Spacecraft Dynamics

Space System Design, MAE 342, Princeton University
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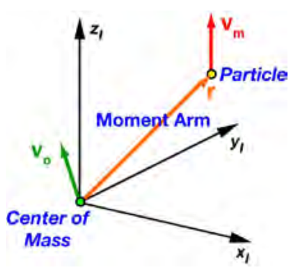
- Angular rate dynamics
- Spinning and non-spinning spacecraft
- Gravity gradient satellites
- Euler Angles and spacecraft attitude
- Rotation matrix
- Precession of spinning axisymmetric spacecraft



Angular Momentum of a Particle

- Moment of linear momentum of differential particles that make up the body
  - Differential mass of a particle times component of its velocity that is perpendicular to the moment arm from the center of rotation to the particle

$$\boxed{\begin{aligned} d\mathbf{h} &= \left(\mathbf{r} \times dm\mathbf{v}\right) = \left(\mathbf{r} \times \mathbf{v}_m\right) dm \\ &= \left(\mathbf{r} \times \left(\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r}\right)\right) dm \end{aligned}}$$

Angular Momentum of an Object

Integrate moment of linear momentum of differential particles over the body

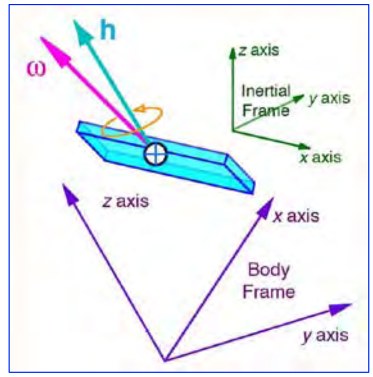
$$\mathbf{h} = \int_{Body} \left(\mathbf{r} \times \left(\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r}\right)\right) dm$$

$$= \int_{Body} \left(\mathbf{r} \times \mathbf{v}_o\right) dm + \int_{Body} \left(\mathbf{r} \times \left(\boldsymbol{\omega} \times \mathbf{r}\right)\right) dm$$

$$= 0 - \int_{Body} \left(\mathbf{r} \times \left(\mathbf{r} \times \boldsymbol{\omega}\right)\right) dm = - \int_{Body} \left(\mathbf{r} \times \mathbf{r}\right) dm \times \boldsymbol{\omega}$$

$$\equiv - \int_{Body} \left(\tilde{\mathbf{r}}\tilde{\mathbf{r}}\right) dm \boldsymbol{\omega}$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}$$



Angular Momentum with Respect to the Center of Mass

Choose center of mass as origin about which angular momentum is calculated (= center of rotation)

Use cross-product-equivalent matrix to define the inertia matrix, $\mathbb{I}$

$$\mathbf{h} = - \int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\, dm\, \boldsymbol{\omega}$$

$$= - \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \rho(x,y,z)\, dx\, dy\, dz$$

$$= \int_{Body} \begin{bmatrix} (y^2 + z^2) & -xy & -xz \\ -xy & (x^2 + z^2) & -yz \\ -xz & -yz & (x^2 + y^2) \end{bmatrix} dm\, \boldsymbol{\omega} = \mathbb{I}\, \boldsymbol{\omega}$$

Inertia Matrix

$$\mathbb{I} = \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Moments of inertia on the diagonal
Products of inertia off the diagonal
If products of inertia are zero, (x, y, z) are principal axes

$$\mathbb{I}_P = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

For fixed mass distribution, inertia matrix is constant in body frame of reference

Inertial-Frame Inertia Matrix is Not Constant if Body is Rotating

Newton's 2nd Law applies to rotational motion in an inertial frame
Rate of change of angular momentum = applied moment (or torque), m

$$\frac{d\mathbf{h}}{dt} = \frac{d(\mathbb{I}\boldsymbol{\omega})}{dt} = \mathbf{M} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

In an inertial frame

$$\frac{d\mathbb{I}}{dt} \neq 0$$

Chain Rule

$$\frac{d(\mathbb{I}\boldsymbol{\omega})}{dt} = \frac{d\mathbf{h}}{dt} = \frac{d\mathbb{I}}{dt}\boldsymbol{\omega} + \mathbb{I}\dot{\boldsymbol{\omega}}$$

Inertial-frame solution for angular rate

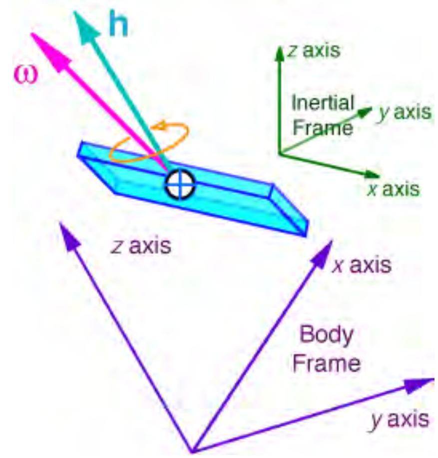
$$\mathbb{I}\dot{\boldsymbol{\omega}} = \mathbf{M} - \frac{d\mathbb{I}}{dt}\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\omega}} = \mathbb{I}^{-1}\left(\mathbf{M} - \frac{d\mathbb{I}}{dt}\boldsymbol{\omega}\right)$$

How Do We Get Rid of $d\mathbb{I}/dt$ in the Angular Momentum Equation?

Write the dynamic equation in a body-referenced frame

- Inertia matrix is ~unchanging in a body frame
- Body-axis frame is rotating
- Dynamic equation must be modified to account for rotation



Expressing Vectors in Different Reference Frames

- Angular momentum and rate are vectors
  - They can be expressed in either the inertial or body frame
  - The 2 frames are related by the rotation matrix (also called the direction cosine matrix)

$\mathbf{H}_I^B$: Rotation transformation from inertial frame $\Rightarrow$ body frame

$\mathbf{H}_B^I$: Rotation transformation from body frame $\Rightarrow$ inertial frame

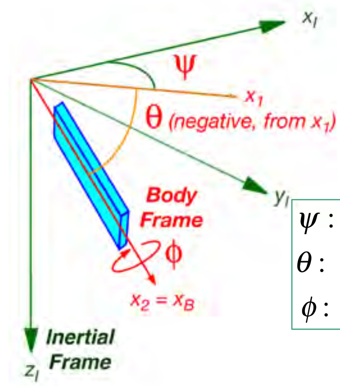
$$\mathbf{h}_B = \mathbf{H}_I^B \mathbf{h}_I$$
$$\boldsymbol{\omega}_B = \mathbf{H}_I^B \boldsymbol{\omega}_I$$

$$\mathbf{h}_I = \mathbf{H}_B^I \mathbf{h}_B$$
$$\boldsymbol{\omega}_I = \mathbf{H}_B^I \boldsymbol{\omega}_B$$

Euler Angles

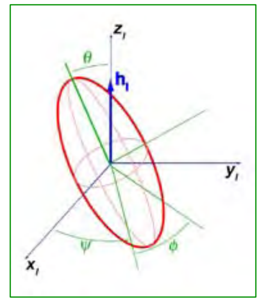
- **Body attitude measured with respect to inertial frame**
- **Three-angle orientation expressed by sequence of three orthogonal single-angle rotations**

$$\boxed{Inertial \Rightarrow Intermediate_1 \Rightarrow Intermediate_2 \Rightarrow Body}$$



ψ :	Yaw angle
θ :	Pitch angle
ϕ :	Roll angle

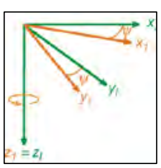
ψ :	Yaw angle$_1$
θ :	Pitch angle
ϕ :	Yaw angle$_2$



- **24 (±12) possible sequences of single-axis rotations**
- **Aircraft convention: 3-2-1, *z* positive down**
- **Spacecraft convention: 3-1-3, *z* positive up**

Reference Frame Rotation from Inertial to Body: Aircraft Convention (1-2-3)

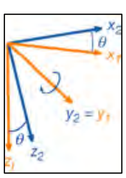
Yaw rotation (ψ) about z_I axis



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_I = \begin{bmatrix} x_I\cos\psi + y_I\sin\psi \\ -x_I\sin\psi + y_I\cos\psi \\ z_I \end{bmatrix}$$

$$\mathbf{r}_1 = \mathbf{H}_I^1 \mathbf{r}_I$$

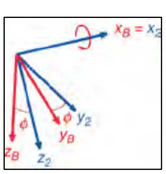
Pitch rotation (θ) about y_1 axis



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_1$$

$$\mathbf{r}_2 = \mathbf{H}_1^2 \mathbf{r}_1 = \left[\mathbf{H}_1^2 \mathbf{H}_I^1\right] \mathbf{r}_I = \mathbf{H}_I^2 \mathbf{r}_I$$

Roll rotation (φ) about x_2 axis



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_2$$

$$\mathbf{r}_B = \mathbf{H}_2^B \mathbf{r}_2 = \left[\mathbf{H}_2^B \mathbf{H}_1^2 \mathbf{H}_I^1\right] \mathbf{r}_I = \mathbf{H}_I^B \mathbf{r}_I$$

Reference Frame Rotation from Inertial to Body: Spacecraft Convention (3-1-3)

Yaw rotation (ψ) about z_I axis

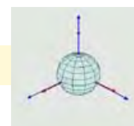
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_I = \begin{bmatrix} x_I \cos\psi + y_I \sin\psi \\ -x_I \sin\psi + y_I \cos\psi \\ z_I \end{bmatrix}$$

$$\boxed{\mathbf{r}_1 = \mathbf{H}_I^1 \mathbf{r}_I}$$

Pitch rotation (θ) about x_1 axis

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_1$$

$$\boxed{\mathbf{r}_2 = \mathbf{H}_1^2 \mathbf{r}_1 = \left[\mathbf{H}_1^2 \mathbf{H}_I^1\right] \mathbf{r}_I = \mathbf{H}_I^2 \mathbf{r}_I}$$



Yaw rotation (φ) about z_2 axis

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_B = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_2$$

$$\boxed{\mathbf{r}_B = \mathbf{H}_2^B \mathbf{r}_2 = \left[\mathbf{H}_2^B \mathbf{H}_1^2 \mathbf{H}_I^1\right] \mathbf{r}_I = \mathbf{H}_I^B \mathbf{r}_I}$$

Rotation Matrix from *I* to *B*
Aircraft Convention (1-2-3)

$$\boxed{\mathbf{H}_I^B(\phi,\theta,\psi) = \mathbf{H}_2^B(\phi)\,\mathbf{H}_1^2(\theta)\,\mathbf{H}_I^1(\psi)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix}$$

Rotation Matrix from *I* to *B*
Spacecraft Convention (3-1-3)

$$\mathbf{H}_I^B(\phi,\theta,\psi) = \mathbf{H}_2^B(\phi)\mathbf{H}_1^2(\theta)\mathbf{H}_I^1(\psi)$$

$$= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \cos\phi\sin\psi + \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ \hline -\sin\phi\cos\psi - \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \cos\phi\sin\theta \\ \hline \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{array} \right]$$

Properties of the Rotation Matrix

$$\mathbf{H}_I^B(\phi,\theta,\psi) = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}_I^B$$

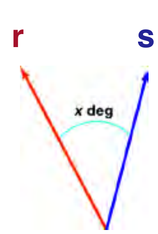
$$\mathbf{H}_I^B(\phi,\theta,\psi) = \mathbf{H}_2^B(\phi)\mathbf{H}_1^2(\theta)\mathbf{H}_I^1(\psi)$$

$$\left[\mathbf{H}_I^B(\phi,\theta,\psi)\right]^{-1} = \left[\mathbf{H}_I^B(\phi,\theta,\psi)\right]^T = \mathbf{H}_B^I(\psi,\theta,\phi)$$

Orthonormal transformation

Angles between vectors are preserved

Lengths are preserved



$$|\mathbf{r}_I| = |\mathbf{r}_B| \quad ; \quad |\mathbf{s}_I| = |\mathbf{s}_B|$$

$$\angle(\mathbf{r}_I,\mathbf{s}_I) = \angle(\mathbf{r}_B,\mathbf{s}_B) = x\,\deg$$

Rotation Matrix Inverse

Inverse relationship: interchange sub- and superscripts

$$\mathbf{r}_B = \mathbf{H}_I^B \mathbf{r}_I$$
$$\mathbf{r}_I = \left(\mathbf{H}_I^B\right)^{-1} \mathbf{r}_B = \mathbf{H}_B^I \mathbf{r}_B$$

Because transformation is orthonormal

Inverse = transpose

Rotation matrix is always non-singular

$$\mathbf{H}_B^I = \left(\mathbf{H}_I^B\right)^{-1} = \left(\mathbf{H}_I^B\right)^T = \mathbf{H}_1^I \mathbf{H}_2^1 \mathbf{H}_B^2$$

$$\mathbf{H}_B^I \mathbf{H}_I^B = \mathbf{H}_I^B \mathbf{H}_B^I = \mathbf{I}$$

Vector Derivative Expressed in a Rotating Frame

$$\mathbf{h}_I(t) = \mathbf{H}_B^I(t)\mathbf{h}_B(t)$$

Chain Rule

Effect of body-frame rotation

$$\dot{\mathbf{h}}_I(t) = \mathbf{H}_B^I(t)\dot{\mathbf{h}}_B(t) + \dot{\mathbf{H}}_B^I(t)\mathbf{h}_B(t)$$

Rate of change expressed in body frame

Alternatively

$$\dot{\mathbf{h}}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \boldsymbol{\omega}_I \times \mathbf{h}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \tilde{\boldsymbol{\omega}}_I \mathbf{h}_I$$

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Angular Rate Derivative in Body Frame of Reference

Angular momentum change

$$\boxed{\begin{aligned}\dot{\mathbf{h}}_B &= \mathbf{H}_I^B \dot{\mathbf{h}}_I + \dot{\mathbf{H}}_I^B \mathbf{h}_I = \mathbf{H}_I^B \dot{\mathbf{h}}_I - \boldsymbol{\omega}_B \times h_B \\ &= \mathbf{H}_I^B \dot{\mathbf{h}}_I - \tilde{\boldsymbol{\omega}}_B h_B = \mathbf{H}_I^B \mathbf{M}_I - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B\end{aligned}}$$

$$\boxed{\dot{\mathbf{h}}_B = \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B}$$

Constant body-axis inertia matrix

$$\boxed{\dot{\mathbf{h}}_B(t) = \mathbb{I}_B \dot{\boldsymbol{\omega}}_B(t) = \mathbf{M}_B(t) - \tilde{\boldsymbol{\omega}}_B(t) \mathbb{I}_B \boldsymbol{\omega}_B(t)}$$

Angular rate change

$$\boxed{\dot{\boldsymbol{\omega}}_B(t) = \mathbb{I}_B^{-1}\left[\mathbf{M}_B(t) - \tilde{\boldsymbol{\omega}}_B(t) \mathbb{I}_B \boldsymbol{\omega}_B(t)\right]}$$

Euler-Angle Rates and Body-Axis Rates

Body-axis angular rate vector (orthogonal)

$$\boxed{\boldsymbol{\omega}_B = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_B}$$

Form a non-orthogonal vector of Euler angles

$$\boxed{\boldsymbol{\Theta} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}_{3-2-1} \quad \text{or} \quad \begin{bmatrix} \psi \\ \theta \\ \phi \end{bmatrix}_{3-1-3}}$$

Euler-angle rate vector

$$\boxed{\dot{\boldsymbol{\Theta}} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}_{3-2-1} \quad \text{or} \quad \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}_{3-1-3} \neq \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_I}$$

Relationship Between (1-2-3) Euler-Angle and Body-Axis Rates

- $\dot{\psi}$ measured in Inertial Frame
- $\dot{\theta}$ measured in Intermediate Frame #1
- $\dot{\phi}$ measured in Intermediate Frame #2

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{I}_3 \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{H}_2^B \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{H}_2^B \mathbf{H}_1^2 \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{L}_I^B \dot{\mathbf{\Theta}}$$

Inverse transformation $[(.)^{-1} \neq (.)^T]$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{L}_B^I \boldsymbol{\omega}_B$$

Relationship Between (3-1-3) Euler-Angle and Body-Axis Rates

- $\dot{\psi}$ measured in Inertial Frame
- $\dot{\theta}$ measured in Intermediate Frame #1
- $\dot{\phi}$ measured in Intermediate Frame #2

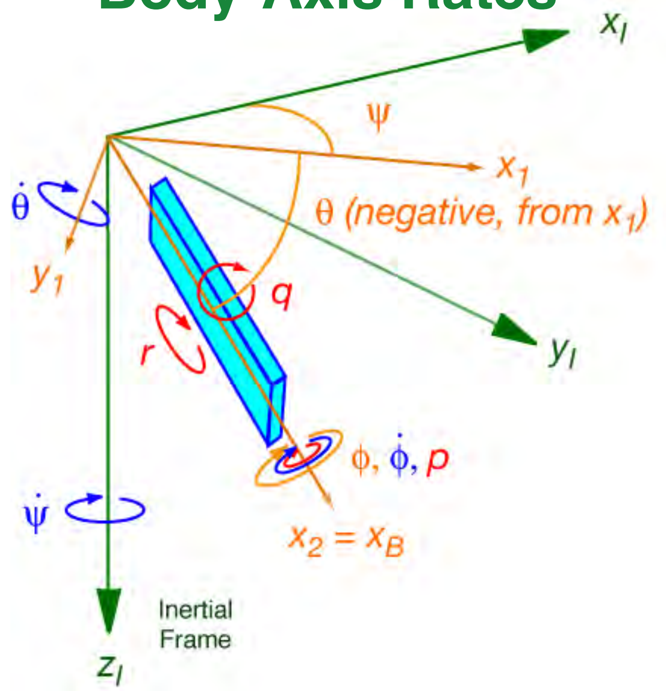
$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_B = \mathbf{I}_3 \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} + \mathbf{H}_2^B \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{H}_2^B \mathbf{H}_1^2 \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_B = \begin{bmatrix} \sin\theta\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & -\sin\phi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \mathbf{L}_I^B \dot{\mathbf{\Theta}}$$

Inverse transformation $[(.)^{-1} \neq (.)^T]$

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{\sin\theta} \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ \cos\phi\sin\theta & -\sin\phi\sin\theta & 0 \\ -\sin\phi\cos\theta & \cos\phi\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_B = \mathbf{L}_B^I \boldsymbol{\omega}_B$$

(1-2-3) Euler-Angle Rates and Body-Axis Rates



Options for Avoiding the Singularity at $\theta = \pm 90°$

- **Don't use Euler angles as primary definition of angular attitude**
- **Alternatives to Euler angles**
  - Direction cosine (rotation) matrix
  - Quaternions *(next lecture)*

Propagation of rotation matrix (1-2-3)
(9 parameters)

$$\boxed{\dot{\mathbf{H}}_B^I \mathbf{h}_B = \tilde{\boldsymbol{\omega}}_I \mathbf{H}_B^I \mathbf{h}_B}$$

Consequently

$$\dot{\mathbf{H}}_I^B(t) = -\tilde{\boldsymbol{\omega}}_B(t)\mathbf{H}_I^B(t) = -\begin{bmatrix} 0 & -r(t) & q(t) \\ r(t) & 0 & -p(t) \\ -q(t) & p(t) & 0(t) \end{bmatrix}_B \mathbf{H}_I^B(t)$$

$$\boxed{\mathbf{H}_I^B(0) = \mathbf{H}_I^B(\phi_0, \theta_0, \psi_0)}$$

Body-Axis Angular Rate Dynamics

[(3-1-3) convention]

$$\dot{\boldsymbol{\omega}}_B = \mathbb{I}_B^{-1}\left[\mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \mathbb{I}_B \boldsymbol{\omega}_B\right]$$

$$\boldsymbol{\omega}_B = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_B$$

For principal axes

$$\begin{bmatrix} \dot{\omega}_x(t) \\ \dot{\omega}_y(t) \\ \dot{\omega}_z(t) \end{bmatrix} = \begin{bmatrix} M_x(t)/I_{xx} \\ M_y(t)/I_{yy} \\ M_z(t)/I_{zz} \end{bmatrix} - \begin{bmatrix} \left(I_{zz} - I_{yy}\right)\omega_y(t)\omega_z(t)/I_{xx} \\ \left(I_{xx} - I_{zz}\right)\omega_x(t)\omega_z(t)/I_{yy} \\ \left(I_{yy} - I_{xx}\right)\omega_x(t)\omega_y(t)/I_{zz} \end{bmatrix}$$

Small Perturbations from Nominal Angular Rate

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \omega_{x_o} + \Delta\omega_x \\ \omega_{y_o} + \Delta\omega_y \\ \omega_{z_o} + \Delta\omega_z \end{bmatrix}$$

$$\begin{bmatrix} d\left(\omega_{x_o} + \Delta\omega_x\right)/dt \\ d\left(\omega_{y_o} + \Delta\omega_y\right)/dt \\ d\left(\omega_{z_o} + \Delta\omega_z\right)/dt \end{bmatrix} = \begin{bmatrix} M_x/I_{xx} \\ M_y/I_{yy} \\ M_z/I_{zz} \end{bmatrix} - \begin{bmatrix} \left(I_{zz} - I_{yy}\right)\left(\omega_{y_o} + \Delta\omega_y\right)\left(\omega_{z_o} + \Delta\omega_z\right)/I_{xx} \\ \left(I_{xx} - I_{zz}\right)\left(\omega_{x_o} + \Delta\omega_x\right)\left(\omega_{z_o} + \Delta\omega_z\right)/I_{yy} \\ \left(I_{yy} - I_{xx}\right)\left(\omega_{x_o} + \Delta\omega_x\right)\left(\omega_{y_o} + \Delta\omega_y\right)/I_{zz} \end{bmatrix}$$

Products of small perturbations are negligible

$$\Delta\omega_x\Delta\omega_y = \Delta\omega_x\Delta\omega_z = \Delta\omega_y\Delta\omega_z \simeq 0$$

Small Perturbation Equations for Spacecraft Spinning about *z* Axis

Assume yaw rate is constant, while pitch and yaw motions are small

$$\boxed{\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \Delta\omega_x \\ \Delta\omega_y \\ \omega_{z_o} \end{bmatrix}}$$

Then

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \Delta\dot{\omega}_x \\ \Delta\dot{\omega}_y \\ \dot{\omega}_{z_o} \end{bmatrix} = \begin{bmatrix} M_x/I_{xx} \\ M_y/I_{yy} \\ 0 \end{bmatrix} - \begin{bmatrix} \left[\omega_{z_o}\left(I_{zz}-I_{yy}\right)\Delta\omega_y\right]/I_{xx} \\ \left[\omega_{z_o}\left(I_{xx}-I_{zz}\right)\Delta\omega_x\right]/I_{yy} \\ 0 \end{bmatrix}$$

2nd-Order Model of Pitch and Yaw Perturbations

$$\boxed{\dot{r}(t) = 0}$$

Linear, Time-Invariant (LTI) Ordinary Differential Equation

$$\begin{bmatrix} \Delta\dot{\omega}_x(t) \\ \Delta\dot{\omega}_y(t) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\omega_{z_o}\left(I_{yy}-I_{zz}\right)}{I_{xx}} \\ \dfrac{\omega_{z_o}\left(I_{zz}-I_{xx}\right)}{I_{yy}} & 0 \end{bmatrix} \begin{bmatrix} \Delta\omega_x(t) \\ \Delta\omega_y(t) \end{bmatrix} + \begin{bmatrix} \dfrac{M_x(t)}{I_{xx}} \\ \dfrac{M_y(t)}{I_{yy}} \end{bmatrix}$$

Laplace Transforms of Scalar Variables

s: Laplace operator, a complex variable

$$\mathcal{L}\left[x(t)\right] = x(s) = \int_0^\infty x(t)e^{-st}\,dt, \quad s = \sigma + j\omega, \quad (j = i = \sqrt{-1})$$

Multiplication by a constant

$$\mathcal{L}\left[a\,x(t)\right] = a\,x(s)$$

Sum of transforms

$$\mathcal{L}\left[x_1(t) + x_2(t)\right] = x_1(s) + x_2(s)$$

Transform of a derivative

$$\mathcal{L}\left[\dot{x}(t)\right] = sx(s) - x(0)$$

Transform of an integral

$$\mathcal{L}\left[\int x(t)\,dt\right] = x_1(s)/s$$

Laplace Transforms of Vectors and Matrices

Laplace transform of a vector variable

$$\mathcal{L}\left[\mathbf{x}(t)\right] = \mathbf{x}(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \\ \ldots \end{bmatrix}$$

Laplace transform of a matrix variable

$$\mathcal{L}\left[\mathbf{A}(t)\right] = \mathbf{A}(s) = \begin{bmatrix} a_{11}(s) & a_{12}(s) & \ldots \\ a_{21}(s) & a_{22}(s) & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

Laplace transform of a time-derivative

$$\mathcal{L}\left[\dot{\mathbf{x}}(t)\right] = s\mathbf{x}(s) - \mathbf{x}(0)$$

Transformation of the Dynamic Equation

$$\Delta\mathbf{x}(t): \quad \text{Dynamic State}$$
$$\Delta\mathbf{u}(t): \quad \text{Control Input}$$

Time-Domain LTI Dynamic Equation

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\,\Delta\mathbf{x}(t) + \mathbf{G}\,\Delta\mathbf{u}(t)$$

Laplace Transform of LTI Dynamic Equation

$$s\Delta\mathbf{x}(s) - \Delta\mathbf{x}(0) = \mathbf{F}\,\Delta\mathbf{x}(s) + \mathbf{G}\Delta\,\mathbf{u}(s)$$

Laplace Transform of the State Vector

Rearrange Laplace Transform of Dynamic Equation

$$s\Delta\mathbf{x}(s) - \mathbf{F}\,\Delta\mathbf{x}(s) = \Delta\mathbf{x}(0) + \mathbf{G}\,\Delta\mathbf{u}(s)$$
$$\left[s\mathbf{I} - \mathbf{F}\right]\Delta\mathbf{x}(s) = \Delta\mathbf{x}(0) + \mathbf{G}\,\Delta\mathbf{u}(s)$$
$$\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\Delta\mathbf{x}(0) + \mathbf{G}\,\Delta\mathbf{u}(s)\right]$$

Inverse of characteristic matrix

$$\left[s\mathbf{I} - \mathbf{F}\right]^{-1} = \frac{Adj\left(s\mathbf{I} - \mathbf{F}\right)}{\left|s\mathbf{I} - \mathbf{F}\right|} \quad (n\ x\ n)$$

$$Adj\left(s\mathbf{I} - \mathbf{F}\right): \quad \textit{Adjoint matrix} \quad (n \times n)$$
$$\left|s\mathbf{I} - \mathbf{F}\right| = \det\left(s\mathbf{I} - \mathbf{F}\right): \quad \textit{Determinant} \quad (1 \times 1)$$

Eigenvalues of the Dynamic System

$$\left[s\mathbf{I}-\mathbf{F}\right]^{-1}=\frac{Adj\left(s\mathbf{I}-\mathbf{F}\right)}{\left|s\mathbf{I}-\mathbf{F}\right|}\quad (n\ x\ n)$$

Characteristic polynomial of the system, [$\Delta(s)$]

$$\left|s\mathbf{I}-\mathbf{F}\right|=\det\left(s\mathbf{I}-\mathbf{F}\right)$$
$$\equiv\Delta(s)=s^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0$$

Characteristic equation of the system, [$\Delta(s)=0$]

$$\Delta(s)=s^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0=0$$
$$=\left(s-\lambda_1\right)\left(s-\lambda_2\right)\left(\ldots\right)\left(s-\lambda_n\right)=0$$

Factors (or roots) of $\Delta(s)$ are the eigenvalues

LTI Dynamic Model of Spacecraft Spinning about *z* Axis *(ODE)*

$$\begin{bmatrix}\Delta\dot{\omega}_x(t)\\ \Delta\dot{\omega}_y(t)\end{bmatrix}=\begin{bmatrix}0 & \dfrac{\omega_{z_o}\left(I_{yy}-I_{zz}\right)}{I_{xx}}\\ \dfrac{\omega_{z_o}\left(I_{zz}-I_{xx}\right)}{I_{yy}} & 0\end{bmatrix}\begin{bmatrix}\Delta\omega_x(t)\\ \Delta\omega_y(t)\end{bmatrix}+\begin{bmatrix}\dfrac{\Delta M_x(t)}{I_{xx}}\\ \dfrac{\Delta M_y(t)}{I_{yy}}\end{bmatrix}$$

$$\Delta\dot{\mathbf{x}}(t)=\mathbf{F}\,\Delta\mathbf{x}(t)+\mathbf{G}\,\Delta\mathbf{u}(t)$$

$$\Delta\mathbf{x}(t)=\begin{bmatrix}\Delta\omega_x(t)\\ \Delta\omega_y(t)\end{bmatrix}\qquad \Delta\mathbf{u}(t)=\begin{bmatrix}\Delta M_x(t)\\ \Delta M_y(t)\end{bmatrix}$$

$$\mathbf{F}=\begin{bmatrix}0 & \dfrac{\omega_{z_o}\left(I_{yy}-I_{zz}\right)}{I_{xx}}\\ \dfrac{\omega_{z_o}\left(I_{zz}-I_{xx}\right)}{I_{yy}} & 0\end{bmatrix}\qquad \mathbf{G}=\begin{bmatrix}1/I_{xx} & 0\\ 0 & 1/I_{yy}\end{bmatrix}$$

LTI Model of Spacecraft Spinning about *z* Axis *(Transform)*

$$\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\Delta\mathbf{x}(0) + \mathbf{G}\,\Delta\mathbf{u}(s)\right]$$

$$\begin{bmatrix} \Delta\omega_x(s) \\ \Delta\omega_y(s) \end{bmatrix} = \begin{bmatrix} s & -\dfrac{\omega_{z_o}\left(I_{yy}-I_{zz}\right)}{I_{xx}} \\ -\dfrac{\omega_{z_o}\left(I_{zz}-I_{xx}\right)}{I_{yy}} & s \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \Delta\omega_x(0) \\ \Delta\omega_y(0) \end{bmatrix} + \begin{bmatrix} \dfrac{\Delta M_x(s)}{I_{xx}} \\ \dfrac{\Delta M_y(s)}{I_{yy}} \end{bmatrix} \right\}$$

$$\Delta\mathbf{x}(s) = \begin{bmatrix} \Delta\omega_x(s) \\ \Delta\omega_y(s) \end{bmatrix}$$

$$\Delta\mathbf{u}(t) = \begin{bmatrix} \Delta M_x(s) \\ \Delta M_y(s) \end{bmatrix}$$

$$\left[s\mathbf{I} - \mathbf{F}\right]^{-1} = \frac{\begin{bmatrix} s & \dfrac{\left(I_{yy}-I_{zz}\right)}{I_{xx}}\omega_{z_o} \\ \dfrac{\left(I_{zz}-I_{xx}\right)}{I_{yy}}\omega_{z_o} & s \end{bmatrix}}{\left[s^2 - \dfrac{\left(I_{zz}-I_{xx}\right)\left(I_{yy}-I_{zz}\right)}{I_{xx}I_{yy}}\omega_{z_o}{}^2\right]}$$

Characteristic Equations and Eigenvalues

Characteristic equation, with $I_{xx} \neq I_{yy} \neq I_{zz}$

$$\Delta(s) = \left[s^2 - \frac{\left(I_{zz}-I_{xx}\right)\left(I_{yy}-I_{zz}\right)}{I_{xx}I_{yy}}\omega_{z_o}{}^2\right] = 0$$

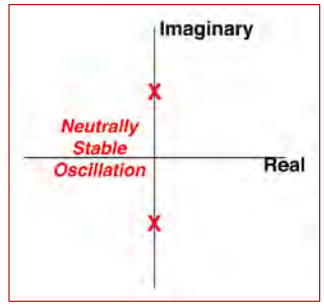
Eigenvalues

$$\lambda_{1,2} = \pm\omega_{z_o}\sqrt{\left(\frac{I_{zz}}{I_{xx}} - 1\right)\left(1 - \frac{I_{zz}}{I_{yy}}\right)} \quad rad/sec$$

Eigenvalues of the Spinning Spacecraft

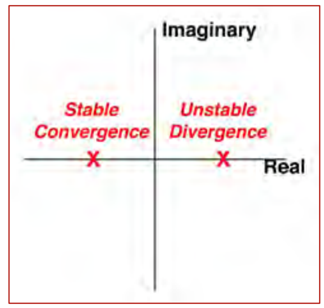
- If $I_{zz} < I_{xx}$ & I_{yy} or $I_{zz} > I_{xx}$ & I_{yy}, eigenvalues are imaginary, and neutrally stable oscillation occurs

$$\boxed{\Delta\omega_x(t) = A\cos(\omega_n t) = \frac{A}{2}\left[e^{-j\omega_n t} + e^{+j\omega_n t}\right]}$$



- If I_{zz} is between I_{xx} and I_{yy}, eigenvalues are real, and one is positive (i.e., unstable)

$$\boxed{\Delta\omega_x(t) = \frac{A}{2}\left(e^{-\sigma t} + Be^{+\sigma t}\right)}$$



Therefore, satellite attitude is stable only if it spins about the axis of maximum or minimum moment of inertia

Axisymmetric Spacecraft Spinning About *z* Axis

$$\boxed{I_{xx} = I_{yy}}$$

$$\lambda_{1,2} = \pm\omega_{z_o}\sqrt{\left(\frac{I_{zz}}{I_{xx}} - 1\right)\left(1 - \frac{I_{zz}}{I_{yy}}\right)} = \pm\omega_{z_o}\sqrt{-\left(1 - \frac{I_{zz}}{I_{xx}}\right)^2}$$

$$= \pm j\omega_{z_o}\left(1 - \frac{I_{zz}}{I_{xx}}\right) rad / sec$$

$$\boxed{j = \sqrt{-1}}$$

Imaginary roots

Neutral, oscillatory stability

Spin Stability

Eigenvalues define natural frequency of an undamped oscillation

$$\boxed{\lambda_{1,2} = \pm j\omega_{z_o}\left(1 - \frac{I_{zz}}{I_{xx}}\right) = \pm j\omega_n rad / sec}$$

Motion is oscillatory but neutrally stable

$$\Delta\omega_x(t) = A\sin\left[\omega_{z_o}\left(1 - \frac{I_{zz}}{I_{xx}}\right)t\right] + B\cos\left[\omega_{z_o}\left(1 - \frac{I_{zz}}{I_{xx}}\right)t\right]$$

$$\Delta\omega_y(t) = A\cos\left[\omega_{z_o}\left(1 - \frac{I_{zz}}{I_{xx}}\right)t\right] - B\sin\left[\omega_{z_o}\left(1 - \frac{I_{zz}}{I_{xx}}\right)t\right]$$

Unfortunate notation overlap:

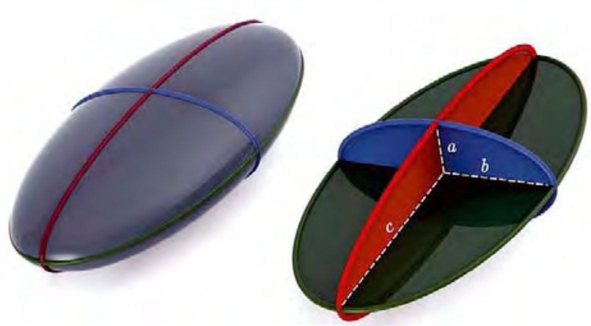
$(\omega_x, \omega_y, \omega_z)$ are components of rotation rate, rad/s

ω and ω_n are oscillatory input frequency and system natural frequency, rad/s

Ellipsoid of Inertia

Properties of the spacecraft mass distribution
For principal axes in the body frame of reference,

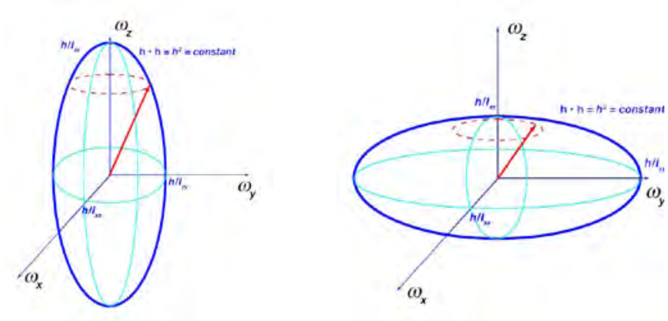
$$\boxed{\begin{gathered}\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \\ I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 = 1\end{gathered}}$$



Angular Momentum Ellipsoid

Locus of all angular rate combinations with <u>constant angular momentum</u> (principal axes, in the <u>body frame or reference</u>)

$$\boxed{I_{xx}{}^2\omega_x{}^2 + I_{yy}{}^2\omega_y{}^2 + I_{zz}{}^2\omega_z{}^2 = h^2 = \mathbf{h}^T\mathbf{h}}$$



… but angular momentum vector is fixed in an inertial reference frame

therefore, body reference frame may rotate even with $M_B = 0$

Angular Momentum Distribution

Magnitude of angular momentum constant and identical in body and inertial reference frames

$$\boxed{\left(I_{xx}{}^2\omega_x{}^2 + I_{yy}{}^2\omega_y{}^2 + I_{zz}{}^2\omega_z{}^2\right)_B = h_B^2 = \mathbf{h}_B^T\mathbf{h}_B = h_I^2 = \text{ Constant}^2}$$

$$\boxed{|h|_B = |h|_I = \text{ Constant}}$$

… but individual components may vary

Axisymmetric spacecraft spinning about *z* axis

$$\boxed{\mathbf{h}_B = \mathbb{I}_B \begin{bmatrix} I_{xx}\omega_x \\ I_{xx}\omega_y \\ I_{zz}\omega_z \end{bmatrix}_B = I_{xx}\begin{bmatrix} \omega_x \\ \omega_y \\ 0 \end{bmatrix} + I_{zz}\begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix} \triangleq \mathbf{h}_{xy} + \mathbf{h}_z}$$

Angular Momentum of Spinning Axisymmetric Spacecraft

$$I_{xx}^{\;2}\left(\omega_x^{\;2}+\omega_y^{\;2}\right)+I_{zz}^{\;2}\omega_z^{\;2}=h_{xy}^2+h_z^2$$

$\theta \triangleq$ Nutation Angle:

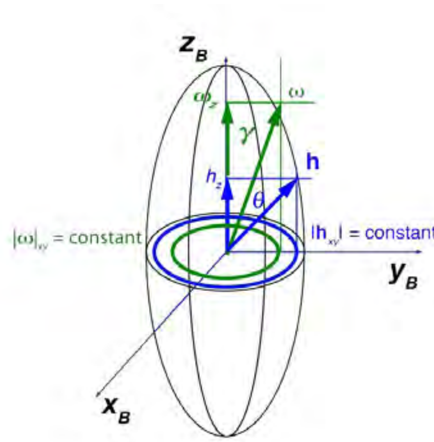
body-axis orientation w.r.t. inertial axes

$$\tan\theta=\frac{h_{xy}}{h_z}=\frac{I_{xx}\sqrt{\omega_x^{\;2}+\omega_y^{\;2}}}{I_{zz}\omega_z}=\text{Constant}$$

$\gamma \triangleq$ Precession Angle:

angular rate orientation w.r.t. body axes

$$\tan\gamma=\frac{\sqrt{\omega_x^{\;2}+\omega_y^{\;2}}}{\omega_z}=\text{Constant}$$

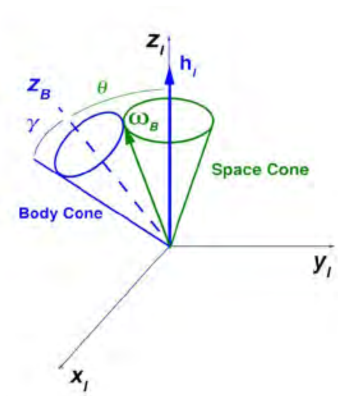


Body and Space Cones

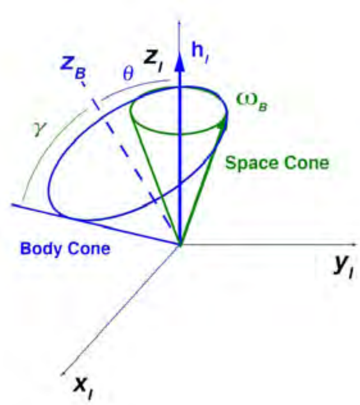
Angular momentum is fixed in inertial frame

Define $\angle\mathbf{h}_I=\angle z_I$ for diagrams

Direct Precession
$I_{zz} < I_{xx}$ (Rod)



Retrograde Precession
$I_{zz} > I_{xx}$ (Disc)

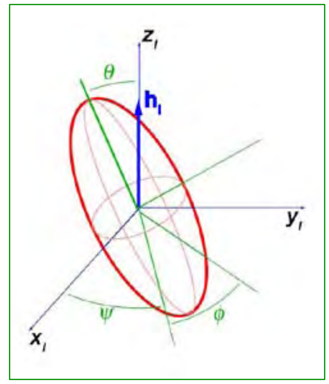


Angular Motion in Space

$$\Theta = \begin{bmatrix} \psi \\ \theta \\ \phi \end{bmatrix}_{3-1-3}$$

Body-axis rates in terms of Euler-angle rates

$$\boldsymbol{\omega}_B = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_B = \left[\begin{array}{c|c|c} \sin\theta\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & -\sin\phi & 0 \\ \cos\theta & 0 & 1 \end{array}\right] \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \mathbf{L}_I^B \dot{\boldsymbol{\Theta}}$$



With $\dot{\theta} = 0$, $\sqrt{\omega_x^2 + \omega_y^2}$ = constant, ω_z = constant

$$\boldsymbol{\omega}_B = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_B = \begin{bmatrix} \dot{\psi}\sin\theta\sin\phi \\ \dot{\psi}\sin\theta\cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta \end{bmatrix}$$

Angular Motion in Space

Body-axis rate is constant

$$\boldsymbol{\omega}_B^T \boldsymbol{\omega}_B = \omega_B^2 = \text{ constant}$$

$$\frac{d\left(\boldsymbol{\omega}_B^T \boldsymbol{\omega}_B\right)}{dt} = 2\left(\boldsymbol{\omega}_B^T \dot{\boldsymbol{\omega}}_B\right) = 0$$

$$\omega_x\dot{\omega}_x + \omega_y\dot{\omega}_y + \omega_z\dot{\omega}_z = 0$$

Body-axis acceleration

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix}_B = \begin{bmatrix} \ddot{\psi}\sin\theta\sin\phi + \dot{\psi}\dot{\phi}\sin\theta\cos\phi \\ \ddot{\psi}\sin\theta\cos\phi - \dot{\psi}\dot{\phi}\sin\theta\sin\phi \\ \ddot{\phi} + \ddot{\psi}\cos\theta \end{bmatrix}$$

Consequently

$$\boldsymbol{\omega}_B^T \dot{\boldsymbol{\omega}}_B = \dot{\psi}\ddot{\psi}\sin^2\theta = 0$$

Angular Motion in Space

… and precession speed is constant

$$\boldsymbol{\omega}_B{}^T \dot{\boldsymbol{\omega}}_B = \dot{\psi}\ddot{\psi}\sin^2\theta = 0$$

$$\text{for } \theta \neq 0, \quad \dot{\psi}\ddot{\psi} = \frac{d}{dt}\left(\frac{\dot{\psi}^2}{2}\right) = 0$$

$$\therefore \dot{\psi} = \text{ constant} = \text{Precession Speed}$$

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix}_B = \begin{bmatrix} \dot{\psi}\dot{\phi}\sin\theta\cos\phi \\ -\dot{\psi}\dot{\phi}\sin\theta\sin\phi \\ \triangleq 0 \end{bmatrix}$$

hence,

$$\ddot{\phi} = 0$$

Angular Motion in Space

$$\text{Recall:} \quad I_{xx}\dot{\omega}_x + \left(I_{zz} - I_{yy}\right)\omega_y\omega_z = 0$$

$$I_{xx} = I_{yy} : I_{xx}\dot{\omega}_x + \left(I_{zz} - I_{xx}\right)\omega_y\omega_z = 0$$

$$I_{xx}\dot{\psi}\dot{\phi}\sin\theta\cos\phi + \left(I_{zz} - I_{xx}\right)\left(\dot{\psi}\sin\theta\cos\phi\right)\left(\dot{\phi} + \dot{\psi}\cos\theta\right) = 0$$

Which reduces to

$$\dot{\psi} = \frac{I_{zz}\dot{\phi}}{\left(I_{xx} - I_{zz}\right)\cos\theta} \text{ or } 0$$

$$\dot{\phi} = \frac{\left(I_{xx} - I_{zz}\right)}{I_{xx}}\omega_z$$

$$\dot{\psi} = \frac{I_{zz}}{I_{xx}\cos\theta}\omega_z \text{ or } 0$$

Direct Precession
$I_{zz} < I_{xx}$ (Rod)

$$\text{sgn}\left(\dot{\psi}\right) = \text{sgn}\left(\dot{\phi}\right)$$

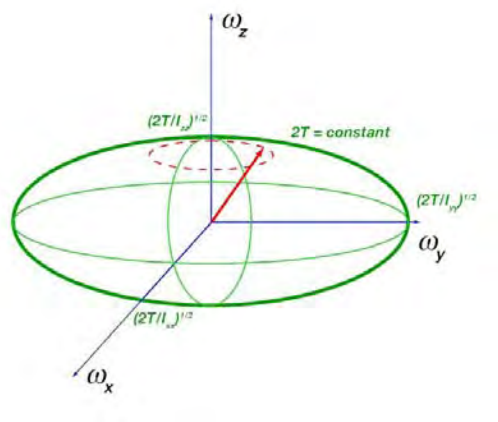
Retrograde Precession
$I_{zz} > I_{xx}$ (Disc)

$$\text{sgn}\left(\dot{\psi}\right) = -\text{sgn}\left(\dot{\phi}\right)$$

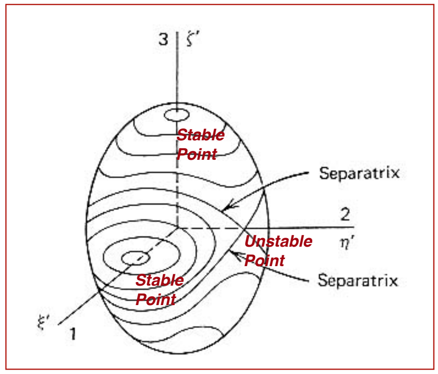
Poinsot (Energy) Ellipsoid

Locus of all angular rate combinations with constant angular energy

$$I_{xx}\omega_x^{\ 2} + I_{yy}\omega_y^{\ 2} + I_{zz}\omega_z^{\ 2} = 2 \times Kinetic\ Energy = 2T = \boldsymbol{\omega}^T \boldsymbol{I}\boldsymbol{\omega}$$



Polhodes: Paths of angular rate oscillations on Inertia Ellipsoid

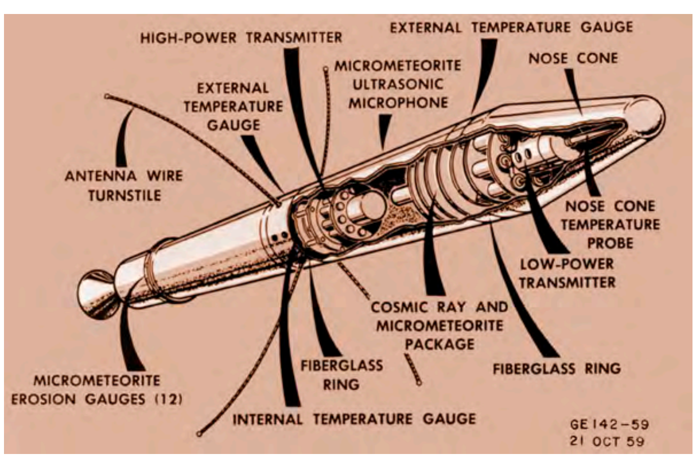


The Strange Case of *Explorer I*

Explorer I **spun about its axis of minimum moment of inertia when inserted into orbit**

Within a short time, it went into a flat spin, rotating about its maximum moment of inertia

Why?

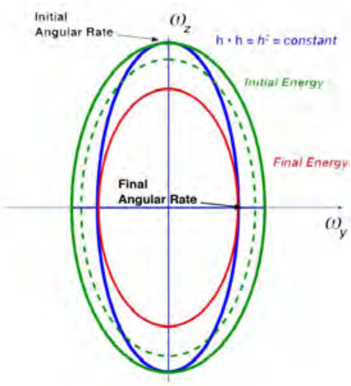


Shifting Rotational Equilibrium of *Explorer I*

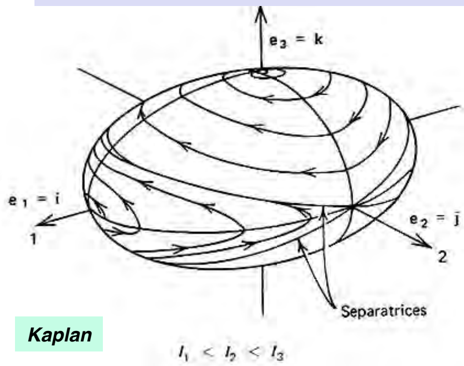
Whipping antennas dissipate angular energy

Angular momentum remains constant

Equilibrium rotational axis shifts from minimum to maximum moment of inertia (i.e., a flat spin)



Polhode of *Explorer I* angular rate perturbations



Dual-Spin Satellite Dynamics

Satellite has spinning and non-spinning components

Angular momentum and rate in non-spinning frame of reference

Angular momentum added by portion spinning about *z* axis

$$\dot{\mathbf{h}}_B = \boldsymbol{I}_B\dot{\boldsymbol{\omega}}_B = \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\left(\boldsymbol{I}_B\boldsymbol{\omega}_B + \mathbf{h}_{rotor}\right)$$

$$\dot{\boldsymbol{\omega}}_B = \boldsymbol{I}_B^{-1}\left[\mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\left(\boldsymbol{I}_B\boldsymbol{\omega}_B + \mathbf{h}_{rotor}\right)\right]$$

$$\mathbf{h}_{rotor} = \begin{bmatrix} 0 \\ 0 \\ h_{rotor} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} - \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \left[\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ h_{rotor} \end{pmatrix} \right] \right\}$$

***I_{xx}, I_{yy}*, and *I_{zz}* are moments of inertia of the entire satellite**

Perturbations in Dual-Spin Satellite Angular Rate

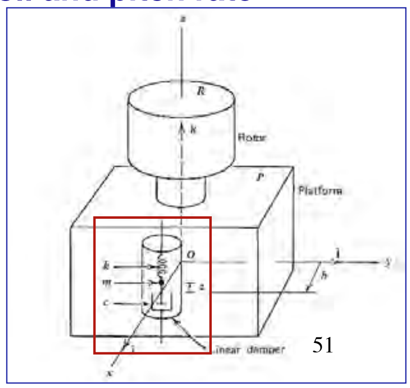
With zero nominal rates

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \omega_{x_o} + \Delta\omega_x \\ \omega_{y_o} + \Delta\omega_y \\ \omega_{z_o} + \Delta\omega_z \end{bmatrix} = \begin{bmatrix} \Delta\omega_x \\ \Delta\omega_y \\ \Delta\omega_z \end{bmatrix}$$

Perturbations (or nutations) in roll and pitch rate

$$\begin{bmatrix} \Delta\dot{\omega}_x \\ \Delta\dot{\omega}_y \\ \Delta\dot{\omega}_z \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}^{-1} \begin{bmatrix} \Delta M_x - h_{rotor}\Delta\omega_y \\ \Delta M_y + h_{rotor}\Delta\omega_x \\ \Delta M_z \end{bmatrix}$$

If *z* is the axis of minimum inertia (i.e., a "prolate" configuration), nutation damping is required to prevent satellite from entering a flat spin



Eigenvalues of Undamped Dual-Spin Satellite

$$\begin{bmatrix} \Delta\dot{\omega}_x \\ \Delta\dot{\omega}_y \\ \Delta\dot{\omega}_z \end{bmatrix} = \begin{bmatrix} 0 & -h_{rotor}/I_{xx} & 0 \\ h_{rotor}/I_{yy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\omega_x \\ \Delta\omega_y \\ \Delta\omega_z \end{bmatrix} + \begin{bmatrix} \Delta M_x(t)/I_{xx} \\ \Delta M_y(t)/I_{yy} \\ \Delta M_z(t)/I_{zz} \end{bmatrix}$$

Eigenvalues

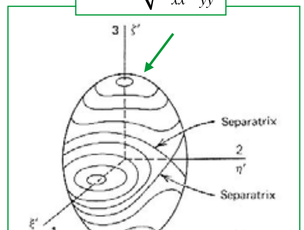
$$\Delta(s) = |s\mathbf{I} - \mathbf{F}| = \begin{vmatrix} s & h_{rotor}/I_{xx} & 0 \\ h_{rotor}/I_{yy} & s & 0 \\ 0 & 0 & s \end{vmatrix}$$

$$= s^3 + \left(h_{rotor}^{\ 2}/I_{xx}I_{yy}\right)s = s\left(s^2 + h_{rotor}^{\ 2}/I_{xx}I_{yy}\right) = 0$$

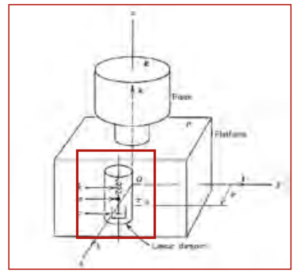
$$\lambda_{1,2,3} = 0, \ \pm j\frac{h_{rotor}}{\sqrt{I_{xx}I_{yy}}}$$

Natural frequency of small nutation orbits about the equilibrium point

$$\omega_n = \frac{h_{rotor}}{\sqrt{I_{xx}I_{yy}}}$$



Dual-Spin Satellite with Nutation Damper



Spring-mass-damper mounted on fixed platform

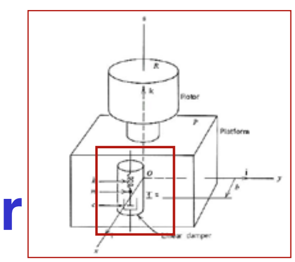
Angular motion about the *x* or *y* axis disturbs the system

$$
\begin{aligned}
m\Delta\ddot{z}_m &= -k_d\left(\Delta\dot{z} - \Delta\dot{z}_{S/C}\right) - k_s\left(\Delta z_m - \Delta z_{S/C}\right) \\
&= -k_d\left(\Delta\dot{z} - b\Delta q_{S/C}\right) - k_s\left(\Delta z_m - b\Delta\theta_{S/C}\right)
\end{aligned}
$$

k_s is a "soft" centering spring. Neglecting k_s, the mass's reaction torque on the spacecraft introduces damping, *d*

$$
\begin{bmatrix} \Delta\dot{\omega}_x \\ \Delta\dot{\omega}_y \\ \Delta\dot{\omega}_z \end{bmatrix} = \begin{bmatrix} 0 & -h_{rotor} / I_{xx} & 0 \\ h_{rotor} / I_{yy} & -d / I_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\omega_x \\ \Delta\omega_y \\ \Delta\omega_z \end{bmatrix}
$$

Natural Frequency and Damping Ratio of Dual-Spin Satellite with Nutation Damper

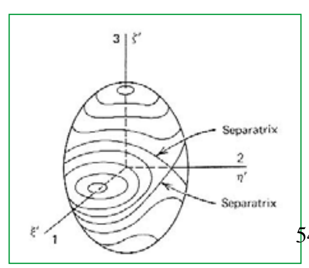


$$
\Delta(s) = \left|s\mathbf{I} - \mathbf{F}\right| = \begin{vmatrix} s & h_{rotor} / I_{xx} & 0 \\ h_{rotor} / I_{yy} & \left(s + d / I_{yy}\right) & 0 \\ 0 & 0 & s \end{vmatrix} = 0
$$

$$
= s^2\left(s + d / I_{yy}\right) + s\left(h_{rotor}{}^2 / I_{xx} I_{yy}\right) = s\left(s^2 + sd / I_{yy} + h_{rotor}{}^2 / I_{xx} I_{yy}\right)
$$

$$
\equiv s\left(s^2 + 2\zeta\omega_n s + \omega_n{}^2\right)
$$

$$
\boxed{\omega_n = \frac{h_{rotor}}{\sqrt{I_{xx} I_{yy}}}} \quad \boxed{\zeta = \frac{d / I_{yy}}{2h_{rotor} / \sqrt{I_{xx} I_{yy}}}}
$$

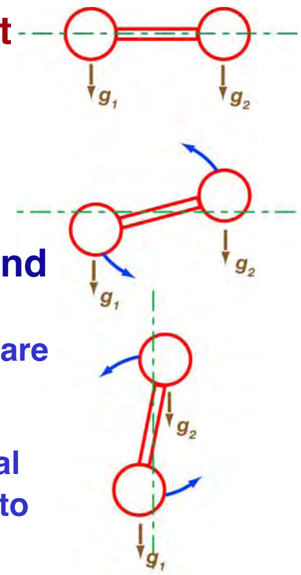


Damper prevents small nutations from becoming large enough to shift equilibrium spin axes

Gravity-Gradient Effect on Spacecraft Attitude

- **Gravitational field and gravity gradient**

$$g = -\frac{\mu}{r^2}; \quad \frac{\partial g}{\partial r} = \frac{2\mu}{r^3}$$



- **Dumbbell satellite (equal masses at end of bar)**
  - At **horizontal** attitude, gravitational effects are equal, and torque is zero
  - At **small angle**, forces are unequal, and torque rotates satellite away from horizontal
  - Near **vertical** attitude, unequal forces tend to align satellite with the vertical

Gravity-Gradient Stabilization about the Local Vertical

Gravitational torques on satellite, *(1-2-3) Euler angles*

$$M_x = \frac{3\mu}{2r^3}\left(I_{zz} - I_{yy}\right)\sin 2\phi \cos^2\theta$$

$$M_y = \frac{3\mu}{2r^3}\left(I_{zz} - I_{xx}\right)\sin 2\theta \cos\phi$$

$$M_z = \frac{3\mu}{2r^3}\left(I_{xx} - I_{yy}\right)\sin 2\theta \sin\phi$$

With $I_{xx} = I_{yy}$, restoring torques produce a librational oscillation with natural frequency

$$\omega_n = \sqrt{\frac{3\mu}{2r^3}\left(1 - \frac{I_{zz}}{I_{xx}}\right)} \; rad/sec$$

Next Time:
Spacecraft Control

Supplemental Material

Cross-Product-Equivalent Matrix

- **Cross product**

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \textit{unit vectors along } (x,y,z)$$

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = \left(yv_z - zv_y\right) i + \left(zv_x - xv_z\right) j + \left(xv_y - yv_x\right) k$$

$$= \left(yv_z - zv_y\right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left(zv_x - xv_z\right) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \left(xv_y - yv_x\right) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(yv_z - zv_y\right) \\ \left(zv_x - xv_z\right) \\ \left(xv_y - yv_x\right) \end{bmatrix} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \tilde{\mathbf{r}}\mathbf{v}$$

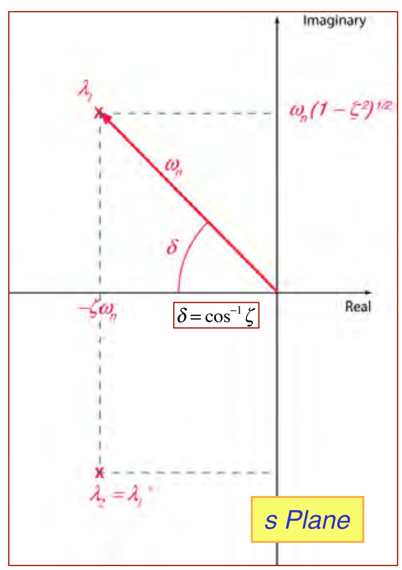
- **Cross-product-equivalent matrix**

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

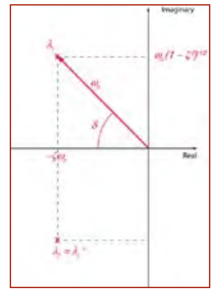
Eigenvalues (or Roots) of the Dynamic System

$$\Delta(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$
$$= \left(s - \lambda_1\right)\left(s - \lambda_2\right)\left(\ldots\right)\left(s - \lambda_n\right) = 0$$

- **Roots may be real or complex**
- **Real and imaginary parts of the eigenvalues can be plotted in the *s* plane**
- **Real roots**
  - **are confined to the real axis**
  - **represent convergent or divergent modes**
- **Complex roots**
  - **occur only in complex-conjugate pairs**
  - **represent oscillatory modes**
  - **natural frequency and damping ratio as shown**



Modes of Motion



$$\mathbf{x}(s) = \frac{Adj\left(s\mathbf{I} - \mathbf{F}\right)}{\Delta(s)}\left[\mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)\right]$$

- Eigenvalues characterize the modes of the system
  - Mode is stable if Real (λ_i) < 0
  - Mode is unstable if Real (λ_i) > 0