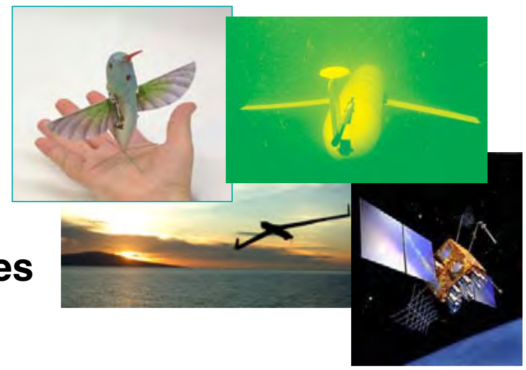
Flying Robots, Motion, and Rigid-Body Dynamics

Robert Stengel
Robotics and Intelligent Systems MAE 345, Princeton University, 2015

- **Aircraft**
- **Aquatic robots**
- **Space robots**
- **Translational motion of particles and rigid bodies**



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http://www.princeton.edu/~stengel/MAE345.html

Assignment # 2

due: End-of-Day, October 5, 2015

Document the physical characteristics and flight behavior of a Syma X11 quadcopter

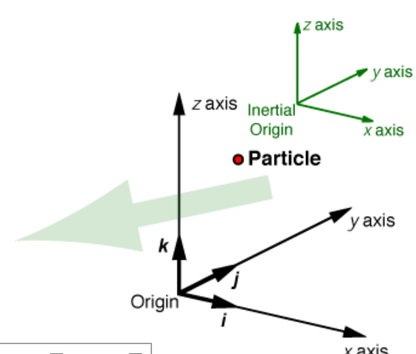
https://www.youtube.com/watch?v=kyiuy2CHzj0

https://www.youtube.com/watch?v=EwO6U7DbqSo

Translational Motion

Reference Frame

- Newtonian (Inertial) Frame of Reference
  - Unaccelerated Cartesian frame
    - Origin referenced to inertial (non-moving) frame
  - Right-hand rule
  - Origin can translate at constant linear velocity
  - Frame cannot rotate with respect to inertial origin



- Position: 3 dimensions
  - What is a non-moving frame?

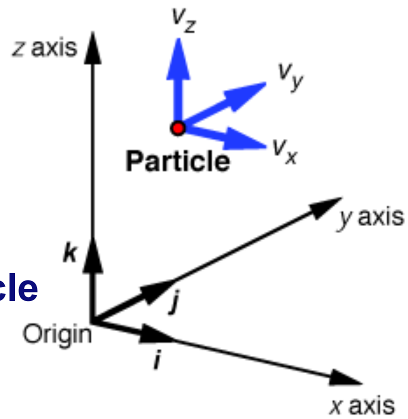
$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Translation = Linear motion

Velocity and Momentum of a Particle

- Velocity of a particle

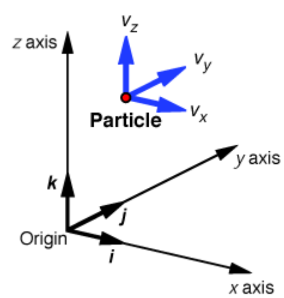
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$



- Linear momentum of a particle

$$\mathbf{p} = m\mathbf{v} = m\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Newton's Laws of Motion: Dynamics of a Particle



First Law

. If no force acts on a particle,

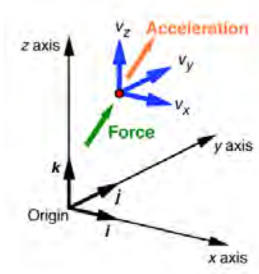
it remains at rest or continues to move in straight line at constant velocity,

. Inertial reference frame

. Momentum is conserved

$$\frac{d}{dt}(m\mathbf{v}) = 0 \quad ; \quad m\mathbf{v}\big|_{t_1} = m\mathbf{v}\big|_{t_2}$$

Newton's Laws of Motion: Dynamics of a Particle



Second Law

- Particle acted upon by <u>force</u>
- <u>Acceleration</u> proportional to and in direction of force
- Inertial reference frame
- Ratio of force to acceleration is particle mass

$$\frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{Force}$$

$$\mathbf{Force} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \textbf{force vector}$$

$$\therefore \quad \frac{d\mathbf{v}}{dt} = \frac{1}{m}\mathbf{Force} = \frac{1}{m}\mathbf{I}_3\mathbf{Force}$$

$$= \begin{bmatrix} 1/m & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 1/m \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

Newton's Laws of Motion: Dynamics of a Particle

Third Law

For every action, there is an equal and opposite reaction

$$\text{Force on rocket motor} = -\text{Force on exhaust gas}$$

$$\mathrm{F}_{\mathbf{R}} = -\mathbf{F}_E$$

One-Degree-of-Freedom Example of Newton's Second Law

2nd-order, linear, time-invariant ordinary differential equation

$$\boxed{\frac{d^2x(t)}{dt^2} \triangleq \ddot{x}(t) = \dot{v}_x(t) = \frac{f_x(t)}{m}}$$

$\boxed{\triangleq \ \text{"Defined as"}}$

Corresponding set of 1st-order equations

(State-Space Model)

$$\begin{aligned} &\textbf{Displacement:} && x_1(t) \triangleq x(t) \\ &\textbf{Rate:} && x_2(t) \triangleq \frac{dx(t)}{dt} \end{aligned}$$

$$\begin{aligned} \frac{dx_1(t)}{dt} &\triangleq \dot{x}_1(t) \triangleq x_2(t) \triangleq v_x(t) \\ \frac{dx_2(t)}{dt} &\triangleq \ddot{x}_1(t) = \dot{x}_2(t) = \dot{v}_x(t) = \frac{f_x(t)}{m} \end{aligned}$$

State-Space Model is a Set of 1st-Order Ordinary Differential Equations

State, control, and output vectors for the example

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; \quad \mathbf{u}(t) = u(t) = f_x(t); \quad \mathbf{y}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Stability and control-effect matrices

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

Dynamic equation

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t)}$$

State-Space Model of the 1-DOF Example

Output equation

$$\boxed{\mathbf{y}(t) = \mathbf{H_x x}(t) + \mathbf{H_u u}(t)}$$

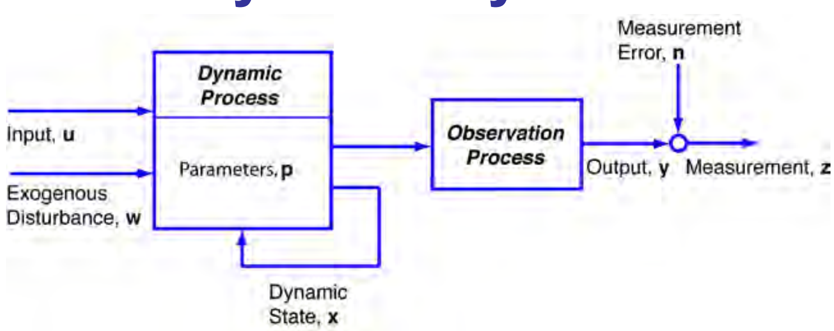
Output coefficient matrices

$$\mathbf{H_x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{H_u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- **Vectors**
  - –**State Vector:** $\mathbf{x}(t)$ *(n x 1)*
  - –**Control Vector:** $\mathbf{u}(t)$ *(m x 1)*
  - –**Output Vector:** $\mathbf{y}(t)$ *(r x 1)*

- **Matrices**
  - –**Stability Matrix:** **F** *(n x n)*
  - –**Control-Effect Matrix;** **G** *(n x m)*
  - –**State-Output Matrix:** $\mathbf{H_x}$ *(r x n)*
  - –**Control-Output Matrix:** $\mathbf{H_u}$ *(r x m)*

Dynamic System



Dynamic Process: Current state may depend on prior state

x : state *dim* = (*n* x 1)
u : input *dim* = (*m* x 1)
w : disturbance *dim* = (*s* x 1)
p : parameter *dim* = (ℓ x 1)

t : time (independent variable, 1 x 1)

Observation Process: Measurement may contain error or be incomplete

y : output (error-free)
dim = (*r* x 1)
n : measurement error
dim = (*r* x 1)
z : measurement
dim = (*r* x 1)

State-Space Model of Three-Degree-of-Freedom Dynamics

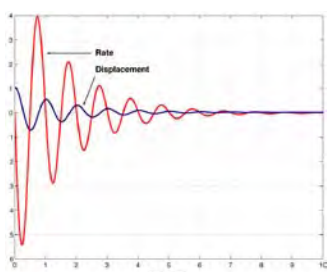
$$\mathbf{x}(t) \triangleq \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{bmatrix} = \left[\begin{array}{c} x \\ y \\ z \\ \hline v_x \\ v_y \\ v_z \end{array}\right]$$

$$\dot{\mathbf{x}}(t) \triangleq \begin{bmatrix} \dot{\mathbf{r}}(t) \\ \dot{\mathbf{v}}(t) \end{bmatrix} = \left[\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{z} \\ \hline \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{array}\right] = \left[\begin{array}{c} v_x \\ v_y \\ v_z \\ \hline f_x / m \\ f_y / m \\ f_z / m \end{array}\right] = \mathbf{Fx}(t) + \mathbf{Gu}(t)$$

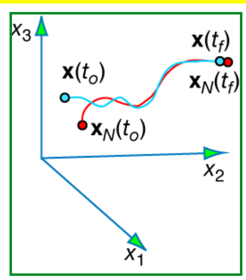
$$= \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x \\ y \\ z \\ \hline v_x \\ v_y \\ v_z \end{array}\right] + \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1/m & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 1/m \end{array}\right] \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

What Use are the Dynamic Equations?

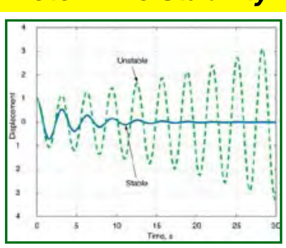
Compute time response



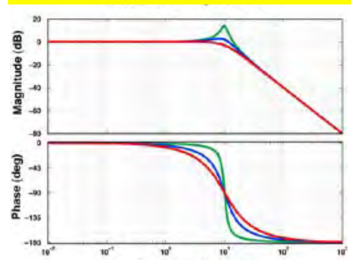
Compute trajectories



Determine stability

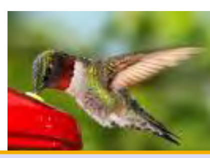


Identify modes of motion



Aircraft

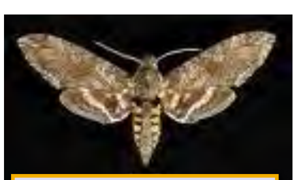
Bio-Inspiration for Flying



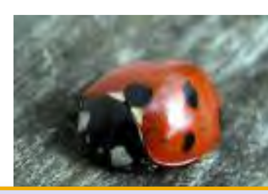
Hummingbird
http://www.youtube.com/watch?v=D8vjYTXglJw&feature=related

Eagle vs. Eagle
http://www.youtube.com/watch?v=tufnqWNP9AA&feature=video_response

Birds Flying
http://www.youtube.com/watch?v=l5GbFgk-EPw



Moth Flying
https://www.youtube.com/watch?v=hD2BjAsvlbI



Lady Bug
http://www.youtube.com/watch?v=fjZobEZJYBc

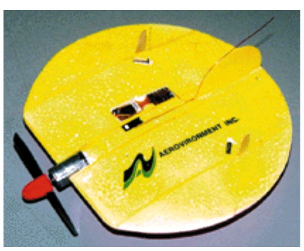
Biomimetic UAVs

Markus Fisher at TED
http://www.youtube.com/watch?v=Fg_JcKSHUtQ

Aerovironment Nano Hummingbird
http://www.avinc.com/nano

Festo Air Ray Dirigible
http://www.youtube.com/watch?v=UxPzodKQays

Harvard Robo-Flies
http://www.youtube.com/watch?v=2lQcKr0A_7c

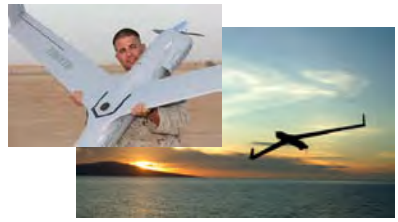


Uninhabited Air Vehicles (UAV)

Uninhabited Aircraft
Tad McGeer, ‘79

Aerosonde
First UAV Transatlantic Crossing, 1998

Boeing (InSitu)
ScanEagle



Aerovel *Flexrotor*

Princeton’s Variable-Stability Airplanes
Holonomic and NonHolonomic Airplanes

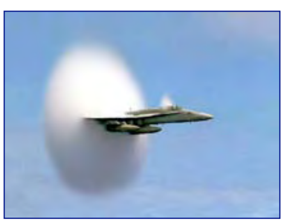
6 degrees of freedom,
6 independent controls

Pitching moment
Yawing moment
Rolling moment

6 degrees of freedom,
5 independent controls

Thrust force
Lift force
~~Side force~~

Forces



External Forces:
Aerodynamic/Hydrodynamic

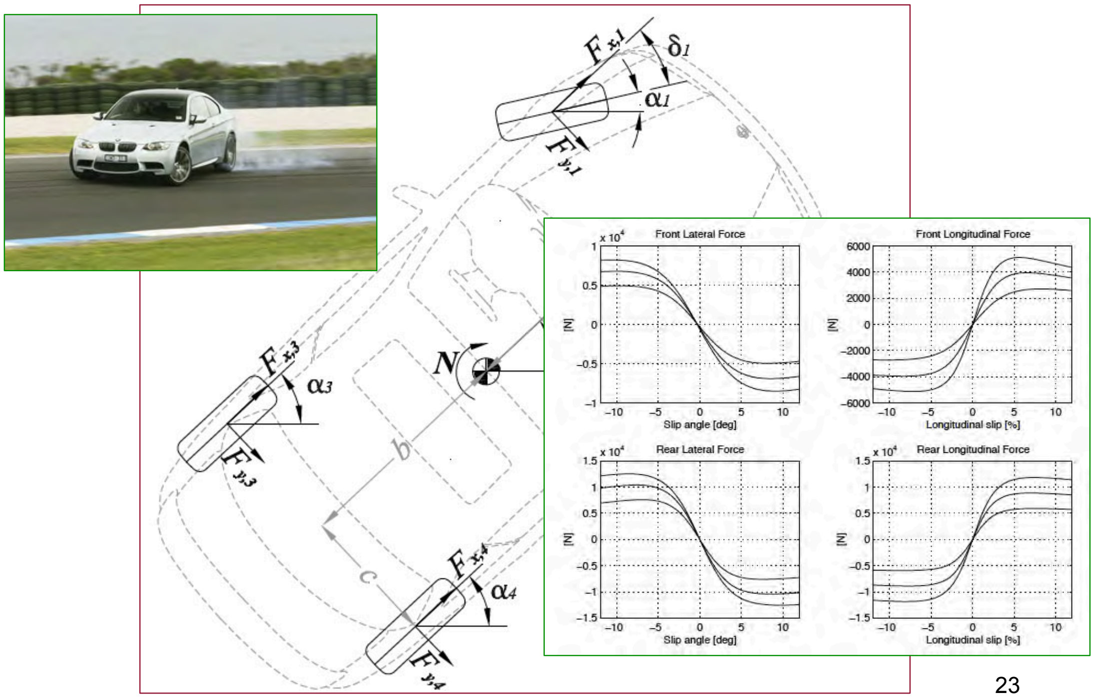
$$\mathbf{f}_{aero} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} \frac{1}{2}\rho V^2 A$$

$$\begin{aligned} \rho &= \text{air density, function of height} \\ &= \rho_{sealevel} e^{-\beta h} \\ V &= \text{airspeed} \\ &= \left[v_x^2 + v_y^2 + v_z^2 \right]^{1/2} = \left[\mathbf{v}^T \mathbf{v} \right]^{1/2} \\ A &= \text{reference area} \end{aligned}$$

$$\begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} = \begin{array}{c} \text{dimensionless} \\ \text{aerodynamic coefficients} \end{array}$$

Inertial frame or body frame?

External Forces: Friction



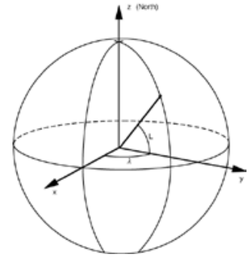
External Forces: Gravity

- **Flat-earth approximation**
  - **g** is gravitational acceleration
  - *m***g** is gravitational force
  - Independent of position
  - *z* measured up

$$m\mathbf{g}_{flat} = m\begin{bmatrix} 0 \\ 0 \\ -g_o \end{bmatrix}; \quad g_o = 9.807\ m/s^2$$

External Forces: Gravity

$$r = \left(x^2 + y^2 + z^2\right)^{1/2}$$
$$L = \text{Latitude}$$
$$\lambda = \text{Longitude}$$



- **Spherical earth, inertial frame**
  - ”Inverse-square” gravitation
  - Non-linear function of position
  - μ = 3.986 x 10^{14} m/s

$$\mathbf{g}_{round} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \mathbf{g}_{gravity}$$

$$= -\frac{\mu}{r^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{\mu}{r^2}\begin{bmatrix} x/r \\ y/r \\ z/r \end{bmatrix} = -\frac{\mu}{r^2}\begin{bmatrix} \cos L \cos \lambda \\ \cos L \sin \lambda \\ \sin L \end{bmatrix}$$

- **Spherical earth, rotating frame**
  - ”Inverse-square” gravitation
  - “Centripetal acceleration”
  - Ω = 7.29 x 10^{-5} rad/s

$$\mathbf{g}_r = \mathbf{g}_{gravity} + \mathbf{g}_{rotation}$$
$$= -\frac{\mu}{r^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \Omega^2 \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

State-Space Model with Round-Earth Gravity Model (Non-Rotating Frame)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} - \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline \mu/r^3 & 0 & 0 \\ 0 & \mu/r^3 & 0 \\ 0 & 0 & \mu/r^3 \end{array}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -\mu/r^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu/r^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu/r^3 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix}$$

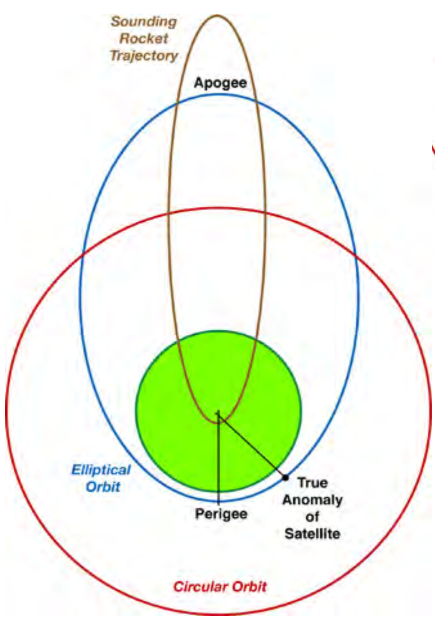
Inverse-square gravity model introduces nonlinearity

Vector-Matrix Form

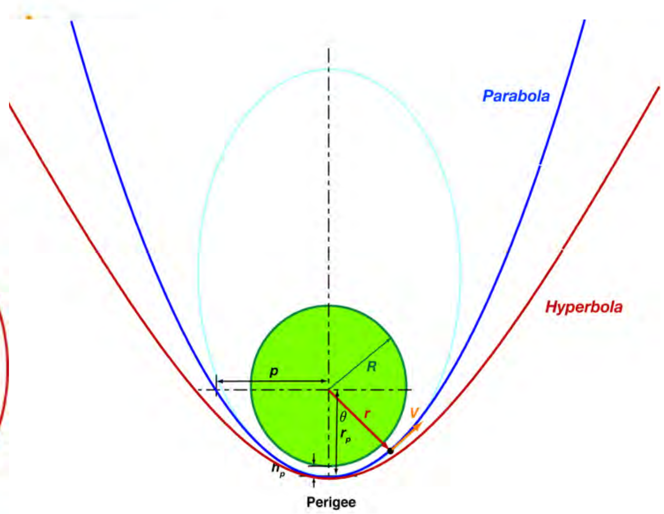
$$\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ \dfrac{-\mu}{r^3}\mathbf{I}_3 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix}$$

Point-Mass Motions of Spacecraft

Satellites



Escape and Capture
(Comets, Meteorites)



Space Robots

Expendable (Rocket) Launch Vehicles

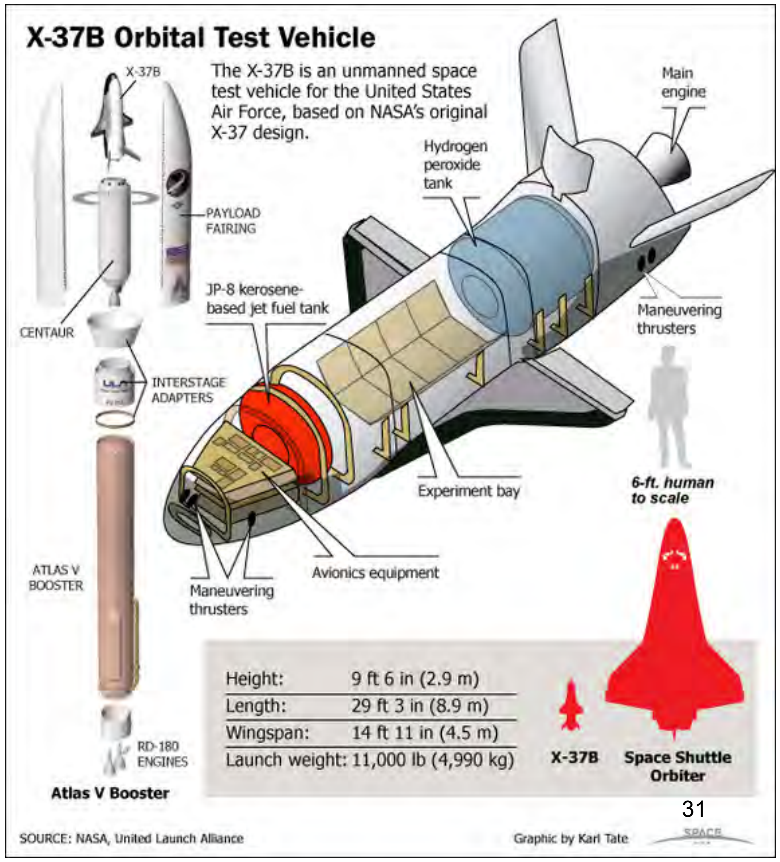
Current space launch vehicles are largely autonomous

Atlas V

http://www.youtube.com/watch?v=KxQbex7LJwg

X-37B

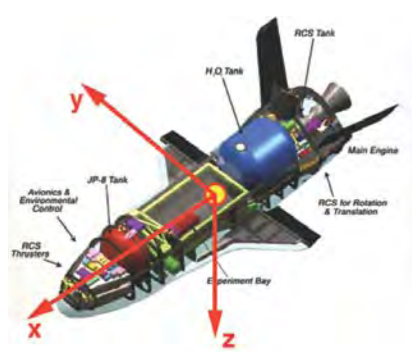
- Reusable experimental/ operational vehicle
- Unmanned “mini-Space Shuttle”
- Highly classified project
- 1st 3 missions: 224, 469, & 675 days in orbit
- 4th mission on-going



Mass of an Object

$$m = \int_{Body} dm = \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} \rho(x,y,z)\,dx\,dy\,dz$$

$$\rho(x,y,z) = density\ of\ the\ body$$



Density of object may vary with *(x,y,z)*

More Forces

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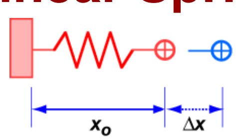
External Forces:
Linear Springs



Scalar, linear spring

$$\boxed{f = -k\Delta x = -k\left(x - x_o\right) \quad ; \quad k = spring\, constant}$$

Uncoupled, linear vector spring

$$\boxed{\mathbf{f}_S = -\begin{bmatrix} k_x \Delta x \\ k_y \Delta y \\ k_z \Delta z \end{bmatrix} = -\begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}}$$

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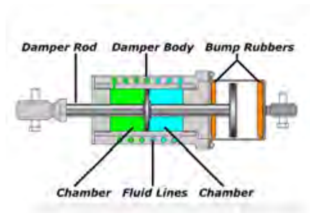


External Forces: Viscous Dampers

Scalar, linear damper

$$\boxed{f = -d\Delta v = -d\left(v - v_o\right) \quad ; \quad d = damping\,constant}$$

Uncoupled, linear vector damper

$$\mathbf{f}_D = -\begin{bmatrix} d_x \Delta v_x \\ d_y \Delta v_y \\ d_z \Delta v_z \end{bmatrix} = -\begin{bmatrix} d_x & 0 & 0 \\ 0 & d_y & 0 \\ 0 & 0 & d_z \end{bmatrix} \begin{bmatrix} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{bmatrix}$$

State-Space Model with Linear Spring and Damping Model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} + \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \frac{-k_x}{m} & 0 & 0 & \frac{-d_x}{m} & 0 & 0 \\ 0 & \frac{-k_y}{m} & 0 & 0 & \frac{-d_y}{m} & 0 \\ 0 & 0 & \frac{-k_z}{m} & 0 & 0 & \frac{-d_z}{m} \end{array}\right] \begin{bmatrix} \left(x - x_o\right) \\ \left(y - y_o\right) \\ \left(z - z_o\right) \\ \left(v_x - v_{x_o}\right) \\ \left(v_y - v_{y_o}\right) \\ \left(v_z - v_{z_o}\right) \end{bmatrix}$$

Spring Effects

Damping Effects

State-Space Model with Linear Spring and Damping Model

Stability Effects

Reference Effects

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-k_x}{m} & 0 & 0 & \frac{-d_x}{m} & 0 & 0 \\ 0 & \frac{-k_y}{m} & 0 & 0 & \frac{-d_y}{m} & 0 \\ 0 & 0 & \frac{-k_z}{m} & 0 & 0 & \frac{-d_z}{m} \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} + \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \frac{k_x}{m} & 0 & 0 & \frac{d_x}{m} & 0 & 0 \\ 0 & \frac{k_y}{m} & 0 & 0 & \frac{d_y}{m} & 0 \\ 0 & 0 & \frac{k_z}{m} & 0 & 0 & \frac{d_z}{m} \end{array}\right] \begin{bmatrix} x_o \\ y_o \\ z_o \\ v_{x_o} \\ v_{y_o} \\ v_{z_o} \end{bmatrix}$$

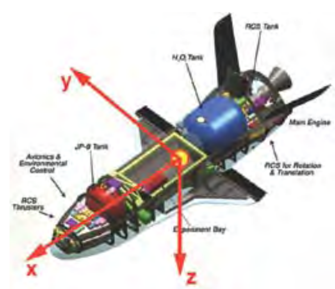
Vector-Matrix Form

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\mathbf{K}/m & -\mathbf{D}/m \end{bmatrix} \begin{bmatrix} (\mathbf{r} - \mathbf{r}_o) \\ (\mathbf{v} - \mathbf{v}_o) \end{bmatrix}$$

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\mathbf{K}/m & -\mathbf{D}/m \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{K}/m & \mathbf{D}/m \end{bmatrix} \begin{bmatrix} \mathbf{r}_o \\ \mathbf{v}_o \end{bmatrix}$$

Rotational Motion

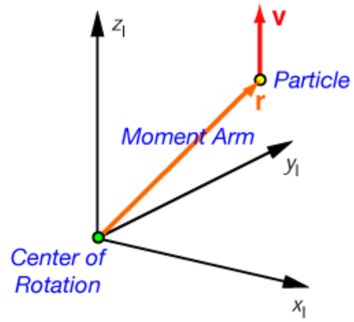
Center of Mass



$$\mathbf{r}_{cm} = \frac{1}{m}\int_{Body} \mathbf{r}\, dm = \int_{x_{\min}}^{x_{\max}}\int_{y_{\min}}^{y_{\max}}\int_{z_{\min}}^{z_{\max}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rho(x,y,z)\, dx\, dy\, dz$$

$$= \begin{bmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \end{bmatrix}$$

Reference point for rotational motion

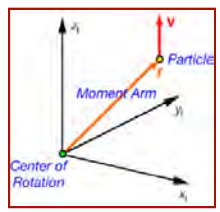


Angular Momentum of a Particle

- **Moment of linear momentum of differential particles that make up the body**
  - **Differential mass of a particle times**
  - **Component of velocity perpendicular to moment arm from center of rotation to particle**

$$\boxed{d\mathbf{h} = (\mathbf{r} \times dm\mathbf{v}) = (\mathbf{r} \times \mathbf{v})dm}$$

Cross Product of Two Vectors



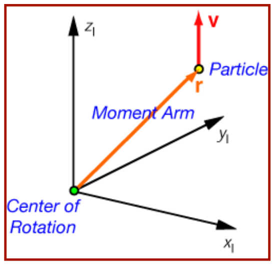
$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = (yv_z - zv_y)i + (zv_x - xv_z)j + (xv_y - yv_x)k$$

$(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$: **Unit vectors** along (x,y,z)

This is equivalent to

$$= \begin{bmatrix} (yv_z - zv_y) \\ (zv_x - xv_z) \\ (xv_y - yv_x) \end{bmatrix} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \tilde{\mathbf{r}}\mathbf{v}$$

Cross-Product-Equivalent Matrix



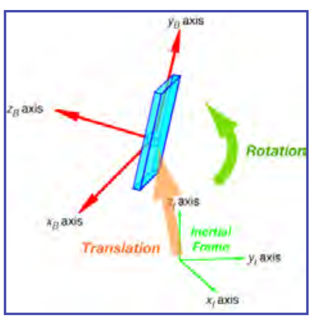
$$\mathbf{r} \times \mathbf{v} = \tilde{\mathbf{r}}\mathbf{v}$$

Cross-product equivalent of radius vector

$$\mathbf{r} \times \triangleq \tilde{\mathbf{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Velocity vector

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

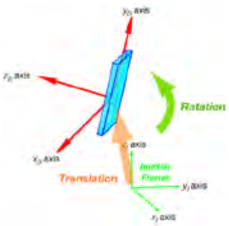


Angular Momentum of an Object

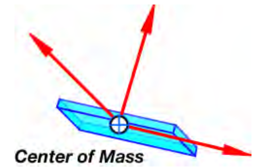
Integrate moment of linear momentum of differential particles over the body

$$\mathbf{h} \triangleq \int_{Body} (\mathbf{r} \times \mathbf{v})\, dm = \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} (\mathbf{r} \times \mathbf{v})\, \rho(x,y,z)\, dx\, dy\, dz$$

$$= \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} (\tilde{\mathbf{r}}\mathbf{v})\, \rho(x,y,z)\, dx\, dy\, dz \triangleq \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}$$



Angular Velocity and Corresponding Velocity Increment

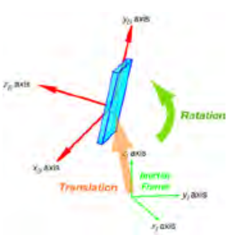


Angular velocity of object with respect to inertial frame of reference

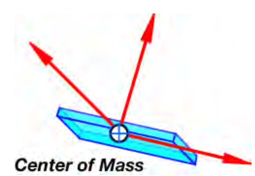
$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_I$$

Linear velocity increment at a point, *(x,y,z)*, due to angular rotation

$$\Delta\mathbf{v}(x,y,z) = \begin{bmatrix} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{bmatrix} = \boldsymbol{\omega} \times \mathbf{r} = -(\mathbf{r} \times \boldsymbol{\omega})$$



Angular Momentum of an Object with Respect to Its Center of Mass



- **Choose center of mass as origin about which angular momentum is calculated (= center of rotation)**
  - **i.e., r is measured from the center of mass**

$$\mathbf{h} = \int_{Body} \left[\mathbf{r} \times (\mathbf{v}_{cm} + \Delta\mathbf{v})\right] dm = \int_{Body} \left[\mathbf{r} \times \mathbf{v}_{cm}\right] dm + \int_{Body} \left[\mathbf{r} \times \Delta\mathbf{v}\right] dm$$

$$= \int_{Body} \mathbf{r}\, dm \times \mathbf{v}_{cm} - \int_{Body} \left[\mathbf{r} \times (\mathbf{r} \times \boldsymbol{\omega})\right] dm$$

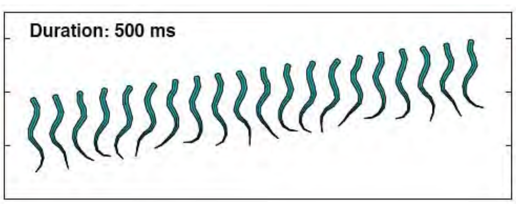
$$= -\int_{Body} \tilde{\mathbf{r}}\tilde{\mathbf{r}}\, \boldsymbol{\omega}\, dm = \left[-\int_{Body} \tilde{\mathbf{r}}\tilde{\mathbf{r}}\, dm\right] \boldsymbol{\omega} = \boldsymbol{I}\boldsymbol{\omega}$$

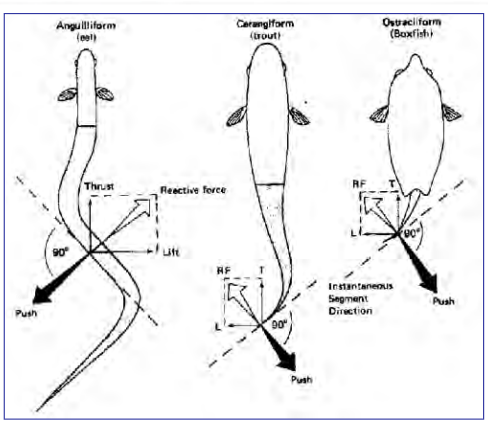
$$\int_{Body} \mathbf{r}\, dm \times \mathbf{v}_{cm} = 0 \quad \textbf{by symmetry}$$

$$\boldsymbol{I} = \text{Inertia Matrix}$$

Undersea Robots

Swimming Gaits





Anguilliform locomotion
Long, slender fish, e.g., lamprey
Amplitude of flexion wave along body ~ constant

Sub-carangiform locomotion
Increase in wave amplitude along the body
Most work done by rear half of fish body
Higher speed, reduced maneuverability

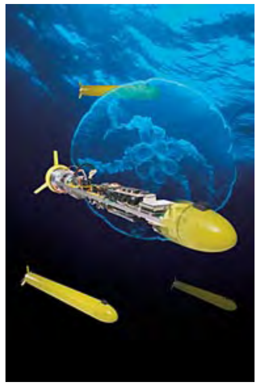
Carangiform locomotion
Stiffer and faster-moving, e.g., trout
Majority of movement rear of body and tail
Rapidly oscillating tails

Thunniform locomotion
High-speed long-distance swimmers, e.g. tuna, shark
Virtually all lateral movement in the tail
Tail itself is large and crescent-shaped

Autonomous Submarines

Autonomous Benthic Explorer

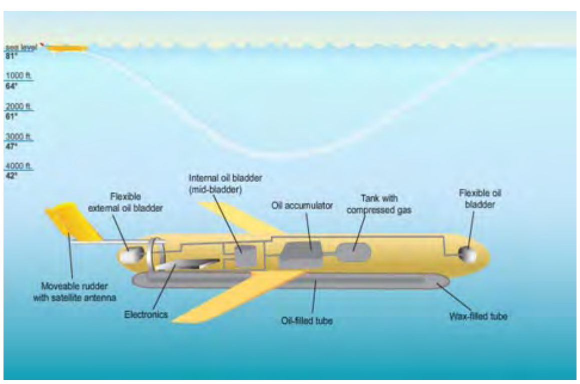
VPI concept



Oberon (U Sydney)

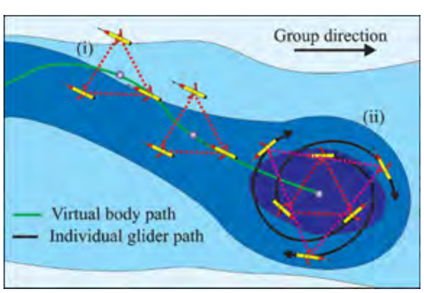
AQUA
http://www.youtube.com/watch?v=9Vm-gQ9_H9I&feature=related

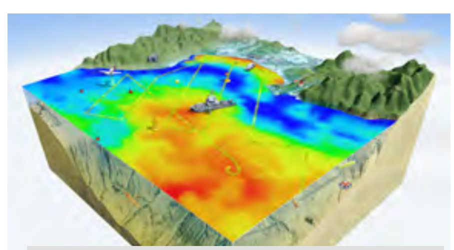
Autonomous Underwater Gliders



- Slocum Glider
  - Variable ballast for climb/dive
  - Adaptive schooling guidance (Leonard et al)

  http://www.youtube.com/watch?v=aRyEDzaogPc

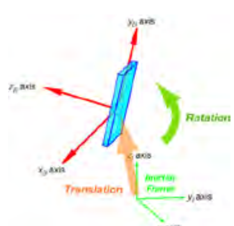




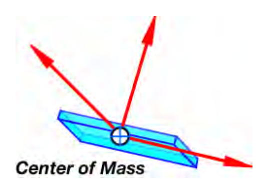
Monterey Bay Experimental Program

https://en.wikipedia.org/wiki/Underwater_glider

Inertia Matrix



Angular Momentum

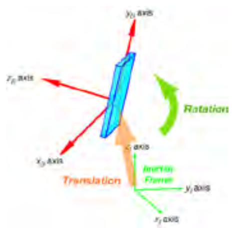


Three components of angular momentum

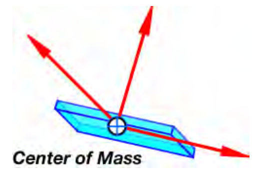
$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \left[-\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\; dm \right] \boldsymbol{\omega} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Vector notation

$$\boxed{\mathbf{h} = \boldsymbol{I}\,\boldsymbol{\omega}}$$



Inertia Matrix



Equal effect of angular rate on all particles

$$\boldsymbol{I} = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\; dm = -\int_{Body} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} dm$$

$$= \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm$$

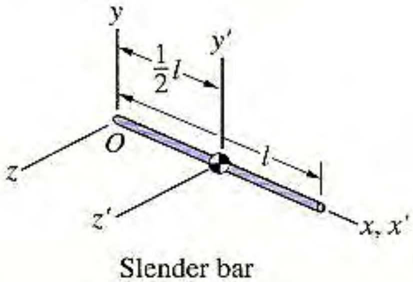
Inertia Matrix

$$\boldsymbol{I} = \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

- **Moments of inertia on the diagonal**
- **Products of inertia off the diagonal**
- **If products of inertia are zero, (x, y, z) are principal axes, and**

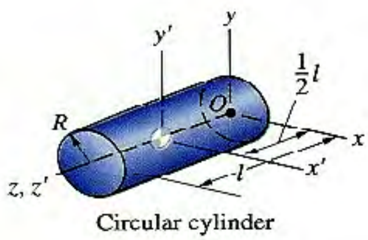
$$\boldsymbol{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

Moments and Products of Inertia for Basic Constant-Density Objects



Slender bar

$$I_{x\,\text{axis}} = 0, \qquad I_{y\,\text{axis}} = I_{z\,\text{axis}} = \frac{1}{3}ml^2,$$
$$I_{xy} = I_{yz} = I_{zx} = 0.$$
$$I_{x'\,\text{axis}} = 0, \qquad I_{y'\,\text{axis}} = I_{z'\,\text{axis}} = \frac{1}{12}ml^2,$$
$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



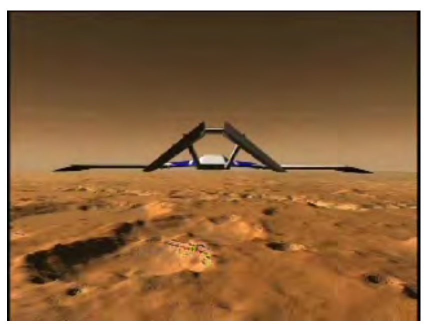
Circular cylinder

$$\text{Volume} = \pi R^2 l$$
$$I_{x\,\text{axis}} = I_{y\,\text{axis}} = m\left(\frac{1}{3}l^2 + \frac{1}{4}R^2\right), \qquad I_{z\,\text{axis}} = \frac{1}{2}mR^2,$$
$$I_{xy} = I_{yz} = I_{zx} = 0.$$
$$I_{x'\,\text{axis}} = I_{y'\,\text{axis}} = m\left(\frac{1}{12}l^2 + \frac{1}{4}R^2\right), \qquad I_{z'\,\text{axis}} = \frac{1}{2}mR^2,$$
$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$

Next Time: Equations of Motion and Articulated Robots

Supplemental Material

Mars Aerial Regional-Scale Environmental Survey (*ARES*) Research Airplane Concept, ~2008



https://www.youtube.com/watch?v=8YutbpJuFiI

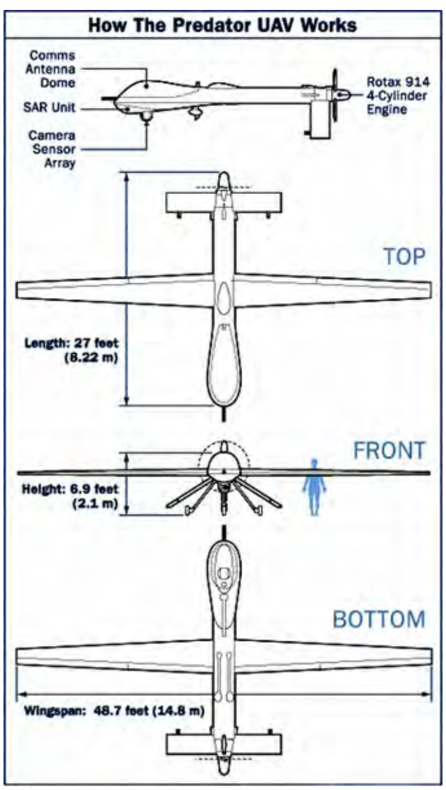
https://www.youtube.com/watch?v=wAOTOmGFs5M

Swimming

- Lift, drag, and vorticity
- Schooling behavior

Human Swimming
http://www.youtube.com/watch?v=ClzBaSiWdRA

Fish Swimming
http://www.youtube.com/watch?v=U_VJ_0wORbM

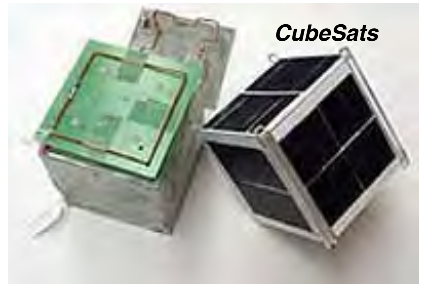


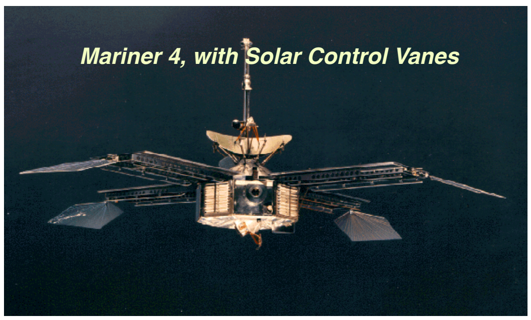
Uninhabited Aircraft

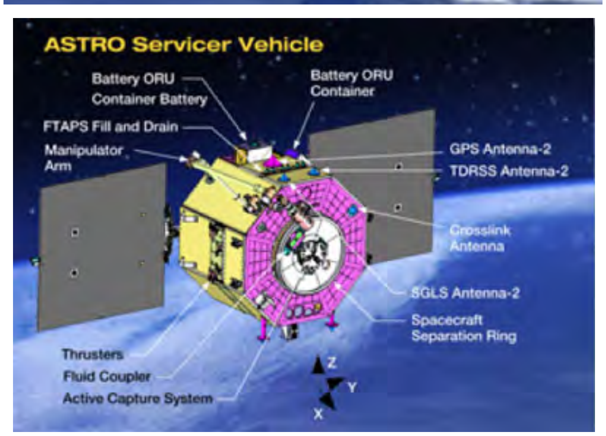
MQ-9 Reaper
http://www.youtube.com/watch?v=kSpOYZR0kIA

Aggressive Quadrotor UAV Maneuvers
http://www.youtube.com/watch?v=MvRTALJp8DM

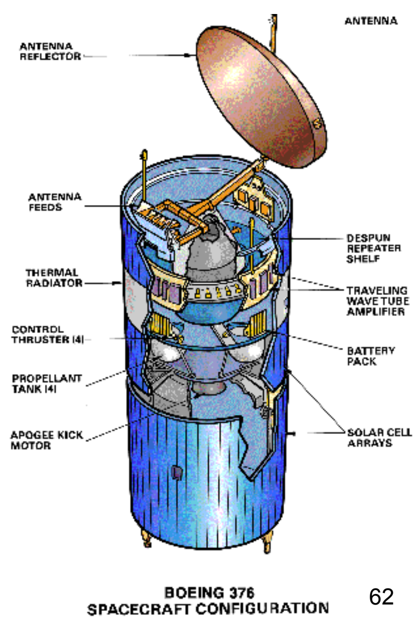
Uninhabited Spacecraft

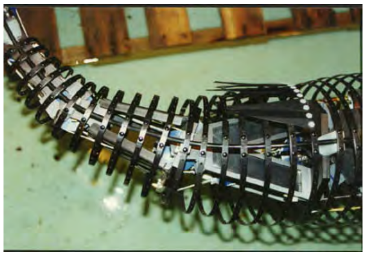






Uninhabited Spacecraft





Autonomous Underwater Vehicles

RoboTuna (MIT)

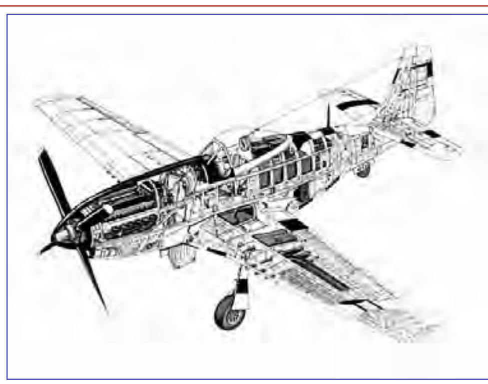
RoboLobster (Northeastern)

$$\boldsymbol{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Construction of Inertia Matrix

Build up moments and products of inertia from components using *parallel-axis theorem*, e.g.,

$$I_{xx_{airplane}} = I_{xx_{wings}} + I_{xx_{fuselage}} + I_{xx_{horizontal\ tail}} + I_{xx_{vertical\ tail}} + \ \ldots$$



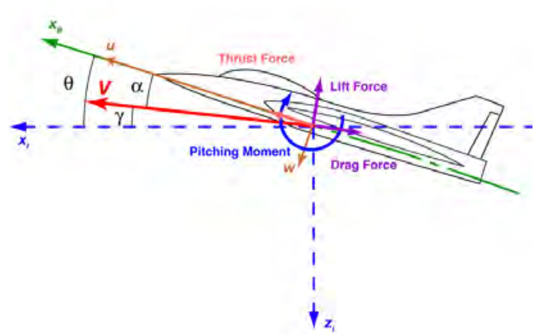
… or use software, e.g., CAD/CAM, *Creo Parametric, SimMechanics*, …

2-D Equations of Motion for a Point Mass

- Restrict motions to a vertical plane (i.e., motions in y direction = 0)
- Cartesian coordinates
- Inertial frame of reference

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{v}_x \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_z \\ \hline f_x / m \\ f_z / m \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ z \\ v_x \\ v_z \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \hline 1/m & 0 \\ 0 & 1/m \end{bmatrix} \begin{bmatrix} f_x \\ f_z \end{bmatrix}$$

Transform Velocity from Cartesian to Polar Coordinates



$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_z \end{bmatrix} = \begin{bmatrix} V\cos\gamma \\ -V\sin\gamma \end{bmatrix} \Rightarrow \begin{bmatrix} V \\ \gamma \end{bmatrix} = \begin{bmatrix} \sqrt{\dot{x}^2+\dot{z}^2} \\ -\sin^{-1}\left(\frac{\dot{z}}{V}\right) \end{bmatrix} = \begin{bmatrix} \sqrt{v_x^2+v_z^2} \\ -\sin^{-1}\left(\frac{v_z}{V}\right) \end{bmatrix}$$

$$\begin{bmatrix} \dot{V} \\ \dot{\gamma} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \sqrt{v_x^2+v_z^2} \\ -\sin^{-1}\left(\frac{v_z}{V}\right) \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}\sqrt{v_x^2+v_z^2} \\ -\frac{d}{dt}\sin^{-1}\left(\frac{v_z}{V}\right) \end{bmatrix}$$

Longitudinal Point-Mass Equations of Motion

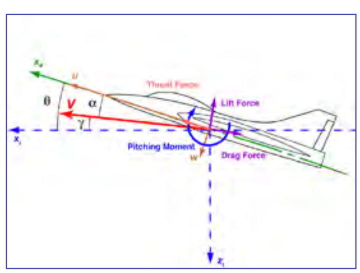
- **Assuming thrust is aligned with the velocity vector**

$$\dot{r}(t) = \dot{x}(t) = v_x = V(t)\cos\gamma(t)$$
$$\dot{h}(t) = -\dot{z}(t) = -v_z = V(t)\sin\gamma(t)$$
$$\dot{V}(t) = \frac{Thrust - Drag - mg\sin\gamma(t)}{m} = \frac{(C_T - C_D)\frac{1}{2}\rho V(t)^2 S - mg\sin\gamma(t)}{m}$$
$$\dot{\gamma}(t) = \frac{Lift - mg\cos\gamma(t)}{mV(t)} = \frac{C_L \frac{1}{2}\rho V(t)^2 S - mg\cos\gamma(t)}{mV(t)}$$

V = *velocity*
γ = *flight path angle*
h = *height (altitude)*
r = *range*

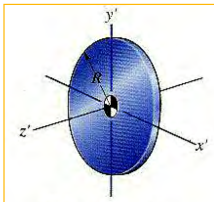
In Steady, Level Flight, $C_T = C_D$

$$\dot{r}(t) = \dot{x}(t) = v_x = V_{constant}\cos(0) = V_{constant}$$
$$\dot{h}(t) = -\dot{z}(t) = -v_z = V_{constant}\sin(0) = 0$$
$$\dot{V}(t) = \frac{Thrust - Drag - mg\sin(0)}{m} = 0$$
$$\dot{\gamma}(t) = \frac{Lift - mg\cos(0)}{mV_{constant}} = 0$$



Moments and Products of Inertia

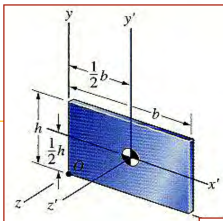
(Bedford & Fowler)



Thin circular plate

$$I_{x'\text{ axis}} = I_{y'\text{ axis}} = \frac{1}{4}mR^2, \qquad I_{z'\text{ axis}} = \frac{1}{2}mR^2,$$

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



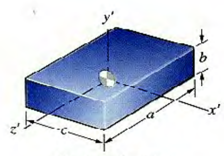
Thin rectangular plate

$$I_{x\text{ axis}} = \frac{1}{3}mh^2, \qquad I_{y\text{ axis}} = \frac{1}{3}mb^2, \qquad I_{z\text{ axis}} = \frac{1}{3}m(b^2 + h^2),$$

$$I_{xy} = \frac{1}{4}mbh, \qquad I_{yz} = I_{zx} = 0.$$

$$I_{x'\text{ axis}} = \frac{1}{12}mh^2, \qquad I_{y'\text{ axis}} = \frac{1}{12}mb^2, \qquad I_{z'\text{ axis}} = \frac{1}{12}m(b^2 + h^2),$$

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



Rectangular prism

Volume $= abc$

$$I_{x'\text{ axis}} = \frac{1}{12}m(a^2 + b^2), \qquad I_{y'\text{ axis}} = \frac{1}{12}m(a^2 + c^2),$$

$$I_{z'\text{ axis}} = \frac{1}{12}m(b^2 + c^2), \qquad I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$