Minimum-Time and -Fuel Optimization

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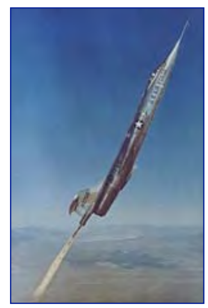
Optimal Control and Estimation MAE 546
Princeton University, 2015

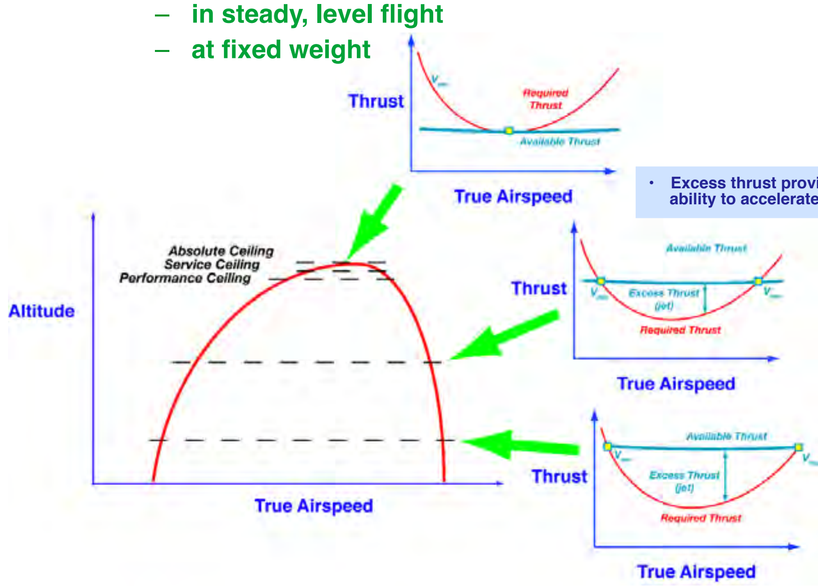
- Climbing flight of an airplane
- Energy and power
- Choice of control variable
- Numerical optimization
- Minimum time-to-climb problem
- Minimum fuel-to-climb problem
- Stability and control *(supplement)*



http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Flight Envelope Determined by Available Thrust

- **Flight Envelope:** Encompasses all altitudes and airspeeds at which an aircraft can fly
  - in steady, level flight
  - at fixed weight



- Excess thrust provides the ability to accelerate or climb

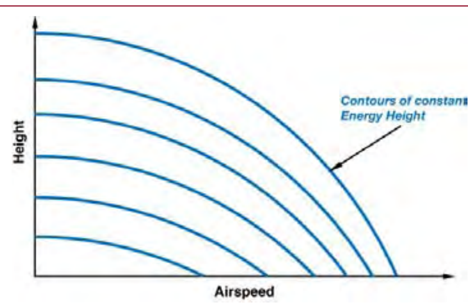
Energy Height and Specific Excess Power

Energy Height

- **Specific Energy**
  - **= (Potential + Kinetic Energy) per Unit Weight**
  - **= Energy Height**

$$\frac{Total\ Energy}{Unit\ Weight} \equiv Specific\ Energy = \frac{mgh + mV^2/2}{mg} = h + \frac{V^2}{2g}$$

$$\equiv \boldsymbol{Energy\ Height}, E_h, \quad ft\ or\ m$$



- **Could trade altitude for airspeed with no change in energy height if thrust and drag were zero**

Specific Excess Power

- **Specific Power**

$$\frac{dE_h}{dt} = \frac{d}{dt}\left(h + \frac{V^2}{2g}\right) = \frac{dh}{dt} + \left(\frac{V}{g}\right)\frac{dV}{dt}$$

$$\dot{V} = (T - D)/m - g\sin\gamma$$
$$\dot{h} = V\sin\gamma$$

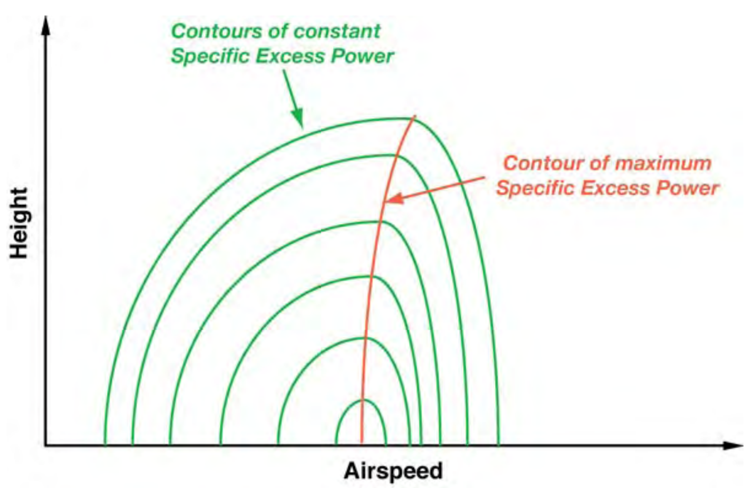
$$\frac{dE_h}{dt} = V\sin\gamma + \left(\frac{V}{g}\right)\left[\frac{(T-D)}{m} - g\sin\gamma\right]$$

$$\frac{dE_h}{dt} = V\frac{(T-D)}{W} = V\frac{(C_T - C_D)\frac{1}{2}\rho(h)V^2 S}{W}$$

$$\boxed{\frac{dE_h}{dt} = \textbf{Specific Excess Power}\ (SEP) = \frac{Excess\ Power}{Unit\ Weight} \equiv \frac{\left(P_{thrust} - P_{drag}\right)}{W}}$$

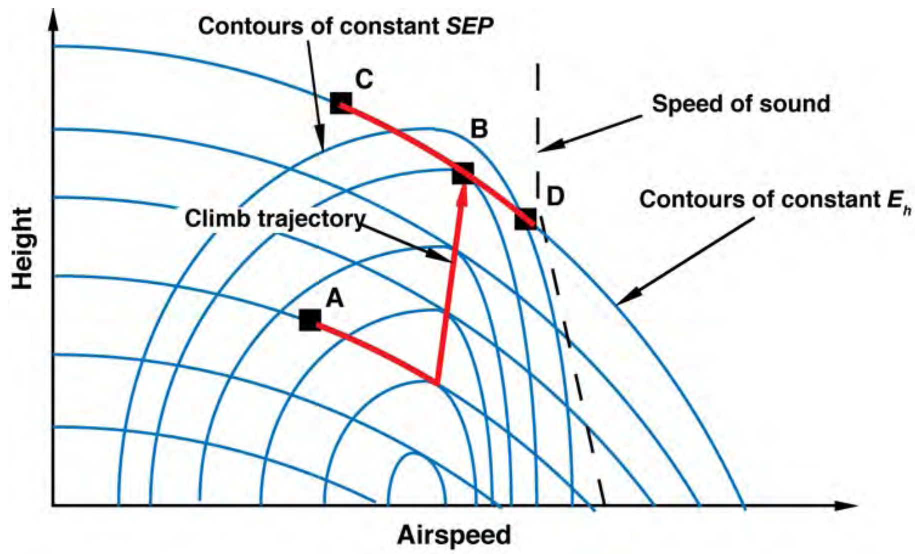
Contours of Constant Specific Excess Power

- **Specific Excess Power is a function of altitude and airspeed**
- ***SEP* is maximized at each altitude, *h*, when** $\boxed{\frac{d[SEP(h)]}{dV} = 0}$



Subsonic Energy Climb

Objective: Minimize time or fuel to climb to desired altitude and airspeed



Approximate Optimal Trajectory produced by a Switching Curve

Angle of Attack Control

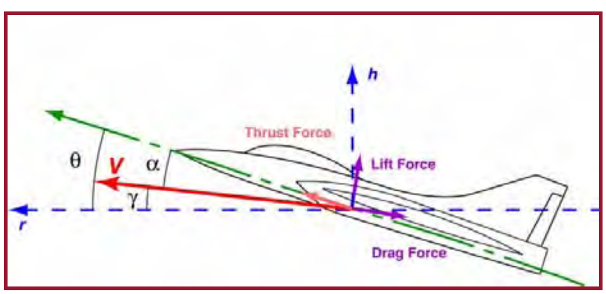
$$\dot{V} = \left[T_{\max} - \left(C_{D_o} + \varepsilon \left[C_{L_\alpha} \alpha \right]^2 \right) \frac{1}{2} \rho V^2 S \right] \Big/ m - g \sin\gamma$$

$$\dot{\gamma} = \frac{1}{V} \left[\left(C_{L_\alpha} \alpha \frac{1}{2} \rho V^2 S \right) \Big/ m - g \cos\gamma \right]$$

$$\dot{h} = V \sin\gamma$$

$$\dot{r} = V \cos\gamma$$

$$\dot{m}_{fuel} = -(SFC)(T_{\max})$$



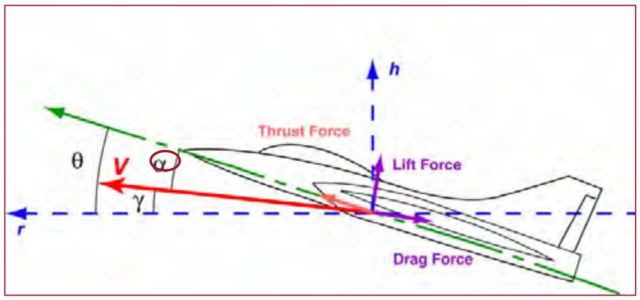
Minimum Time-to-Climb Problem

- **Initial and final conditions**

$$\boxed{V_o = 100\text{ m/s};\quad \gamma_o = 0\text{ rad};\quad h_o = 0\text{ m (sea level)};\quad r_o = open}$$

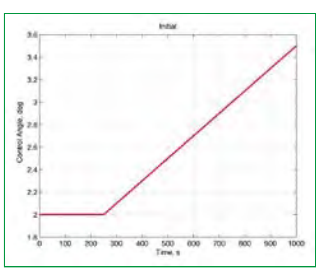
$$\boxed{V_f = 200\text{ m/s};\quad \gamma_f = open;\quad h_f = 10{,}000\text{ m};\quad r_f = open}$$

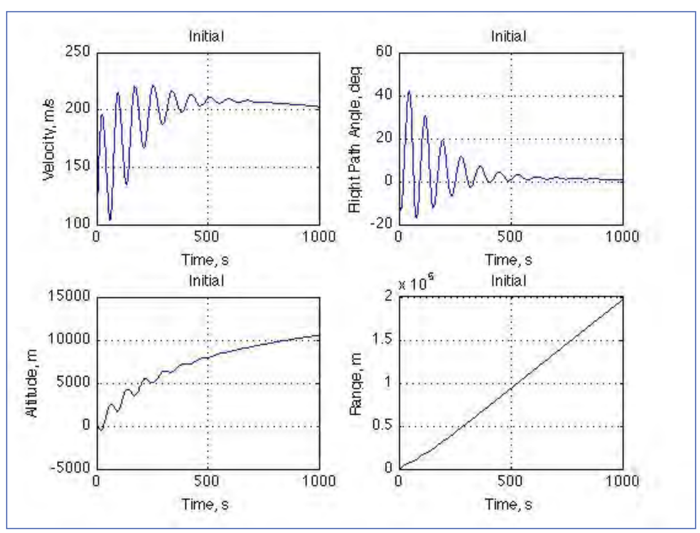
- **End time, t_f, is open, thrust takes maximum value, and control variable is angle of attack, $\alpha(t)$**
- **Fuel expended, but vehicle mass held constant**



1,000-sec Trajectory, Simple Angle of Attack Control

Initial Angle of Attack History

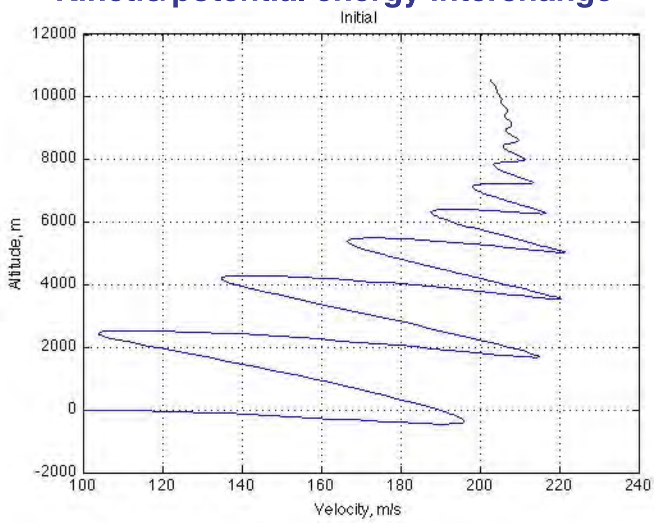




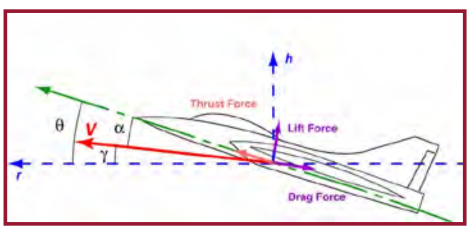
- **No attempt to optimize**
- **Interchange of kinetic and potential energy**
- **Lightly damped "phugoid" mode**

Altitude vs. Velocity, Simple Angle of Attack Control, t_f = 1,000 sec

- Lightly damped, long-period oscillation
- Kinetic/potential energy interchange



Alternative: Pitch Angle Control



$$\alpha = \theta - \gamma$$

$\alpha =$ Angle of Attack, rad
$\theta =$ Pitch Angle, rad
$\gamma =$ Flight Path Angle, rad

$$\dot{\gamma} = \frac{1}{V}\left[\left(C_{L_\alpha}(\theta - \gamma)\frac{1}{2}\rho V^2 S\right)\Big/ m - g\cos\gamma\right]$$

Controlling pitch angle introduces flight path angle damping

(see Supplemental Material for details)

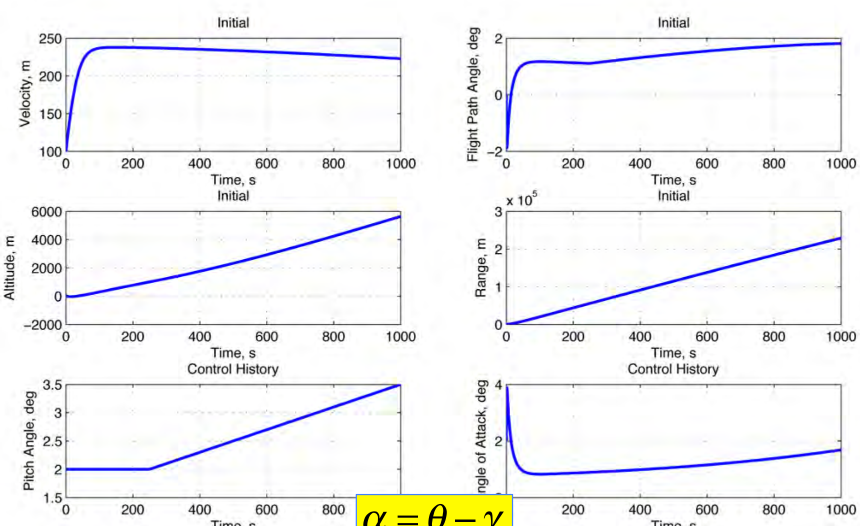
$$\dot{\gamma} = \frac{1}{V}\left[\left(C_{L_\alpha}\frac{1}{2}\rho V^2 S\theta - C_{L_\alpha}\frac{1}{2}\rho V^2 S\gamma\right)\Big/ m - g\cos\gamma\right]$$

Angle of Attack or Pitch Angle Control?

$\alpha = \theta - \gamma$

1,000-sec Trajectory, Simple Pitch Angle Control

- **Note difference in pitch-angle and angle-of-attack profiles**

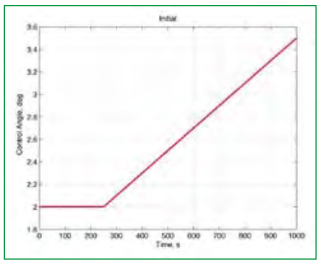


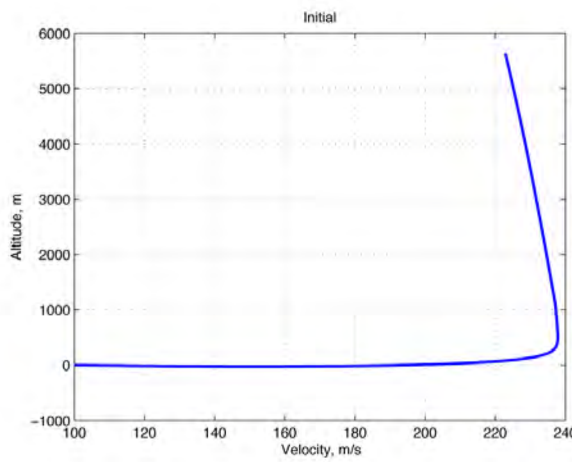
$\alpha = \theta - \gamma$

Altitude vs. Velocity Simple Pitch Angle Control, t_f = 1,000 sec

- Increased damping eliminates oscillation
- Kinetic energy increase followed by potential energy increase

Initial Pitch Angle History





Minimum Time-to-Climb Optimization Problem

Minimum-Time Cost Function

$$J(t_f) = \frac{1}{2}\left\{\left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]^T \mathbf{P}_f \left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]\right\}$$
$$+ \frac{1}{2}\int_{t_o}^{t_f} \left\{1 + \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]\right\} dt$$

- **Terminal cost provides trajectory objective**
- **Integrand**
  - **"1" is the integrand for minimizing time**
  - **Small quadratic term**
    - **provides non-singular trajectory control with *ad hoc* damping and regulation [good], but**
    - **penalizes non-zero values of state and control [not so good]**

Minimum-Time Cost Function with Augmented Trajectory Damping

$$J(t_f) = \frac{1}{2}\left\{\left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]^T \mathbf{P}_f \left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]\right\}$$
$$+ \frac{1}{2}\int_{t_o}^{t_f} \left\{1 + \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \underline{\dot{\mathbf{x}}^T(t)\mathbf{Q}_{\dot{\mathbf{x}}}\dot{\mathbf{x}}(t)} + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]\right\} dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t)\right]$$

- **<u>State rate weighting</u>**
  - **is unbiased by non-zero state or control**
  - **provides damping**

$$J(t_f) = \frac{1}{2}\left\{\left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]^T \mathbf{P}_f \left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]\right\}$$
$$+ \frac{1}{2}\int_{t_o}^{t_f} \left\{1 + \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{f}^T\left[\mathbf{x}(t), \mathbf{u}(t)\right]\mathbf{Q}_{\dot{\mathbf{x}}}\mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t)\right] + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]\right\} dt$$

Typical Weights Used in Minimum-Time Optimization

Terminal Penalty

$$\mathbf{P} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

State Penalty

$$\mathbf{Q} = \begin{bmatrix} 10^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

State-Rate Penalty

$$\mathbf{Q}_{\dot{\mathbf{x}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Control Penalty

$$\mathbf{R} = 1$$

$x_1 = V$: Velocity, m/s

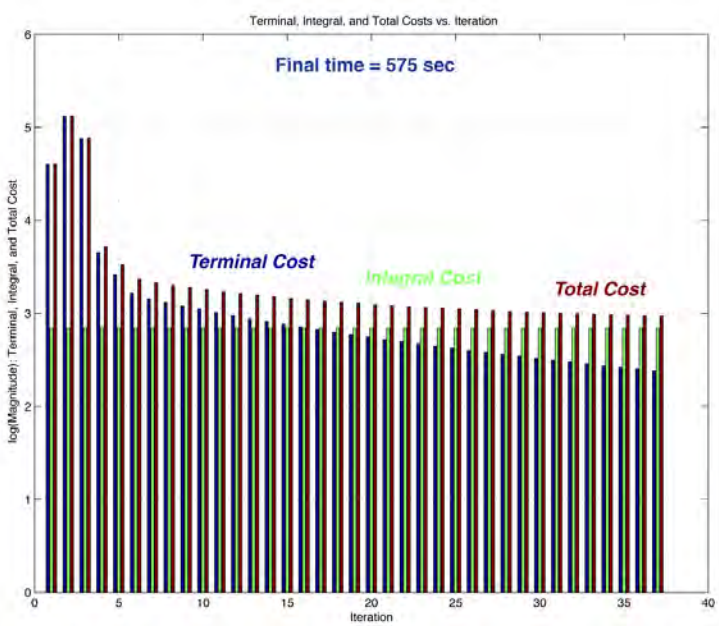
$x_2 = \gamma$: Flight path angle, rad

$x_3 = h$: Height, m

$x_4 = r$: Range, m

$u = \theta$: Pitch angle, rad

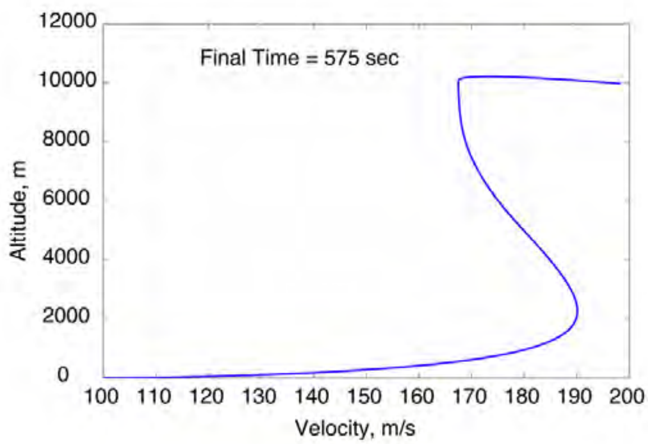
Minimum-Time Cost History, t_f = 575 sec, 36 Iterations



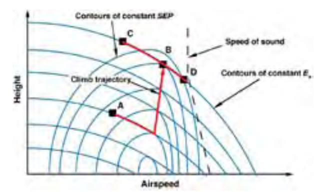
- Adaptive steepest-descent optimization algorithm
- Optimization algorithm is still minimizing at iteration cutoff

~Minimum-Time Altitude vs. Velocity, t_f = 575 sec, 36 Iterations

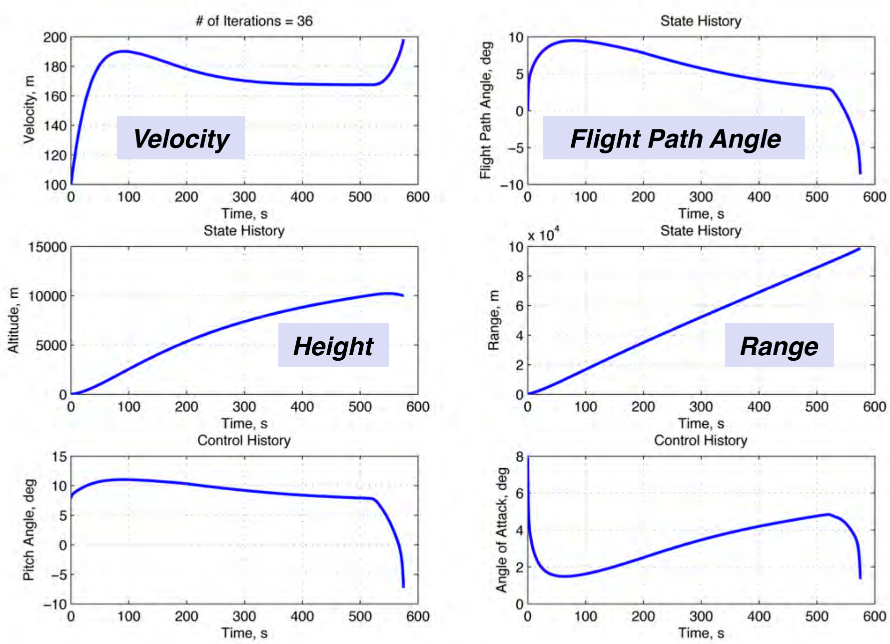
- Kinetic energy increase
- Trade for potential energy increase
- Velocity Increase and shallow dive to satisfy terminal condition



Recall Energy-Height Approximation



~Minimum-Time Trajectory, t_f = 575 sec, 36 Iterations



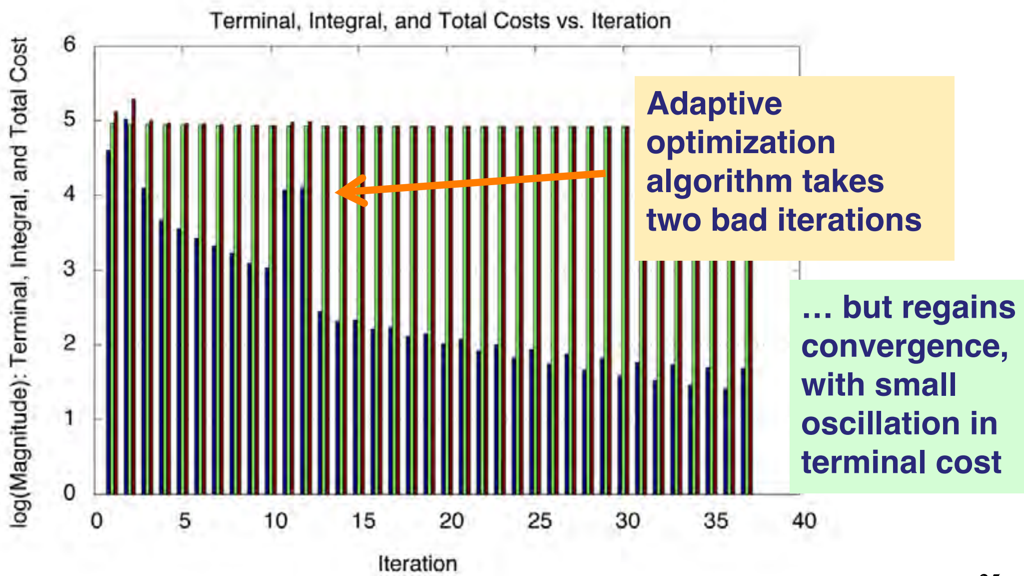
Minimum Fuel-to-Climb Optimization Problem

Minimum-Fuel Cost Function with Augmented Trajectory Damping

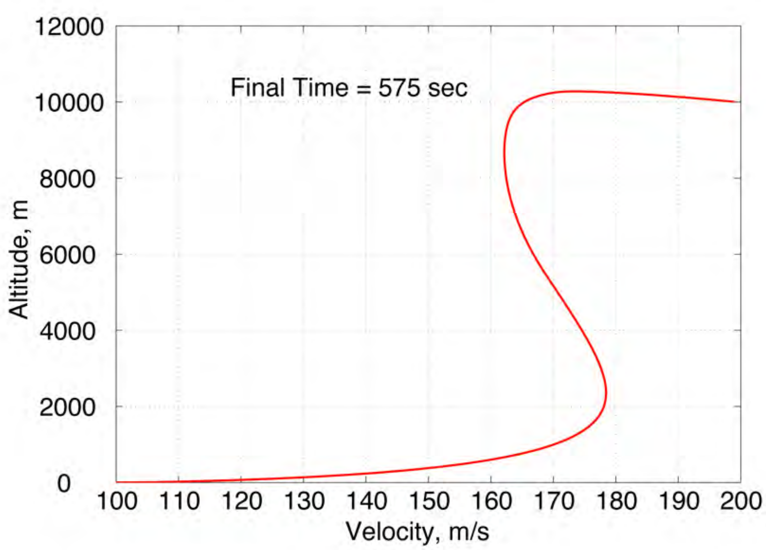
$$J(t_f) = \frac{1}{2}\left\{\left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]^T \mathbf{P}_f \left[\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)\right]\right\}$$
$$+\frac{1}{2}\int_{t_o}^{t_f}\left\{\dot{m} + \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \dot{\mathbf{x}}^T(t)\mathbf{Q}_{\dot{\mathbf{x}}}\dot{\mathbf{x}}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]\right\} dt$$

- **Same cost-function weights**
- **Integrand**
  - **Fuel-flow rate in the integrand for minimizing total fuel used**

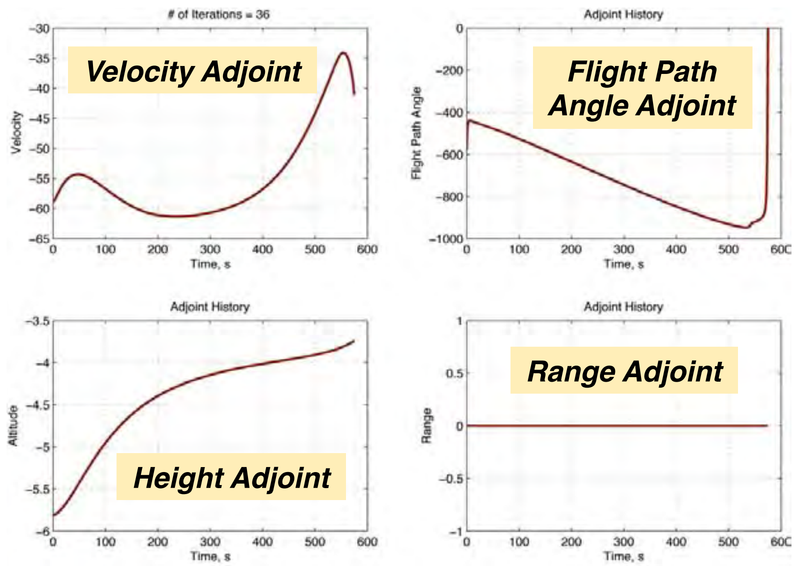
Minimum-Fuel Cost History, t_f = 575 sec, 36 Iterations



Minimum-Fuel Altitude vs. Velocity, t_f = 575 sec, 36 Iterations



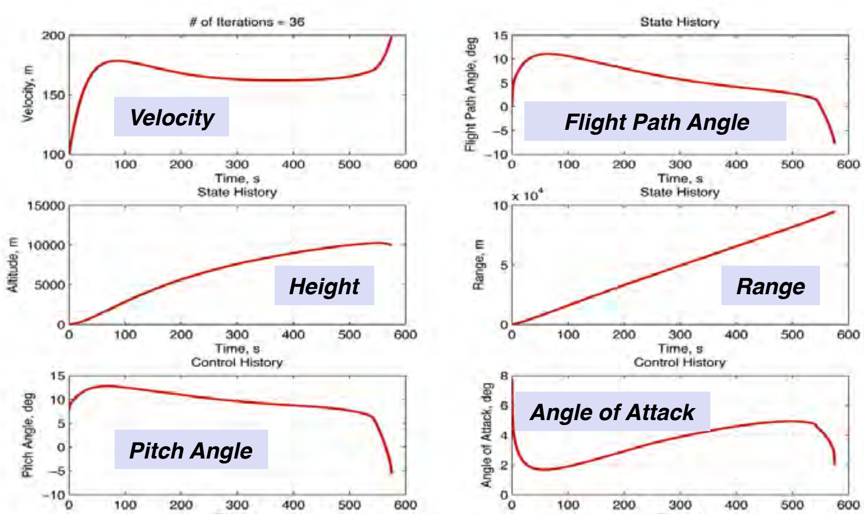
Minimum-Fuel Adjoint Vector History, $\lambda(t)$, t_f = 575 sec, 36 Iterations



Cost sensitivity to state perturbations over time

$$\lambda(t_f) = \left[\frac{\partial\phi}{\partial \mathbf{x}}(t_f)\right]^T = \Delta\mathbf{x}^T\mathbf{P}_f$$

~Minimum-Fuel Trajectory, t_f = 575 sec, 36 Iterations



Virtually identical to minimum-time solution

Fuel used
. Minimum time: 849 kg
. Minimum fuel: 831 kg

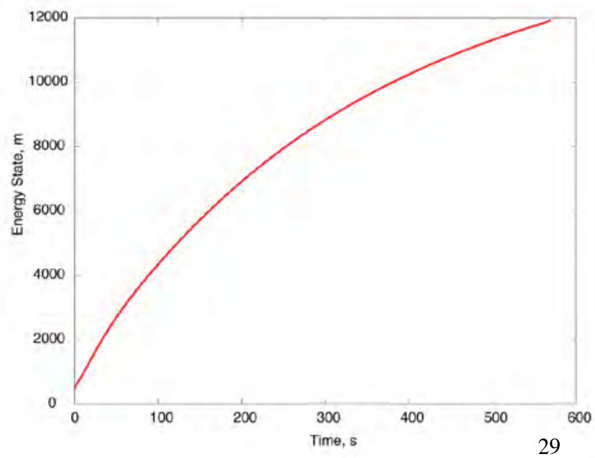
Energy State Profile,
t_f = 570 sec

$$\textbf{Energy State: } E = \frac{1}{mg}\left(\frac{mV^2}{2} + mgh\right) = \frac{V^2}{2g} + h, \quad \textit{meters}$$

- Optimized Energy State is monotonic and always increasing
- Rate of change decreases with altitude

Specific Excess Power:

$$SEP = \frac{V}{mg}(Thrust - Drag)$$



Next Time:
Neighboring-Optimal Control via Linear-Quadratic Feedback

Reading
OCE: Section 3.7

Supplemental Material

Definitions and Numerical Values for Variables and Constants

$V =$ Velocity, m/s

$\gamma =$ Flight path angle, rad

$h =$ Altitude (or height), m

$r =$ Range, m

$m =$ Mass, kg

$\alpha =$ **Angle of attack**, rad; $\alpha_{\max} = 10°$

$\rho =$ Air density $= \rho_{SL} e^{-\beta h} = 1.225 e^{-h/9{,}042} \text{kg/m}^3$

$g =$ Gravitational acceleration $= 9.801 \text{ m/s}^2$

$T = T_{SL}\left(\dfrac{\rho}{\rho_{SL}}\right)\delta T = T_{SL}\left(e^{-\beta h}\right)\delta T =$ Thrust, N;

$\delta T =$ Throttle setting, %, = 100% for minimum-time/fuel problem

$SFC =$ Specific Fuel Consumption = 10 g/kN-s

$$C_D = \frac{\left(C_{D_o} + \varepsilon C_L^2\right)}{\sqrt{1-\left(V/V_{sound}\right)^2}} = \text{Drag coefficient}$$

$$= \left(0.025 + 0.072 C_L^2\right) \Big/ \sqrt{1-\left(V/V_{sound}\right)^2}$$

$C_L = C_{L_\alpha}\alpha =$ Lift coefficient $= 5.7\alpha$

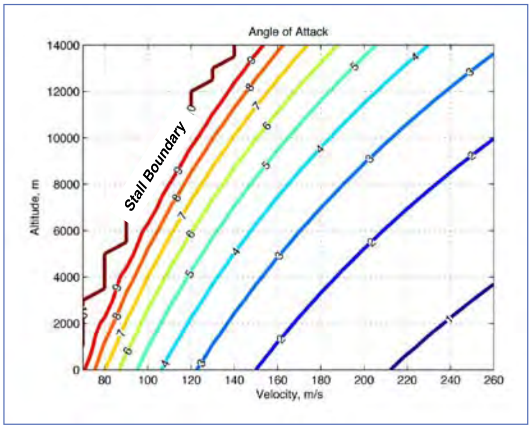
$S =$ Reference area $= 21.5 \text{ m}^2$

$m_E =$ Vehicle mass = 4,550 kg ~ constant

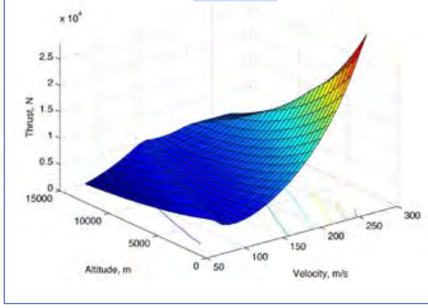
- **Angle of attack has linear effect on lift and quadratic effect on drag**

Angle of Attack, Thrust, and Throttle for Steady Flight vs. Altitude and Velocity

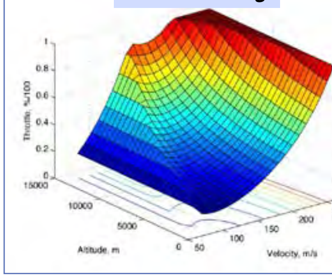
Angle of Attack



Thrust



Throttle Setting



33

Modal Properties of the System

Linearized Equations for Velocity and Flight Path Angle Perturbations, Using Angle of Attack as the Control

$$\begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\gamma} \end{bmatrix} = \begin{bmatrix} (T_V - D_V) & -g \\ L_V/V_o & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix} + \begin{bmatrix} \sim 0 \\ L_\alpha/V_o \end{bmatrix} \Delta\alpha$$

$$State: \quad \Delta\mathbf{x} = \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix}$$

$$Control: \quad \Delta\mathbf{u} = \Delta\alpha$$

- **where**

$$T_V = \frac{\partial(Thrust/m)}{\partial V} = \text{Sensitivity of acceleration-due-to-thrust to velocity variation}$$

$$= 0 \text{ for this problem because } [\![\textbf{thrust = maximum thrust}]\!]$$

$$D_V = \frac{\partial(Drag/m)}{\partial V} = \text{Sensitivity of acceleration-due-to-drag to velocity variation } > 0$$

$$L_V = \frac{\partial(Lift/m)}{\partial V} = \text{Sensitivity of acceleration-due-to-lift to velocity variation } > 0$$

$$L_\alpha = \frac{\partial(Lift/m)}{\partial \alpha} = \text{Sensitivity of acceleration-due-to-lift to angle-of-attack variation } > 0$$

Characteristic Equation and Stability, Using Angle of Attack as the Control Variable

Characteristic Equation

$$|s\mathbf{I} - \mathbf{F}| = \left| s\mathbf{I} - \begin{bmatrix} -D_V & -g \\ L_V/V_o & 0 \end{bmatrix} \right| = \begin{vmatrix} (s + D_V) & g \\ -L_V/V_o & s \end{vmatrix}$$

$$= s(s + D_V) + g\,L_V/V_o$$

$$= s^2 + D_V s + g\,L_V/V_o$$

$$= s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Natural Frequency and Damping Ratio

$$s^2 + 2\zeta\omega_n s + \omega_n{}^2 = 0$$

$$\omega_n = \sqrt{g\,L_V/V_o} \simeq \sqrt{2}\,\frac{g}{V_0} \simeq \frac{13.9}{V_o(m/s)}; \quad Period \simeq 0.453\,V_o, \sec$$

$$\zeta = \frac{D_V}{2\sqrt{g\,L_V/V_o}} \simeq \frac{\sqrt{2}}{2}\left(\frac{C_D}{C_L}\right)$$

$$\text{"Total Damping"} = 2\zeta\omega_n = D_V$$

–(Trace of the matrix)

Linearized Equations for Velocity and Flight Path Angle Perturbations, Using Pitch Angle as the Control

$$\begin{bmatrix} \Delta \dot{V} \\ \Delta \dot{\gamma} \end{bmatrix} = \begin{bmatrix} -D_V & -g \\ L_V / V_o & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix} + \begin{bmatrix} \sim 0 \\ L_\alpha / V_o \end{bmatrix} \Delta \alpha$$

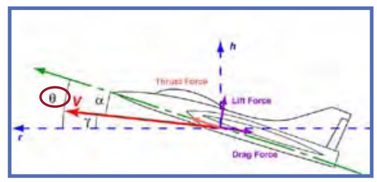
$$\textit{State}: \quad \Delta \mathbf{x} = \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix}$$

$$\textit{Control}: \quad \Delta \mathbf{u} = \Delta \theta = \Delta \alpha + \Delta \gamma$$

- **Replace angle of attack by pitch angle for control**

$$\begin{bmatrix} \Delta \dot{V} \\ \Delta \dot{\gamma} \end{bmatrix} = \begin{bmatrix} -D_V & -g \\ L_V / V_o & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix} + \begin{bmatrix} \sim 0 \\ L_\alpha / V_o \end{bmatrix} (\Delta \theta - \Delta \gamma)$$

$$= \begin{bmatrix} -D_V & -g \\ L_V / V_o & -L_\alpha / V_o \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix} + \begin{bmatrix} \sim 0 \\ L_\alpha / V_o \end{bmatrix} \Delta \theta$$



Characteristic Equation and Stability, Using Pitch Angle as the Control Variable

$$|s\mathbf{I} - \mathbf{F}| = \left| s\mathbf{I} - \begin{bmatrix} -D_V & -g \\ L_V / V_o & -L_\alpha / V_o \end{bmatrix} \right| = \begin{vmatrix} (s + D_V) & g \\ -L_V / V_o & \left(s + L_\alpha / V_o\right) \end{vmatrix}$$

$$= \left(s + L_\alpha / V_o\right)(s + D_V) + g L_V / V_o$$

$$= s^2 + \left(L_\alpha / V_o + D_V\right) s + \left(g L_V / V_o + D_V L_\alpha / V_o\right)$$

$$= s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\omega_n = \sqrt{\left(g L_V / V_o + D_V L_\alpha / V_o\right)}$$

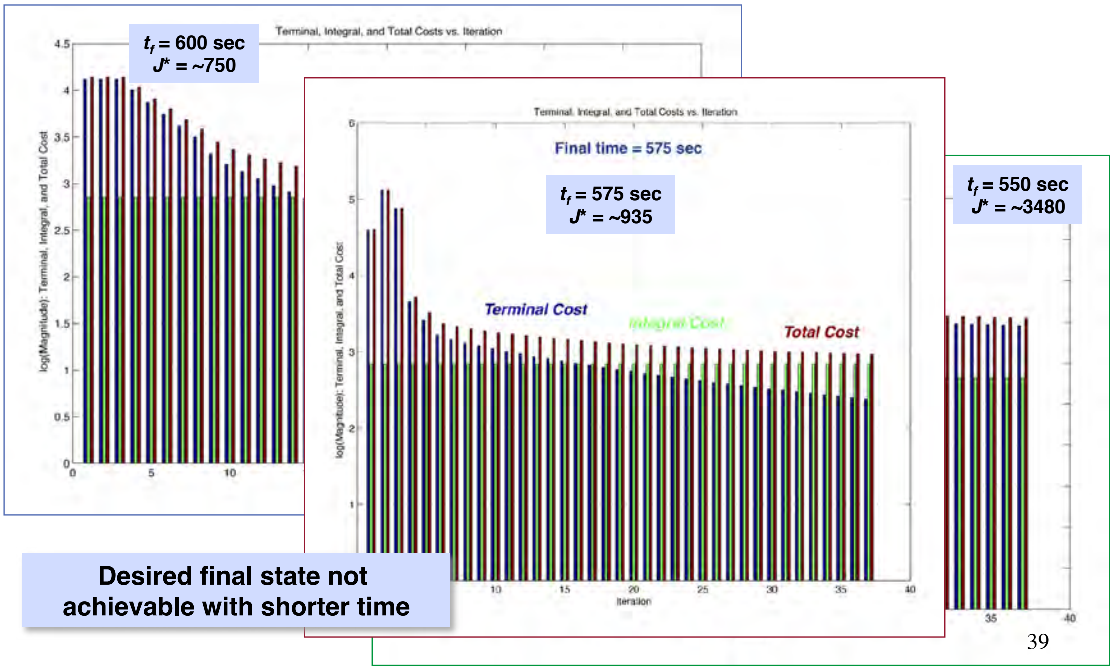
$$\zeta = \frac{\left(L_\alpha / V_o + D_V\right)}{2\sqrt{\left(g L_V / V_o + D_V L_\alpha / V_o\right)}}$$

$$\text{"Total Damping"} = 2\zeta\omega_n = \left(L_\alpha / V_o + D_V\right)$$

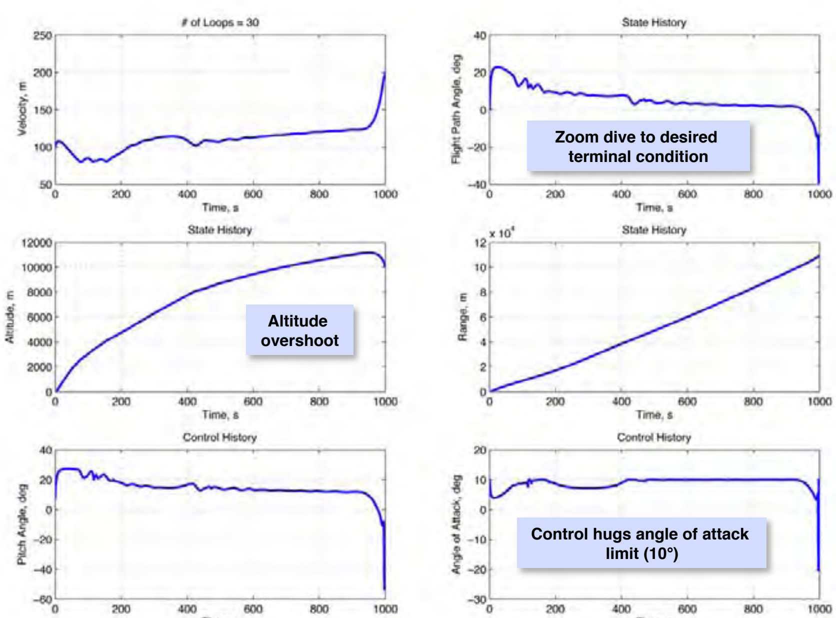
- **Natural frequency is increased**
- **Damping is increased**

– (Trace of the matrix)

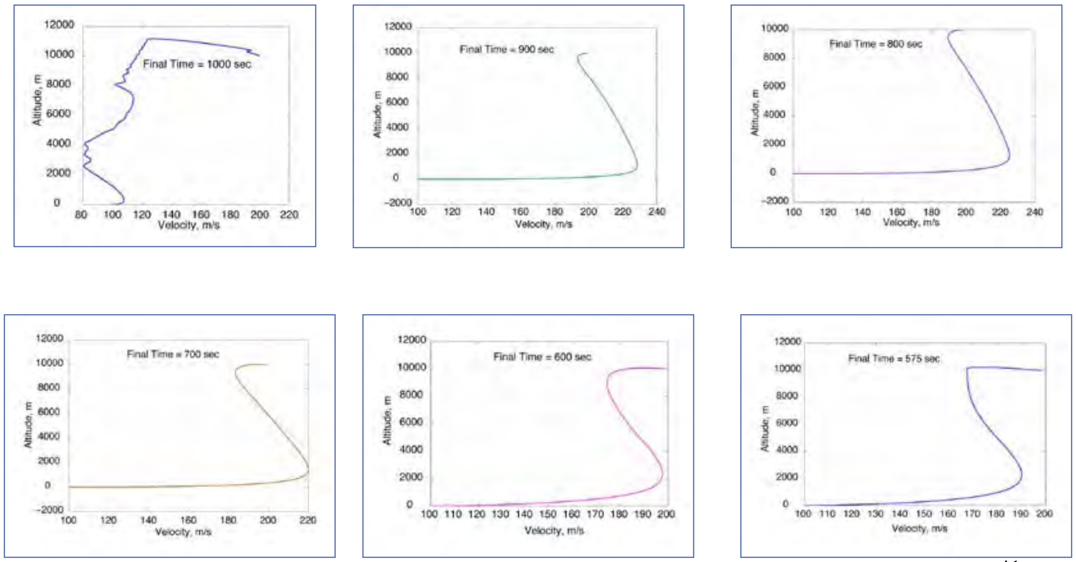
Why is t_f = 575 sec About the Minimum-Time Solution?



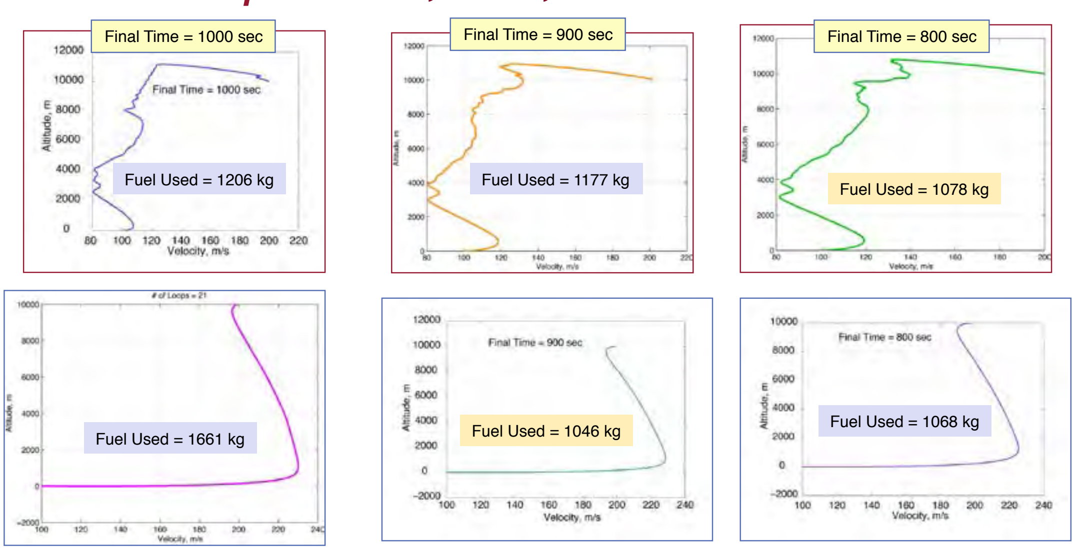
Local-Optimal State History for t_f = 1000 sec



Optimal Altitude vs. Velocity Progression for Various Final Times, t_f = 1000 to 575 sec



<u>Local Optimal</u>* Fuel-Use with Various Fixed End Times
t_f = 1000, 900, and 800 sec



* Not global optima; compare to earlier result for t_f = 575 s