Linear-Optimal Estimation for Continuous-Time Systems

Robert Stengel
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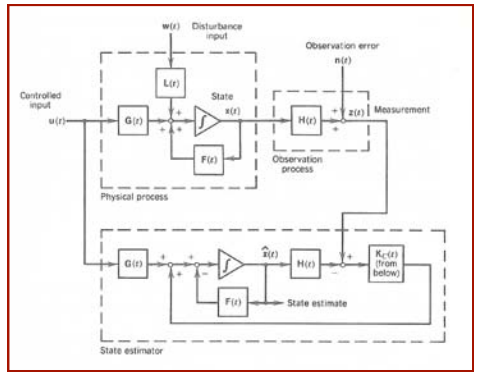
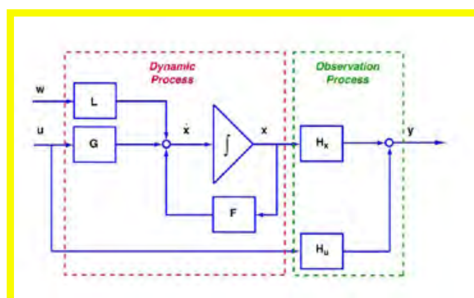
- Propagation of uncertainty in continuous-time systems
- Linear-optimal Gaussian estimator (Kalman-Bucy filter)
- Linear-optimal prediction
- Asymptotic stability of the state estimate
- Duality between estimation and control
- Filter divergence
- Square root filtering
- Correlated disturbance and measurement noise



http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Uncertain Linear, Time-Varying (LTV) Dynamic Model



- Continuous-time LTV model with known coefficients

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t), \quad \mathbf{x}(t_o)\ given$$

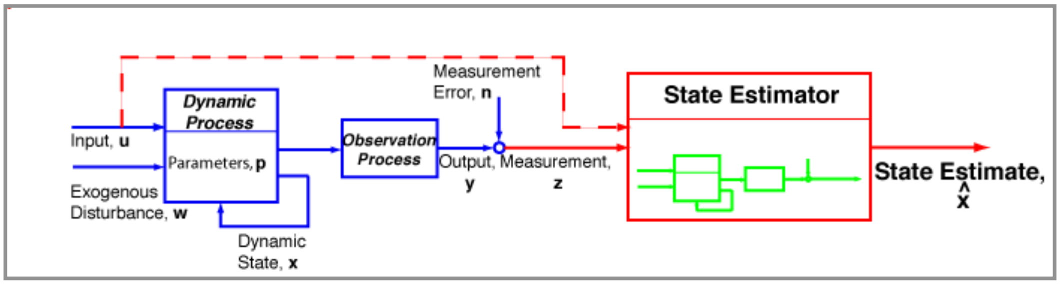
$$\mathbf{x}(t) = \mathbf{x}(t_o) + \int_{t_o}^{t} \left[\mathbf{F}(\tau)\mathbf{x}(\tau) + \mathbf{G}(\tau)\mathbf{u}(\tau) + \mathbf{L}(\tau)\mathbf{w}(\tau)\right] d\tau$$

$$\mathbf{y}(t) = \mathbf{H_x}\mathbf{x}(t) + \mathbf{H_u}\mathbf{u}(t)$$

$$\dim\left[\mathbf{w}(t)\right] = s \times 1$$
$$\dim\left[\mathbf{z}(t)\right] = r \times 1$$

- Initial condition and disturbance inputs are not known precisely
- Measurement of state is transformed and is subject to error

Continuous-Time State Estimation



Same general structure as discrete-time estimator

Propagation of Mean Value Estimate in Continuous-Time Systems

Assumed mean value statistics

$$E\left[\mathbf{x}(0)\right]=\mathbf{m}(0); \quad E\left[\mathbf{x}(t)\right]=\mathbf{m}(t); \quad E\left[\dot{\mathbf{x}}(t)\right]=\dot{\mathbf{m}}(t)$$
$$E\left[\mathbf{w}(t)\right]=\mathbf{0}; \quad E\left[\mathbf{u}(t)\right]=\mathbf{u}(t)$$

By substitution in the system equation, the expected mean value is

$$\begin{aligned} E\left[\dot{\mathbf{x}}(t)\right] &= E\left[\mathbf{F}(t)\mathbf{x}(t)+\mathbf{G}(t)\mathbf{u}(t)+\mathbf{L}(t)\mathbf{w}(t)\right] \\ &= \mathbf{F}(t)E\left[\mathbf{x}(t)\right]+\mathbf{G}(t)\mathbf{u}(t) \end{aligned}$$

$$\boxed{\dot{\mathbf{m}}(t)=\mathbf{F}(t)\mathbf{m}(t)+\mathbf{G}(t)\mathbf{u}(t)}$$

Alternative Derivation of Mean Value Estimate Propagation

The sampled-data case

$$\mathbf{m}(t_k) = \mathbf{\Phi}\left(t_k - t_{k-1}\right)\mathbf{m}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \mathbf{\Phi}\left(t_k - \tau\right)\left[\mathbf{G}(\tau)\mathbf{u}(\tau)\right]dt$$

$$\mathbf{\Phi}\left(\Delta t\right) = \mathbf{I}_n + \mathbf{F}\Delta t + \frac{1}{2!}\mathbf{F}^2\Delta t^2 + \frac{1}{3!}\mathbf{F}^3\Delta t^3 + \cdots$$

For small Δt

$$\mathbf{m}(t_k) = \left(\mathbf{I}_n + \mathbf{F}_{k-1}\Delta t\right)\mathbf{m}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \left[\mathbf{I}_n + \mathbf{F}_{k-1}\left(t_k - \tau\right)\right]\left[\mathbf{G}(\tau)\mathbf{u}(\tau)\right]d\tau$$

Mean Value Estimate Propagation

- **Rearranging terms**

$$\frac{\mathbf{m}(t_k) - \mathbf{m}(t_{k-1})}{\Delta t} = \mathbf{F}_{k-1}\mathbf{m}(t_{k-1}) + \left[\mathbf{I}_n + \mathbf{F}_{k-1}\frac{\Delta t}{2}\right]\mathbf{G}_{k-1}\mathbf{u}_{k-1}$$

- **In the limit, the difference equation converges to a differential equation**
- **Propagation of the expected state**

$$\text{As } \Delta t \to 0, \quad t_{k-1} \to t_k \to t, \text{ and}$$

$$\lim_{\Delta t \to 0} \frac{\mathbf{m}(t_k) - \mathbf{m}(t_{k-1})}{\Delta t} = \frac{d\mathbf{m}(t)}{dt} = \dot{\mathbf{m}}(t) = \mathbf{F}(t)\mathbf{m}(t) + \mathbf{G}(t)\mathbf{u}(t)$$

Covariance Estimate Propagation

Assumed covariance statistics

$$E\left\{\left[\mathbf{x}(t)-\mathbf{m}(t)\right]\left[\mathbf{x}(\tau)-\mathbf{m}(\tau)\right]^T\right\}=\mathbf{P}(t)$$

$$E\left[\mathbf{w}(t)\mathbf{w}^T(\tau)\right]=\mathbf{Q}'_C\,\delta(t-\tau);\quad E\left\{\left[\mathbf{u}(t)-\bar{\mathbf{u}}(t)\right]\left[\mathbf{u}(\tau)-\bar{\mathbf{u}}(\tau)\right]^T\right\}=\mathbf{0}$$

For small Δt

$$\mathbf{P}_k=\left(\mathbf{I}_n+\mathbf{F}_{k-1}\Delta t\right)\mathbf{P}_{k-1}\left(\mathbf{I}_n+\mathbf{F}_{k-1}\Delta t\right)^T+\mathbf{Q}_{k-1}$$

$$=\mathbf{P}_{k-1}+\mathbf{F}_{k-1}\mathbf{P}_{k-1}\Delta t+\left(\mathbf{F}_{k-1}\mathbf{P}_{k-1}\right)^T\Delta t+\mathbf{F}_{k-1}\mathbf{P}_{k-1}\mathbf{F}_{k-1}^T\Delta t^2+\mathbf{Q}_{k-1}$$

Covariance Estimate Propagation

Rearranging

$$\frac{\mathbf{P}_k-\mathbf{P}_{k-1}}{\Delta t}=\mathbf{F}_{k-1}\mathbf{P}_{k-1}+\left(\mathbf{F}_{k-1}\mathbf{P}_{k-1}\right)^T+\mathbf{F}_{k-1}\mathbf{P}_{k-1}\mathbf{F}_{k-1}^T\Delta t+\frac{\mathbf{Q}_{k-1}}{\Delta t}$$

Covariance rate of change

As $\Delta t \to 0,\quad t_{k-1}\to t_k \to t$

$$\lim_{\Delta t\to 0}\frac{\mathbf{P}_k-\mathbf{P}_{k-1}}{\Delta t}=\frac{d\mathbf{P}(t)}{dt}=\dot{\mathbf{P}}(t)$$

Disturbance uncertainty

$$\mathbf{Q}_{k-1}\approx\mathbf{L}(t)\mathbf{Q}'_C\mathbf{L}^T(t)\Delta t$$

$$\therefore\ \frac{\mathbf{Q}_{k-1}}{\Delta t}\xrightarrow[\Delta t\to 0]{}\mathbf{L}(t)\mathbf{Q}'_C(t)\mathbf{L}^T(t)$$

$$\dot{\mathbf{P}}(t)=\mathbf{F}(t)\mathbf{P}(t)+\mathbf{P}(t)\mathbf{F}(t)^T+\mathbf{L}(t)\mathbf{Q}'_C(t)\mathbf{L}^T(t)$$

Kalman-Bucy Filter

Kalman-Bucy* Filter

- Optimal estimator for linear systems with Gaussian uncertainty
- Three equations
  1) State estimate extrapolation and update
  2) Covariance estimate extrapolation and "update"
  3) Filter gain computation

* Rudolf Kalman, RIAS, and Richard C. Bucy, Johns Hopkins Applied Physics Laboratory, 1961. Related developments were made by Weiner (1949), Bode and Shannon (1950), Robbins and Munro (1951), Kiefer and Wolfowitz (1952), Blum(1958), Zadeh and Ragazzini (1952), Carlton and Follin (1956), Berkson (1956), Swerling (1959), Kolmogorov(1962), and Yaglom (1962)

Covariance Estimate (Part 1)

Substitute

$$\mathbf{P}_k(-) = \mathbf{\Phi}_{k-1}\,\mathbf{P}_{k-1}(+)\mathbf{\Phi}_{k-1}^T + \mathbf{Q}_{k-1}$$

In

$$\mathbf{P}_k(+) = \left[\mathbf{P}_k^{-1}(-) + \mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k\right]^{-1}$$

To obtain the rate of change of the prior covariance estimate

$$\frac{\mathbf{P}_k(-) - \mathbf{P}_{k-1}(-)}{\Delta t} = \mathbf{F}_{k-1}\mathbf{P}_{k-1}(-) + \mathbf{P}_{k-1}(-)\mathbf{F}_{k-1}^T + \frac{\mathbf{Q}_{k-1}}{\Delta t} - \frac{\mathbf{K}_{k-1}}{\Delta t}\mathbf{H}_{k-1}\mathbf{P}_{k-1}(-)$$
$$- \mathbf{F}_{k-1}\mathbf{K}_{k-1}\mathbf{H}_{k-1}\mathbf{P}_{k-1}(-) - \mathbf{K}_{k-1}\mathbf{H}_{k-1}\mathbf{P}_{k-1}(-)\mathbf{F}_{k-1}^T$$

Disturbance and Gain Matrices

Disturbance spectral density matrix

$$\mathbf{Q}_{k-1} \approx \mathbf{L}(t)\mathbf{Q}'_C\,\mathbf{L}^T(t)\Delta t$$
$$\therefore \frac{\mathbf{Q}_{k-1}}{\Delta t} \xrightarrow[\Delta t \to 0]{} \mathbf{L}(t)\mathbf{Q}'_C(t)\mathbf{L}^T(t)$$

Gain matrix

$$\mathbf{K}_{k-1} = \mathbf{P}_{k-1}(+)\mathbf{H}_{k-1}^T\mathbf{R}_{k-1}^{-1}$$

Measurement error covariance matrix

$$\mathbf{R}_{k-1}\Delta t \approx \int_{-\Delta t/2}^{\Delta t/2} \mathbf{R}_C\delta(t_{k-1} - \tau)\,d\tau$$

$$\mathbf{R}_{k-1} \approx \frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} \mathbf{R}_C\delta(t_{k-1} - \tau)\,d\tau = \frac{\mathbf{R}_C(t_{k-1})}{\Delta t}$$

$$\mathbf{R}_{k-1}^{-1} = \mathbf{R}_C^{-1}(t_{k-1})\Delta t$$

Covariance Estimate (Part 2)

$$\boxed{\mathbf{K}_{k-1} = \mathbf{P}_{k-1}(+)\mathbf{H}_{k-1}^T\mathbf{R}_C^{-1}\Delta t}$$

$$\begin{aligned}\frac{\mathbf{P}_k(-)-\mathbf{P}_{k-1}(-)}{\Delta t} &= \mathbf{F}_{k-1}\mathbf{P}_{k-1}(-)+\mathbf{P}_{k-1}(-)\mathbf{F}_{k-1}^T \\ &+\frac{\mathbf{Q}_{k-1}}{\Delta t}-\mathbf{P}_{k-1}(+)\mathbf{H}_{k-1}^T\mathbf{R}_C^{-1}\mathbf{H}_{k-1}\mathbf{P}_{k-1}(-) \\ &-\begin{bmatrix}\mathbf{F}_{k-1}\mathbf{P}_{k-1}(+)\mathbf{H}_{k-1}^T\mathbf{R}_C^{-1}(t_{k-1})\mathbf{H}_{k-1}\mathbf{P}_{k-1}(-) \\ +\mathbf{P}_{k-1}(+)\mathbf{H}_{k-1}^T\mathbf{R}_C^{-1}(t_{k-1})\mathbf{H}_{k-1}\mathbf{P}_{k-1}(-)\mathbf{F}_{k-1}^T\end{bmatrix}\Delta t\end{aligned}$$

(+) and (–) values coalesce as $\Delta t \to 0$

propagation and update become <u>concurrent</u>

$$\boxed{\mathbf{P}_{k-1}(+)\xrightarrow[\Delta t\to 0]{}\mathbf{P}_{k-1}(-)=\mathbf{P}(t_{k-1})}$$

Covariance Estimate (Part 3)

$$\begin{aligned}&\lim_{\Delta t\to 0}\frac{\mathbf{P}_k(-)-\mathbf{P}_{k-1}(-)}{\Delta t}=\dot{\mathbf{P}}(t) \\ &=\mathbf{F}(t)\mathbf{P}(t)+\mathbf{P}(t)\mathbf{F}^T(t)+\mathbf{L}(t)\mathbf{Q}'_C(t)\mathbf{L}^T(t)-\mathbf{P}(t)\mathbf{H}^T(t)\mathbf{R}_C^{-1}(t)\mathbf{H}(t)\mathbf{P}(t)\end{aligned}$$

$$\boxed{\dot{\mathbf{P}}(t)=\mathbf{F}(t)\mathbf{P}(t)+\mathbf{P}(t)\mathbf{F}^T(t)+\mathbf{L}(t)\mathbf{Q}'_C(t)\mathbf{L}^T(t)-\mathbf{K}_C(t)\mathbf{H}(t)\mathbf{P}(t)}$$

The continuous-time filter gain matrix is

$$\boxed{\mathbf{K}_C(t)=\mathbf{P}(t)\mathbf{H}^T(t)\mathbf{R}_C^{-1}(t)}$$

State Estimate-1

Combine propagation and update equations

$$\begin{aligned}\hat{\mathbf{x}}_k(-) &= \mathbf{\Phi}_{k-1}\,\hat{\mathbf{x}}_{k-1}(+) + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1}\\ &\approx \left[\mathbf{I}_n + \mathbf{F}(t_{k-1})\Delta t\right]\hat{\mathbf{x}}_{k-1}(+) + \mathbf{G}(t_{k-1})\Delta t\,\mathbf{u}(t_{k-1})\end{aligned}$$

$$\hat{\mathbf{x}}_k(+) = \hat{\mathbf{x}}_k(-) + \mathbf{K}_k\left[\mathbf{z}_k - \mathbf{H}_k\hat{\mathbf{x}}_k(-)\right]$$

$$\begin{gathered}\hat{\mathbf{x}}_k(+) = \left[\mathbf{I}_n + \mathbf{F}(t_{k-1})\Delta t\right]\hat{\mathbf{x}}_{k-1}(+)\\ +\mathbf{G}(t_{k-1})\Delta t\,\mathbf{u}(t_{k-1}) + \mathbf{K}_k\left\{\mathbf{z}_k - \mathbf{H}_k\left[\mathbf{I}_n + \mathbf{F}(t_{k-1})\Delta t\right]\hat{\mathbf{x}}_{k-1}(+) + \mathbf{G}(t_{k-1})\Delta t\,\mathbf{u}(t_{k-1})\right\}\end{gathered}$$

State Estimate-2

$$\begin{gathered}\hat{\mathbf{x}}_k(+) = \left[\mathbf{I}_n + \mathbf{F}(t_{k-1})\Delta t\right]\hat{\mathbf{x}}_{k-1}(+)\\ +\mathbf{G}(t_{k-1})\Delta t\,\mathbf{u}(t_{k-1}) + \mathbf{K}_k\left\{\mathbf{z}_k - \mathbf{H}_k\left[\mathbf{I}_n + \mathbf{F}(t_{k-1})\Delta t\right]\hat{\mathbf{x}}_{k-1}(+) + \mathbf{G}(t_{k-1})\Delta t\,\mathbf{u}(t_{k-1})\right\}\end{gathered}$$

In the limit, the difference equation converges to a differential equation

$$\text{As } \Delta t \to 0, \quad t_{k-1} \to t_k \to t \quad \text{and} \quad \lim_{\Delta t \to 0}\frac{\hat{\mathbf{x}}(t_k) - \hat{\mathbf{x}}(t_{k-1})}{\Delta t} = \frac{d\hat{\mathbf{x}}(t)}{dt}$$

State estimator

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{K}_C(t)\underline{\left[\mathbf{z}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)\right]}$$

Residual acts as a driving term in the state estimate equation

Summary of the Kalman-Bucy Filter

State Estimator

$$\boxed{\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{K}_C(t)\left[\mathbf{z}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)\right]}$$

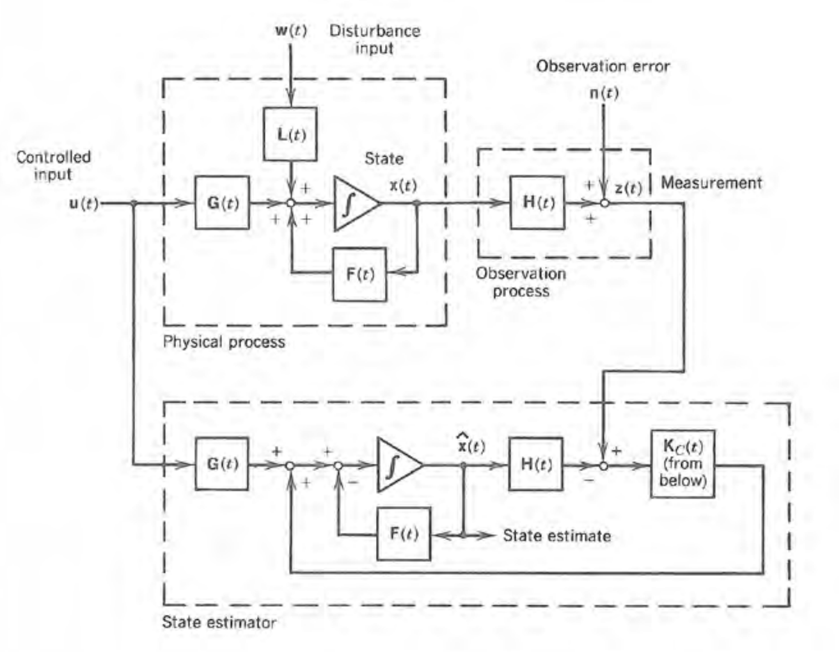
Filter Gain Matrix

$$\boxed{\mathbf{K}_C(t) = \mathbf{P}(t)\mathbf{H}^T(t)\mathbf{R}_C^{-1}(t)}$$

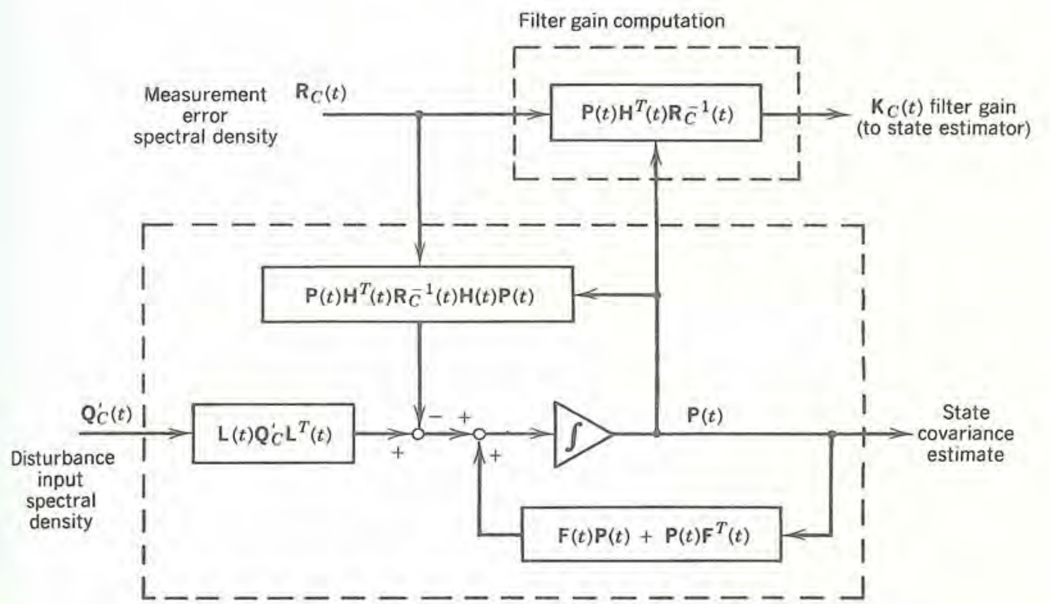
Covariance Estimator

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{L}(t)\mathbf{Q}'_C(t)\mathbf{L}^T(t) - \mathbf{K}_C(t)\mathbf{H}(t)\mathbf{P}(t),$$
$$\mathbf{P}(0) \text{ given}$$

Block Diagram of the Kalman-Bucy Filter



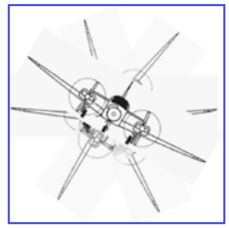
Block Diagram of the Covariance Estimate and Gain Computation for the Kalman-Bucy Filter



19

Second-Order Example

Second-Order Example of Kalman Filter



Rolling motion of an airplane, continuous-time

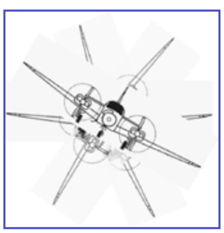
$$\begin{bmatrix} \Delta\dot{p}(t) \\ \Delta\dot{\phi}(t) \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta p(t) \\ \Delta\phi(t) \end{bmatrix} + \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix} \Delta\delta A(t) + \begin{bmatrix} L_p \\ 0 \end{bmatrix} \Delta p_w(t)$$

State estimator

$$\begin{bmatrix} \Delta\dot{\hat{p}}(t) \\ \Delta\dot{\hat{\phi}}(t) \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\hat{p}(t) \\ \Delta\hat{\phi}(t) \end{bmatrix} + \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix} \Delta\delta A(t) + \begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{bmatrix} \left\{ \begin{bmatrix} \Delta p_M(t) \\ \Delta\phi_M(t) \end{bmatrix} - \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \Delta\hat{p}(t) \\ \Delta\hat{\phi}(t) \end{bmatrix} \right\}$$

Filter gain matrix

$$\begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{bmatrix}^T \begin{bmatrix} r_{11}(t) & 0 \\ 0 & r_{22}(t) \end{bmatrix}^{-1}$$



Second-Order Example of Kalman Filter

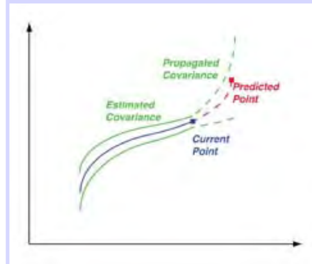
$$\begin{bmatrix} \dot{p}_{11}(t) & \dot{p}_{12}(t) \\ \dot{p}_{21}(t) & \dot{p}_{22}(t) \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix}^T + \begin{bmatrix} L_p \sigma_{p_w}^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{bmatrix} \begin{bmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}$$

Linear-Optimal Predictor

t_k : Current time, sec
t_K : Future time, sec



State estimate extrapolation (or propagation)

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\mathbf{u}(t), \quad \hat{\mathbf{x}}(t_k) \text{ from Kalman-Bucy filter}$$

$$\hat{\mathbf{x}}(t_K) = \hat{\mathbf{x}}(t_k) + \int_{t_k}^{t_K} \left[\mathbf{F}(\tau)\hat{\mathbf{x}}(\tau) + \mathbf{G}(\tau)\Delta\mathbf{u}(\tau)\right] d\tau$$

Covariance estimate extrapolation (or propagation)

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{L}(t)\mathbf{Q}'_C\mathbf{L}^T(t), \quad \mathbf{P}(t_k) \text{ from Kalman-Bucy filter}$$

$$\mathbf{P}(t_K) = \mathbf{P}(t_k) + \int_{t_k}^{t_K} \left[\mathbf{F}(\tau)\mathbf{P}(\tau) + \mathbf{P}(\tau)\mathbf{F}^T(\tau) + \mathbf{L}(\tau)\mathbf{Q}'_C(\tau)\mathbf{L}^T(\tau)\right] d\tau$$

Duality Between Estimation and Control

Duality Between Linear-Optimal Estimation and Control

Linear-Gaussian Estimator

$$\boxed{\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{K}_C(t)\left[\mathbf{z}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)\right]}$$

$$\boxed{\mathbf{K}_C(t) = \mathbf{P}(t)\mathbf{H}^T(t)\mathbf{R}_C^{-1}(t)}$$

$$\boxed{\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{L}(t)\mathbf{Q}'_C(t)\mathbf{L}^T(t) - \mathbf{P}(t)\mathbf{H}^T(t)\mathbf{R}_C^{-1}(t)\mathbf{H}(t)\mathbf{P}(t), \quad \mathbf{P}(0) \text{ given}}$$

Linear-Quadratic Controller

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) - \mathbf{G}(t)\mathbf{C}(t)\mathbf{x}(t)}$$

$$\boxed{\mathbf{C}(t) = \mathbf{R}^{-1}(t)\mathbf{G}^T(t)\mathbf{S}(t)}$$

$$\boxed{\dot{\mathbf{S}}(t) = -\mathbf{S}(t)\mathbf{F}(t) - \mathbf{F}(t)^T\mathbf{S}(t) - \mathbf{Q}(t) + \mathbf{S}(t)\mathbf{G}(t)\mathbf{R}^{-1}(t)\mathbf{G}^T(t)\mathbf{S}(t), \quad \mathbf{S}(t_f) \text{ given}}$$

Dual Matrix Definitions for Solution of the Riccati Equation

Linear-Quadratic Controller	*Linear-Gaussian Estimator*
$\mathbf{F}(t)$	$\mathbf{F}^T(t)$
$\mathbf{G}(t)$	$\mathbf{H}^T(t)$
$\mathbf{Q}(t)$	$\mathbf{L}(t)\mathbf{Q}'_C\mathbf{L}^T(t)$
$\mathbf{R}$	$\mathbf{R}_C$
$\mathbf{S}(t)$	$\mathbf{P}(t)$
$\mathbf{S}(t_f)$	$\mathbf{P}(0)$
$\mathbf{C}(t)$	$\mathbf{K}_C^T(t)$

Controller Riccati equation propagates **backward** in time

Estimator Riccati equation propagates **forward** in time

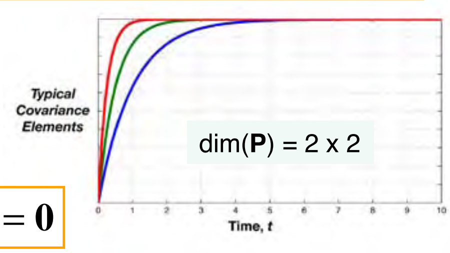
Stability of the Kalman-Bucy Filter

Stochastic Equilibrium of the Constant-Gain Kalman-Bucy Filter

- For linear, time-invariant systems, covariance estimate approaches a non-zero steady state

$$\boxed{\dot{\mathbf{P}}(t) = \mathbf{F}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T + \mathbf{L}\mathbf{Q}'_C\mathbf{L}^T - \mathbf{K}_C(t)\mathbf{H}\mathbf{P}(t) \xrightarrow[\Delta t \to 0]{} 0}$$

- Steady-state covariance matrix is the positive-definite solution to an algebraic Riccati equation



$$\boxed{\mathbf{F}\mathbf{P}_{SS} + \mathbf{P}_{SS}\mathbf{F}^T + \mathbf{L}\mathbf{Q}'_C\mathbf{L}^T - \mathbf{K}_C\mathbf{H}\mathbf{P}_{SS} = \mathbf{0}}$$

- Corresponding filter gain is constant

$$\boxed{\mathbf{K}_C = \mathbf{P}_{SS}\mathbf{H}^T\mathbf{R}_C^{-1}}$$

Asymptotic Stability of the Constant-Gain Kalman-Bucy Filter

Estimation error

$$\boldsymbol{\varepsilon}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}_k(t); \quad \dot{\boldsymbol{\varepsilon}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}_k(t)$$

Linear, time-invariant state estimator

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{K}_C\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right] \\ &= \left(\mathbf{F} - \mathbf{K}_C\mathbf{H}\right)\hat{\mathbf{x}}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{K}_C\mathbf{z}(t) \end{aligned}$$

Constant filter gain matrix

$$\mathbf{K}_C = \mathbf{P}_{SS}\mathbf{H}^T\mathbf{R}_C^{-1}$$

Algebraic Riccati equation

$$\mathbf{F}\mathbf{P}_{SS} + \mathbf{P}_{SS}\mathbf{F}^T + \mathbf{L}\mathbf{Q}'_C\mathbf{L}^T - \mathbf{K}_C\mathbf{H}\mathbf{P}_{SS} = \mathbf{0}$$

Asymptotic Stability of the Steady-State Kalman-Bucy Filter

LTI state equation

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{L}\mathbf{w}(t)$$

Measurement equation

$$\mathbf{z}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{n}(t)$$

Estimation error propagation

$$\begin{aligned} &\dot{\boldsymbol{\varepsilon}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}_k(t) \\ &= \left[\mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{L}\mathbf{w}(t)\right] - \left\{\mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\mathbf{u}(t) + \mathbf{K}_C\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right]\right\} \end{aligned}$$

$$\dot{\boldsymbol{\varepsilon}}(t) = (\mathbf{F} - \mathbf{K}\mathbf{H})\boldsymbol{\varepsilon}(t) + \mathbf{L}\mathbf{w}(t) - \mathbf{K}_C\mathbf{n}(t)$$

Estimator stability governed by (F – KH)

Estimator response driven by w(*t*) and n(*t*)

Information Matrix

Information matrix and its time rate of change

$$\mathscr{I}(t) = \mathbf{P}^{-1}(t)$$
$$\dot{\mathscr{I}}(t) = \dot{\mathbf{P}}^{-1}(t) = -\mathbf{P}^{-1}(t)\dot{\mathbf{P}}(t)\mathbf{P}^{-1}(t) = -\mathscr{I}(t)\dot{\mathbf{P}}(t)\mathscr{I}(t)$$

Substitute information matrix in matrix Riccati equation

$$\dot{\mathscr{I}}(t) = -\left[\mathscr{I}(t)\mathbf{F} + \mathbf{F}^T\mathscr{I}(t) + \mathscr{I}(t)\mathbf{L}\mathbf{Q}'_C\mathbf{L}^T\mathscr{I}(t) - \mathbf{H}^T\mathbf{R}_C^{-1}\mathbf{H}\right]$$

Steady-state solution for the information matrix

$$\mathbf{0} = \mathscr{I}_{SS}\mathbf{F} + \mathbf{F}^T\mathscr{I}_{SS} + \mathscr{I}_{SS}\mathbf{L}\mathbf{Q}'_C\mathbf{L}^T\mathscr{I}_{SS} - \mathbf{H}^T\mathbf{R}_C^{-1}\mathbf{H}$$

Lyapunov Function for Estimator Error

Lyapunov function for estimation error

$$\mathscr{V}\left[\boldsymbol{\varepsilon}(t)\right] = \boldsymbol{\varepsilon}^T(t)\mathscr{I}_{SS}\boldsymbol{\varepsilon}(t)$$

Homogeneous error dynamics

$$\dot{\boldsymbol{\varepsilon}}(t) = (\mathbf{F} - \mathbf{K}\mathbf{H})\boldsymbol{\varepsilon}(t)$$

Time rate of change of the Lyapunov function must be negative to assure stability

$$\frac{d\mathscr{V}\left[\boldsymbol{\varepsilon}(t)\right]}{dt} = 2\boldsymbol{\varepsilon}^T(t)\mathscr{I}_{SS}\dot{\boldsymbol{\varepsilon}}(t)$$
$$= \boldsymbol{\varepsilon}^T(t)\left[\left(\mathscr{I}_{SS}\mathbf{F} - \mathbf{H}^T\mathbf{R}_C^{-1}\mathbf{H}\right) + \left(\mathscr{I}_{SS}\mathbf{F} - \mathbf{H}^T\mathbf{R}_C^{-1}\mathbf{H}\right)^T\right]\boldsymbol{\varepsilon}^T(t)$$
$$= -\boldsymbol{\varepsilon}^T(t)\left[\mathscr{I}_{SS}\mathbf{L}\mathbf{Q}_C{}'\mathbf{L}^T\mathscr{I}_{SS} + \mathbf{H}^T\mathbf{R}_C^{-1}\mathbf{H}\right]\boldsymbol{\varepsilon}^T(t) < \mathbf{0}$$

$\left[\mathscr{I}_{SS}\mathbf{L}\mathbf{Q}_C{}'\mathbf{L}^T\mathscr{I}_{SS} + \mathbf{H}^T\mathbf{R}_C^{-1}\mathbf{H}\right]$ **is positive definite**

Eigenvalues of the Constant-Gain Kalman-Bucy Filter

LTI state estimation error

$$\boxed{\dot{\boldsymbol{\varepsilon}}(t) = (\mathbf{F} - \mathbf{KH})\boldsymbol{\varepsilon}(t) + \mathbf{Lw}(t) - \mathbf{K}_C\mathbf{n}(t)}$$

Stability indicated by eigenvalues of (F – KH)

$$\boxed{\left|s\mathbf{I}_n - (\mathbf{F} - \mathbf{KH})\right| = \Delta_{estimator}(s) = 0}$$

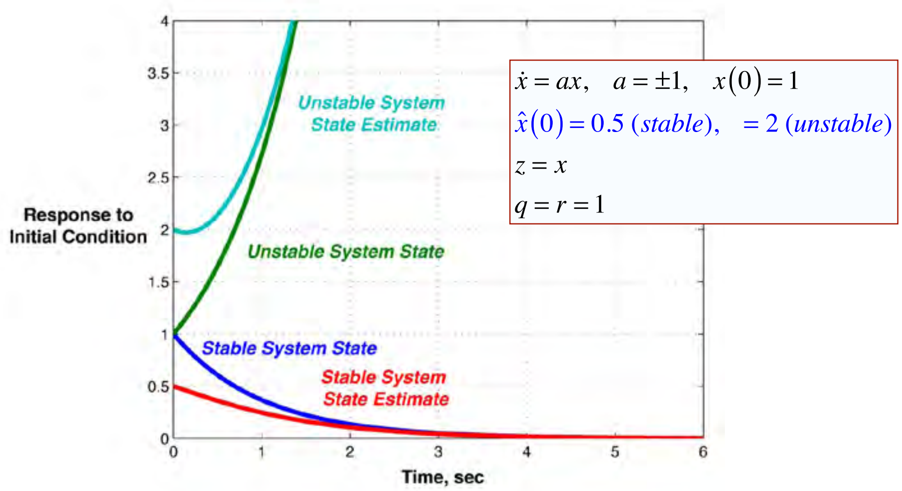
By duality to the LQ regulator, stability of the estimate is guaranteed if the model is correct and

$$\boxed{\begin{array}{c} (\mathbf{F},\mathbf{H}): \text{ Detectable pair} \\ (\mathbf{F},\mathbf{D}): \text{ Stabilizable pair, where } \mathbf{LWL}^T = \mathbf{D}^T\mathbf{D} \end{array}}$$

$$\boxed{\begin{array}{c} \mathbf{LWL}^T: \text{ Positive semi-definite matrix} \\ \mathbf{R}_C: \text{ Positive definite matrix} \end{array}}$$

Stability of the Estimate vs. Stability of the System

Estimate error is stable even if the system is not



Filter Divergence

Could the state estimate diverge from its most likely mean value?

- Yes, if there are
  - Discrepancies in the dynamic model, e.g., nonlinear system with linear model
  - Errors in the assumed covariance matrices
  - Measurement biases
  - Numerical errors in calculation

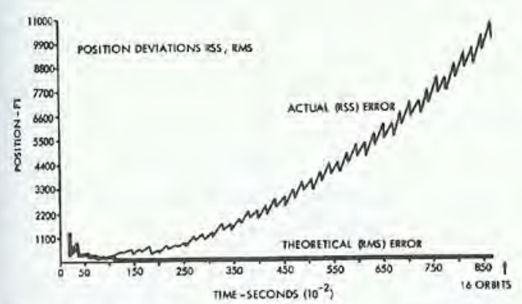


Fig. 1. Kalman filter divergence caused by 25% error in drag acceleration.

- RMS error = “root-mean-square” error = estimated standard deviation
- RSS error = “root-of-sum-of-square” errors = square root of the empirical sum of the squares of the difference between actual and estimated state components

Examples of Filter Divergence

(Schlee, Standish, Toda, 1967)

Divergence due to Modeling Error

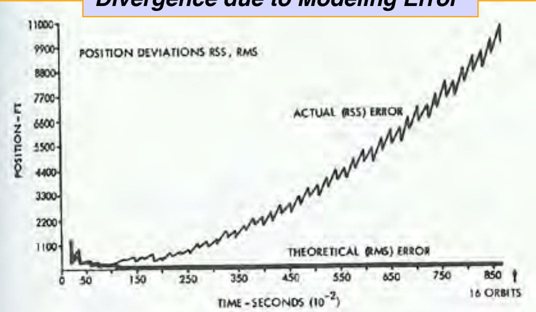


Fig. 1. Kalman filter divergence caused by 25% error in drag acceleration.

Divergence due to Computational Error

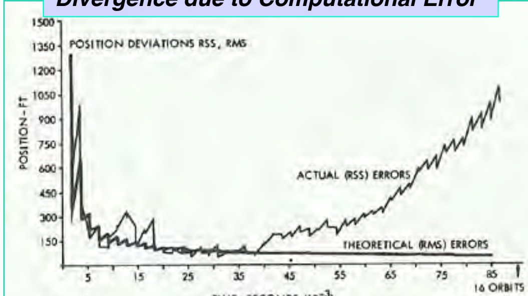


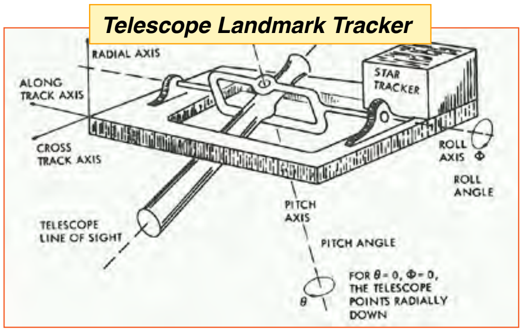
Fig. 2 Divergence of the Kalman filter caused by computational errors resulting from single precision arithmetic (IBM 7090).

- Satellite altitude estimation using simplified 1-D model
  - Filter designed assuming constant altitude
  - Drag and gravitational effects change as altitude increases or decreases

Examples of Filter Divergence

(Schlee, Standish, Toda, 1967)

Telescope Landmark Tracker



Effect of Model Detail on Filter Divergence

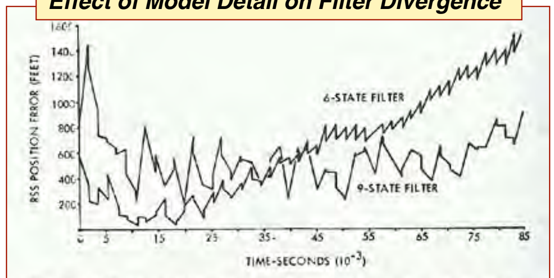


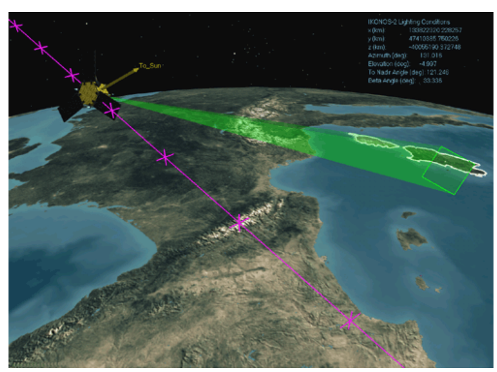
Fig. 5 Effect of filter state dimension upon divergence.

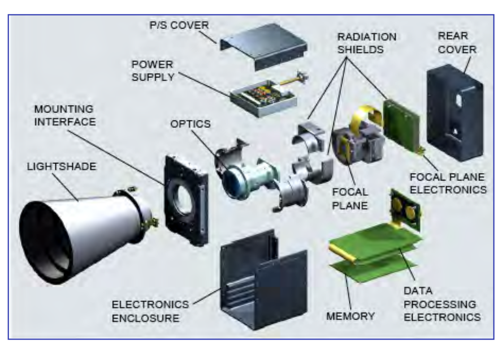
Orbit determination for a space vehicle
Measurements are angle sightings of known terrestrial landmarks
Increased model precision can reduce divergence rate

6 - state filter :	Spacecraft position & velocity
9 - state filter :	Spacecraft position & velocity, Landmark position

Landmark and Star Tracking

- **Landmark and star tracking are functionally similar**
- **Instrument has narrow field of view**
- **Star/geographic location catalogs help identify targets**
- ***x* and *y* location of landmark or star on focal plane determines angles to the target**





Divergence Occurs When Filter Gains are Too Low

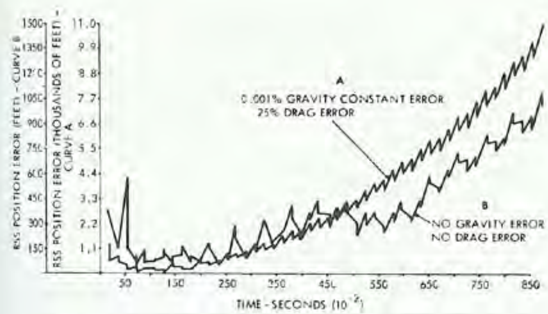


Fig. 6 Effect of dynamic bias errors upon divergence.

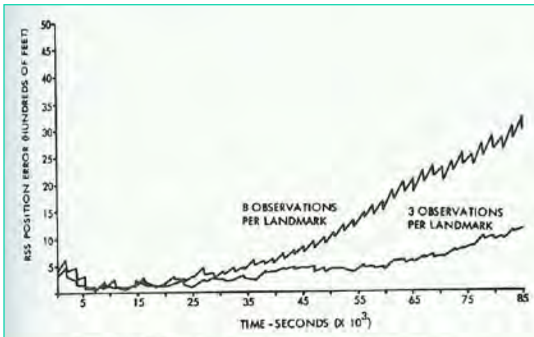


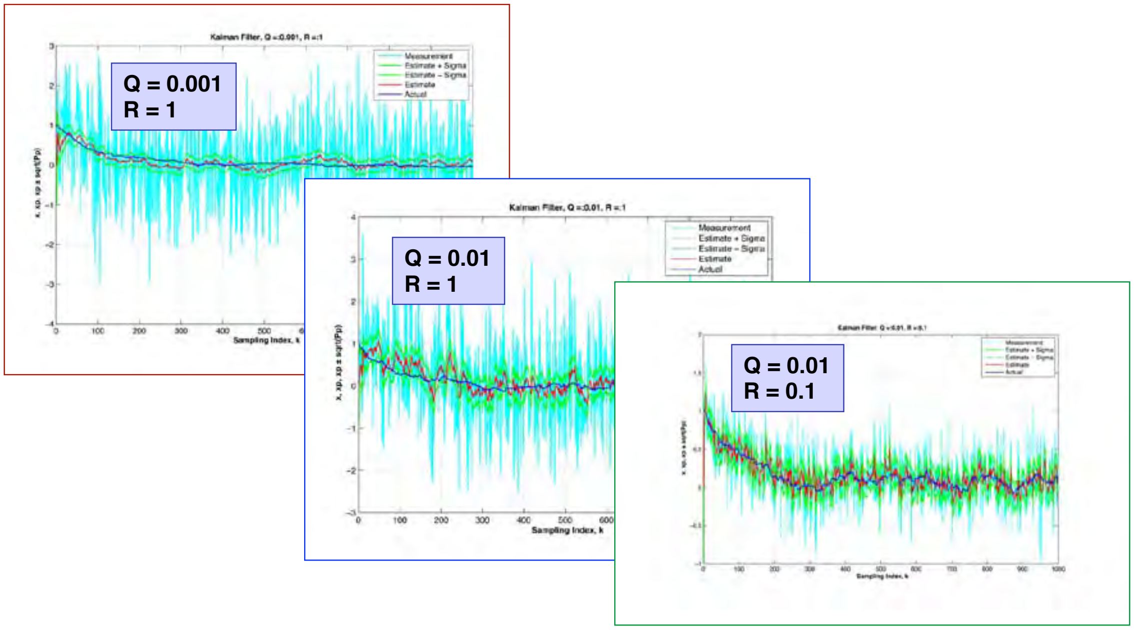
Fig. 7 Effect of data rate upon divergence.

1) Dynamic model is incorrect
2) State-error covariance estimate is incorrect
3) Filter gains become too low to properly weight new information

Solutions for Filter Divergence

- Increase "process noise" used for estimator design (i.e., assumed disturbance covariance)
- Improve system modeling, e.g.,
  - Estimate measurement bias and scale factor
  - Include higher-order terms
  - Model nonlinearity
- Use higher precision or square-root filtering
- Adapt estimator to changing conditions

Recall Effects of Assumed "Process Noise" and Measurement Error



Adding "Process Noise" to Eliminate Divergence*

- **Satellite orbit determination**
  - Aerodynamic drag produced unmodeled bias
  - Optimal filter did not estimate bias
- **"Process noise" increased for filter design**
  - Divergence was contained

* Fitzgerald, 1971

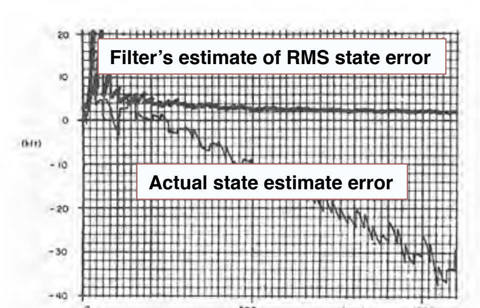


Fig. 6. Divergence due to ignoring drag in satellite navigation (forward component of position error).

Fig. 7. Elimination of divergence by increasing assumed process noise.

Square Root Filtering

Square-Root Filtering

Improved precision by reducing condition number

Define

$$\mathbf{P}(t) = \mathbf{S}(t)\mathbf{S}^T(t)$$

where

$\mathbf{S}(t)$: Lower triangular square root of $\mathbf{P}(t)$

Use Cholesky decomposition to compute $\mathbf{S}(t)$ [see text]

Riccati equation

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{L}(t)\mathbf{Q}'_C(t)\mathbf{L}^T(t) - \mathbf{P}(t)\mathbf{H}^T(t)\mathbf{R}_C^{-1}(t)\mathbf{H}(t)\mathbf{P}(t)$$

Substitute

$$\begin{gathered}\dot{\mathbf{S}}(t)\mathbf{S}^T(t) + \mathbf{S}(t)\dot{\mathbf{S}}^T(t) = \\ \mathbf{F}(t)\mathbf{S}(t)\mathbf{S}^T(t) + \mathbf{S}(t)\mathbf{S}^T(t)\mathbf{F}^T(t) + \mathbf{L}(t)\mathbf{Q}'_C(t)\mathbf{L}^T(t) \\ -\mathbf{S}(t)\mathbf{S}^T(t)\mathbf{H}^T(t)\mathbf{R}_C^{-1}(t)\mathbf{H}(t)\mathbf{S}(t)\mathbf{S}^T(t)\end{gathered}$$

Matrix Decomposition Examples

Cholesky Decomposition

$$\begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix} = \begin{pmatrix} 2 & & \\ 6 & 1 & \\ -8 & 5 & 3 \end{pmatrix} \begin{pmatrix} 2 & 6 & -8 \\ & 1 & 5 \\ & & 3 \end{pmatrix}$$

UDU^T *(or LDL^T)* Decomposition

$$\begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 3 & 1 & \\ -4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 4 & & \\ & 1 & \\ & & 9 \end{pmatrix} \begin{pmatrix} 1 & 3 & -4 \\ & 1 & 5 \\ & & 1 \end{pmatrix}$$

Examples from Wikipedia
http://en.wikipedia.org/wiki/Cholesky_decomposition

Square-Root Filtering

Pre-multiply by $\mathbf{S}^{-1}$; post-multiply by $\mathbf{S}^{-T}$

$$\mathbf{S}^{-1}\dot{\mathbf{S}} + \dot{\mathbf{S}}^T\mathbf{S}^{-T} = \mathbf{S}^{-1}\mathbf{F}\mathbf{S} + \mathbf{S}^T\mathbf{F}^T\mathbf{S}^{-T} + \mathbf{S}^{-1}\mathbf{L}\mathbf{Q}'_C\mathbf{L}^T\mathbf{S}^{-T} - \mathbf{S}^T\mathbf{H}^T\mathbf{R}_C^{-1}\mathbf{H}\mathbf{S}$$
$$\triangleq \mathbf{M}(t) = \mathbf{M}_{LT}(t) + \mathbf{M}_{UT}(t)$$

Elements of $\mathbf{M}_{LT}(t)$

$$\left(m_{ij}\right)_{LT} = \begin{cases} m_{ij}, & i > j \\ m_{ij}/2, & i = j \\ 0, & i < j \end{cases}$$

$\mathbf{M}_{LT}(t)$: Lower triangular portion of $\mathbf{M}(t)$
$\mathbf{M}_{UT}(t)$: Upper triangular portion of $\mathbf{M}(t)$
$\mathbf{M}_{UT}(t) = \mathbf{M}_{LT}^T(t)$

Because $\mathbf{S}$ is lower triangular
$\dot{\mathbf{S}}$ and $\mathbf{S}^{-1}$ are lower triangular

Square-Root Filtering

Hence, the Riccati equation becomes

$$\boxed{\dot{\mathbf{S}}(t) = \mathbf{S}(t)\mathbf{M}_{LT}(t), \qquad \mathbf{S}(0)\mathbf{S}^T(0) = \mathbf{P}(0) > \mathbf{0}}$$

Estimator gain matrix

$$\boxed{\mathbf{K}_C(t) = \mathbf{S}(t)\mathbf{S}^T(t)\mathbf{H}^T(t)\mathbf{R}_C^{-1}(t), \qquad \mathbf{S}(0)\mathbf{S}^T(0) = \mathbf{P}(0) > \mathbf{0}}$$

Square root required to define S(0) but not for calculation of S(*t*)

Kalman Filter for Correlated Disturbance Inputs and Measurement Noise*

Disturbance process produces error in measurements

Correlation expressed by expected value

$$E\left\{\begin{bmatrix}\mathbf{w}(t)\\ \mathbf{n}(t)\end{bmatrix}\begin{bmatrix}\mathbf{w}(\tau) & \mathbf{n}(\tau)\end{bmatrix}\right\} = \begin{bmatrix}\mathbf{Q}_C(t) & \mathbf{M}_C(t)\\ \mathbf{M}_C^T(t) & \mathbf{R}_C(t)\end{bmatrix}\delta(t-\tau)$$

Riccati equation

$$\boxed{\begin{aligned}\dot{\mathbf{P}}(t) &= \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{L}(t)\mathbf{Q}'_C(t)\mathbf{L}^T(t)\\ &-\left[\mathbf{P}(t)\mathbf{H}^T(t) + \mathbf{L}(t)\mathbf{M}_C(t)\right]\mathbf{R}_C^{-1}(t)\left[\mathbf{H}(t)\mathbf{P}(t) + \mathbf{M}_C^T(t)\mathbf{L}^T(t)\right]\end{aligned}}$$

Estimator gain matrix

$$\boxed{\mathbf{K}_C(t) = \left[\mathbf{P}(t)\mathbf{H}^T(t) + \mathbf{L}(t)\mathbf{M}_C(t)\right]\mathbf{R}_C^{-1}(t)}$$

** Bryson and Johansen*

Next Time:
Nonlinear State Estimation