Analog and Digital Control Systems

Robert Stengel
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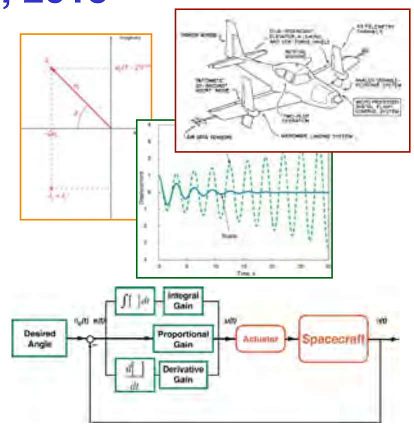
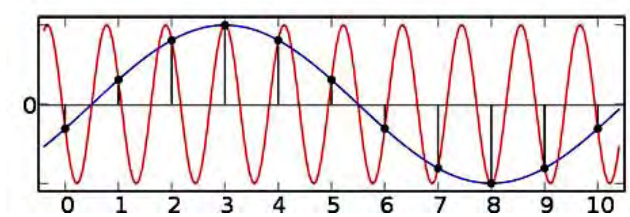
- Frequency Response
- Transfer Functions
- Bode Plots
- Root Locus
- Proportional-Integral-Derivative (PID) Control
- Discrete-time Dynamic Models



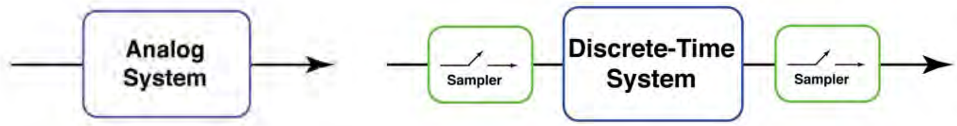
Analog vs. Digital Signals



- Signal: A physical indicator that conveys information, e.g., position, voltage, temperature, or pressure
- Analog signal: A signal that is continuous in time, with infinite precision and infinitesimal time spacing between indication points
- Discrete-time signal: A signal with infinite precision and discrete spacing between indication points
- Digital signal: A signal with finite precision and discrete spacing between indication points

Analog vs. Digital Systems

- **System: Assemblage of parts with structure, connectivity, and behavior that responds to input signals and produces output signals**

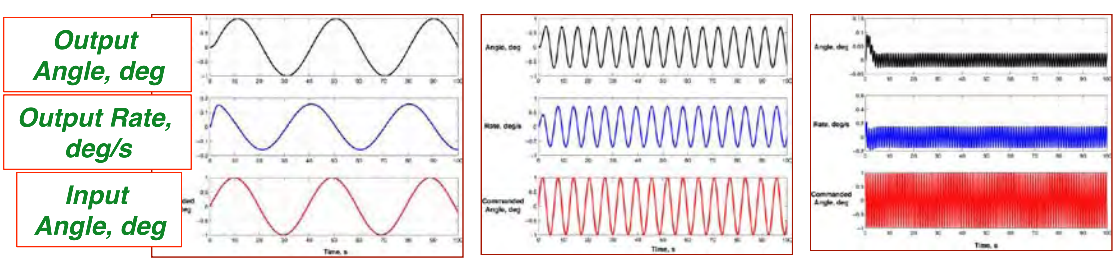


- **Analog system: A system that operates continuously, with infinite precision and infinitesimal time spacing between signaling points**
- **Discrete-time system: A system that operates continuously, with infinite precision and discrete spacing between signaling points**

- **Digital system: A system that operates continuously, with finite precision and discrete spacing between signaling points**

Frequency Response of a Continuous-Time Dynamic System

$\omega < \omega_n$ | $\omega = \omega_n$ | $\omega > \omega_n$



ω : Input frequency, rad/s

ω_n : Natural frequency of the system, rad/s

Hz : **Hertz**, frequency in cycles per sec

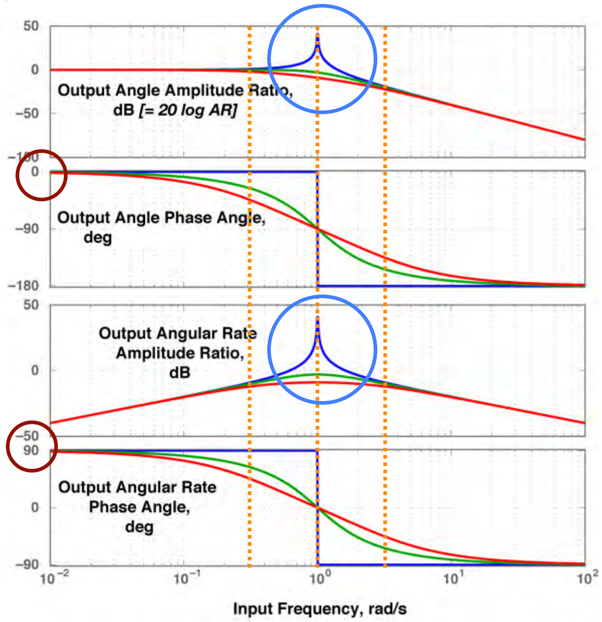
$$\omega(Hz) = \frac{\omega(rad/s)}{2\pi}$$

- **Sinusoidal command input (i.e., Desired rotational angle)**
- **Long-term output of the dynamic system:**
  - **Sinusoid with same frequency as input**
  - **Output/input amplitude ratio dependent on input frequency**
  - **Output/input phase shift dependent on input frequency**
- **Bandwidth: Input frequency below which output amplitude and phase angle variations are negligibly small**

Amplitude-Ratio and Phase-Angle Frequency Response of a 2nd-Order System

Output Angle, deg

Output Rate, deg/s

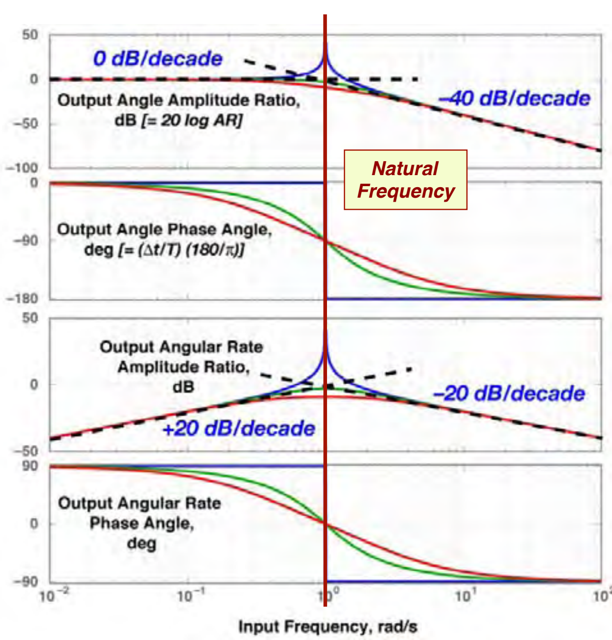


Straight-line asymptotes at high and low frequency

Straight-line asymptotes at high and low frequency

$$20 \log AR \ and \ \phi \ vs. \log \omega$$

Asymptotes of Frequency Response Amplitude Ratio



$20 \log_{10}\left(\omega^n\right)$ has slope of $20n$ dB/decade on Bode plot

$$20 \log_{10} \boldsymbol{H}_{CL_i}(j\omega) =$$

$$= 20 \log_{10}\left[AR_i(\omega)\, e^{j\phi_i(\omega)}\right]$$

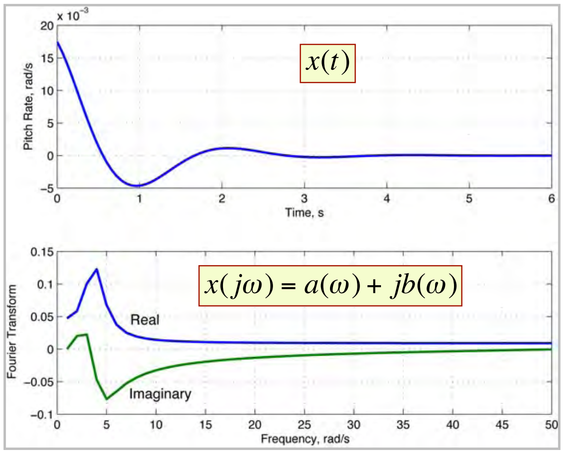
$$= 20\left[\log_{10} AR_i + \log_{10} e^{j\phi_i(\omega)}\right]$$
$$= 20 \log_{10} AR_i + j\phi_i(\omega)\left(20 \log_{10} e\right)$$

- **Logarithm of transfer function**
  - **Amplitude ratio and phase angle are orthogonal**
  - **Decibels (dB): 20 log *AR***
  - **Decade: power of 10**

Fourier Transform of a Scalar Variable

$$\boxed{\mathcal{F}\left[x(t)\right] = x(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \quad \omega = frequency, rad/s}$$

$$j \triangleq i \triangleq \sqrt{-1}$$



$x(t)$: **real** variable

$x(j\omega)$: **complex** variable

$= a(\omega) + jb(\omega)$

$= A(\omega)e^{j\varphi(\omega)}$

A : **amplitude**

φ : **phase angle**

Laplace Transforms of Scalar Variables

- Laplace transform of a scalar variable is a **complex number**
- ***s*** is the ***Laplace operator***, a complex variable

$$\boxed{\mathcal{L}\left[x(t)\right] = x(s) = \int_{0}^{\infty} x(t)e^{-st}dt, \quad s = \sigma + j\omega}$$

$x(t)$: **real** variable

$x(s)$: **complex** variable

$= a(\omega) + jb(\omega)$

$= A(\omega)e^{j\varphi(\omega)}$

Multiplication by a constant

$$\mathcal{L}\left[a\,x(t)\right] = a\,x(s)$$

Sum of Laplace transforms

$$\mathcal{L}\left[x_1(t) + x_2(t)\right] = x_1(s) + x_2(s)$$

Laplace Transforms of Vectors and Matrices

Laplace transform of a vector variable

$$\mathcal{L}[\mathbf{x}(t)] = \mathbf{x}(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \\ \ldots \end{bmatrix}$$

Laplace transform of a matrix variable

$$\mathcal{L}[\mathbf{A}(t)] = \mathbf{A}(s) = \begin{bmatrix} a_{11}(s) & a_{12}(s) & \ldots \\ a_{21}(s) & a_{22}(s) & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

Laplace transform of a derivative w.r.t. time

$$\mathcal{L}\left[\frac{d\mathbf{x}(t)}{dt}\right] = s\mathbf{x}(s) - \mathbf{x}(0)$$

Laplace transform of an integral over time

$$\mathcal{L}\left[\int \mathbf{x}(\tau)\,d\tau\right] = \frac{\mathbf{x}(s)}{s}$$

Laplace Transforms of the System Equations

Time-Domain System Equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t)$$

Dynamic Equation

$$\mathbf{y}(t) = \mathbf{H}_\mathbf{x}\mathbf{x}(t) + \mathbf{H}_\mathbf{u}\mathbf{u}(t)$$

Output Equation

Laplace Transforms of System Equations

$$s\mathbf{x}(s) - \mathbf{x}(0) = \mathbf{F}\,\mathbf{x}(s) + \mathbf{G}\,\mathbf{u}(s)$$

Dynamic Equation

$$\mathbf{y}(s) = \mathbf{H}_\mathbf{x}\mathbf{x}(s) + \mathbf{H}_\mathbf{u}\mathbf{u}(s)$$

Output Equation

Example of System Transformation

DC Motor Equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u(t)$$

Dynamic Equation

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Output Equation

Laplace Transforms of Motor Equations

$$\begin{bmatrix} sx_1(s) - x_1(0) \\ sx_2(s) - x_2(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u(s)$$

Dynamic Equation

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix}$$

Output Equation

Laplace Transform of State Response to Initial Condition and Control

Rearrange

$$s\mathbf{x}(s) - \mathbf{F}\,\mathbf{x}(s) = \mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)$$

$$\left[s\mathbf{I} - \mathbf{F}\right]\mathbf{x}(s) = \mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)$$

$$\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)\right]$$

Matrix inverse

$$\left[s\mathbf{I} - \mathbf{F}\right]^{-1} = \frac{Adj\left(s\mathbf{I} - \mathbf{F}\right)}{\left|s\mathbf{I} - \mathbf{F}\right|} \quad (n\ x\ n)$$

$Adj\left(s\mathbf{I} - \mathbf{F}\right)$: **Adjoint matrix** $(n \times n)$: **Transpose of matrix of cofactors**

$\left|s\mathbf{I} - \mathbf{F}\right| = \det\left(s\mathbf{I} - \mathbf{F}\right)$: **Determinant** (1×1)

Characteristic Polynomial of a Dynamic System

Characteristic matrix

$$
(s\mathbf{I}-\mathbf{F}) = \begin{pmatrix} (s-f_{11}) & -f_{12} & \dots & -f_{1n} \\ -f_{21} & (s-f_{22}) & \dots & -f_{2n} \\ \dots & \dots & \dots & \dots \\ -f_{n1} & -f_{n2} & \dots & (s-f_{nn}) \end{pmatrix} \quad (n \; x \; n)
$$

Characteristic polynomial

$$
\begin{aligned} |s\mathbf{I}-\mathbf{F}| &= \det(s\mathbf{I}-\mathbf{F}) \\ \equiv \Delta(s) &= s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 \end{aligned}
$$

Eigenvalues (or Roots) of the Dynamic System

Characteristic equation

$$
\begin{aligned} \Delta(s) &= s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 = 0 \\ &= (s-\lambda_1)(s-\lambda_2)(\dots)(s-\lambda_n) = 0 \end{aligned}
$$

λ_i are eigenvalues of $\mathbf{F}$ or roots of the characteristic polynomial, $\Delta(s)$

2 x 2 Eigenvalue Example

Characteristic matrix

$$(s\mathbf{I}-\mathbf{F}) = \begin{bmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{bmatrix}$$

Determinant of characteristic matrix

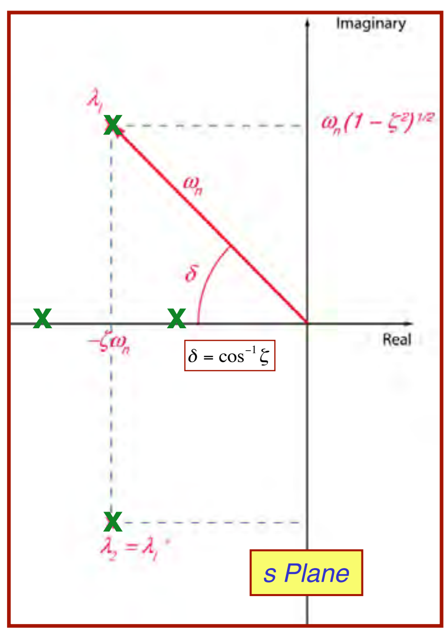
$$\begin{aligned} |s\mathbf{I}-\mathbf{F}| &= \begin{vmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{vmatrix} \\ &= (s-f_{11})(s-f_{22}) - f_{12}f_{21} \\ &= s^2 - (f_{11}+f_{22})s + (f_{11}f_{22} - f_{12}f_{21}) \end{aligned}$$

Factors of the 2nd-Degree Characteristic Equation

$$\Delta(s) = s^2 - (f_{12}+f_{21})s + (f_{11}f_{22} - f_{12}f_{21}) = 0$$

$$\Delta(s) = (s-\lambda_1)(s-\lambda_2) = 0$$

- Solutions of the equation are *eigenvalues* of the system, either
  - 2 real roots, or
  - Complex-conjugate pair



$$\lambda_1 = \sigma_1 + j\omega_1, \quad \lambda_2 = \sigma_1 - j\omega_1$$

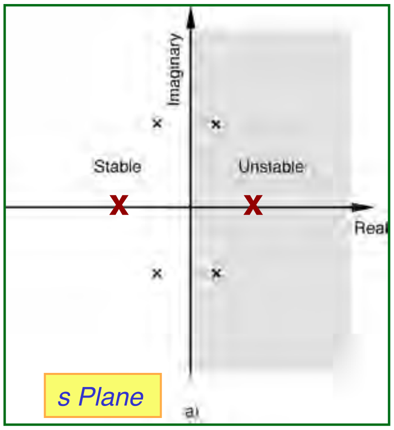
$$\lambda_1 = \sigma_1, \quad \lambda_2 = \sigma_2$$

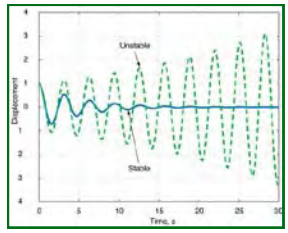
$$\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Eigenvalues Determine the Stability of the LTI System

Positive real part represents instability

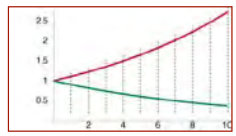
Envelope of time response converges or diverges





$$\Delta x_1(t) = e^{-\zeta\omega_n t}\cos\left[\omega_n\sqrt{1-\zeta^2}\,t + \varphi_1\right]\Delta x_1(0)$$

Same criterion for real roots



$$\Delta x(t) = e^{at}\Delta x(0)$$

Numerator of the Matrix Inverse

Matrix Inverse

$$\left[s\mathbf{I}-\mathbf{F}\right]^{-1} = \frac{Adj\left(s\mathbf{I}-\mathbf{F}\right)}{\left|s\mathbf{I}-\mathbf{F}\right|} \quad (n\ x\ n)$$

Adjoint matrix is the transpose of the matrix of cofactors*

$$Adj\left(s\mathbf{I}-\mathbf{F}\right) = \mathbf{C}^T \quad (n\ x\ n)$$

2 x 2 example

$$\left(s\mathbf{I}-\mathbf{F}\right) = \begin{bmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{bmatrix}$$

$$Adj\left(s\mathbf{I}-\mathbf{F}\right) = \begin{bmatrix} (s-f_{22}) & f_{21} \\ f_{12} & (s-f_{11}) \end{bmatrix}^T = \begin{bmatrix} (s-f_{22}) & f_{12} \\ f_{21} & (s-f_{11}) \end{bmatrix}$$

** Cofactors = signed minor determinants of the matrix*

Analog Control

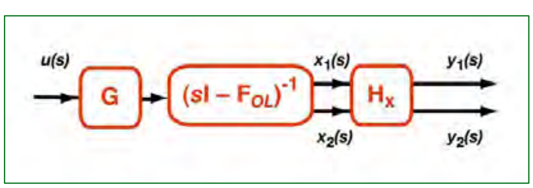
Transfer Function Matrix

Laplace Transform of Output Vector
from control input to system output
(neglect initial condition, and $H_u = 0$)

$$\mathbf{y}(s) = \mathbf{H_x x}(s) = \mathbf{H_x}\left(s\mathbf{I} - \mathbf{F}\right)^{-1}\mathbf{G}\,\mathbf{u}(s) \triangleq \boldsymbol{H}(s)\mathbf{u}(s)$$

Transfer Function Matrix

$$\boldsymbol{H}(s) \triangleq \mathbf{H_x}\left(s\mathbf{I} - \mathbf{F}\right)^{-1}\mathbf{G} \quad (r\ x\ m)$$



Open-Loop Dynamic Equation of Direct-Current Motor with Load Inertia, *J*

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u(t)$$

Same as simplified pitching dynamics for quadcopter, with J = I_{yy}

Open-Loop Transfer Function Matrix for DC Motor

Transfer Function Matrix

$$\boldsymbol{H}_{OL}(s) = \mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}_{OL}\right]^{-1}\mathbf{G} \quad (r \times m)$$

where

$$\mathbf{F}_{OL} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \left[s\mathbf{I} - \mathbf{F}_{OL}\right] = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$$

$$\mathbf{H}_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2; \quad \mathbf{G} = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

Matrix Inverse

$$\left[s\mathbf{I} - \mathbf{F}_{OL}\right]^{-1} = \frac{Adj\left(s\mathbf{I} - \mathbf{F}_{OL}\right)}{\left|s\mathbf{I} - \mathbf{F}_{OL}\right|} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}}{s^2} = \boxed{\begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}}$$

Open-Loop Transfer Function Matrix for DC Motor

Dimension = 2 x 1

$$\boldsymbol{H}_{OL}(s) = \mathbf{H}_\mathbf{x}\left[s\mathbf{I}-\mathbf{F}_{OL}\right]^{-1}\mathbf{G} = \mathbf{H}_\mathbf{x}\frac{Adj\left(s\mathbf{I}-\mathbf{F}_{OL}\right)}{\left|s\mathbf{I}-\mathbf{F}_{OL}\right|}\mathbf{G}$$

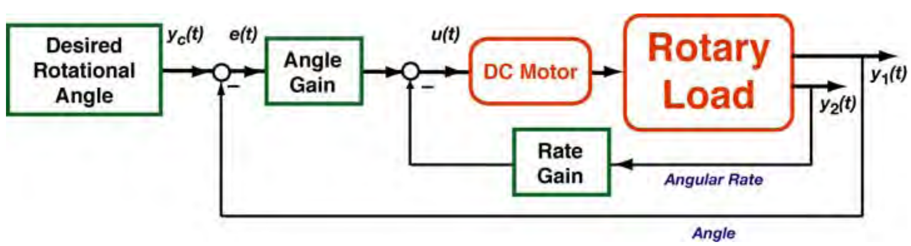
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}\begin{bmatrix} 0 \\ 1/J \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/Js^2 \\ 1/Js \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{Js^2} \\ \dfrac{1}{Js} \end{bmatrix} = \begin{bmatrix} \dfrac{y_1(s)}{u(s)} \\ \dfrac{y_2(s)}{u(s)} \end{bmatrix}$$

Angle = Double integral of input torque, u(s)

Angular rate = Integral of input torque, u(s)

Closed-Loop System



Closed-loop dynamic equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c_1/J & -c_2/J \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ c_1/J \end{bmatrix} y_C$$

$$\dot{\mathbf{x}}(t) = \mathbf{F}_{CL}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t)$$

Closed-Loop Transfer Function Matrix

$$\boldsymbol{H}_{CL}(s) = \mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}_{CL}\right]^{-1}\mathbf{G} = \mathbf{H_x}\frac{Adj\left(s\mathbf{I} - \mathbf{F}_{CL}\right)}{\left|s\mathbf{I} - \mathbf{F}_{CL}\right|}\mathbf{G}$$

Dimension = 2 x 1

$$= \frac{\begin{bmatrix} \left(s + c_2/J\right) & 1 \\ -c_1/J & s \end{bmatrix}}{\left(s^2 + \frac{c_2}{J}s + \frac{c_1}{J}\right)}\begin{bmatrix} 0 \\ c_1/J \end{bmatrix}$$

Closed-loop transfer functions

Scalar components differ only in numerators

$$\begin{bmatrix} \dfrac{y_1(s)}{y_C(s)} \\ \dfrac{y_2(s)}{y_C(s)} \end{bmatrix} = \frac{\begin{bmatrix} c_1/J \\ \left(c_1/J\right)s \end{bmatrix}}{\left(s^2 + \frac{c_2}{J}s + \frac{c_1}{J}\right)} = \frac{\begin{bmatrix} n_1(s) \\ n_2(s) \end{bmatrix}}{\Delta(s)}$$

Frequency Response Functions

Substitute $s = j\omega$ in transfer functions

$$\frac{\textbf{Output Angle}}{\textbf{Commanded Angle}} = \frac{y_1(j\omega)}{y_C(j\omega)} = \frac{c_1/J}{(j\omega)^2 + \left(c_2/J\right)(j\omega) + c_1/J}$$

$$= \frac{\omega_n^{\ 2}}{-\omega^2 + 2\zeta\omega_n(j\omega) + \omega_n^{\ 2}}$$

$$\triangleq a_1(\omega) + jb_1(\omega) \rightarrow AR_1(\omega)\,e^{j\phi_1(\omega)}$$

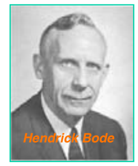
$$\frac{\textbf{Output Rate}}{\textbf{Commanded Angle}} = \frac{y_2(j\omega)}{y_C(j\omega)} = \frac{(j\omega)\,c_1/J}{(j\omega)^2 + \left(c_2/J\right)(j\omega) + c_1/J}$$

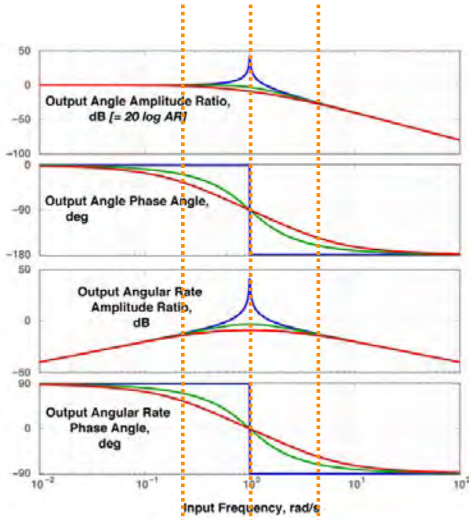
$$= \frac{(j\omega)\omega_n^{\ 2}}{-\omega^2 + 2\zeta\omega_n(j\omega) + \omega_n^{\ 2}} \rightarrow AR_2(\omega)\,e^{j\phi_2(\omega)}$$

Natural Frequency: $\omega_n = \sqrt{c_1/J}$

Damping Ratio:
$$\zeta = \left(c_2/J\right)/2\omega_n = \left(c_2/J\right)/2\sqrt{c_1/J}$$

Bode Plot Depicts Frequency Response





$20\log AR \text{ and } \phi \text{ vs. } \log\omega$

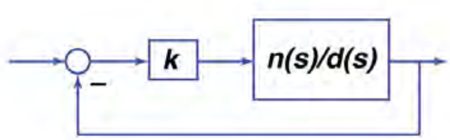
```
%  Frequency Response of DC Motor Angle Control

F1  =  [0 1;-1 0];
G1  =  [0;1];

F1a =  [0 1;-1 -1.414];
F1b =  [0 1;-1 -2.828];

Hx  =  [1 0;0 1];

Sys1   =  ss(F1,G1,Hx,0);
Sys2   =  ss(F1a,G1,Hx,0);
Sys3   =  ss(F1b,G1,Hx,0);

w  =  logspace(-2,2,1000);
bode(Sys1,Sys2,Sys3,w), grid
```


Root (Eigenvalue) Locus

$$\begin{bmatrix} \dfrac{y_1(s)}{y_C(s)} \\ \dfrac{y_2(s)}{y_C(s)} \end{bmatrix} = \frac{\begin{bmatrix} c_1/J \\ (c_1/J)s \end{bmatrix}}{\left(s^2 + \dfrac{c_2}{J}s + \dfrac{c_1}{J}\right)} = \frac{\begin{bmatrix} \omega_n^2 \\ \omega_n^2 s \end{bmatrix}}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{\begin{bmatrix} 1 \\ s \end{bmatrix}}{\left(s^2 + 1.414\omega_n s + 1\right)}$$

- Variation of roots as scalar gain, ***k***, goes from 0 to ∞
- With nominal gains, c_1 and c_2,

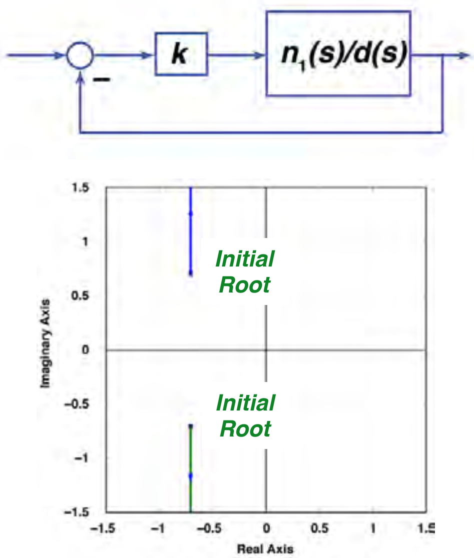
$$\omega_n = \sqrt{\frac{c_1}{J}} = 1\, rad/s, \quad \zeta = \frac{c_2/J}{2\omega_n} = 0.707$$



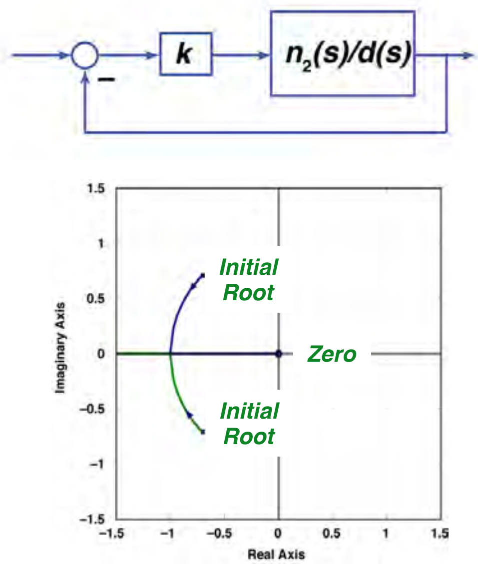

```
%  Root Locus of DC Motor Angle Control

F   =  [0 1;-1 -1.414];
G   =  [0;1];

Hx1 =  [1 0];  %  Angle Output
Hx2 =  [0 1];  %  Angular Rate Output

Sys1   =  ss(F,G,Hx1,0);
Sys2   =  ss(F,G,Hx2,0);

rlocus(Sys1), grid
figure
rlocus(Sys2), grid
```

Root (Eigenvalue) Locus

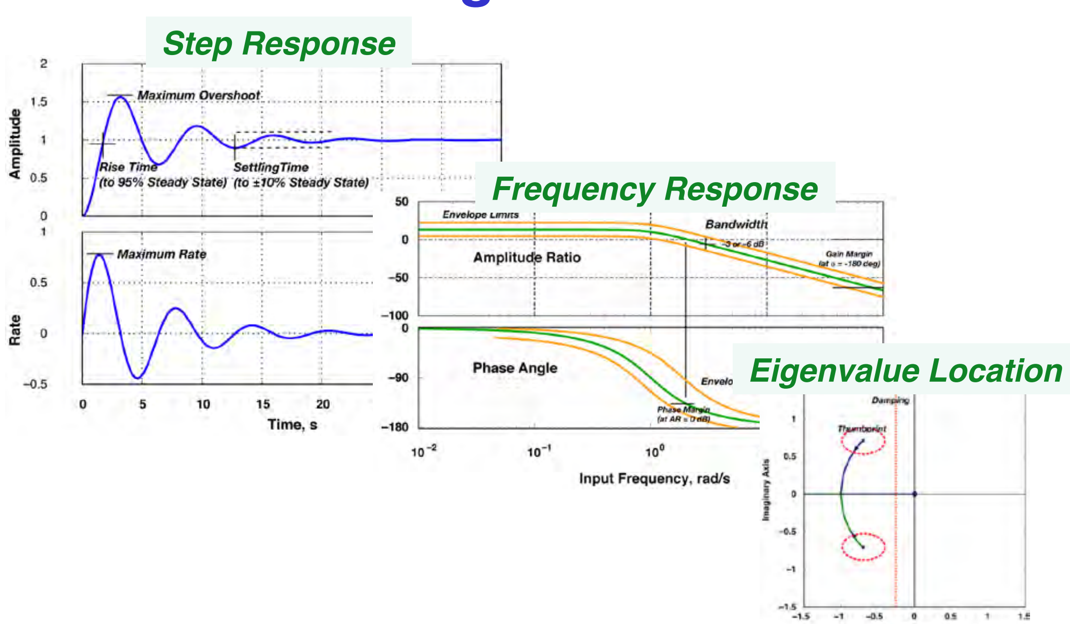
Increase Angle Feedback Gain

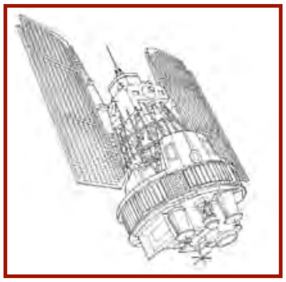


Increase Rate Feedback Gain



Classical Control System Design Criteria





Single-Axis Angular Control of Non-Spinning Spacecraft

Pitching motion (about the *y* axis) is to be controlled

$$\begin{bmatrix} \dot{\theta}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/I_{yy} \end{bmatrix} M_y(t)$$

$$\begin{bmatrix} \theta(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} Pitch\ Angle \\ Pitch\ Rate \end{bmatrix}$$

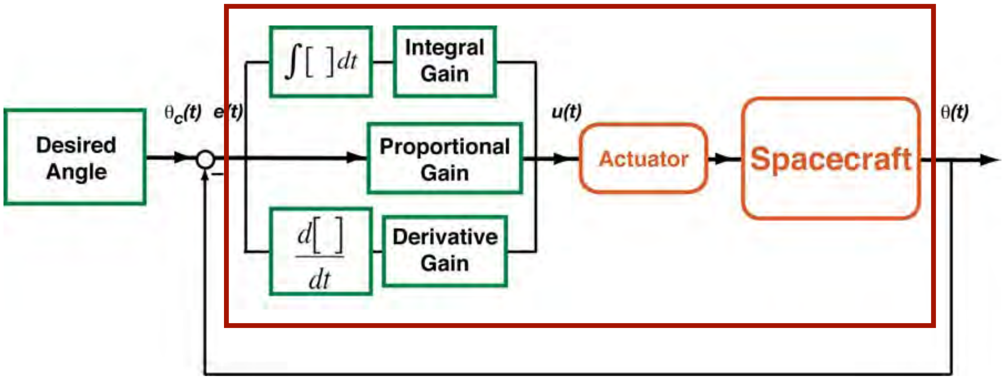
Identical to the DC Motor control problem

Proportional-Integral-Derivative (PID) Controller

Control Error	$e(s) = \theta_C(s) - \theta(s)$
Control Command	$u(s) = c_P e(s) + c_I \dfrac{e(s)}{s} + c_D s e(s)$
Control Law Transfer Function (w/common denominator)	$\dfrac{u(s)}{e(s)} = \dfrac{c_P s + c_I + c_D s^2}{s}$

- **Proportional term weights control error directly**
- **Integrator compensates for persistent (bias) disturbance**
- **Differentiator produces rate term for damping**

Proportional-Integral-Derivative (PID) Controller

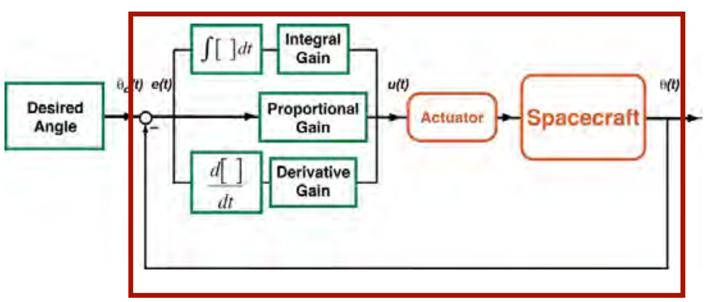


Forward-Loop Angle Transfer Function

g_A = Actuator Gain

$$\frac{\theta(s)}{e(s)} = \left[\frac{c_I + c_P s + c_D s^2}{s}\right]\left[\frac{g_A}{I_{yy}s^2}\right]$$

Closed-Loop Spacecraft Control Transfer Function w/PID Control



Closed-Loop Angle Transfer Function

Ziegler-Nichols PID Tuning Method
http://en.wikipedia.org/wiki/Ziegler–Nichols_method

$$\frac{\theta(s)}{\theta_c(s)} = \frac{\dfrac{\theta(s)}{e(s)}}{1+\dfrac{\theta(s)}{e(s)}} = \frac{\left[\dfrac{c_I + c_P s + c_D s^2}{I_{yy}s^3}g_A\right]}{1+\left[\dfrac{c_I + c_P s + c_D s^2}{I_{yy}s^3}g_A\right]}$$

$$= \frac{c_D s^2 + c_P s + c_I}{I_{yy}s^3 / g_A + c_D s^2 + c_P s + c_I}$$

Closed-Loop Frequency Response w/PID Control

$$\frac{\theta(s)}{\theta_c(s)} = \frac{c_I + c_P s + c_D s^2}{c_I + c_P s + c_D s^2 + I_{yy} s^3 / g_A}$$

Let $s = j\omega$. As $\omega \to 0$

$$\frac{\theta(j\omega)}{\theta_c(j\omega)} \to \frac{c_I}{c_I} = 1$$

Steady-state output = desired steady-state input

As $\omega \to \infty$

$$\frac{\theta(j\omega)}{\theta_c(j\omega)} \to \frac{-c_D\omega^2}{-jI_{yy}\omega^3} g_A = \frac{c_D}{jI_{yy}\omega} g_A = -\frac{jc_D}{I_{yy}\omega} g_A$$

$$AR \to \frac{c_D}{I_{yy}\omega} g_A; \quad \varphi \to -90 \text{ deg}$$

High-frequency response "rolls off" and lags input

Digital Control

Stengel, *Optimal Control and Estimation,* 1994, pp. 79-84, E-Reserve

From Continuous- to Discrete-Time Systems

- Continuous-time systems are described by <u>differential</u> equations, e.g.,

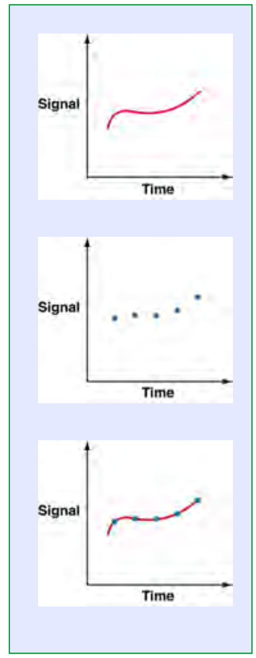
$$\boxed{\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)}$$

- Discrete-time systems are described by <u>difference</u> equations, e.g.,

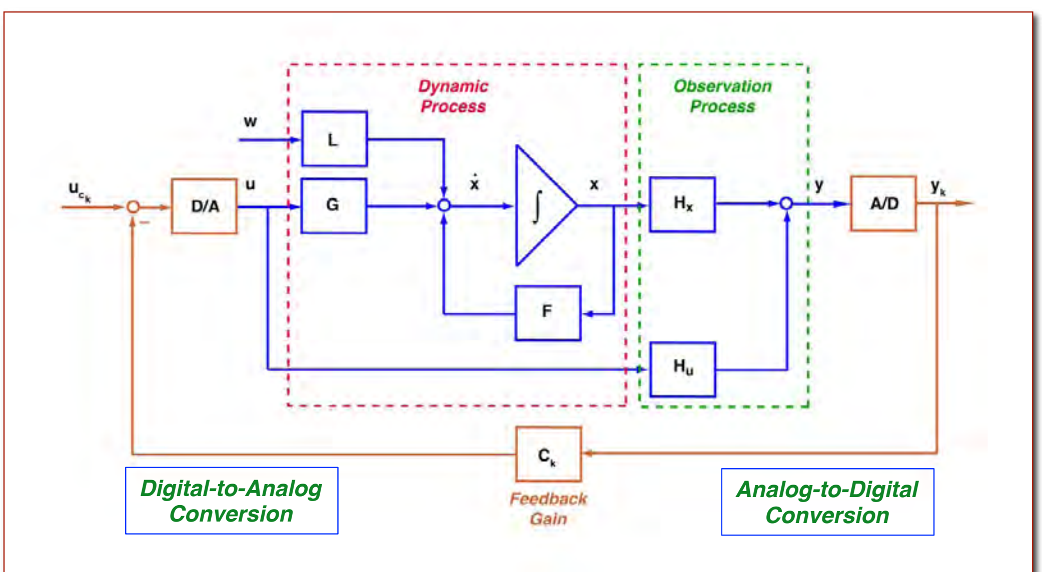
$$\boxed{\Delta\mathbf{x}(t_{k+1}) = \mathbf{\Phi}\Delta\mathbf{x}(t_k) + \mathbf{\Gamma}\Delta\mathbf{u}(t_k)}$$

- Discrete-time systems that are meant to describe continuous-time systems are called **sampled-data systems**

$$\boxed{\mathbf{\Phi} = fcn(\mathbf{F}); \quad \mathbf{\Gamma} = fcn(\mathbf{F},\mathbf{G})}$$



Sampled-Data (*Digital*) Feedback Control



Integration of Linear, Time-Invariant (LTI) Systems

$$\boxed{\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)}$$

$$\boxed{\Delta\mathbf{x}(t) = \Delta\mathbf{x}(0) + \int_0^t \Delta\dot{\mathbf{x}}(\tau)\,d\tau = \Delta\mathbf{x}(0) + \int_0^t \left[\mathbf{F}\Delta\mathbf{x}(\tau) + \mathbf{G}\Delta\mathbf{u}(\tau)\right]d\tau}$$

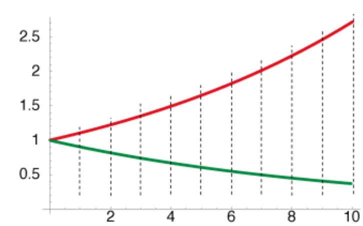
Homogeneous solution (no input), Δu = 0

$$\Delta\mathbf{x}(t) = \Delta\mathbf{x}(0) + \int_0^t \left[\mathbf{F}\Delta\mathbf{x}(\tau)\right]d\tau$$
$$= e^{\mathbf{F}(t-0)}\Delta\mathbf{x}(0) = \mathbf{\Phi}(t,0)\Delta\mathbf{x}(0)$$

$$\boxed{\mathbf{\Phi}(t,0) = e^{\mathbf{F}(t-0)} = e^{\mathbf{F}t} = \textbf{State transition matrix } \textit{from } 0 \textit{ to } t}$$

Equivalence of 1st-Order Continuous- and Discrete-Time Systems

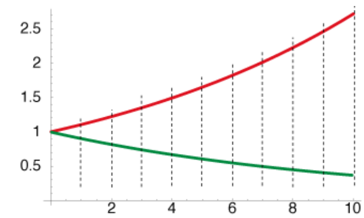
Example demonstrates stability (green) and instability (red), defined by sign of *a*



$$\dot{x}(t) = ax(t)\,,\ x(0)\ \text{given}$$
$$x(t) = x(0) + \int_0^t \dot{x}(t)\,dt = e^{at}x(0)$$

Choose small time interval, Δ*t*
Propagate state to time, *t* + *n*Δ*t*

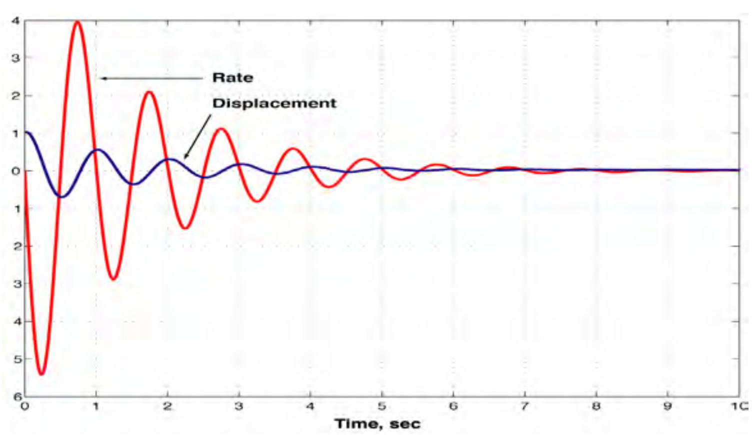
Equivalence of 1st-Order Continuous- and Discrete-Time Systems



$$x_0 = x(0)$$
$$x_1 = x(\Delta t) = e^{a\Delta t} x_0 = \phi(\Delta t) x_0$$
$$x_2 = x(2\Delta t) = e^{2a\Delta t} x_0 = e^{a\Delta t} x_1 = \phi(\Delta t) x_1$$
$$\ldots\ldots.$$
$$x_k = x(k\Delta t) = e^{ka\Delta t} x_0 = e^{a\Delta t} x_{k-1} = \phi(\Delta t) x_{k-1}$$
$$\ldots\ldots.$$
$$x_n = x(n\Delta t) = x(t) = e^{na\Delta t} x_0 = \phi(\Delta t) x_{n-1}$$

Initial Condition Response of Continuous-and Discrete-Time Models

Identical response at the sampling instants



$$\Delta \mathbf{x}(t_{k+1}) = \Delta \mathbf{x}(t_k) + \int_{t_k}^{t_{k+1}} \mathbf{F} \Delta \mathbf{x}(t)\,dt = e^{\mathbf{F}(t_{k+1}-t_k)\Delta t} \Delta \mathbf{x}(t_k) = \mathbf{\Phi}(\Delta t) \Delta \mathbf{x}(t_k)$$

Input Response of LTI System

Inhomogeneous (forced) solution

$$\Delta \mathbf{x}(t) = e^{\mathbf{F}t}\Delta \mathbf{x}(0) + \int_0^t \left[e^{\mathbf{F}(t-\tau)}\mathbf{G}\Delta \mathbf{u}(\tau) \right] d\tau$$

$$= \mathbf{\Phi}(t-0)\Delta \mathbf{x}(0) + \mathbf{\Phi}(t-0)\int_0^t \left[\mathbf{\Phi}(t-\tau)\mathbf{G}\Delta \mathbf{u}(\tau) \right] d\tau$$

Sampled-Data System

(Stepwise application of prior results)

Assume $(t_k - t_{k+1}) = \Delta t$ = constant

$$\mathbf{\Phi}\left(t_{k+1} - t_k\right) = e^{\mathbf{F}(t_{k+1}-t_k)} = e^{\mathbf{F}(\Delta t)} = \mathbf{\Phi}(\Delta t)$$

Assume Δu = constant from t_k to t_{k+1}

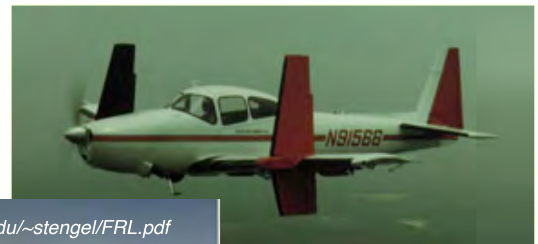
$$\Delta \mathbf{x}(t_{k+1}) = \mathbf{\Phi}(\Delta t)\Delta \mathbf{x}(t_k) + \mathbf{\Phi}(\Delta t)\int_0^{\Delta t} \left[\mathbf{\Phi}(\Delta t - \tau)\mathbf{G} \right] d\tau \; \Delta \mathbf{u}_k$$

Evaluate the integral

$$\Delta \mathbf{x}(t_{k+1}) = \mathbf{\Phi}(\Delta t)\Delta \mathbf{x}(t_k) + \mathbf{\Phi}(\Delta t)\left[\mathbf{I} - \mathbf{\Phi}^{-1}(\Delta t) \right]\mathbf{F}^{-1}\mathbf{G}\Delta \mathbf{u}(t_k)$$
$$\equiv \mathbf{\Phi}(\Delta t)\Delta \mathbf{x}(t_k) + \mathbf{\Gamma}(\Delta t)\Delta \mathbf{u}(t_k)$$

Digital Flight Control

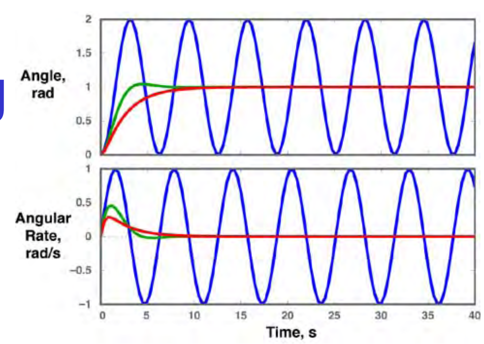
- **Historical notes:**
  - *NASA Fly-By-Wire F-8C (Apollo GNC system, 1972)*
    - 1st conventional aircraft with digital flight control
    - NASA Dryden Research Center
  - *Princeton's Variable-Response Research Aircraft (Z-80 microprocessor, 1978)*
    - 2nd conventional aircraft with digital flight control
    - Princeton University's Flight Research Laboratory



Next Time:
Sensors and Actuators

SUPPLEMENTAL MATERIAL

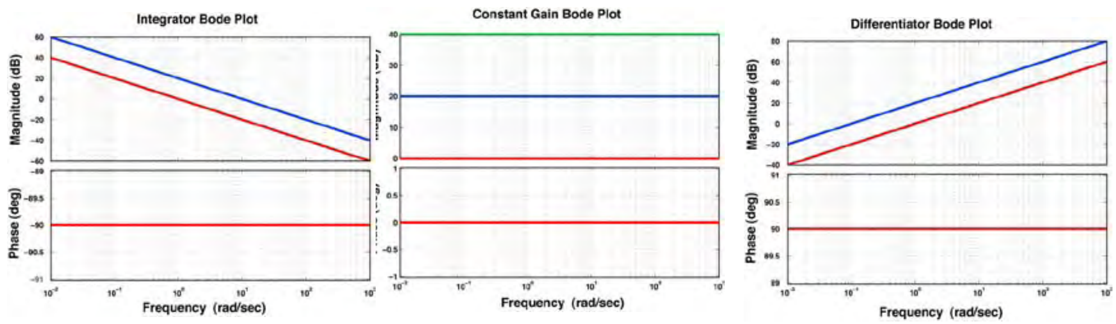
Eigenvalues, Damping Ratio, and Natural Frequency




```
%   Eigenvalues, Damping Ratio,
and Natural Frequency of Angle Control

   F1  =  [0 1;-1 0];
   G1  =  [0;1];

   F1a =  [0 1;-1 -1.414];
   F1b =  [0 1;-1 -2.828];

   Hx  =  [1 0;0 1];

   Sys1   =  ss(F1,G1,Hx,0);
   Sys2   =  ss(F1a,G1,Hx,0);
   Sys3   =  ss(F1b,G1,Hx,0);

   eig(F1)
   eig(F1a)
   eig(F1b)

   damp(Sys1)
   damp(Sys2)
```

Eigenvalues λ_1, λ_2	Damping Ratio ζ	Natural Frequency ω_n
0 + 1i 0 - 1i	0	1
-0.707 + 0.707i -0.707 - 0.707i	0.707	1
-0.414 -2.414	*Overdamped*	

Bode Plots of Proportional, Integral, and Derivative Compensation



$\boxed{H(j\omega) = 1}$

$\boxed{H(j\omega) = 10}$

$\boxed{H(j\omega) = 100}$

- **Amplitude ratio, *AR(dB)* = constant**

$\boxed{\text{Phase Angle, } \phi(\text{deg}) = 0}$

$\boxed{H(j\omega) = \frac{1}{j\omega}}$

$\boxed{H(j\omega) = \frac{10}{j\omega}}$

- ***Slope of AR(dB)* = –20 dB/decade**

$\boxed{\text{Phase Angle, } \phi = -90 \text{ deg}}$

$\boxed{H(j\omega) = j\omega}$

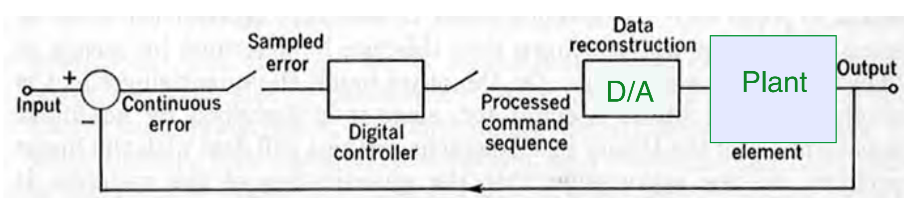
$\boxed{H(j\omega) = 10\,j\omega}$

- ***Slope of AR(dB)* = +20 dB/decade**

$\boxed{\text{Phase Angle, } \phi = +90 \text{ deg}}$

Single-Input/Single-Output Sampled-Data Control

- **Sampler is an analog-to-digital (A/D) converter**
- **Reconstructor is a digital-to-analog (D/A) converter**
- **Sampling of input and feedback signal could occur separately, before the summing junction**
- **Sampling may be periodic or aperiodic**



Equilibrium Response of Linear Discrete-Time Model

- **Dynamic model**

$$\boxed{\Delta\mathbf{x}_{k+1} = \mathbf{\Phi}\Delta\mathbf{x}_k + \mathbf{\Gamma}\Delta\mathbf{u}_k + \mathbf{\Lambda}\Delta\mathbf{w}_k}$$

- **At equilibrium**

$$\boxed{\Delta\mathbf{x}_{k+1} = \Delta\mathbf{x}_k = \Delta\mathbf{x}^* = \text{constant}}$$

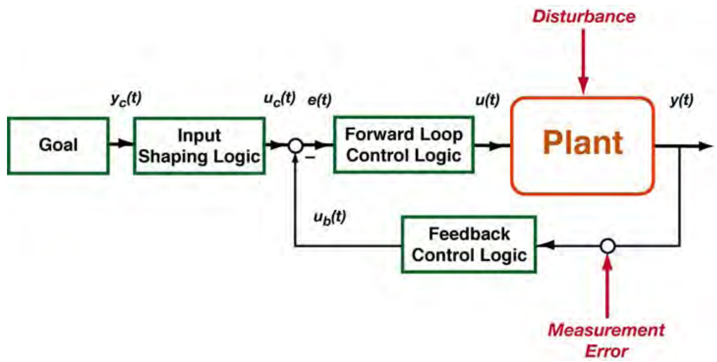
- **Equilibrium response**

$$\boxed{\begin{aligned}(\mathbf{I}-\mathbf{\Phi})\Delta\mathbf{x}^* &= \mathbf{\Gamma}\Delta\mathbf{u}^* + \mathbf{\Lambda}\Delta\mathbf{w}^* \\ \Delta\mathbf{x}^* &= (\mathbf{I}-\mathbf{\Phi})^{-1}(\mathbf{\Gamma}\Delta\mathbf{u}^* + \mathbf{\Lambda}\Delta\mathbf{w}^*)\end{aligned}}$$

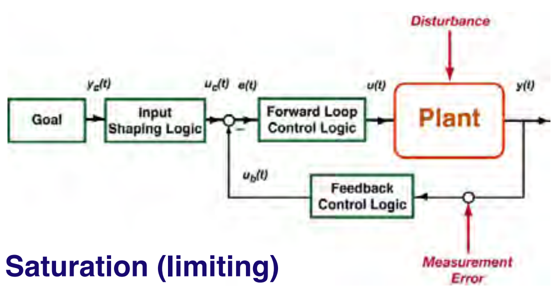
(.)* = *constant*

Factors That Complicate Precise Control

Error, Uncertainty, and Incompleteness May Have Significant Effects

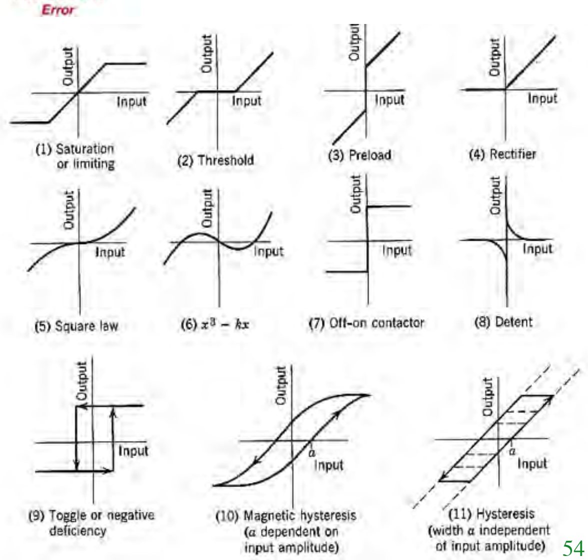


- **Scale factors and biases:** $y = kx + b$
- **Disturbances:** Constant/variable forces
- **Measurement error:** Noise, bias
- **Modeling error**
  - Parameters: e.g., Inertia
  - Structure: e.g., Higher-order modes



Nonlinearity Complicates Response

- **Saturation (limiting)**
  - Displacement limit, maximum force or rate
- **Threshold**
  - Dead zone, slop
- **Preload**
  - Breakout force
- **Friction**
  - Sliding surface resistance
- **Rectifier**
  - Hard constraint, absolute value (double)
- **Hysteresis**
  - Backlash, slop
- **Higher-degree terms**
  - Cubic spring, separation

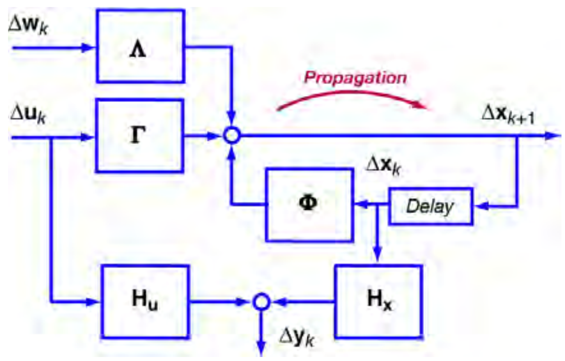


Discrete-Time and Sampled-Data Models

Linear, time-invariant, discrete-time model

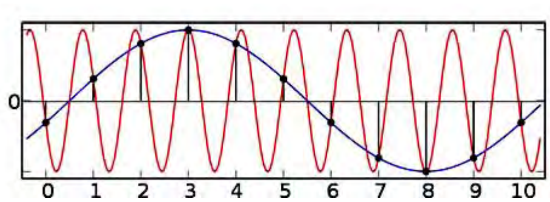
$$\boxed{\Delta \mathbf{x}_{k+1} = \Phi \Delta \mathbf{x}_k + \Gamma \Delta \mathbf{u}_k + \Lambda \Delta \mathbf{w}_k}$$

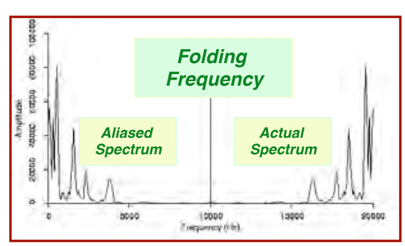
Discrete-time model is a sampled-data model if it represents a continuous-time system



Sampling Effect on Continuous Signal

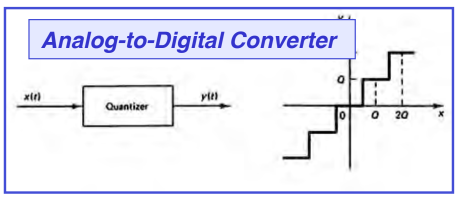
- **Two waves, different frequencies, indistinguishable in periodically sampled data**
- **Frequencies above the sampling frequency *aliased* to appear at lower frequency *(frequency folding)***
- **Solutions: Either**
  - ***Sampling at higher rate* or**
  - ***Analog low-pass filtering***

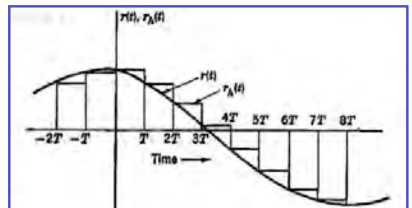




Quantization and Delay Effects

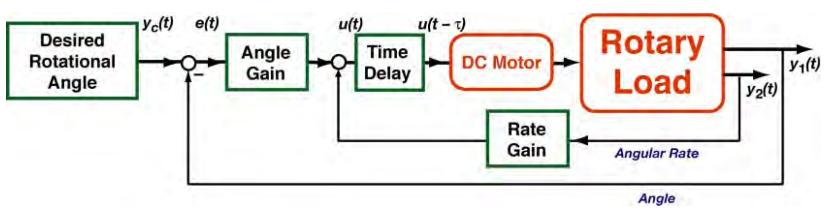
- Continuous signal sampled with finite precision (***quantized***)
  - Solution: ***More bits*** in A/D and D/A conversion
- ***Effective delay*** of sampled signal
  - Described as ***phase shift*** or ***lag***
  - Solution:
    - ***Higher sampling rate*** or
    - ***Lead compensation***





r_h illustrates zero-order hold effect

Time Delay Effect



- Control command delayed by τ sec
- Laplace transform of pure time delay

$$\boldsymbol{L}\left[u(t-\tau)\right] = e^{-\tau s}u(s)$$

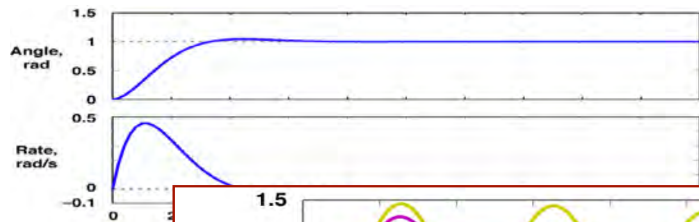
- Delay introduces frequency-dependent phase lag with no change in amplitude

$$AR\left(e^{-j\tau\omega}\right) = 1$$
$$\phi\left(e^{-j\tau\omega}\right) = -\omega$$

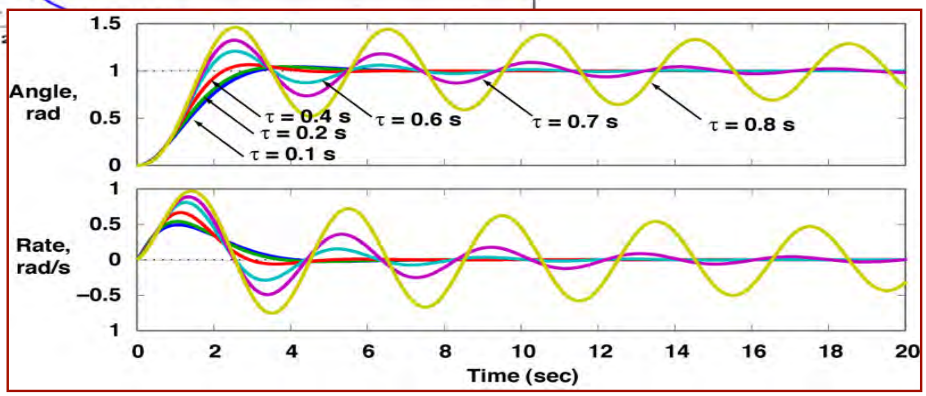
- Phase lag reduces closed-loop stability

Effect of Closed-Loop Time Delay on Step Response

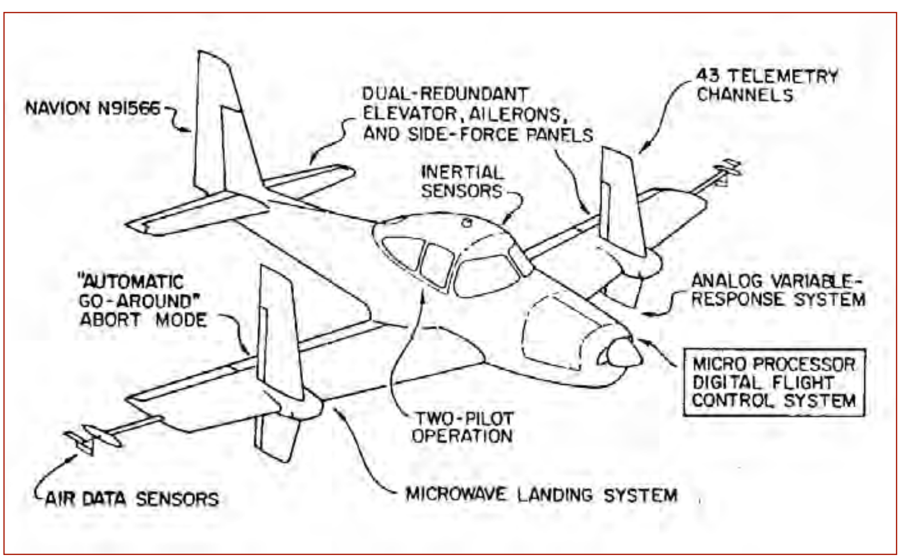
- **With no delay** $\omega_n = 1\,rad/s, \quad \zeta = 0.707$



- **With varying delays**



Princeton's Variable-Response Research Aircraft



Discrete-Time Frequency Response Can be Evaluated Using the z Transform

$$\boxed{\Delta\mathbf{x}_{k+1} = \mathbf{\Phi}\,\Delta\mathbf{x}_k + \mathbf{\Gamma}\,\Delta\mathbf{u}_k}$$

z transform is the Laplace transform of a periodic sequence

System *z* transform is

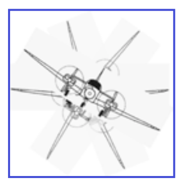
$$\boxed{z\Delta\mathbf{x}(z) - \Delta\mathbf{x}(0) = \mathbf{\Phi}\,\Delta\mathbf{x}(z) + \mathbf{\Gamma}\Delta\,\mathbf{u}(z)}$$

which leads to

$$\boxed{\Delta\mathbf{x}(z) = \left(z\mathbf{I} - \mathbf{\Phi}\right)^{-1}\left[\Delta\mathbf{x}(0) + \mathbf{\Gamma}\Delta\,\mathbf{u}(z)\right]}$$

Further details are beyond the present scope

Second-Order Example: Airplane Roll Motion



$$\begin{bmatrix} \Delta p \\ \Delta\phi \end{bmatrix} = \begin{bmatrix} \text{Roll rate, rad/s} \\ \text{Roll angle, rad} \end{bmatrix}$$

$\Delta\delta A = $ Aileron deflection, rad

L_p : *roll damping coefficient*

$L_{\delta A}$: *aileron control effect*

Rolling motion of an airplane, continuous-time

$$\begin{bmatrix} \Delta\dot{p} \\ \Delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \Delta p \\ \Delta\phi \end{bmatrix} + \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix}\Delta\delta A$$

Rolling motion of an airplane, discrete-time

$$\begin{bmatrix} \Delta p_k \\ \Delta\phi_k \end{bmatrix} = \begin{bmatrix} e^{L_pT} & 0 \\ \dfrac{\left(e^{L_pT} - 1\right)}{L_p} & 1 \end{bmatrix}\begin{bmatrix} \Delta p_{k-1} \\ \Delta\phi_{k-1} \end{bmatrix} + \begin{bmatrix} \sim L_{\delta A}T \\ 0 \end{bmatrix}\Delta\delta A_{k-1}$$

T = sampling interval, s