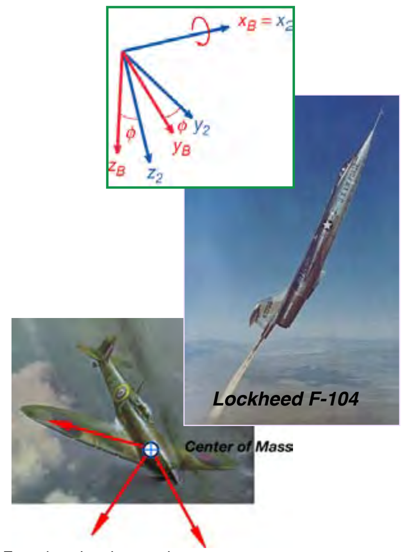
Aircraft Equations of Motion - 1

Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2014

Learning Objectives

- What use are the equations of motion?
- How is the angular orientation of the airplane described?
- What is a cross-product-equivalent matrix?
- What is angular momentum?
- How are the inertial properties of the airplane described?
- How is the rate of change of angular momentum calculated?



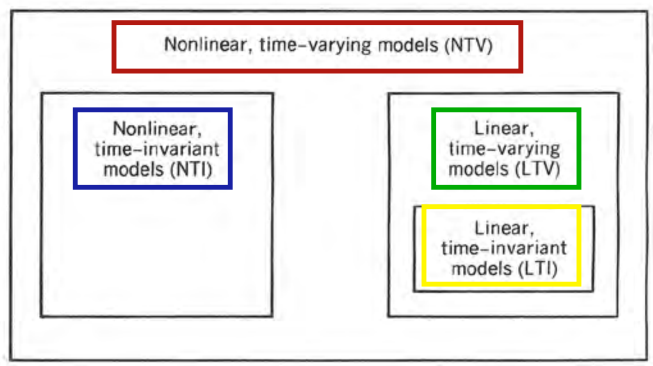
Reading:
Flight Dynamics
155-161

http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

Assignment #5
due: End of day, Oct 24, 2014

- *Takeoff from Princeton Airport, fly over Princeton and Lake Carnegie, and land at Princeton Airport*
- *"HotSeat" cockpit simulation of the Cessna 172*
- *3- and 4-member teams; each member successfully flies the circuit*
- *Individual flight testing reports*

Ordinary Differential Equations Fall Into 4 Categories



$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t\right]$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t)$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t)\right]$$

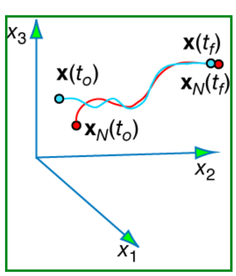
$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t) + \mathbf{L}\,\mathbf{w}(t)$$

What Use are the Equations of Motion?

- Nonlinear equations of motion
  - Compute "exact" flight paths and motions
    - Simulate flight motions
    - Optimize flight paths
    - Predict performance
  - Provide basis for approximate solutions

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t\right]$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t) + \mathbf{L}\,\mathbf{w}(t)$$

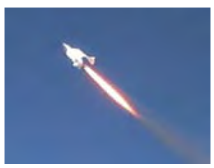


- Linear equations of motion
  - Simplify computation of flight paths and solutions
  - Define modes of motion
  - Provide basis for control system design and flying qualities analysis

Examples of Airplane Dynamic System Models

- **Nonlinear, Time-Varying**
  - Large amplitude motions
  - Significant change in mass

- **Linear, Time-Varying**
  - Small amplitude motions
  - Perturbations from a dynamic flight path



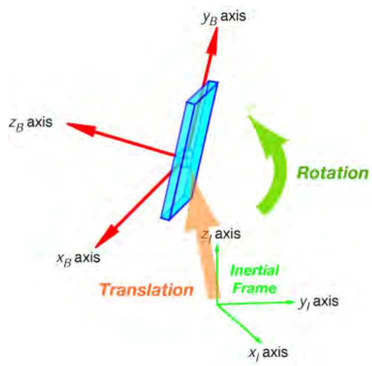
- **Nonlinear, Time-Invariant**
  - Large amplitude motions
  - Negligible change in mass

- **Linear, Time-Invariant**
  - Small amplitude motions
  - Perturbations from an equilibrium flight path

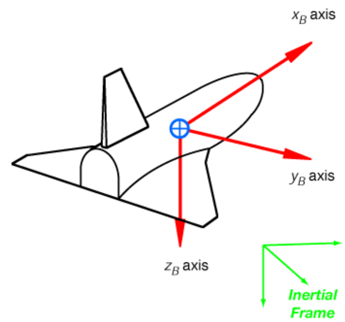
Translational Position

Cartesian Frames of Reference

- Two reference frames of interest
  - *I*: Inertial frame (fixed to inertial space)
  - *B*: Body frame (fixed to body)
- Translation
  - Relative linear positions of origins
- Rotation
  - Orientation of the body frame with respect to the inertial frame



Common convention (*z* up)



Aircraft convention (*z* down)

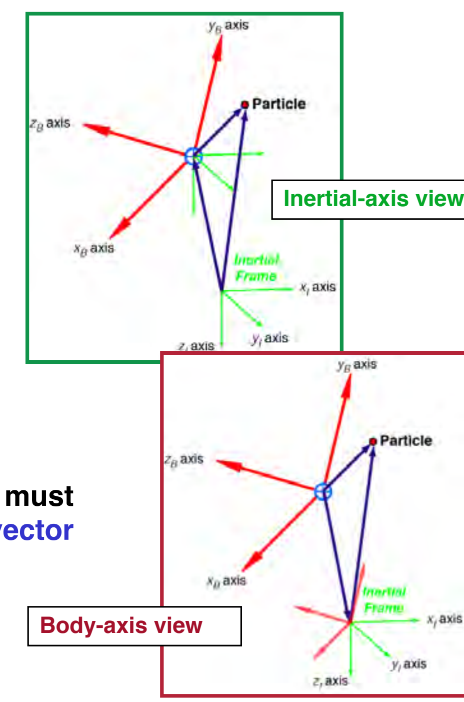
Measurement of Position in Alternative Frames - 1

- Two reference frames of interest
  - *I*: Inertial frame (fixed to inertial space)
  - *B*: Body frame (fixed to body)

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

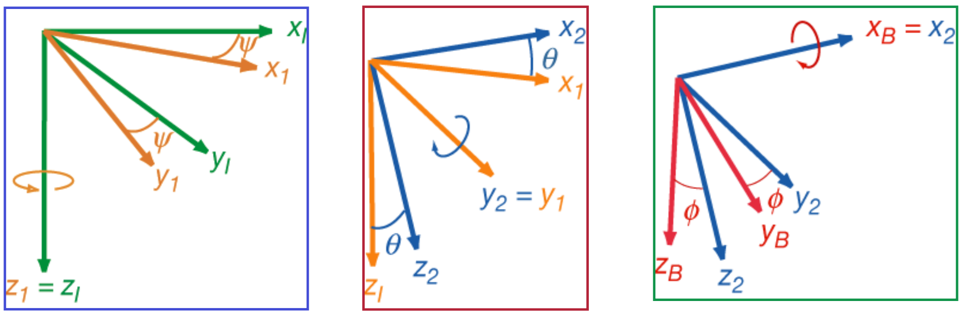
$$\mathbf{r}_{particle} = \mathbf{r}_{origin} + \Delta\mathbf{r}_{w.r.t.origin}$$

- Differences in frame orientations must be taken into account in adding vector components



Euler Angles Measure the Orientation of One Frame with Respect to the Other

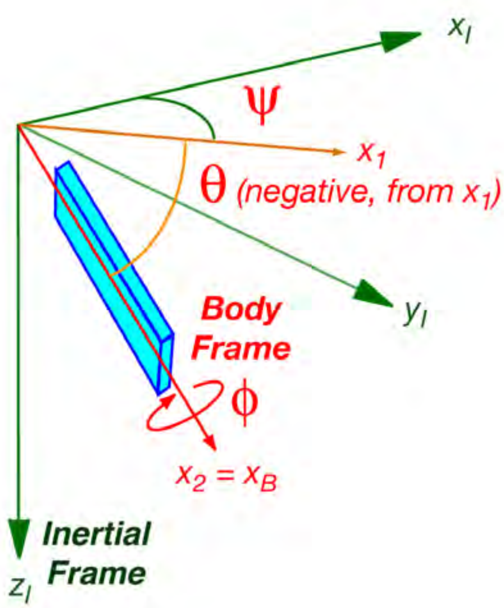
- **Conventional sequence of rotations from inertial to body frame**
  - **Each rotation is about a single axis**
  - **Right-hand rule**
  - **Yaw, then pitch, then roll**
  - **These are called Euler Angles**



Yaw rotation (ψ) about z_I **Pitch rotation (θ) about y_1** **Roll rotation (φ) about x_2**

Other sequences of 3 rotations can be chosen; however, once sequence is chosen, it must be retained

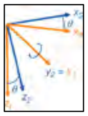
Euler Angles



Effects of Rotation on Vector Transformation from Inertial to Body Frame of Reference

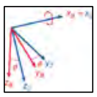
Yaw rotation (ψ) about z_I – *Intermediate Frame 1*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_I = \begin{bmatrix} x_I\cos\psi + y_I\sin\psi \\ -x_I\sin\psi + y_I\cos\psi \\ z_I \end{bmatrix}; \quad \mathbf{r}_1 = \mathbf{H}_I^1 \mathbf{r}_I$$



Pitch rotation (θ) about y_1 – *Intermediate Frame 2*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_1; \quad \mathbf{r}_2 = \mathbf{H}_1^2 \mathbf{r}_1 = \left[\mathbf{H}_1^2 \mathbf{H}_I^1\right] \mathbf{r}_I = \mathbf{H}_I^2 \mathbf{r}_I$$



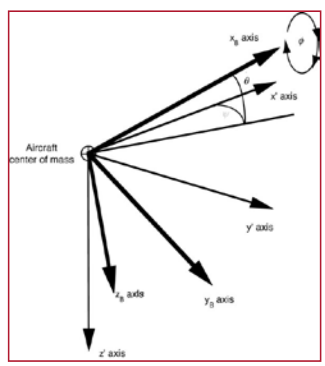
Roll rotation (φ) about x_2 - *Body Frame*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_2; \quad \mathbf{r}_B = \mathbf{H}_2^B \mathbf{r}_2 = \left[\mathbf{H}_2^B \mathbf{H}_1^2 \mathbf{H}_I^1\right] \mathbf{r}_I = \mathbf{H}_I^B \mathbf{r}_I$$

The Rotation Matrix

- **The three-angle rotation matrix is the product of 3 single-angle rotation matrices:**

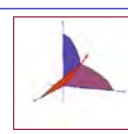
$$\boxed{\mathbf{H}_I^B(\phi,\theta,\psi) = \mathbf{H}_2^B(\phi)\,\mathbf{H}_1^2(\theta)\,\mathbf{H}_I^1(\psi)}$$



$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix}$$

also called *Direction Cosine Matrix (see supplement)*



Properties of the Rotation Matrix

$$\mathbf{H}_I^B(\phi,\theta,\psi) = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}_I^B$$

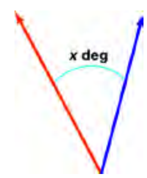
$$\mathbf{H}_I^B(\phi,\theta,\psi) = \mathbf{H}_2^B(\phi)\mathbf{H}_1^2(\theta)\mathbf{H}_I^1(\psi)$$

$$\left[\mathbf{H}_I^B(\phi,\theta,\psi)\right]^{-1} = \left[\mathbf{H}_I^B(\phi,\theta,\psi)\right]^T = \mathbf{H}_B^I(\psi,\theta,\phi)$$

- **The rotation matrix produces an orthonormal transformation**
  - **Angles are preserved**
  - **Lengths are preserved**

$$\left|\mathbf{r}_I\right| = \left|\mathbf{r}_B\right| \quad ; \quad \left|\mathbf{s}_I\right| = \left|\mathbf{s}_B\right|$$

$$\angle(\mathbf{r}_I,\mathbf{s}_I) = \angle(\mathbf{r}_B,\mathbf{s}_B) = x\,\text{deg}$$



Properties of the Rotation Matrix

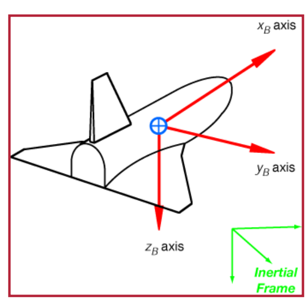
- **Inverse relationship; interchange sub- and superscripts**

$$\mathbf{r}_B = \mathbf{H}_I^B\mathbf{r}_I \quad ; \quad \mathbf{r}_I = \left(\mathbf{H}_I^B\right)^{-1}\mathbf{r}_B = \mathbf{H}_B^I\mathbf{r}_B$$

- **Because transformation is orthonormal,**
  - **Inverse = transpose**
  - **Rotation matrix is always non-singular**

$$\mathbf{H}_B^I = \left(\mathbf{H}_I^B\right)^{-1} = \left(\mathbf{H}_I^B\right)^T = \mathbf{H}_1^I\mathbf{H}_2^1\mathbf{H}_B^2$$

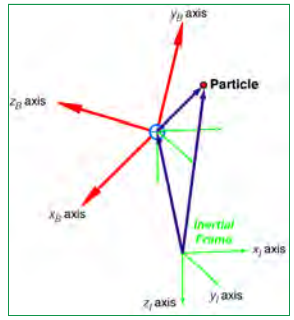
$$\mathbf{H}_B^I\mathbf{H}_I^B = \mathbf{H}_I^B\mathbf{H}_B^I = \mathbf{I}$$



Measurement of Position in Alternative Frames - 2

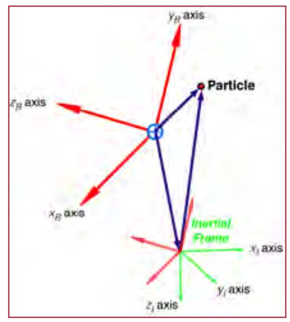
Inertial-axis view

$$\mathbf{r}_{particle_I} = \mathbf{r}_{origin-B_I} + H_B^I \Delta\mathbf{r}_B$$



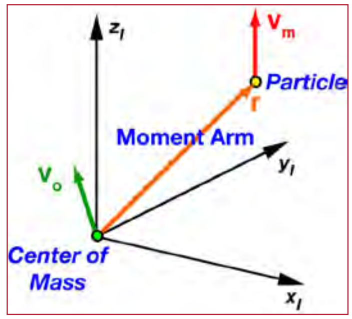
Body-axis view

$$\mathbf{r}_{particle_B} = \mathbf{r}_{origin-I_B} + H_I^B \Delta\mathbf{r}_I$$



Angular Momentum

Angular Momentum of a Particle



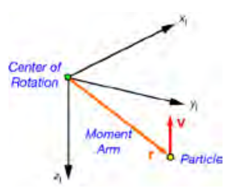
- Moment of linear momentum of differential particles that make up the body
  - (Differential masses) x components of the velocity that are perpendicular to the moment arms

$$d\mathbf{h} = (\mathbf{r} \times dm\mathbf{v}) = (\mathbf{r} \times \mathbf{v}_m)\,dm$$
$$= (\mathbf{r} \times (\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r}))\,dm$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

- Cross Product: Evaluation of a determinant with unit vectors (*i, j, k*) along axes, (*x, y, z*) and (v_x, v_y, v_z) projections on to axes

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = (yv_z - zv_y)\boldsymbol{i} + (zv_x - xv_z)\boldsymbol{j} + (xv_y - yv_x)\boldsymbol{k}$$



Cross-Product-Equivalent Matrix

Cross product

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = (yv_z - zv_y)\boldsymbol{i} + (zv_x - xv_z)\boldsymbol{j} + (xv_y - yv_x)\boldsymbol{k}$$

$$= \begin{bmatrix} (yv_z - zv_y) \\ (zv_x - xv_z) \\ (xv_y - yv_x) \end{bmatrix} = \tilde{\mathbf{r}}\mathbf{v} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Cross-product-equivalent matrix

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Angular Momentum of the Aircraft

- **Integrate moment of linear momentum of differential particles over the body**

$$\mathbf{h} = \int_{Body} \left(\mathbf{r} \times \left(\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r}\right)\right) dm = \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} \left(\mathbf{r} \times \mathbf{v}\right) \rho(x,y,z)\, dx\, dy\, dz = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}$$

$\rho(x,y,z) =$ Density of the body

- **Choose the center of mass as the rotational center**

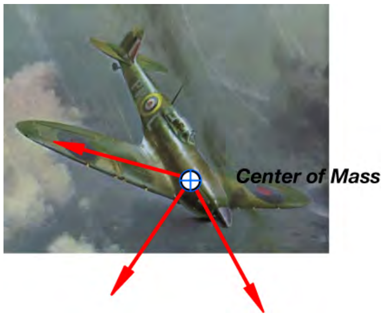
$$\mathbf{h} = \int_{Body} \left(\mathbf{r} \times \mathbf{v}_o\right) dm + \int_{Body} \left(\mathbf{r} \times \left(\boldsymbol{\omega} \times \mathbf{r}\right)\right) dm$$

$$= 0 - \int_{Body} \left(\mathbf{r} \times \left(\mathbf{r} \times \boldsymbol{\omega}\right)\right) dm = -\int_{Body} \left(\mathbf{r} \times \mathbf{r}\right) dm \times \boldsymbol{\omega}$$

$$\equiv -\int_{Body} \left(\tilde{\mathbf{r}}\tilde{\mathbf{r}}\right) dm \boldsymbol{\omega}$$

Location of the Center of Mass

$$\mathbf{r}_{cm} = \frac{1}{m} \int_{Body} \mathbf{r}\, dm = \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} \mathbf{r} \rho(x,y,z)\, dx\, dy\, dz = \begin{bmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \end{bmatrix}$$



The Inertia Matrix

The Inertia Matrix

$$\mathbf{h} = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,\boldsymbol{\omega}\,dm = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,dm\,\boldsymbol{\omega} = \boldsymbol{I}\boldsymbol{\omega}$$

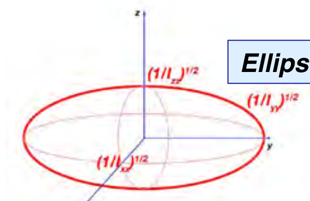
$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where

$$\boldsymbol{I} = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,dm = -\int_{Body} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} dm$$

$$= \int_{Body} \begin{bmatrix} (y^2 + z^2) & -xy & -xz \\ -xy & (x^2 + z^2) & -yz \\ -xz & -yz & (x^2 + y^2) \end{bmatrix} dm$$

- **Inertia matrix derives from equal effect of angular rate on all particles of the aircraft**



Ellipsoid of Inertia

$$I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 = 1$$

Moments and Products of Inertia

- **Inertia matrix**

$$\mathbf{I} = \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

  - **Moments of inertia on the diagonal**
  - **Products of inertia off the diagonal**

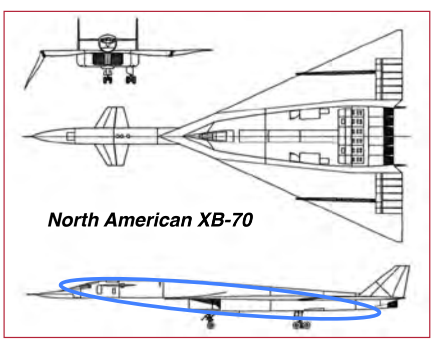
- **If products of inertia are zero, (x, y, z) are principal axes --->**
- **All rigid bodies have a set of principal axes**

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

Inertia Matrix of an Aircraft with Mirror Symmetry

$$\mathbf{I} = \int_{Body} \begin{bmatrix} (y^2+z^2) & 0 & -xz \\ 0 & (x^2+z^2) & 0 \\ -xz & 0 & (x^2+y^2) \end{bmatrix} dm = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}$$

- **Nose high/low product of inertia, I_{xz}**



North American XB-70

Nominal Configuration

Tips folded, 50% fuel, W = 38,524 lb

x_{cm} @0.218 $\bar{c}$

$I_{xx} = 1.8 \times 10^6$ slug-ft^2

$I_{yy} = 19.9 \times 10^6$ slug-ft^2

$I_{xx} = 22.1 \times 10^6$ slug-ft^2

$I_{xz} = -0.88 \times 10^6$ slug-ft^2

Historical Factoids

Technology of World War II Aviation

- **1938-45:** Analytical and experimental approach to design
  - Many configurations designed and flight-tested
  - Increased specialization; radar, navigation, and communication
  - Approaching the "sonic barrier"
- **Aircraft Design**
  - Large, powerful, high-flying aircraft
  - Turbocharged engines
  - Oxygen and Pressurization

Power Effects on Stability and Control

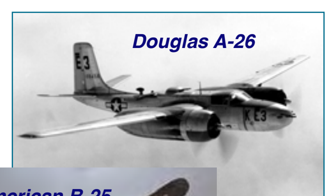
- ***Brewster Buffalo***: over-armored and under-powered
- During W.W.II, the size of fighters remained about the same, but installed horsepower doubled (*F4F* vs. *F8F*)
- Use of flaps means high power at low speed, increasing relative significance of thrust effects

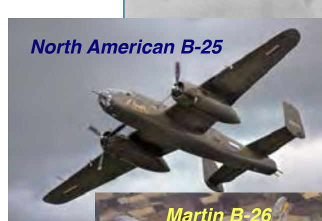
World War II Carrier-Based Airplanes

- Takeoff without catapult, relatively low landing speed
  http://www.youtube.com/watch?v=4dySbhK1vNk
- Tailhook and arresting gear
- Carrier steams into wind
- Design for storage (short tail length, folding wings) affects stability and control

Multi-Engine Aircraft of World War II

- Large W.W.II aircraft had unpowered controls:
  - High foot-pedal force
  - Rudder stability problems arising from balancing to reduce pedal force
- Severe engine-out problem for twin-engine aircraft





WW II Military Flying Boats

Seaplanes proved useful during World War II

Rate of Change of Angular Momentum

Newton's 2nd Law, Applied to Rotational Motion

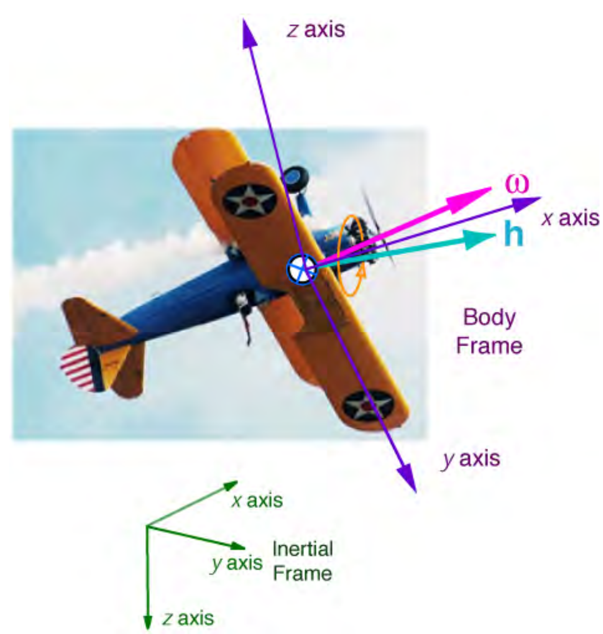
- In inertial frame, rate of change of angular momentum = applied moment (or torque), **M**

$$\frac{d\mathbf{h}}{dt} = \frac{d(\boldsymbol{I}\omega)}{dt} = \frac{d\boldsymbol{I}}{dt}\omega + \boldsymbol{I}\frac{d\omega}{dt}$$
$$= \dot{\boldsymbol{I}}\omega + \boldsymbol{I}\dot{\omega} = \mathbf{M} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}$$

Angular Momentum and Rate

- Angular momentum and rate vectors are not necessarily aligned

$$\boxed{\mathbf{h} = \boldsymbol{I}\omega}$$



How Do We Get Rid of *dI/dt* in the Angular Momentum Equation?

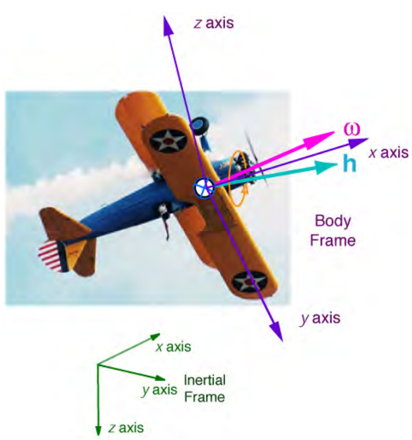
- Chain Rule

$$\boxed{\frac{d(\boldsymbol{I}\omega)}{dt} = \dot{\boldsymbol{I}}\omega + \boldsymbol{I}\dot{\omega}}$$

... and in an inertial frame

$$\boxed{\dot{\boldsymbol{I}} \neq 0}$$

- Dynamic equation in a body-referenced frame
  - Inertial properties of a constant-mass, rigid body are unchanging in a body frame of reference
  - ... but a body-referenced frame is "non-Newtonian" or "non-inertial"
  - Therefore, dynamic equation must be modified for expression in a rotating frame



Angular Momentum Expressed in Two Frames of Reference

- Angular momentum and rate are vectors
  - Expressed in either the inertial or body frame
  - Two frames related algebraically by the rotation matrix

$$\boxed{\mathbf{h}_B(t) = \mathbf{H}_I^B(t)\mathbf{h}_I(t); \qquad \mathbf{h}_I(t) = \mathbf{H}_B^I(t)\mathbf{h}_B(t)}$$

$$\boxed{\omega_B(t) = \mathbf{H}_I^B(t)\omega_I(t); \qquad \omega_I(t) = \mathbf{H}_B^I(t)\omega_B(t)}$$

Vector Derivative Expressed in a Rotating Frame

- Chain Rule

Effect of body-frame rotation

$$\dot{\mathbf{h}}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \dot{\mathbf{H}}_B^I \mathbf{h}_B$$

Rate of change expressed in body frame

- Alternatively

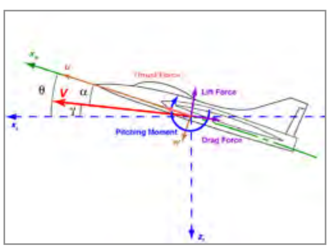
$$\dot{\mathbf{h}}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \boldsymbol{\omega}_I \times \mathbf{h}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \tilde{\boldsymbol{\omega}}_I \mathbf{h}_I$$

- Consequently, the 2nd term is

$$\dot{\mathbf{H}}_B^I \mathbf{h}_B = \tilde{\boldsymbol{\omega}}_I \mathbf{h}_I = \tilde{\boldsymbol{\omega}}_I \mathbf{H}_B^I \mathbf{h}_B$$

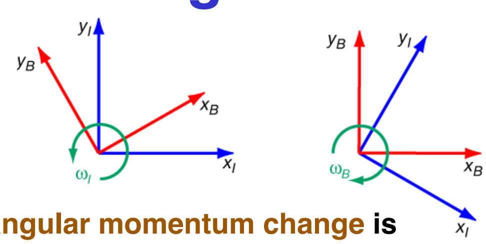
... where the cross-product-equivalent matrix of angular rate is

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$



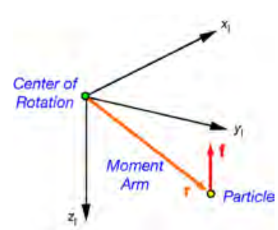
External Moment Causes Change in Angular Rate

Positive rotation of Frame B w.r.t. Frame A is a negative rotation of Frame A w.r.t. Frame B



- In the body frame of reference, the angular momentum change is

$$\begin{aligned}\dot{\mathbf{h}}_B &= \mathbf{H}_I^B \dot{\mathbf{h}}_I + \dot{\mathbf{H}}_I^B \mathbf{h}_I = \mathbf{H}_I^B \dot{\mathbf{h}}_I - \boldsymbol{\omega}_B \times h_B \\ &= \mathbf{H}_I^B \dot{\mathbf{h}}_I - \tilde{\boldsymbol{\omega}}_B h_B = \mathbf{H}_I^B \mathbf{M}_I - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B \\ &= \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B\end{aligned}$$



- Moment = torque = force x moment arm

$$\mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}_I ; \quad \mathbf{M}_B = \mathbf{H}_I^B \mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

Rate of Change of Body-Referenced Angular Rate due to External Moment

- In the body frame of reference, the angular momentum change is

$$\boxed{\begin{aligned}\dot{\mathbf{h}}_B &= \mathbf{H}_I^B\dot{\mathbf{h}}_I + \dot{\mathbf{H}}_I^B\mathbf{h}_I = \mathbf{H}_I^B\dot{\mathbf{h}}_I - \boldsymbol{\omega}_B \times h_B \\ &= \mathbf{H}_I^B\dot{\mathbf{h}}_I - \tilde{\boldsymbol{\omega}}_B h_B = \mathbf{H}_I^B\mathbf{M}_I - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B \\ &= \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B\end{aligned}}$$

- For constant body-axis inertia matrix

$$\boxed{\dot{\mathbf{h}}_B = \boldsymbol{I}_B\dot{\boldsymbol{\omega}}_B = \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B}$$

- Consequently, the differential equation for angular rate of change is

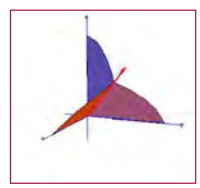
$$\boxed{\dot{\boldsymbol{\omega}}_B = \boldsymbol{I}_B^{-1}\left(\mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \boldsymbol{I}_B \boldsymbol{\omega}_B\right)}$$

Next Time: Aircraft Equations of Motion – 2

Reading:
Flight Dynamics
161-180

SUPPLEMENTAL MATERIAL

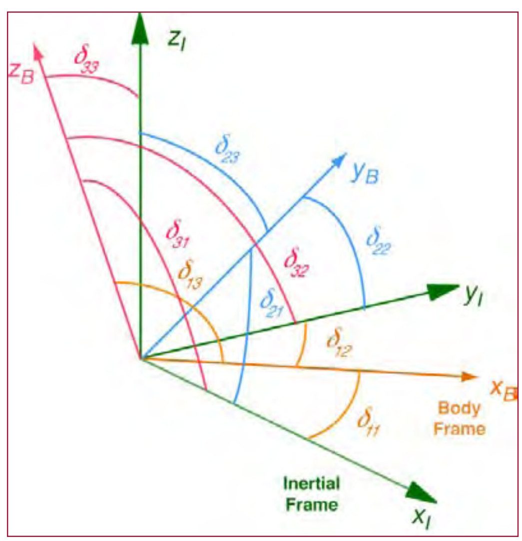
Direction Cosine Matrix (also called Rotation Matrix)

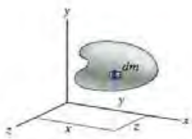


$$\mathbf{H}_I^B = \begin{bmatrix} \cos\delta_{11} & \cos\delta_{21} & \cos\delta_{31} \\ \cos\delta_{12} & \cos\delta_{22} & \cos\delta_{32} \\ \cos\delta_{13} & \cos\delta_{23} & \cos\delta_{33} \end{bmatrix}$$

- **Cosines of angles between each *I* axis and each *B* axis**
- **Projections of vector components**

$$\mathbf{r}_B = \mathbf{H}_I^B \mathbf{r}_I$$

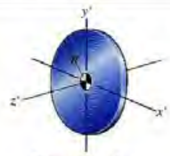




$$I_{x\,\text{axis}} = I_{xx} = \int_m (y^2 + z^2)\, dm,$$
$$I_{y\,\text{axis}} = I_{yy} = \int_m (x^2 + z^2)\, dm,$$
$$I_{z\,\text{axis}} = I_{zz} = \int_m (x^2 + y^2)\, dm,$$
$$I_{xy} = \int_m xy\, dm, \qquad I_{yz} = \int_m yz\, dm,$$
$$I_{zx} = \int_m zx\, dm.$$

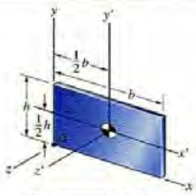
Slender bar

$$I_{x\,\text{axis}} = 0, \qquad I_{y\,\text{axis}} = I_{z\,\text{axis}} = \frac{1}{3} ml^2,$$
$$I_{xy} = I_{yz} = I_{zx} = 0.$$
$$I_{x'\,\text{axis}} = 0, \qquad I_{y'\,\text{axis}} = I_{z'\,\text{axis}} = \frac{1}{12} ml^2,$$
$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



Thin circular plate

$$I_{x'\,\text{axis}} = I_{y'\,\text{axis}} = \frac{1}{4} mR^2, \qquad I_{z'\,\text{axis}} = \frac{1}{2} mR^2,$$
$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



Thin rectangular plate

$$I_{x\,\text{axis}} = \frac{1}{3} mh^2, \qquad I_{y\,\text{axis}} = \frac{1}{3} mb^2, \qquad I_{z\,\text{axis}} = \frac{1}{3} m(b^2 + h^2),$$
$$I_{xy} = \frac{1}{4} mbh, \qquad I_{yz} = I_{zx} = 0.$$
$$I_{x'\,\text{axis}} = \frac{1}{12} mh^2, \qquad I_{y'\,\text{axis}} = \frac{1}{12} mb^2, \qquad I_{z'\,\text{axis}} = \frac{1}{12} m(b^2 + h^2),$$
$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$

Moments and Products of Inertia

(Bedford & Fowler)

- Moments and products of inertia tabulated for geometric shapes with uniform density
- Construct aircraft moments and products of inertia from components using parallel-axis theorem
- Model in *Creo*, etc.