Orbital Mechanics

Space System Design, MAE 342, Princeton University
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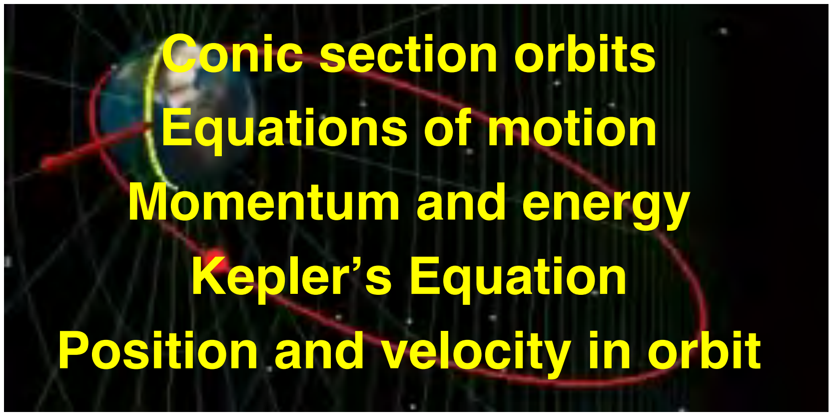
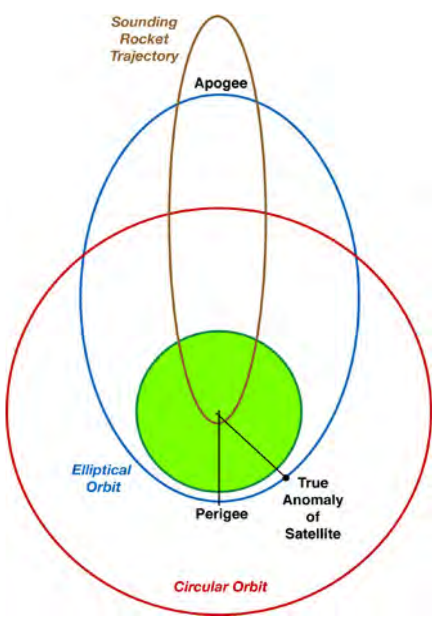
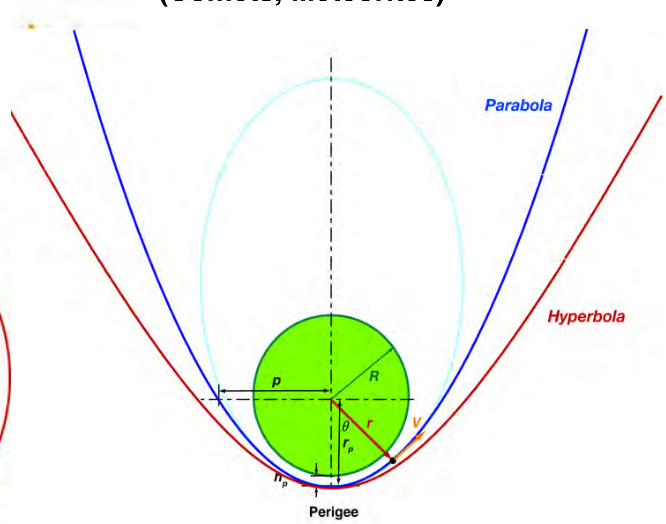


Orbits 101

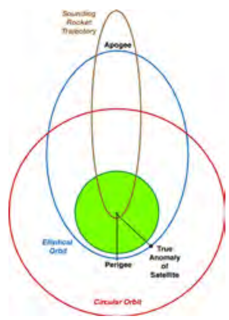
Satellites

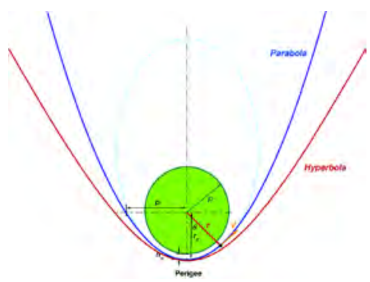


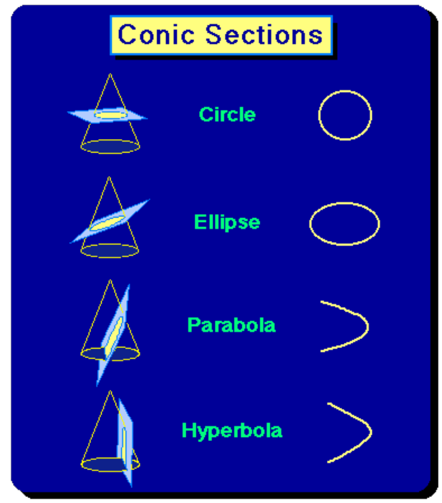
Escape and Capture (Comets, Meteorites)



Two-Body Orbits are Conic Sections







Classical Orbital Elements

Dimension and Time

a : Semi-major axis

e : Eccentricity

t_p : Time of perigee passage

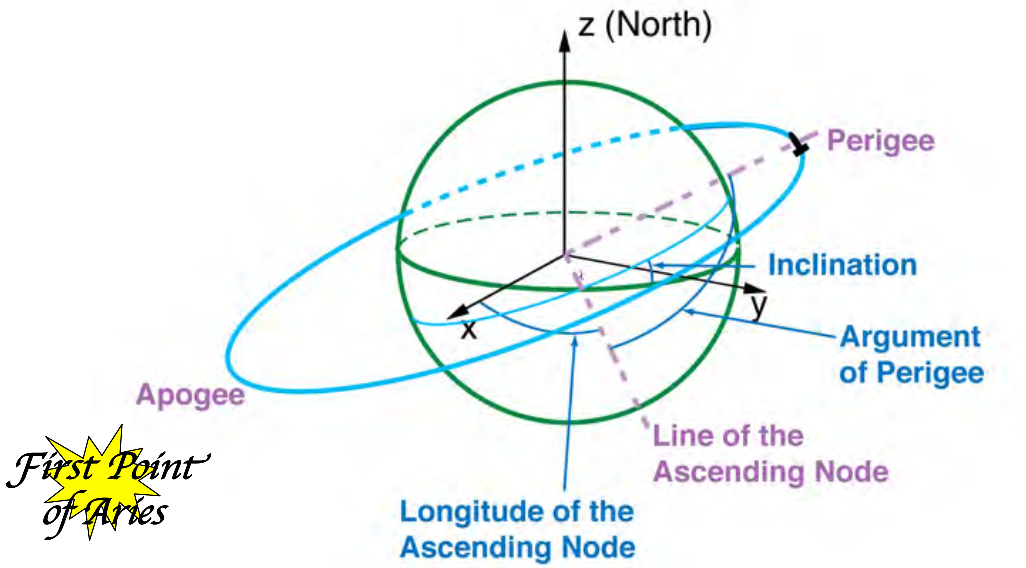
Orientation

Ω : Longitude of the Ascending/Descending Node

i : Inclination of the Orbital Plane

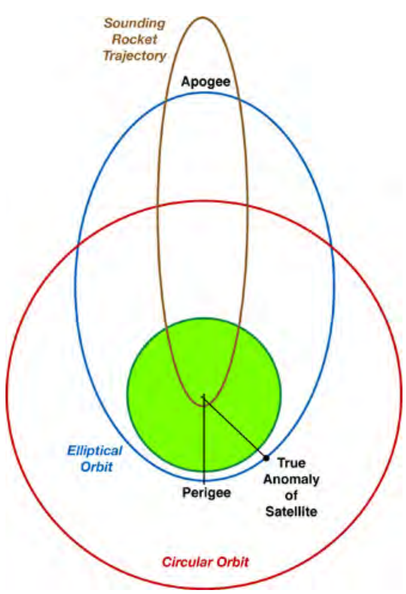
ω: Argument of Perigee

Orientation of an Elliptical Orbit



Orbits 102

(2-Body Problem)



- e.g.,
  - Sun and Earth or
  - Earth and Moon or
  - Earth and Satellite
- Circular orbit: radius and velocity are constant
  - Low Earth orbit: 17,000 mph = 24,000 ft/s = 7.3 km/s
- Super-circular velocities
  - Earth to Moon: 24,550 mph = 36,000 ft/s = 11.1 km/s
  - Escape: 25,000 mph = 36,600 ft/s = 11.3 km/s
- Near escape velocity, small changes have huge influence on apogee

Newton's 2nd Law

- Particle of fixed mass (also called a point mass) acted upon by a force changes velocity with
  - acceleration proportional to and in direction of force
- Inertial reference frame
- Ratio of force to acceleration is the mass of the particle: **F** = *m* **a**

$$\frac{d}{dt}\left[m\mathbf{v}(t)\right] = m\frac{d\mathbf{v}(t)}{dt} = m\mathbf{a}(t) = \mathbf{F}$$

$$\mathbf{F} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \text{force vector}$$

$$m\frac{d}{dt}\begin{bmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

Equations of Motion for a Particle

Integrating the acceleration (Newton's 2nd Law) allows us to solve for the velocity of the particle

$$\frac{d\mathbf{v}(t)}{dt} = \dot{\mathbf{v}}(t) = \frac{1}{m}\mathbf{F} = \begin{bmatrix} f_x/m \\ f_y/m \\ f_z/m \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$$\mathbf{v}(T) = \int_0^T \frac{d\mathbf{v}(t)}{dt}dt + \mathbf{v}(0) = \int_0^T \mathbf{a}(t)dt + \mathbf{v}(0) = \int_0^T \frac{1}{m}\mathbf{F}\,dt + \mathbf{v}(0)$$

3 components of velocity

$$\begin{bmatrix} v_x(T) \\ v_y(T) \\ v_z(T) \end{bmatrix} = \int_0^T \begin{bmatrix} a_x(t) \\ a_y(t) \\ a_z(t) \end{bmatrix} dt + \begin{bmatrix} v_x(0) \\ v_y(0) \\ v_z(0) \end{bmatrix} = \int_0^T \begin{bmatrix} f_x(t)/m \\ f_y(t)/m \\ f_z(t)/m \end{bmatrix} dt + \begin{bmatrix} v_x(0) \\ v_y(0) \\ v_z(0) \end{bmatrix}$$

Equations of Motion for a Particle

Integrating the velocity allows us to solve for the position of the particle

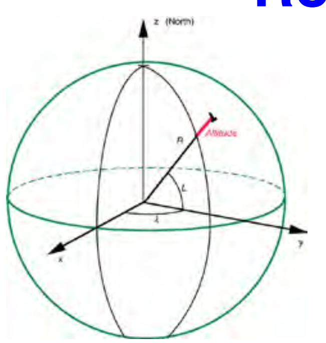
$$\frac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t) = \mathbf{v}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix}$$

$$\mathbf{r}(T) = \int_0^T \frac{d\mathbf{r}(t)}{dt} dt + \mathbf{r}(0) = \int_0^T \mathbf{v}\, dt + \mathbf{r}(0)$$

3 components of position

$$\begin{bmatrix} x(T) \\ y(T) \\ z(T) \end{bmatrix} = \int_0^T \begin{bmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix} dt + \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}$$

Spherical Model of the Rotating Earth



Spherical model of earth's surface, earth-fixed (rotating) coordinates

$$\mathbf{R}_E = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_E = \begin{bmatrix} \cos L_E \cos \lambda_E \\ \cos L_E \sin \lambda_E \\ \sin L_E \end{bmatrix} R$$

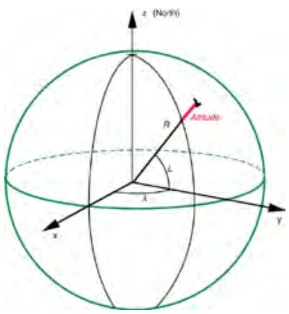
L_E : Latitude (from Equator), deg

λ_E : Longitude (from Prime Meridian), deg

R : Radius (from Earth's center), deg

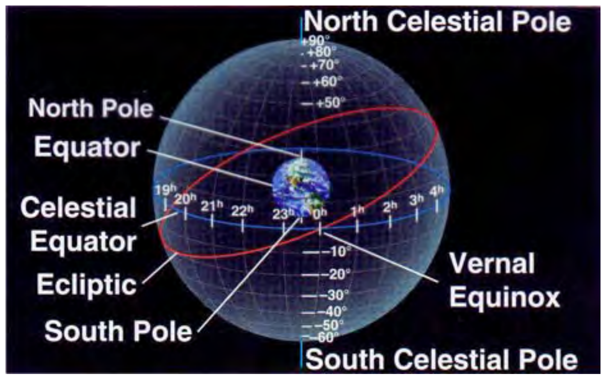
Earth's rotation rate, Ω, is 15.04 deg/hr

Non-Rotating (Inertial) Reference Frame for the Earth



Celestial longitude, λ_C, measured from First Point of Aries on the Celestial Sphere at Vernal Equinox

$$\lambda_C = \lambda_E + \Omega\left(t - t_{epoch}\right) = \lambda_E + \Omega\ \Delta t$$



Transformation Effects of Rotation

Transformation from inertial frame, *I*, to Earth's rotating frame, *E*

$$\mathbf{R}_E = \begin{bmatrix} \cos\Omega\Delta t & \sin\Omega\Delta t & 0 \\ -\sin\Omega\Delta t & \cos\Omega\Delta t & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{R}_I = \begin{bmatrix} \cos\Omega\Delta t & \sin\Omega\Delta t & 0 \\ -\sin\Omega\Delta t & \cos\Omega\Delta t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}_I$$

Location of satellite, rotating and inertial frames

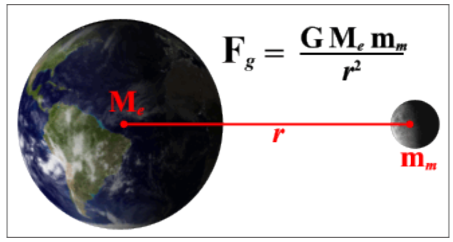
$$\mathbf{r}_E = \begin{bmatrix} \cos L_E \cos\lambda_E \\ \cos L_E \sin\lambda_E \\ \sin L_E \end{bmatrix} (R + Altitude); \quad \mathbf{r}_I = \begin{bmatrix} \cos L_E \cos\lambda_C \\ \cos L_E \sin\lambda_C \\ \sin L_E \end{bmatrix} (R + Altitude)$$

Orbital calculations generally are made in an inertial frame of reference

Gravity Force Between Two Point Masses, e.g., Earth and Moon

Magnitude of gravitational attraction

$$F = \frac{Gm_1 m_2}{r^2}$$



G : Gravitational constant $= 6.67 \times 10^{-11}\,\mathrm{Nm^2/kg^2}$

m_1 : Mass of 1st body $= 5.98 \times 10^{24}$ kg for Earth

m_2 : Mass of 2nd body $= 7.35 \times 10^{22}$ kg for Moon

r : Distance between centers of mass of m_1 and m_2, m

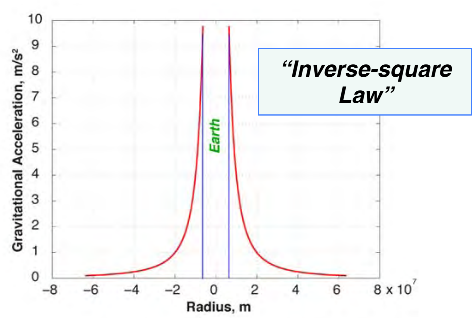
Acceleration Due To Gravity

$$F_2 = m_2 a_{1on2} = \frac{Gm_1 m_2}{r^2}$$

$$a_{1on2} = \frac{Gm_1}{r^2} \triangleq \frac{\mu_1}{r^2}$$

$$\mu_1 = Gm_1$$

Gravitational parameter of 1st mass



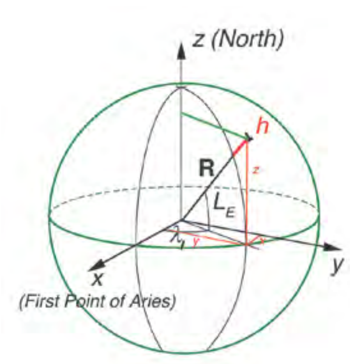
At Earth's surface, acceleration due to gravity is

$$a_g \triangleq g_{o_{Earth}} = \frac{\mu_E}{R^2_{surface}} = \frac{3.98 \times 10^{14}\, m^3/s^2}{(6{,}378{,}137\,m)^2} = 9.798\,\mathrm{m/s^2}$$

Gravitational Force Vector of the Spherical Earth

Force always directed toward the Earth's center

$$\mathbf{F}_g = -m\frac{\mu_E}{r_I^2}\left(\frac{\mathbf{r}_I}{|\mathbf{r}_I|}\right) = -m\frac{\mu_E}{r_I^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_I (vector), \quad as\,|\mathbf{r}_I| = r_I$$



(*x, y, z*) establishes the direction of the local vertical

$$\frac{\mathbf{r}_I}{|\mathbf{r}_I|} = \frac{\begin{bmatrix} x \\ y \\ z \end{bmatrix}_I}{\sqrt{x_I^2 + y_I^2 + z_I^2}} = \begin{bmatrix} \cos L_E \cos\lambda_I \\ \cos L_E \sin\lambda_I \\ \sin L_E \end{bmatrix}$$

Equations of Motion for a Particle in an Inverse-Square-Law Field

Integrating the acceleration (Newton's 2nd Law) allows us to solve for the velocity of the particle

$$\frac{d\mathbf{v}(t)}{dt} = \dot{\mathbf{v}}(t) = \frac{1}{m}\mathbf{F}_g = -\frac{\mu_E}{r_I^2}\left(\frac{\mathbf{r}_I}{|\mathbf{r}_I|}\right) = -\frac{\mu_E}{r_I^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{v}(T) = \int_0^T \frac{d\mathbf{v}(t)}{dt}dt + \mathbf{v}(0) = \int_0^T \mathbf{a}(t)dt + \mathbf{v}(0) = \int_0^T \frac{1}{m}\mathbf{F}\,dt + \mathbf{v}(0)$$

3 components of velocity

$$\begin{bmatrix} v_x(T) \\ v_y(T) \\ v_z(T) \end{bmatrix} = -\mu_E\int_0^T \begin{bmatrix} x/r_I^3 \\ y/r_I^3 \\ z/r_I^3 \end{bmatrix} dt + \begin{bmatrix} v_x(0) \\ v_y(0) \\ v_z(0) \end{bmatrix}$$

Equations of Motion for a Particle in an Inverse-Square-Law Field

As before; Integrating the velocity allows us to solve for the position of the particle

$$\frac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t) = \mathbf{v}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix}$$

$$\mathbf{r}(T) = \int_0^T \frac{d\mathbf{r}(t)}{dt} dt + \mathbf{r}(0) = \int_0^T \mathbf{v}\, dt + \mathbf{r}(0)$$

3 components of position

$$\begin{bmatrix} x(T) \\ y(T) \\ z(T) \end{bmatrix} = \int_0^T \begin{bmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix} dt + \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}$$

Dynamic Model with Inverse-Square-Law Gravity

No aerodynamic or thrust force

Neglect motions in the *z* direction

$m_{satellite} \ll m_{Earth}$

Dynamic Equations

$$\dot{v}_x(t) = -\mu_E x_I(t) / r_I^{\,3}(t)$$

$$\dot{v}_y(t) = -\mu_E y_I(t) / r_I^{\,3}(t)$$

$$\dot{x}_I(t) = v_x(t)$$

$$\dot{y}_I(t) = v_y(t)$$

$$\text{where } r_I(t) = \sqrt{x_I^2(t) + y_I^2(t)}$$

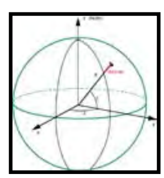
Example: Initial Conditions at Equator

$$v_x(0) = 7.5, 8, 8.5 \text{ km/s}$$

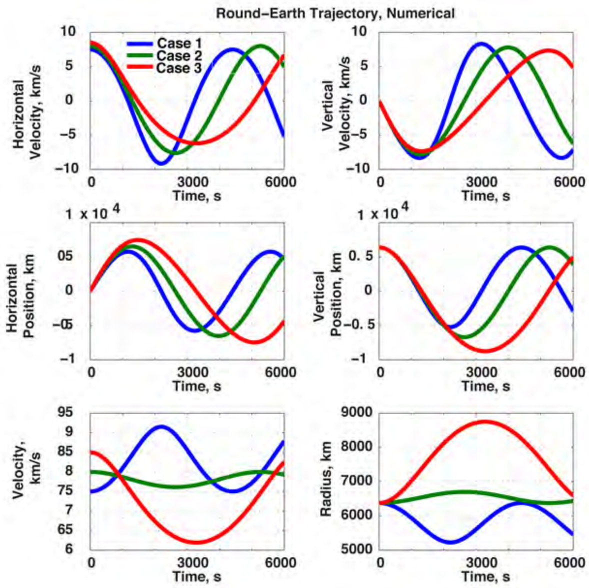
$$v_y(0) = 0$$

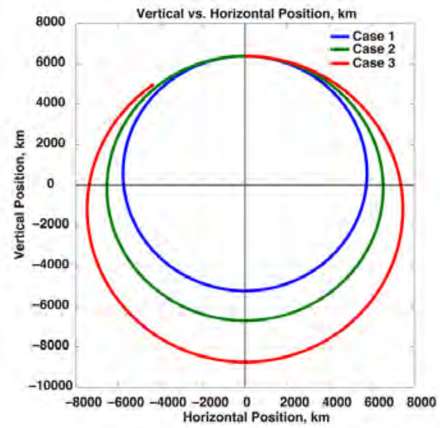
$$x(0) = 0$$

$$y(0) = 6{,}378 \text{ km} = R$$



Equatorial Orbits Calculated with Inverse-Square-Law Model





Work

"Work" is a scalar measure of change in energy

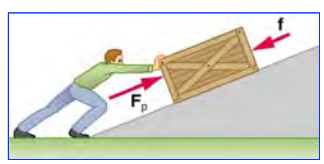
With constant force,

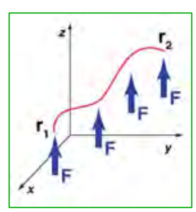
In one dimension

$$W_{12} = F\left(r_2 - r_1\right) = F\Delta r$$

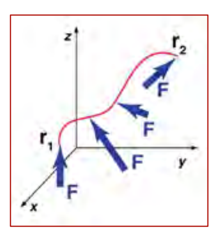
In three dimensions

$$W_{12} = \mathbf{F}^T\left(\mathbf{r}_2 - \mathbf{r}_1\right) = \mathbf{F}^T\Delta\mathbf{r}$$





With varying force, work is the integral



$$W_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}^T\, d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2}\left(f_x dx + f_y dy + f_z dz\right), \quad d\mathbf{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

Conservative Force

- **Assume that the 3-D force field is a function of position**

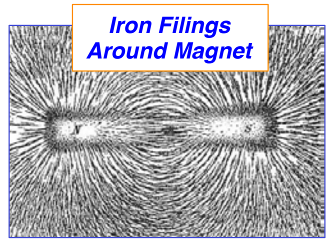
$$\boxed{\mathbf{F} = \mathbf{F}(\mathbf{r})}$$

- **The force field is <u>conservative</u> if**

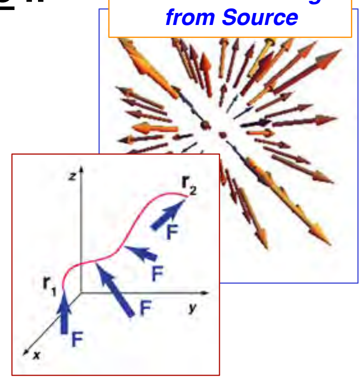
$$\boxed{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}^T(\mathbf{r})\,d\mathbf{r} + \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{F}^T(\mathbf{r})\,d\mathbf{r} = 0}$$

… for any path between $\mathbf{r}_1$ and $\mathbf{r}_2$ and back

Iron Filings Around Magnet



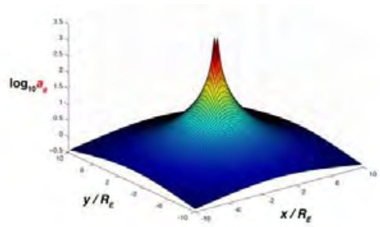
Force Emanating from Source

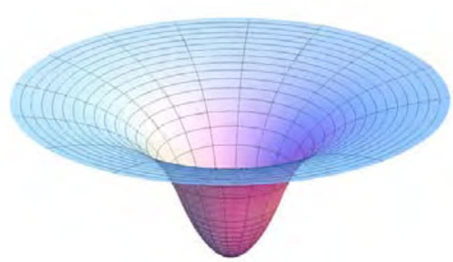


Gravitational Force is Gradient of a Potential, $V(\mathbf{r})$

Gravity potential, $V(\mathbf{r})$, is a function only of position

$$\boxed{\mathbf{F}_g = -m\frac{\mu_E}{r_I^3}\mathbf{r}_I = \frac{\partial}{\partial \mathbf{r}}\left(m\frac{\mu_E}{|\mathbf{r}|}\right) \triangleq \frac{\partial}{\partial \mathbf{r}}V(\mathbf{r})}$$

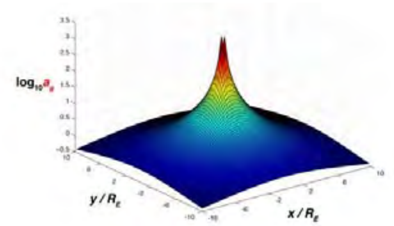




Gravitational Force Field

Gravitational force field

$$\boxed{\mathbf{F}_g = -m\frac{\mu_E}{r_I^{\ 3}}\mathbf{r}_I}$$

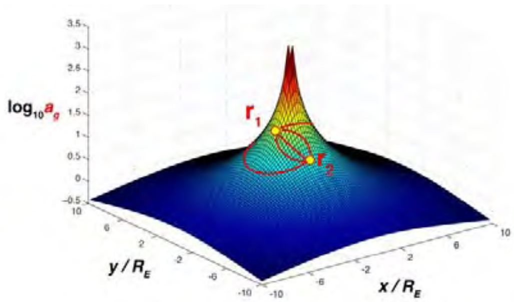


Gravitational force field is <u>conservative</u> because

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2}\frac{\partial}{\partial \mathbf{r}}V(\mathbf{r})\,d\mathbf{r}_I - \int_{\mathbf{r}_2}^{\mathbf{r}_1}\frac{\partial}{\partial \mathbf{r}}V(\mathbf{r})\,d\mathbf{r}_I =$$

$$-\int_{\mathbf{r}_1}^{\mathbf{r}_2} m\frac{\mu_E}{r_I^{\ 3}}\mathbf{r}_I\,d\mathbf{r}_I + \int_{\mathbf{r}_1}^{\mathbf{r}_2} m\frac{\mu_E}{r_I^{\ 3}}\mathbf{r}_I\,d\mathbf{r}_I = 0$$

… for any path between $\mathbf{r}_1$ and $\mathbf{r}_2$ and back



Potential Energy in Gravitational Force Field

Potential energy, *V* or *PE*, is defined with respect to a reference point, $\mathbf{r}_0$

$$\boxed{PE(\mathbf{r}_0) = V(\mathbf{r}_0) = V_0 \left(\equiv -U_0\right)}$$

$$\boxed{\Delta PE \triangleq V(\mathbf{r}_2) - V(\mathbf{r}_1) = -\left(m\frac{\mu}{r_2} + V_0\right) + \left(m\frac{\mu}{r_1} + V_0\right) = -m\frac{\mu}{r_2} + m\frac{\mu}{r_1}}$$

Kinetic Energy

Apply Newton's 2nd Law to the definition of Work

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}; \quad d\mathbf{r} = \mathbf{v}dt$$

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

$$W_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}^T \, d\mathbf{r} = \int_{t_1}^{t_2} m\left(\frac{d\mathbf{v}}{dt}\right)^T \mathbf{v}\,dt$$
$$= \frac{1}{2}m\int_{t_1}^{t_2} \frac{d}{dt}\left(\mathbf{v}^T\mathbf{v}\right)dt = \frac{1}{2}m\int_{t_1}^{t_2} \frac{d}{dt}\left(v^2\right)dt$$

$$= \frac{1}{2}mv^2\Big|_{t_1}^{t_2} = \frac{1}{2}m\left[v^2(t_2) - v^2(t_1)\right]$$
$$= \frac{1}{2}m\left[v_2^2 - v_1^2\right] \triangleq T_2 - T_1 \triangleq \Delta KE$$

Work = integral from 1st to 2nd time

T **(= *KE*) is the kinetic energy of the point mass, *m***

Total Energy in Point-Mass Gravitational Field

- **Potential energy of mass, *m*, depends only on the gravitational force field**

$$V = PE = -m\frac{\mu}{r}$$

- **Kinetic energy of mass, *m*, depends only on the velocity magnitude measured in an inertial frame of reference**

$$T \triangleq KE = \frac{1}{2}mv^2$$

- **Total energy, $\mathbb{E}$, is the sum of the two:**

$$\mathbb{E} = PE + KE$$
$$= -m\frac{\mu}{r} + \frac{1}{2}mv^2$$
$$= \textbf{Constant}$$

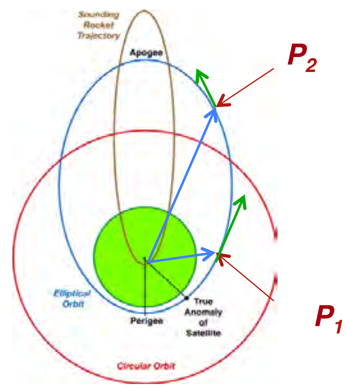
Interchange Between Potential and Kinetic Energy in a Conservative System

$$\mathbb{E}_2 - \mathbb{E}_1 = 0$$

$$\left(-m\frac{\mu}{r_2} + \frac{1}{2}mv_2^2\right) - \left(-m\frac{\mu}{r_1} + \frac{1}{2}mv_1^2\right) = 0$$

$$\left(-m\frac{\mu}{r_2} + m\frac{\mu}{r_1}\right) = \left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right)$$

$$PE_2 - PE_1 = KE_2 - KE_1$$



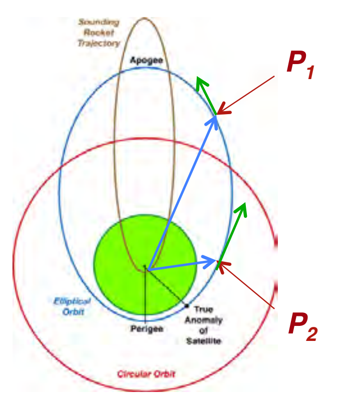
Specific Energy…

Energy per unit of the satellite's mass

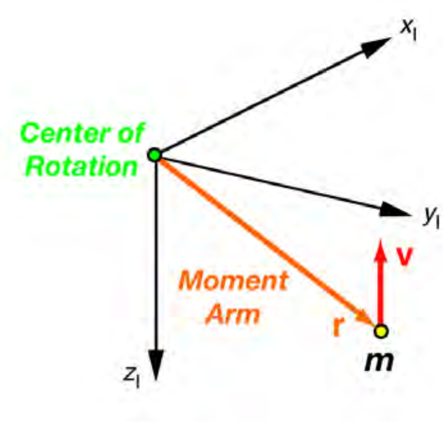
$$\mathbb{E}_S = PE_S + KE_S$$

$$= \frac{1}{m}\left(-\frac{m\mu}{r} + \frac{1}{2}mv^2\right)$$

$$= -\frac{\mu}{r} + \frac{1}{2}v^2$$

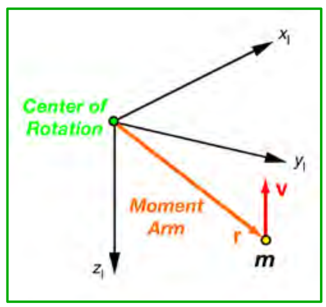


Angular Momentum of a Particle (Point Mass)



$$\boxed{\mathbf{h} = (\mathbf{r} \times m\mathbf{v}) = m(\mathbf{r} \times \mathbf{v}) = m(\mathbf{r} \times \dot{\mathbf{r}})}$$

Angular Momentum of a Particle

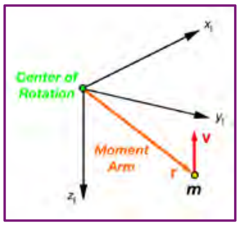


- Moment of linear momentum of a particle
  - Mass times components of the velocity that are perpendicular to the moment arm

$$\boxed{\mathbf{h} = (\mathbf{r} \times m\mathbf{v}) = m(\mathbf{r} \times \mathbf{v})}$$

- Cross Product: Evaluation of a determinant with unit vectors (*i, j, k*) along axes, (*x, y, z*) and (v_x, v_y, v_z) projections on to axes

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = (yv_z - zv_y)\boldsymbol{i} + (zv_x - xv_z)\boldsymbol{j} + (xv_y - yv_x)\boldsymbol{k}$$



Cross Product in Column Notation

Cross product identifies perpendicular components of r and v

$$\mathbf{r}\times\mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = \left(yv_z - zv_y\right)\boldsymbol{i} + \left(zv_x - xv_z\right)\boldsymbol{j} + \left(xv_y - yv_x\right)\boldsymbol{k}$$

Column notation

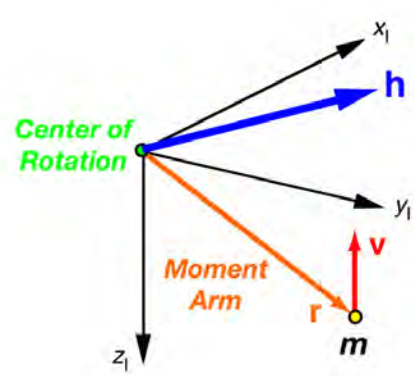
$$\mathbf{r}\times\mathbf{v} = \begin{bmatrix} \left(yv_z - zv_y\right) \\ \left(zv_x - xv_z\right) \\ \left(xv_y - yv_x\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Angular Momentum Vector is Perpendicular to Both Moment Arm and Velocity

$$\mathbf{h} = m\mathbf{r}\times\mathbf{v} = m\begin{bmatrix} \left(yv_z - zv_y\right) \\ \left(zv_x - xv_z\right) \\ \left(xv_y - yv_x\right) \end{bmatrix}$$

$$= m\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = m\tilde{\mathbf{r}}\mathbf{v}$$

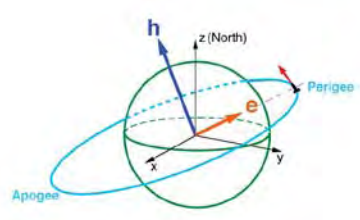


Specific Angular Momentum Vector of a Satellite

... is the angular momentum per unit of the satellite's mass, referenced to the center of attraction

$$\mathbf{h}_S = \frac{m}{m}\mathbf{r}\times\mathbf{v} = \mathbf{r}\times\mathbf{v} = \mathbf{r}\times\dot{\mathbf{r}}$$

Perpendicular to the orbital plane



Equations of Motion for a Particle in an Inverse-Square-Law Field

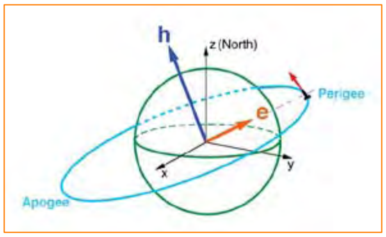
Acceleration is

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t) = -\frac{\mu}{r^2(t)}\left(\frac{\mathbf{r}_I(t)}{|\mathbf{r}(t)|}\right) = -\frac{\mu}{r^3(t)}\mathbf{r}(t)$$

... or

$$\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}$$

Cross Products of Radius and Radius Rate



Then

$$\boxed{\mathbf{r} \times \left[\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} \right] = \mathbf{0}}$$

$$\boxed{\mathbf{r} \times \mathbf{r} = \mathbf{0}} \qquad \boxed{\dot{\mathbf{r}} \times \dot{\mathbf{r}} = \mathbf{0}}$$

… because they are parallel

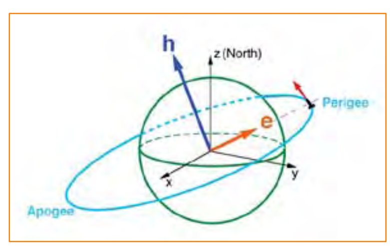
Chain Rule for Differentiation

$$\boxed{\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = (\dot{\mathbf{r}} \times \dot{\mathbf{r}}) + (\mathbf{r} \times \ddot{\mathbf{r}}) = (\mathbf{r} \times \ddot{\mathbf{r}})}$$

Specific Angular Momentum

$$\boxed{\mathbf{r} \times \left[\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r} \right] = (\mathbf{r} \times \ddot{\mathbf{r}}) + \frac{\mu}{r^3}\underbrace{\cancel{(\mathbf{r} \times \mathbf{r})}}_{0}}$$

$$\boxed{= \frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \frac{d\mathbf{h}_S}{dt} = \mathbf{0}}$$



Consequently

$$\boxed{\mathbf{h}_S = \text{ Constant}}$$

$$\boxed{\mathbf{h}_S \triangleq \mathbf{h} = (\mathbf{r} \times \dot{\mathbf{r}}) \quad \text{(Perpendicular to the plane of motion)}}$$

Orbital plane is fixed in inertial space

Eccentricity Vector is a Constant of Integration

$$\boxed{\left[\ddot{\mathbf{r}} + \frac{\mu}{r^3}\mathbf{r}\right] \times \mathbf{h} = \ddot{\mathbf{r}} \times \mathbf{h} + \frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = \mathbf{0}}$$

$$\boxed{\ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})}$$

With triple vector product identity *(see Supplement)*

$$\boxed{\ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = -\frac{\mu}{r^2}(\dot{r}\mathbf{r} - r\dot{\mathbf{r}}) = \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)}$$

Integrating

$$\boxed{\int(\ddot{\mathbf{r}} \times \mathbf{h})dt = \dot{\mathbf{r}} \times \mathbf{h} = \mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right)}$$

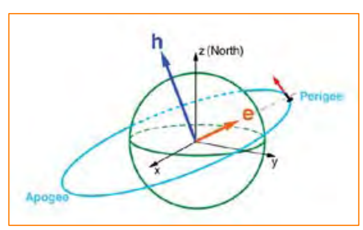
$$\boxed{\mathbf{e} = \quad \textbf{Eccentricity vector} \quad (\textit{Constant of integration})}$$

Significance of Eccentricity Vector

$$\boxed{\left[\dot{\mathbf{r}} \times \mathbf{h} - \mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right)\right]^T \mathbf{h} = \mathbf{0} \text{ because } \left[\dot{\mathbf{r}} \times \mathbf{h} - \mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right)\right] = \mathbf{0}}$$

$$\boxed{\therefore \underbrace{\cancel{(\dot{\mathbf{r}} \times \mathbf{h})^T \mathbf{h}}}_{0} - \underbrace{\cancel{\frac{\mu\mathbf{r}^T\mathbf{h}}{r}}}_{0} - \mu\mathbf{e}^T\mathbf{h} = \mathbf{0}}$$

$$\boxed{\therefore -\mu\mathbf{e}^T\mathbf{h} = \mathbf{0}}$$



- **e is perpendicular to angular momentum,**
- **which means it lies in the orbital plane**
- **Its angle provides a reference direction for the perigee**

General Polar Equation of a Conic Section

$$\boxed{\mathbf{r}^T\left[\dot{\mathbf{r}}\times\mathbf{h}-\mu\left(\frac{\mathbf{r}}{r}+\mathbf{e}\right)\right]=\mathbf{0}}$$

1st term is angular momentum squared

$$\boxed{\mathbf{r}^T\left(\dot{\mathbf{r}}\times\mathbf{h}\right)=\mathbf{h}^T\left(\mathbf{r}\times\dot{\mathbf{r}}\right)=\mathbf{h}^T\mathbf{h}=h^2}$$

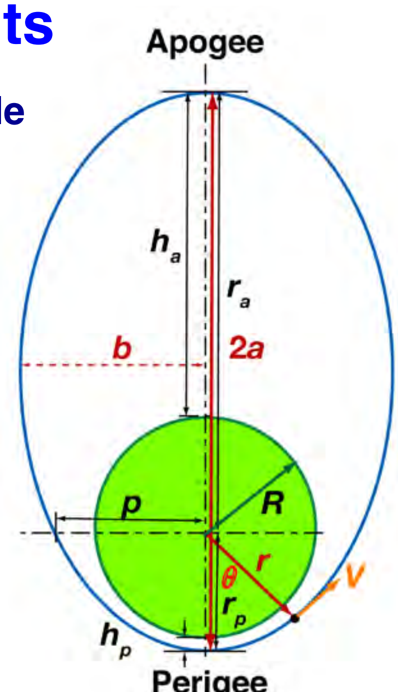
Then

$$\boxed{h^2-\mu\left(\frac{\mathbf{r}^T\mathbf{r}}{r}+\mathbf{r}^T\mathbf{e}\right)=0} \quad \boxed{h^2=\mu\left(r+\mathbf{r}^T\mathbf{e}\right)=\mu\left(r+re\cos\theta\right)=0}$$

$$\boxed{r=\frac{h^2/\mu}{1+e\cos\theta}}$$

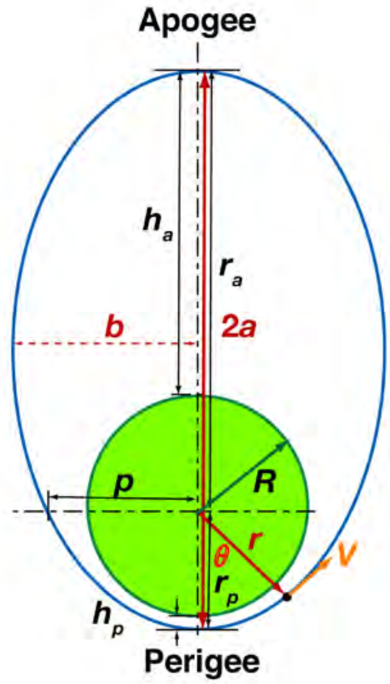
Elliptical Planetary Orbits

- **Assume satellite mass is negligible compared to Earth's mass**
- **Then**
  - **Center of mass of the 2 bodies is at Earth's center of mass**
  - **Center of mass is at <u>one of ellipse's focal points</u>**
  - **Other focal point is "vacant"**



$$\boxed{\begin{gathered}r=\frac{p}{1+e\cos\theta}=\frac{h^2/\mu}{1+e\cos\theta},\quad \text{m or km}\\ \theta:\ \textbf{True Anomaly}=\\ \text{Angle from perigee direction, deg or rad}\end{gathered}}$$

Properties of Elliptical Orbits



Eccentricity **can be determined from apogee and perigee radii**

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{r_a - r_p}{2a}$$

$$r_p = a/(1-e) \qquad r_a = a/(1+e)$$

Semi-major axis **is the average of the two**

$$a = \frac{r_a + r_p}{2}$$

Properties of Elliptical Orbits

- ***Semi-latus rectum, p,*** **can be expressed as a function of *h* or *a* and *e***

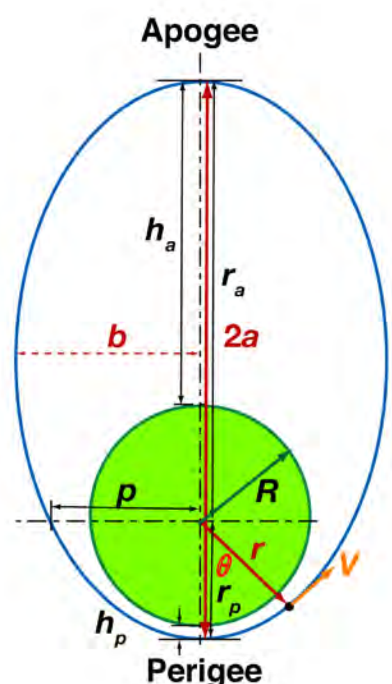
$$p = h^2/\mu = a\left(1-e^2\right)$$

- ***Semi-minor axis, b,*** **can be expressed as a function of r_a and r_p**

$$b = \sqrt{r_a r_p}$$

- ***Area of the ellipse, A,*** **is**

$$A = \pi\, a\, b = \pi a^2 \sqrt{1-e^2}, \quad \mathrm{m}^2$$



Energy is Inversely Proportional to the Semi-Major Axis

At the periapsis, r_p

$$\dot{r}_p = 0 \text{ and } v = r_p \dot{\theta}_p$$

$$\mathbb{E}_S \triangleq \mathbb{E} = \frac{1}{2}\left(r_p \dot{\theta}_p\right)^2 - \frac{\mu}{r_p} = \frac{1}{2}\frac{h^2}{r_p^2} - \frac{\mu}{r_p}$$

$$p = h^2/\mu = a\left(1-e^2\right)$$

$$r_p = a\left(1-e\right)$$

$$\mathbb{E} = \frac{1}{2r_p^2}\left(\mu p - 2\mu r_p\right) = \frac{\mu}{2a\left(1-e\right)^2}\left[\left(1-e^2\right) - 2\left(1-e\right)\right]$$

$$= -\frac{\mu\left(1-e\right)}{2a\left(1-e\right)}$$

$$\boxed{\mathbb{E} = -\frac{\mu}{2a}}$$

Classification of Conic Section Orbits

Orbit Shape	Eccentricity, e	Energy, E	Semi-Major Axis, a	Semi-Latus Rectum, p
Circle	0	< 0	> 0	a
Ellipse	0 < e < 1	< 0	> 0	$a(1-e^2)$
Parabola	1	0	Undefined ($\rightarrow\infty$)	$2r_p$
Hyperbola	>1	> 0	< 0	$a(1-e^2)$

"Vis Viva *(Living Force)* Integral"

Velocity is a function of radius and specific energy

$$\boxed{\frac{1}{2}v^2 = \frac{\mu}{r} + \mathbb{E}} \qquad \boxed{v = \sqrt{2\left(\frac{\mu}{r} + \mathbb{E}\right)}}$$

- **Specific total energy, $\mathbb{E}$, is inversely proportional to the semi-major axis**

$$\boxed{\mathbb{E} = -\frac{\mu}{2a}}$$

- **Velocity is a function of radius and semi-major axis**

$$\boxed{v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}}$$

Maximum and Minimum Velocities on an Ellipse

Velocity at periapsis

$$\boxed{\begin{aligned} v_p &= \sqrt{\mu\left(\frac{2}{r_p} - \frac{1}{a}\right)} = \sqrt{\mu\left(\frac{2}{a(1-e)} - \frac{1}{a}\right)} \\ &= \sqrt{\frac{\mu}{a}\frac{(1+e)}{(1-e)}} \end{aligned}}$$

Velocity at apoapsis

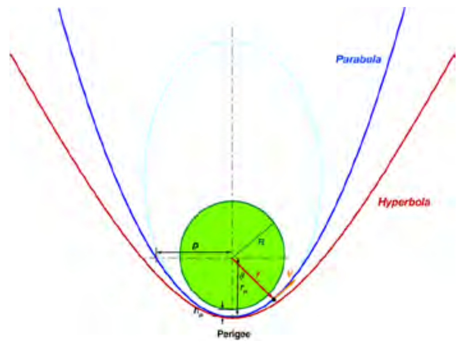
$$\boxed{v_a = \sqrt{\mu\left(\frac{2}{r_a} - \frac{1}{a}\right)} = \sqrt{\frac{\mu}{a}\frac{(1-e)}{(1+e)}}}$$

Velocities at Periapses and Infinity of Parabola and Hyperbola

Parabola (a ➔ ∞)

$$v_p = \sqrt{\frac{2\mu}{r_p}}$$

$$v_\infty = 0$$



Hyperbola (a < 0)

$$v_p = \sqrt{\mu\left(\frac{2}{r_p} - \frac{1}{a}\right)}$$

$$v_\infty = \sqrt{-\frac{\mu}{a}}$$

Relating Time and Position in Elliptical Orbit

Rearrange and integrate angular momentum over time and angle

$$h = \sqrt{\mu p} = r^2 \frac{d\theta}{dt} = 2\frac{d(\text{ Ellipse Area})}{dt}$$

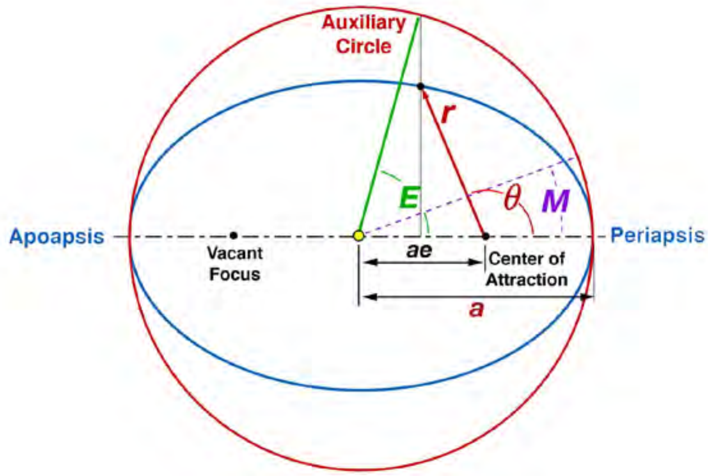
$$\int_{t_o}^{t_f} dt = \int_{\theta_o}^{\theta_f} \frac{r^2}{\sqrt{\mu p}} d\theta = \sqrt{\frac{p^3}{\mu}} \int_{\theta_o}^{\theta_f} \frac{d\theta}{(1 + e\cos\theta)^2}$$

… but difficult to integrate analytically

Anomalies of an Elliptical Orbit

Angles measured from last periapsis

θ (or ν): **True Anomaly**
E (or ψ): **Eccentric Anomaly**
M: **Mean Anomaly**



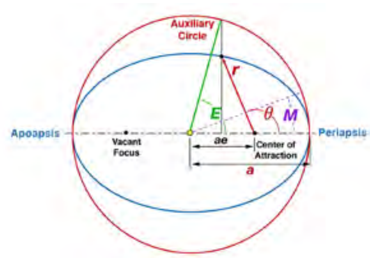
Kepler's Equation for the Mean Anomaly

$$\boxed{M = E - e\sin E}$$

From the diagram,

$$\cos E = \frac{ae + (p-r)/e}{a} = \frac{a-r}{ae}$$

$$r = a(1-\cos E)$$
$$\dot{r} = \dot{E}\, ae\sin E$$



Relationship of Time to Eccentric Anomaly

$$\mathbb{E} = -\frac{\mu}{2a} = \frac{1}{2}\left(\dot{r}^2 + \frac{h^2}{r^2}\right) - \mu r$$

$$\text{where} \quad h^2 = \mu a\left(1-e^2\right)$$

then $\quad r^2\dot{r}^2 = \mu\left[\left(\frac{2}{r}-\frac{1}{a}\right)r^2 - a\left(1-e^2\right)\right]$

leading to $\quad \frac{a^3}{\mu}(1-e\cos E)^2\left(\frac{dE}{dt}\right)^2 = 1$

or

$$\sqrt{\frac{\mu}{a^3}}dt = \left(1-e\cos E\right)dE$$

Integrating the Prior Result …

$$\int_{t_0}^{t_1}\sqrt{\frac{\mu}{a^3}}\,dt = \int_{E_0}^{E_1}\left(1-e\cos E\right)dE$$

$$\sqrt{\frac{\mu}{a^3}}\left(t_1-t_0\right) = \left(E-e\sin E\right)\Big|_{E_0}^{E_1} = M_1 - M_0$$

Time is proportional to Mean Anomaly

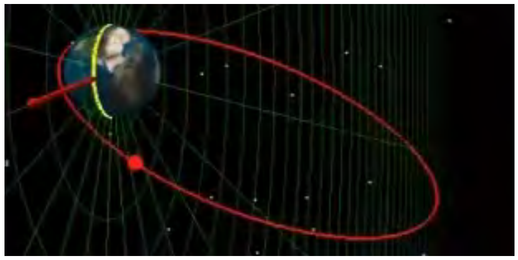
$$\left(t_1-t_0\right) = \frac{M_1-M_0}{\sqrt{\mu/a^3}} \quad \text{or} \quad t_1 = t_0 + \frac{M_1-M_0}{\sqrt{\mu/a^3}}$$

Orbital Period, *P*

$$\sqrt{\frac{\mu}{a^3}}\left(P-0\right) = \left(E-e\sin E\right)\Big|_0^{2\pi} = 2\pi$$

$$P = 2\pi\sqrt{a^3/\mu}$$

Orbital Period



Orbital period is related to the total energy

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} = \pi\sqrt{\frac{-\mu^2}{2\mathbb{E}^3}} \quad \text{where } \mathbb{E} < 0 \text{ for an ellipse}$$

Mean Motion, *n*, is the inverse of the Period

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \triangleq \frac{2\pi}{n} \quad \text{where } n \text{ is the } \textbf{Mean Motion}$$

Position and Velocity in Orbit at Time, *t*

Mean Anomaly, given time from perigee passage, t_p

$$M(t) = \sqrt{\frac{\mu}{a^3}}\left(t - t_p\right)$$

Eccentric Anomaly, *E(t)*, given Mean Anomaly, *M(t)*

$$E(t) - e\sin E(t) = M(t)$$

Newton's method of successive approximation, using M(t) as starting guess for E(t)

$$E_o(t) = M(t) + e\sin M(t)$$

$$\text{Iterate until } \Delta M_i < |\text{Tolerance}|$$

$$\Delta M_i = M(t) - \left[E_i(t) - e\sin E_i(t)\right]$$

$$\Delta E_{i+1} = \Delta M_i \Big/ \left[1 - e\cos E_i(t)\right]$$

$$E_{i+1}(t) = E_i(t) + \Delta E_{i+1}$$

Position and Velocity in Orbit at Time, *t*

Calculate True Anomaly, given Eccentric Anomaly

$$\boxed{\theta(t) = 2\tan^{-1}\left[\sqrt{\frac{1+e}{1-e}}\tan\frac{E(t)}{2}\right]}$$

Compute magnitude of radius

$$\boxed{r(t) = \frac{a\left(1-e^2\right)}{1+e\cos\theta(t)}}$$

Position and Velocity in Orbit at Time, *t*

Radius vector, in the orbital plane

$$\boxed{\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} r(t)\cos\theta(t) \\ r(t)\sin\theta(t) \end{bmatrix}}$$

Velocity vector, in the orbital plane

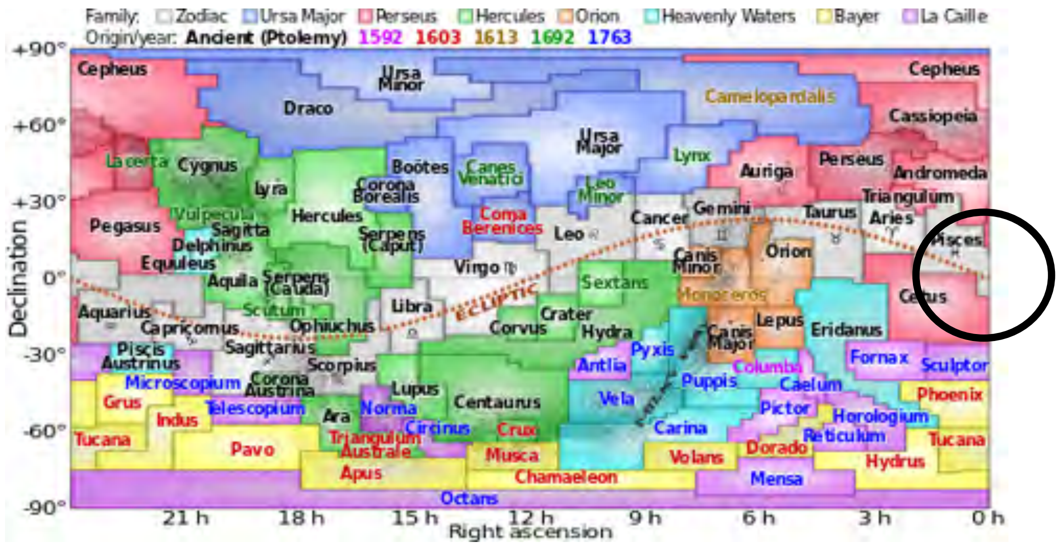
$$\boxed{\mathbf{v}(t) = \begin{bmatrix} v_x(t) \\ v_y(t) \end{bmatrix} = \sqrt{\frac{\mu}{p}}\begin{bmatrix} -\sin\theta(t) \\ e+\cos\theta(t) \end{bmatrix}}$$

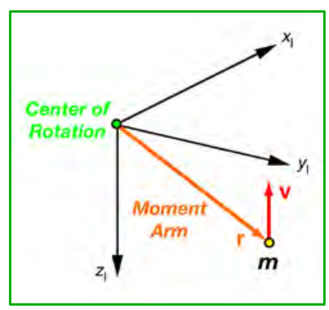
see Weisel, *Spaceflight Dynamics*, 1997, pp. 64-66

Next Time:
Planetary Defense

Supplemental Material

First Point of Aries
(Ecliptic Intercept at Right)





Dimension of energy?

Scalar (1 x 1)

Dimension of linear momentum?

Vector (3 x 1)

Dimension of angular momentum?

Vector (3 x 1)

Sub-Orbital (Sounding) Rockets

1945 - Present

MATLAB Code for Flat-Earth Trajectories

Script for Analytic Solution

```
g   =   9.8;
t   =   0:0.1:40;

vx0 =   10;
vz0 =   100;
x0  =   0;
z0  =   0;

vx1 =   vx0;
vz1 =   vz0 – g*t;
x1  =   x0 + vx0*t;
z1  =   z0 + vz0*t - 0.5*g*t.* t;
```

Script for Numerical Solution

```
tspan    = 40;  % Time span, s
xo       = [10;100;0;0];  % Init. Cond.
[t1,x1] = ode45('FlatEarth',tspan,xo);
```

Function for Numerical Solution

```
function xdot = FlatEarth(t,x)
%   x(1)    =   vx
%   x(2)    =   vz
%   x(3)    =   x
%   x(4)    =   z
    g        =    9.8;
    xdot(1) =   0;
    xdot(2) =   -g;
    xdot(3) =   x(1);
    xdot(4) =   x(2);
    xdot     =   xdot';
end
```

Trajectories Calculated with Flat-Earth Model

- Constant gravity, ***g***, is the only force in the model, i.e., no aerodynamic or thrust force
- Can neglect motions in the ***y*** direction

Dynamic Equations

$$
\begin{aligned}
v_x(t) &= v_{x_0} \\
\dot{v}_z(t) &= -g \quad (z \text{ positive up}) \\
\dot{x}(t) &= v_x(t) \\
\dot{z}(t) &= v_z(t)
\end{aligned}
$$

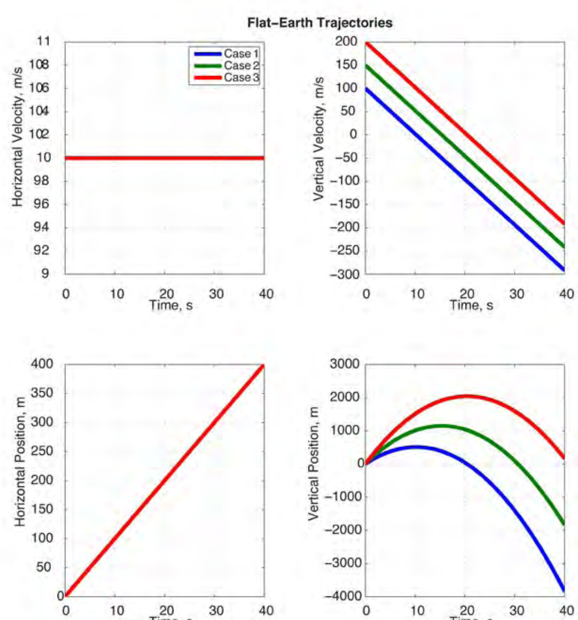
Initial Conditions

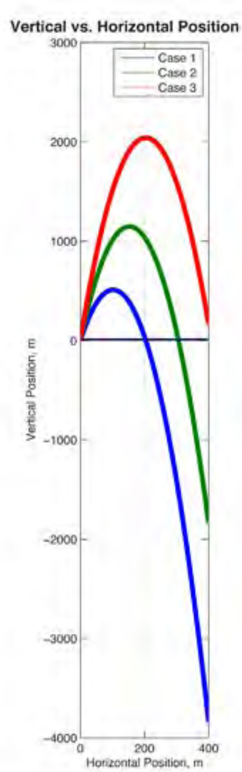
$$
\begin{aligned}
v_x(0) &= v_{x_0} \\
v_z(0) &= v_{z_0} \\
x(0) &= x_0 \\
z(0) &= z_0
\end{aligned}
$$

Analytic (Closed-Form) Solution

$$
\begin{aligned}
v_x(T) &= v_{x_0} \\
v_z(T) &= v_{z_0} - \int_0^T g\,dt = v_{z_0} - gT \\
x(T) &= x_0 + v_{x_0} T \\
z(T) &= z_0 + v_{z_0} T - \int_0^T gt\,dt = z_0 + v_{z_0} T - gT^2/2
\end{aligned}
$$

Trajectories Calculated with Flat-Earth Model





MATLAB Code for Spherical-Earth Trajectories

Script for Numerical Solution

```
R           =   6378;  %   Earth Surface Radius, km
tspan       =   6000;  %   seconds
options     =   odeset('MaxStep', 10)
xo          =   [7.5;0;0;R];
[t1,x1]     =   ode15s('RoundEarth',tspan,xo,options);
for i = 1:length(t1)
    v1(i)       =   sqrt(x1(i,1)*x1(i,1) + x1(i,2)*x1(i,2));
    r1(i)       =   sqrt(x1(i,3)*x1(i,3) + x1(i,4)*x1(i,4));
end
```

Function for Numerical Solution

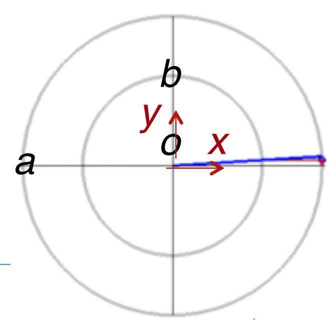
```
function xdot = RoundEarth(t,x)
%   x(1)    =   vx
%   x(2)    =   vy
%   x(3)    =   x
%   x(4)    =   y
    mu  =   3.98*10^5;  %   km^2/s^2
    r   =   sqrt(x(3)^2 + x(4)^2);

    xdot(1) =   -mu * x(3) / r^3;
    xdot(2) =   -mu * x(4) / r^3;
    xdot(3) =   x(1);
    xdot(4) =   x(2);
    xdot    =   xdot';
end
```


Equations that Describe Ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

a : **Semi-major axis**, m or km

b : **Semi-minor axis**, m or km



$$x(\theta) = a\cos(\theta)$$
$$y(\theta) = b\sin(\theta)$$

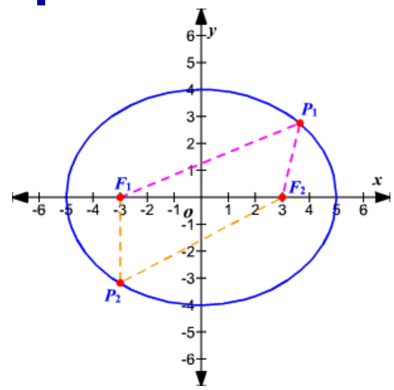
θ : **Angle from x-axis (origin at center)** rad

Constructing Ellipses

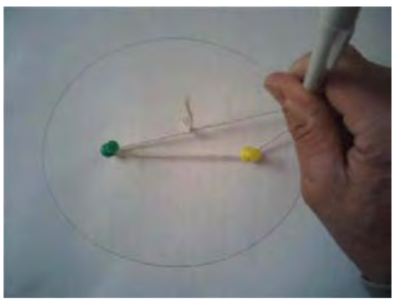
$$\boxed{F_1P_1 + F_2P_1 = F_1P_2 + F_2P_2 = 2a}$$

Foci (from center),

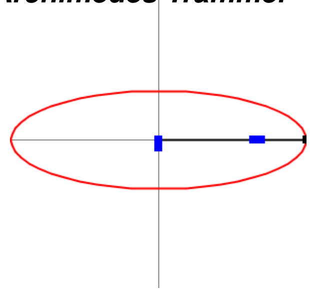
$$\begin{bmatrix} x_f \\ y_f \end{bmatrix}_{1,2} = \begin{bmatrix} -\sqrt{a^2-b^2} \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{a^2-b^2} \\ 0 \end{bmatrix}$$



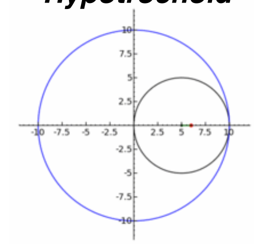
String, Tacks, and Pen



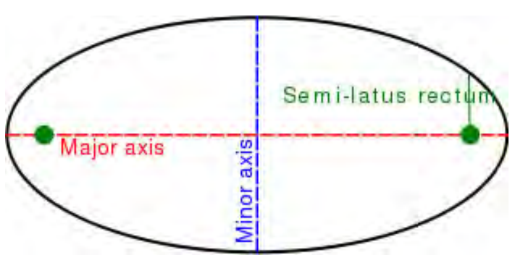
Archimedes Trammel



Hypotrochoid



Ellipses

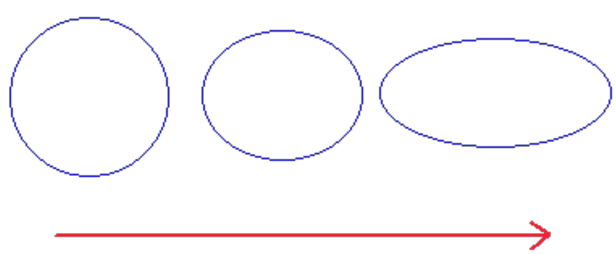


Semi-latus rectum ("The Parameter"),

$$p = \frac{b^2}{a}, \quad m$$

Eccentricity, $e = \sqrt{1 - \frac{b^2}{a^2}}$

$$\frac{b^2}{a^2} = 1 - e; \quad b = a\sqrt{1-e}$$



Increasing Eccentricity

How Do We Know that Gravitational Force is Conservative?

Because the force is the derivative (with respect to r) of a <u>scalar function</u> of r called the potential, *V*(r):

$$V(\mathbf{r}) = -m\frac{\mu}{r} + V_o = -m\frac{\mu}{\left(\mathbf{r}^T\mathbf{r}\right)^{1/2}} + V_o$$

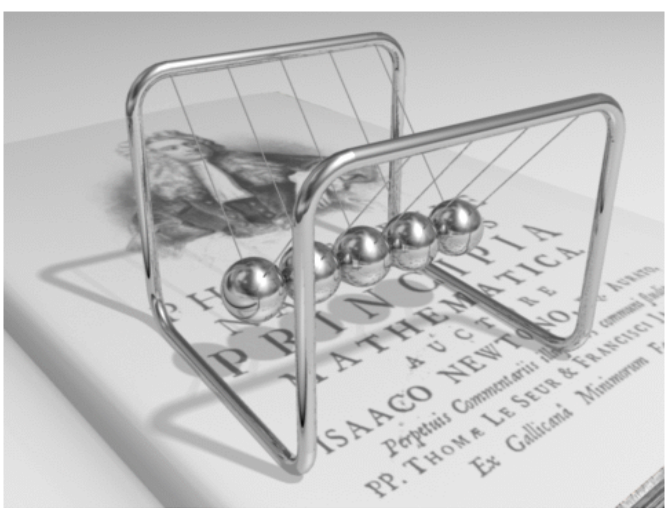
$$\frac{\partial V(\mathbf{r})}{\partial \mathbf{r}} = \begin{bmatrix} \partial V/\partial x \\ \partial V/\partial y \\ \partial V/\partial z \end{bmatrix} = m\frac{\mu}{r^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\mathbf{F}_g$$

This derivative is also called the gradient of *V* with respect to r

Conservation of Energy

Energy is conserved in an elastic collision, i.e. no losses due to friction, air drag, etc.

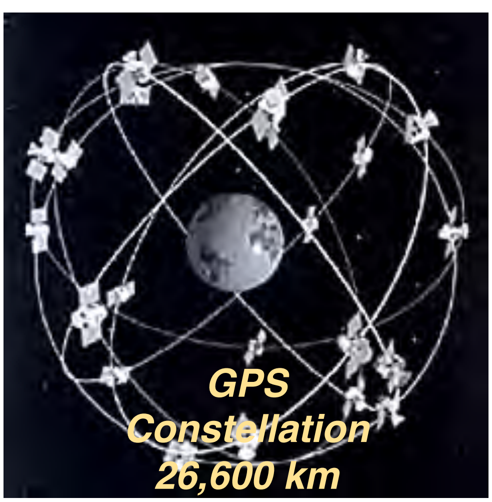
"Newton's Cradle" illustrates interchange of potential and kinetic energy in a gravitational field

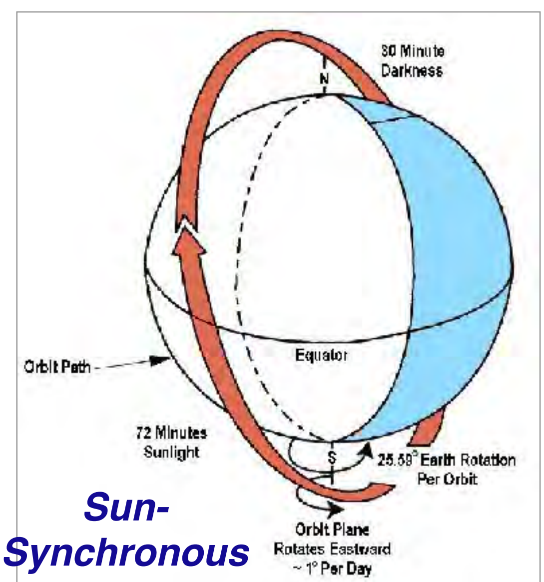


Examples of Circular Orbit Periods for Earth and Moon

Altitude above Surface, km	Period, min Earth	Period, min Moon
0	84.5	108.5
100	86.5	118
1000	105.1	214.6
10000	347.7	1905

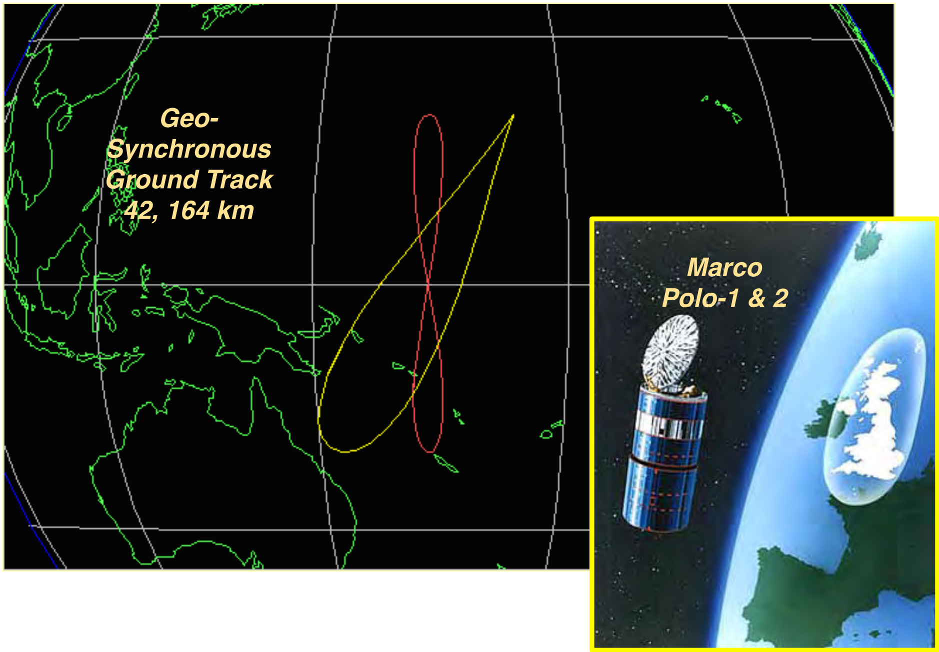
Typical Satellite Orbits





Sun-Synchronous Orbit

Geo-Synchronous Ground Track



Background Math

Triple Vector Product Identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\equiv(\mathbf{a}\bullet\mathbf{c})\mathbf{b}-(\mathbf{a}\bullet\mathbf{b})\mathbf{c}$$
$$=(\mathbf{a}^T\mathbf{c})\mathbf{b}-(\mathbf{a}^T\mathbf{b})\mathbf{c}$$

Dot Product of Radius and Rate

$$\mathbf{r}^T\dot{\mathbf{r}}=\mathbf{r}\bullet\dot{\mathbf{r}}=r\frac{dr}{dt}$$