State Estimation

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Learning Objectives

- Compute least-squares estimates of a constant vector
  - Unweighted and weighted batch processing of noisy data
  - Recursive processing to incorporate new data
- Estimate the state of an uncertain linear dynamic system with incomplete, noisy measurements
  - Discrete-time Kalman filter
  - Continuous-time Kalman-Bucy filter

Estimate Constant Vector by Inverse Transformation

- Given
  - Measurements, $\mathbf{y}$, of a constant vector, $\mathbf{x}$
- Estimate $\mathbf{x}$
- Assume that output, $\mathbf{y}$, is a perfect measurement and $\mathbf{H}$ is invertible

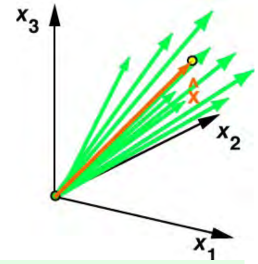
$$\boxed{\mathbf{y} = \mathbf{H}\,\mathbf{x}}$$

- $\mathbf{y}$: (n x 1) output vector
- $\mathbf{H}$: (n x n) output matrix
- $\mathbf{x}$: (n x 1) vector to be estimated

- Estimate is based on inverse transformation

$$\boxed{\hat{\mathbf{x}} = \mathbf{H}^{-1}\,\mathbf{y}}$$

Imperfect Measurement of a Constant Vector



- Given
  - "Noisy" measurements, **z**, of a constant vector, **x**
- Effects of error can be reduced if measurement is redundant
- Noise-free output, **y**

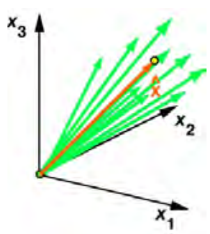
$$\boxed{\mathbf{y} = \mathbf{H}\,\mathbf{x}}$$

- **y**: (k x 1) output vector
- **H**: (k x n) output matrix, $k > n$
- **x** : (n x 1) vector to be estimated

- Measurement of output with error, **z**

$$\boxed{\mathbf{z} = \mathbf{y} + \mathbf{n} = \mathbf{H}\,\mathbf{x} + \mathbf{n}}$$

- **z**: (k x 1) measurement vector
- **n** : (k x 1) error vector



Least-Squares Estimate of a Constant Vector

- Measurement-error <u>residual</u>

$$\boxed{\boldsymbol{\varepsilon} = \mathbf{z} - \mathbf{H}\,\hat{\mathbf{x}} = \mathbf{z} - \hat{\mathbf{y}}}$$

$$\dim(\boldsymbol{\varepsilon}) = (k \times 1)$$

- Squared measurement error = cost function, J

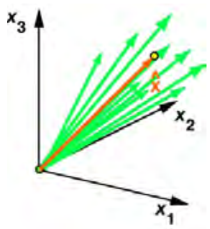
$$J = \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{z} - \mathbf{H}\,\hat{\mathbf{x}})^T(\mathbf{z} - \mathbf{H}\,\hat{\mathbf{x}})$$
$$= \frac{1}{2}\left(\mathbf{z}^T\mathbf{z} - \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{z} - \mathbf{z}^T\mathbf{H}\,\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{H}\,\hat{\mathbf{x}}\right)$$

Quadratic norm

- What is the control parameter?

The estimate of x $\boxed{\hat{\mathbf{x}}}$

$$\dim(\hat{\mathbf{x}}) = (n \times 1)$$



Least-Squares Estimate of a Constant Vector

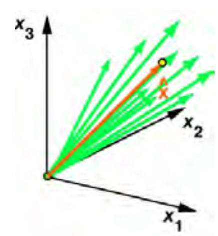
$$J = \frac{1}{2}\left(\mathbf{z}^T\mathbf{z} - \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{z} - \mathbf{z}^T\mathbf{H}\,\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{H}\,\hat{\mathbf{x}}\right)$$

- Necessary condition for a minimum

$$\frac{\partial J}{\partial \hat{\mathbf{x}}} = \mathbf{0} = \frac{1}{2}\left[\mathbf{0} - \left(\mathbf{H}^T\mathbf{z}\right)^T - \mathbf{z}^T\mathbf{H} + \left(\mathbf{H}^T\mathbf{H}\,\hat{\mathbf{x}}\right)^T + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{H}\right]$$

- The 2nd and 4th terms are transposes of the 3rd and 5th terms

Least-Squares Estimate of a Constant Vector

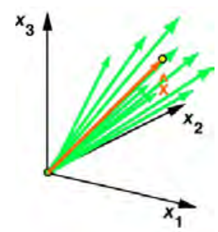


The derivative of a scalar, J, with respect to a vector, $\mathbf{x}$, (i.e., the gradient) is defined to be a row vector; thus,

$$\frac{\partial J}{\partial \hat{\mathbf{x}}} = \left[\begin{array}{cccc} \dfrac{\partial J}{\partial \hat{x}_1} & \dfrac{\partial J}{\partial \hat{x}_2} & \cdots & \dfrac{\partial J}{\partial \hat{x}_n} \end{array}\right]$$

$$= \frac{1}{2}\left[\mathbf{0} - \left(\mathbf{H}^T\mathbf{z}\right)^T - \mathbf{z}^T\mathbf{H} + \left(\mathbf{H}^T\mathbf{H}\,\hat{\mathbf{x}}\right)^T + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{H}\right]$$

$$= \left[-\mathbf{z}^T\mathbf{H} + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{H}\right] = \mathbf{0}$$



Optimal Estimate of x

Rearranging

$$\left[\hat{\mathbf{x}}^T \mathbf{H}^T \mathbf{H} - \mathbf{z}^T \mathbf{H}\right] = \mathbf{0}$$

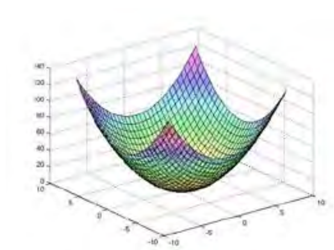
$$\hat{\mathbf{x}}^T \mathbf{H}^T \mathbf{H} = \mathbf{z}^T \mathbf{H}$$

Then, the optimal estimate is

$$\hat{\mathbf{x}}^T \left(\mathbf{H}^T \mathbf{H}\right)\left(\mathbf{H}^T \mathbf{H}\right)^{-1} = \hat{\mathbf{x}}^T (\mathbf{I}) = \hat{\mathbf{x}}^T = \mathbf{z}^T \mathbf{H}\left(\mathbf{H}^T \mathbf{H}\right)^{-1} \quad (row)$$

$$\hat{\mathbf{x}} = \left(\mathbf{H}^T \mathbf{H}\right)^{-1} \mathbf{H}^T \mathbf{z} \quad (column)$$

$(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T$ is called the *pseudoinverse* of H



Is the Least-Squares Solution a Minimum or a Maximum?

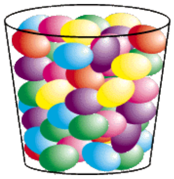
Gradient

$$\frac{\partial J}{\partial \hat{\mathbf{x}}} = \left[\begin{array}{cccc} \dfrac{\partial J}{\partial \hat{x}_1} & \dfrac{\partial J}{\partial \hat{x}_2} & \dots & \dfrac{\partial J}{\partial \hat{x}_n} \end{array}\right] = \left[-\mathbf{z}^T \mathbf{H} + \hat{\mathbf{x}}^T \mathbf{H}^T \mathbf{H}\right] = \mathbf{0}$$

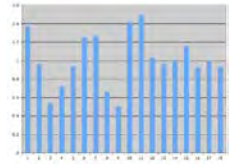
Hessian matrix

$$\frac{\partial^2 J}{\partial \hat{\mathbf{x}}^2} = \mathbf{H}^T \mathbf{H} > \mathbf{0}, \quad \dim = (n \times n)$$

A minimum



Estimation of a Scalar Constant: Average Weight of the Jelly Beans



Measurements are equally uncertain

$$z_i = x + n_i \ , i = 1 \ to \ k$$

Express measurements as

$$\mathbf{z} = \mathbf{H}x + \mathbf{n}$$

Output matrix

$$\mathbf{H} = \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}$$

Optimal estimate

$$\hat{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\,\mathbf{z}$$

Estimation of a Scalar Constant: Average Weight of the Jelly Beans

Optimal estimate

$$\hat{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\,\mathbf{z}$$

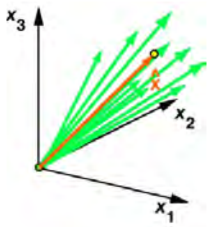
$$(1\times 1) = \left[(1\times k)(k\times 1)\right]^{-1}(1\times k)(k\times 1)$$

$$\hat{x} = \left(\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_k \end{bmatrix}$$

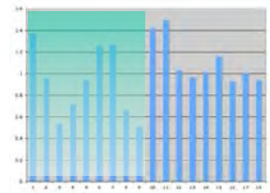
...is the average

$$\hat{x} = (k)^{-1}\left(z_1 + z_2 + \dots + z_k\right)$$

$$\hat{x} = \frac{1}{k}\sum_{i=1}^{k} z_i \quad \left[\textit{sample mean value}\right]$$



Measurements of Differing Quality



Original cost function, J, and optimal estimate of x

$$J = \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{z} - \mathbf{H}\,\hat{\mathbf{x}})^T(\mathbf{z} - \mathbf{H}\,\hat{\mathbf{x}})$$
$$= \frac{1}{2}\left(\mathbf{z}^T\mathbf{z} - \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{z} - \mathbf{z}^T\mathbf{H}\,\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{H}\,\hat{\mathbf{x}}\right)$$

$$\hat{\mathbf{x}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\,\mathbf{z}$$

Suppose some elements of the measurement, z, are more uncertain than others

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

$\mathbf{n}$: Error vector

Give the more uncertain measurements less weight in arriving at the minimum-cost estimate

Error-Weighted Cost Function

Measurement uncertainty matrix, R (large is worse)

$$\mathbf{R} = \begin{bmatrix} \text{(large error)} & 0 & \ldots & 0 \\ 0 & \text{(small error)} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \text{(medium error)} \end{bmatrix}$$

Error-weighting matrix, $\mathbf{R}^{-1}$

$$\mathbf{R}^{-1} = \begin{bmatrix} \text{(low weight)} & 0 & \ldots & 0 \\ 0 & \text{(high weight)} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \text{(medium weight)} \end{bmatrix}$$

Weighted cost function, J, reduces significance of poorer measurements

$$J = \frac{1}{2}\boldsymbol{\varepsilon}^T\mathbf{R}^{-1}\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{z} - \mathbf{H}\,\hat{\mathbf{x}})^T\mathbf{R}^{-1}(\mathbf{z} - \mathbf{H}\,\hat{\mathbf{x}})$$
$$= \frac{1}{2}\left(\mathbf{z}^T\mathbf{R}^{-1}\mathbf{z} - \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{z} - \mathbf{z}^T\mathbf{R}^{-1}\mathbf{H}\,\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\,\hat{\mathbf{x}}\right)$$

Weighted Least-Squares Estimate of a Constant Vector

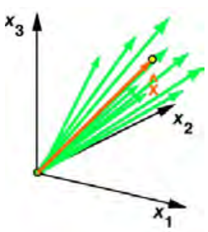
Weighted cost function, *J*

$$J = \frac{1}{2}\boldsymbol{\varepsilon}^T \mathbf{R}^{-1}\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{z} - \mathbf{H}\,\hat{\mathbf{x}}\right)^T \mathbf{R}^{-1}\left(\mathbf{z} - \mathbf{H}\,\hat{\mathbf{x}}\right)$$

$$= \frac{1}{2}\left(\mathbf{z}^T\mathbf{R}^{-1}\mathbf{z} - \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{z} - \mathbf{z}^T\mathbf{R}^{-1}\mathbf{H}\,\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\,\hat{\mathbf{x}}\right)$$

Necessary condition for a minimum

$$\frac{\partial J}{\partial \hat{\mathbf{x}}} = \mathbf{0} = \frac{1}{2}\left[\mathbf{0} - \left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{z}\right)^T - \mathbf{z}^T\mathbf{R}^{-1}\mathbf{H} + \left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\,\hat{\mathbf{x}}\right)^T + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right]$$



Weighted Least-Squares Estimate of a Constant Vector

Necessary condition for a minimum

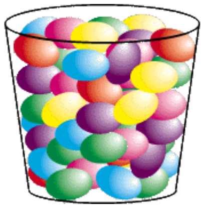
$$\frac{\partial J}{\partial \hat{\mathbf{x}}} = \mathbf{0} = \frac{1}{2}\left[\mathbf{0} - \left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{z}\right)^T - \mathbf{z}^T\mathbf{R}^{-1}\mathbf{H} + \left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\,\hat{\mathbf{x}}\right)^T + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right]$$

$$\left[\hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} - \mathbf{z}^T\mathbf{R}^{-1}\mathbf{H}\right] = \mathbf{0}$$

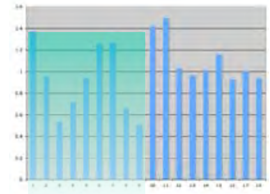
$$\hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} = \mathbf{z}^T\mathbf{R}^{-1}\mathbf{H}$$

The weighted optimal estimate is

$$\hat{\mathbf{x}} = \left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{R}^{-1}\,\mathbf{z}$$



Weighted Estimate of Average Jelly Bean Weight



Error-weighting matrix based on standard deviations

$$\mathbf{R}^{-1} = \mathbf{A} = \begin{bmatrix} 1/\sigma_{n_1}^2 & 0 & \ldots & 0 \\ 0 & 1/\sigma_{n_2}^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1/\sigma_{n_k}^2 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{kk} \end{bmatrix}$$

Optimal estimate of average jelly bean weight

$$\hat{\mathbf{x}} = \left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{z}$$

$$\hat{x} = \left(\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{kk} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{kk} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \ldots \\ z_k \end{bmatrix}$$

$$\hat{x} = \frac{\sum_{i=1}^{k} a_{ii} z_i}{\sum_{i=1}^{k} a_{ii}}$$

Recursive Least-Squares Estimation of Constant Vector, x

- "Batch-processing" approach
  - All information is gathered prior to processing
  - All information is processed at once

- Recursive approach
  - Optimal estimate has been made from prior measurement set
  - New measurement set is obtained
  - Optimal estimate is improved by incremental change (or correction) to the prior optimal estimate

Addition of New Measurement

Initial measurement set and state estimate

$$\mathbf{z}_1 = \mathbf{H}_1\mathbf{x} + \mathbf{n}_1, \quad \dim(\mathbf{z}_1) = k_1 \times 1$$

$$\hat{\mathbf{x}}_1 = \left(\mathbf{H}_1^T \mathbf{R}_1^{-1} \mathbf{H}_1\right)^{-1} \mathbf{H}_1^T \mathbf{R}_1^{-1} \mathbf{z}_1$$

$\mathbf{R}_1$: Error covariance of 1st measurement

New measurement set

$$\mathbf{z}_2 = \mathbf{H}_2\mathbf{x} + \mathbf{n}_2, \quad \dim(\mathbf{z}_2) = k_2 \times 1$$

$\mathbf{R}_2$: Error covariance of 2nd measurement

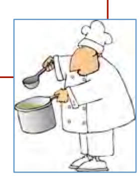
Improved Estimate Incorporating New Measurement Set

$$\mathbf{P}_1^{-1} \triangleq \mathbf{H}_1^T \mathbf{R}_1^{-1} \mathbf{H}_1 \qquad \hat{\mathbf{x}}_1 = \mathbf{P}_1 \mathbf{H}_1^T \mathbf{R}_1^{-1} \mathbf{z}_1$$

New estimate is a correction to the old

$$\hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_1 - \mathbf{P}_1 \mathbf{H}_2^T \left(\mathbf{H}_2 \mathbf{P}_1 \mathbf{H}_2^T + \mathbf{R}_2\right)^{-1} \left(\mathbf{z}_2 - \mathbf{H}_2 \hat{\mathbf{x}}_1\right)$$

$$\triangleq \hat{\mathbf{x}}_1 - \mathbf{K}\left(\mathbf{z}_2 - \mathbf{H}_2 \hat{\mathbf{x}}_1\right)$$



$\mathbf{K}$: Estimator gain matrix

$$= \mathbf{P}_1 \mathbf{H}_2^T \left(\mathbf{H}_2 \mathbf{P}_1 \mathbf{H}_2^T + \mathbf{R}_2\right)^{-1}$$

See reading for details

Recursive Optimal Estimate of Constant Vector, x

- Prior estimate may be based on prior incremental estimate, and so on
- Generalize to a recursive form, with sequential index *i*

$$\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_{i-1} - \mathbf{K}_i\left(\mathbf{z}_i - \mathbf{H}_i\hat{\mathbf{x}}_{i-1}\right)$$

with

$$\mathbf{K}_i = \mathbf{P}_{i-1}\mathbf{H}_i^T\left(\mathbf{H}_i\mathbf{P}_{i-1}\mathbf{H}_i^T + \mathbf{R}_i\right)^{-1}$$

$$\mathbf{P}_i = \left(\mathbf{P}_{i-1}^{-1} + \mathbf{H}_i^T\mathbf{R}_i^{-1}\mathbf{H}_i\right)^{-1}$$

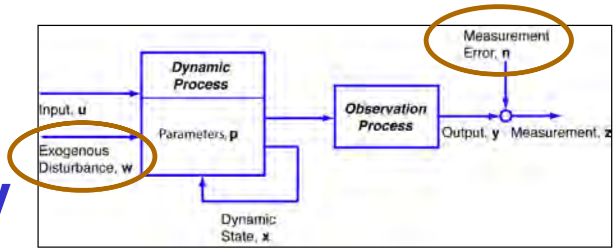
$\dim(\mathbf{x}) = n \times 1;\quad \dim(\mathbf{P}) = n \times n$

$\dim(\mathbf{z}) = r \times 1;\quad \dim(\mathbf{R}) = r \times r$

$\dim(\mathbf{H}) = r \times n;\quad \dim(\mathbf{K}) = n \times r$

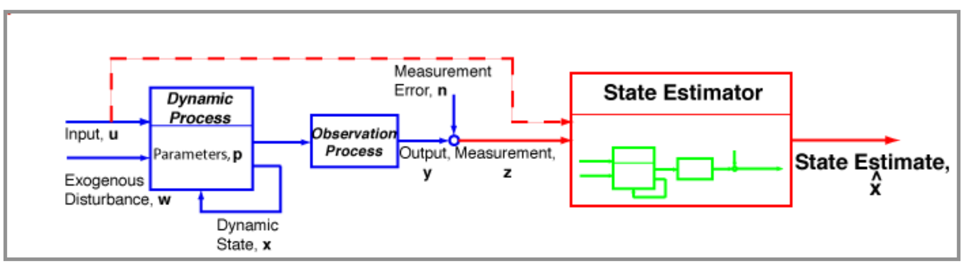
Dynamic Sampled-Data Systems with Uncertain Inputs and Disturbances

Systems with Uncertainty



- **x** is not constant in a dynamic system
- Dynamic systems may have uncertain
  - Initial conditions
  - Inputs
  - Measurements
  - System parameters or dynamic structure
- **Design goal:** estimate the state with minimum expected error
  - Mean value ➔ actual value of the state
  - Expected value of estimate error as small as possible

State Estimation



- **Goals**
  - Minimize effects of <u>measurement error</u> on knowledge of the state
  - Recontruct full state from <u>reduced measurement set</u> ($r \leq n$)
  - Average <u>redundant measurements</u> ($r \geq n$) to produce estimate of the full state
- **Method**
  - Provide optimal balance between measurements and estimates based on the dynamic model alone
  - Continuous- or discrete-time implementation

Uncertain Continuous-Time Linear Dynamic Model

Continuous-time LTI model with known coefficients

$$\dot{\mathbf{x}}(t)=\mathbf{F}\mathbf{x}(t)+\mathbf{G}\mathbf{u}(t)+\mathbf{L}\mathbf{w}(t), \quad \mathbf{x}(t_o)\ given$$

$$\mathbf{x}(t)=\mathbf{x}(t_o)+\int_{t_o}^{t}\left[\mathbf{F}\mathbf{x}(\tau)+\mathbf{G}\mathbf{u}(\tau)+\mathbf{L}\mathbf{w}(\tau)\right]d\tau$$

$$\mathbf{y}(t)=\mathbf{H}_{\mathbf{x}}\mathbf{x}(t)+\mathbf{H}_{\mathbf{u}}\mathbf{u}(t): \ \text{Output vector}$$

$$\mathbf{z}(t)=\mathbf{y}(t)+\mathbf{n}(t): \ \text{Measurement vector}$$

Initial condition and disturbance inputs are not known precisely

Measurement of state is transformed and is subject to error

Uncertain Sampled-Data Linear Dynamic Model

Discrete-time LTI model with known coefficients

$$\mathbf{x}_k=\mathbf{\Phi}\mathbf{x}_{k-1}+\mathbf{\Gamma}\mathbf{u}_{k-1}+\mathbf{\Lambda}\mathbf{w}_{k-1}$$

$$\mathbf{y}_k=\mathbf{H}_{\mathbf{x}}\mathbf{x}_k+\mathbf{H}_u\mathbf{u}_k$$

$$\mathbf{z}_k=\mathbf{y}_k+\mathbf{n}_k$$

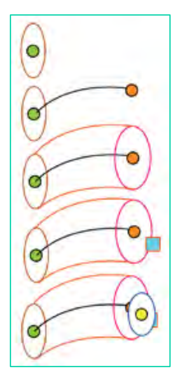
Equivalent to the continuous-time model at sampling instants

Optimal Sampled-Data State Estimation

see Supplemental Material for continuous-time filter and example

Discrete-Time Linear-Optimal State Estimation

- **Kalman filter is the optimal estimator for <u>discrete-time</u> linear systems with Gaussian uncertainty**
- **It has five equations**
  1) **State estimate extrapolation**
  2) **Covariance estimate extrapolation**
  3) **Filter gain computation**
  4) **State estimate update**
  5) **Covariance estimate "update"**



- **Notation**

$\hat{\mathbf{x}}_k(-)$: Estimate at k^{th} instant **before** measurement update

$\hat{\mathbf{x}}_k(+)$: Estimate at k^{th} instant **after** measurement update

Equations of the Kalman Filter

1) State estimate extrapolation (or propagation)

$$\boxed{\hat{\mathbf{x}}_k(-) = \mathbf{\Phi}_{k-1}\,\hat{\mathbf{x}}_{k-1}(+) + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1}}$$

2) Covariance estimate extrapolation (or propagation)

$$\boxed{\mathbf{P}_k(-) = \mathbf{\Phi}_{k-1}\,\mathbf{P}_{k-1}(+)\mathbf{\Phi}_{k-1}^T + \mathbf{Q}_{k-1}}$$

Equations of the Kalman Filter

3) Filter gain computation

$$\boxed{\mathbf{K}_k = \mathbf{P}_k(-)\mathbf{H}_k^T\left[\mathbf{H}_k\mathbf{P}_k(-)\mathbf{H}_k^T + \mathbf{R}_k\right]^{-1}}$$

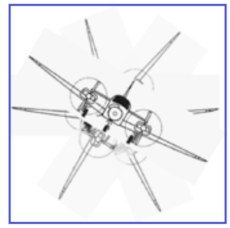
4) State estimate update

$$\boxed{\hat{\mathbf{x}}_k(+) = \hat{\mathbf{x}}_k(-) + \mathbf{K}_k\left[\mathbf{z}_k - \mathbf{H}_k\hat{\mathbf{x}}_k(-)\right]}$$

5) Covariance estimate “update”

$$\boxed{\mathbf{P}_k(+) = \left[\mathbf{P}_k^{-1}(-) + \mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k\right]^{-1}}$$

Example: Estimate Rolling Motion of an Airplane



Continuous-time model

$$\begin{bmatrix} \Delta\dot{p} \\ \Delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta\phi \end{bmatrix} + \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix} \Delta\delta A + \begin{bmatrix} L_p \\ 0 \end{bmatrix} \Delta p_w$$

$\begin{bmatrix} \Delta p \\ \Delta\phi \end{bmatrix} = \begin{bmatrix} \text{Roll rate, rad/s} \\ \text{Roll angle, rad} \end{bmatrix}$

$\Delta\delta A$ = Aileron deflection, rad

Δp_w = Turbulence disturbance, rad/s

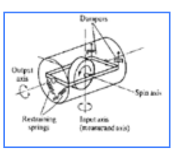
Discrete-time model

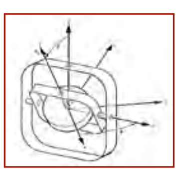
$$\begin{bmatrix} \Delta p_k \\ \Delta\phi_k \end{bmatrix} = \begin{bmatrix} e^{L_p T} & 0 \\ \dfrac{\left(e^{L_p T} - 1\right)}{L_p} & 1 \end{bmatrix} \begin{bmatrix} \Delta p_{k-1} \\ \Delta\phi_{k-1} \end{bmatrix} + \begin{bmatrix} \sim L_{\delta A} T \\ 0 \end{bmatrix} \Delta\delta A_{k-1} + \begin{bmatrix} \sim L_p T \\ 0 \end{bmatrix} \Delta p_{w_{k-1}}$$

$$= \begin{bmatrix} \varphi_{11} & 0 \\ \varphi_{21} & 1 \end{bmatrix} \begin{bmatrix} \Delta p_{k-1} \\ \Delta\phi_{k-1} \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ 0 \end{bmatrix} \Delta\delta A_{k-1} + \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} \Delta p_{w_{k-1}}$$

T = sampling interval, s

Kalman Filter Example





Rate and Angle Measurement

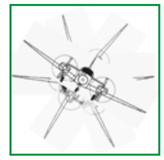
$$\begin{matrix} \text{Roll rate gyro, rad/s} \\ \text{Roll angle gyro, rad} \end{matrix} \begin{bmatrix} \Delta p_M \\ \Delta\phi_M \end{bmatrix}_k = \begin{bmatrix} \Delta p + \Delta n_p \\ \Delta\phi + \Delta n_\phi \end{bmatrix}_k$$

$$= \mathbf{I}\Delta\mathbf{x}_k + \Delta\mathbf{n}_k$$

1) State Estimate Extrapolation

$$\begin{bmatrix} \Delta\hat{p}_k(-) \\ \Delta\hat{\phi}_k(-) \end{bmatrix} = \begin{bmatrix} \varphi_{11} & 0 \\ \varphi_{21} & 1 \end{bmatrix} \begin{bmatrix} \Delta\hat{p}_{k-1}(+) \\ \Delta\hat{\phi}_{k-1}(+) \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ 0 \end{bmatrix} \Delta\delta A_{k-1}$$

Kalman Filter Example



2) Covariance Extrapolation

$$\begin{bmatrix} p_{11}(-) & p_{12}(-) \\ p_{21}(-) & p_{22}(-) \end{bmatrix}_k = \begin{bmatrix} \varphi_{11} & 0 \\ \varphi_{21} & 1 \end{bmatrix} \begin{bmatrix} p_{11}(+) & p_{12}(+) \\ p_{21}(+) & p_{22}(+) \end{bmatrix}_{k-1} \begin{bmatrix} \varphi_{11} & \varphi_{21} \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_p^2 & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$\mathbf{Q}_{k-1} \approx \begin{bmatrix} L_p \\ 0 \end{bmatrix} Q'_C \begin{bmatrix} L_p & 0 \end{bmatrix} T = \begin{bmatrix} \sigma_p^2 & 0 \\ 0 & 0 \end{bmatrix}$$

3) Gain Computation

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_k = \begin{bmatrix} p_{11}(-) & p_{12}(-) \\ p_{21}(-) & p_{22}(-) \end{bmatrix}_k \left\{ \begin{bmatrix} p_{11}(-) & p_{12}(-) \\ p_{21}(-) & p_{22}(-) \end{bmatrix}_k + \begin{bmatrix} \sigma_{p_M}^2 & 0 \\ 0 & \sigma_{\phi_M}^2 \end{bmatrix}_k \right\}^{-1}$$

where

$$\mathbf{R}_k \delta_{jk} = \begin{bmatrix} \sigma_{p_M}^2 & 0 \\ 0 & \sigma_{\phi_M}^2 \end{bmatrix}_k$$

Kalman Filter Example

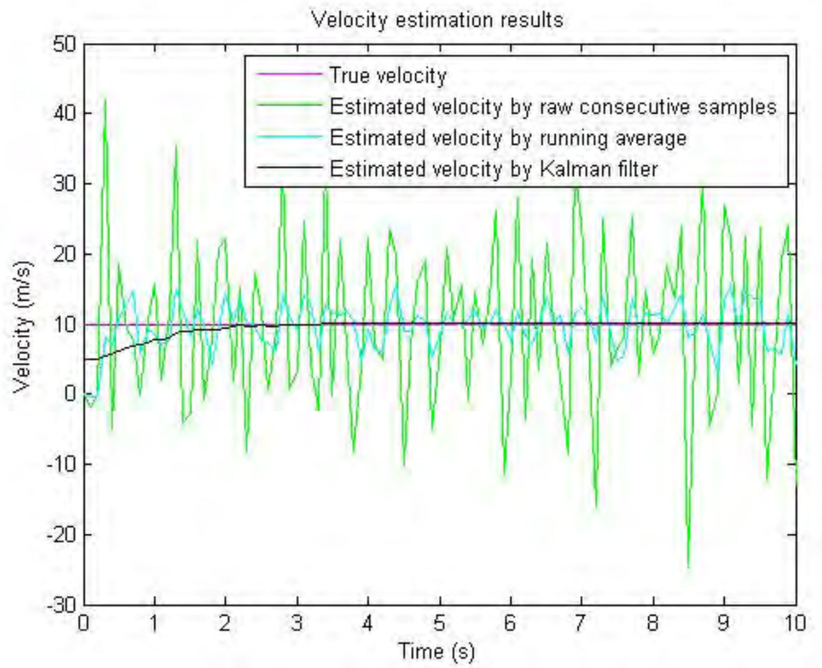
4) State Estimate Update

$$\begin{bmatrix} \Delta\hat{p}_k(+) \\ \Delta\hat{\phi}_k(+) \end{bmatrix} = \begin{bmatrix} \Delta\hat{p}_k(-) \\ \Delta\hat{\phi}_k(-) \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_k \left\{ \begin{bmatrix} \Delta p_{M_k} \\ \Delta\phi_{M_k} \end{bmatrix} - \begin{bmatrix} \Delta\hat{p}_k(-) \\ \Delta\hat{\phi}_k(-) \end{bmatrix} \right\}$$

5) Covariance “Update”

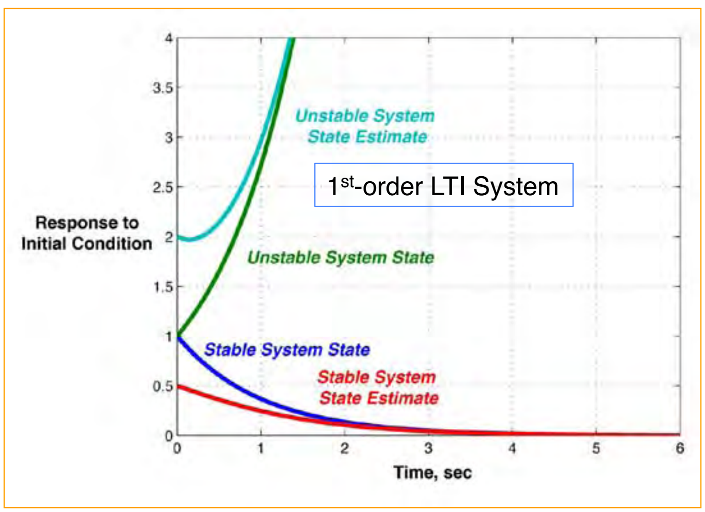
$$\begin{bmatrix} p_{11}(+) & p_{12}(+) \\ p_{21}(+) & p_{22}(+) \end{bmatrix}_k = \left\{ \begin{bmatrix} p_{11}(-) & p_{12}(-) \\ p_{21}(-) & p_{22}(-) \end{bmatrix}_k + \begin{bmatrix} \dfrac{1}{\sigma_{p_M}^2} & 0 \\ 0 & \dfrac{1}{\sigma_{\phi_M}^2} \end{bmatrix}_k \right\}^{-1}$$

Comparison of Running Average and Kalman Estimate of Velocity from Position Measurement



Kalman Filter Estimate is Stable

Estimate is stable even if the dynamic system is unstable



Next Time:
Stochastic Control

Supplemental Material

Continuous-Time Linear-Optimal State Estimation

Continuous-time linear dynamic process with random disturbance

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t)}$$

Measurement with random error

$$\boxed{\mathbf{z}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{n}(t)}$$

Uncertainty model for initial condition, disturbance input, and measurement error

$$\begin{aligned}
&\bar{\mathbf{x}}(t_0) = E\left[\mathbf{x}(t_0)\right]; \quad \mathbf{P}(t_0) = E\left\{\left[\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)\right]\left[\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)\right]^T\right\} \\
&\mathbf{u}(t) = E\left[\mathbf{u}(t)\right]; \quad \mathbf{U}(t_0) = \mathbf{0} \\
&\bar{\mathbf{w}}(t) = \mathbf{0}; \quad \mathbf{W}(t) = E\left\{\left[\mathbf{w}(t)\right]\left[\mathbf{w}(\tau)\right]^T\right\} \\
&\bar{\mathbf{n}}(t) = \mathbf{0}; \quad \mathbf{N}(t) = E\left\{\left[\mathbf{n}(t)\right]\left[\mathbf{n}(\tau)\right]^T\right\}
\end{aligned}$$

Linear-Optimal State Estimator (Kalman-Bucy Filter)

- **Optimal estimate of state**

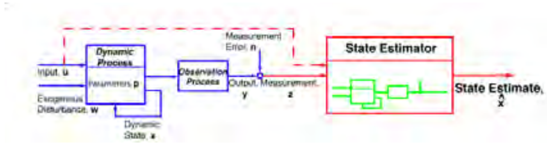
$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{K}(t)\left[\mathbf{z}(t) - \mathbf{H}\hat{\mathbf{x}}(t)\right], \quad \hat{\mathbf{x}}(t_o) = \bar{\mathbf{x}}(t_o)$$

$$\mathbf{K}(t): \text{ Optimal estimator gain matrix } (n \times r)$$

- **Two parts to the optimal state estimator**
  - **Propagation of the expected value of x**
  - **Least-squares correction to the model-based estimate**

$$\boxed{\Delta\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\Delta\hat{\mathbf{x}}(t) + \mathbf{G}\Delta\mathbf{u}(t)} \quad \boxed{+\mathbf{K}(t)\left[\Delta\mathbf{z}(t) - \mathbf{H}\Delta\hat{\mathbf{x}}(t)\right]}$$

LTI System $\boxed{\Delta\dot{\hat{\mathbf{x}}}(t) = \left[\mathbf{F} - \mathbf{K}\mathbf{H}\right]\Delta\hat{\mathbf{x}}(t) + \mathbf{G}\Delta\mathbf{u}(t) + \mathbf{K}\Delta\mathbf{z}(t)}$



Estimator Gain for the Kalman-Bucy Filter

Optimal filter gain matrix

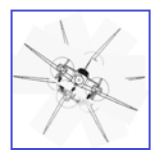
$$\boxed{\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^T\mathbf{N}^{-1}(t)}$$

Matrix Riccati equation for estimator covariance

$$\begin{aligned}\dot{\mathbf{P}}(t) &= \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{L}(t)\mathbf{W}(t)\mathbf{L}^T(t)\\ &\quad - \mathbf{P}(t)\mathbf{H}^T\mathbf{N}^{-1}\mathbf{H}\mathbf{P}(t), \quad \mathbf{P}(t_o) = \mathbf{P}_o\end{aligned}$$

- Same equations as those that define LQ control gain, except
  - Solution matrix, **P**, propagated forward in time
  - Matrices are modified

Continuous-Time 2nd-Order Example of Kalman-Bucy Filter



Rolling motion of an airplane

$$\begin{bmatrix} \dot{p}(t) \\ \dot{\phi}(t) \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ \phi(t) \end{bmatrix} + \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix} \delta A(t) + \begin{bmatrix} L_p \\ 0 \end{bmatrix} p_w(t)$$

$$\begin{bmatrix} p \\ \phi \end{bmatrix} = \begin{bmatrix} \text{Roll rate, rad/s} \\ \text{Roll angle, rad} \end{bmatrix}$$

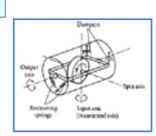
δA = Aileron deflection, rad

p_w = Turbulence disturbance, rad/s

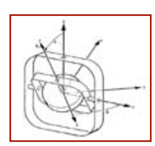
$$L_p : \begin{cases} \textit{Roll-rate damping} \\ \textit{Turbulence sensitivity} \end{cases}$$

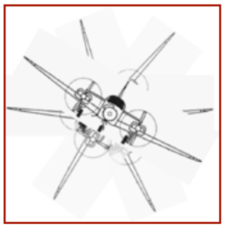
$L_{\delta A}$: *Control effectiveness*

Measurement of roll rate and angle



$$\begin{bmatrix} p_M(t) \\ \phi_M(t) \end{bmatrix}_k = \begin{bmatrix} p(t) + n_p(t) \\ \phi(t) + n_\phi(t) \end{bmatrix}_k = \mathbf{I}_2\mathbf{x}(t) + \mathbf{n}(t)$$





Second-Order Example of Kalman-Bucy Filter

Covariance extrapolation

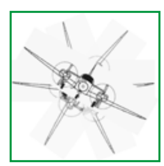
$$\begin{bmatrix} \dot{p}_{11}(t) & \dot{p}_{12}(t) \\ \dot{p}_{12}(t) & \dot{p}_{22}(t) \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} + \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} L_p & 1 \\ 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} L_p^2 \sigma_{p_W}^2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} \sigma_{p_M}^2 & 0 \\ 0 & \sigma_{\phi_M}^2 \end{bmatrix}^{-1} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}$$

Estimator gain computation

$$\begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} \sigma_{p_M}^2 & 0 \\ 0 & \sigma_{\phi_M}^2 \end{bmatrix}^{-1}$$

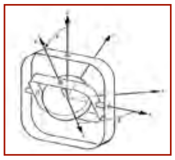
Kalman-Bucy Filter with Two Measurements



State estimate with roll rate and angle measurements

$$\begin{bmatrix} \dot{\hat{p}}(t) \\ \dot{\hat{\phi}}(t) \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{p}(t) \\ \hat{\phi}(t) \end{bmatrix} + \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix} \delta A(t)$$

$$+ \begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{bmatrix} \begin{bmatrix} p_M(t) - \hat{p}(t) \\ \phi_M(t) - \hat{\phi}(t) \end{bmatrix}$$



State Estimate with Angle Measurement Only

Covariance extrapolation

$$\begin{bmatrix} \dot p_{11}(t) & \dot p_{12}(t) \\ \dot p_{12}(t) & \dot p_{22}(t) \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} + \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} L_p & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_p^2 \sigma_{p_W}^2 & 0 \\ 0 & 0 \end{bmatrix}$$
$$- \frac{1}{\sigma_{\phi_M}^2} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}^T \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}$$

Gain computation

$$\begin{bmatrix} k_{11}(t) \\ k_{21}(t) \end{bmatrix} = \frac{1}{\sigma_{\phi_M}^2} \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{bmatrix}$$

State estimate with roll angle measurement

$$\begin{bmatrix} \dot{\hat p}(t) \\ \dot{\hat\phi}(t) \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat p(t) \\ \hat\phi(t) \end{bmatrix} + \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix} \delta A(t) + \begin{bmatrix} k_{11}(t) \\ k_{21}(t) \end{bmatrix} \left[\phi_M(t) - \hat\phi(t) \right]$$

State and Output Vectors for the Quadrotor Helicopter

Longitudinal State

$$\mathbf{x} = \begin{bmatrix} x \\ z \\ u \\ w \\ q \\ \theta \end{bmatrix} \quad \begin{matrix} \textit{Range} \\ \textit{Height} \\ \textit{Axial Velocity} \\ \textit{Normal Velocity} \\ \textit{Pitch Rate} \\ \textit{Pitch Angle} \end{matrix}$$

Longitudinal Output

$$\mathbf{y} = \begin{bmatrix} x \\ -z \\ -z \\ -z \\ \dot x \\ \dot u \\ \dot w \\ q \\ \theta \end{bmatrix} \quad \begin{matrix} \textit{Range(GPS)} \\ \textit{Height(GPS)} \\ \textit{Height(Ultrasound)} \\ \textit{Height(Pressure Sensor)} \\ \textit{Ground Speed(QVGA Camera)} \\ \textit{Axial Acceleration} \\ \textit{Normal Acceleration} \\ \textit{Pitch Rate(Gyro)} \\ \textit{Pitch Angle(Magnetometer)} \end{matrix}$$

$$\mathbf{y} \neq \mathbf{x}$$

Output Vector and Matrix for the Quadrotor Helicopter

Neglect GPS and Pressure Sensor

Longitudinal Output, Linearized at $\theta = 0$

$$\mathbf{y} = \mathbf{H_x x} + \mathbf{H_u u}$$

$$= \begin{bmatrix} -z \\ \dot{x} \\ \dot{u} \\ \dot{w} \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -g \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -2/m & -2/m \\ d/I_{yy} & -d/I_{yy} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_f \\ T_a \end{bmatrix}$$

How would you design the Kalman Filter?