Linearized Equations of Motion

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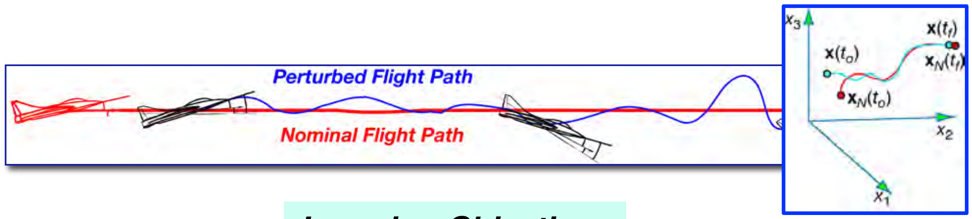
MAE 331, 2014



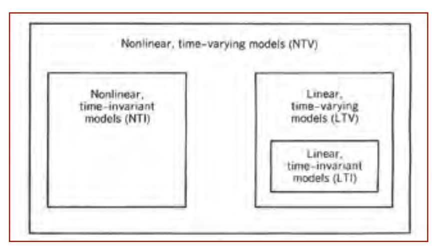
Learning Objectives

- Develop linear equations to describe small perturbational motions
- Apply to aircraft dynamic equations

Reading:
Flight Dynamics
234-242, 255-266, 274-297, 321-325, 329-330

http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

How Is System Response Calculated?

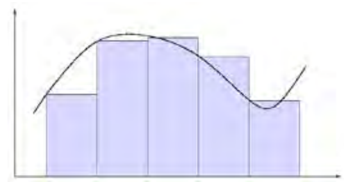


- **Linear and nonlinear, time-varying and time-invariant dynamic models**
  - **Numerical integration ("time domain")**
- **Linear, time-invariant (LTI) dynamic models**
  - **Numerical integration ("time domain")**
  - **State transition ("time domain")**
  - **Transfer functions ("frequency domain")**

Integration Algorithms

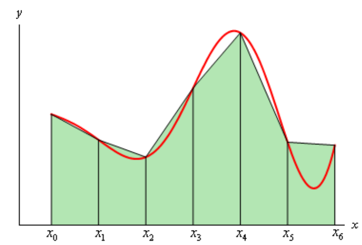
- **Exact** $$\boxed{\mathbf{x}(T) = \mathbf{x}(0) + \int_0^T \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t)\right]dt}$$
- **Rectangular (Euler) Integration**

$$\begin{gathered}\mathbf{x}(t_k) = \mathbf{x}(t_{k-1}) + \delta\mathbf{x}(t_{k-1},t_k) \\ \approx \mathbf{x}(t_{k-1}) + \mathbf{f}\left[\mathbf{x}(t_{k-1}),\mathbf{u}(t_{k-1}),\mathbf{w}(t_{k-1})\right]\delta t \\ \delta t = t_k - t_{k-1}\end{gathered}$$



- **Trapezoidal (modified Euler) Integration** *(~MATLAB's ode23)*

$$\begin{gathered}\mathbf{x}(t_k) \approx \mathbf{x}(t_{k-1}) + \frac{1}{2}\left[\delta\mathbf{x}_1 + \delta\mathbf{x}_2\right] \\ where \\ \delta\mathbf{x}_1 = \mathbf{f}\left[\mathbf{x}(t_{k-1}),\mathbf{u}(t_{k-1}),\mathbf{w}(t_{k-1})\right]\delta t \\ \delta\mathbf{x}_2 = \mathbf{f}\left[\mathbf{x}(t_{k-1}) + \delta\mathbf{x}_1,\mathbf{u}(t_k),\mathbf{w}(t_k)\right]\delta t\end{gathered}$$



See MATLAB manual for descriptions of *ode45* and *ode15s*

Numerical Integration: MATLAB Ordinary Differential Equation Solvers*

- Explicit Runge-Kutta Algorithm
- Adams-Bashforth-Moulton Algorithm
- Numerical Differentiation Formula
- Modified Rosenbrock Method
- Trapezoidal Rule
- Trapezoidal Rule w/Back Differentiation

Solver	Problem Type	Order of Accuracy	When to Use
`ode45`	Nonstiff	Medium	Most of the time. This should be the first solver you try.
`ode23`	Nonstiff	Low	For problems with crude error tolerances or for solving moderately stiff problems.
`ode113`	Nonstiff	Low to high	For problems with stringent error tolerances or for solving computationally intensive problems.
`ode15s`	Stiff	Low to medium	If `ode45` is slow because the problem is stiff.
`ode23s`	Stiff	Low	If using crude error tolerances to solve stiff systems and the mass matrix is constant.
`ode23t`	Moderately Stiff	Low	For moderately stiff problems if you need a solution without numerical damping.
`ode23tb`	Stiff	Low	If using crude error tolerances to solve stiff systems.

* http://www.mathworks.com/access/helpdesk/help/techdoc/index.html?/access/helpdesk/help/techdoc/ref/ode23.html.
Shampine, L. F. and M. W. Reichelt, "The MATLAB ODE Suite," *SIAM Journal on Scientific Computing*, Vol. 18, 1997, pp 1-22.

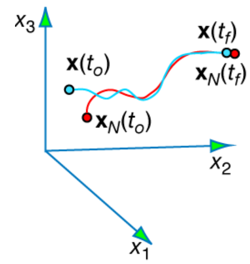
Nominal and Actual Trajectories

- **Nominal (or reference) trajectory and control history**

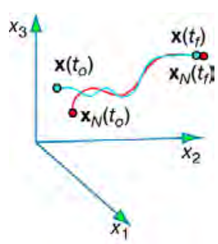
$\mathbf{x}$: *dynamic state*
$\mathbf{u}$: *control input*
$\mathbf{w}$: *disturbance input*

$$\left\{\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t)\right\} \quad \text{for } t \text{ in } [t_o, t_f]$$

- **Actual trajectory perturbed by**
  - Small initial condition variation, $\Delta\mathbf{x}_o(t_o)$
  - Small control variation, $\Delta\mathbf{u}(t)$



$$\left\{\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)\right\} \quad \text{for } t \text{ in } [t_o, t_f]$$
$$= \left\{\mathbf{x}_N(t) + \Delta\mathbf{x}(t), \mathbf{u}_N(t) + \Delta\mathbf{u}(t), \mathbf{w}_N(t) + \Delta\mathbf{w}(t)\right\}$$



Both Paths Satisfy the Dynamic Equations

Dynamic models for the actual and the nominal problems are the same

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t)], \quad \mathbf{x}_N(t_o) \text{ given}$$
$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)], \quad \mathbf{x}(t_o) \text{ given}$$

Differences in initial condition and forcing ...

$$\Delta\mathbf{x}(t_o) = \mathbf{x}(t_o) - \mathbf{x}_N(t_o)$$
$$\left\{\begin{array}{l} \Delta\mathbf{u}(t) = \mathbf{u}(t) - \mathbf{u}_N(t) \\ \Delta\mathbf{w}(t) = \mathbf{w}(t) - \mathbf{w}_N(t) \end{array}\right\} \text{ in } [t_o, t_f]$$

... perturb rate of change and the state

$$\left\{\begin{array}{l} \dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t) \\ \mathbf{x}(t) = \mathbf{x}_N(t) + \Delta\mathbf{x}(t) \end{array}\right\} \text{ in } [t_o, t_f]$$

Approximate Neighboring Trajectory as a Linear Perturbation to the Nominal Trajectory

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t]$$
$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}_N(t) + \Delta\mathbf{x}(t), \mathbf{u}_N(t) + \Delta\mathbf{u}(t), \mathbf{w}_N(t) + \Delta\mathbf{w}(t), t]$$

Approximate the new trajectory as the sum of the nominal path plus a linear perturbation

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t)$$
$$\approx \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t] + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\Delta\mathbf{x}(t) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\Delta\mathbf{u}(t) + \frac{\partial \mathbf{f}}{\partial \mathbf{w}}\Delta\mathbf{w}(t)$$

Linearized Equation Approximates Perturbation Dynamics

- **Solve for the nominal and perturbation trajectories <u>separately</u>**

Nominal Equation

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t], \quad \mathbf{x}_N(t_o) \text{ given}$$

$\dim(\mathbf{x}) = n \times 1$
$\dim(\mathbf{u}) = m \times 1$
$\dim(\mathbf{w}) = s \times 1$

Perturbation Equation

$$\Delta\dot{\mathbf{x}}(t) \approx \left[\left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} \Delta\mathbf{x}(t) \right] + \left[\left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} \Delta\mathbf{u}(t) \right] + \left[\left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} \Delta\mathbf{w}(t) \right]$$
$$\triangleq \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t) + \mathbf{L}(t)\Delta\mathbf{w}(t), \quad \Delta\mathbf{x}(t_o) \text{ given}$$

$\dim(\Delta\mathbf{x}) = n \times 1$
$\dim(\Delta\mathbf{u}) = m \times 1$
$\dim(\Delta\mathbf{w}) = s \times 1$

Jacobian Matrices Express Solution Sensitivity to Small Perturbations

Stability matrix, F, is square

$$\mathbf{F}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \begin{bmatrix} \partial f_1/\partial x_1 & \dots & \partial f_1/\partial x_n \\ \dots & \dots & \dots \\ \partial f_n/\partial x_1 & \dots & \partial f_n/\partial x_n \end{bmatrix}_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

$$\boxed{\dim(\mathbf{F}) = n \times n}$$

Sensitivity to Control Perturbations, G

$$\mathbf{G}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 & \dots & \partial f_1/\partial u_m \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 & \dots & \partial f_2/\partial u_m \\ \dots & \dots & \dots & \dots \\ \partial f_n/\partial u_1 & \partial f_n/\partial u_2 & \dots & \partial f_n/\partial u_m \end{bmatrix}_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

$$\boxed{\dim(\mathbf{G}) = n \times m}$$

Sensitivity to Disturbance Perturbations, G

$$\mathbf{L}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \left[\begin{array}{cccc} \partial f_1/\partial w_1 & \partial f_1/\partial w_2 & \cdots & \partial f_1/\partial w_s \\ \partial f_2/\partial w_1 & \partial f_2/\partial w_2 & \cdots & \partial f_2/\partial w_s \\ \cdots & \cdots & \cdots & \cdots \\ \partial f_n/\partial w_1 & \partial f_n/\partial w_2 & \cdots & \partial f_n/\partial w_s \end{array}\right]_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

$\dim(\mathbf{L}) = n \times s$

Scalar Example

Actual System $\dot{x}(t) = ax(t) + bx^2(t) + cu(t) + dw^3(t)$

Nominal System $\dot{x}_N(t) = ax_N(t) + bx_N^2(t) + cu_N(t) + dw_N^3(t)$

Perturbation System $\Delta\dot{x}(t) = a\Delta x(t) + 2bx_N\Delta x(t) + c\Delta u(t) + 3dw_N^2\Delta w(t)$

Numerical Example

$$a = 1, b = 2, c = 3, d = 4$$
$$\dot{x}(t) = x(t) + 2x^2(t) + 3u(t) + 4w^3(t)$$

$$\Delta\dot{x}(t) = \Delta x(t) + 4x_N\Delta x(t) + 3\Delta u(t) + 12w_N^2\Delta w(t)$$

Comparison of Damped Linear and Nonlinear Systems

Linear Spring

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) - x_2(t)$$

Displacement

Rate of Change

Spring **Damper**

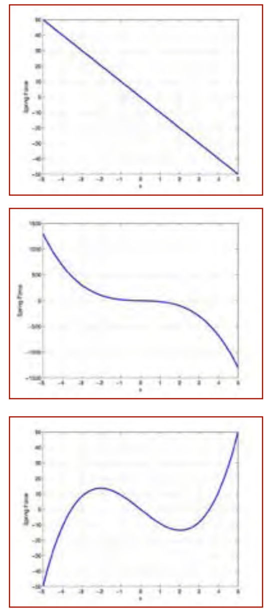
Linear plus Stiffening Cubic Spring

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) - 10x_1^{\,3}(t) - x_2(t)$$

Linear plus Weakening Cubic Spring

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) + 0.8x_1^{\,3}(t) - x_2(t)$$

Spring Force vs. Displacement



MATLAB Simulation of Linear and Nonlinear Dynamic Systems

MATLAB Main Script

```
%   Nonlinear and Linear Examples
    clear
    tspan   =   [0 10];
    xo =   [0, 10];
    [t1,x1 = ode23('NonLin',tspan,xo);
    xo = [0, 1];
    [t2,x2] = ode23('NonLin',tspan,xo);
    xo = [0, 10];
    [t3,x3] =  ode23('Lin',tspan,xo);
    xo = [0, 1];
    [t4,x4] = ode23('Lin',tspan,xo);

    subplot(2,1,1)
    plot(t1,x1(:,1),'k',t2,x2(:,1),'b',t3,x3(:,1),'r',t4,x4(:,1),'g')
    ylabel('Position'), grid
    subplot(2,1,2)
    plot(t1,x1(:,2),'k',t2,x2(:,2),'b',t3,x3(:,2),'r',t4,x4(:,2),'g')
    xlabel('Time'), ylabel('Rate'), grid
```

Linear System

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) - x_2(t)$$

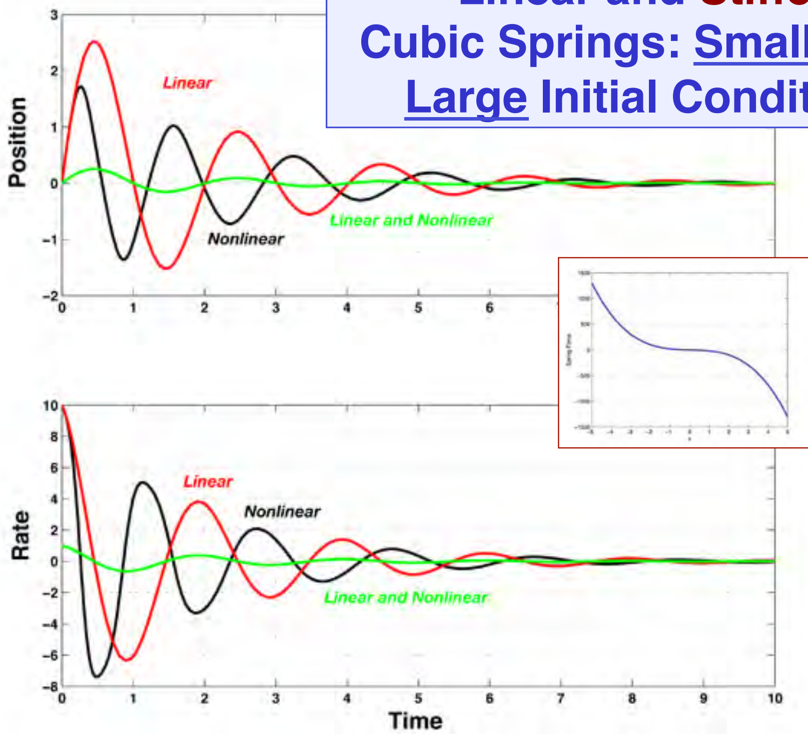
```
function    xdot    =   Lin(t,x)
%   Linear Ordinary Differential Equation
% x(1) = Position
% x(2) = Rate
xdot = [x(2)
        -10*x(1) - x(2)];
```

Nonlinear System

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) + 0.8x_1^{\,3}(t) - x_2(t)$$

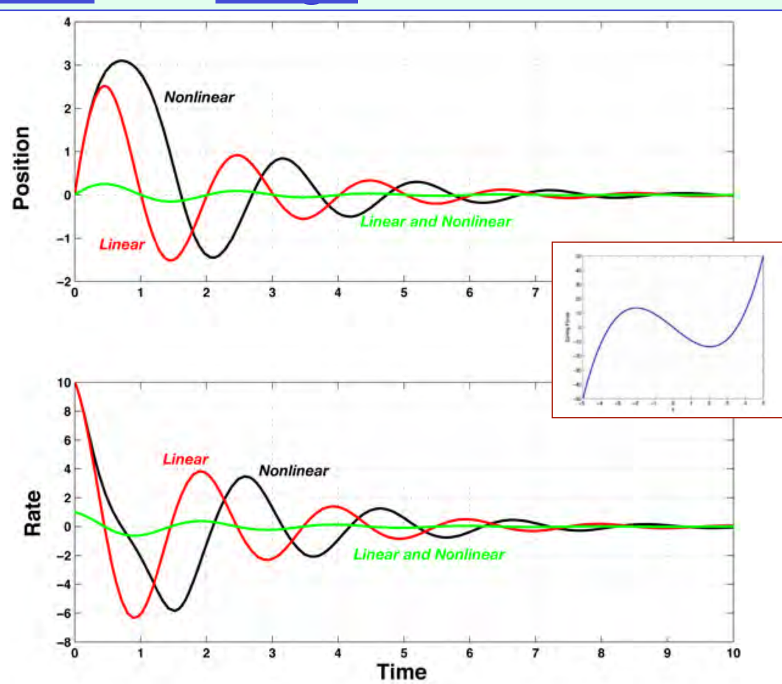
```
function    xdot    =   NonLin(t,x)
%   Nonlinear Ordinary Differential Equation
% x(1) = Position
% x(2) = Rate
xdot = [x(2)
        -10*x(1) + 0.8*x(1)^3 - x(2)];
```

Linear and Stiffening Cubic Springs: Small and Large Initial Conditions



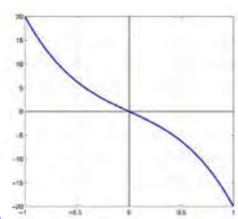
Linear and nonlinear responses are indistinguishable with small initial condition

Linear and Weakening Cubic Springs: Small and Large Initial Conditions



Linear, Time-Varying (LTV) Approximation of Perturbation Dynamics

Stiffening Linear-Cubic Spring Example



Nonlinear, time-invariant (NTI) equation

$$\dot{x}_1(t) = f_1 = x_2(t)$$
$$\dot{x}_2(t) = f_2 = -10x_1(t) - 10x_1{}^3(t) - x_2(t)$$

Integrate equations to produce nominal path

$$\begin{bmatrix} x_{1_N}(0) \\ x_{2_N}(0) \end{bmatrix} \Rightarrow \int_0^{t_f} \begin{bmatrix} f_{1_N} \\ f_{2_N} \end{bmatrix} dt \Rightarrow \begin{bmatrix} x_{1_N}(t) \\ x_{2_N}(t) \end{bmatrix} \quad in \quad [0, t_f]$$

Analytical evaluation of partial derivatives

$$\partial f_1 / \partial x_1 = 0; \quad \partial f_1 / \partial x_2 = 1$$
$$\partial f_2 / \partial x_1 = -10 - 30x_{1_N}^2(t); \quad \partial f_2 / \partial x_2 = -1$$

$$\partial f_1 / \partial u = 0; \quad \partial f_1 / \partial w = 0$$
$$\partial f_2 / \partial u = 0; \quad \partial f_2 / \partial w = 0$$

Nominal (NTI) and Perturbation (LTV) Dynamic Equations

Nonlinear, time-invariant (NTI) nominal equation

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t)], \quad \mathbf{x}_N(0) \text{ given}$$
$$\dot{x}_{1_N}(t) = x_{2_N}(t)$$
$$\dot{x}_{2_N}(t) = -10x_{1_N}(t) - 10x_{1_N}^3(t) - x_{2_N}(t)$$

Example

$$\begin{bmatrix} x_{1_N}(0) \\ x_{2_N}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

Perturbations approximated by linear, time-varying (LTV) equation

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}(t)\Delta\mathbf{x}(t), \quad \Delta\mathbf{x}(0) \text{ given}$$
$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\left(10 + 30x_{1_N}^2(t)\right) & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

Example

$$\begin{bmatrix} \Delta x_1(0) \\ \Delta x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Comparison of Approximate and Exact Solutions

Initial Conditions

$x_{2_N}(0) = 9$

$\Delta x_2(0) = 1$

$x_{2_N}(t) + \Delta x_2(t) = 10$

$x_2(t) = 10$

$\mathbf{x}_N(t)$

$\Delta\mathbf{x}(t)$

$\mathbf{x}_N(t) + \Delta\mathbf{x}(t)$

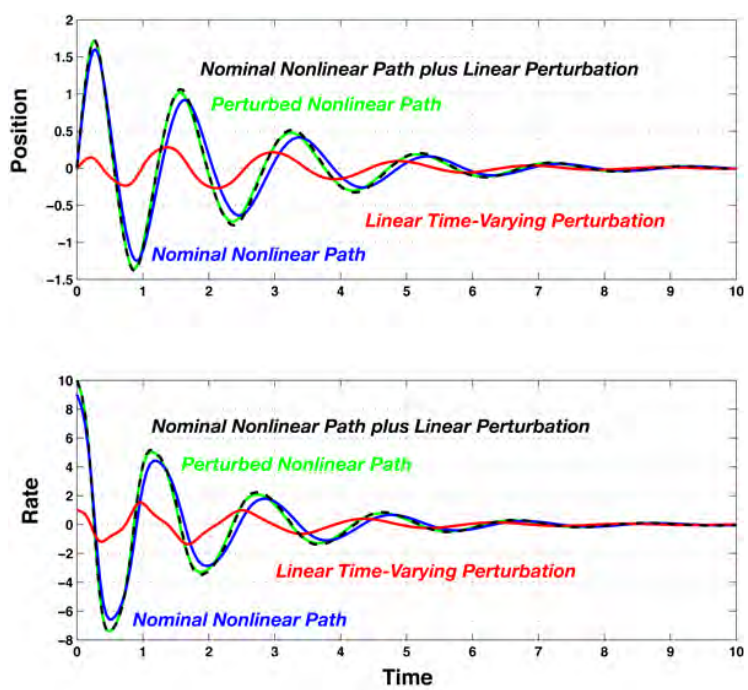
$\mathbf{x}(t)$

$\dot{\mathbf{x}}_N(t)$

$\Delta\dot{\mathbf{x}}(t)$

$\dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t)$

$\dot{\mathbf{x}}(t)$



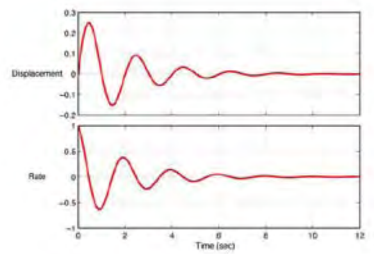
Suppose Nominal Initial Condition is Zero

Nominal solution remains at equilibrium

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t)], \quad \mathbf{x}_N(0) = 0, \quad \mathbf{x}_N(t) = 0 \text{ in } \left[0,\infty\right]$$

Perturbation equation is linear and time-invariant (LTI)

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \left[-10-\cancel{30(0)}\right] & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$



Separation of the Equations of Motion into Longitudinal and Lateral-Directional Sets

Rigid-Body Equations of Motion (Scalar Notation)

- **Rate of change of Translational Velocity**

$$\dot{u} = X/m - g\sin\theta + rv - qw$$
$$\dot{v} = Y/m + g\sin\phi\cos\theta - ru + pw$$
$$\dot{w} = Z/m + g\cos\phi\cos\theta + qu - pv$$

- **Rate of change of Translational Position**

$$\dot{x}_I = (\cos\theta\cos\psi)u + (-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi)v + (\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi)w$$
$$\dot{y}_I = (\cos\theta\sin\psi)u + (\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)v + (-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi)w$$
$$\dot{z}_I = (-\sin\theta)u + (\sin\phi\cos\theta)v + (\cos\phi\cos\theta)w$$

- **Rate of change of Angular Velocity ($I_{xy} = I_{yz} = 0$)**

$$\dot{p} = \left(I_{zz}L + I_{xz}N - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)p + \left[I_{xz}^2 + I_{zz}\left(I_{zz} - I_{yy}\right)\right]r\right\}q\right) \div \left(I_{xx}I_{zz} - I_{xz}^2\right)$$
$$\dot{q} = \left[M - \left(I_{xx} - I_{zz}\right)pr - I_{xz}\left(p^2 - r^2\right)\right] \div I_{yy}$$
$$\dot{r} = \left(I_{xz}L + I_{xx}N - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)r + \left[I_{xz}^2 + I_{xx}\left(I_{xx} - I_{yy}\right)\right]p\right\}q\right) \div \left(I_{xx}I_{zz} - I_{xz}^2\right)$$

- **Rate of change of Angular Position**

$$\dot{\phi} = p + (q\sin\phi + r\cos\phi)\tan\theta$$
$$\dot{\theta} = q\cos\phi - r\sin\phi$$
$$\dot{\psi} = (q\sin\phi + r\cos\phi)\sec\theta$$

State Vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \hline x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ x \\ y \\ z \\ \hline p \\ q \\ r \\ \phi \\ \theta \\ \psi \end{bmatrix}$$

Reorder the State Vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \hline x_4 \\ x_5 \\ x_6 \\ \hline x_7 \\ x_8 \\ x_9 \\ \hline x_{10} \\ x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ \hline x \\ y \\ z \\ \hline p \\ q \\ r \\ \hline \phi \\ \theta \\ \psi \end{bmatrix}$$

First six elements of the state are longitudinal variables

Second six elements of the state are lateral-directional variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \hline x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{bmatrix}_{new} = \begin{bmatrix} \mathbf{x}_{Lon} \\ \hline \mathbf{x}_{Lat-Dir} \end{bmatrix} = \begin{bmatrix} u \\ w \\ x \\ z \\ q \\ \theta \\ \hline v \\ y \\ p \\ r \\ \phi \\ \psi \end{bmatrix}$$

Longitudinal Equations of Motion

Dynamics of velocity, position, angular rate, and angle primarily in the vertical plane

$$
\begin{aligned}
\dot{u} &= X/m - g\sin\theta + rv - qw && \triangleq \dot{x}_1 = f_1 \\
\dot{w} &= Z/m + g\cos\phi\cos\theta + qu - pv && \triangleq \dot{x}_2 = f_2
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_I &= (\cos\theta\cos\psi)u + \\
&\quad (-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi)v + (\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi)w && \triangleq \dot{x}_3 = f_3 \\
\dot{z}_I &= (-\sin\theta)u + (\sin\phi\cos\theta)v + (\cos\phi\cos\theta)w && \triangleq \dot{x}_4 = f_4
\end{aligned}
$$

$$
\begin{aligned}
\dot{q} &= \left[M - (I_{xx} - I_{zz})pr - I_{xz}\left(p^2 - r^2\right)\right] \div I_{yy} && \triangleq \dot{x}_5 = f_5 \\
\dot{\theta} &= q\cos\phi - r\sin\phi && \triangleq \dot{x}_6 = f_6
\end{aligned}
$$

$$
\dot{\mathbf{x}}_{Lon}(t) = \mathbf{f}[\mathbf{x}_{Lon}(t), \mathbf{u}_{Lon}(t), \mathbf{w}_{Lon}(t)]
$$

Lateral-Directional Equations of Motion

Dynamics of velocity, position, angular rate, and angle primarily out of the vertical plane

$$
\begin{aligned}
\dot{v} &= Y/m + g\sin\phi\cos\theta - ru + pw && \triangleq \dot{x}_7 = f_7 \\
\dot{y}_I &= (\cos\theta\sin\psi)u + (\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)v + \\
&\quad (-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi)w && \triangleq \dot{x}_8 = f_8
\end{aligned}
$$

$$
\begin{aligned}
\dot{p} &= \left(I_{zz}L + I_{xz}N - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)p + \left[I_{xz}^2 + I_{zz}\left(I_{zz} - I_{yy}\right)\right]r\right\}q\right) \div \left(I_{xx}I_{zz} - I_{xz}^2\right) && \triangleq \dot{x}_9 = f_9 \\
\dot{r} &= \left(I_{xz}L + I_{xx}N - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)r + \left[I_{xz}^2 + I_{xx}\left(I_{xx} - I_{yy}\right)\right]p\right\}q\right) \div \left(I_{xx}I_{zz} - I_{xz}^2\right) && \triangleq \dot{x}_{10} = f_{10}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\phi} &= p + (q\sin\phi + r\cos\phi)\tan\theta && \triangleq \dot{x}_{11} = f_{11} \\
\dot{\psi} &= (q\sin\phi + r\cos\phi)\sec\theta && \triangleq \dot{x}_{12} = f_{12}
\end{aligned}
$$

$$
\dot{\mathbf{x}}_{LD}(t) = \mathbf{f}[\mathbf{x}_{LD}(t), \mathbf{u}_{LD}(t), \mathbf{w}_{LD}(t)]
$$

Sensitivity to Small Motions

(12 x 12) stability matrix for the entire system

$$\mathbf{F}(t) = \begin{bmatrix} \partial f_1/\partial u & \partial f_1/\partial w & \ldots & \partial f_1/\partial \psi \\ \partial f_2/\partial u & \partial f_2/\partial w & \ldots & \partial f_2/\partial \psi \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_{12}/\partial u & \partial f_{12}/\partial w & \ldots & \partial f_{12}/\partial \psi \end{bmatrix}$$

Four (6 x 6) blocks distinguish longitudinal and lateral-directional effects

Effects of longitudinal perturbations on longitudinal motion

Effects of lateral-directional perturbations on longitudinal motion

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{F}_{Lon} & \mathbf{F}_{Lat-Dir}^{Lon} \\ \hline \mathbf{F}_{Lon}^{Lat-Dir} & \mathbf{F}_{Lat-Dir} \end{array}\right]$$

Effects of longitudinal perturbations on lateral-directional motion

Effects of lateral-directional perturbations on lateral-directional motion

Sensitivity to Small Control Inputs

(12 x 6) control matrix for the entire system

$$\mathbf{G}(t) = \left[\begin{array}{ccc|ccc} \partial f_1/\partial \delta E & \partial f_1/\partial \delta T & \partial f_1/\partial \delta F & \partial f_1/\partial \delta A & \partial f_1/\partial \delta R & \partial f_1/\partial \delta SF \\ \partial f_2/\partial \delta E & \partial f_2/\partial \delta T & \partial f_2/\partial \delta F & \partial f_2/\partial \delta A & \partial f_2/\partial \delta R & \partial f_2/\partial \delta SF \\ \hline \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \partial f_{12}/\partial \delta E & \partial f_{12}/\partial \delta T & \partial f_{12}/\partial \delta F & \partial f_{12}/\partial \delta A & \partial f_{12}/\partial \delta R & \partial f_{12}/\partial \delta SF \end{array}\right]$$

Four (6 x 3) blocks distinguish longitudinal and lateral-directional control effects

Effects of longitudinal controls on longitudinal motion

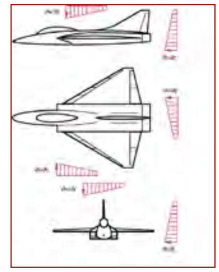
Effects of lateral-directional controls on longitudinal motion

$$\mathbf{G} = \left[\begin{array}{c|c} \mathbf{G}_{Lon} & \mathbf{G}_{Lat-Dir}^{Lon} \\ \hline \mathbf{G}_{Lon}^{Lat-Dir} & \mathbf{G}_{Lat-Dir} \end{array}\right]$$

Effects of longitudinal controls on lateral-directional motion

Effects of lateral-directional controls on lateral-directional motion

Sensitivity to Small Disturbance Inputs



- **Disturbance input vector and perturbation**

$$\mathbf{w}(t) = \begin{bmatrix} u_w(t) \\ w_w(t) \\ q_w(t) \\ \hline v_w(t) \\ p_w(t) \\ r_w(t) \end{bmatrix} \begin{array}{c} Axial\ wind,\ m/s \\ Normal\ wind,\ m/s \\ Pitching\ wind\ shear,\ deg/s\ or\ rad/s \\ \hline Lateral\ wind,\ m/s \\ Rolling\ wind\ shear,\ deg/s\ or\ rad/s \\ Yawing\ wind\ shear,\ deg/s\ or\ rad/s \end{array}$$

$$\Delta\mathbf{w}(t) = \begin{bmatrix} \Delta u_w(t) \\ \Delta w_w(t) \\ \Delta q_w(t) \\ \hline \Delta v_w(t) \\ \Delta p_w(t) \\ \Delta r_w(t) \end{bmatrix}$$

- **Four (6 x 3) blocks distinguish longitudinal and lateral-directional effects**

Effects of longitudinal disturbances on longitudinal motion

Effects of lateral-directional disturbances on longitudinal motion

$$\mathbf{L} = \left[\begin{array}{c|c} \mathbf{L}_{Lon} & \mathbf{L}_{Lat-Dir}^{Lon} \\ \hline \mathbf{L}_{Lon}^{Lat-Dir} & \mathbf{L}_{Lat-Dir} \end{array}\right]$$

Effects of longitudinal disturbances on lateral-directional motion

Effects of lateral-directional disturbances on lateral-directional motion

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Decoupling Approximation for Small Perturbations from Steady, Level Flight

Restrict the <u>Nominal</u> Flight Path to the Vertical Plane

- **<u>Nominal</u> lateral-directional motions are zero**

$$\dot{\mathbf{x}}_{Lat-Dir_N} = \mathbf{0}$$
$$\mathbf{x}_{Lat-Dir_N} = \mathbf{0}$$

- **Nominal longitudinal equations reduce to**

$$\begin{aligned}
\dot{u}_N &= X/m - g\sin\theta_N - q_N w_N \\
\dot{w}_N &= Z/m + g\cos\theta_N + q_N u_N \\
\dot{x}_{I_N} &= \left(\cos\theta_N\right)u_N + \left(\sin\theta_N\right)w_N \\
\dot{z}_{I_N} &= \left(-\sin\theta_N\right)u_N + \left(\cos\theta_N\right)w_N \\
\dot{q}_N &= \frac{M}{I_{yy}} \\
\dot{\theta}_N &= q_N
\end{aligned}$$

Nominal State Vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \hline x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{bmatrix}_N = \begin{bmatrix} \mathbf{x}_{Lon} \\ \hline \mathbf{x}_{Lat-Dir} \end{bmatrix}_N = \begin{bmatrix} u_N \\ w_N \\ x_N \\ z_N \\ q_N \\ \theta_N \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Restrict the <u>Nominal</u> Flight Path to Steady, Level Flight

- **Specify nominal airspeed *(V_N)* and altitude *($h_N = -z_N$)***
- **Calculate conditions for trimmed (equilibrium) flight**
  - **See *Flight Dynamics* and *FLIGHT* program for a solution method**

$$\begin{aligned}
0 &= X/m - g\sin\theta_N - q_N w_N \\
0 &= Z/m + g\cos\theta_N + q_N u_N \\
V_N &= \left(\cos\theta_N\right)u_N + \left(\sin\theta_N\right)w_N \\
0 &= \left(-\sin\theta_N\right)u_N + \left(\cos\theta_N\right)w_N \\
0 &= \frac{M}{I_{yy}} \\
0 &= q_N
\end{aligned}$$

Trimmed State Vector is constant

$$\begin{bmatrix} u \\ w \\ x \\ z \\ q \\ \theta \end{bmatrix}_{Trim} = \begin{bmatrix} u_{Trim} \\ w_{Trim} \\ V_N\left(t - t_0\right) \\ z_N \\ 0 \\ \theta_{Trim} \end{bmatrix}$$

Small Longitudinal and Lateral-Directional Perturbation Effects are Uncoupled in Steady, Symmetric, Level Flight

- Assume the airplane is symmetric and its nominal path is steady, level flight
  - Small longitudinal and lateral-directional perturbations are approximately uncoupled from each other
  - (12 x 12) system is
    - block diagonal
    - constant, i.e., linear, time-invariant (LTI)
    - decoupled into two separate (6 x 6) systems

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{F}_{Lon} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{F}_{Lat-Dir} \end{array}\right] \qquad \mathbf{G} = \left[\begin{array}{c|c} \mathbf{G}_{Lon} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{G}_{Lat-Dir} \end{array}\right] \qquad \mathbf{L} = \left[\begin{array}{c|c} \mathbf{L}_{Lon} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{L}_{Lat-Dir} \end{array}\right]$$

(6 x 6) LTI Longitudinal Perturbation Model

Dynamic Equation

$$\Delta\dot{\mathbf{x}}_{Lon}(t) = \mathbf{F}_{Lon}\Delta\mathbf{x}_{Lon}(t) + \mathbf{G}_{Lon}\Delta\mathbf{u}_{Lon}(t) + \mathbf{L}_{Lon}\Delta\mathbf{w}_{Lon}(t)$$

State Vector

$$\Delta\mathbf{x}_{Lon} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta x_5 \\ \Delta x_6 \end{bmatrix}_{Lon} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta x \\ \Delta z \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

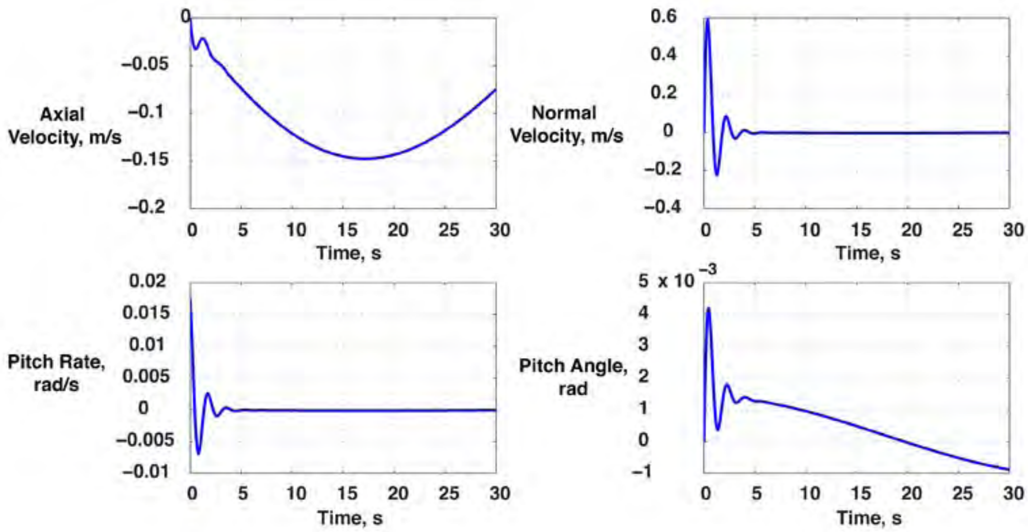
Control Vector

$$\Delta\mathbf{u}_{Lon} = \begin{bmatrix} \Delta\delta T \\ \Delta\delta E \\ \Delta\delta F \end{bmatrix}$$

Disturbance Vector

$$\Delta\mathbf{w}_{Lon} = \begin{bmatrix} \Delta u_{wind} \\ \Delta w_{wind} \\ \Delta q_{wind} \end{bmatrix}$$

LTI Longitudinal Response to Initial Pitch Rate



(6 x 6) LTI Lateral-Directional Perturbation Model

Dynamic Equation

$$\Delta\dot{\mathbf{x}}_{Lat-Dir}(t) = \mathbf{F}_{Lat-Dir}\Delta\mathbf{x}_{Lat-Dir}(t) + \mathbf{G}_{Lat-Dir}\Delta\mathbf{u}_{Lat-Dir}(t) + \mathbf{L}_{Lat-Dir}\Delta\mathbf{w}_{Lat-Dir}(t)$$

State Vector

$$\Delta\mathbf{x}_{Lat-Dir} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta x_5 \\ \Delta x_6 \end{bmatrix}_{Lat-Dir} = \begin{bmatrix} \Delta v \\ \Delta y \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{bmatrix}$$

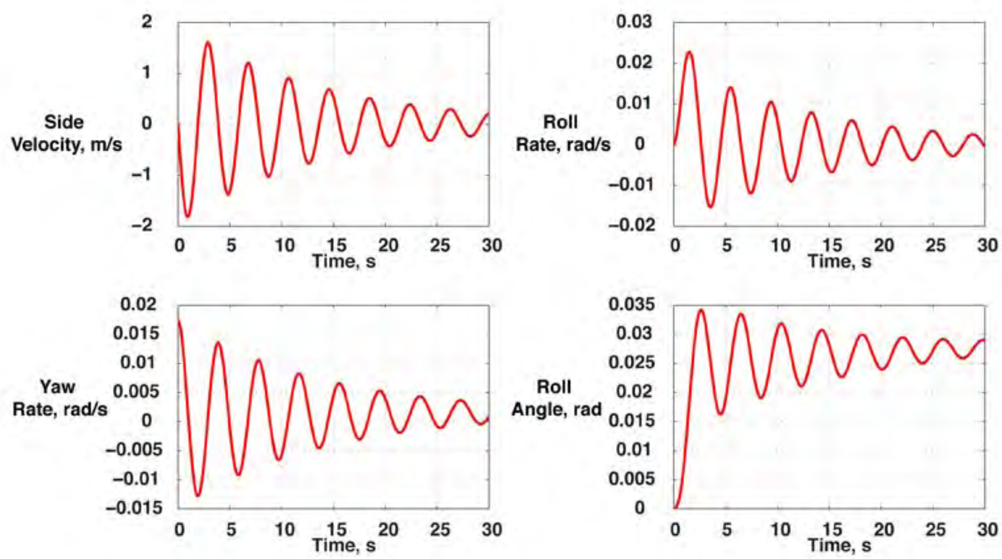
Control Vector

$$\Delta\mathbf{u}_{Lat-Dir} = \begin{bmatrix} \Delta\delta A \\ \Delta\delta R \\ \Delta\delta SF \end{bmatrix}$$

Disturbance Vector

$$\Delta\mathbf{w}_{Lon} = \begin{bmatrix} \Delta v_{wind} \\ \Delta p_{wind} \\ \Delta r_{wind} \end{bmatrix}$$

LTI Lateral-Directional Response to Initial Yaw Rate



Next Time: Longitudinal Dynamics

Reading:
Flight Dynamics
452-464, 482-486
Airplane Stability and Control
Chapter 7

Supplemental Material

How Do We Calculate the Partial Derivatives?

$$\boxed{\mathbf{F}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}} \quad \boxed{\mathbf{G}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}} \quad \boxed{\mathbf{L}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}$$

- **Numerically**
  - **First differences in f(x,u,w)**
- **Analytically**
  - **Symbolic evaluation of analytical models of F, G, and L**

Numerical Estimation of the Jacobian Matrix, F(t)

$$\frac{\partial f_1}{\partial x_1}(t) \approx \frac{f_1\begin{bmatrix}(x_1+\Delta x_1)\\ x_2\\ \dots\\ x_n\end{bmatrix} - f_1\begin{bmatrix}(x_1-\Delta x_1)\\ x_2\\ \dots\\ x_n\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta x_1}; \quad \frac{\partial f_1}{\partial x_2}(t) \approx \frac{f_1\begin{bmatrix}x_1\\ (x_2+\Delta x_2)\\ \dots\\ x_n\end{bmatrix} - f_1\begin{bmatrix}x_1\\ (x_2-\Delta x_2)\\ \dots\\ x_n\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta x_2}$$

$$\frac{\partial f_2}{\partial x_1}(t) \approx \frac{f_2\begin{bmatrix}(x_1+\Delta x_1)\\ x_2\\ \dots\\ x_n\end{bmatrix} - f_2\begin{bmatrix}(x_1-\Delta x_1)\\ x_2\\ \dots\\ x_n\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta x_1}; \quad \frac{\partial f_2}{\partial x_2}(t) \approx \frac{f_2\begin{bmatrix}x_1\\ (x_2+\Delta x_2)\\ \dots\\ x_n\end{bmatrix} - f_2\begin{bmatrix}x_1\\ (x_2-\Delta x_2)\\ \dots\\ x_n\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta x_2}$$

Continue for all *n* x *n* elements of F(t)

Numerical Estimation of the Jacobian Matrix, G(t)

$$\frac{\partial f_1}{\partial u_1}(t) \approx \frac{f_1\begin{bmatrix}(u_1+\Delta u_1)\\ u_2\\ \dots\\ u_m\end{bmatrix} - f_1\begin{bmatrix}(u_1-\Delta u_1)\\ u_2\\ \dots\\ u_m\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta u_1}; \quad \frac{\partial f_1}{\partial u_2}(t) \approx \frac{f_1\begin{bmatrix}u_1\\ (u_2+\Delta u_2)\\ \dots\\ u_m\end{bmatrix} - f_1\begin{bmatrix}u_1\\ (u_2-\Delta u_2)\\ \dots\\ u_m\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta u_2}$$

$$\frac{\partial f_2}{\partial u_1}(t) \approx \frac{f_2\begin{bmatrix}(u_1+\Delta u_1)\\ u_2\\ \dots\\ u_m\end{bmatrix} - f_2\begin{bmatrix}(u_1-\Delta u_1)\\ u_2\\ \dots\\ u_m\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta u_1}; \quad \frac{\partial f_2}{\partial u_2}(t) \approx \frac{f_2\begin{bmatrix}u_1\\ (u_2+\Delta u_2)\\ \dots\\ u_m\end{bmatrix} - f_2\begin{bmatrix}u_1\\ (u_2-\Delta u_2)\\ \dots\\ u_m\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta u_2}$$

Continue for all *n* x *m* elements of G(t)

Numerical Estimation of the Jacobian Matrix, L(*t*)

$$\frac{\partial f_1}{\partial w_1}(t) \approx \frac{f_1\begin{bmatrix}(w_1+\Delta w_1)\\ w_2\\ \dots\\ w_s\end{bmatrix} - f_1\begin{bmatrix}(w_1-\Delta w_1)\\ w_2\\ \dots\\ w_s\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta w_1}; \quad \frac{\partial f_1}{\partial w_2}(t) \approx \frac{f_1\begin{bmatrix}w_1\\ (w_2+\Delta w_2)\\ \dots\\ w_s\end{bmatrix} - f_1\begin{bmatrix}w_1\\ (w_2-\Delta w_2)\\ \dots\\ w_s\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta w_2}$$

$$\frac{\partial f_2}{\partial w_1}(t) \approx \frac{f_2\begin{bmatrix}(w_1+\Delta w_1)\\ w_2\\ \dots\\ w_s\end{bmatrix} - f_2\begin{bmatrix}(w_1-\Delta w_1)\\ w_2\\ \dots\\ w_s\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta w_1}; \quad \frac{\partial f_2}{\partial w_2}(t) \approx \frac{f_2\begin{bmatrix}w_1\\ (w_2+\Delta w_2)\\ \dots\\ w_s\end{bmatrix} - f_2\begin{bmatrix}w_1\\ (w_2-\Delta w_2)\\ \dots\\ w_s\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta w_2}$$

Continue for all *n* x *s* elements of L(*t*)