Spectral Properties of Linear-Quadratic Regulators

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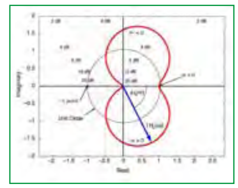
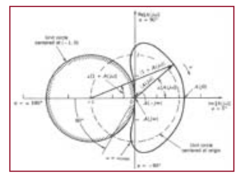
Optimal Control and Estimation MAE 546
Princeton University, 2015

- Stability margins of single-input/single-output (SISO) systems
- Characterizations of frequency response
- Loop transfer function
- Return difference function
- Kalman inequality
- Stability margins of scalar linear-quadratic regulators

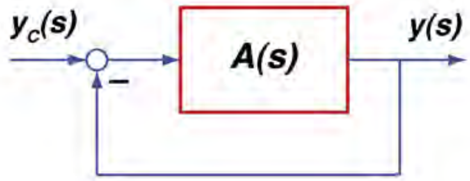




http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Return Difference Function and Closed-Loop Roots Single-Input/Single-Output Control Systems

SISO Transfer Function and Return Difference Function

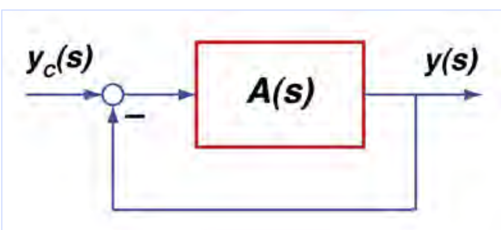


- **Unit feedback control law**
- **Block diagram algebra**

$$y(s) = A(s)\left[y_C(s) - y(s)\right]$$
$$\left[1 + A(s)\right]y(s) = A(s)y_C(s)$$

$$\frac{y(s)}{y_C(s)} = \frac{A(s)}{1 + A(s)} : \textbf{ Closed - Loop Transfer Function}$$

$$A(s) : \textbf{ Open - Loop Transfer Function}$$
$$\left[1 + A(s)\right] : \textbf{ Return Difference Function}$$



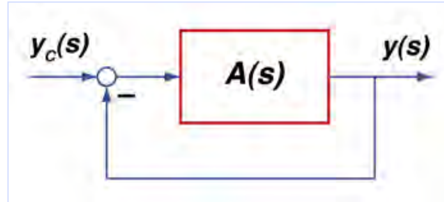
Return Difference Function and Root Locus

$$A(s) = \frac{kn(s)}{d(s)}$$

$$1 + A(s) = 1 + \frac{kn(s)}{d(s)} = 0 \text{ defines locus of roots}$$

$$d(s) + kn(s) = 0 \text{ defines locus of closed-loop roots}$$

Return Difference Example



$$A(s)=\frac{kn(s)}{d(s)}=\frac{k(s-z)}{s^2+2\zeta\omega_n s+\omega_n{}^2}=\frac{1.25(s+40)}{s^2+2(0.3)(7)s+(7)^2}$$

$$1+A(s)=1+\frac{k(s-z)}{s^2+2\zeta\omega_n s+\omega_n{}^2}=1+\frac{1.25(s+40)}{s^2+2(0.3)(7)s+(7)^2}=0$$

$$\left[s^2+2(0.3)(7)s+(7)^2\right]+1.25(s+40)=0$$

Closed-Loop Transfer Function Example

$$\frac{A(s)}{1+A(s)}=\frac{\left[\dfrac{1.25(s+40)}{s^2+2(0.3)(7)s+(7)^2}\right]}{1+\left[\dfrac{1.25(s+40)}{s^2+2(0.3)(7)s+(7)^2}\right]}$$

$$\frac{A(s)}{1+A(s)}=\frac{1.25(s+40)}{\left[s^2+2(0.3)(7)s+(7)^2\right]+1.25(s+40)}$$

$$=\frac{1.25(s+40)}{s^2+\left[2(0.3)(7)+1.25\right]s+\left[(7)^2+1.25(40)\right]}$$

$$=\frac{kn(s)}{d(s)+kn(s)}$$

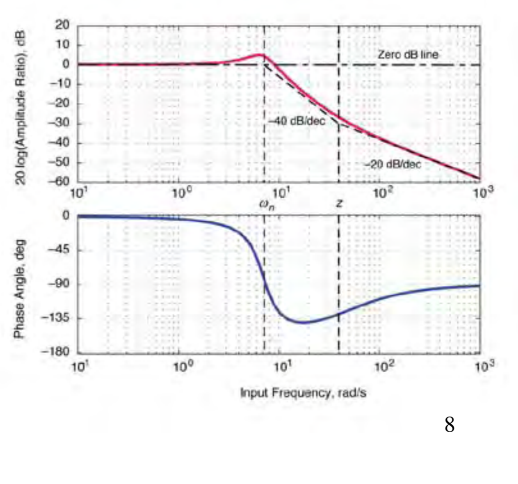
SISO Frequency-Response Plots

Open-Loop Frequency Response: Bode Plot

$$A(j\omega) = \frac{K(j\omega - z)}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{1.25(j\omega + 40)}{(j\omega)^2 + 2(0.3)(7)j\omega + (7)^2}$$

Two plots

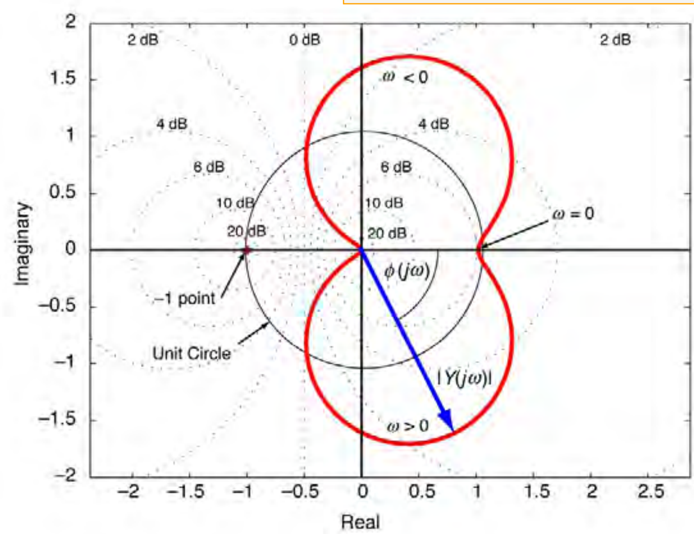
- $20\log|A(j\omega)|$ vs. $\log\omega$
- $\measuredangle[A(j\omega)]$ vs. $\log\omega$



- **Gain Margin**
  - Referenced to 0 dB line
  - Evaluated where phase angle = –180°
- **Phase Margin**
  - Referenced to –180°
  - Evaluated where amplitude ratio = 0 dB

Open-Loop Frequency Response: Nyquist Plot

$$\mathrm{Re}\big(A(j\omega)\big) \text{ vs. } \mathrm{Imag}\big(A(j\omega)\big)$$



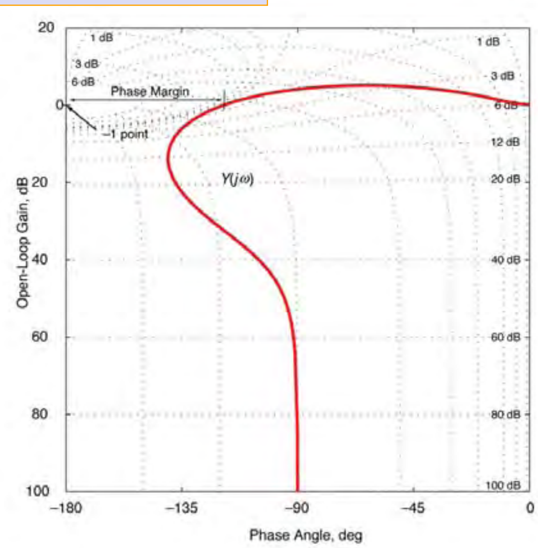
Only positive frequencies need be considered

- Single plot; input frequency not shown explicitly
- Gain and Phase Margins referenced to (–1) point
- GM and PM represented as length and angle

Open-Loop Frequency Response: Nichols Chart

$$20\log_{10}\big[A(j\omega)\big] \text{ vs. } \measuredangle\big[A(j\omega)\big]$$

- Single plot
- Gain and Phase Margins shown directly



b)

Algebraic Riccati Equation in the Frequency Domain

Linear-Quadratic Control

- **Quadratic cost function for infinite final time**

$$J = \frac{1}{2}\int_{t_o}^{\infty}\begin{bmatrix} \Delta\mathbf{x}^T(t) & \Delta\mathbf{u}^T(t) \end{bmatrix}\begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t) \end{bmatrix} dt = \frac{1}{2}\int_{t_o}^{\infty}\left[\Delta\mathbf{x}^T(t)\mathbf{Q}\Delta\mathbf{x}(t) + \Delta\mathbf{u}(t)\mathbf{R}\Delta\mathbf{u}^T(t)\right]dt$$

- **Linear, time-invariant dynamic system**

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)$$

- **Constant-gain optimal control law**

$$\begin{aligned} \Delta\mathbf{u}(t) &= -\mathbf{R}^{-1}\mathbf{G}^T\mathbf{P}\Delta\mathbf{x}(t) \\ &= -\mathbf{C}\Delta\mathbf{x}(t) \end{aligned}$$

Algebraic Riccati equation

$$\begin{aligned} \mathbf{0} &= -\mathbf{Q} - \mathbf{F}^T\mathbf{P} - \mathbf{P}\mathbf{F} + \mathbf{P}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T\mathbf{P} \\ \mathbf{Q} &= -\mathbf{F}^T\mathbf{P} - \mathbf{P}\mathbf{F} + \mathbf{C}^T\mathbf{R}\mathbf{C} \end{aligned}$$

Frequency Characteristics of the Algebraic Riccati Equation

Add and subtract *sP* such that

$$\boxed{\mathbf{P}(-\mathbf{F})+(-\mathbf{F}^T)\mathbf{P}+\mathbf{C}^T\mathbf{R}\mathbf{C}=\mathbf{Q}}$$

$$\boxed{\mathbf{P}(s\mathbf{I}_n-\mathbf{F})+(-s\mathbf{I}_n-\mathbf{F}^T)\mathbf{P}+\mathbf{C}^T\mathbf{R}\mathbf{C}=\mathbf{Q}}$$

State Characteristic Matrix	Adjoint Characteristic Matrix

Frequency Characteristics of the Algebraic Riccati Equation

$$\boxed{\mathbf{P}(s\mathbf{I}_n-\mathbf{F})+(-s\mathbf{I}_n-\mathbf{F}^T)\mathbf{P}+\mathbf{C}^T\mathbf{R}\mathbf{C}=\mathbf{Q}}$$

Pre-multiply each term by

$$\boxed{\mathbf{G}^T(-s\mathbf{I}_n-\mathbf{F}^T)^{-1}}$$

Post-multiply each term by

$$\boxed{(s\mathbf{I}_n-\mathbf{F})^{-1}\mathbf{G}}$$

$$\mathbf{G}^T(-s\mathbf{I}_n-\mathbf{F}^T)^{-1}\mathbf{P}\mathbf{G}+\mathbf{G}^T\mathbf{P}(s\mathbf{I}_n-\mathbf{F})^{-1}\mathbf{G}+\mathbf{G}^T(-s\mathbf{I}_n-\mathbf{F}^T)^{-1}\mathbf{C}^T\mathbf{R}\mathbf{C}(s\mathbf{I}_n-\mathbf{F})^{-1}\mathbf{G}$$
$$=\mathbf{G}^T(-s\mathbf{I}_n-\mathbf{F}^T)^{-1}\mathbf{Q}(s\mathbf{I}_n-\mathbf{F})^{-1}\mathbf{G}$$

Frequency Characteristics of the Algebraic Riccati Equation

Substitute with the control gain matrix

$$\mathbf{C} = \mathbf{R}^{-1}\mathbf{G}^T\mathbf{P}$$
$$\mathbf{G}^T\mathbf{P} = \mathbf{R}\mathbf{C}$$

$$\mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{C}^T\mathbf{R} + \mathbf{R}\mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G} + \mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{C}^T\mathbf{R}\mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$
$$= \mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{Q}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$

Add **R** to both sides

$$\mathbf{R} + \mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{C}^T\mathbf{R} + \mathbf{R}\mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G} + \mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{C}^T\mathbf{R}\mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$
$$= \mathbf{R} + \mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{Q}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$

Frequency Characteristics of the Algebraic Riccati Equation

$$\mathbf{R} + \mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{C}^T\mathbf{R} + \mathbf{R}\mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G} + \mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{C}^T\mathbf{R}\mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$
$$= \mathbf{R} + \mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{Q}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$

The left side can be factored as*

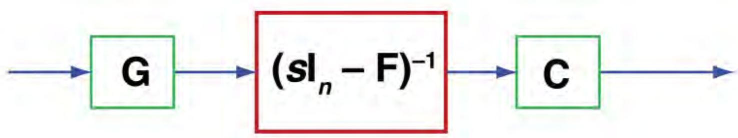
$$\left[\mathbf{I}_m + \mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{C}^T\right]\mathbf{R}\left[\mathbf{I}_m + \mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}\right]$$
$$= \mathbf{R} + \mathbf{G}^T\left(-s\mathbf{I}_n - \mathbf{F}^T\right)^{-1}\mathbf{Q}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$

** Verify by multiplying the factored form*

Modal Expression of Algebraic Riccati Equation

Define the loop transfer function matrix

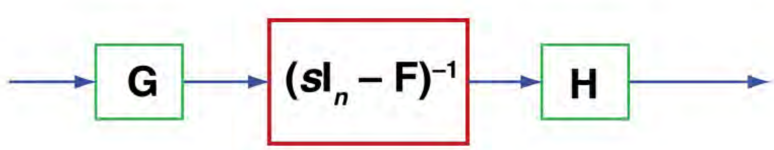
$$\boxed{\mathbf{A}(s) = \mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}}$$



Modal Expression of Algebraic Riccati Equation

Recall the cost function transfer matrix

$$\boxed{\mathbf{Y}(s) = \mathbf{H}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}, \quad \text{where } \mathbf{Q} = \mathbf{H}^T\mathbf{H}}$$



Laplace transform of algebraic Riccati equation becomes

$$\boxed{\left[\mathbf{I}_m + \mathbf{A}(-s)\right]^T \mathbf{R}\left[\mathbf{I}_m + \mathbf{A}(s)\right] = \mathbf{R} + \mathbf{Y}^T(-s)\mathbf{Y}(s)}$$

Algebraic Riccati Equation

$$\boxed{\left[\mathbf{I}_m + \mathbf{A}(-s)\right]^T \mathbf{R}\left[\mathbf{I}_m + \mathbf{A}(s)\right] = \mathbf{R} + \mathbf{Y}^T(-s)\mathbf{Y}(s)}$$

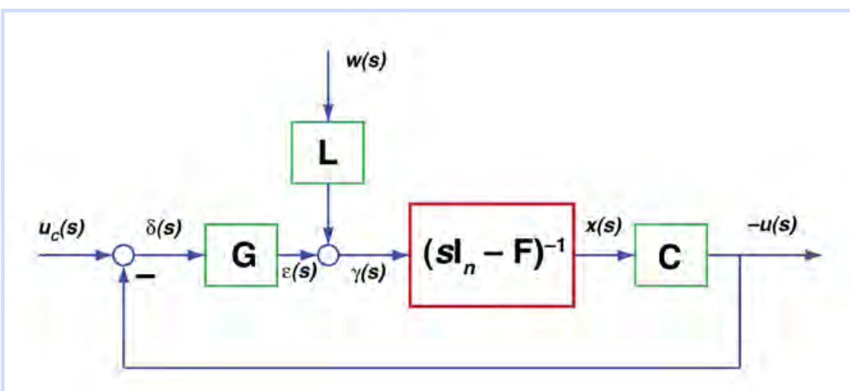
- **Cost function transfer matrix, Y(s)**
  - **Reflects control-induced state variations in the cost function**
  - **Governs closed-loop modal properties as R becomes small**
  - **Does not depend on R or P**

$$\mathbf{Y}(s) = \mathbf{H}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$

- **Loop transfer function matrix, A(s)**
  - **Defines the modal control vector when $s = s_i$**

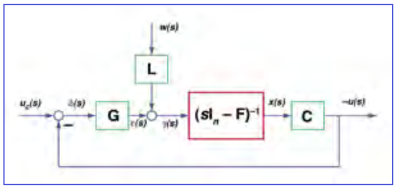
$$\Delta\mathbf{u}_i = -\mathbf{C}\left(s_i\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}\Delta\mathbf{u}_i$$
$$= -\mathbf{A}(s_i)\Delta\mathbf{u}_i, \quad i = 1, n$$

LQ Regulator Portrayed as a Unit-Feedback System



$$\mathbf{A}(s) = \mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$

$$\mathbf{I}_m + \mathbf{A}(s) = \mathbf{I}_m + \mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G} : \quad \textbf{Return Difference Matrix}$$



Determinant of Return Difference Matrix Defines Closed-Loop Eigenvalues

$$\left|\mathbf{I}_m + \mathbf{A}(s)\right| = \left|\mathbf{I}_m + \mathbf{C}\left(s\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}\right|$$

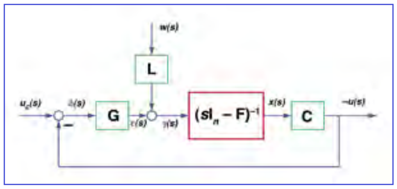
$$= \left|\mathbf{I}_m + \frac{\mathbf{C}Adj\left(s\mathbf{I}_n - \mathbf{F}\right)\mathbf{G}}{\left|s\mathbf{I}_n - \mathbf{F}\right|}\right| = \left|\mathbf{I}_m + \frac{\mathbf{C}Adj\left(s\mathbf{I}_n - \mathbf{F}\right)\mathbf{G}}{\Delta_{\mathbf{OL}}(s)}\right|$$

Characteristic Equation

$$\Delta_{\mathbf{OL}}(s)\left|\mathbf{I}_m + \mathbf{A}(s)\right| = \Delta_{\mathbf{OL}}(s)\left|\mathbf{I}_m + \frac{\mathbf{C}Adj\left(s\mathbf{I}_n - \mathbf{F}\right)\mathbf{G}}{\Delta_{\mathbf{OL}}(s)}\right|$$

$$= \left|\Delta_{\mathbf{OL}}(s)\mathbf{I}_m + \mathbf{C}Adj\left(s\mathbf{I}_n - \mathbf{F}\right)\mathbf{G}\right| = \Delta_{\mathbf{CL}}(s) = 0$$

Stability Margins and Robustness of Scalar LQ Regulators



Scalar Case

Multivariable algebraic Riccati equation

$$\left[\mathbf{I}_m + \mathbf{A}(-s)\right]^T \mathbf{R}\left[\mathbf{I}_m + \mathbf{A}(s)\right] = \mathbf{R} + \mathbf{Y}^T(-s)\mathbf{Y}(s)$$

Algebraic Riccati equation with <u>scalar control</u>

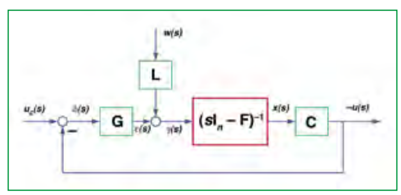
$$\left[1 + A(-s)\right] r \left[1 + A(s)\right] = r + Y^T(-s)Y(s)$$

where

$$A(s) = \mathbf{C}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G} \quad (1 \times 1)$$
$$\dim(\mathbf{C}) = (1 \times n)$$
$$\dim(\mathbf{F}) = (n \times n)$$
$$\dim(\mathbf{G}) = (n \times 1)$$

$$Y(s) = \mathbf{H}(s\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}$$
$$\dim\left[Y(s)\right] = (p \times 1)$$
$$\dim(\mathbf{H}) = (p \times n)$$

Scalar Case



Let $s = j\omega$

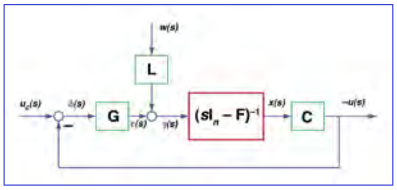
$$\left[1 + A(-j\omega)\right] r \left[1 + A(j\omega)\right] = r + Y^T(-j\omega)Y(j\omega)$$

or

$$\left[1 + A(-j\omega)\right]\left[1 + A(j\omega)\right] = 1 + \frac{Y^T(-j\omega)Y(j\omega)}{r}$$

$A(j\omega)$ is a complex variable

$$\left[1 + A(-j\omega)\right]\left[1 + A(j\omega)\right] = \left\{\left[1 + c(\omega)\right] - jd(\omega)\right\}\left\{\left[1 + c(\omega)\right] + jd(\omega)\right\}$$
$$= \left\{\left[1 + c(\omega)\right]^2 + d^2(\omega)\right\} = \left|1 + A(j\omega)\right|^2 \quad \text{(absolute value)}$$



Kalman Inequality

In frequency domain, cost transfer function becomes

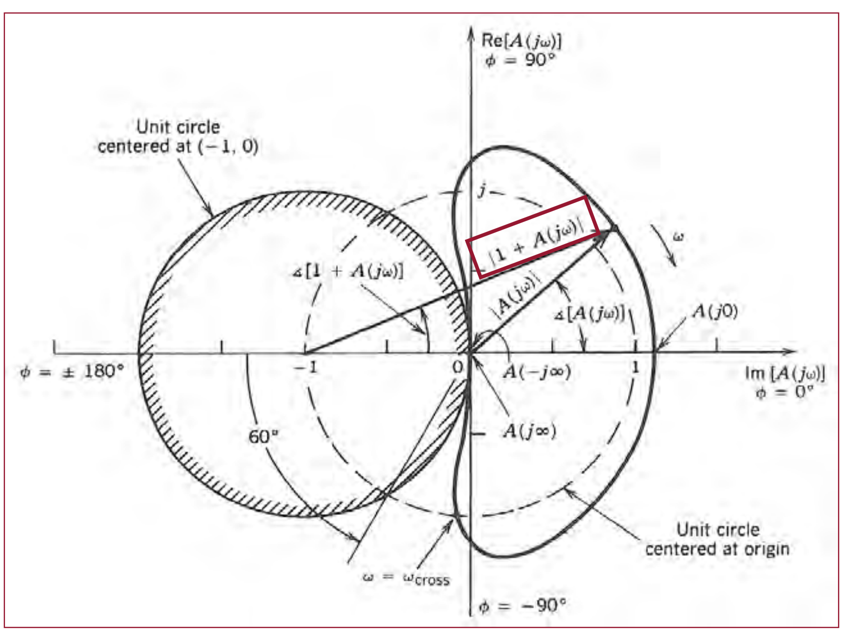
$$Y_i(j\omega) = \left[l_i(\omega) + jm_i(\omega) \right], \quad i = 1, p$$

$$1 + \frac{Y^T(-j\omega)Y(j\omega)}{r} = 1 + \sum_{i=1}^{p} \frac{\left[l_i^2(\omega) + m_i^2(\omega) \right]}{r}$$

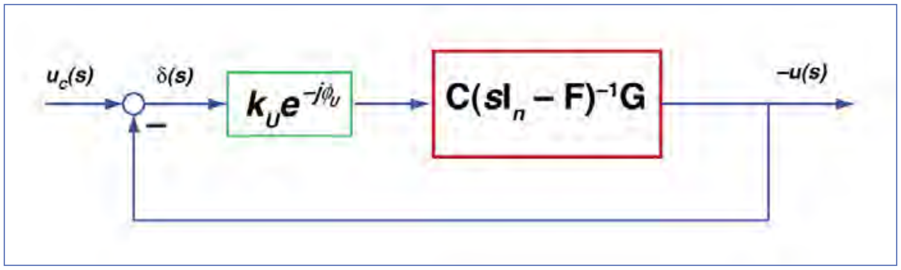
Consequently, the return difference function <u>magnitude</u> is greater than one

$$\left| 1 + A(j\omega) \right| \geq 1 \quad \textbf{Kalman Inequality}$$

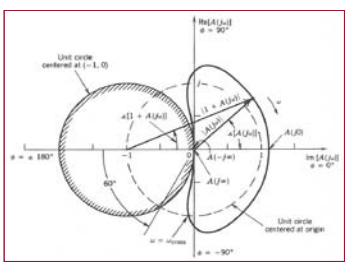
Nyquist Plot Showing Consequences of Kalman Inequality



Uncertain Gain and Phase Modifications to the LQ Feedback Loop

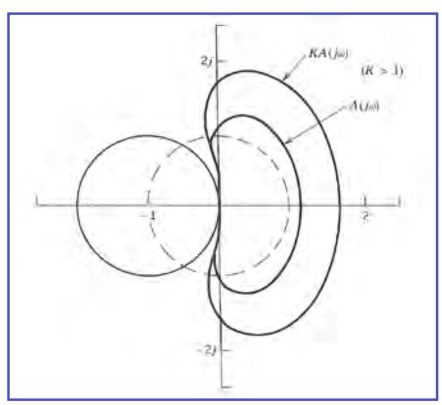


How large an uncertainty in loop gain or phase angle can be tolerated by the LQ regulator?



LQ Gain Margin Revealed by Kalman Inequality

- **Stability is preserved if**
  - No encirclements of the –1 point, or
  - Number of counterclockwise encirclements of the –1 point equals the number of unstable open-loop roots
- **Loop gain change expands or shrinks entire Nyquist plot**



Gain Change Expands or Shrinks Entire Plot

$$k_U \triangleq \text{ Uncertain gain}$$

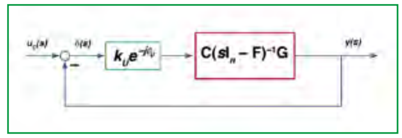
$$A_{optimal}(j\omega) = \mathbf{C}_{LQ}(j\omega\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}$$

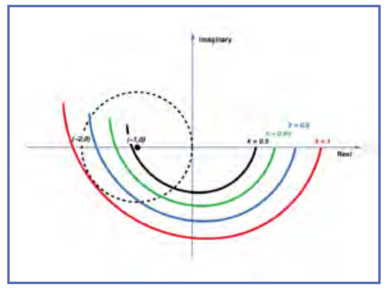
$$A_{non-optimal}(j\omega) = k_U A_{optimal}(j\omega)$$

$$= k_U \mathbf{C}_{LQ}(j\omega\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}$$

$$\left|A_{non-optimal}(j\omega)\right| = k_U \left|A_{optimal}(j\omega)\right|$$

- **Closed-Loop LQ system is stable until –1 point is reached, and # of encirclements changes**

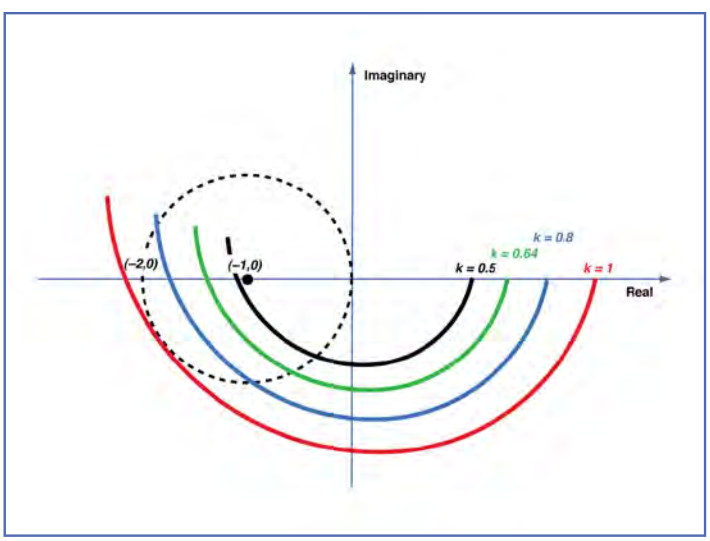




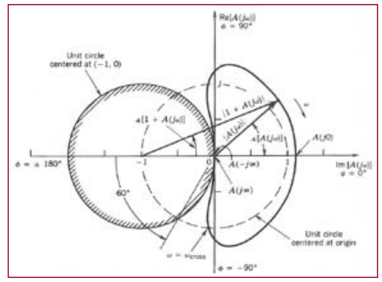
Scalar LQ Regulator Gain Margin

Increased gain margin = Infinity

Decreased gain margin = 50%



LQ Phase Margin Revealed by Kalman Inequality



- Stability is preserved if
  - No encirclements of the –1 point
  - Number of counterclockwise encirclements of the –1 point equals the number of unstable open-loop roots

Return Difference Function, $1 + \mathrm{A}(j\omega)$, is excluded from a unit circle centered at $(-1,0)$

$\left|\mathrm{A}(j\omega)\right| = 1$ intersects a unit circle centered at the origin

Intersection of the unit circles occurs where the phase angle of $A(j\omega) = \left(-180^\circ \pm 60^\circ\right)$

Therefore, Phase Margin of LQ regulator ≥ 60°

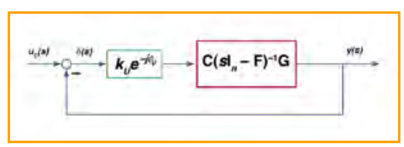
LQ Regulator Preserves Stability with Phase Uncertainties of At Least –60°

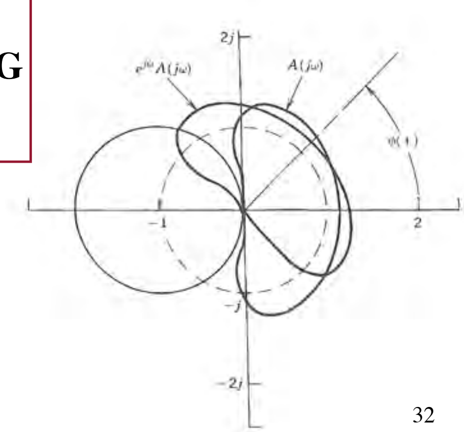
$$\varphi_U \triangleq \text{ Uncertain phase angle, deg}$$

$$A_{optimal}(j\omega) = \mathbf{C}_{LQ}\left(j\omega\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$

$$A_{non-optimal}(j\omega) = e^{j\varphi_U} A_{optimal}(j\omega) = e^{j\varphi_U}\mathbf{C}_{LQ}\left(j\omega\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$

$$\varphi_{non-optimal} = \varphi_{optimal} + \varphi_U$$

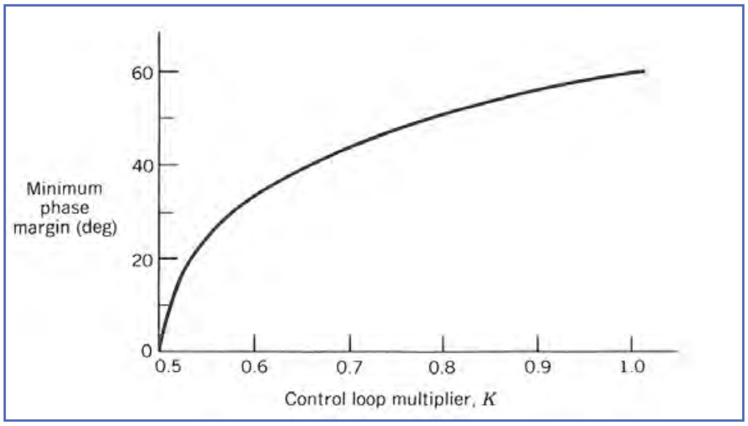




- Phase-angle change rotates entire Nyquist plot
- Closed-Loop LQ system is stable until –1 point is reached

Reduced-Gain-/Phase-Margin Tradeoff

Reduced loop gain decreases allowable phase lag while retaining closed-loop stability

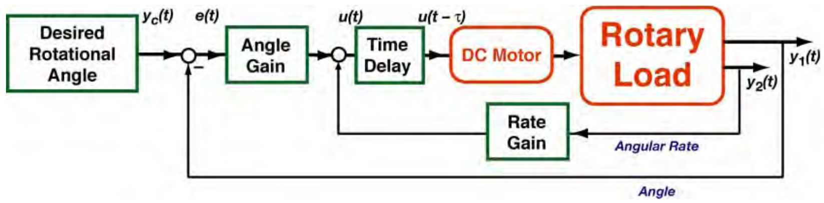


Next Time:
Singular Value Analysis of LQ Systems

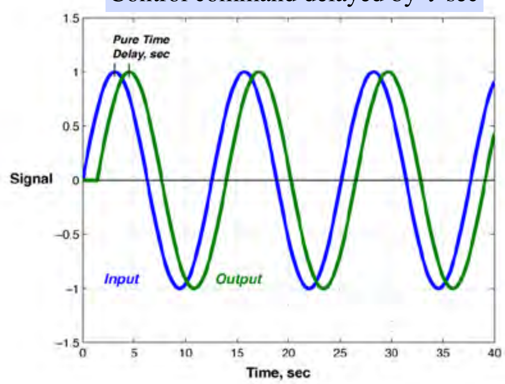
Supplemental Material

Effect of Time Delay

Time Delay Example: DC Motor Control

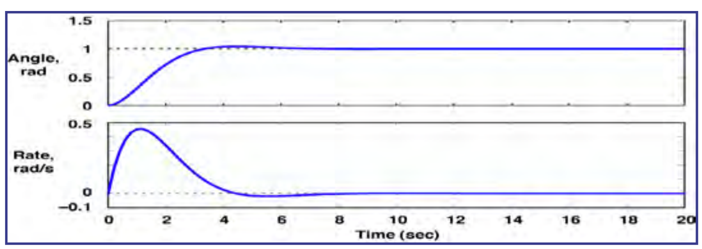


Control command delayed by τ sec

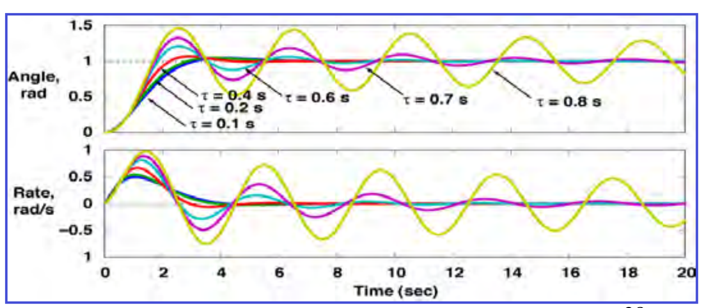


Effect of Time Delay on Step Response

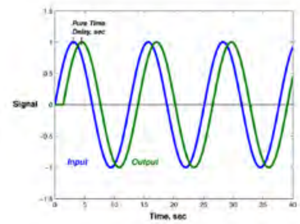
With no delay

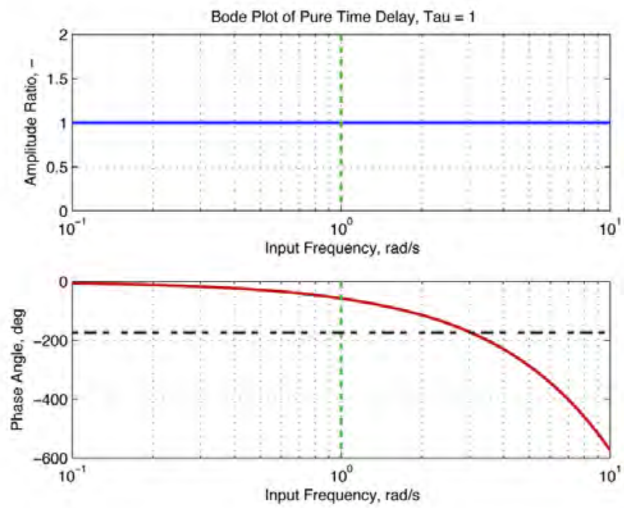


Phase lag due to time delay reduces closed-loop stability



Bode Plot of Pure Time Delay





$$AR\left(e^{-j\tau\omega}\right) = 1$$
$$\phi\left(e^{-j\tau\omega}\right) = -\tau\omega$$

As input frequency increases, phase angle eventually exceeds –180°

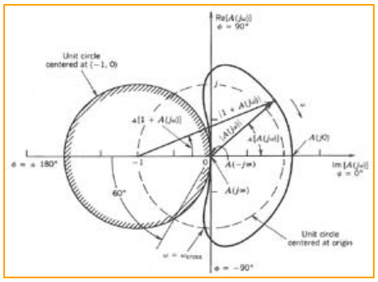
Effect of Pure Time Delay on LQ Regulator Loop Transfer Function

Laplace transform of time-delayed signal:

$$L\left[u(t-\tau)\right] = e^{-\tau s} L\left[u(t)\right] = e^{-\tau s} u(s)$$

$\tau_U \triangleq$ Uncertain time delay, sec

$$A_{optimal}(j\omega) = \mathbf{C}_{LQ}\left(j\omega\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$
$$A_{non-optimal}(j\omega) = e^{-\tau_U j\omega} A_{optimal}(j\omega)$$
$$= e^{-\tau_U j\omega}\mathbf{C}_{LQ}\left(j\omega\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}$$



Effect of Pure Time Delay on LQ Regulator Loop Transfer Function

Crossover frequency, ω_{cross}, is frequency for which

$$\left|A_{optimal}\left(j\omega_{cross}\right)\right| = \left|\mathbf{C}_{LQ}\left(j\omega_{cross}\mathbf{I}_n - \mathbf{F}\right)^{-1}\mathbf{G}\right| = 1$$

Time delay that produces 60° phase lag

$$\tau_U = \frac{60°}{\omega_{cross}}\left(\frac{\pi}{180°}\right) = \frac{\pi}{3\omega_{cross}}, \text{sec}$$