Formal Logic, Algorithms, and Incompleteness

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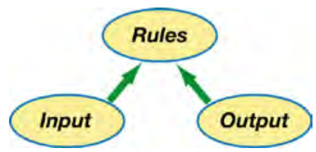
Robotics and Intelligent Systems MAE 345,
Princeton University, 2015

Learning Objectives

- Principles of axiomatic systems and formal logic
- Application of logic in computing machines
- Algorithms and numbering systems
- Gödel's Theorems: What axiomatic systems can't do

X	Y	$X \wedge Y$	$X \vee Y$	$X \to Y$	$X \equiv Y$	$X \wedge (\neg Y)$	...
T	T	T	T	$\boxed{T}$	T	F	
T	F	F	T	$\boxed{F}$	F	T	
F	T	F	T	$\boxed{T}$	F	F	
F	F	F	F	$\boxed{T}$	T	F	



http://www.princeton.edu/~stengel/MAE345.html

Intelligent Systems

- **Perform useful functions driven by desired goals and current knowledge**
  - **Emulate** biological and cognitive processes
  - **Process** information to achieve objectives
  - **Learn** by example or from experience
  - **Adapt** functions to a changing environment

Should robots be "More like us?"

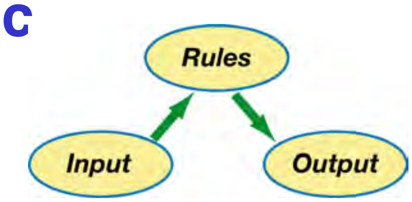
- **Semantics: The study of meaning**
- **Syntax: Orderly or systematic arrangement of parts or elements**

Cognitive Paradigms for Intelligent Systems

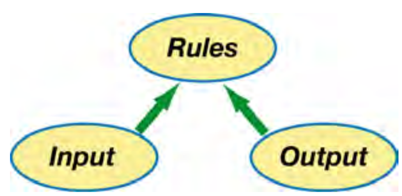
- Thinking
  - Syntax
  - Algorithmic behavior
  - Comparison
  - Reflection
- Consciousness
  - Understanding and judgment of truth
- Intelligence
  - Flexible response
  - Recognition of similarity and contradiction
  - Ranking of information
  - Synthesis of solutions
  - Reasoning

Underlying structure: Logic

Formal Logic



- Deduction
  - Shows that a proposition follows from one or more other propositions
  - Establishes the validity of a claim or argument
  - Reasons from input to rules to output



- Induction
  - Infers a general law or principle from the observation of particular instances
  - Reasons from input and output to rules

- Inference
  - Derivation of conclusions from information, as by
    - Deduction
    - Induction
  - Reasoning from something known or assumed, as by
    - Application of rules or meta-rules (i.e., rules about rules)
    - Probability and statistics

“Forms of Inference” Lead to “Formulas”

- **Formulas**
  - Symbols
  - Operations
  - Rules
- **Axioms**
  - Unproved but assumed formulas
  - Starting point for proofs of formulas
- **Theorems**
  - Formulas proved to be true based on
    - Axioms
    - Other theorems
- **Algorithms**
  - Systematic procedures for using formulas
- **Calculus**
  - A system or method of calculation
  - A method of assessment

Propositional Calculus - 1

- **Proposition:** A statement that may be either true or false
- Complete, unanalyzed propositions and combinations
  - What can be said -- formal relations and implications -- axioms of the system
  - Deductive structure: Rules of Inference
  - Concern with form or syntax of statements
  - Meaning of a statement may not be self-evident; for example,

$$\boxed{(2+3),\quad (+\,2\;3),\quad (2\;3\,+)}$$

  - may be different notations for the same statement

Infix Prefix Postfix

Examples of Propositions

Princeton's colors are orange and black (true) … are red and gray (false)

6 + 6 = 12; 6 + 7 = 12

"I have a bridge to sell to you …."

Operators
(or *Sentential Connectives*)

And	$\wedge$ or &	*Conjunction*
Or	$\vee$	*Disjunction*
Not	$\neg$ or ~	*Negation*
Implies	$\rightarrow$ or $\supset$	*Material Implication* (*If*)
Equivalent	$\equiv$ or $\leftrightarrow$	*Material Equivalence* (*If and only if*)

- **Sentential variables may be either true or false**
- **Operators connect sentential (or propositional) variables**
- **A proposition (or sentence) is a formula containing variables and operators**

Dyadic Operations - 1

- Operations involving two arguments (i.e., sentential variables)
- Arguments of operators = Propositions
  - *X* represents "Socrates is a man"
  - *Y* represents "All men are mortal"
- Examples of formulas or connective expressions [dyadic operations (2 arguments)]

$$X \wedge Y$$

$$X \vee Y$$

- "Socrates is a man" and "All men are mortal"
- "Socrates is a man" or "All men are mortal"

Dyadic Operations - 2

$$X \rightarrow Y$$

$$X \equiv Y$$

- "Socrates is a man" implies that "All men are mortal"
- "Socrates is a man" is equivalent to "All men are mortal"

- 1st argument is the antecedent; 2nd argument is the consequent

- "IF-THEN-ELSE" interpretation of dyadic operations
  - If *X* is true and *Y* is true, then $X \wedge Y$ is true; else $X \wedge Y$ is false
  - If *X* is true or *Y* is true, then $X \vee Y$ is true; else $X \vee Y$ is false

Monadic Operations and Syntactic Propositions

- Negation is a monadic (single argument) operation
  - If X is true, then $\neg X$ is false
  - If X is false, then $\neg X$ is true
- Brackets group propositions to form Syntactic Propositions (i.e., propositions based on propositions)
- Incorporation of negation in dyadic operations:

$$\boxed{\begin{array}{l} X \wedge (\neg Y) \\ X \vee (\neg Y) \end{array}}$$

If X is true and Y is false, then $\boxed{X \wedge (\neg Y)}$ is true; else …

If X is true or Y is false, then …

Truth Tables for Dyadic Expressions

X	Y	$X \wedge Y$	$X \vee Y$	$X \rightarrow Y$	$X \equiv Y$	$X \wedge (\neg Y)$	…
T	T	T	T	T	T	F	
T	F	F	T	F	F	T	
F	T	F	T	T	F	F	
F	F	F	F	T	T	F	

- Syntactic combinations build sentences
- Tautology (repetitive statement) is always true
  - "X implies Y and Z" is the same as "X implies Y and X implies Z"

$$\boxed{\left(X \rightarrow (Y \wedge Z)\right) \equiv \left((X \rightarrow Y) \wedge (X \rightarrow Z)\right)}$$

More Concepts in Propositional Calculus

- **Fallacy** or **Contradiction**
  - Saying that [*X* or *Y* is false is the same as saying that "*X* is false and *Y* is false" is false)] is a fallacy or contradiction

$$\neg(X \vee Y) \equiv \neg(\neg Y \wedge \neg X)$$

  - Liar's paradox: "I am lying." True or false? Sentence refers to its own truth.
- Truth depends on the propositions described by *X*, *Y*, and *Z*

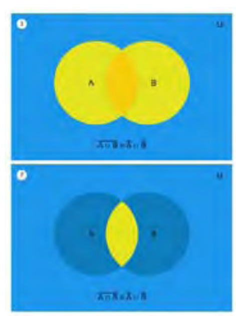
$$(X \wedge Y) \vee (\neg Y \wedge Z)$$

- **Well-formed formulas** (*WFFs*) make sense and are unambiguous

$$(X \wedge Y) \vee (\neg YY(Z)) \quad \text{Not a } WFF$$

More Concepts in Propositional Calculus

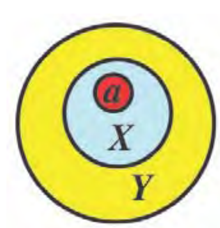
- **Decisions** are based on testing the validity of *WFFs*
- **De Morgan's Laws**
  - Two propositions are jointly true only if neither is false



$$\neg(X \wedge Y) \equiv \neg X \vee \neg Y$$

$$\neg(X \vee Y) \equiv \neg X \wedge \neg Y$$

- ***Modus Ponens*** rule (rule of detachment or elimination)
  - If *X* is true and *X* implies *Y*, then we can infer that *Y* is true



$$(X \wedge (X \rightarrow Y)) \rightarrow Y$$

Modus Ponens Rule

- **Rule of detachment, elimination, definition, or substitution**
  - **If *X* is true and *X* implies *Y*, then we can infer that *Y* is true**

$$(X \wedge (X \rightarrow Y)) \rightarrow Y$$

  - ***X* is true and *X* implies *Y*, then (*X* is true and *X* implies *Y)* implies that *Y* is true**
- **Example from Wikipedia:**
  - **If it's raining, I'll meet you at the movie theater.**
  - **It's raining.**
  - **Therefore, I'll meet you at the movie theater**

Material Implication

- ***X* -> *Y***
- **Same as "¬*X* or *Y"***
- ***X* is false does not imply that *Y* is not true**
- **"If", not "If and only if", which is material equivalency**
- **Double negative**

- **Example:**
  - ***X*: Anyone can be caught in the rain**
  - ***Y*: That person is wet**
  - ***X* -> *Y*, or (if *X Y*)**
  - **Suppose Dave is wet; was he caught in the rain?**
  - **Dave went under a sprinkler and got wet; he was not caught in the rain, but he is wet**
  - **Therefore [(false) -> (true)] is true**
  - **Material implication does not indicate causality**

Material Implication *(if)* vs. Material Equivalence *(iff)*

- $X \equiv Y$
- "If and only if": *iff*
- The truth of *X* requires the truth of *Y*
- If: I will eat lunch if the E-Quad Café has tuna salad
- Iff: I will eat lunch if and only if the E-Quad Café has tuna salad

Toward Predicate Calculus

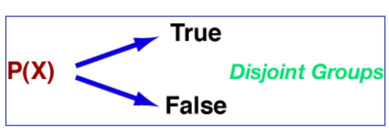
- Sentence
  - Series of words forming a grammatically complete expression of a single thought
  - Normally contains (at least) a subject and a predicate
- Predicate
  - That which is predicated (or said) of the subject in a proposition
  - Second term of a proposition, e.g.,
    - Socrates is a man
  - The statement made about the subject, e.g.,
    - The main verb, its object, and modifiers

Predicate Calculus

- Extensions to propositional calculus
  - Predicates
  - Flexible variables, i.e., more states than only true or false
  - Quantification
    - Conversion of words to numbers
    - Introduction of degrees of value
  - Inference rules for quantifiers
    - First-order logic
    - Productive use of predicates, variables, and quantification
- Building blocks for expert systems

Predicates

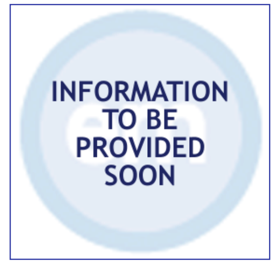
- Predicate, P(X)
  - A statement (or proposition) about individuals (or arguments) that is either true or false*
  - One argument:
    Example: "is-red"
  - Two arguments:
    Example: "is-greater-than"
  - QUEEN OF HEARTS is-red (true)
  - LIVE GRASS is-red (false)
  - SEVEN is-greater-than FOUR
- One-argument predicate, P(X), performs a sort

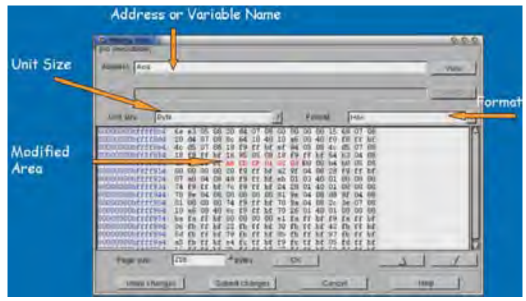


* also called an atomic formula

Variable

- A placeholder that is to be filled with a constant, e.g., X in P(X)
- A slot that receives a value
- A symbolic address for information





Quantification

- "Universal quantifiers say something that is true for all possible values of a variable."*

$$\boxed{(\mathit{forall}\ (x)\ f)}$$

$x:$ *variable*
$f:$ *formula; specifies* ***scope*** *of* x

$$\boxed{(\mathit{forall}\ (x)\ (\mathit{if}\ (\mathit{inst}\ x\ \mathit{fire-engine})\ (\mathit{color}\ x\ \mathit{red})))}$$

- Existential quantifiers
  - state conditions under which a variable exists
  - predicate properties or relationships of one or more variables

$$\boxed{(\mathit{exists}\ (x)\ f)}$$

$$\boxed{(\mathit{forall}\ (x)\ (\mathit{if}\ (\mathit{person}\ x)\ (\mathit{exists}\ (y)\ (\mathit{head-of}\ x\ y))))}$$

**** Charniak and McDermott, 1985***

Inference Rules for Quantifiers

- Well-formed formula (WFF)
  - Syntactically correct combination of connectives, predicates, constants, variables, and quantifiers
- Universal Quantification (or Elimination or Instantiation)
  - Man(Socrates) -> Mortal(Socrates)
  - or "The man, Socrates, is mortal" ["given any", "for all"]
- Existential Quantification (or Elimination or Instantiation)
  - Man(person) -> Happy(person)
  - Someone is happy ["there exists at least one"]
- Existential Introduction (Generalization)
  - Man(Jerry) -> Likes_ice_cream(Jerry)
  - Someone likes ice cream ["general to specific" or v.v.]

Examples of Sentences

- LISP-like terms and prefix notation
  - (catch-object jack-1 block-1)
    - Jack-1 catches the object called Block-1
  - (inst block-1 block)
    - Block-1 is an instantiation of a block
  - (color block-1 blue)
    - Block-1 is blue
- With connectives
  - (and (color block-1 yellow) (inst block-1 elephant))
    - Block-1 is a yellow elephant
  - (if (supports block-2 block-1) (on block-1 block-2))
    - If block-2 supports block-1, then block-1 is on block-2
  - (if (and (inst clyde elephant) (color elephant gray)) (color clyde gray))
    - If clyde is an elephant and an elephant is gray, then clyde is gray

First-Order Logic

- Further extensions to predicate calculus
- Functions
  - Fixed number of arguments
  - Rather than returning TRUE or FALSE, functions return objects, e.g.,
    - "uncle-of" Mary returns John
  - Functions of functions, e.g.,
    - (father-of (father-of (John)) returns John's paternal grandfather

First-Order Logic

- Equals
  - Two individuals are equal if and only if (equivalence) they are indistinguishable under all predicates and functions

$$\boxed{X \equiv Y} \qquad \text{if and only if}$$

$$\boxed{P(X) \equiv P(Y), \quad F(X) \equiv F(Y), \quad \forall P \wedge F}$$

- Axiomatization
  - Axioms: necessary relationships between objects in a domain
  - Formal expression in sentences of first-order logic (emphasis on syntax over semantics)

Apollo Guidance Computer Commands

- **Display/Keyboard (DSKY)**
- **Sentence**
  - **Subject and predicate**
  - **Subject is implied**
    - **Astronaut, or**
    - **GNC system**
  - **Sentence describes action to be taken employing or involving an object**
- **Predicate**
  - **Verb = Action**
  - **Noun = Variable or Program (i.e., the object)**

See http://www.ibiblio.org/apollo/ for simulation

Numerical Codes for Verbs and Nouns in Apollo Guidance Computer Programs

Verb Code	Description	Remarks
01	Display 1st component of	Octal display of data on REGISTER 1
02	Display 2nd component of	Octal display of data on REGISTER 1
03	Display 3rd component of	Octal display of data on REGISTER 1

Noun Code	Description	Scale/Units
01	Specify machine address	XXXXX
02	Specify machine address	XXXXX
03	(Spare)	
04	(Spare)	
05	Angular error	XXX.XX degrees
06	Pitch angle	XXX.XX degrees
	Heads up-down	+/- 00001
07	Change of program or major mode	
11	Engine ON enable	

Verbs and Nouns in Apollo Guidance Computer Programs

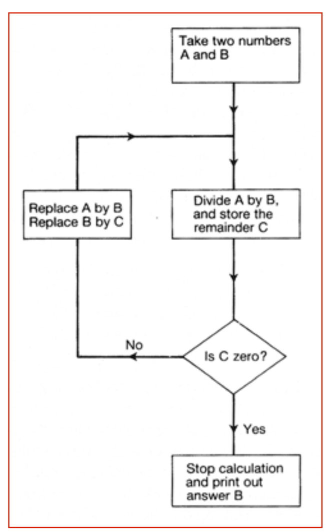
- Verbs *(Actions)*
  - Display
  - Enter
  - Monitor
  - Write
  - Terminate
  - Start
  - Change
  - Align
  - Lock
  - Set
  - Return
  - Test
  - Calculate
  - Update
- Selected Nouns *(Variables)*
  - Checklist
  - Self-test ON/OFF
  - Star number
  - Failure register code
  - Event time
  - Inertial velocity
  - Altitude
  - Latitude
  - Miss distance
  - Delta time of burn
  - Velocity to be gained
- Selected Programs (CM)
  - AGC Idling
  - Gyro Compassing
  - LET Abort
  - Landmark Tracking
  - Ground Track Determination
  - Return to Earth
  - SPS Minimum Impulse
  - CSM/IMU Align
  - Final Phase
  - First Abort Burn

Algorithms

- Systematic procedures for using formulas
- Computer programs contain algorithms
- Euclid's Algorithm
  - Highest common denominator (HCD) of 2 numbers
  - In example, HCD = 21
  - Operations based on natural numbers (positive integers)
- Procedure is completed in a finite number of steps

$$\begin{array}{rl}
3654 \div 1365 & \text{gives remainder } 924 \\
1365 \div 924 & \text{gives remainder } 441 \\
924 \div 441 & \text{gives remainder } 42 \\
441 \div 42 & \text{gives remainder } 21 \\
42 \div 21 & \text{gives remainder } 0.
\end{array}$$

- Flow charts
  - Operations
  - Conditions
  - Sub-routines



Some Natural Numbering Systems

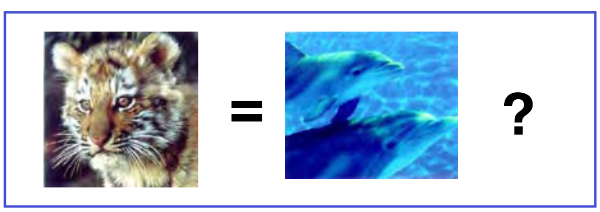
Natural numbers: non-negative, whole numbers

Denary (*Base 10*)	Binary (*Base 2*)	Unary (*Base 1*)
0	0	?
1	1	1
2	10	11
3	11	111
4	100	1111
5	101	11111
6	110	111111
7	111	1111111
8	1000	11111111
9	1001	111111111
10	1010	1111111111
11	1011	11111111111
Digits	Binary Digits "Bits" (John Tukey)	Marks
Two 5-finger hands	True-False	Chalk and a rock
One 10-finger hand	Yes-No	Abacus
	Present-Absent	"Chisenbop"

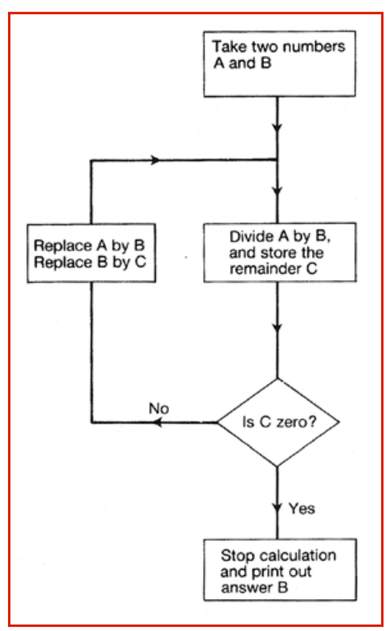
- Other number systems
  - DNA (*Base 4*) *[ATCG]*
  - Octal (*Base 8*)
  - Hexadecimal (*Base 16*)

$$F3 = (15\times16^1)+(3\times16^0) = 243$$

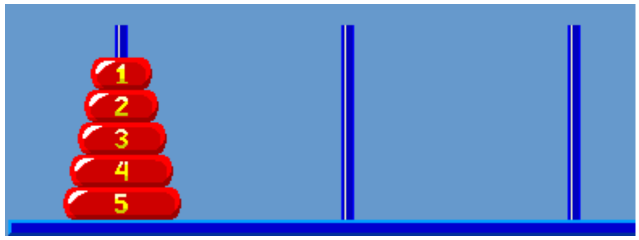
Algorithms are Independent of Numbering System



- Logical algorithms may deal with objects or symbols directly
- For computation, objects or symbols ultimately are represented by numbers (e.g., 0s and 1s) or alphabet
- Mathematical logical algorithms are independent of the numbering system



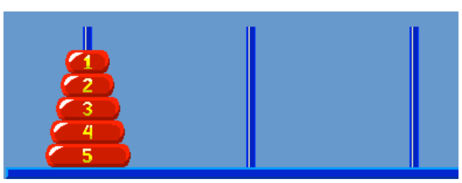
Towers of Hanoi:
An Axiomatic System



Problem: Move all disks (one at a time) from 1st peg to 3rd peg without putting a larger disk on a smaller disk

- **Objects**
  - Disks: 1, 2, 3, 4, 5
  - Pegs: A, B, C
- **Predicates**
  - Sorting: DISK, PEG
    - DISK(A) is FALSE
    - PEG(A) is TRUE
  - Comparison: SMALLER
    - SMALLER(1,2) is TRUE

Barr and Feigenbaum, 1982



Towers of Hanoi

- **First axiom**

$$\boxed{\forall XYZ.(\mathrm{SMALLER}(X,Y) \wedge (\mathrm{SMALLER}(Y,Z)) \rightarrow \mathrm{SMALLER}(X,Z)}$$

- **Premise**

$$\boxed{\mathrm{SMALLER}(1,2) \wedge \mathrm{SMALLER}(2,3)}$$

- **Situational constant, *S***
  - Identifies state of system after a series of moves
- **More predicates**
  - Vertical relationship: ON
    - ON(*X*,*Y*,*S*) asserts that disc *X* is on disk *Y* in situation *S*
  - Nothing on top of disk: FREE
    - FREE*(X*,*S*) indicates that no disc is on *X*

Towers of Hanoi

- **Second axiom***

$$\boxed{\forall\, X\, S.\mathrm{FREE}(X,S) \equiv \neg \exists\, Y.\big(\mathrm{ON}(Y,X,S)\big)}$$

* "For all disks *X* and situation *S*, *X* is free in situation *S* if and only if there does not exist a disk *Y* such that *Y* is ON *X* in situation *S*."

- **More Predicates**
  - **Acceptable (legal) move: LEGAL (*X*,*Y*,*S*)**
  - **Act of moving disk: MOVE(*X*,*Y*,*S*)**
- **Object of analysis**
  - **Find a situation that is TRUE if a move is legal and is accomplished**
- **More Axioms**
  - **See *Handbook of AI* for additional steps**

Example of theorem proving, i.e., of theory that a goal state can be reached

Gödel's Incompleteness Theorems (1931)

http://en.wikipedia.org/wiki/Gödel%27s_incompleteness_theorems

- **1st Theorem:** "No consistent system of axioms whose theorems can be listed by an 'effective procedure' (e.g., a computer program …) is capable of proving all truths about the relations of the natural numbers (arithmetic)."
  - "There will always be statements about the natural numbers that are true, but that are unprovable within the system."
- **2nd Theorem:** "Such a system cannot demonstrate its own consistency."
- **~ "Liar's Paradox", replacing "provability" for "truth"**

http://mathworld.wolfram.com/GoedelsIncompletenessTheorem.html

- **1st Theorem:** " Informally, Gödel's incompleteness theorem states that all consistent axiomatic formulations of number theory include undecidable propositions (Hofstadter 1989)."
- **2nd Theorem:** "If number theory is consistent, then a proof of this fact does not exist using the methods of first-order predicate calculus."

Thomas Kuhn: *The Structure of Scientific Revolutions*, 1962

- **Advances in Science**
  - **Not a steady, cumulative acquisition of knowledge**
  - **Peaceful interludes punctuated by intellectually violent revolutions**
- **Paradigm**
  - **Pre-Kuhn: A pattern, exemplar, or example *(OED, 1483)***
  - **Post-Kuhn: "A collection of procedures or ideas that instruct scientists, implicitly, what to believe and how to work." *(Horgan, 2012)***
- **Paradigm Shift**
  - **One world view is replaced by another**
  - **Gödel's theorem: for any axiomatic system there exist propositions that are either undecidable or not provably consistent**
  - **Theory rests on subjective framework**
  - **Propositions are true or false only within the context of a paradigm**

http://blogs.scientificamerican.com/cross-check/2012/05/23/what-thomas-kuhn-really-thought-about-scientific-truth/

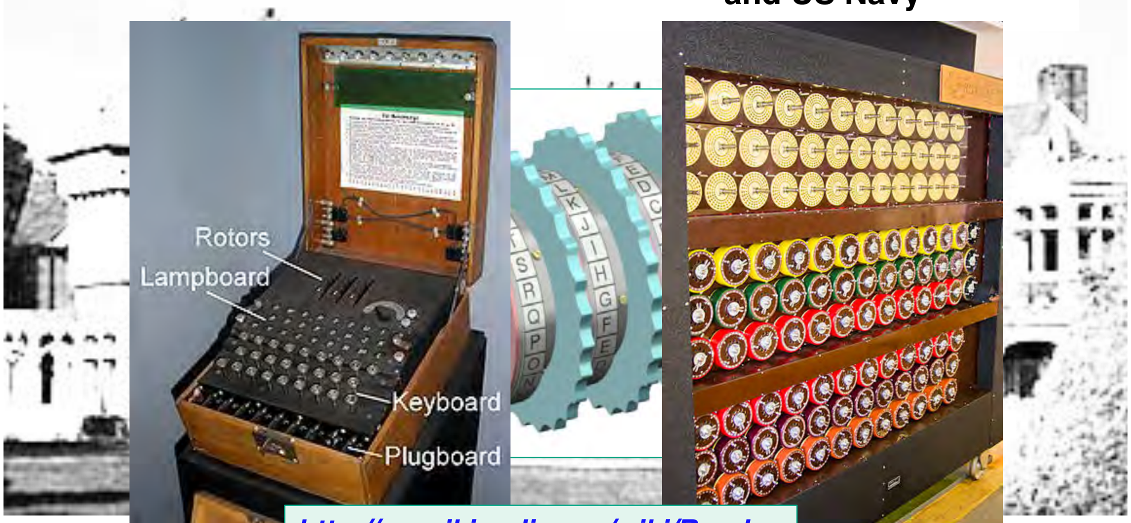
Next Time: Computers, Computing, and Sets

Supplemental Material

Enigma and the Bletchley Park Bombe

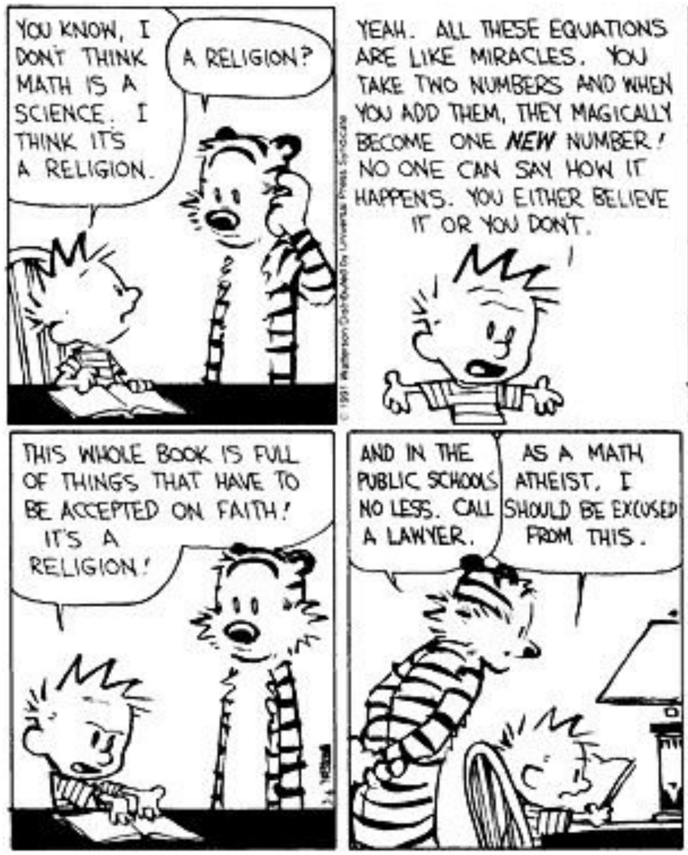
26-letter, 3- or 4-rotor encryption device used by German military during WWII

Algorithmic decyphering computer designed by Polish mathematicians, Alan Turing, and US Navy



http://en.wikipedia.org/wiki/Bombe

Calvin and Hobbes

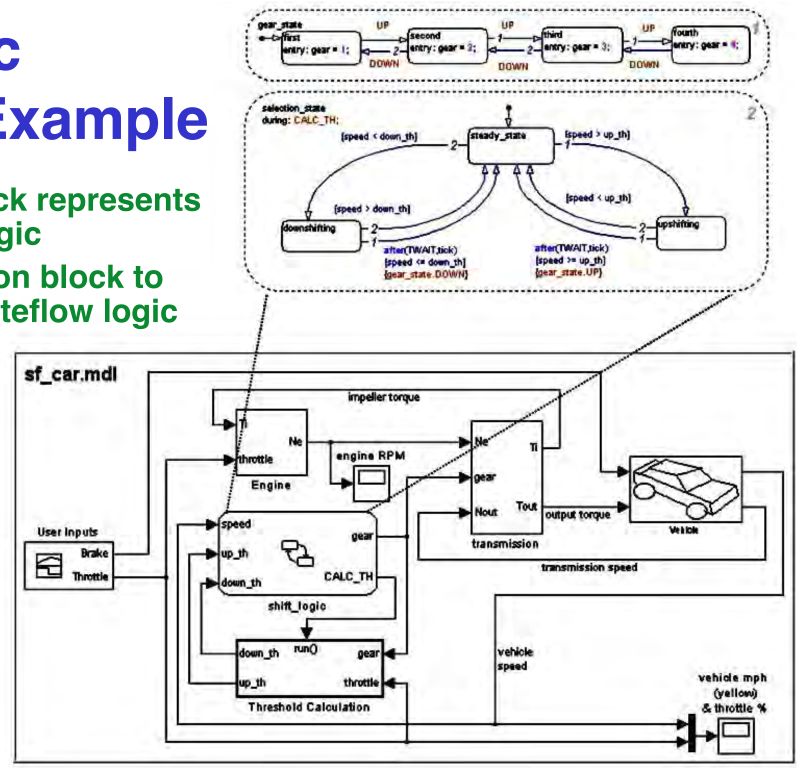


MATLAB Stateflow

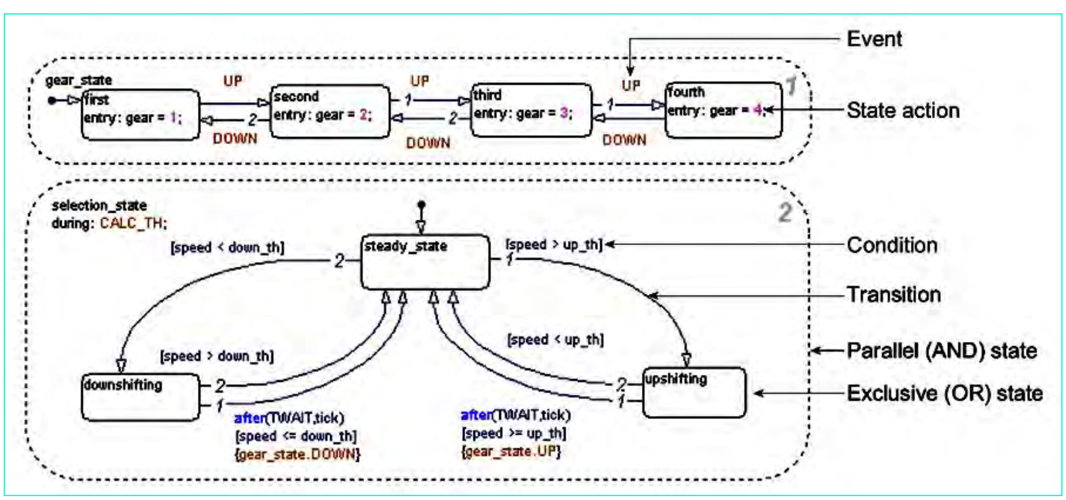
- Incorporation of event-driven logic in a control system
  - Simulink operates within the MATLAB environment
  - Stateflow implements logic blocks within Simulink

Automatic Shifting Example

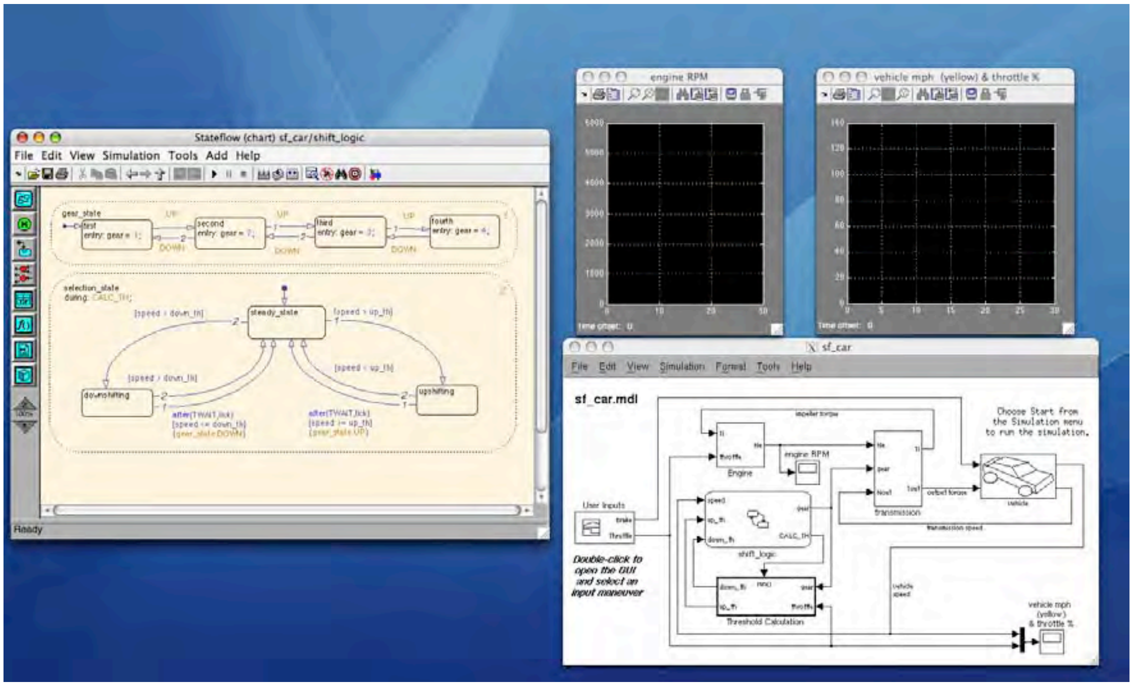
- Stateflow block represents the control logic
- Double-click on block to reveal the Stateflow logic



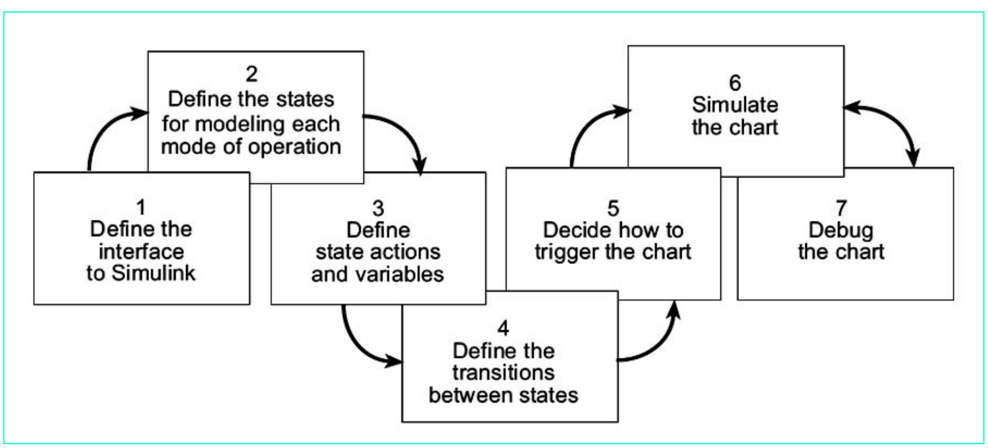
Stateflow Chart for an Automatic Transmission



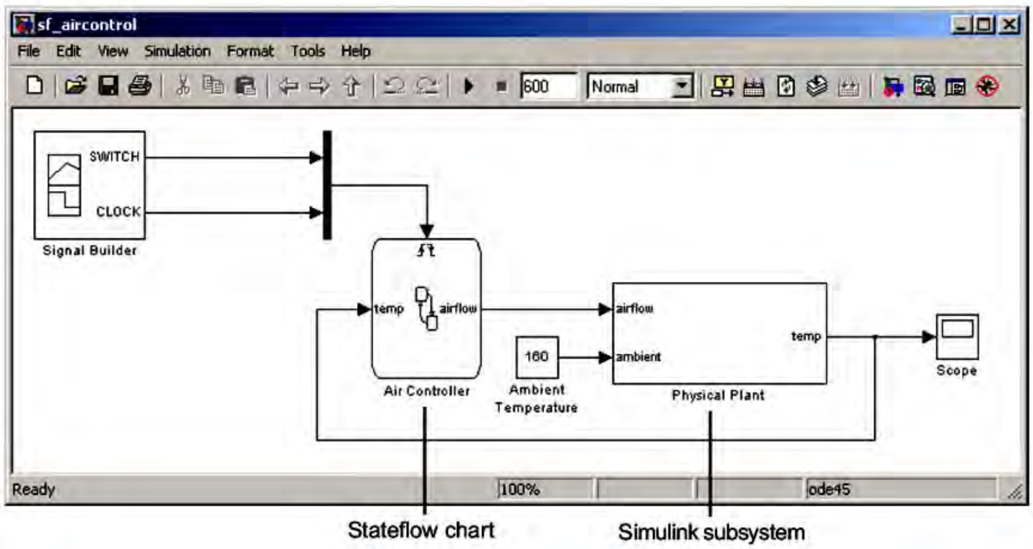
Automatic Shifting Simulation



Combining Discrete-Event Logic with the Dynamic Model



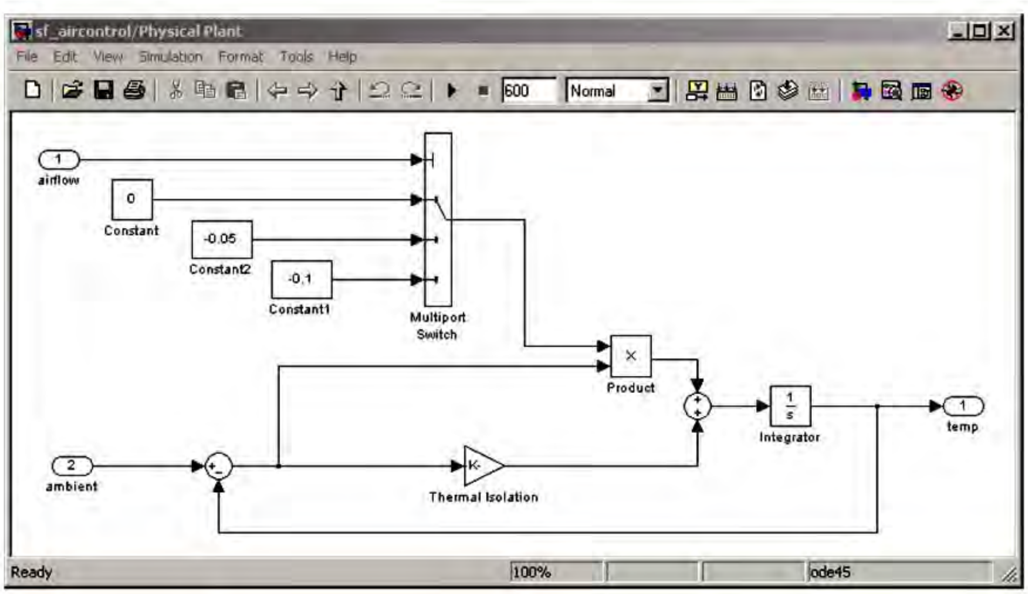
Temperature Control Example



See MATLAB Manual, *Getting Started*, Simulink, for details of model building *(http://www.mathworks.com/access/helpdesk/help/toolbox/stateflow/)*

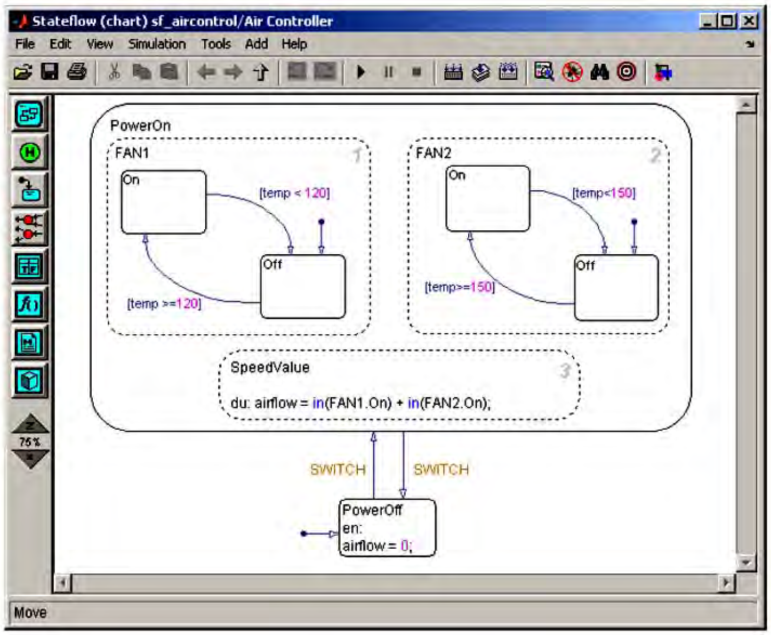
Physical Plant Model

Contents of *Physical Plant*

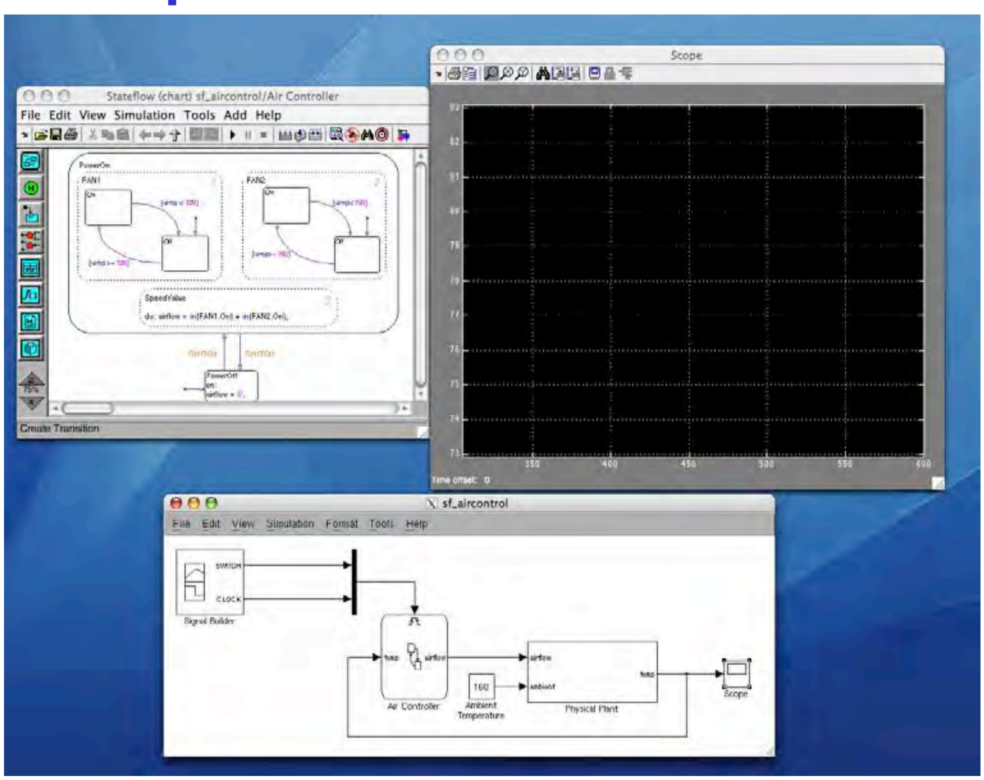


Air Control Logic

Contents of *Air Controller*



Temperature Control Simulation



Solving Rubik's Cube:

An algorithm
http://www.cs.swarthmore.edu/~knerr/helps/rcube.html

This makes me humble
http://www.youtube.com/watch?v=Z9Jq15NqNuQ&feature=related