf{x} \right) dt < \infty$$

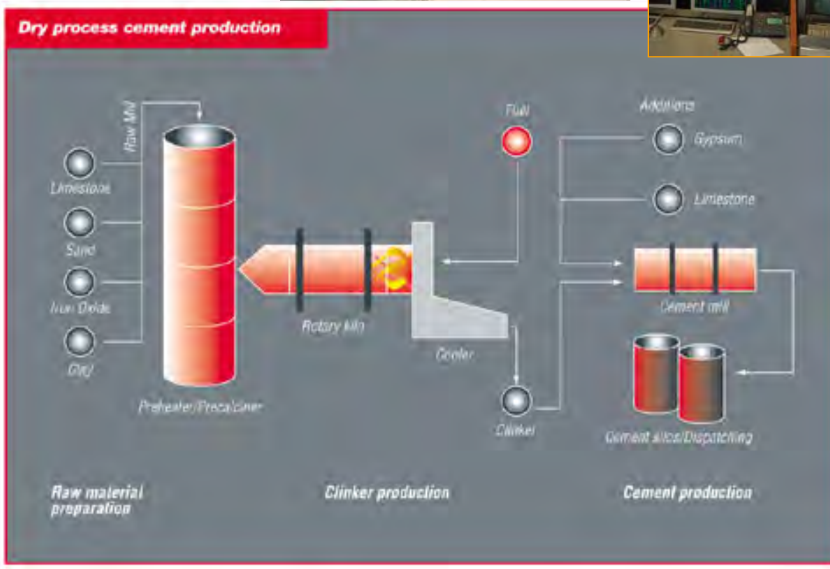




- *No penalty for control use*
- *Why not use infinite control?*

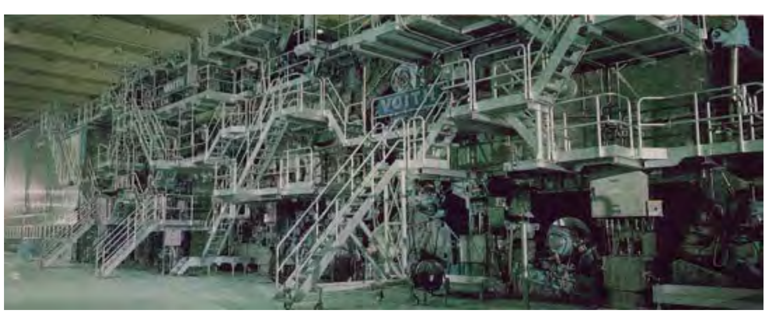
Cement Kiln





Pulp & Paper Machines

- **Machine length: ~ 2 football fields**
- **Paper speed ≤ 2,200 m/min = 80 mph**
- **Maintain 3-D paper quality**
- **Avoid paper breaks at all cost!**



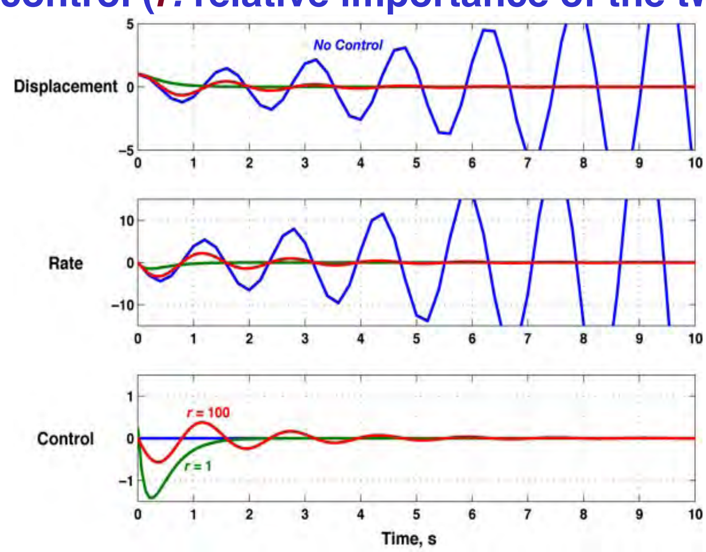
Hazardous Waste Generated by Large Industrial Plants

- **Cement dust**
- **Coal fly ash**
- **Metal emissions**
- **Dioxin**
- **“Electroscrap” and other hazardous waste**
- **Waste chemicals**
- **Ground water contamination**
- **Ancillary mining and logging issues**
- **“Greenhouse” gasses**
- **Need to optimize <u>total</u> cost-benefit of production processes (including environmental cost)**

Tradeoffs Between Performance and Control in Integrated Cost Function

Trade performance against control usage

Minimize a cost function that contains state and control (*r:* relative importance of the two)



Dynamic Optimization:
The Optimal Control Problem

Minimize a scalar function, *J*, of terminal and integral costs

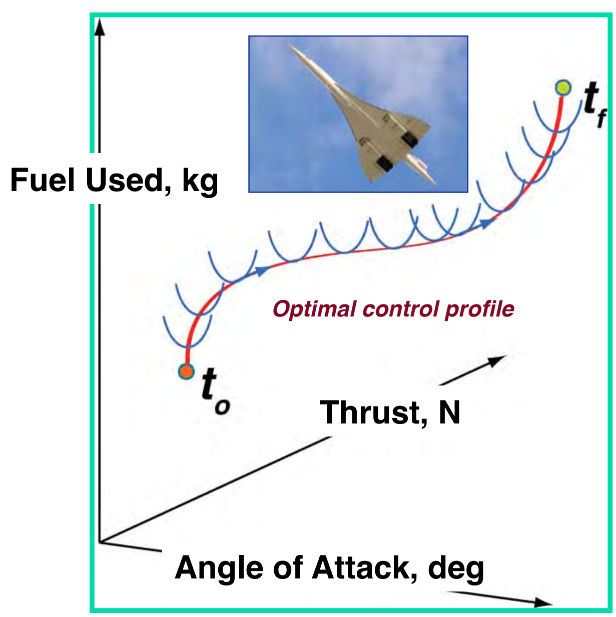
$$\min_{\mathbf{u}(t)} J = \min_{\mathbf{u}(t)} \left\{ \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right] dt \right\}$$

with respect to the control, $\mathbf{u}(t)$, in (t_o, t_f), subject to a dynamic constraint

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)], \quad \mathbf{x}(t_o)\ given$$

dim($\mathbf{x}$) = n x 1
dim($\mathbf{f}$) = n x 1
dim($\mathbf{u}$) = m x 1

Example of Dynamic Optimization



Any deviation from optimal thrust and angle-of-attack profiles would increase total fuel used

Components of the Cost Function

Integral cost is a function of the state and control from start to finish

$$\int_{t_o}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right]dt \quad \textbf{positive scalar function of two vectors}$$

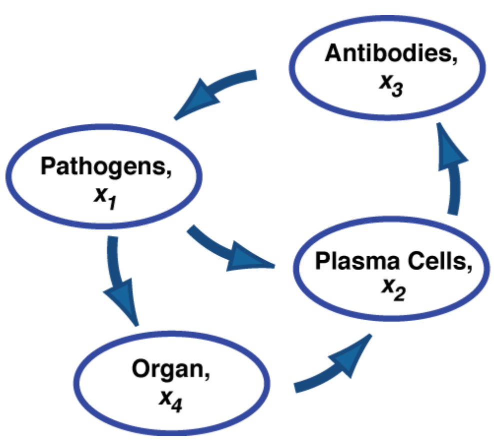
$$L\left[\mathbf{x}(t),\mathbf{u}(t)\right]: \quad \textbf{Lagrangian} \text{ of the cost function}$$

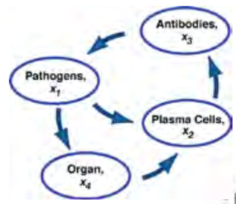
Terminal cost is a function of the state at the final time

$$\phi\left[\mathbf{x}(t_f)\right] \quad \textbf{positive scalar function of a vector}$$

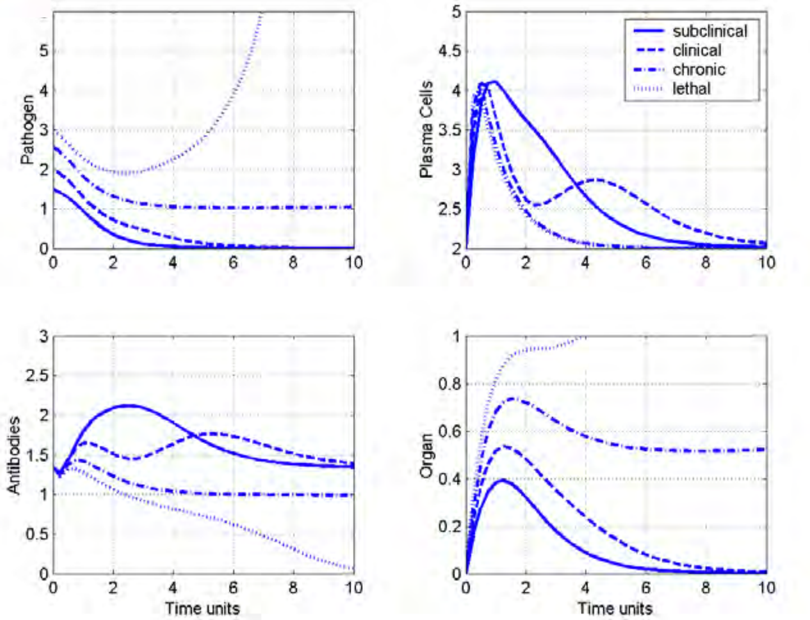
Example: Dynamic Model of Infection and Immune Response

- x_1 = Concentration of a pathogen, which displays antigen
- x_2 = Concentration of plasma cells, which are carriers and producers of antibodies
- x_3 = Concentration of antibodies, which recognize antigen and kill pathogen
- x_4 = Relative characteristic of a damaged organ [0 = healthy, 1 = dead]





Natural Response to Pathogen Assault (No Therapy)



Cost Function Considers Infection, Organ Health, and Drug Usage

$$\min_{\mathbf{u}(t)} J = \min_{\mathbf{u}(t)} \left\{ \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right] dt \right\}$$

$$= \min_{u} \left[\frac{1}{2}\left(s_{11}x_{1_f}{}^2 + s_{44}x_{4_f}{}^2\right) + \frac{1}{2}\int_{t_o}^{t_f}\left(q_{11}x_1{}^2 + q_{44}x_4{}^2 + ru^2\right)dt \right]$$

- **Tradeoffs between final values, integral values over a fixed time interval, state, and control**
- **Cost function includes weighted square values of**
  - **Final concentration of the pathogen**
  - **Final health of the damaged organ (0 is good, 1 is bad)**
  - **Integral of pathogen concentration**
  - **Integral health of the damaged organ (0 is good, 1 is bad)**
  - **Integral of drug usage**
- **Drug cost may reflect physiological or financial cost**

Necessary Conditions for Optimal Control

Augment the Cost Function

- **Must express sensitivity of the cost to the dynamic response**
- **Adjoin *dynamic constraint* to *integrand* using *Lagrange multiplier*, $\boldsymbol{\lambda}(t)$**
  - **Same dimension as the dynamic constraint, [*n* x 1]**
  - **Constraint = 0 if the dynamic equation is satisfied**

$$J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} \left\{ L\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \boldsymbol{\lambda}^T(t)\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] - \frac{d\mathbf{x}(t)}{dt}\right]\right\} dt$$

Define Hamiltonian, *H*[.]

$$\boxed{H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) \triangleq L(\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x},\mathbf{u})}$$

Substitute the Hamiltonian in the Cost Function

Substitute the Hamiltonian in the cost function

$$J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} \left\{ H\left[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)\right] - \boldsymbol{\lambda}^T(t) \frac{d\mathbf{x}(t)}{dt} \right\} dt$$

The optimal cost, *J, is produced by the optimal histories of state, control, and Lagrange multiplier**

$$\min_{\mathbf{u}(t)} J = J^* = \phi\left[\mathbf{x}^*(t_f)\right] + \int_{t_o}^{t_f} \left\{ H\left[\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)\right] - \boldsymbol{\lambda}^{*T}(t) \frac{d\mathbf{x}^*(t)}{dt} \right\} dt$$

Integration by Parts

- **Scalar indefinite integral**

$$\int u\, dv = uv - \int v\, du$$

- **Vector definite integral**

$$u = \boldsymbol{\lambda}^T(t)$$
$$dv = \dot{\mathbf{x}}(t)dt = d\mathbf{x}$$

$$\int_{t_0}^{t_f} \boldsymbol{\lambda}^T(t) \frac{d\mathbf{x}(t)}{dt} dt = \boldsymbol{\lambda}^T(t)\mathbf{x}(t)\Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d\boldsymbol{\lambda}^T(t)}{dt} \mathbf{x}(t)\, dt$$

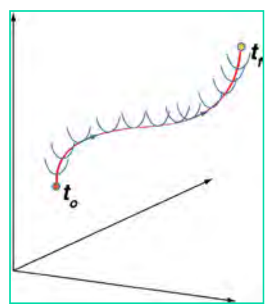
The Optimal Control Solution

- **Along the optimal trajectory, the cost, *J****, **should be insensitive to small variations in control policy**
  - **To first order,**

$$\Delta J^* = \left\{\left[\frac{\partial \phi}{\partial x} - \boldsymbol{\lambda}^T\right]\right\}\Delta\mathbf{x}(\Delta\mathbf{u})\Big|_{t=t_f} + \left[\boldsymbol{\lambda}^T\Delta\mathbf{x}(\Delta\mathbf{u})\right]\Big|_{t=t_o}$$
$$+\int_{t_o}^{t_f}\left\{\frac{\partial H}{\partial \mathbf{u}}\Delta\mathbf{u} + \left[\frac{\partial H}{\partial \mathbf{x}} + \frac{d\boldsymbol{\lambda}^T}{dt}\right]\Delta\mathbf{x}(\Delta\mathbf{u})\right\}dt = 0$$

Setting $\Delta J^* = 0$ leads to
three necessary conditions for optimality

Three Conditions for Optimality



Individual terms should remain zero for arbitrary variations in $\Delta\mathbf{x}(t)$ and $\Delta\mathbf{u}(t)$

Solution for Lagrange Multiplier

$$1)\quad \left[\frac{\partial \phi}{\partial \mathbf{x}} - \boldsymbol{\lambda}^T\right]\Bigg|_{t=t_f} = \mathbf{0}$$

$$2)\quad \left[\frac{\partial H}{\partial \mathbf{x}} + \frac{d\boldsymbol{\lambda}^T}{dt}\right] = \mathbf{0} \quad in\left(t_0, t_f\right)$$

$$\Rightarrow \lambda^*(t) \text{ in } \left(t_o, t_f\right)$$

Insensitivity to Control Variation

$$3)\quad \frac{\partial H}{\partial \mathbf{u}} = \mathbf{0} \quad in\left(t_0, t_f\right)$$

$$\Rightarrow \mathbf{u}^*(t) \text{ in } \left(t_o, t_f\right)$$

Iterative Numerical Optimization Using Steepest-Descent

- **Forward solution to find the state, x(*t*)**
- Backward solution to find the Lagrange multiplier, $\lambda(t)$
- Steepest-descent adjustment of control history, **u(*t*)**
- Iterate to find the optimal solution

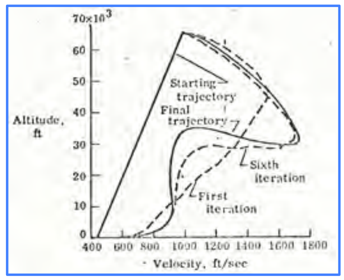
$$\dot{\mathbf{x}}_k(t) = \mathbf{f}[\mathbf{x}_k(t), \mathbf{u}_{k-1}(t)],$$

with

$\mathbf{x}(t_o)$ given

$\mathbf{u}_{k-1}(t)$ prescribed in (t_o, t_f)

$k =$ Iteration index



Use educated guess for u(t) on first iteration

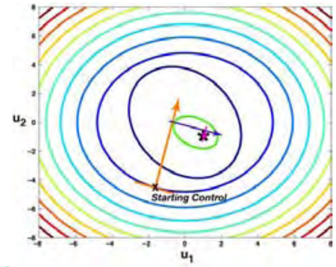
Numerical Optimization Using Steepest-Descent

- Forward solution to find the state, **x(*t*)**
- **Backward solution to find the Lagrange multiplier,** $\lambda(t)$
- Steepest-descent adjustment of control history, **u(*t*)**
- Iterate to optimal solution

$$\boldsymbol{\lambda}_k(t_f) = \left\{ \frac{\partial \phi[\mathbf{x}_k(t_f)]}{\partial \mathbf{x}} \right\}^T \qquad \left[\begin{array}{c} \text{Boundary condition at final time} \\ \text{Calculated from terminal value of the state} \end{array} \right]$$

$$\frac{d\boldsymbol{\lambda}_k(t)}{dt} = -\left[\frac{\partial H(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\lambda}_k)}{\partial \mathbf{x}} \right]_k^T = -\left[\left. \frac{\partial L(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} \right|_{\substack{\mathbf{x}(t)=\mathbf{x}_k(t) \\ \mathbf{u}(t)=\mathbf{u}_{k-1}(t)}} + \boldsymbol{\lambda}_k^T(t) \left. \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}} \right|_{\substack{\mathbf{x}(t)=\mathbf{x}_k(t) \\ \mathbf{u}(t)=\mathbf{u}_{k-1}(t)}} \right]_k^T$$

Numerical Optimization Using Steepest-Descent



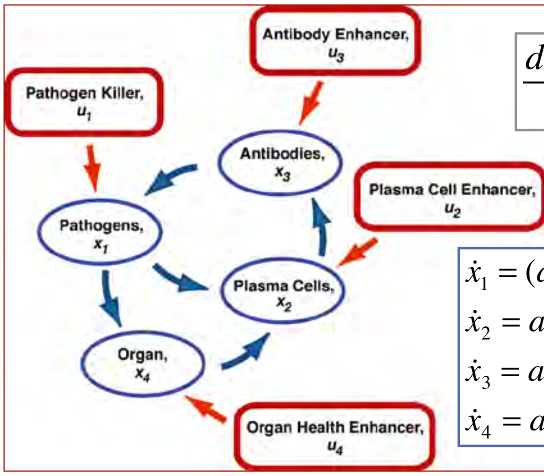
- Forward solution to find the state, $\mathbf{x}(t)$
- Backward solution to find the Lagrange multiplier, $\boldsymbol{\lambda}(t)$
- **Steepest-descent adjustment of control history, $\mathbf{u}(t)$**
- Iterate to optimal solution

$$
\begin{aligned}
\mathbf{u}_k(t) &= \mathbf{u}_{k-1}(t) - \varepsilon \left[\frac{\partial H}{\partial \mathbf{u}}\right]_k^T \\
&= \mathbf{u}_{k-1}(t) - \varepsilon \left[\left.\frac{\partial L}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}(t)=\mathbf{x}_k(t) \\ \mathbf{u}(t)=\mathbf{u}_{k-1}(t)}} + \boldsymbol{\lambda}_k^T(t) \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}(t)=\mathbf{x}_k(t) \\ \mathbf{u}(t)=\mathbf{u}_{k-1}(t)}} \right]^T
\end{aligned}
$$

ε : Steepest-descent gain

Optimal Treatment of an Infection

Dynamic Model for the Infection Treatment Problem

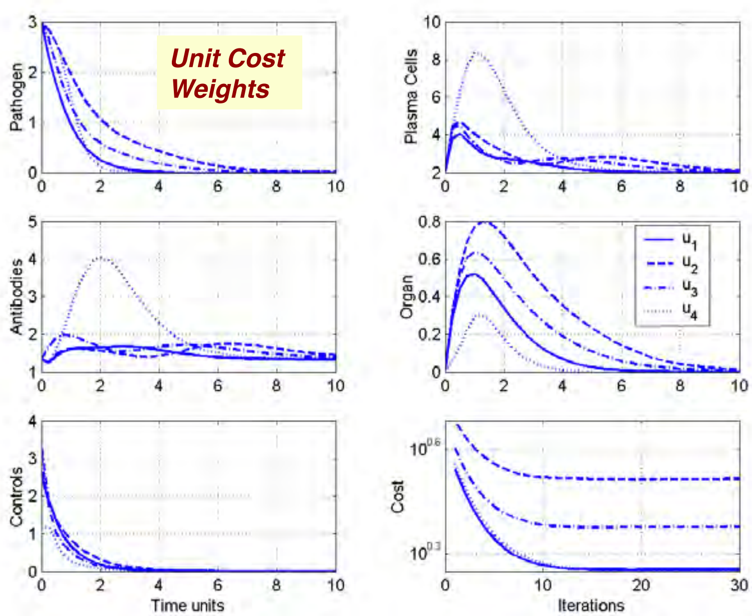


$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)], \quad \mathbf{x}(t_o)\ given$$

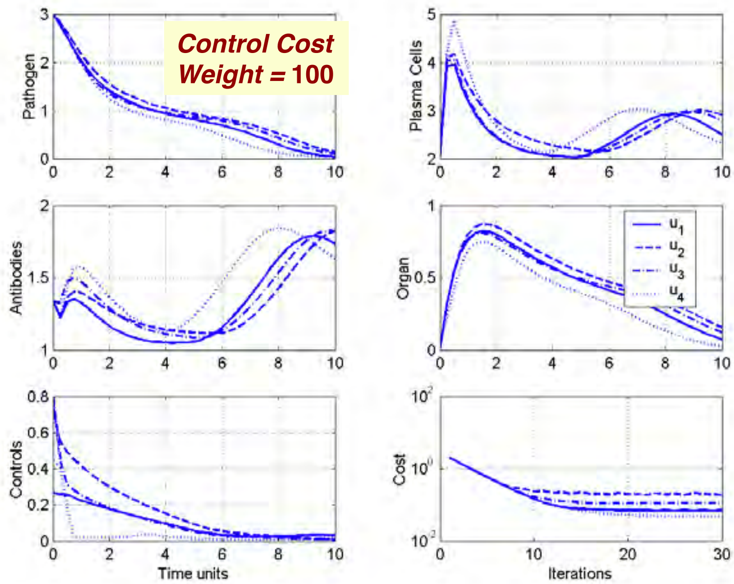
Nonlinear Dynamics of Innate Immune Response and Drug Effect

$$\begin{aligned}
\dot{x}_1 &= (a_{11} - a_{12}x_3)x_1 + b_1u_1 \\
\dot{x}_2 &= a_{21}(x_4)a_{22}x_1x_3 - a_{23}(x_2 - x_2{*}) + b_2u_2 \\
\dot{x}_3 &= a_{31}x_2 - (a_{32} + a_{33}x_1)x_3 + b_3u_3 \\
\dot{x}_4 &= a_{41}x_1 - a_{42}x_4 + b_4u_4
\end{aligned}$$

Optimal Treatment with Four Drugs (separately)



Increased Cost of Drug Use

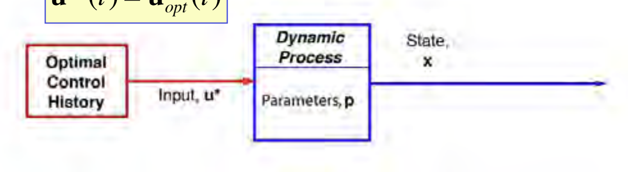


Accounting for Uncertainty in Initial Condition

Account for Uncertainty in Initial Condition and Unknown Disturbances

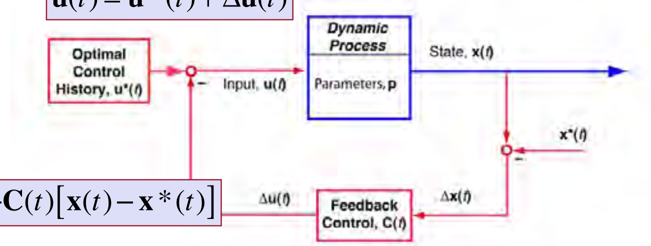
Nominal, Open-Loop Optimal Control

$$\mathbf{u}^*(t) = \mathbf{u}_{opt}(t)$$



Neighboring-Optimal (*Feedback*) Control

$$\mathbf{u}(t) = \mathbf{u}^*(t) + \Delta\mathbf{u}(t)$$



$$\Delta\mathbf{u}(t) = -\mathbf{C}(t)\left[\mathbf{x}(t) - \mathbf{x}^*(t)\right]$$

Neighboring-Optimal Control

Linearize dynamic equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \dot{\mathbf{x}}^*(t) + \Delta\dot{\mathbf{x}}(t) \\ &= \mathbf{f}\{[\mathbf{x}^*(t) + \Delta\mathbf{x}(t)],[\mathbf{u}^*(t) + \Delta\mathbf{u}(t)]\} \\ &\approx \mathbf{f}[\mathbf{x}^*(t),\mathbf{u}^*(t)] + \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t)\end{aligned}$$

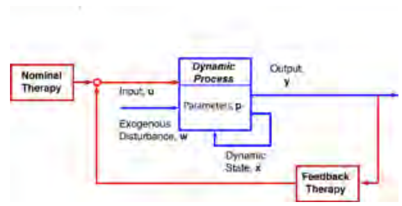
- Nominal optimal control history
- Optimal perturbation control
- Sum the two for neighboring-optimal control

$$\mathbf{u}^*(t) = \mathbf{u}_{opt}(t)$$

$$\Delta\mathbf{u}(t) = -\mathbf{C}(t)\left[\mathbf{x}(t) - \mathbf{x}_{opt}(t)\right]$$

$$\mathbf{u}(t) = \mathbf{u}_{opt}(t) + \Delta\mathbf{u}(t)$$

Optimal Feedback Gain, C(*t*)



- Solution of Euler-Lagrange equations for
  - Linear dynamic system
  - Quadratic cost function
- leads to linear, time-varying (LTV) optimal feedback controller

$$\boxed{\Delta\mathbf{u}^*(t) = -\mathbf{C}^*(t)\,\Delta\mathbf{x}(t)}$$

where

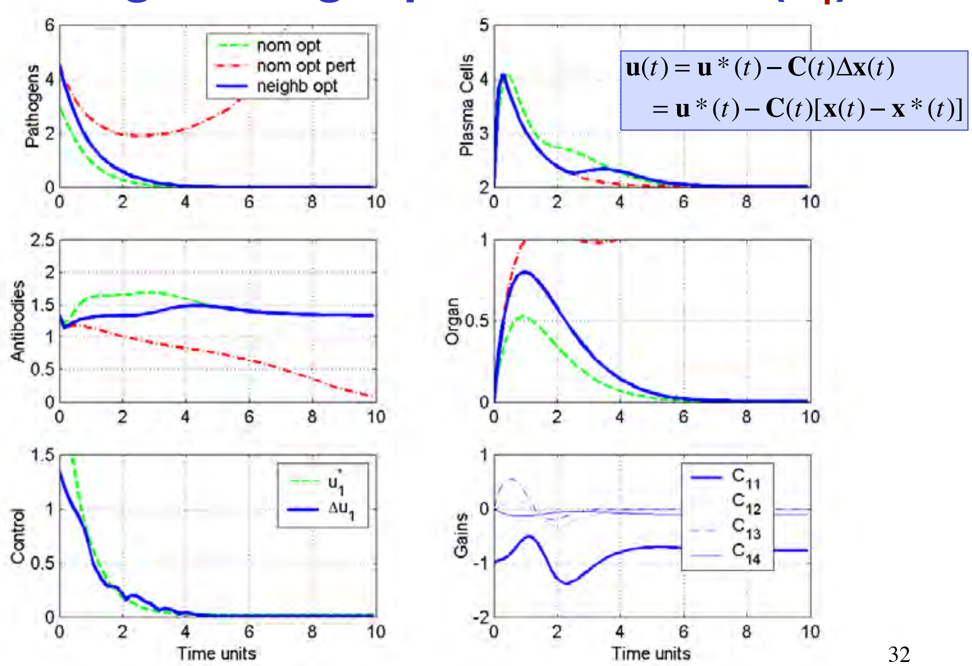
$$\mathbf{C}^*(t) = \mathbf{R}^{-1}\mathbf{G}^T(t)\mathbf{S}(t)$$

$$\dot{\mathbf{S}}(t) = -\mathbf{F}^T(t)\mathbf{S}(t) - \mathbf{S}(t)\mathbf{F}(t) + \mathbf{S}(t)\mathbf{G}(t)\mathbf{R}^{-1}\mathbf{G}^T(t)\mathbf{S}(t) - \mathbf{Q}$$

$$\mathbf{S}(t_f) = \mathbf{S}_f$$

Matrix Riccati equation (see Supplemental Material)

50% Increased Initial Infection and Scalar Neighboring-Optimal Control (u_1)

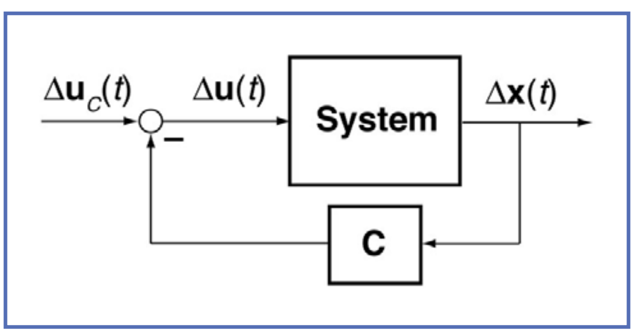


$$\mathbf{u}(t) = \mathbf{u}^*(t) - \mathbf{C}(t)\Delta\mathbf{x}(t)$$
$$= \mathbf{u}^*(t) - \mathbf{C}(t)[\mathbf{x}(t) - \mathbf{x}^*(t)]$$

Optimal, Constant Gain Feedback Control for Linear, Time-Invariant Systems

Linear-Quadratic (LQ) Optimal Control Law with Command Input

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(\mathbf{t})\Delta\mathbf{u}(t)$$



$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(\mathbf{t})\left[\Delta\mathbf{u}_C(t) - \mathbf{C}^*(t)\Delta\mathbf{x}(t)\right]$$

Optimal Control for Linear, Time-Invariant Dynamic Process

Original system is linear and time-invariant (LTI)

$$\boxed{\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t), \quad \Delta\mathbf{x}(0) \;\; \text{given}}$$

Minimize quadratic cost function for $\boxed{t_f \to \infty}$

Terminal cost is of no concern

$$\boxed{\min_u J = J^* = \lim_{t_f \to \infty} \frac{1}{2} \int_0^{t_f} \left[\Delta\mathbf{x}^T(t)\mathbf{Q}\Delta\mathbf{x}(t) + \Delta\mathbf{u}^T(t)\mathbf{R}\Delta\mathbf{u}(t) \right] dt}$$

Dynamic constraint is the linear, time-invariant (LTI) plant

Linear-Quadratic (LQ) Optimal Control for LTI System, and $\boxed{t_f \to \infty}$

Optimal control

$$\boxed{\Delta\mathbf{u}(t) = -\mathbf{C}^* \Delta\,\mathbf{x}(t)}$$

Optimal control gain matrix

$$\boxed{\begin{gathered}\mathbf{C}^* = \mathbf{R}^{-1}\mathbf{G}^T\mathbf{S}^* \\ (m \times n) = (m \times m)(m \times n)(n \times n)\end{gathered}}$$

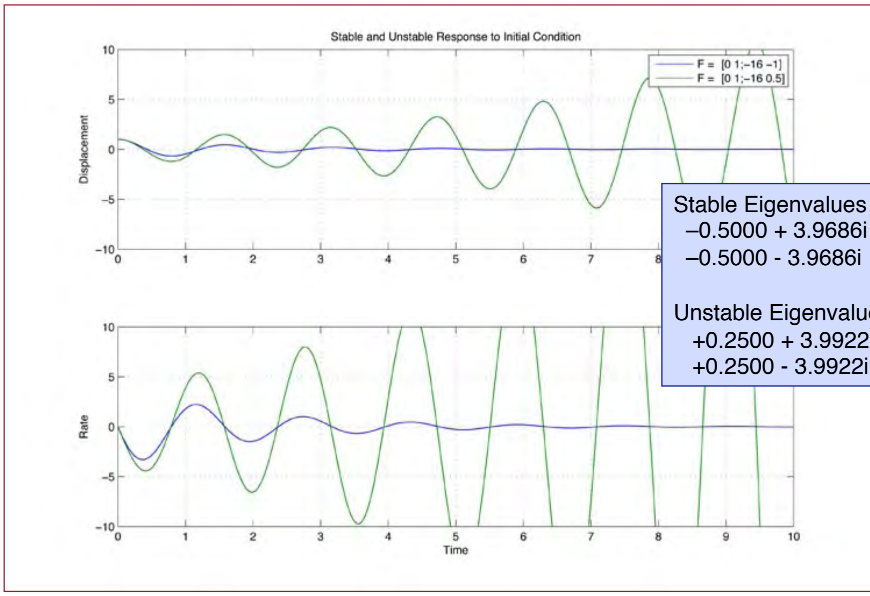
Steady-state solution of the matrix Riccati equation = Algebraic Riccati Equation

$$\boxed{-\mathbf{F}^T\mathbf{S}^* - \mathbf{S}^*\mathbf{F} + \mathbf{S}^*\mathbf{G}^*\mathbf{R}^{-1}\mathbf{G}^T\mathbf{S}^* - \mathbf{Q} = \mathbf{0}}$$

$$\boxed{\dot{\mathbf{S}}^*(0) \xrightarrow[t_f \to \infty]{} \mathbf{0}}$$

MATLAB function: lqr

Example: Stable and Unstable Second-Order System Response to Initial Condition



Stable Eigenvalues =
–0.5000 + 3.9686i
–0.5000 - 3.9686i

Unstable Eigenvalues =
+0.2500 + 3.9922i
+0.2500 - 3.9922i

Example: LQ Regulator Stabilizes Unstable System, r = 1 and 100

$$\min_{\Delta u} J = \min_{\Delta u}\left[\frac{1}{2}\int_0^\infty \left(\Delta x_1^{\,2} + \Delta x_2^{\,2} + r\Delta u^2\right)dt\right]$$

$$\Delta u(t) = -\begin{bmatrix} c_1 & c_2 \end{bmatrix}\begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} = -c_1\Delta x_1(t) - c_2\Delta x_2(t)$$

r = 1

Control Gain (**C***) =
0.2620 1.0857

Riccati Matrix (**S***) =
2.2001 0.0291
0.0291 0.1206

Closed-Loop Eigenvalues =
–6.4061
–2.8656

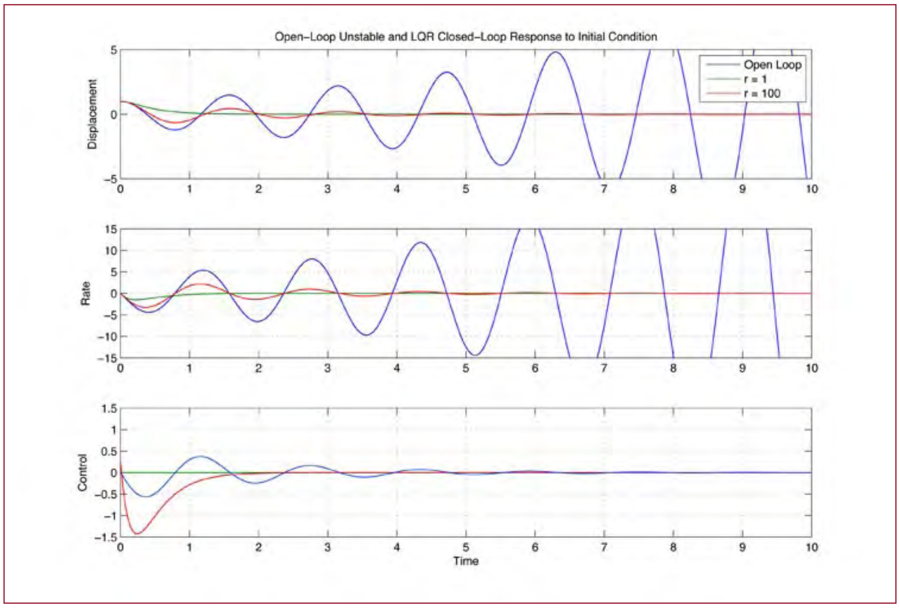
r = 100

Control Gain (**C***) =
0.0028 0.1726

Riccati Matrix (**S***) =
30.7261 0.0312
0.0312 1.9183

Closed-Loop Eigenvalues =
–0.5269 + 3.9683j
–0.5269 - 3.9683j

Example: LQ Regulator Stabilizes Unstable System, r = 1 and 100

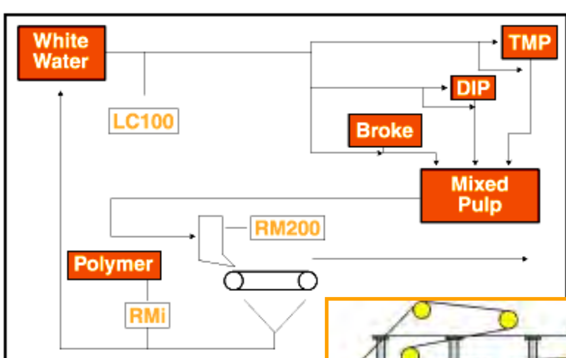


Next Time: Formal Logic, Algorithms, and Incompleteness

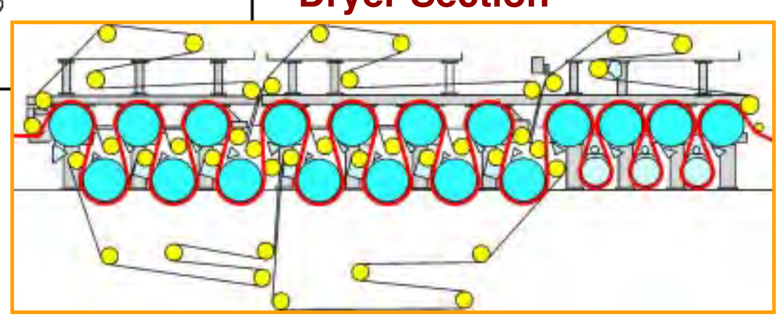
Supplemental Material

Pulp & Paper Machine Control

Wet End



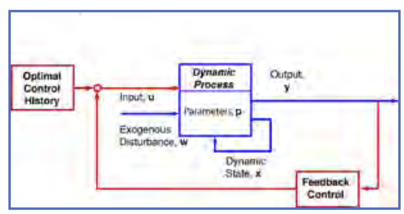
Dryer Section



Refrigerator Recycling "Robot"

- **Dismantles one refrigerator every 60 sec**
- **Captures refrigerant ("greenhouse gas") trapped in insulation**

Linearized Model of Infection Dynamics



Locally linearized (time-varying) dynamic equation

$$
\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \\ \Delta\dot{x}_3 \\ \Delta\dot{x}_4 \end{bmatrix}
=
\begin{bmatrix}
(a_{11} - a_{12}x_3^*) & 0 & -a_{12}x_1^* & 0 \\
a_{21}(x_4^*)a_{22}x_3^* & a_{23} & a_{21}(x_4^*)a_{22}x_1^* & \dfrac{\partial a_{21}}{\partial x_4}a_{22}x_1^* x_3^* \\
-a_{33}x_3^* & a_{31} & a_{31}x_1^* & 0 \\
a_{41} & 0 & 0 & -a_{42}
\end{bmatrix}
\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix}
+
\begin{bmatrix}
b_1 & 0 & 0 & 0 \\
0 & b_2 & 0 & 0 \\
0 & 0 & b_3 & 0 \\
0 & 0 & 0 & b_4
\end{bmatrix}
\begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{bmatrix}
+
\begin{bmatrix} \Delta w_1 \\ \Delta w_2 \\ \Delta w_3 \\ \Delta w_4 \end{bmatrix}
$$

Expand Optimal Control Function

- **Expand optimized cost function to second degree**

$$\begin{aligned} &J\left\{\left[\mathbf{x}^*(t_o)+\Delta\mathbf{x}(t_o)\right],\left[\mathbf{x}^*(t_f)+\Delta\mathbf{x}(t_f)\right]\right\} \simeq \\ &J^*\left[\mathbf{x}^*(t_o),\mathbf{x}^*(t_f)\right]+\cancel{\Delta J\left[\Delta\mathbf{x}(t_o),\Delta\mathbf{x}(t_f)\right]}+\Delta^2 J\left[\Delta\mathbf{x}(t_o),\Delta\mathbf{x}(t_f)\right]\end{aligned}$$

$$= J^*\left[\mathbf{x}^*(t_o),\mathbf{x}^*(t_f)\right]+\Delta^2 J\left[\Delta\mathbf{x}(t_o),\Delta\mathbf{x}(t_f)\right]$$

as **First Variation**, $\Delta J\left[\Delta\mathbf{x}(t_o),\Delta\mathbf{x}(t_f)\right]=0$

- **Nominal optimized cost, plus nonlinear dynamic constraint**

$$J^*\left[\mathbf{x}^*(t_o),\mathbf{x}^*(t_f)\right]=\phi\left[\mathbf{x}^*(t_f)\right]+\int_{t_o}^{t_f} L\left[\mathbf{x}^*(t),\mathbf{u}^*(t)\right]dt$$

subject to nonlinear dynamic equation

$$\dot{\mathbf{x}}^*(t)=\mathbf{f}\left[\mathbf{x}^*(t),\mathbf{u}^*(t)\right],\ \mathbf{x}(t_o)=\mathbf{x}_o$$

Second Variation of the Cost Function

Objective: Minimize second-variational cost subject to linear dynamic constraint

$$\min_{\Delta\mathbf{u}} \Delta^2 J=\frac{1}{2}\Delta\mathbf{x}^T(t_f)\phi_{\mathbf{xx}}(t_f)\Delta\mathbf{x}(t_f)+\frac{1}{2}\left\{\int_{t_o}^{t_f}\begin{bmatrix}\Delta\mathbf{x}^T(t) & \Delta\mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}L_{\mathbf{xx}}(t) & L_{\mathbf{xu}}(t)\\ L_{\mathbf{ux}}(t) & L_{\mathbf{uu}}(t)\end{bmatrix}\begin{bmatrix}\Delta\mathbf{x}(t)\\ \Delta\mathbf{u}(t)\end{bmatrix}dt\right\}$$

subject to perturbation dynamics

$$\Delta\dot{\mathbf{x}}(t)=\mathbf{F}(t)\Delta\mathbf{x}(t)+\mathbf{G}(t)\Delta\mathbf{u}(t),\ \Delta\mathbf{x}(t_o)=\Delta\mathbf{x}_o$$

Cost weighting matrices expressed as

$$\mathbf{S}(t_f)\triangleq\phi_{\mathbf{xx}}(t_f)=\frac{\partial^2\phi}{\partial\mathbf{x}^2}(t_f)$$

$$\begin{bmatrix}\mathbf{Q}(t) & \mathbf{M}(t)\\ \mathbf{M}^T(t) & \mathbf{R}(t)\end{bmatrix}\triangleq\begin{bmatrix}L_{\mathbf{xx}}(t) & L_{\mathbf{xu}}(t)\\ L_{\mathbf{ux}}(t) & L_{\mathbf{uu}}(t)\end{bmatrix}$$

$$\begin{aligned}&\dim\left[\mathbf{S}(t_f)\right]=\dim\left[\mathbf{Q}(t)\right]=n\times n\\ &\dim\left[\mathbf{R}(t)\right]=m\times m\\ &\dim\left[\mathbf{M}(t)\right]=n\times m\end{aligned}$$

Second Variational Hamiltonian

Variational cost function

$$\Delta^2 J = \frac{1}{2}\Delta\mathbf{x}^T(t_f)\mathbf{S}(t_f)\Delta\mathbf{x}(t_f) + \frac{1}{2}\left\{\int_{t_o}^{t_f}\begin{bmatrix}\Delta\mathbf{x}^T(t) & \Delta\mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}\mathbf{Q}(t) & \mathbf{M}(t)\\ \mathbf{M}^T(t) & \mathbf{R}(t)\end{bmatrix}\begin{bmatrix}\Delta\mathbf{x}(t)\\ \Delta\mathbf{u}(t)\end{bmatrix}dt\right\}$$

$$= \frac{1}{2}\Delta\mathbf{x}^T(t_f)\mathbf{S}(t_f)\Delta\mathbf{x}(t_f) + \frac{1}{2}\left\{\int_{t_o}^{t_f}\left[\Delta\mathbf{x}^T(t)\mathbf{Q}(t)\Delta\mathbf{x}(t) + 2\Delta\mathbf{x}^T(t)\mathbf{M}(t)\Delta\mathbf{u}(t) + \Delta\mathbf{u}^T(t)\mathbf{R}(t)\Delta\mathbf{u}(t)\right]dt\right\}$$

Variational Lagrangian plus adjoined dynamic constraint

$$H\left[\Delta\mathbf{x}(t),\Delta\mathbf{u}(t),\Delta\boldsymbol{\lambda}(t)\right] = L\left[\Delta\mathbf{x}(t),\Delta\mathbf{u}(t)\right] + \Delta\boldsymbol{\lambda}^T(t)\mathbf{f}\left[\Delta\mathbf{x}(t),\Delta\mathbf{u}(t)\right]$$
$$= \frac{1}{2}\left[\Delta\mathbf{x}^T(t)\mathbf{Q}(t)\Delta\mathbf{x}(t) + 2\Delta\mathbf{x}^T(t)\mathbf{M}(t)\Delta\mathbf{u}(t) + \Delta\mathbf{u}^T(t)\mathbf{R}(t)\Delta\mathbf{u}(t)\right]$$
$$+\Delta\boldsymbol{\lambda}^T(t)\left[\mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t)\right]$$

Second Variational Euler-Lagrange Equations

$$H = \frac{1}{2}\left[\Delta\mathbf{x}^T(t)\mathbf{Q}(t)\Delta\mathbf{x}(t) + 2\Delta\mathbf{x}^T(t)\mathbf{M}(t)\Delta\mathbf{u}(t) + \Delta\mathbf{u}^T(t)\mathbf{R}(t)\Delta\mathbf{u}(t)\right]$$
$$+\Delta\boldsymbol{\lambda}^T(t)\left[\mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t)\right]$$

Terminal condition, solution for adjoint vector, and optimality condition

$$\Delta\boldsymbol{\lambda}(t_f) = \phi_{\mathbf{xx}}(t_f)\Delta\mathbf{x}(t_f) = \mathbf{S}(t_f)\Delta\mathbf{x}(t_f)$$

$$\Delta\dot{\boldsymbol{\lambda}}(t) = -\left\{\frac{\partial H\left[\Delta\mathbf{x}(t),\Delta\mathbf{u}(t),\Delta\boldsymbol{\lambda}(t)\right]}{\partial\mathbf{x}}\right\}^T = -\mathbf{Q}(t)\Delta\mathbf{x}(t) - \mathbf{M}(t)\Delta\mathbf{u}(t) - \mathbf{F}^T(t)\Delta\boldsymbol{\lambda}(t)$$

$$\left\{\frac{\partial H\left[\Delta\mathbf{x}(t),\Delta\mathbf{u}(t),\Delta\boldsymbol{\lambda}(t)\right]}{\partial\mathbf{u}}\right\}^T = \mathbf{M}^T(t)\Delta\mathbf{x}(t) + \mathbf{R}(t)\Delta\mathbf{u}(t) - \mathbf{G}^T(t)\Delta\boldsymbol{\lambda}(t) = \mathbf{0}$$

Use Control Law to Solve the Two-Point Boundary-Value Problem

From $H_u = 0$ $\quad \Delta\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\left[\mathbf{M}^T(t)\Delta\mathbf{x}(t) + \mathbf{G}^T(t)\Delta\boldsymbol{\lambda}(t)\right]$

Substitute for control in system and adjoint equations

Two-point boundary-value problem

$$\begin{bmatrix} \Delta\dot{\mathbf{x}}(t) \\ \Delta\dot{\boldsymbol{\lambda}}(t) \end{bmatrix} = \left\{ \begin{array}{cc} \left[\mathbf{F}(t) - \mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{M}^T(t)\right] & -\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{G}^T(t) \\ \left[-\mathbf{Q}(t) + \mathbf{M}(t)\mathbf{R}^{-1}(t)\mathbf{M}^T(t)\right] & -\left[\mathbf{F}(t) - \mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{M}^T(t)\right]^T \end{array} \right\} \begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\boldsymbol{\lambda}(t) \end{bmatrix}$$

Boundary conditions at initial and final times

$$\begin{bmatrix} \Delta\mathbf{x}(t_o) \\ \Delta\boldsymbol{\lambda}(t_f) \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{x}_o \\ \mathbf{S}_f\Delta\mathbf{x}_f \end{bmatrix}$$

Perturbation state vector
Perturbation adjoint vector

Use Control Law to Solve the Two-Point Boundary-Value Problem

Suppose that the terminal adjoint relationship applies over the entire interval

$$\Delta\boldsymbol{\lambda}(t) = \mathbf{S}(t)\Delta\mathbf{x}(t)$$

Feedback control law becomes

$$\begin{aligned} \Delta\mathbf{u}(t) &= -\mathbf{R}^{-1}(t)\left[\mathbf{M}^T(t)\Delta\mathbf{x}(t) + \mathbf{G}^T(t)\mathbf{S}(t)\Delta\mathbf{x}(t)\right] \\ &= -\mathbf{R}^{-1}(t)\left[\mathbf{M}^T(t) + \mathbf{G}^T(t)\mathbf{S}(t)\right]\Delta\mathbf{x}(t) \\ &\triangleq -\mathbf{C}(t)\Delta\mathbf{x}(t) \qquad \dim(\mathbf{C}) = m \times n \end{aligned}$$

Linear-Quadratic (LQ) Optimal Control Gain Matrix

$$\boxed{\Delta\mathbf{u}(t) = -\mathbf{C}(t)\Delta\mathbf{x}(t)}$$

- Optimal feedback gain matrix

$$\boxed{\mathbf{C}(t) \quad = \mathbf{R}^{-1}(t)\left[\mathbf{G}^T(t)\mathbf{S}(t) + \mathbf{M}^T(t)\right]}$$

- Properties of feedback gain matrix
  - Full state feedback *(m x n)*
  - Time-varying matrix
  - **R**, **G**, and **M** given
    - Control weighting matrix, **R**
    - State-control weighting matrix, **M**
    - Control effect matrix, **G**
  - **S(*t*)** remains to be determined

Solution for the Adjoining Matrix, S(*t*)

Time-derivative of adjoint vector

$$\boxed{\Delta\dot{\boldsymbol{\lambda}}(t) = \dot{\mathbf{S}}(t)\Delta\mathbf{x}(t) + \mathbf{S}(t)\Delta\dot{\mathbf{x}}(t)}$$

Rearrange

$$\boxed{\dot{\mathbf{S}}(t)\Delta\mathbf{x}(t) = \Delta\dot{\boldsymbol{\lambda}}(t) - \mathbf{S}(t)\Delta\dot{\mathbf{x}}(t)}$$

Recall

$$\begin{bmatrix} \Delta\dot{\mathbf{x}}(t) \\ \Delta\dot{\boldsymbol{\lambda}}(t) \end{bmatrix} = \left\{ \begin{matrix} \left[\mathbf{F}(t) - \mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{M}^T(t)\right] & -\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{G}^T(t) \\ \left[-\mathbf{Q}(t) + \mathbf{M}(t)\mathbf{R}^{-1}(t)\mathbf{M}^T(t)\right] & -\left[\mathbf{F}(t) - \mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{M}^T(t)\right]^T \end{matrix} \right\} \begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\boldsymbol{\lambda}(t) \end{bmatrix}$$

Solution for the Adjoining Matrix, S(*t*)

Substitute

$$\dot{\mathbf{S}}(t)\Delta\mathbf{x}(t)=\left[-\mathbf{Q}(t)+\mathbf{M}(t)\mathbf{R}^{-1}(t)\mathbf{M}^{T}(t)\right]\Delta\mathbf{x}(t)-\left[\mathbf{F}(t)-\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{M}^{T}(t)\right]^{T}\Delta\boldsymbol{\lambda}(t)$$
$$-\mathbf{S}(t)\left\{\left[\mathbf{F}(t)-\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{M}^{T}(t)\right]\Delta\mathbf{x}(t)-\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{G}^{T}(t)\Delta\boldsymbol{\lambda}(t)\right\}$$

Substitute

$$\Delta\boldsymbol{\lambda}(t)=\mathbf{S}(t)\Delta\mathbf{x}(t)$$

$$\dot{\mathbf{S}}(t)\underline{\Delta\mathbf{x}(t)}=\left[-\mathbf{Q}(t)+\mathbf{M}(t)\mathbf{R}^{-1}(t)\mathbf{M}^{T}(t)\right]\underline{\Delta\mathbf{x}(t)}-\left[\mathbf{F}(t)-\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{M}^{T}(t)\right]^{T}\mathbf{S}(t)\underline{\Delta\mathbf{x}(t)}$$
$$-\mathbf{S}(t)\left\{\left[\mathbf{F}(t)-\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{M}^{T}(t)\right]\underline{\Delta\mathbf{x}(t)}-\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{G}^{T}(t)\mathbf{S}(t)\underline{\Delta\mathbf{x}(t)}\right\}$$

Δx(*t*) can be eliminated

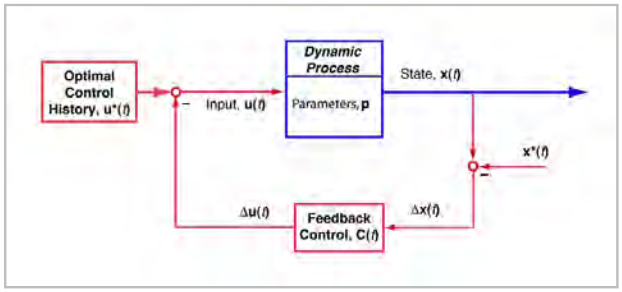
Matrix Riccati Equation for S(*t*)

The result is a nonlinear, ordinary differential equation for S(*t*), with terminal boundary conditions

$$\dot{\mathbf{S}}(t)=\left[-\mathbf{Q}(t)+\mathbf{M}(t)\mathbf{R}^{-1}(t)\mathbf{M}^{T}(t)\right]-\left[\mathbf{F}(t)-\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{M}^{T}(t)\right]^{T}\mathbf{S}(t)$$
$$-\mathbf{S}(t)\left[\mathbf{F}(t)-\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{M}^{T}(t)\right]+\mathbf{S}(t)\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{G}^{T}(t)\mathbf{S}(t)$$
$$\mathbf{S}(t_f)=\phi_{\mathbf{xx}}(t_f)$$

- **Characteristics of the Riccati matrix, S(*t*)**
  - **S(t_f) is symmetric, *n* x *n*, and typically positive semi-definite**
  - **Matrix Riccati equation is symmetric**
  - **Therefore, S(*t*) is symmetric and positive semi-definite throughout**
- **Once S(*t*) has been determined, optimal feedback control gain matrix, C(*t*) can be calculated**

Neighboring-Optimal (LQ) Feedback Control Law

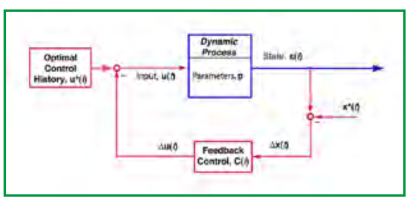


- **Full state is fed back to all available controls**

$$\Delta\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\left[\mathbf{M}^{T}(t) + \mathbf{G}^{T}(t)\mathbf{S}(t)\right]\Delta\mathbf{x}(t) = -\mathbf{C}(t)\Delta\mathbf{x}(t)$$

- **Optimal control history plus feedback correction**

$$\mathbf{u}(t) = \mathbf{u}^*(t) - \mathbf{C}(t)\Delta\mathbf{x}(t) = \mathbf{u}^*(t) - \mathbf{C}(t)\left[\mathbf{x}(t) - \mathbf{x}^*(t)\right]$$



Nonlinear System with Neighboring-Optimal Feedback Control

Nonlinear dynamic system

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t)\right]$$

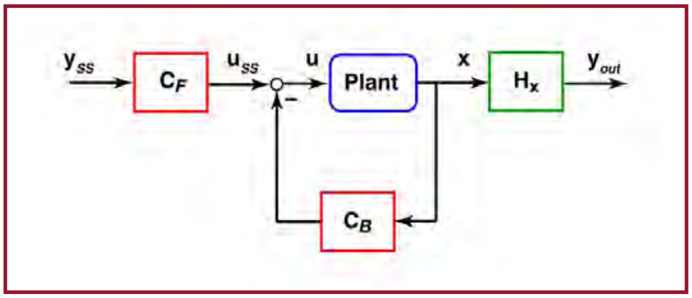
Neighboring-optimal control law

$$\mathbf{u}(t) = \mathbf{u}^*(t) - \mathbf{C}(t)\Delta\mathbf{x}(t) = \mathbf{u}^*(t) - \mathbf{C}(t)\left[\mathbf{x}(t) - \mathbf{x}^*(t)\right]$$

Nonlinear dynamic system with neighboring-optimal feedback control

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left\{\mathbf{x}(t), \left[\mathbf{u}^*(t) - \mathbf{C}(t)\left[\mathbf{x}(t) - \mathbf{x}^*(t)\right]\right]\right\}$$

Linear-Quadratic (LQ) Optimal Control Law with Command Input

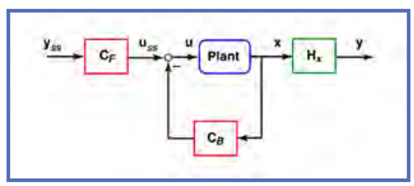


$$\begin{aligned}\Delta\dot{\mathbf{x}}(t) &= \mathbf{F}(t)\Delta\mathbf{x}(t)+G(\mathbf{t})\Delta\mathbf{u}(t)\\ &= \mathbf{F}(t)\Delta\mathbf{x}(t)+G(\mathbf{t})\left[\mathbf{u}^*(t)-\mathbf{C}(t)\left[\mathbf{x}(t)-\mathbf{x}^*(t)\right]\right]\end{aligned}$$

How can C_F be chosen so that

$\boxed{\mathbf{y}_{out}(t) \underset{t\to\infty}{\longrightarrow} \mathbf{y}_{SS}}$ **? See Supplemental Material**

Command Input Gain Matrix



- **In steady state**

$$\begin{aligned}&[\mathbf{F}-\mathbf{G}\mathbf{C}_B]\mathbf{x}_{ss}+\mathbf{G}\mathbf{u}_{SS}=0\\ &\mathbf{x}_{ss}=-[\mathbf{F}-\mathbf{G}\mathbf{C}_B]^{-1}\mathbf{G}\mathbf{u}_{SS}\end{aligned}$$

- **… and the steady-state command is**

$$\mathbf{y}_{SS}=\mathbf{H}_\mathbf{x}\mathbf{x}_{ss}=-\mathbf{H}_\mathbf{x}[\mathbf{F}-\mathbf{G}\mathbf{C}_B]^{-1}\mathbf{G}\mathbf{u}_{SS}$$

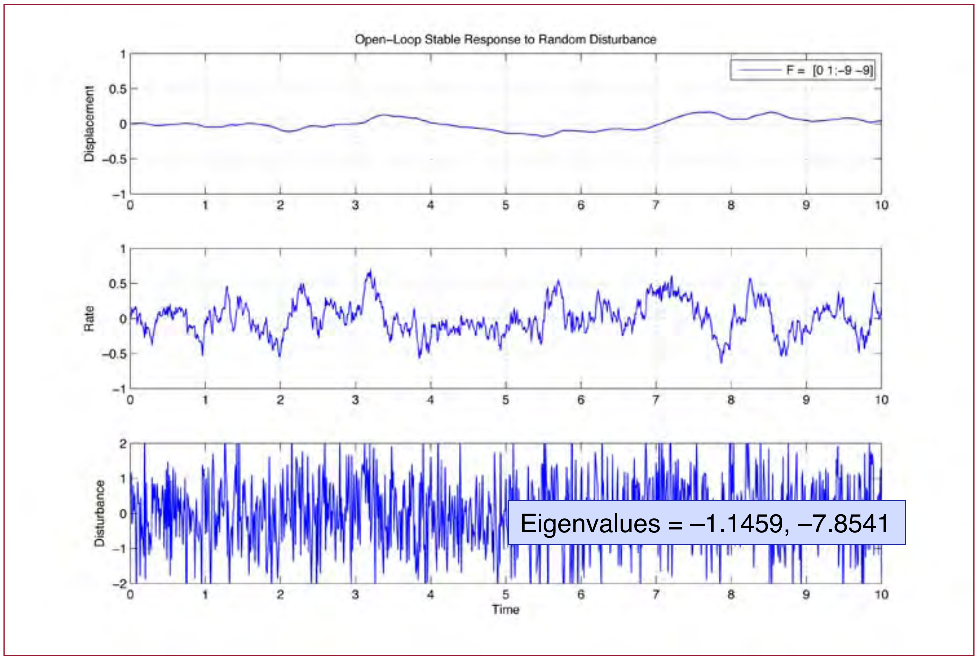
- **The steady-state control is**

$$\begin{aligned}\mathbf{u}_{SS}&=\left\{-\mathbf{H}_\mathbf{x}[\mathbf{F}-\mathbf{G}\mathbf{C}_B]^{-1}\mathbf{G}\right\}^{-1}\mathbf{y}_{SS}\\ &\triangleq \mathbf{C}_F\mathbf{y}_{SS}\end{aligned}$$

$$\begin{aligned}\dim(y)&=r\times 1\\ \dim(u)&=m\times 1\end{aligned}$$

- **{.} must be invertible, which requires that dim(y) = dim(u), and closed-loop system has a steady state**

Example: Response of Stable Second-Order System to Random Disturbance



Example: Disturbance Response of Unstable System with LQ Regulators, r = 1 and 100