Principles for Optimal Control of Dynamic Systems

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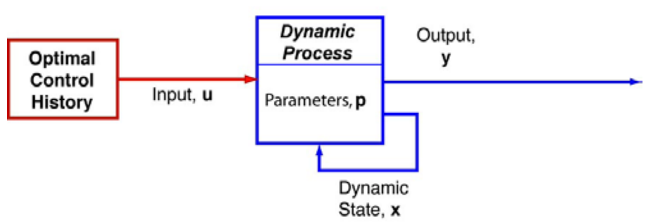
Optimal Control and Estimation, MAE 546,
Princeton University, 2015

- Dynamic systems
- Cost functions
- Problems of Lagrange, Mayer, and Bolza
- Necessary conditions for optimality
  - Euler-Lagrange equations
- Sufficient conditions for optimality
  - Convexity, normality, and uniqueness

The Dynamic Process

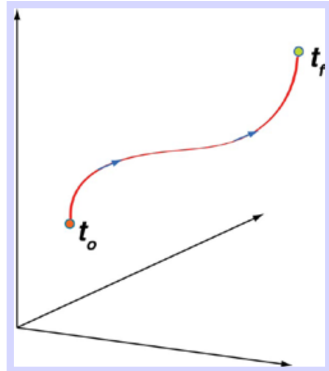


- Dynamic Process
  - Neglect disturbance effects, $\mathbf{w}(t)$
  - Subsume $\mathbf{p}(t)$ and explicit dependence on t in the definition of $\mathbf{f}[.]$

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)]$$

Trajectory of the System

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)]$$



Integrate the dynamic equation to determine the trajectory from original time, t_o, to final time, t_f

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{f}[\mathbf{x}(\tau), \mathbf{u}(\tau)]\, d\tau,$$

given $\mathbf{u}(t)$ for $t_0 \le t \le t_f$

What Cost Function Might Be Minimized?

- **Minimize time required to go from A to B**

$$J = \int_0^{\text{final time}} dt = \textbf{Final time}$$

- **Minimize fuel used to go from A to B**

$$J = \int_0^{\text{final range}} (\text{Fuel-use Efficiency})\, dR = \textbf{Fuel Used}$$

- **Minimize financial cost of producing a product**

$$J = \int_0^{\text{final time}} (\text{Cost per hour})\, dt = \$\$$$

Optimal System Regulation

Minimize mean-square state deviations over a time interval

Scalar variation of a single component

$$J = \frac{1}{T}\int_0^T \left(x^2(t)\right) dt$$

dim(x) = 1 x 1

Sum of variation of all state elements

$$J = \frac{1}{T}\int_0^T \left[\mathbf{x}^T(t)\mathbf{x}(t)\right] dt = \frac{1}{T}\int_0^T \left[x_1^2 + x_1^2 + \cdots + x_n^2\right] dt$$

dim($\mathbf{x}$) = n x 1

Weighted sum of state element variations

$$J = \frac{1}{T}\int_0^T \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t)\right] dt = \frac{1}{T}\int_0^T \left\{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right\} dt$$

$n = 3$
dim($\mathbf{x}$) = n x 1
dim($\mathbf{Q}$) = n x n

Why not use infinite control?

Tradeoffs Between State and Control Variations

Trade performance, x, against control usage, u

$$J = \int_0^T \left(x^2(t) + ru^2(t)\right) dt, \quad r > 0$$

dim(u) = 1 x 1

Minimize a cost function that contains state and control vectors

$$J = \int_0^T \left(\mathbf{x}^T(t)\mathbf{x}(t) + r\mathbf{u}^T(t)\mathbf{u}(t)\right) dt, \quad r > 0$$

dim($\mathbf{u}$) = m x 1

Weight the relative importance of state and control components

$$J = \int_0^T \left(\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right) dt, \quad \mathbf{Q},\mathbf{R} > 0$$

dim($\mathbf{R}$) = m x m

Examples

Effects of Control Weighting in Optimal Control of LTI System

$$\min_u J = \int_0^T \left(\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + ru^2(t) \right) dt, \quad \mathbf{Q}, r > 0$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}\mathbf{x}(t) + \mathbf{G}u(t)$$

$$\mathbf{x} = \begin{bmatrix} x_1, & \textit{displacement} \\ x_2, & \textit{rate} \end{bmatrix}$$

Example

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -a & b \end{bmatrix}, \quad a, b > 0 \ \ \text{[unstable]}$$

$$\mathbf{G} = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

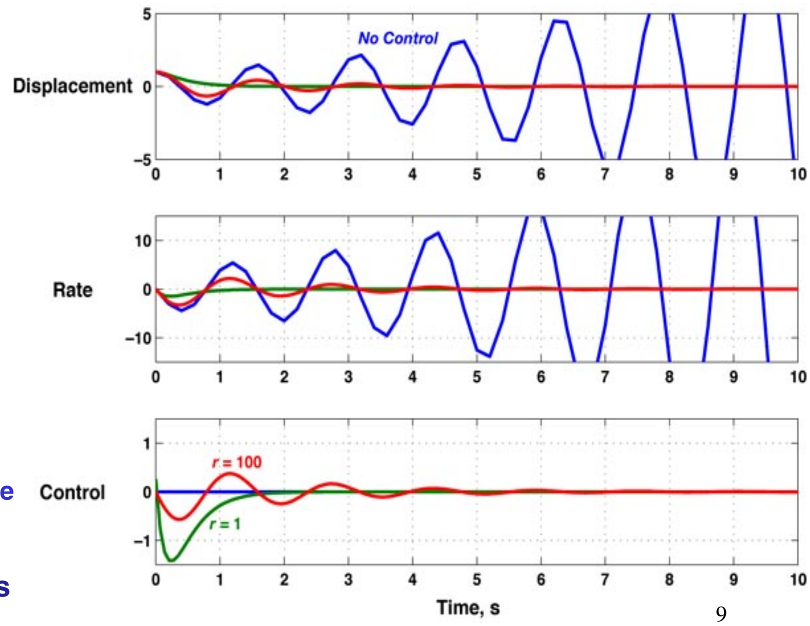
$r = 1$ or 100

Effects of Control Weighting in Optimal Control of LTI System

- **Optimal feedback control *(TBD)* stabilizes unstable system response to initial condition**

$$\frac{d\mathbf{x}}{dt} = \mathbf{Fx} + \mathbf{G}u_{optimal} = \mathbf{Fx} - \mathbf{GCx} = (\mathbf{F} - \mathbf{GC})\mathbf{x}$$

- **Smaller control weight**
  - **Allows larger control response**
  - **Decreases state variation**
- **Larger control weight conserves control energy**

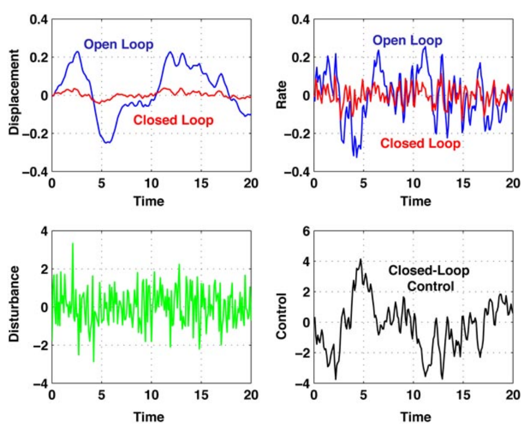


$$\mathbf{Q} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$
$$\mathbf{R} = 1$$

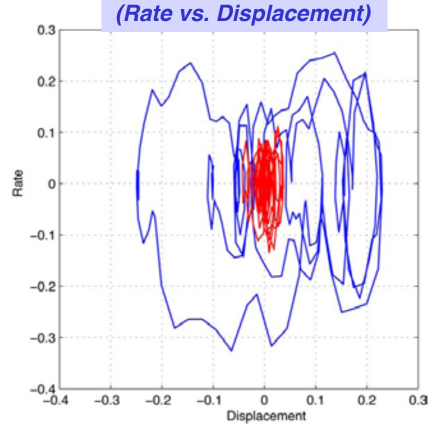
Open- and Optimal Closed-Loop Response to Disturbance

- **Stable 2nd-order linear dynamic system:** $d\mathbf{x}(t)/dt = \mathbf{Fx}(t) + \mathbf{Gu}(t) + \mathbf{Lw}(t)$
- **Optimal feedback control *(TBD)* reduces response to disturbances**

Time Response



Phase-Plane Plot
(Rate vs. Displacement)



Classical Cost Functions for Optimizing Dynamic Systems

The Problem of Lagrange
(*c. 1780*)

$$\boxed{\min_{\mathbf{u}(t)} J = \int_{t_o}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right]dt}$$

subject to
$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]\,, \quad \mathbf{x}(t_o)\ given$$

$$\dim(\mathbf{x}) = n \times 1$$
$$\dim(\mathbf{f}) = n \times 1$$
$$\dim(\mathbf{u}) = m \times 1$$

Examples of Integral Cost: the Lagrangian

$L\left[\mathbf{x}(t),\mathbf{u}(t)\right] = \left[\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right]$ **Quadratic trade between state and control**

$= 1$ **Minimum time problem**

$= \dot{m}(t) = fcn\left[\mathbf{x}(t),\mathbf{u}(t)\right]$ **Minimum fuel use problem**

$L\left[\mathbf{x}(s),\mathbf{u}(s)\right] =$ Change in area with respect to differential length, e.g., fencing, ds [Maximize]

The Problem of Mayer (*c. 1890*)

$$\boxed{\min_{\mathbf{u}(t)} J = \phi\left[\mathbf{x}(t_f)\right]}$$

subject to
$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]\,, \quad \mathbf{x}(t_o)\ given$

Examples of Terminal Cost

$$\begin{aligned}\phi\left[\mathbf{x}(t_f)\right] &= \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)\Big|_{t=t_f} && \textbf{Weighted square-error in final state}\\ &= \left|\left(t_{final} - t_{initial}\right)\right| && \textbf{Minimum time problem}\\ &= \left|\left(m_{initial} - m_{final}\right)\right| && \textbf{Minimum fuel problem}\end{aligned}$$

The Problem of Bolza (*c. 1900*)
The Modern Optimal Control Problem*

Combine the Problems of Lagrange and Mayer

- **Minimize the sum of terminal and integral costs**
  - **By choice of u(*t*)**
  - **Subject to dynamic constraint**

$$\boxed{\min_{\mathbf{u}(t)} J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right] dt}$$

subject to
$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]\,, \quad \mathbf{x}(t_o)\ given$
and with fixed end time, t_f

Augmented Cost Function

Adjoin the dynamic constraint to the integrand using a Lagrange multiplier* to form the Augmented Cost Function, J_A:

$$J_A = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} \left\{ L\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \underline{\boldsymbol{\lambda}^T(t)\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] - \dot{\mathbf{x}}(t)\right]} \right\} dt$$

$$\dim\left[\boldsymbol{\lambda}(t)\right] = \dim\left\{\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),t\right]\right\} = n \times 1$$

The Dynamic Constraint

$$\dim\left\{\boldsymbol{\lambda}^T(t)\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] - \dot{\mathbf{x}}(t)\right]\right\} = (1 \times n)(n \times 1) = 1$$

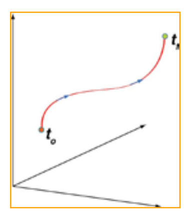
The constraint = 0 when the dynamic equation is satisfied

$$\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] - \dot{\mathbf{x}}(t)\right] = 0 \text{ when } \dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] \text{ in } \left[t_0, t_f\right]$$

* Lagrange multiplier is also called
- Adjoint vector
- Costate vector

Necessary Conditions for a Minimum

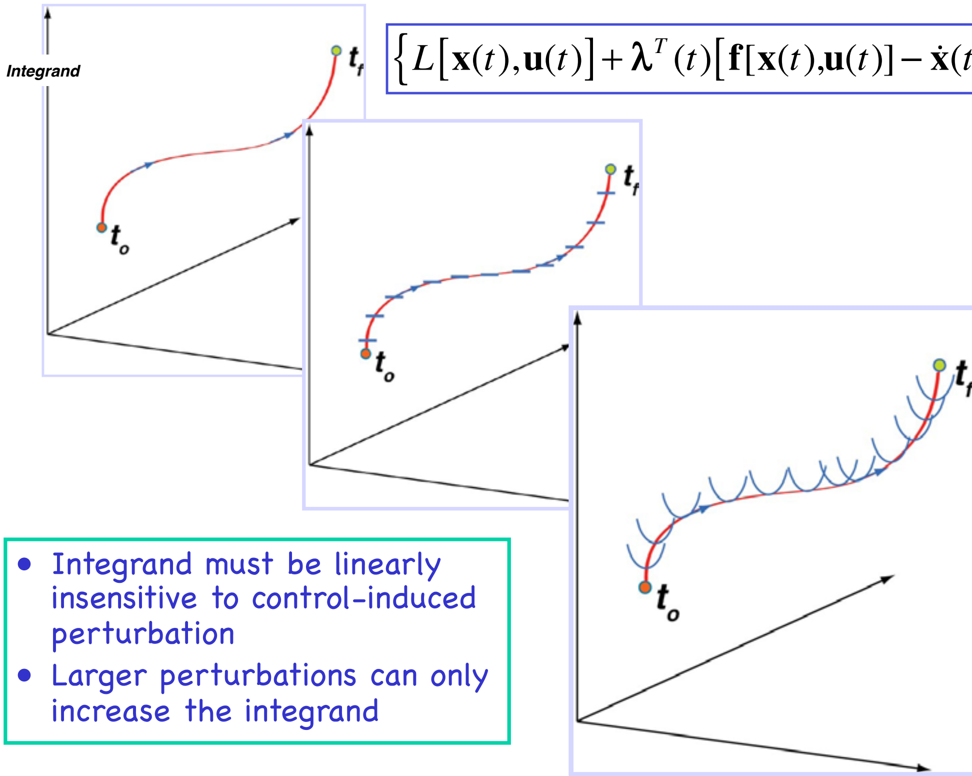
Necessary Conditions for a Minimum



- Satisfy *necessary conditions for stationarity* along entire trajectory, from t_o to t_f

- For integral to be minimized, integrand takes lowest possible value at every time
  - Linear insensitivity to small control-induced perturbations
  - Large perturbations can only increase the integral cost

- Cost is insensitive to control-induced perturbations at the final time



$$\left\{L\left[\mathbf{x}(t),\mathbf{u}(t)\right]+\boldsymbol{\lambda}^T(t)\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]-\dot{\mathbf{x}}(t)\right]\right\}$$

- Integrand must be linearly insensitive to control-induced perturbation
- Larger perturbations can only increase the integrand

The Hamiltonian

Re-phrase the integrand by introducing the Hamiltonian

$$H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right]=L\left[\mathbf{x}(t),\mathbf{u}(t)\right]+\boldsymbol{\lambda}^T(t)\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right]$$

$$\left\{L\left[\mathbf{x}(t),\mathbf{u}(t)\right]+\boldsymbol{\lambda}^T(t)\left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]-\dot{\mathbf{x}}(t)\right]\right\}=$$
$$\left\{H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right]-\boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}(t)\right\}$$

The Hamiltonian is a function of the Lagrangian, adjoint vector, and system dynamics

Incorporate the Hamiltonian in the Cost Function

- **Variations in the Hamiltonian reflect**
  - integral cost
  - constraining effect of system dynamics
- **Substitute the Hamiltonian in the cost function**

$$J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} \left\{ H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}(t) \right\} dt$$

- **The optimal cost, J^*, is produced by the optimal histories of state, control, and Lagrange multiplier: $\mathbf{x}^*(t)$, $\mathbf{u}^*(t)$, and $\lambda^*(t)$**

$$\min_{\mathbf{u}(t)} J = J^* = \phi\left[\mathbf{x}^*(t_f)\right] + \int_{t_o}^{t_f} \left\{ H\left[\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t)\right] - \boldsymbol{\lambda}^{*T}(t)\dot{\mathbf{x}}^*(t) \right\} dt$$

Integration by Parts

Scalar indefinite integral

$$\int u\,dv = uv - \int v\,du$$

Vector definite integral

$$u = \boldsymbol{\lambda}^T(t)$$
$$dv = \dot{\mathbf{x}}(t)dt = d\mathbf{x}$$

$$\int_{t_0}^{t_f} \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}(t)\,dt = \boldsymbol{\lambda}^T(t)\mathbf{x}(t)\Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \dot{\boldsymbol{\lambda}}^T(t)\mathbf{x}(t)\,dt$$

Integrate the Cost Function By Parts

$$J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} \left\{ H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}(t) \right\} dt$$

$$u = \boldsymbol{\lambda}^T(t)$$
$$dv = \dot{\mathbf{x}}(t)dt = d\mathbf{x}$$

Cost function can be re-written as

$$J = \phi\left[\mathbf{x}(t_f)\right] + \left[\boldsymbol{\lambda}^T(t_0)\mathbf{x}(t_0) - \boldsymbol{\lambda}^T(t_f)\mathbf{x}_f(t)\right]$$
$$+ \int_{t_o}^{t_f} \left\{ H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] + \dot{\boldsymbol{\lambda}}^T(t)\mathbf{x}(t) \right\} dt$$

First-Order Variations

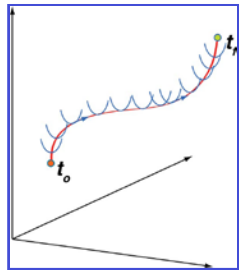
First variations in a quantity induced by control variations

$$\Delta(.) = \frac{\partial(.)}{\partial \mathbf{u}}\Delta\mathbf{u} + \frac{\partial(.)}{\partial \mathbf{x}}\Delta\mathbf{x}(\Delta\mathbf{u}) + \frac{\partial(.)}{\partial \boldsymbol{\lambda}}\Delta\boldsymbol{\lambda}(\Delta\mathbf{u})$$
$$= \frac{\partial(.)}{\partial \mathbf{u}}\Delta\mathbf{u} + \frac{\partial(.)}{\partial \mathbf{x}}\Delta\mathbf{x}(\Delta\mathbf{u}) + \frac{\partial(.)}{\partial \boldsymbol{\lambda}}(0)$$

$$\Delta(.) = \frac{\partial(.)}{\partial \mathbf{u}}\Delta\mathbf{u} + \frac{\partial(.)}{\partial \mathbf{x}}\Delta\mathbf{x}(\Delta\mathbf{u})$$

(The adjoint vector is a function of time alone)

Stationarity of the Cost Function

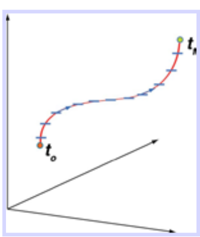


Cost must be insensitive to small variations in control policy along the optimal trajectory

First variation of the cost function due to control

$$\boxed{\begin{aligned}\Delta J^* = \left\{\left[\frac{\partial \phi}{\partial x} - \boldsymbol{\lambda}^T\right]\right\}\Delta \mathbf{x}(\Delta \mathbf{u})\Big|_{t=t_f} + \left[\boldsymbol{\lambda}^T \Delta \mathbf{x}(\Delta \mathbf{u})\right]\Big|_{t=t_o} + \int_{t_o}^{t_f}\left\{\frac{\partial H}{\partial \mathbf{u}}\Delta \mathbf{u} + \left[\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T\right]\Delta \mathbf{x}(\Delta \mathbf{u})\right\}dt = 0 \\ \equiv \Delta J(t_f) + \Delta J(t_0) + \Delta J(t_0 \rightarrow t_f)\end{aligned}}$$

Three, independent, necessary conditions for stationarity (Euler-Lagrange equations) $\boxed{\Delta J^* = 0}$



First-Order Insensitivity to Control Perturbations

Individual terms of ΔJ^* must remain zero for arbitrary variations in $\Delta \mathbf{u}(t)$

$$1) \quad \left[\frac{\partial \phi}{\partial \mathbf{x}} - \boldsymbol{\lambda}^T\right]\Big|_{t=t_f} = \mathbf{0}$$

$\dot{\mathbf{x}}(0) = \mathbf{f}\left[\mathbf{x}(0), \mathbf{u}(0)\right]$ need not be zero, but

$\mathbf{x}(0)$ cannot change instantaneously unless control is infinite

$$\therefore \left[\Delta \mathbf{x}(\Delta \mathbf{u})\right]\Big|_{t=t_0} \equiv 0, \text{ so } \Delta J\big|_{t=0} = 0$$

$$2) \quad \left[\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T\right] = \mathbf{0} \quad in\ (t_0, t_f)$$

$$3) \quad \frac{\partial H}{\partial \mathbf{u}} = \mathbf{0} \quad in\ (t_0, t_f)$$

Euler-Lagrange Equations

Euler-Lagrange Equations

Boundary condition for adjoint vector

$$1)\quad \boldsymbol{\lambda}(t_f) = \left\{ \frac{\partial \phi[\mathbf{x}(t_f)]}{\partial \mathbf{x}} \right\}^T$$

Ordinary differential equation for adjoint vector

$$2)\quad \dot{\boldsymbol{\lambda}}(t) = -\left\{ \frac{\partial H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t]}{\partial \mathbf{x}} \right\}^T$$

$$= -\left[\frac{\partial L}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T(t) \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]^T = -\left[L_{\mathbf{x}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{F}(t) \right]^T$$

Jacobian matrices

$$\mathbf{F}(t) \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t)$$

$$\mathbf{G}(t) \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(t)$$

Optimality condition

$$3)\quad \frac{\partial H[\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t]}{\partial \mathbf{u}} = \left[\frac{\partial L}{\partial \mathbf{u}} + \boldsymbol{\lambda}^T(t) \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right] = \left[L_{\mathbf{u}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{G}(t) \right] = \mathbf{0}$$

Jacobian Matrices

Jacobian Matrices Express the Solution Sensitivity to Small Perturbations

Nominal (reference) dynamic equation

$$\dot{\mathbf{x}}_N(t) = \frac{d\mathbf{x}_N(t)}{dt} = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t)]$$

Sensitivity to state perturbations: stability matrix

$$\mathbf{F}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \ldots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \ldots & \partial f_2/\partial x_n \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \ldots & \partial f_n/\partial x_n \end{bmatrix}_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

Sensitivity to Small Control Perturbations

Control-effect matrix

$$\mathbf{G}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \left[\begin{array}{cccc} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 & \cdots & \partial f_1/\partial u_m \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 & \cdots & \partial f_2/\partial u_m \\ \cdots & \cdots & \cdots & \cdots \\ \partial f_n/\partial u_1 & \partial f_n/\partial u_2 & \cdots & \partial f_n/\partial u_m \end{array}\right]_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

Jacobian Matrix Example

Original nonlinear equation describes nominal dynamics

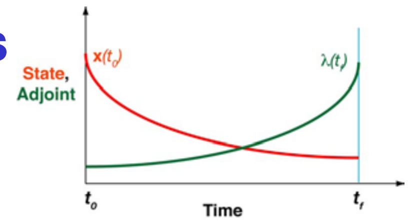
$$\dot{\mathbf{x}}_N(t) = \begin{bmatrix} \dot{x}_{1_N}(t) \\ \dot{x}_{2_N}(t) \\ \dot{x}_{3_N}(t) \end{bmatrix} = \begin{bmatrix} x_{2_N}(t) \\ a_2\left[x_{3_N}(t) - x_{2_N}(t)\right] + a_1\left[x_{3_N}(t) - x_{1_N}(t)\right]^2 + b_1 u_{1_N}(t) \ + b_2 u_{2_N}(t) \\ c_2 x_{3_N}(t)^3 + c_1\left[x_{1_N}(t) + x_{2_N}(t)\right] + b_3 x_{1_N}(t) u_{1_N}(t) \end{bmatrix}$$

Jacobian matrices are time-varying

$$\mathbf{F}(t) = \begin{bmatrix} 0 & 1 & 0 \\ -2a_1\left[x_{3_N}(t) - x_{1_N}(t)\right] & -a_2 & a_2 + 2a_1\left[x_{3_N}(t) - x_{1_N}(t)\right] \\ \left[c_1 + b_3 u_{1_N}(t)\right] & c_1 & 3c_2 x_{3_N}^2(t) \end{bmatrix}$$

$$\mathbf{G}(t) = \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \\ b_3 x_{1_N}(t) & 0 \end{bmatrix}$$

Dynamic Optimization is a Two-Point Boundary Value Problem



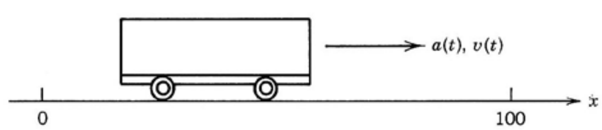
Boundary condition for the state equation is specified at t_0

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]\,, \quad \mathbf{x}(t_o)\ given$$

Boundary condition for the adjoint equation is specified at t_f

$$\dot{\boldsymbol{\lambda}}(t) = -\left[\frac{\partial L}{\partial \mathbf{x}}(t) + \boldsymbol{\lambda}^T(t)\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t)\right]^T, \quad \boldsymbol{\lambda}(t_f) = \left\{\frac{\partial \phi[\mathbf{x}(t_f)]}{\partial \mathbf{x}}\right\}^T$$

Sample Two-Point Boundary Value Problem
Move Cart 100 Meters in 10 Seconds



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Position \\ Velocity \end{bmatrix}$$

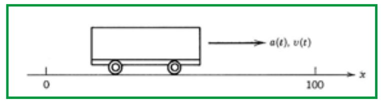
- **Cost function: tradeoff between**
  - **Terminal error squared**
  - **Integral cost of control squared**

$$J = q\left(x_{1_f} - 100\right)^2 + \int_{t_o}^{t_f} ru^2\,dt$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}; \quad L = ru^2; \quad \phi = q\left(x_{1_f} - 100\right)^2$$

$$H[\mathbf{x},\mathbf{u},\boldsymbol{\lambda}] = L[\mathbf{x},\mathbf{u}] + \boldsymbol{\lambda}^T \mathbf{f}[\mathbf{x},\mathbf{u}]$$
$$= ru^2 + \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}$$

Solution for Adjoint Vector

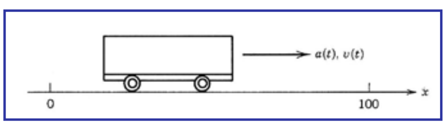


$$\dot{\boldsymbol{\lambda}}(t) = -\left\{\frac{\partial H}{\partial \mathbf{x}}\right\}^T = -\left[\frac{\partial L}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]^T = -\left[0 + \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]^T$$

$$\boldsymbol{\lambda}(t_f) = \left\{\frac{\partial \phi[\mathbf{x}(t_f)]}{\partial \mathbf{x}}\right\}^T = \begin{bmatrix} 2q\left(x_{1_f} - 100\right) & 0 \end{bmatrix}^T$$

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ \lambda_1 \end{bmatrix}; \quad \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}_{t=t_f} = \begin{bmatrix} 2q\left(x_{1_f} - 100\right) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1(t_f) \\ \lambda_1(t_f)\left(t_f - t\right) \end{bmatrix} = \begin{bmatrix} 2q\left(x_{1_f} - 100\right) \\ 2q\left(x_{1_f} - 100\right)\left(t_f - t\right) \end{bmatrix}$$



Solution for Control History

Optimality condition

$$\left(\frac{\partial H}{\partial \mathbf{u}}\right)^T = \left[\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)^T \boldsymbol{\lambda}(t)\right] = \mathbf{0}$$

Optimal control strategy

$$2ru(t) + \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{bmatrix} 2q\left(x_{1_f} - 100\right) \\ 2q\left(x_{1_f} - 100\right)\left(t_f - t\right) \end{bmatrix} = 0$$

$$u(t) = -\frac{q}{r}\left(x_{1_f} - 100\right)\left(t_f - t\right) \triangleq k_1 + k_2 t$$

Cost Weighting Effects on Optimal Solution

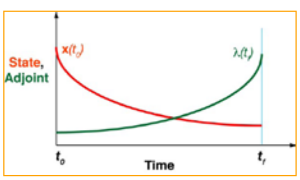
$$\mathbf{x}(t) = \mathbf{x}(t_o) + \int_{t_o}^{t} \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]\,dt, \quad t_o \to t_f$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} k_1 t^2/2 + k_2 t^3/6 \\ k_1 t + k_2 t^2/2 \end{bmatrix}$$

$$u(t) = -\frac{q}{r}\left(x_{1_f} - 100\right)\left(t_f - t\right) \triangleq k_1 + k_2 t$$

$$\text{For } t = 10s,\ x_{1_f} = \frac{100}{1 + 0.003\dfrac{r}{q}}$$

q	100	1	1
r	1	1	100
k_1	3.000	2.991	2.308
k_2	−0.300	−0.299	−0.231
x_{1_f}	99.997	99.701	76.923
x_{2_f}	15.000	14.955	11.538
$\int u^2\, dt$	29.998	29.821	17.751
J	32.794	29.923	2307.7



Typical Iteration to Find Optimal Trajectory

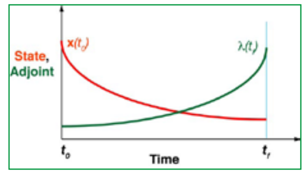
Calculate x(*t*) using prior estimate of u(*t*) , i.e., starting guess

$$\mathbf{x}(t) = \mathbf{x}(t_o) + \int_{t_o}^{t} \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)]\,dt, \quad t_o \to t_f$$

Calculate adjoint vector using prior estimate of x(*t*) and u(*t*)

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}(t_f) - \int_{t_f}^{t} \left[\frac{\partial L}{\partial \mathbf{x}}(t) + \boldsymbol{\lambda}^T(t)\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(t)\right]^T dt, \quad t_f \to t_o$$

Typical Iteration to Find Optimal Trajectory



Calculate $H(t)$ and $\partial H/\partial \mathbf{u}$ using prior estimates of state, control, and adjoint vector

$$H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] = L\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \boldsymbol{\lambda}^T(t)\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right]$$

$$\frac{\partial H}{\partial \mathbf{u}} = \left[\frac{\partial L}{\partial \mathbf{u}} + \boldsymbol{\lambda}^T(t)\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right], \quad t_o \to t_f$$

Estimate new $\mathbf{u}(t)$

$$\mathbf{u}_{new}(t) = \mathbf{u}_{old}(t) + \Delta\mathbf{u}\left[\frac{\partial H(t)}{\partial \mathbf{u}}\right], \quad t_o \to t_f$$

Alternative Necessary Condition for Time-Invariant Problem

Time-Invariant Optimization Problem

Time-invariant problem: Neither *L* nor *f* is explicitly dependent on time

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t),\mathbf{p}(t),t] = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t),\mathbf{p}]$$

$$L\left[\mathbf{x}(t),\mathbf{u}(t),t\right] = L\left[\mathbf{x}(t),\mathbf{u}(t)\right]$$

Then, the Hamiltonian is

$$\begin{aligned} H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t\right] &= L\left[\mathbf{x}(t),\mathbf{u}(t)\right] + \boldsymbol{\lambda}^T(t)\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right] \\ &= H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] \end{aligned}$$

Time-Rate-of-Change of the Hamiltonian for Time-Invariant System

$$\frac{dH[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)]}{dt} = \cancel{\frac{\partial H}{\partial t}} + \frac{\partial H}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial H}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial H}{\partial \boldsymbol{\lambda}}\frac{\partial \boldsymbol{\lambda}}{\partial t}$$

$$\frac{dH}{dt} = \left[L_{\mathbf{x}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{F}(t)\right]\dot{\mathbf{x}} + \left[L_{\mathbf{u}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{G}(t)\right]\dot{\mathbf{u}} + \mathbf{f}^T\dot{\boldsymbol{\lambda}}$$

$$\begin{aligned} \frac{dH}{dt} &= \left[\left(L_{\mathbf{x}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{F}(t)\right) + \dot{\boldsymbol{\lambda}}^{\mathrm{T}}\right]\dot{\mathbf{x}} + \left[L_{\mathbf{u}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{G}(t)\right]\dot{\mathbf{u}} \\ &= [\mathbf{0}]\dot{\mathbf{x}} + [\mathbf{0}]\dot{\mathbf{u}} = \mathbf{0} \textbf{ on optimal trajectory} \end{aligned}$$

from Euler-Lagrange Equations #2 and #3

Hamiltonian is Constant on the Optimal Trajectory

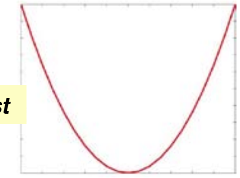
For time-invariant system dynamics and Lagrangian

$$\boxed{\frac{dH}{dt} = 0 \rightarrow H^* = \text{ constant on optimal trajectory}}$$

***H** = constant** **is an alternative scalar necessary condition for optimality**

Open-End-Time Optimization Problem

Open End-Time Problem



Final time, t_f, is free to vary

$$J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{\boxed{t_f}} \left\{ H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}(t) \right\} dt$$

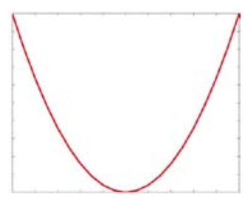
t_f is an <u>additional control variable</u> for minimizing J

$$\Delta J = \Delta J(t_f) + \Delta J(t_0) + \Delta J(t_0 \rightarrow t_f)$$

$$\Delta J(t_f) = \Delta J(t_f)\Big|_{fixed\ t_f} + \left.\frac{dJ}{dt}\right|_{t=t_f} \Delta t_f$$

Goal: t_f for which sensitivity to perturbation in final time is zero

Additional Necessary Condition for Open End-Time Problem



Cost sensitivity to final time should be zero

$$\left.\frac{dJ}{dt}\right|_{t=t_f} = \left.\left\{ \left[\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \mathbf{x}} \dot{\mathbf{x}} \right] + \left[H - \boldsymbol{\lambda}^T \dot{\mathbf{x}} \right] \right\}\right|_{t=t_f}$$

$$= \left.\left\{ \left[\frac{\partial \phi}{\partial t} + \boldsymbol{\lambda}^T \dot{\mathbf{x}} \right] + \left[H - \boldsymbol{\lambda}^T \dot{\mathbf{x}} \right] \right\}\right|_{t=t_f}$$

$$= \left.\left\{ \frac{\partial \phi}{\partial t} + H \right\}\right|_{t=t_f} = 0$$

Additional necessary condition for stationarity

$$\frac{\partial \phi^*}{\partial t} = -H^* \text{ at } t = t_f \quad \text{for open end time}$$

H^* = 0 with Open End-Time

If terminal cost is independent of time, and final time is open

$$\left.\frac{dJ}{dt}\right|_{t=t_f} = \left.\left\{\frac{\partial\phi}{\partial t} + H\right\}\right|_{t=t_f} = \left.\left\{(0) + H\right\}\right|_{t=t_f} = 0$$

$$\therefore \left. H^* \right|_{t=t_f} = 0$$

H^* = 0 with Open End-Time and Time-Invariant System

If terminal and integral costs are independent of time, and final time is open

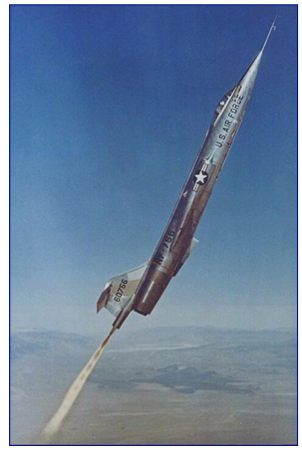
$$\therefore \left. H^* \right|_{t=t_f} = 0$$

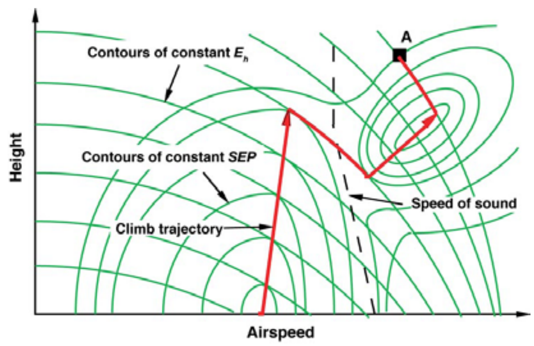
$$\frac{dH}{dt} = 0 \rightarrow H^* = \text{ constant on optimal trajectory}$$

$$H^* = 0 \quad in \quad t_0 \leq t \leq t_f$$

Examples of Open End-Time Problems

- Minimize elapsed time to achieve an objective
- Minimize fuel to go from one place to another
- Achieve final objective using a fixed amount of energy





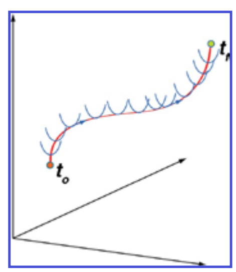
Sufficient Conditions for a Minimum

Sufficient Conditions for a Minimum

- **Euler-Lagrange equations are satisfied (necessary conditions for stationarity), plus proof of**
  - **Convexity**
  - **Controllability <--> Normality**
  - **Uniqueness**
- **Singular optimal control**
  - **Higher-order conditions**

Convexity

Legendre-Clebsch Condition



"Strengthened" condition

$$\frac{\partial^2 H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*)}{\partial \mathbf{u}^2} > 0 \text{ in } (t_0, t_f)$$

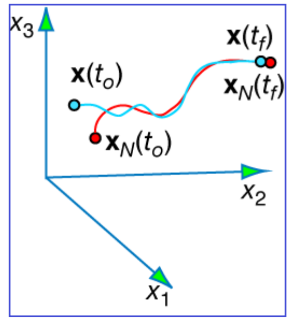
Positive definite (m x m) Hessian matrix throughout trajectory

"Weakened" condition

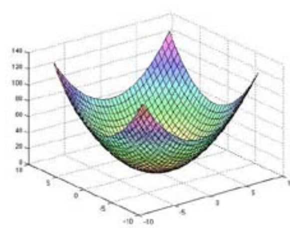
$$\frac{\partial^2 H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*)}{\partial \mathbf{u}^2} \geq 0 \text{ in } (t_0, t_f)$$

Hessian may equal zero at isolated points

Normality and Controllability



- **Normality: Existence of neighboring-optimal solutions**
  - **Neighboring vs. neighboring-optimal trajectories**
- **Controllability: Ability to satisfy a terminal equality constraint**
- **Legendre-Clebsch condition satisfied**



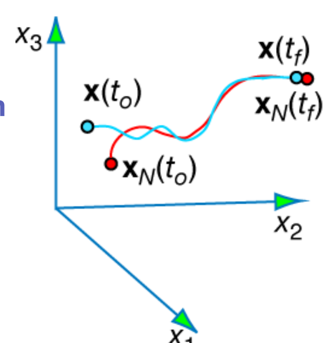
Neighboring vs. Neighboring-Optimal Trajectories

- **Nominal (or reference) trajectory and control history**

$$\boxed{\left\{\mathbf{x}_N(t), \mathbf{u}_N(t)\right\} \quad \text{for } t \text{ in } [t_o, t_f]}$$

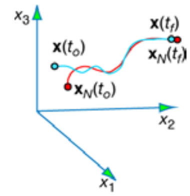
- **Trajectory perturbed by**
  - **Small initial condition variation**
  - **Small control variation**

$$\boxed{\begin{aligned}&\left\{\mathbf{x}(t), \mathbf{u}(t)\right\} \quad \text{for } t \text{ in } [t_o, t_f]\\ &= \left\{\mathbf{x}_N(t) + \Delta\mathbf{x}(t), \mathbf{u}_N(t) + \Delta\mathbf{u}(t)\right\}\end{aligned}}$$



- **This a neighboring trajectory**
- **… but it is not necessarily optimal**

Both Paths Satisfy the Dynamic Equations



$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t)], \quad x_N(t_o) \text{ given}$$
$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)], \quad x(t_o) \text{ given}$$

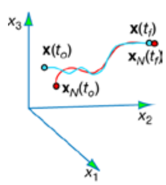
Alternative notation

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t)]$$
$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}_N(t) + \Delta\mathbf{x}(t), \mathbf{u}_N(t) + \Delta\mathbf{u}(t)]$$

$$\Delta\mathbf{x}(t_o) = \mathbf{x}(t_o) - \mathbf{x}_N(t_o)$$
$$\Delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_N(t)$$
$$\Delta\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_N(t)$$

$$\Delta\mathbf{u}(t) = \mathbf{u}(t) - \mathbf{u}_N(t)$$

Neighboring-Optimal Trajectories



$\mathbf{x}_N^*(t)$ is an optimal solution to a cost function

$$\dot{\mathbf{x}}_N^*(t) = \mathbf{f}[\mathbf{x}_N^*(t), \mathbf{u}_N^*(t)], \quad \mathbf{x}_N(t_o) \text{ given}$$
$$J_N^* = \phi\left[\mathbf{x}_N^*(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}_N^*(t), \mathbf{u}_N^*(t)\right] dt$$

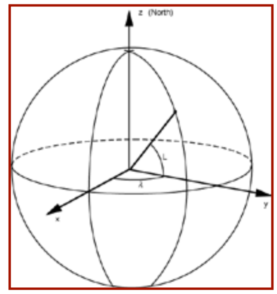
If $\mathbf{x}^*(t)$ is an optimal solution to the same cost function

$$\dot{\mathbf{x}}^*(t) = \mathbf{f}[\mathbf{x}^*(t), \mathbf{u}^*(t)], \quad \mathbf{x}(t_o) \text{ given}$$
$$J^* = \phi\left[\mathbf{x}^*(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}^*(t), \mathbf{u}^*(t)\right] dt$$

Then $\mathbf{x}_N$ and $\mathbf{x}$ are neighboring-optimal trajectories

Uniqueness

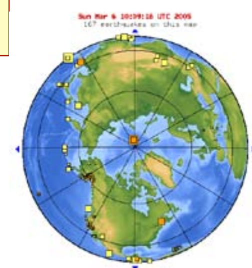
Jacobi Condition



$$\boxed{\{\Delta\mathbf{x}(t) < \infty\} \Leftrightarrow \{\Delta\mathbf{u}(t) < \infty\}}$$

- **Finite state perturbation implies finite control perturbation**
- **No conjugate points**
- **Example: Minimum distance from the north pole to the equator**

Conjugate Point at North Pole



http://en.wikipedia.org/wiki/Conjugate_points

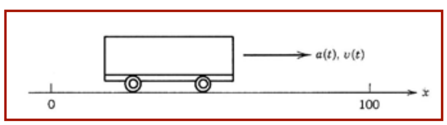
http://www.encyclopediaofmath.org/index.php/Jacobi_condition

Next Time: Principles for Optimal Control, Part 2

Reading: OCE: pp. 222-231

Supplemental Material

Time-Invariant Example with Scalar Control
Cart on a Track

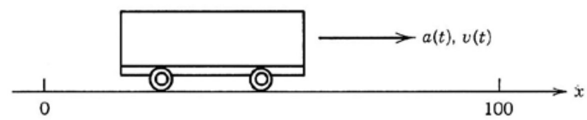


$$H[\mathbf{x},\mathbf{u},\boldsymbol{\lambda}] = L[\mathbf{x},\mathbf{u}] + \boldsymbol{\lambda}^T \mathbf{f}[\mathbf{x},\mathbf{u}] = \text{ Constant}$$
$$= ru(t)^2 + \begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix} \begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}$$
$$= ru(t)^2 + \lambda_1(t)x_2(t) + \lambda_1(t)\left(t_f - t\right)u(t) = \text{ Constant}$$

$$H[\mathbf{x},\mathbf{u},\boldsymbol{\lambda}] = ru(t)^2 + \begin{bmatrix} 2q\left(x_{1_f} - 100\right) & 2q\left(x_{1_f} - 100\right)\left(t_f - t\right) \end{bmatrix} \begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}$$

$$ru(t)^2 + 2q\left(x_{1_f} - 100\right)\left(t_f - t\right)u(t) + 2q\left(x_{1_f} - 100\right)x_2(t) = \textbf{ Constant}\left(\textbf{TBD}\right)$$

Cart on a Track with Scalar Control and Open End Time



$$\boxed{H^* = ru(t)^2 + \lambda_1(t)x_2(t) + \lambda_1(t)\left(t_f - t\right)u(t) = 0}$$

- **Fixed end-time results (t_f = 10 s)**
- **Open end-time would be important only if *q/r* is small**

q	100	1	1
r	1	1	100
k_1	3.000	2.991	2.308
k_2	−0.300	−0.299	−0.231
x_{1_f}	99.997	99.701	76.923
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