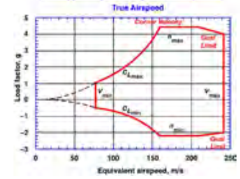
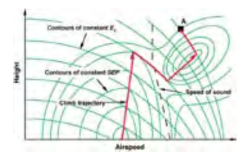
Gliding, Climbing, and Turning Flight Performance

Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2014





Learning Objectives

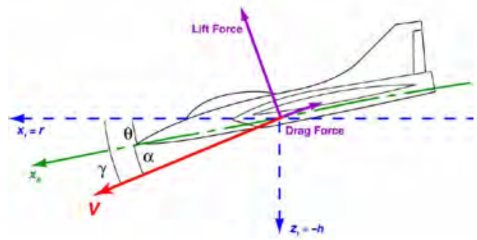
- Conditions for gliding flight
- Parameters for maximizing climb angle and rate
- Review the *V-n* diagram
- Energy height and specific excess power
- Alternative expressions for steady turning flight
- The *Herbst maneuver*

Reading:
Flight Dynamics
Aerodynamic Coefficients, 130-141

http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

Gliding Flight

Equilibrium Gliding Flight



$$D = C_D \frac{1}{2}\rho V^2 S = -W \sin\gamma$$

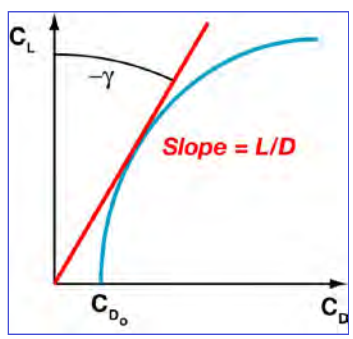
$$C_L \frac{1}{2}\rho V^2 S = W \cos\gamma$$

$$\dot{h} = V \sin\gamma$$

$$\dot{r} = V \cos\gamma$$

Gliding Flight

- Thrust = 0
- Flight path angle < 0 in gliding flight
- Altitude is decreasing
- Airspeed ~ constant
- Air density ~ constant



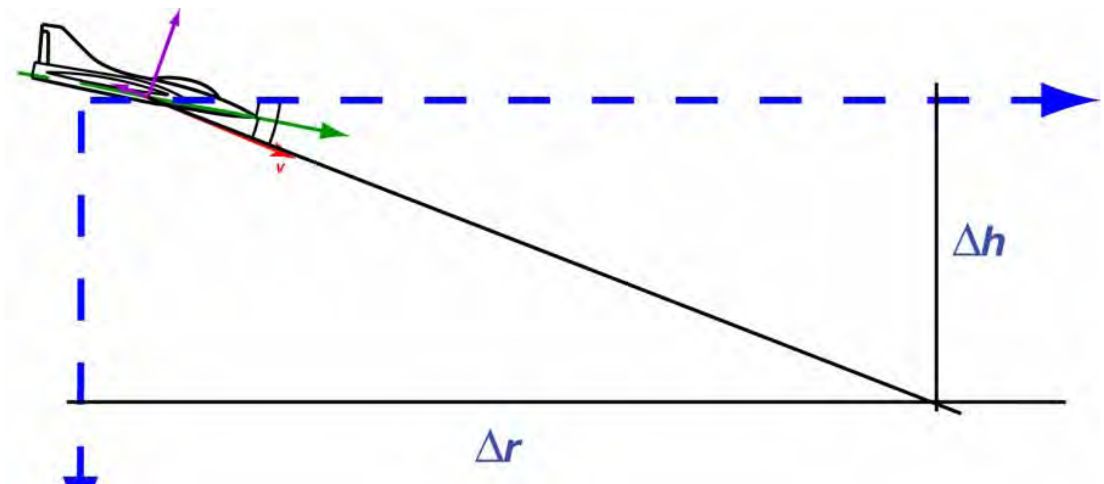
Gliding flight path angle

$$\tan\gamma = -\frac{D}{L} = -\frac{C_D}{C_L} = \frac{\dot{h}}{\dot{r}} = \frac{dh}{dr}; \quad \gamma = -\tan^{-1}\left(\frac{D}{L}\right) = -\cot^{-1}\left(\frac{L}{D}\right)$$

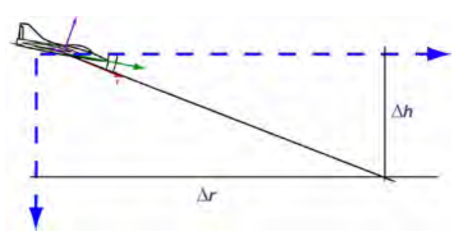
Corresponding airspeed

$$V_{glide} = \sqrt{\frac{2W}{\rho S \sqrt{C_D^2 + C_L^2}}}$$

Maximum Steady Gliding Range



- **Glide range is maximum when γ is least negative, i.e., most positive**
- **This occurs at $(L/D)_{max}$**



Maximum Steady Gliding Range

- **Glide range is maximum when γ is least negative, i.e., most positive**
- **This occurs at $(L/D)_{max}$**

$$\gamma_{max} = -\tan^{-1}\left(\frac{D}{L}\right)_{min} = -\cot^{-1}\left(\frac{L}{D}\right)_{max}$$

$$\tan\gamma = \frac{\dot{h}}{\dot{r}} = \textit{negative constant} = \frac{(h-h_o)}{(r-r_o)}$$

$$\Delta r = \frac{\Delta h}{\tan\gamma} = \frac{-\Delta h}{-\tan\gamma} = \textit{maximum when } \frac{L}{D} = \textit{maximum}$$

Sink Rate, m/s

- **Lift and drag define γ and V in gliding equilibrium**

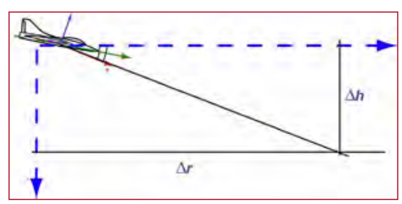
$$D = C_D \frac{1}{2}\rho V^2 S = -W \sin\gamma$$
$$\sin\gamma = -\frac{D}{W}$$

$$L = C_L \frac{1}{2}\rho V^2 S = W \cos\gamma$$
$$V = \sqrt{\frac{2W\cos\gamma}{C_L \rho S}}$$

- **Sink rate = altitude rate, *dh/dt* (negative)**

$$\begin{aligned}\dot{h} &= V\sin\gamma \\ &= -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\left(\frac{D}{W}\right) = -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\left(\frac{L}{W}\right)\left(\frac{D}{L}\right) \\ &= -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\cos\gamma\left(\frac{1}{L/D}\right)\end{aligned}$$

Conditions for Minimum Steady Sink Rate



- **Minimum sink rate provides maximum endurance**
- **Minimize sink rate by setting $\partial(dh/dt)/\partial C_L = 0$ ($\cos\gamma \sim 1$)**

$$\begin{aligned}\dot{h} &= -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\cos\gamma\left(\frac{C_D}{C_L}\right) \\ &= -\sqrt{\frac{2W\cos^3\gamma}{\rho S}}\left(\frac{C_D}{C_L^{3/2}}\right) \approx -\sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}\left(\frac{C_D}{C_L^{3/2}}\right)\end{aligned}$$

$$C_{L_{ME}} = \sqrt{\frac{3C_{D_o}}{\varepsilon}} \quad and \quad C_{D_{ME}} = 4C_{D_o}$$

L/D and V_{ME} for Minimum Sink Rate

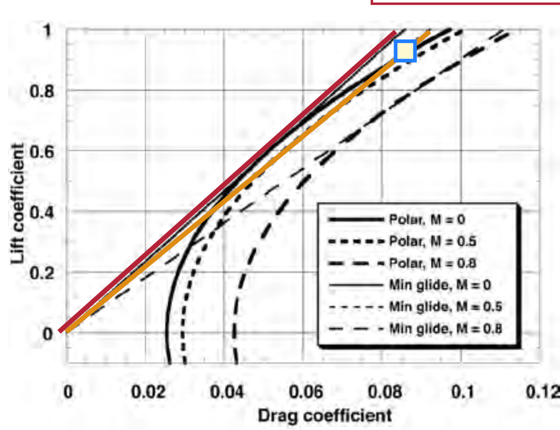
$$\left(L/D\right)_{ME} = \frac{1}{4}\sqrt{\frac{3}{\varepsilon C_{D_o}}} = \frac{\sqrt{3}}{2}\left(L/D\right)_{max} \approx 0.86\left(L/D\right)_{max}$$

$$V_{ME} = \sqrt{\frac{2W}{\rho S\sqrt{C_{D_{ME}}^2 + C_{L_{ME}}^2}}} \approx \sqrt{\frac{2\left(W/S\right)}{\rho}\sqrt{\frac{\varepsilon}{3C_{D_o}}}} \approx 0.76 V_{L/D_{max}}$$

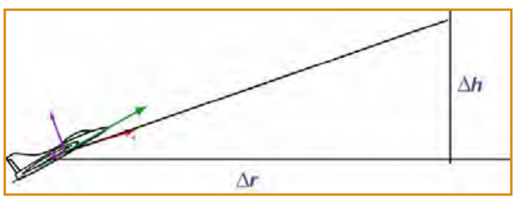
L/D for Minimum Sink Rate

- **For $L/D < L/D_{max}$, there are two solutions**
- **Which one produces minimum sink rate?**

$$\left(L/D\right)_{ME} \approx 0.86\left(L/D\right)_{max}$$
$$V_{ME} \approx 0.76 V_{L/D_{max}}$$



Climbing Flight



Climbing Flight

- **Flight path angle**

$$\dot{V} = 0 = \frac{(T - D - W\sin\gamma)}{m}$$
$$\sin\gamma = \frac{(T-D)}{W}; \quad \gamma = \sin^{-1}\frac{(T-D)}{W}$$

- **Required lift**

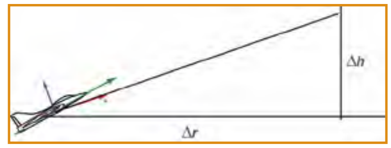
$$\dot{\gamma} = 0 = \frac{(L - W\cos\gamma)}{mV}$$
$$L = W\cos\gamma$$

- **Rate of climb, *dh/dt* = Specific Excess Power**

$$\dot{h} = V\sin\gamma = V\frac{(T-D)}{W} = \frac{(P_{thrust} - P_{drag})}{W}$$

$$\textit{Specific Excess Power (SEP)} = \frac{\textit{Excess Power}}{\textit{Unit Weight}} \equiv \frac{(P_{thrust} - P_{drag})}{W}$$

Steady Rate of Climb



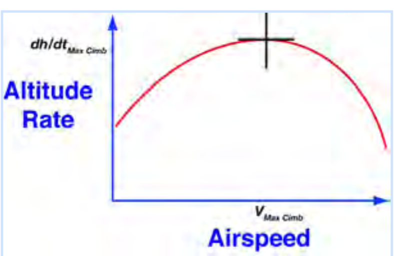
- **Climb rate**

$$\dot{h} = V \sin\gamma = V\left[\left(\frac{T}{W}\right) - \frac{\left(C_{D_o} + \varepsilon C_L^2\right)\bar{q}}{\left(W/S\right)}\right]$$

$$L = C_L \bar{q} S = W \cos\gamma$$
$$C_L = \left(\frac{W}{S}\right)\frac{\cos\gamma}{\bar{q}}$$
$$V = \sqrt{2\left(\frac{W}{S}\right)\frac{\cos\gamma}{C_L \rho}}$$

- **Note significance of thrust-to-weight ratio and wing loading**

$$\dot{h} = V\left[\left(\frac{T}{W}\right) - \frac{C_{D_o}\bar{q}}{\left(W/S\right)} - \frac{\varepsilon\left(W/S\right)\cos^2\gamma}{\bar{q}}\right]$$
$$= V\left(\frac{T(h)}{W}\right) - \frac{C_{D_o}\rho(h)V^3}{2\left(W/S\right)} - \frac{2\varepsilon\left(W/S\right)\cos^2\gamma}{\rho(h)V}$$

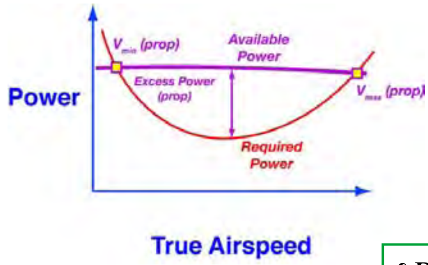


Condition for Maximum Steady Rate of Climb

$$\dot{h} = V\left(\frac{T}{W}\right) - \frac{C_{D_o}\rho V^3}{2\left(W/S\right)} - \frac{2\varepsilon\left(W/S\right)\cos^2\gamma}{\rho V}$$

- **Necessary condition for a maximum with respect to airspeed**

$$\frac{\partial\dot{h}}{\partial V} = 0 = \left[\left(\frac{T}{W}\right) + V\left(\frac{\partial T/\partial V}{W}\right)\right] - \frac{3C_{D_o}\rho V^2}{2\left(W/S\right)} + \frac{2\varepsilon\left(W/S\right)\cos^2\gamma}{\rho V^2}$$



Maximum Steady Rate of Climb:
Propeller-Driven Aircraft

- At constant power

$$\frac{\partial P_{thrust}}{\partial V} = 0 = \left[\left(\frac{T}{W}\right) + V\left(\frac{\partial T / \partial V}{W}\right)\right]$$

- With $\cos^2\gamma \sim 1$, optimality condition reduces to

$$\frac{\partial \dot{h}}{\partial V} = 0 = -\frac{3C_{D_o}\rho V^2}{2(W/S)} + \frac{2\varepsilon(W/S)}{\rho V^2}$$

- Airspeed for maximum rate of climb at maximum power, P_{max}

$$V^4 = \left(\frac{4}{3}\right)\frac{\varepsilon(W/S)^2}{C_{D_o}\rho^2}; \quad V = \sqrt{2\frac{(W/S)}{\rho}\sqrt{\frac{\varepsilon}{3C_{D_o}}}} = V_{ME}$$

Maximum Steady Rate of Climb:
Jet-Driven Aircraft

- Condition for a maximum at constant thrust and $\cos^2\gamma \sim 1$

$$\frac{\partial \dot{h}}{\partial V} = 0$$

$$= -\frac{3C_{D_o}\rho}{2(W/S)}V^4 + \left(\frac{T}{W}\right)V^2 + \frac{2\varepsilon(W/S)}{\rho}$$

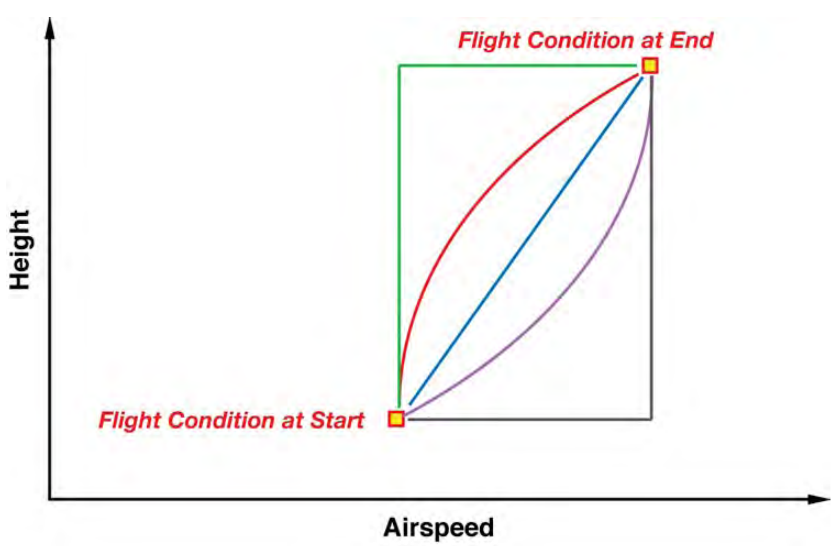
$$= -\frac{3C_{D_o}\rho}{2(W/S)}\left(V^2\right)^2 + \left(\frac{T}{W}\right)\left(V^2\right) + \frac{2\varepsilon(W/S)}{\rho}$$

- Airspeed for maximum rate of climb at maximum thrust, T_{max}

$$0 = ax^2 + bx + c \text{ and } \quad V = +\sqrt{x}$$

Optimal Climbing Flight

What is the Fastest Way to Climb from One Flight Condition to Another?

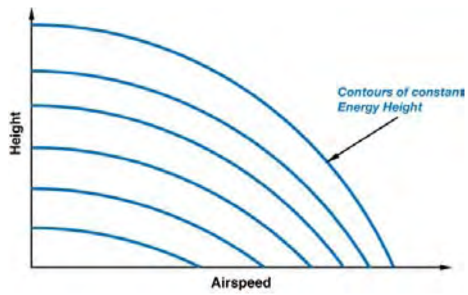


Energy Height

- **Specific Energy**
  - **= (Potential + Kinetic Energy) per Unit Weight**
  - **= Energy Height**

$$\frac{Total\ Energy}{Unit\ Weight} \equiv Specific\ Energy = \frac{mgh + mV^2/2}{mg} = h + \frac{V^2}{2g}$$

$$\equiv \textbf{\textit{Energy Height}}, E_h, \quad ft\ or\ m$$



- **Could trade altitude with airspeed with no change in energy height if thrust and drag were zero**

Specific Excess Power

Rate of change of Specific Energy

$$\frac{dE_h}{dt} = \frac{d}{dt}\left(h + \frac{V^2}{2g}\right) = \frac{dh}{dt} + \left(\frac{V}{g}\right)\frac{dV}{dt}$$

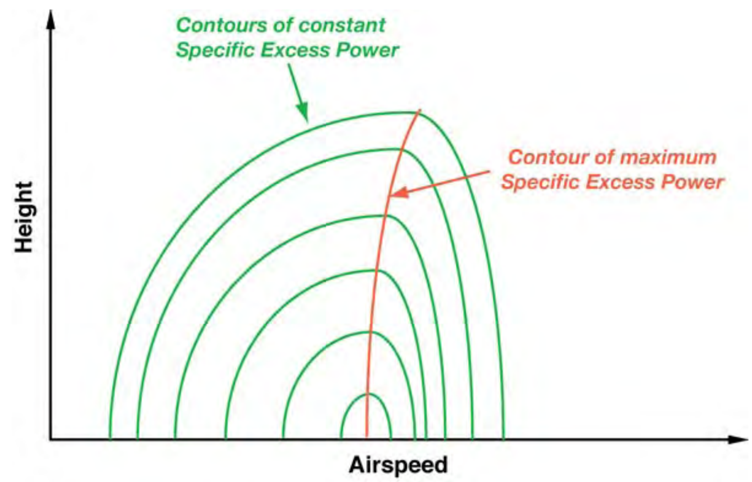
$$= V\sin\gamma + \left(\frac{V}{g}\right)\left(\frac{T - D - mg\sin\gamma}{m}\right) = V\frac{(T-D)}{W} = V\frac{(C_T - C_D)\frac{1}{2}\rho(h)V^2 S}{W}$$

$$= \textbf{\textit{Specific Excess Power}}\ (SEP)$$

$$= \frac{Excess\ Power}{Unit\ Weight} \equiv \frac{\left(P_{thrust} - P_{drag}\right)}{W}$$

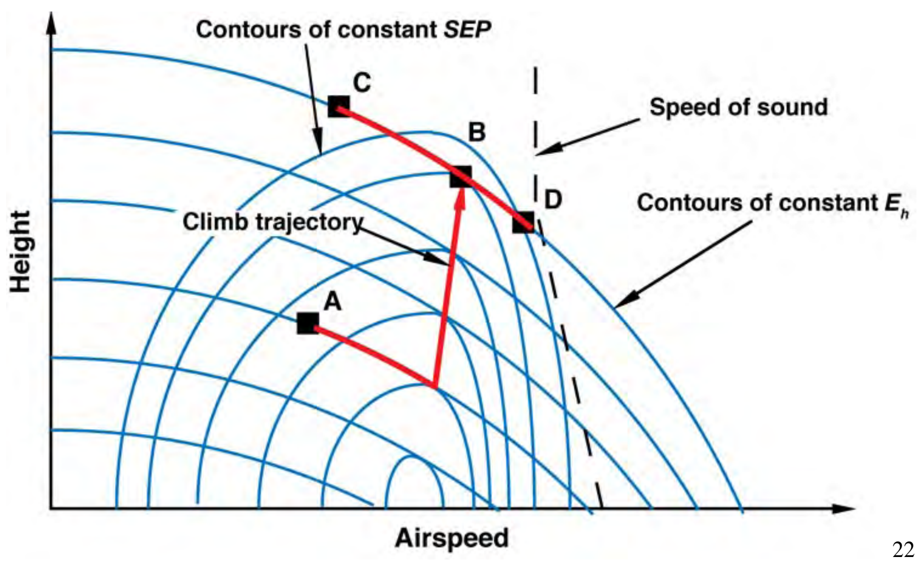
Contours of Constant Specific Excess Power

- Specific Excess Power is a function of altitude and airspeed
- *SEP* is maximized at each altitude, *h*, when $\boxed{\dfrac{d\left[SEP(h)\right]}{dV} = 0}$



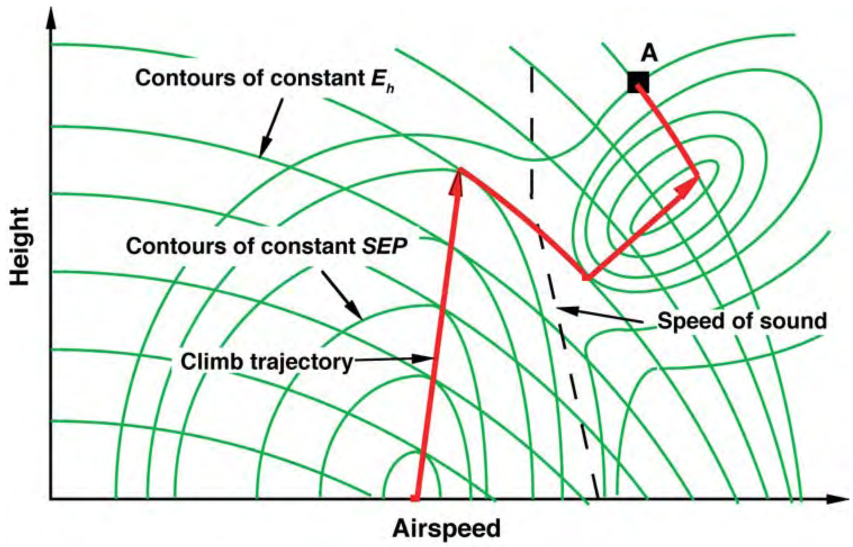
Subsonic Energy Climb

- **Objective:** Minimize time or fuel to climb to desired altitude and airspeed



Supersonic Energy Climb

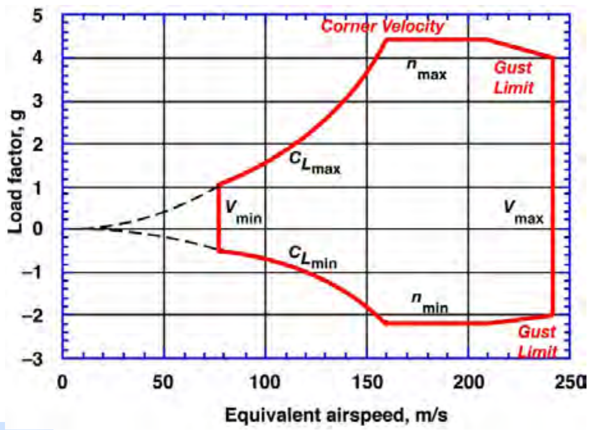
- **Objective:** **Minimize time or fuel to climb to desired altitude and airspeed**



The Maneuvering Envelope

Typical Maneuvering Envelope: *V-n* Diagram

- **Maneuvering envelope**: limits on normal load factor and allowable equivalent airspeed
  - Structural factors
  - Maximum and minimum achievable lift coefficients
  - Maximum and minimum airspeeds
  - Protection against overstressing due to gusts
  - Corner Velocity: Intersection of maximum lift coefficient and maximum load factor

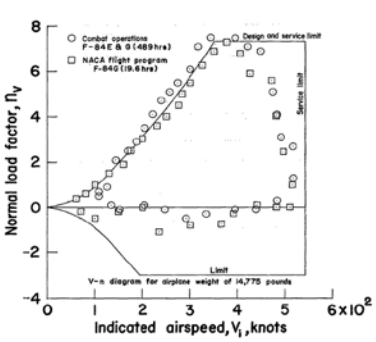


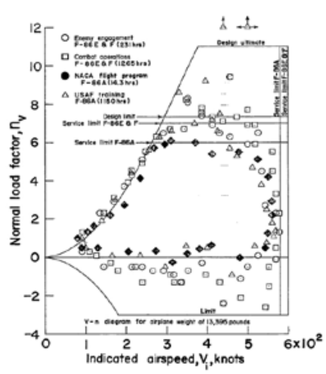
- Typical positive load factor limits
  - Transport: > 2.5
  - Utility: > 4.4
  - Aerobatic: > 6.3
  - Fighter: > 9

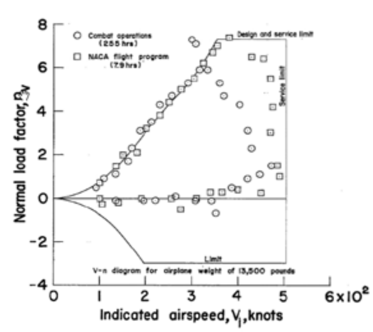
- Typical negative load factor limits
  - Transport: < –1
  - Others: < –1 to –3

C-130 exceeds maneuvering envelope
http://www.youtube.com/watch?v=4bDNCac2N1o&feature=related

Maneuvering Envelopes (*V-n Diagrams*) for Three Fighters of the Korean War Era







Turning Flight

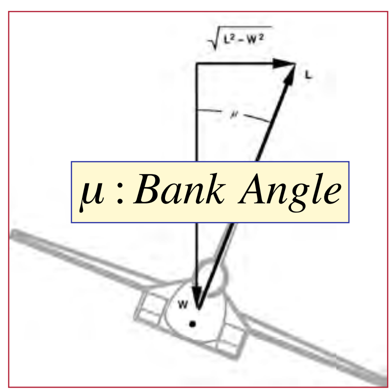
Level Turning Flight

- **Level flight = constant altitude**
- **Sideslip angle = 0**
- **Vertical force equilibrium**

$$\boxed{L\cos\mu = W}$$

- **Load factor**

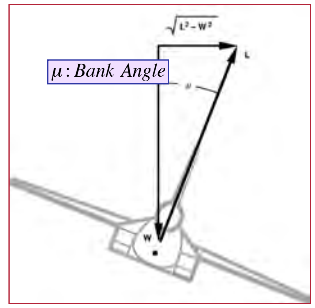
$$\boxed{n = L/W = L/mg = \sec\mu, \text{"}g\text{"}s}$$



- **Thrust required to maintain level flight**

$$\boxed{\begin{aligned} T_{req} &= \left(C_{D_o} + \varepsilon C_L^2\right)\frac{1}{2}\rho V^2 S = D_o + \frac{2\varepsilon}{\rho V^2 S}\left(\frac{W}{\cos\mu}\right)^2 \\ &= D_o + \frac{2\varepsilon}{\rho V^2 S}(nW)^2 \end{aligned}}$$

Maximum Bank Angle in Level Flight



- **Bank angle**

$$\cos\mu = \frac{W}{C_L\bar{q}S} = \frac{1}{n} = W\sqrt{\frac{2\varepsilon}{\left(T_{req} - D_o\right)\rho V^2 S}}$$

$$\mu = \cos^{-1}\left(\frac{W}{C_L\bar{q}S}\right) = \cos^{-1}\left(\frac{1}{n}\right) = \cos^{-1}\left[W\sqrt{\frac{2\varepsilon}{\left(T_{req} - D_o\right)\rho V^2 S}}\right]$$

- **Bank angle is limited by**

$$C_{L_{\max}} \;\; or \; T_{\max} \;\; or \; n_{\max}$$

Turning Rate and Radius in Level Flight

- **Turning rate**

$$\dot{\xi} = \frac{C_L\bar{q}S\sin\mu}{mV} = \frac{W\tan\mu}{mV} = \frac{g\tan\mu}{V} = \frac{\sqrt{L^2 - W^2}}{mV}$$

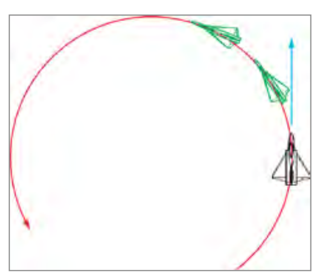
$$= \frac{W\sqrt{n^2 - 1}}{mV} = \frac{\sqrt{\left(T_{req} - D_o\right)\rho V^2 S/2\varepsilon - W^2}}{mV}$$

- **Turning rate is limited by**

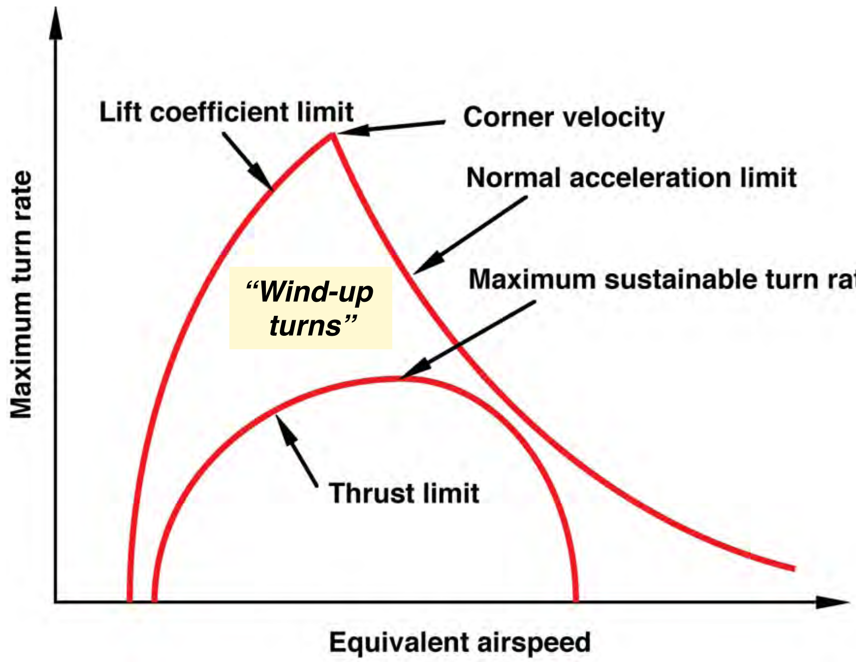
$$C_{L_{\max}} \;\; or \; T_{\max} \;\; or \; n_{\max}$$

- **Turning radius**

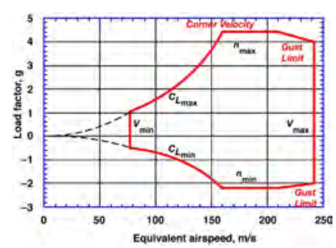
$$R_{turn} = \frac{V}{\dot{\xi}} = \frac{V^2}{g\sqrt{n^2 - 1}}$$



Maximum Turn Rates



Corner Velocity Turn



- **Corner velocity**

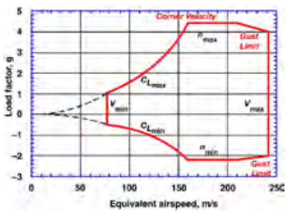
$$V_{corner} = \sqrt{\frac{2 n_{max} W}{C_{L_{mas}} \rho S}}$$

- **For steady climbing or diving flight**

$$\sin\gamma = \frac{T_{max} - D}{W}$$

- **Turning radius**

$$R_{turn} = \frac{V^2 \cos^2\gamma}{g\sqrt{n_{max}^2 - \cos^2\gamma}}$$



Corner Velocity Turn

- **Turning rate**

$$\dot{\xi} = \sqrt{\frac{g\left(n_{max}^2 - \cos^2\gamma\right)}{V\cos\gamma}}$$

- **Time to complete a full circle**

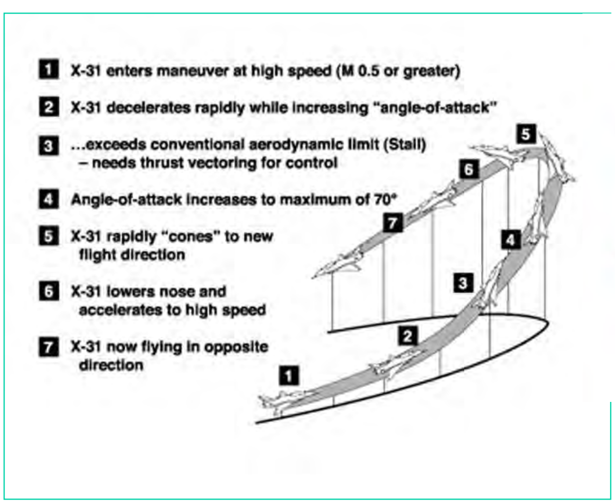
$$t_{2\pi} = \frac{V\cos\gamma}{g\sqrt{n_{max}^2 - \cos^2\gamma}}$$

- **Altitude gain/loss**

$$\Delta h_{2\pi} = t_{2\pi} V \sin\gamma$$

Herbst Maneuver

- Minimum-time reversal of direction
- Kinetic-/potential-energy exchange
- Yaw maneuver at low airspeed
- *X-31* performing the maneuver



Next Time:
Aircraft Equations of Motion

Reading:
Flight Dynamics,
Section 3.1, 3.2, pp. 155-161
Airplane Stability and Control
Chapter 5

Supplemental Material

Gliding Flight of the P-51 Mustang

Maximum Range Glide

$$Loaded\ Weight = 9{,}200\ lb\ (3{,}465\ kg)$$

$$(L/D)_{\max} = \frac{1}{2\sqrt{\varepsilon C_{D_o}}} = 16.31$$

$$\gamma_{MR} = -\cot^{-1}\left(\frac{L}{D}\right)_{\max} = -\cot^{-1}(16.31) = -3.51°$$

$$(C_D)_{L/D_{\max}} = 2C_{D_o} = 0.0326$$

$$(C_L)_{L/D_{\max}} = \sqrt{\frac{C_{D_o}}{\varepsilon}} = 0.531$$

$$V_{L/D_{\max}} = \frac{76.49}{\sqrt{\rho}}\ m/s$$

$$\dot{h}_{L/D_{\max}} = V\sin\gamma = -\frac{4.68}{\sqrt{\rho}}\ m/s$$

$$R_{h_o=10km} = (16.31)(10) = 163.1\ km$$

Maximum Endurance Glide

$$Loaded\ Weight = 9{,}200\ lb\ (3{,}465\ kg)$$

$$S = 21.83\ m^2$$

$$C_{D_{ME}} = 4C_{D_o} = 4(0.0163) = 0.0652$$

$$C_{L_{ME}} = \sqrt{\frac{3C_{D_o}}{\varepsilon}} = \sqrt{\frac{3(0.0163)}{0.0576}} = 0.921$$

$$(L/D)_{ME} = 14.13$$

$$\dot{h}_{ME} = -\sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}\left(\frac{C_{D_{ME}}}{C_{L_{ME}}^{3/2}}\right) = -\frac{4.11}{\sqrt{\rho}}\ m/s$$

$$\gamma_{ME} = -4.05°$$

$$V_{ME} = \frac{58.12}{\sqrt{\rho}}\ m/s$$