Transfer Functions and Frequency Response

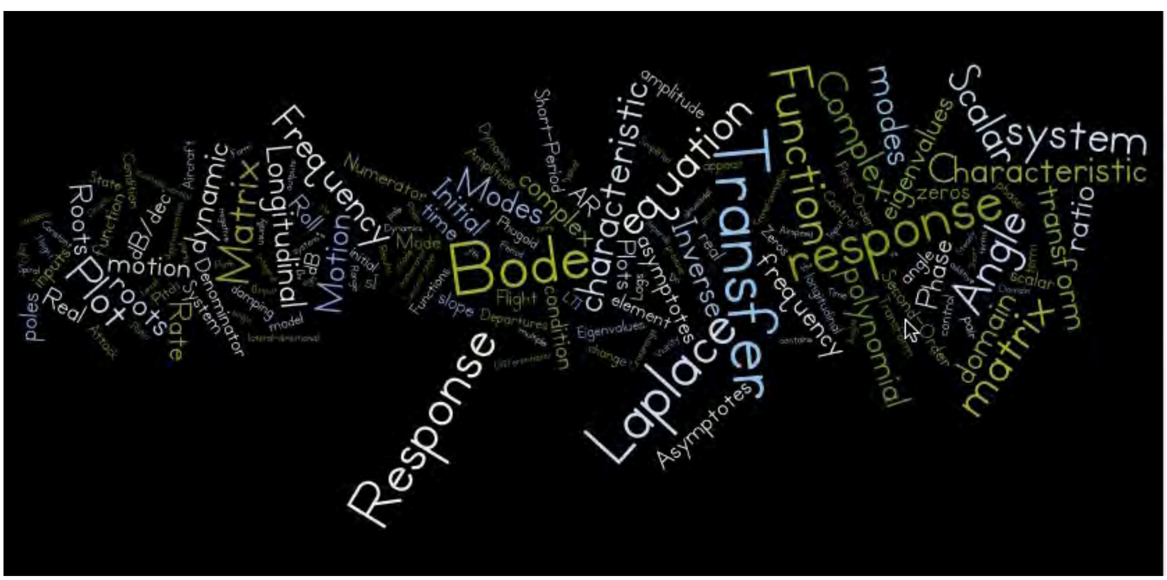
Robert Stengel, Aircraft Flight Dynamics
MAE 331, 2014

Learning Objectives

- Frequency domain view of initial condition response
- Response of dynamic systems to sinusoidal inputs
- Transfer functions
- Bode plots

Reading:
Flight Dynamics
342-357
Airplane Stability and Control
Chapter 20

http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html



Fourier and Laplace Transforms

Fourier Transform of a Scalar Variable

Transformation from "time domain" to "frequency domain"

$$F\left[\Delta x(t)\right] = \Delta x(j\omega) = \int_{-\infty}^{\infty} \Delta x(t) e^{-j\omega t} dt, \quad \omega = frequency, rad/s$$

$j\omega$: Imaginary operator, rad/s

$\Delta x(t)$: **real variable**

$\Delta x(j\omega)$: **complex variable**

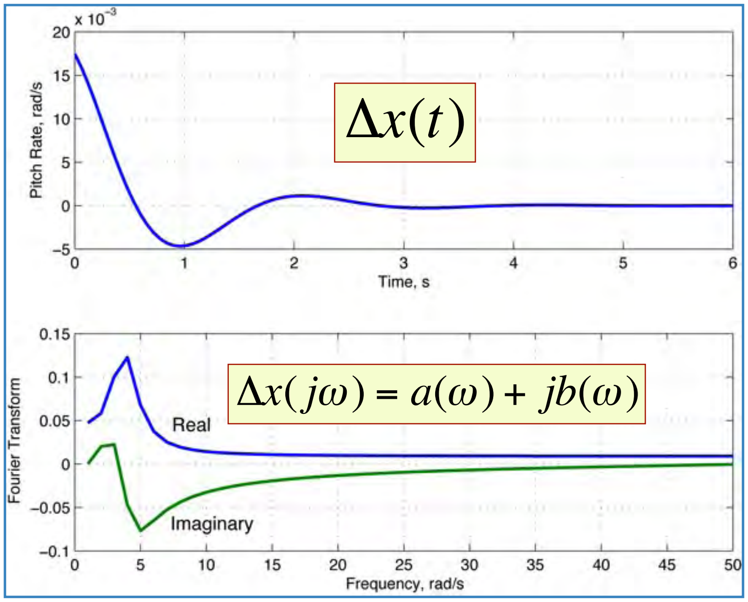
$$= a(\omega) + jb(\omega)$$

$$= A(\omega) e^{j\varphi(\omega)}$$

A : **amplitude**

φ : **phase angle**

Fourier Transform of a Scalar Variable



Laplace Transform of a Scalar Variable

Laplace transformation from "time domain" to "frequency domain"

$$L\left[\Delta x(t)\right] = \Delta x(s) = \int_0^\infty \Delta x(t) e^{-st} dt$$

$$s = \sigma + j\omega$$
$$= \text{Laplace (complex) operator, rad/s}$$

$\Delta x(t)$: **real variable**

$\Delta x(s)$: **complex variable**

$$= a(s) + jb(s)$$
$$= A(s) e^{j\varphi(s)}$$

Laplace Transformation is a Linear Operation

Sum of Laplace transforms

$$\boxed{L\left[\Delta x_1(t)+\Delta x_2(t)\right]=L\left[\Delta x_1(t)\right]+L\left[\Delta x_2(t)\right]=\Delta x_1(s)+\Delta x_2(s)}$$

Multiplication by a constant

$$\boxed{L\left[a\Delta x(t)\right]=aL\left[\Delta x(t)\right]=a\Delta x(s)}$$

Laplace Transforms of Vectors and Matrices

Laplace transform of a vector variable

$$\boxed{L\left[\Delta\mathbf{x}(t)\right]=\Delta\mathbf{x}(s)=\begin{bmatrix}\Delta x_1(s)\\ \Delta x_2(s)\\ \ldots\end{bmatrix}}$$

Laplace transform of a matrix variable

$$\boxed{L\left[\mathbf{F}(t)\right]=\mathbf{F}(s)=\begin{bmatrix}f_{11}(s) & f_{12}(s) & \ldots\\ f_{21}(s) & f_{22}(s) & \ldots\\ \ldots & \ldots & \ldots\end{bmatrix}}$$

Laplace transform of a time-derivative

$$\boxed{L\left[\Delta\dot{\mathbf{x}}(t)\right]=s\Delta\mathbf{x}(s)-\Delta\mathbf{x}(0)}$$

Laplace Transform of a Dynamic System

System equation

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\,\Delta\mathbf{x}(t) + \mathbf{G}\,\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t)$$

$\dim(\Delta\mathbf{x}) = (n \times 1)$
$\dim(\Delta\mathbf{u}) = (m \times 1)$
$\dim(\Delta\mathbf{w}) = (s \times 1)$

Laplace transform of system equation

$$s\Delta\mathbf{x}(s) - \Delta\mathbf{x}(0) = \mathbf{F}\,\Delta\mathbf{x}(s) + \mathbf{G}\Delta\,\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)$$

Laplace Transform of a Dynamic System

Rearrange Laplace transform of dynamic equation

F to left, I.C. to right

$$s\Delta\mathbf{x}(s) - \mathbf{F}\Delta\,\mathbf{x}(s) = \Delta\mathbf{x}(0) + \mathbf{G}\Delta\,\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)$$

Combine terms

$$\left[s\mathbf{I} - \mathbf{F}\right]\Delta\mathbf{x}(s) = \Delta\mathbf{x}(0) + \mathbf{G}\Delta\,\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)$$

Multiply both sides by <u>inverse</u> of (sI – F)

$$\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\Delta\mathbf{x}(0) + \mathbf{G}\,\Delta\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)\right]$$

Matrix Inverse

Forward / Inverse

$$\mathbf{y} = \mathbf{A}\mathbf{x}; \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$$

$$\dim(\mathbf{x}) = \dim(\mathbf{y}) = (n \times 1)$$
$$\dim(\mathbf{A}) = (n \times n)$$

$$[\mathbf{A}]^{-1} = \frac{\mathrm{Adj}(\mathbf{A})}{|\mathbf{A}|} = \frac{\mathrm{Adj}(\mathbf{A})}{\det \mathbf{A}} \quad \frac{(n \times n)}{(1 \times 1)}$$

$$= \frac{\mathbf{C}^T}{\det \mathbf{A}}; \quad \mathbf{C} = matrix\ of\ cofactors$$

Cofactors** are signed minors of **A

*ij^{th} **minor** of **A** is the determinant of **A** with the i^{th} row and j^{th} column removed*

Numerator is a square matrix of cofactor transposes
Denominator is a scalar

Matrix Inverse Examples

$\dim(\mathbf{A}) = (1 \times 1)$

$$\mathbf{A} = a; \quad \mathbf{A}^{-1} = \frac{1}{a}$$

$\dim(\mathbf{A}) = (2 \times 2)$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad \mathbf{A}^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}^T}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{a_{11}a_{22} - a_{12}a_{21}}$$

$\dim(\mathbf{A}) = (3 \times 3)$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad \mathbf{A}^{-1} = \frac{\begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{21}a_{33} - a_{23}a_{31}) & (a_{21}a_{32} - a_{22}a_{31}) \\ -(a_{12}a_{33} - a_{13}a_{32}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) \\ (a_{12}a_{23} - a_{13}a_{22}) & -(a_{11}a_{23} - a_{13}a_{21}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}^T}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}}$$

$$= \frac{\begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{12}a_{33} - a_{13}a_{32}) & (a_{12}a_{23} - a_{13}a_{22}) \\ -(a_{21}a_{33} - a_{23}a_{31}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{23} - a_{13}a_{21}) \\ (a_{21}a_{32} - a_{22}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}}$$

Matrix Inverse Examples

$$\mathbf{A} = 5; \quad \mathbf{A}^{-1} = \frac{1}{5} = 0.2$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}; \quad \mathbf{A}^{-1} = \frac{\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}}{-2} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 8 & 12 & 9 \end{bmatrix}; \quad \mathbf{A}^{-1} = \frac{\begin{bmatrix} -30 & 18 & 4 \\ 20 & -15 & 5 \\ 0 & 4 & -2 \end{bmatrix}}{10} = \begin{bmatrix} -3 & 1.8 & 0.4 \\ 2 & -1.5 & 0.5 \\ 0 & 0.4 & -0.2 \end{bmatrix}$$

Characteristic Matrix Inverse

Characteristic matrix

(short-period model as example)

$$\left[s\mathbf{I} - \mathbf{F}_{SP} \right]$$

Inverse of characteristic matrix

$$\left[s\mathbf{I} - \mathbf{F}_{SP} \right]^{-1} = \frac{Adj\left(s\mathbf{I} - \mathbf{F}_{SP} \right)}{\left| s\mathbf{I} - \mathbf{F}_{SP} \right|} = \frac{\mathbf{C}_{SP}^{T}(s)}{\Delta_{SP}(s)} \quad \frac{(2 \times 2)}{(1 \times 1)}$$

Denominator is characteristic polynomial, a scalar

$$\left| s\mathbf{I} - \mathbf{F}_{SP} \right| \equiv \Delta_{SP}(s) = s^2 + c_1 s + c_0$$

Numerator of the Characteristic Matrix Inverse

Numerator is an (n x n) matrix of polynomials

$$Adj\left(s\mathbf{I}-\mathbf{F}_{SP}\right)=\begin{bmatrix} n_q^q(s) & n_\alpha^q(s) \\ n_q^\alpha(s) & n_\alpha^\alpha(s) \end{bmatrix}$$

For example,

$$n_q^q(s)=k(s-z)$$

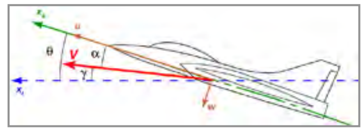
$(s\mathbf{I}-\mathbf{F})^{-1}$ Distributes and Shapes the Effects of Initial Conditions

$$\left[s\mathbf{I}-\mathbf{F}_{SP}\right]^{-1}=\frac{\begin{bmatrix} n_q^q(s) & n_\alpha^q(s) \\ n_q^\alpha(s) & n_\alpha^\alpha(s) \end{bmatrix}}{s^2+c_1 s+c_0} \qquad \frac{(2\times 2)}{(1\times 1)}$$

Denominator determines the modes of motion
Numerator distributes each element of the initial condition to each element of the state

$$\Delta\mathbf{x}(s)=\frac{Adj\left(s\mathbf{I}-\mathbf{F}_{SP}\right)}{\left|s\mathbf{I}-\mathbf{F}_{SP}\right|}\Delta\mathbf{x}(0) \qquad (2\times 1)$$

Initial Condition Response in Frequency Domain



$$\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1} \Delta\mathbf{x}(0)$$

Longitudinal dynamic model (time domain)

$$\begin{bmatrix} \Delta\dot{q}(t) \\ \Delta\dot{\alpha}(t) \end{bmatrix} = \begin{bmatrix} M_q & M_\alpha \\ \left(1 - L_q/V_N\right) & -L_\alpha/V_N \end{bmatrix} \begin{bmatrix} \Delta q(t) \\ \Delta\alpha(t) \end{bmatrix}, \qquad \begin{bmatrix} \Delta q(0) \\ \Delta\alpha(0) \end{bmatrix} \textit{given}$$

Longitudinal model (frequency domain)

$$\begin{bmatrix} \Delta q(s) \\ \Delta\alpha(s) \end{bmatrix} = \left[s\mathbf{I} - \mathbf{F}_{SP}\right]^{-1} \begin{bmatrix} \Delta q(0) \\ \Delta\alpha(0) \end{bmatrix}$$

Transfer Function Matrix

- Frequency-domain effect of all inputs on all outputs
- Assume control effects do not appear directly in the output: $\mathbf{H_u} = 0$
- Transfer function matrix

$$H(s) = \mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}\right]^{-1}\mathbf{G}$$

$$(r \times n)(n \times n)(n \times m) = (r \times m)$$

1^{st}-Order Transfer Function

Scalar dynamic system

$$\dot{x}(t) = fx(t) + gu(t)$$
$$y(t) = hx(t)$$

Scalar transfer function (= first-order lag)

$$\frac{y(s)}{u(s)} = H(s) = h[s-f]^{-1} g = \frac{hg}{(s-f)} \qquad (n = m = r = 1)$$

2^{nd}-Order Transfer Function

Second-order dynamic system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} u(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \cdots$$

Second-order transfer function matrix

$$\boldsymbol{H}(s) = \mathbf{H_x}(s\mathbf{I}-\mathbf{F})^{-1}(s)\mathbf{G} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \frac{\operatorname{adj}\begin{bmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{bmatrix}}{\det\begin{pmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{pmatrix}} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$(r \times n)(n \times n)(n \times m) = (r \times m) = (2 \times 2)$$

Numerator and Denominator of 2nd-Order $(s\mathbf{I} - \mathbf{F})^{-1}$

$$\text{adj}\begin{bmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{bmatrix} = \begin{bmatrix} (s-f_{22}) & f_{12} \\ f_{21} & (s-f_{11}) \end{bmatrix}$$

$$\begin{aligned} \det\begin{pmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{pmatrix} &= (s-f_{11})(s-f_{22}) - f_{12}f_{21} \\ &= s^2 - (f_{11}+f_{22})s + (f_{11}f_{22} - f_{12}f_{21}) \\ &\triangleq s^2 + 2\zeta\omega_n s + \omega_n^2 \triangleq \Delta(s) \end{aligned}$$

2nd-Order Transfer Function

$$\boldsymbol{H}(s) = \mathbf{H_x}(s\mathbf{I} - \mathbf{F})^{-1}(s)\mathbf{G} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \frac{\begin{bmatrix} (s-f_{22}) & f_{12} \\ f_{21} & (s-f_{11}) \end{bmatrix}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{H}(s) &= \frac{\begin{bmatrix} [h_{11}(s-f_{22}) + h_{12}f_{21}] & [h_{11}f_{12} + h_{12}(s-f_{11})] \\ [h_{21}(s-f_{22}) + h_{22}f_{21}] & [h_{21}f_{12} + h_{22}(s-f_{11})] \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{\begin{bmatrix} [h_{11}(s-f_{22}) + h_{12}f_{21}]g_1 + [h_{11}f_{12} + h_{12}(s-f_{11})]g_2 \\ [h_{21}(s-f_{22}) + h_{22}f_{21}]g_1 + [h_{21}f_{12} + h_{22}(s-f_{11})]g_2 \end{bmatrix}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

2^{nd}-Order Transfer Function

$$\boldsymbol{H}(s)=\frac{\begin{bmatrix}\left[h_{11}\left(s-f_{22}\right)+h_{12}f_{21}\right]g_1+\left[h_{11}f_{12}+h_{12}\left(s-f_{11}\right)\right]g_2\\ \left[h_{21}\left(s-f_{22}\right)+h_{22}f_{21}\right]g_1+\left[h_{21}f_{12}+h_{22}\left(s-f_{11}\right)\right]g_2\end{bmatrix}}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\triangleq\frac{\begin{bmatrix}k_1\left(s-z_1\right)\\ k_2\left(s-z_2\right)\end{bmatrix}}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Transfer Function Matrix for Short-Period Approximation

Dynamic Equation

$$\Delta\dot{\mathbf{x}}_{SP}(t)=\begin{bmatrix}\Delta\dot{q}(t)\\ \Delta\dot{\alpha}(t)\end{bmatrix}\approx\begin{bmatrix}M_q & M_\alpha\\ \left(1-L_q/V_N\right) & -L_\alpha/V_N\end{bmatrix}\begin{bmatrix}\Delta q(t)\\ \Delta\alpha(t)\end{bmatrix}+\begin{bmatrix}M_{\delta E}\\ -L_{\delta E}/V_N\end{bmatrix}\Delta\delta E(t)$$

Transfer Function Matrix (with H_x = I, H_u = 0)

$$H_{SP}(s)=\mathbf{I}_2\left(s\mathbf{I}-\mathbf{F}\right)_{SP}^{-1}(s)\mathbf{G}_{SP}=\begin{bmatrix}\left(s-M_q\right) & -M_\alpha\\ -\left(1-L_q/V_N\right) & \left(s+L_\alpha/V_N\right)\end{bmatrix}^{-1}\begin{bmatrix}M_{\delta E}\\ -L_{\delta E}/V_N\end{bmatrix}$$

Transfer Function Matrix for Short-Period Approximation

Transfer Function Matrix (with H_x = I, H_u = 0)

$$H_{SP}(s) = \left[s\mathbf{I} - \mathbf{F}_{Lon}\right]^{-1}\mathbf{G}_{SP} = \frac{\begin{bmatrix} \left(s + L_\alpha / V_N\right) & M_\alpha \\ \left(1 - L_q / V_N\right) & \left(s - M_q\right) \end{bmatrix} \begin{bmatrix} M_{\delta E} \\ -L_{\delta E} / V_N \end{bmatrix}}{\left(s - M_q\right)\left(s + L_\alpha / V_N\right) - M_\alpha\left(1 - L_q / V_N\right)}$$

Transfer Function Matrix for Short-Period Approximation

$$H_{SP}(s) = \frac{\begin{bmatrix} \left[M_{\delta E}\left(s + L_\alpha / V_N\right) - L_{\delta E}M_\alpha / V_N\right] \\ \left[M_{\delta E}\left(1 - L_q / V_N\right) - \left(L_{\delta E} / V_N\right)\left(s - M_q\right)\right] \end{bmatrix}}{s^2 + \left(-M_q + L_\alpha / V_N\right)s - \left[M_\alpha\left(1 - L_q / V_N\right) + M_q L_\alpha / V_N\right]}$$

$$= \frac{\begin{bmatrix} M_{\delta E}\left[s + \left(L_\alpha / V_N - L_{\delta E}M_\alpha / V_N M_{\delta E}\right)\right] \\ -\left(L_{\delta E} / V_N\right)\left\{s + \left[\frac{V_N M_{\delta E}}{L_{\delta E}}\left(1 - L_q / V_N\right) - M_q\right]\right\} \end{bmatrix}}{\Delta_{SP}(s)}$$

Transfer Function Matrix for Short-Period Approximation

$$H_{SP}(s) \triangleq \frac{\begin{bmatrix} k_q n_{\delta E}^q(s) \\ k_\alpha n_{\delta E}^\alpha(s) \end{bmatrix}}{s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega_{n_{SP}}^2} = \begin{bmatrix} \dfrac{\Delta q(s)}{\Delta\delta E(s)} \\ \dfrac{\Delta\alpha(s)}{\Delta\delta E(s)} \end{bmatrix}$$

dim = 2 x 1

Scalar Transfer Functions for Short-Period Approximation

Pitch Rate Transfer Function

$$\frac{\Delta q(s)}{\Delta\delta E(s)} = \frac{M_{\delta E}\left[s + \left(L_\alpha/V_N - L_{\delta E}M_\alpha/V_N M_{\delta E}\right)\right]}{s^2 + \left(-M_q + L_\alpha/V_N\right)s - \left[M_\alpha\left(1 - L_q/V_N\right) + M_q\, L_\alpha/V_N\right]} = \frac{k_q\left(s - z_q\right)}{s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega_{n_{SP}}^2}$$

Angle of Attack Transfer Function

$$\frac{\Delta\alpha(s)}{\Delta\delta E(s)} = \frac{-\left(L_{\delta E}/V_N\right)\left\{s + \left[\dfrac{V_N M_{\delta E}}{L_{\delta E}}\left(1 - L_q/V_N\right) - M_q\right]\right\}}{s^2 + \left(-M_q + L_\alpha/V_N\right)s - \left[M_\alpha\left(1 - L_q/V_N\right) + M_q\, L_\alpha/V_N\right]} = \frac{k_\alpha\left(s - z_\alpha\right)}{s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega_{n_{SP}}^2}$$

Relationship of $(s\mathbf{I} - \mathbf{F})^{-1}$ to State Transition Matrix, $\Phi(t,0)$

Initial condition response

Time Domain	$\Delta\mathbf{x}(t) = \mathbf{\Phi}(t,0)\Delta\mathbf{x}(0)$
Frequency Domain	$\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\Delta\mathbf{x}(0) =$

Δx(s) is the Laplace transform of Δx(t)

$$\Delta\mathbf{x}(s) = L\left[\Delta\mathbf{x}(t)\right] = L\left[\mathbf{\Phi}(t,0)\Delta\mathbf{x}(0)\right] = L\left[\mathbf{\Phi}(t,0)\right]\Delta\mathbf{x}(0)$$

Relationship of $(s\mathbf{I} - \mathbf{F})^{-1}$ to State Transition Matrix, F(t,0)

Therefore,

$$\left[s\mathbf{I} - \mathbf{F}\right]^{-1} = L\left[\mathbf{\Phi}(t,0)\right]$$

$$= \textbf{ Laplace transform of the state transition matrix}$$

Initial Condition Response of a Single State Element (Frequency Domain)

$$\boxed{\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\Delta\mathbf{x}(0)}$$

$$\begin{bmatrix} \Delta x_1(s) \\ \Delta x_2(s) \\ \cdots \\ \Delta x_n(s) \end{bmatrix} = \frac{\begin{bmatrix} n_{11}(s) & n_{12}(s) & \cdots & n_{1n}(s) \\ n_{21}(s) & n_{22}(s) & \cdots & n_{2n}(s) \\ \cdots & \cdots & \cdots & \cdots \\ n_{n1}(s) & n_{n2}(s) & \cdots & n_{n2}(s) \end{bmatrix}}{\Delta(s)} \begin{bmatrix} \Delta x_1(0) \\ \Delta x_2(0) \\ \cdots \\ \Delta x_n(0) \end{bmatrix}$$

Initial Condition Response of a Single State Element

Initial condition response of $\Delta x_2(s)$

$$\Delta x_2(s) = \frac{n_{21}(s)}{\Delta(s)}\Delta x_1(0) + \frac{n_{22}(s)}{\Delta(s)}\Delta x_2(0) + \cdots + \frac{n_{2n}(s)}{\Delta(s)}\Delta x_n(0)$$

$$\triangleq \frac{p_2(s)}{\Delta(s)}$$

Partial Fraction Expansion of the Initial Condition Response

Scalar response can be expressed with *n* parts, each containing a single mode

$$\Delta x_i(s) = \frac{p_i(s)}{\Delta(s)}$$

$$= \left(\frac{d_1}{(s-\lambda_1)} + \frac{d_2}{(s-\lambda_2)} + \cdots \frac{d_n}{(s-\lambda_n)} \right)_i, \quad i = 1, n$$

For each i, the coefficients are

$$d_j = (s-\lambda_j) \left. \frac{p_i(s)}{\Delta(s)} \right|_{s=\lambda_j}, \quad j = 1, n$$

Partial Fraction Expansion of the Initial Condition Response

Time response is the inverse Laplace transform

$$\Delta x_i(t) = L^{-1}\left[\Delta x_i(s)\right]$$

$$= L^{-1}\left[\frac{d_1}{(s-\lambda_1)} + \frac{d_2}{(s-\lambda_2)} + \cdots \frac{d_n}{(s-\lambda_n)} \right]_i$$

$$= \left(d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t} + \cdots + d_n e^{\lambda_n t} \right)_i, \quad i = 1, n$$

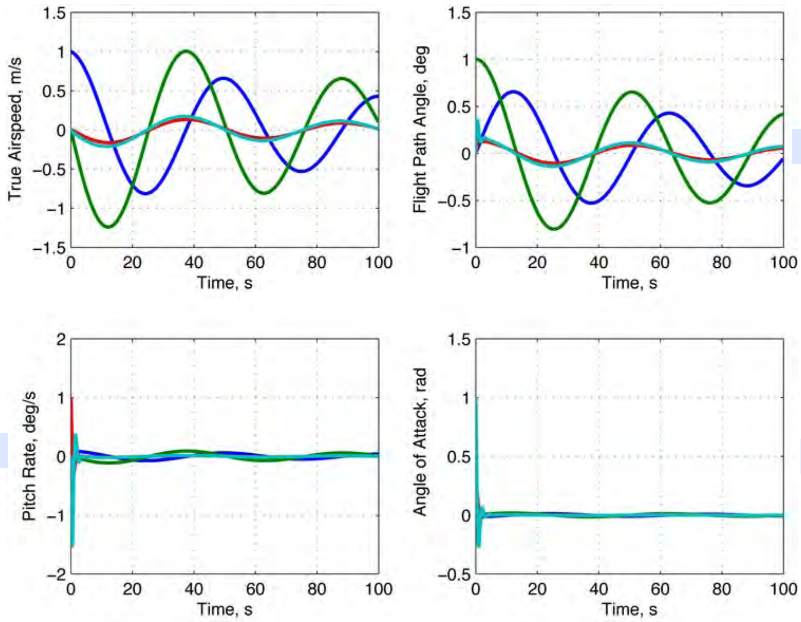
Each element's time response contains every mode of the system (although some coefficients may be negligible)

Longitudinal Motions Contain Both Modes

Phugoid (Long-Period) Mode

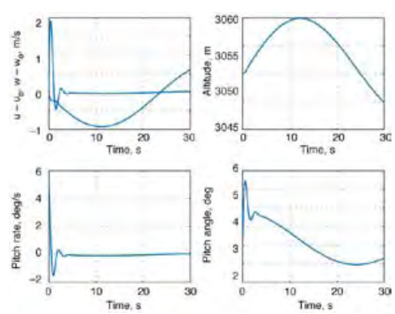
Airspeed

Flight Path Angle

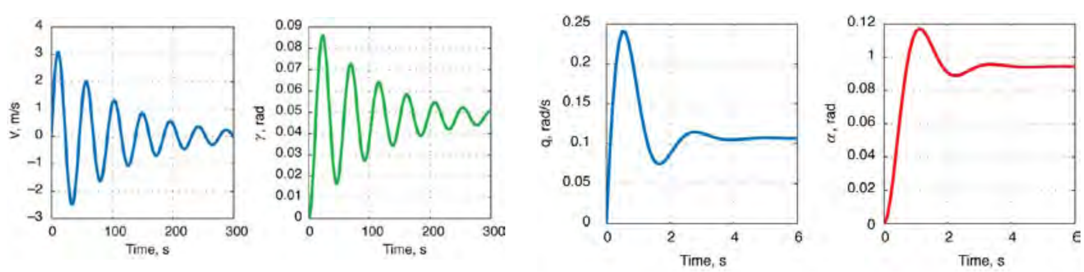


Pitch Rate

Angle of Attack



Aircraft Modes of Motion



Characteristic Polynomial of a LTI Dynamic System

$$\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\Delta\mathbf{x}(0) + \mathbf{G}\,\Delta\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)\right]$$

Inverse of characteristic matrix

$$\left[s\mathbf{I} - \mathbf{F}\right]^{-1} = \frac{Adj\left(s\mathbf{I} - \mathbf{F}\right)}{\left|s\mathbf{I} - \mathbf{F}\right|} \quad (n \; x \; n)$$

- **Characteristic polynomial of the system**
  - **is a scalar**
  - **defines the system's modes of motion**

$$\begin{aligned}\left|s\mathbf{I} - \mathbf{F}\right| &= \det\left(s\mathbf{I} - \mathbf{F}\right) \equiv \Delta(s) \\ &= s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0\end{aligned}$$

Eigenvalues (or Roots) of a Dynamic System

Characteristic equation of the system

$$\begin{aligned}\Delta(s) = \left|s\mathbf{I} - \mathbf{F}\right| &= s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0 = 0 \\ &= \left(s - \lambda_1\right)\left(s - \lambda_2\right)\left(\ldots\right)\left(s - \lambda_n\right) = 0\end{aligned}$$

... where λ_i are the eigenvalues of F or the roots of the characteristic polynomial

Eigenvalues (or Roots) of a Dynamic System

Eigenvalues are real or complex numbers that can be plotted in the ***s* plane**

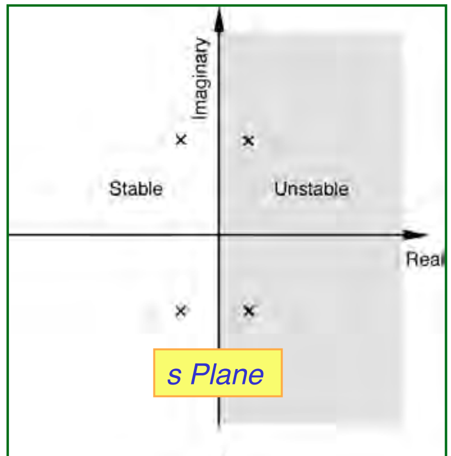
- Real root

$$\lambda_i = \sigma_i$$

- Complex roots occur in conjugate pairs

$$\lambda_i = \sigma_i + j\omega_i$$

$$\lambda_i^* = \sigma_i - j\omega_i$$



Positive real part indicates instability

Roots of the Aircraft Dynamics Characteristic Equation

- 12th-order system of LTI equations
- 12 eigenvalues of the stability matrix, **F**
- 12 roots of the characteristic equation
- Characteristic equation of the system

$$\Delta(s) = s^{12} + c_{11}s^{11} + \ldots + c_1 s + c_0 = 0$$
$$= (s-\lambda_1)(s-\lambda_2)(\ldots)(s-\lambda_{12}) = 0$$

Up to 12 modes of motion

In steady, level flight, longitudinal and lateral-directional LTI perturbation models are uncoupled

$$\Delta(s) = \left[(s-\lambda_1)\cdots(s-\lambda_6)\right]_{long}\left[(s-\lambda_1)\cdots(s-\lambda_6)\right]_{lat-dir} = 0$$

Lateral-Directional Modes of Motion in Steady, Level Flight

$$\Delta\dot{\mathbf{x}}_{Lat-Dir}(t) = \mathbf{F}_{Lat-Dir}\Delta\mathbf{x}_{Lat-Dir}(t) + \mathbf{G}_{Lat-Dir}\Delta\mathbf{u}_{Lat-Dir}(t) + \mathbf{L}_{Lat-Dir}\Delta\mathbf{w}_{Lat-Dir}(t)$$

Roots of the lateral-directional characteristic equation

$$\Delta_{LD}(s) = (s-\lambda_1)(s-\lambda_2)(\ldots)(s-\lambda_6) = 0$$
$$= (s-\lambda_{CR})(s-\lambda_{Head})(s-\lambda_S)(s-\lambda_R)\left[(s-\lambda_{DR})(s-\lambda^*{}_{DR})\right]$$

5 modes of motion (typical)

$$\Delta_{LD}(s) = (s-\lambda_{CR})(s-\lambda_{Head})(s-\lambda_S)(s-\lambda_R)\left(s^2 + 2\zeta_{DR}\omega_{n_{DR}}s + \omega_{n_{DR}}^2\right) = 0$$

Crossrange | Heading | Spiral | Roll | Dutch Roll

Longitudinal Modes of Motion in Steady, Level Flight

$$\Delta\dot{\mathbf{x}}_{Lon}(t) = \mathbf{F}_{Lon}\Delta\mathbf{x}_{Lon}(t) + \mathbf{G}_{Lon}\Delta\mathbf{u}_{Lon}(t) + \mathbf{L}_{Lon}\Delta\mathbf{w}_{Lon}(t)$$

6 roots of the longitudinal characteristic equation

$$\Delta_{Lon}(s) = (s-\lambda_1)(s-\lambda_2)(\ldots)(s-\lambda_6) = 0$$
$$= (s-\lambda_R)(s-\lambda_H)\left[(s-\lambda_P)(s-\lambda^*{}_P)\right]\left[(s-\lambda_{SP})(s-\lambda^*{}_{SP})\right]$$

Real | Real | Complex | Complex | Complex | Complex

4 modes of motion (typical)

$$\Delta_{Lon}(s) = (s-\lambda_R)(s-\lambda_H)\left(s^2 + 2\zeta_P\omega_{nP}s + \omega_{n_P}{}^2\right)\left(s^2 + 2\zeta_{SP}\omega_{n_{SP}}s + \omega_{n_{SP}}^2\right) = 0$$

Range | Height | Phugoid | Short Period

Complex Conjugate Roots Form a Single Oscillatory Mode of Motion

Phugoid Roots

$$
\begin{aligned}
&(s-\lambda_P)(s-\lambda^*_P) \\
&= \left[s-(\sigma_P + j\omega_P)\right]\left[s-(\sigma_P - j\omega_P)\right] \\
&= \left(s^2 + 2\zeta_P\omega_{n_P}s + \omega_{n_P}^2\right)
\end{aligned}
$$

ω_n : **Natural frequency**, rad/s

ζ : **Damping ratio**, -

Short Period Roots

$$
\begin{aligned}
&(s-\lambda_{SP})(s-\lambda^*_{SP}) \\
&= \left[s-(\sigma_{SP} + j\omega_{SP})\right]\left[s-(\sigma_{SP} - j\omega_{SP})\right] \\
&= \left(s^2 + 2\zeta_{SP}\omega_{n_{SP}}s + \omega_{n_{SP}}^2\right)
\end{aligned}
$$

Response to a Control Input

Neglect initial condition

State response to control

$$
s\Delta\mathbf{x}(s) = \mathbf{F}\Delta\mathbf{x}(s) + \mathbf{G}\Delta\mathbf{u}(s) + \Delta\mathbf{x}(0), \quad \Delta\mathbf{x}(0) \triangleq \mathbf{0}
$$

$$
\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\mathbf{G}\,\Delta\mathbf{u}(s)
$$

Output response to control

$$
\begin{aligned}
\Delta\mathbf{y}(s) &= \mathbf{H_x}\Delta\mathbf{x}(s) + \mathbf{H_u}\Delta\mathbf{u}(s) \\
&= \mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}\right]^{-1}\mathbf{G}\,\Delta\mathbf{u}(s) + \mathbf{H_u}\Delta\mathbf{u}(s) \\
&= \left\{\mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}\right]^{-1}\mathbf{G} + \mathbf{H_u}\right\}\Delta\mathbf{u}(s)
\end{aligned}
$$

Longitudinal Transfer Function Matrix

- With $\mathbf{H_x = I}$, and assuming
  - Elevator produces only a pitching moment
  - Throttle affects only the rate of change of velocity
  - Flaps produce only lift

$$\boldsymbol{H}_{Lon}(s) = \mathbf{H}_{\mathbf{x}_{Lon}}\left[s\mathbf{I} - \mathbf{F}_{Lon}\right]^{-1}\mathbf{G}_{Lon}$$

$$= \frac{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} n_V^V(s) & n_\gamma^V(s) & n_q^V(s) & n_\alpha^V(s) \\ n_V^\gamma(s) & n_\gamma^\gamma(s) & n_q^\gamma(s) & n_\alpha^\gamma(s) \\ n_V^q(s) & n_\gamma^q(s) & n_q^q(s) & n_\alpha^q(s) \\ n_V^\alpha(s) & n_\gamma^\alpha(s) & n_q^\alpha(s) & n_\alpha^\alpha(s) \end{bmatrix}\begin{bmatrix} 0 & T_{\delta T} & 0 \\ 0 & 0 & L_{\delta F}/V_N \\ M_{\delta E} & 0 & 0 \\ 0 & 0 & -L_{\delta F}/V_N \end{bmatrix}}{\Delta_{Lon}(s)}$$

Longitudinal Transfer Function Matrix

- There are 4 outputs and 3 inputs

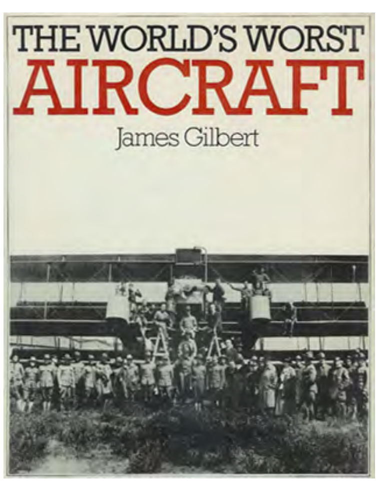
$$\boldsymbol{H}_{Lon}(s) = \frac{\begin{bmatrix} n_{\delta E}^V(s) & n_{\delta T}^V(s) & n_{\delta F}^V(s) \\ n_{\delta E}^\gamma(s) & n_{\delta T}^\gamma(s) & n_{\delta F}^\gamma(s) \\ n_{\delta E}^q(s) & n_{\delta T}^q(s) & n_{\delta F}^q(s) \\ n_{\delta E}^\alpha(s) & n_{\delta T}^\alpha(s) & n_{\delta F}^\alpha(s) \end{bmatrix}}{\left(s^2 + 2\zeta_P\omega_{nP}s + \omega_{n_P}^2\right)\left(s^2 + 2\zeta_{SP}\omega_{n_{SP}}s + \omega_{n_{SP}}^2\right)}$$

Longitudinal Transfer Function Matrix

- **Input-output relationship**

$$\boxed{\begin{bmatrix} \Delta V(s) \\ \Delta\gamma(s) \\ \Delta q(s) \\ \Delta\alpha(s) \end{bmatrix} = \boldsymbol{H}_{Lon}(s) \begin{bmatrix} \Delta\delta E(s) \\ \Delta\delta T(s) \\ \Delta\delta F(s) \end{bmatrix}}$$

Historical Factoids
Unusual Aircraft

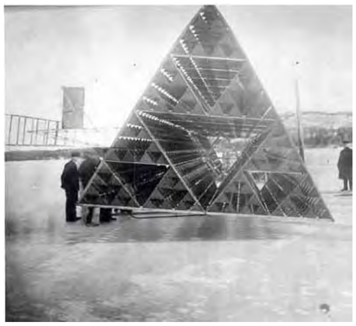


Forssman bomber (?)

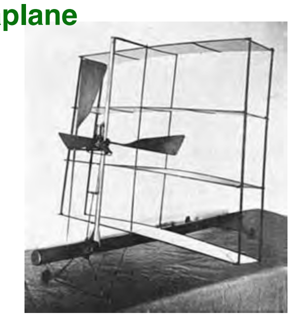
Westland P.12 Lysander

Multiplanes-1

- AEA Cygnet II, Alexander Graham Bell, Glenn Curtiss, 1909



- Hargrave quadraplane (model), 1889

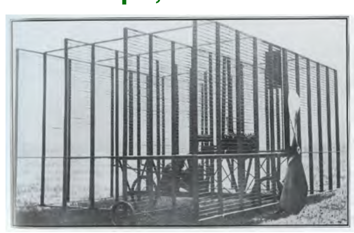


- D'Equevillery, 1908

Multiplanes-2

- Phillips, 1904

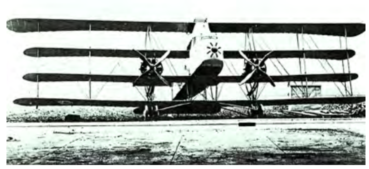
- Phillips, 1907



- Vedo Villi, 1911

- Wight Quadraplane, 1916

- Pemberton-Billings Nighthawk, 1916



- John Septaplane, 1919

Flying House Boat

- **Caproni Ca 60, 1920**

Miraculously, this machine DID fly the first time in 1921- it reached a height of 60 feet, collapsed, and plummeted toward the lake just after take off, killing both pilots.

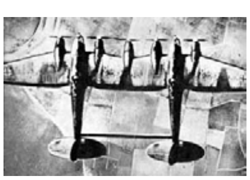
Wings derived from Ca.42 bomber

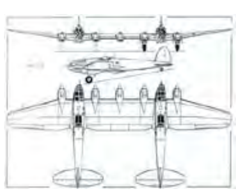
Unusual Engine Layouts

- **Farman 3-engine Jabiru**

- **Tarrant 6-engine Tabor, 1919**

- **Heinkel 5-engine He111Z**





- **Farman 4-engine Jabiru, 1923**

Scalar Transfer Function from Δu_j to Δy_i

- Just one element of the matrix, $H(s)$
- Each numerator term is a polynomial with q zeros, where q varies from term to term and $\leq n-1$

$$\boxed{H_{ij}(s) = \frac{n_{ij}(s)}{\Delta(s)} = \frac{k_{ij}\left(s^q + b_{q-1}s^{q-1} + \ldots + b_1 s + b_0\right)}{\left(s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0\right)}}$$

- Denominator polynomial contains n roots

$$\boxed{= k_{ij}\frac{\left(s-z_1\right)_{ij}\left(s-z_2\right)_{ij}\ldots\left(s-z_q\right)_{ij}}{\left(s-\lambda_1\right)\left(s-\lambda_2\right)\ldots\left(s-\lambda_n\right)}}$$

\# zeros = q
\# poles = n

Control Response of a Single State Element

$$\boxed{\Delta y_i\left(s\right) = k_{ij}\frac{n_{ij}(s)}{\Delta(s)}\Delta u_j\left(s\right)}$$

Bode Plot

(Frequency Response of a Scalar Transfer Function)

Scalar Frequency Response Function

Substitute: $s = j\omega$

$$H_{ij}(j\omega) = \frac{k_{ij}\left(j\omega - z_1\right)_{ij}\left(j\omega - z_2\right)_{ij}\ldots\left(j\omega - z_q\right)_{ij}}{\left(j\omega - \lambda_1\right)\left(j\omega - \lambda_2\right)\ldots\left(j\omega - \lambda_n\right)}$$

$$= a(\omega) + jb(\omega) \rightarrow AR(\omega)\, e^{j\phi(\omega)}$$

- **Frequency response is a complex function of input frequency, ω**
  - **Real and imaginary parts, or**
  - **** Amplitude ratio and phase angle ****

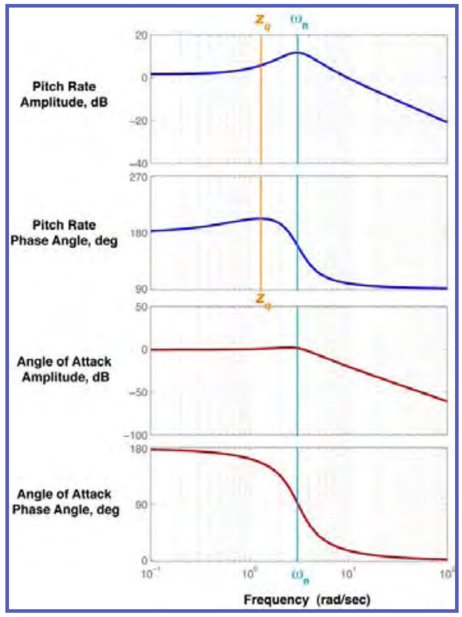
Short-Period Frequency Response ($s = j\omega$) Expressed as Amplitude Ratio and Phase Angle

Pitch-rate frequency response

$$\frac{\Delta q(j\omega)}{\Delta\delta E(j\omega)} = \frac{k_q\left(j\omega - z_q\right)}{-\omega^2 + 2\zeta_{SP}\omega_{n_{SP}} j\omega + \omega_{n_{SP}}^2} = AR_q(\omega)\, e^{j\phi_q(\omega)}$$

Angle-of-attack frequency response

$$\frac{\Delta\alpha(j\omega)}{\Delta\delta E(j\omega)} = \frac{k_\alpha\left(j\omega - z_\alpha\right)}{-\omega^2 + 2\zeta_{SP}\omega_{n_{SP}} j\omega + \omega_{n_{SP}}^2} = AR_\alpha(\omega)\, e^{j\phi_\alpha(\omega)}$$



Bode Plot Portrays Response to Sinusoidal Control Input

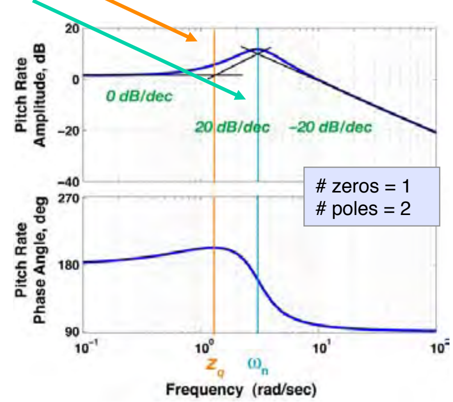
$$\frac{\Delta q(j\omega)}{\Delta\delta E(j\omega)} = \frac{k_q\left(j\omega - z_q\right)}{-\omega^2 + 2\zeta_{SP}\omega_{n_{SP}} j\omega + \omega_{n_{SP}}^2} = AR_q(\omega)\, e^{j\phi_q(\omega)}$$

Express amplitude ratio in decibels

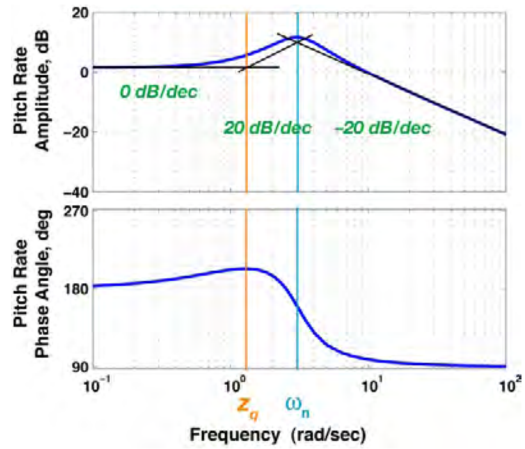
$$AR(dB) = 20\log_{10}\left[AR\left(original\ units\right)\right]$$

20 dB = factor of 10

Products in original units are sums in decibels



Bode Plot Portrays Response to Sinusoidal Control Input



zeros = 1
poles = 2

Plot ***AR(dB)*** vs. $\log_{10}(\omega_{input})$

Plot phase angle, φ(deg) vs. $\log_{10}(\omega_{input})$

Asymptotes form "skeleton" of response amplitude ratio

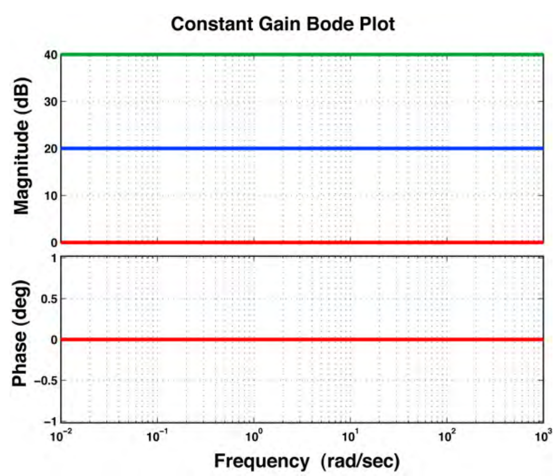
Constant Gain Bode Plot

$$y(t) = hu(t)$$

$$H(j\omega) = 1$$

$$H(j\omega) = 10$$

$$H(j\omega) = 100$$



$$Slope = 0dB/dec, \quad \text{Amplitude Ratio} = \text{constant}$$
$$Phase\ Angle = 0°$$

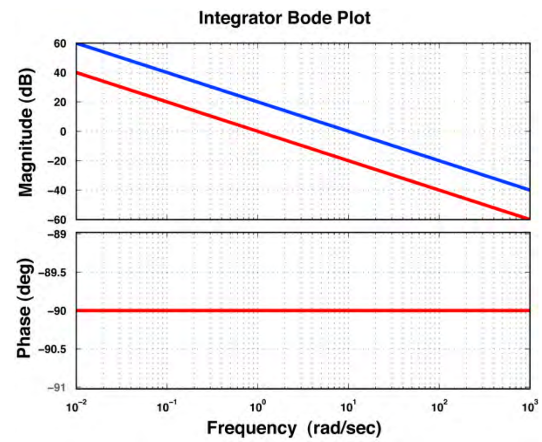
Integrator Bode Plot

$$y(t) = h\int_0^t u(t)\,dt$$

$$H(j\omega) = \frac{1}{j\omega}$$

$$H(j\omega) = \frac{10}{j\omega}$$

$$Slope = -20dB/dec$$
$$Phase\ Angle = -90°$$



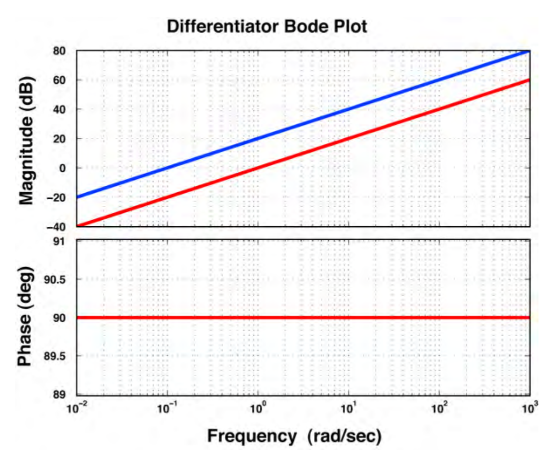
Differentiator Bode Plot

$$y(t) = h\frac{du(t)}{dt}$$

$$H(j\omega) = j\omega$$

$$H(j\omega) = 10\,j\omega$$

$$Slope = +20dB/dec$$
$$Phase\ Angle = +90°$$



Sign Change

Integral

$$\boxed{y(t) = -h\int_0^t u(t)\,dt}$$

$$\boxed{H(j\omega) = -\frac{h}{j\omega}}$$

$$\boxed{\begin{array}{l} Slope = -20dB/dec \\ Phase\ Angle = +90° \end{array}}$$

Derivative

$$\boxed{y(t) = -h\frac{du(t)}{dt}}$$

$$\boxed{H(j\omega) = -j\omega}$$

$$\boxed{\begin{array}{l} Slope = +20dB/dec \\ Phase\ Angle = -90° \end{array}}$$

Multiple Integrators and Differentiators

Double Integral

$$\boxed{y(t) = h\int_0^t\int_0^t u(t)\,dt^2}$$

$$\boxed{H(j\omega) = \frac{h}{(j\omega)^2}}$$

$$\boxed{\begin{array}{l} Slope = -40dB/dec \\ Phase\ Angle = -180° \end{array}}$$

Double Derivative

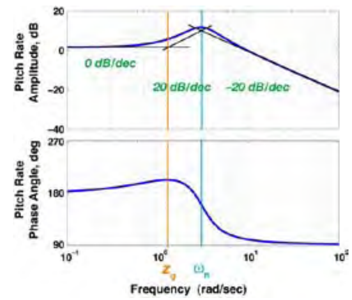
$$\boxed{y(t) = h\frac{d^2u(t)}{dt^2}}$$

$$\boxed{H(j\omega) = h(j\omega)^2}$$

$$\boxed{\begin{array}{l} Slope = +40dB/dec \\ Phase\ Angle = +180° \end{array}}$$

Why Plot Vertical Lines where $\omega = z$ and ω_n?

AR **Asymptotes change at frequencies corresponding to poles and zeros**



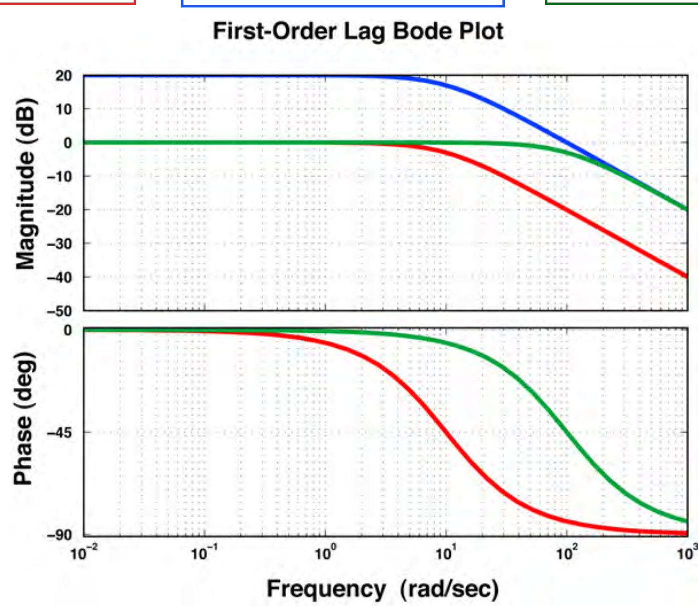
$$\frac{\Delta q(j\omega)}{\Delta \delta E(j\omega)} = \frac{k_q\left(j\omega - z_q\right)}{-\omega^2 + 2\zeta_{SP}\omega_{n_{SP}} j\omega + \omega_{n_{SP}}{}^2}$$

$$\text{When } \omega = -z_q \left(\text{for negative } z_q\right),$$
$$k_q\left(j\omega - z_q\right) = k_q z_q(-j-1) = -k_q z_q(j+1) = k_q\left|z_q\right| e^{+45^\circ}$$

$$\text{When } \omega = \omega_{n_{SP}}, -\omega_{n_{SP}}^2 + 2\zeta_{SP} j\omega_{n_{SP}}^2 + \omega_{n_{SP}}^2 = j2\zeta_{SP}\omega_{n_{SP}}^2$$
$$= \frac{1}{j2\zeta_{SP}\omega_{n_{SP}}^2} = \frac{-j}{2\zeta_{SP}\omega_{n_{SP}}^2} = \frac{1}{2\zeta_{SP}\omega_{n_{SP}}^2} e^{-90^\circ} \text{ for positive } \zeta_{SP}$$

Bode Plots of First-Order Lags

$$H_{red}(j\omega) = \frac{10}{(j\omega + 10)} \qquad H_{blue}(j\omega) = \frac{100}{(j\omega + 10)} \qquad H_{green}(j\omega) = \frac{100}{(j\omega + 100)}$$

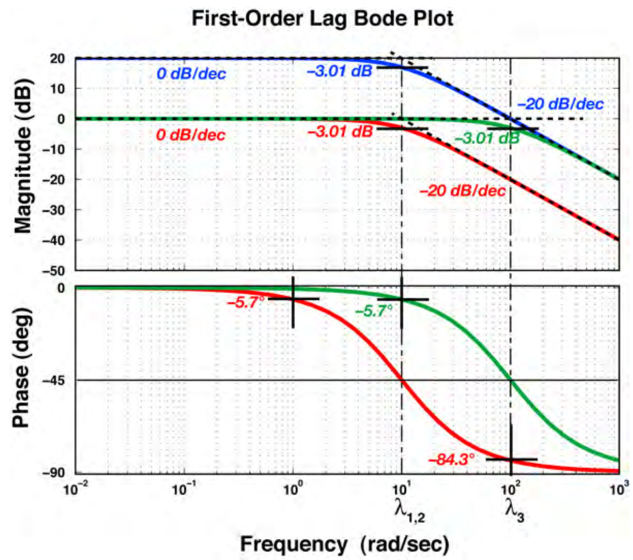


Bode Plot Asymptotes, Departures, and Phase Angles for First-Order Lags

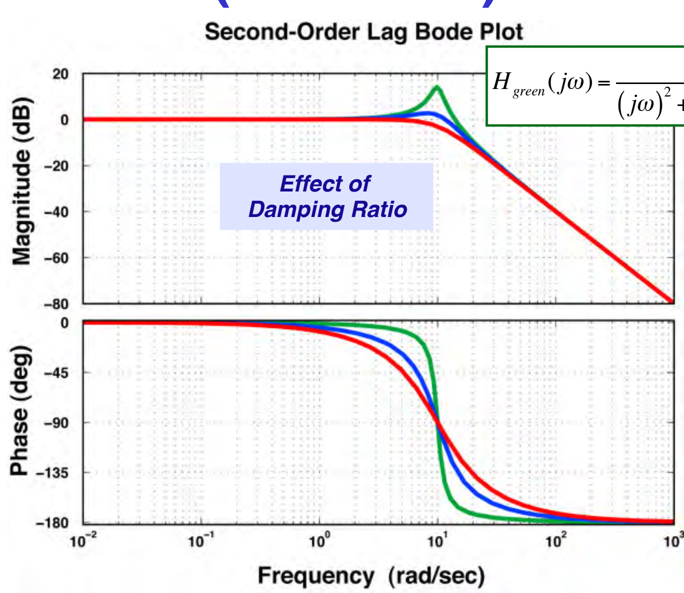
- General shape of amplitude ratio governed by asymptotes
- Slope of asymptotes changes by multiples of ±20 dB/dec at poles or zeros
- Actual AR departs from asymptotes

- AR asymptotes of a real pole
  - When $\omega = 0$, slope = 0 dB/dec
  - When $\omega \geq \lambda$, slope = –20 dB/dec

- Phase angle of a real, negative pole
  - When $\omega = 0$, $\varphi = 0°$
  - When $\omega = \lambda$, $\varphi = –45°$
  - When $\omega \to \infty$, $\varphi \to –90°$



Bode Plots of Second-Order Lags (No Zeros)



$$H_{green}(j\omega) = \frac{10^2}{(j\omega)^2 + 2(0.1)(10)(j\omega) + 10^2}$$

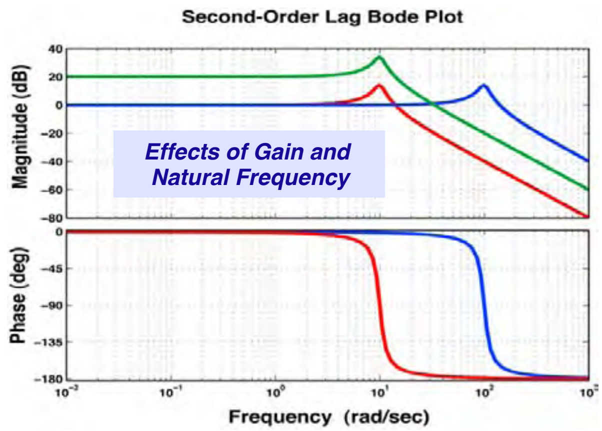
$$H_{blue}(j\omega) = \frac{10^2}{(j\omega)^2 + 2(0.4)(10)(j\omega) + 10^2}$$

$$H_{red}(j\omega) = \frac{10^2}{(j\omega)^2 + 2(0.707)(10)(j\omega) + 10^2}$$

Bode Plots of Second-Order Lags (No Zeros)

$$H_{red}(j\omega) = \frac{10^2}{(j\omega)^2 + 2(0.1)(10)(j\omega) + 10^2}$$

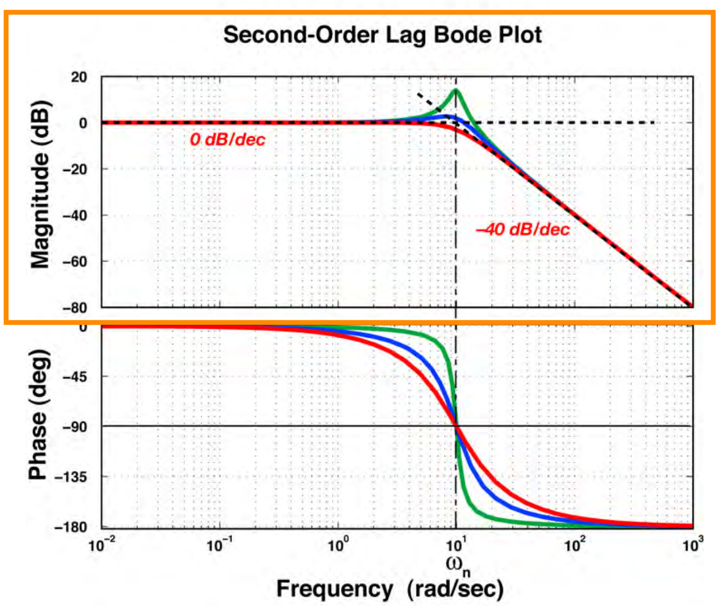
$$H_{green}(j\omega) = \frac{10^3}{(j\omega)^2 + 2(0.1)(10)(j\omega) + 10^2}$$



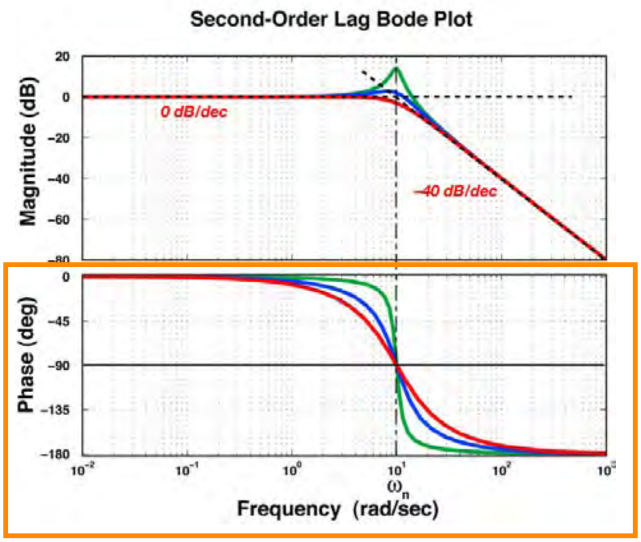
$$H_{blue}(j\omega) = \frac{100^2}{(j\omega)^2 + 2(0.1)(100)(j\omega) + 100^2}$$

Amplitude Ratio Asymptotes and Departures of Second-Order Bode Plots (No Zeros)

- AR asymptotes of a pair of complex poles
  - When $\omega = 0$, slope = 0 dB/dec
  - When $\omega \geq \omega_n$, slope = –40 dB/dec
- Height of resonant peak depends on damping ratio



Phase Angles of Second-Order Bode Plots (No Zeros)



- **Phase angle of a pair of complex negative poles**
  - When $\omega = 0$, $\varphi = 0°$
  - When $\omega = \omega_n$, $\varphi = -90°$
  - When $\omega \to \infty$, $\varphi \to -180°$
- **Abruptness of phase shift depends on damping ratio**

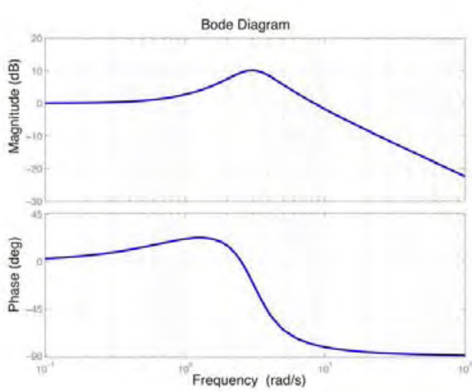
MATLAB Bode Plot with asymp.m

http://www.mathworks.com/matlabcentral/

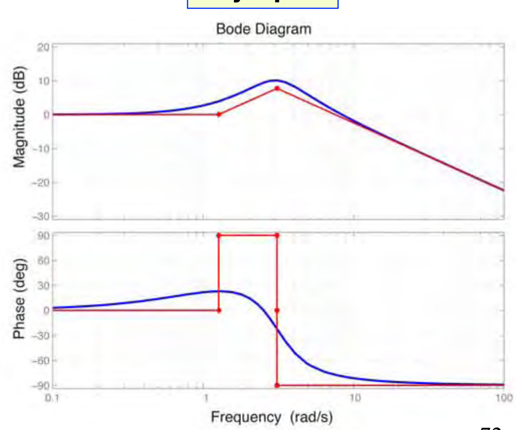
http://www.mathworks.com/matlabcentral/fileexchange/10183-bode-plot-with-asymptotes

2nd-Order Pitch Rate Frequency Response

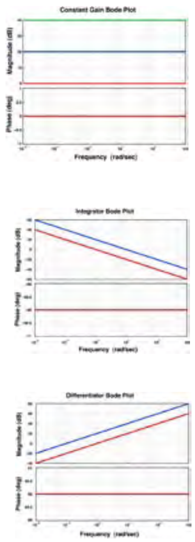
bode.m

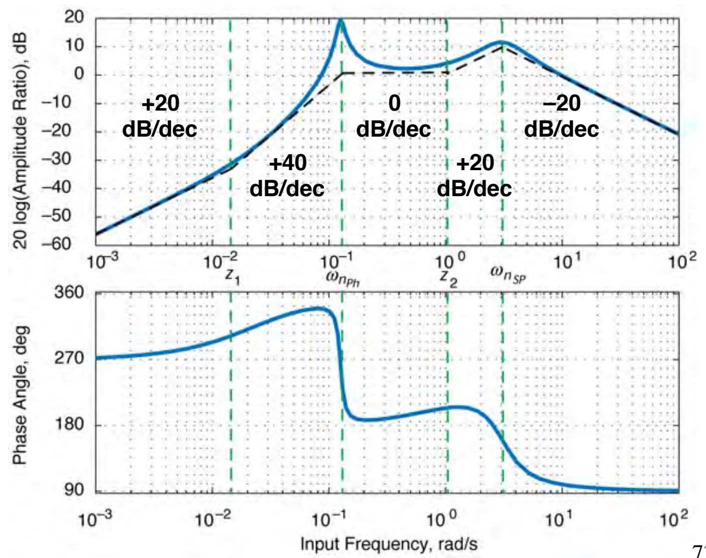


asymp.m



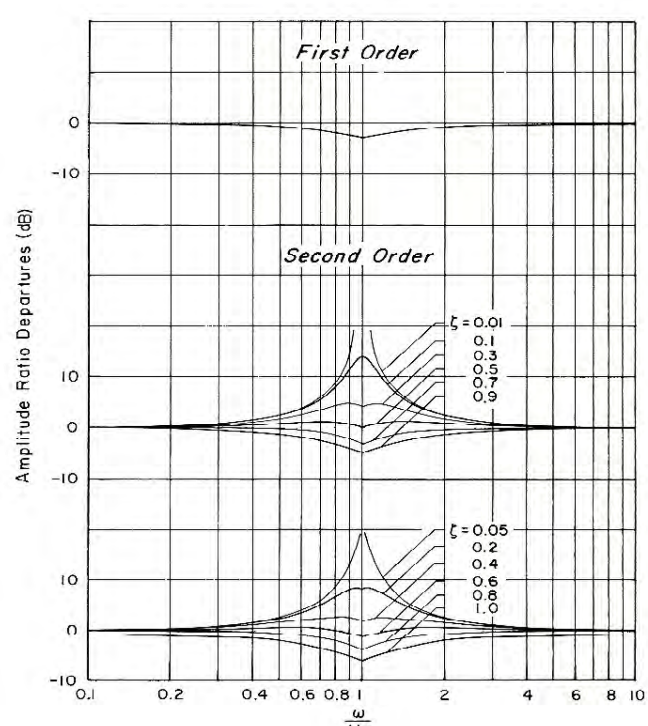
Constant Gain, Integrator, and Differentiator Bode Plots Form Asymptotes for More Complex Transfer Functions





Frequency Response AR Departures in the Vicinity of Poles

First- and Second-Order Departures from Amplitude Ratio Asymptotes



- Difference between actual amplitude ratio (*dB*) and asymptote = departure (dB)
- Results for multiple roots are additive
- Zero departures have opposite sign

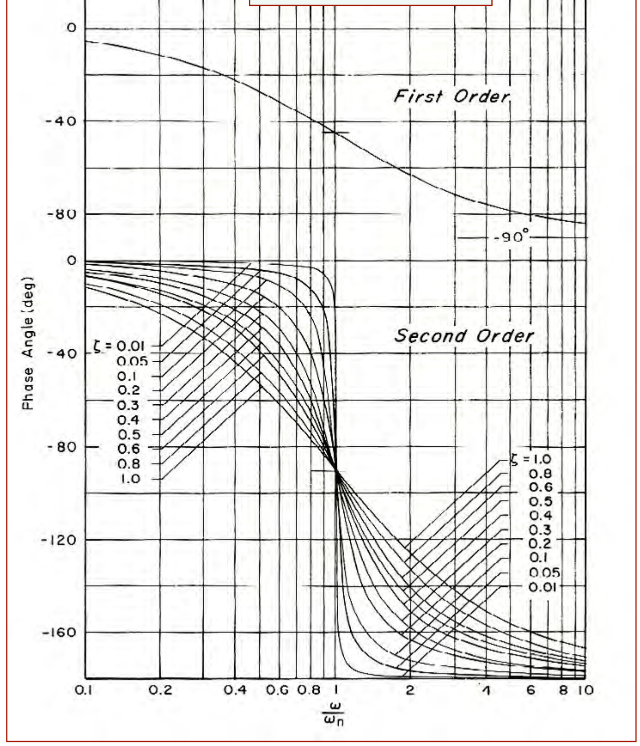
McRuer, Ashkenas, and Graham, ***Aircraft Dynamics and Automatic Control***, Princeton University Press, 1973

Phase Angle Variations in the Vicinity of Poles

- Results for multiple roots are additive
- LHP zero variations have opposite sign
- RHP zeros have same sign

McRuer, Ashkenas, and Graham, *Aircraft Dynamics and Automatic Control*

First- and Second-Order Phase Angles



Historical Factoids

Flying Cars-1

Curtiss Autocar, 1917

Waterman Aerobile, 1935

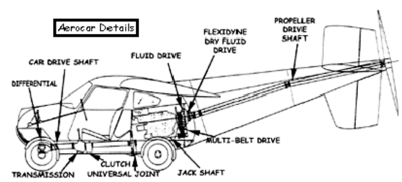
Stout Skycar, 1931

ConsolidatedVultee 111, 1940s

Flying Cars-2

ConvAIRCAR 116 (w/ Crosley auto), 1940s

Taylor AirCar, 1950s



Hallock Road Wing , 1957

Flying Cars-3

“Mitzar” SkyMaster Pinto, 1970s

Lotus Elise Aerocar, concept, 2002

Haynes Skyblazer, concept, 2004

Aeromobil, 2014

Flying Cars-4

Terrafugia Transition

Terrafugia TF-X, concept

… or, for the same price

PLUS

Next Time:
Root Locus Analysis

Reading:
Flight Dynamics
357-361, 465-467, 488-490,
509-514

Supplementary Material

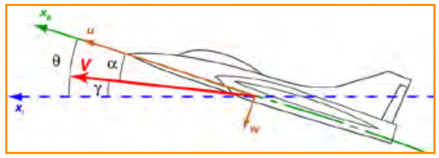
Longitudinal Modes of Motion

- **Eigenvalues determine the damping and natural frequencies of the linear system's modes of motion**
- **Longitudinal characteristic equation has 6 eigenvalues**
  - 4 eigenvalues normally appear as 2 complex pairs
  - Range and height modes usually inconsequential

$$\lambda_{ran}: range\ mode \approx 0$$
$$\lambda_{hgt}: height\ mode \approx 0$$
$$\left(\zeta_P, \omega_{n_P}\right): phugoid\ mode$$
$$\left(\zeta_{SP}, \omega_{n_{SP}}\right): short\text{ - }period\ mode$$

Simplified Longitudinal Modes of Motion

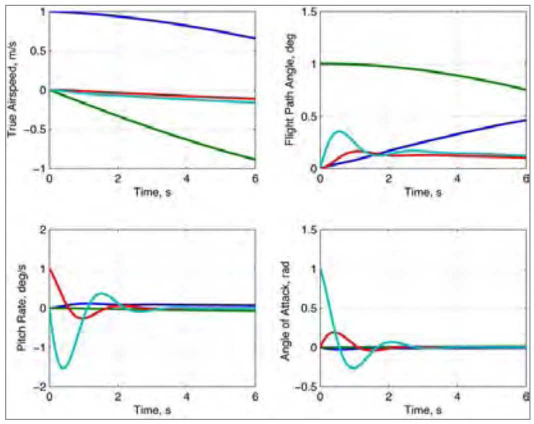
Short-Period Mode



- Note change in time scale

Airspeed

Flight Path Angle



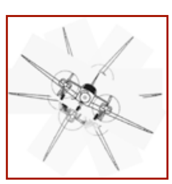
Pitch Rate

Angle of Attack

Lateral-Directional Modes of Motion

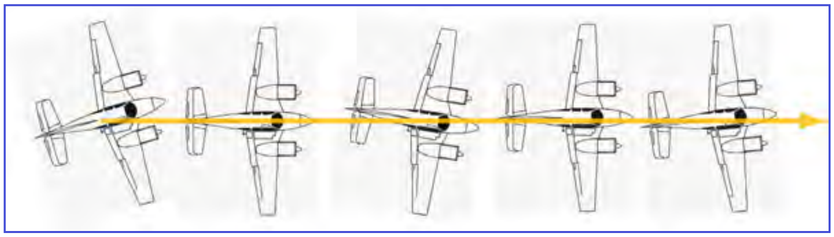
- **Lateral-directional characteristic equation has 6 eigenvalues**
  - 2 eigenvalues normally appear as a complex pair
  - Crossrange and heading modes usually inconsequential

$$\lambda_{cr} : crossrange\ mode \approx 0$$
$$\lambda_{head} : heading\ mode \approx 0$$
$$\lambda_S : spiral\ mode$$
$$\lambda_R : roll\ mode$$
$$\left(\zeta_{DR}, \omega_{n_{DR}}\right) : Dutch\ roll\ mode$$

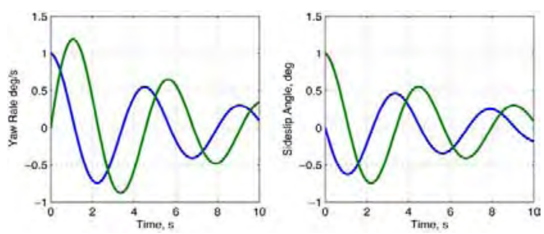


Simplified Lateral Modes of Motion

Dutch-Roll Mode



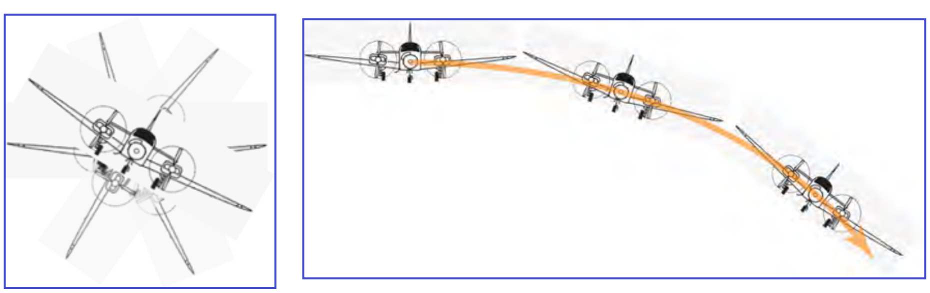
Yaw Rate



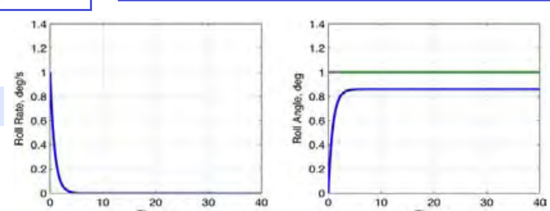
Sideslip Angle

Simplified Lateral Modes of Motion

Roll and Spiral Modes

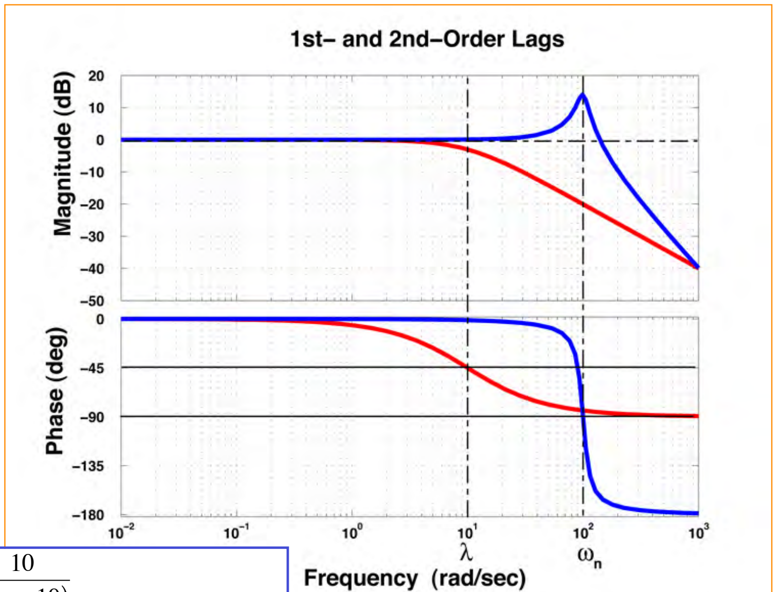


Roll Rate



Roll Angle

Bode Plots of 1st- and 2nd-Order Lags



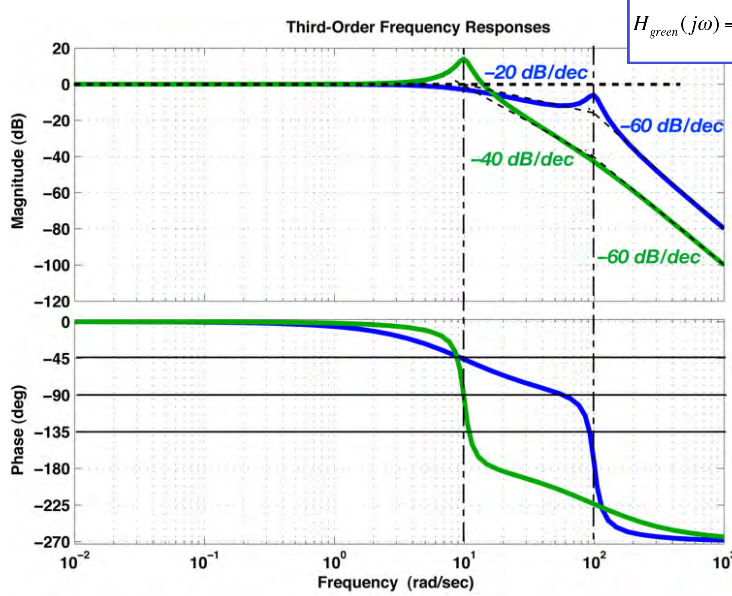
$$H_{red}(j\omega) = \frac{10}{(j\omega + 10)}$$

$$H_{blue}(j\omega) = \frac{100^2}{(j\omega)^2 + 2(0.1)(100)(j\omega) + 100^2}$$

Bode Plots of 3rd-Order Lags

$$H_{blue}(j\omega) = \left[\frac{10}{(j\omega + 10)}\right]\left[\frac{100^2}{(j\omega)^2 + 2(0.1)(100)(j\omega) + 100^2}\right]$$

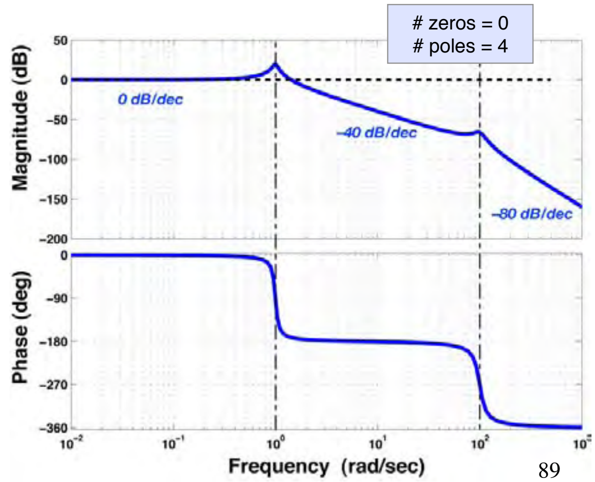
$$H_{green}(j\omega) = \left[\frac{10^2}{(j\omega)^2 + 2(0.1)(10)(j\omega) + 10^2}\right]\left[\frac{100}{(j\omega + 100)}\right]$$



Bode Plot of a 4th-Order System with No Zeros

$$H(j\omega)=\left[\frac{1^2}{(j\omega)^2+2(0.05)(1)(j\omega)+1^2}\right]\left[\frac{100^2}{(j\omega)^2+2(0.1)(100)(j\omega)+100^2}\right]$$

- Resonant peaks and large phase shifts at each natural frequency
- Additive AR slope shifts at each natural frequency

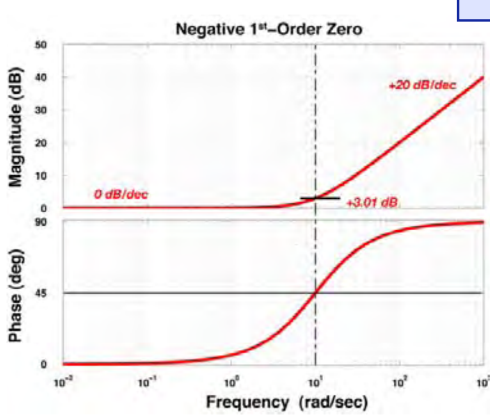


Left-Half-Plane Transfer Function Zero

Zeros are numerator singularities

$$H(j\omega)=\left(j\omega+10\right)$$

$$H(j\omega)=\frac{k\left(j\omega-z_1\right)\left(j\omega-z_2\right)...}{\left(j\omega-\lambda_1\right)\left(j\omega-\lambda_2\right)...\left(j\omega-\lambda_n\right)}$$

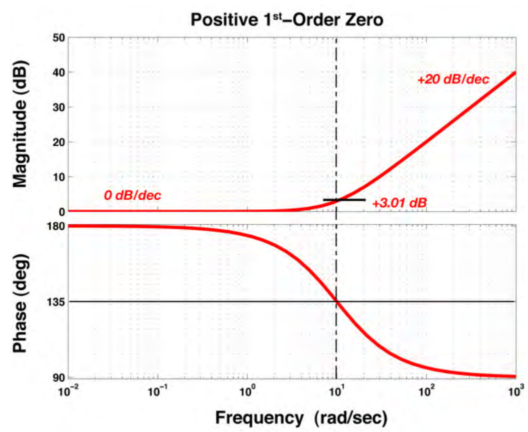


- Single zero in left half plane
- Introduces a +20 dB/dec slope
- Produces phase lead in vicinity of zero

Right-Half-Plane Transfer Function Zero

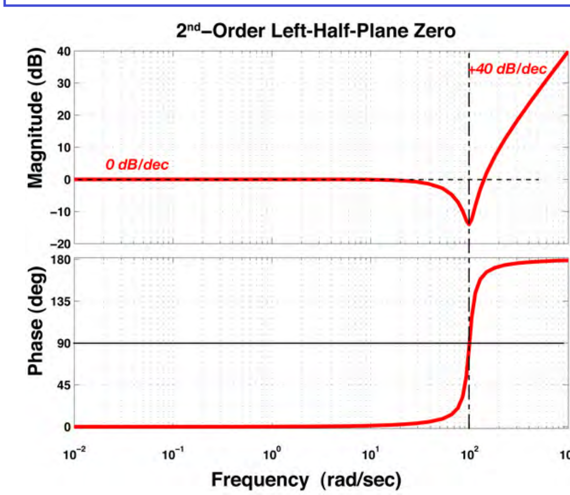
$$\boxed{H(j\omega) = -\left(j\omega - 10\right)}$$

- Single zero in right half plane
- Introduces a +20 dB/dec slope
- Produces phase lag in vicinity of zero



Second-Order Transfer Function Zero

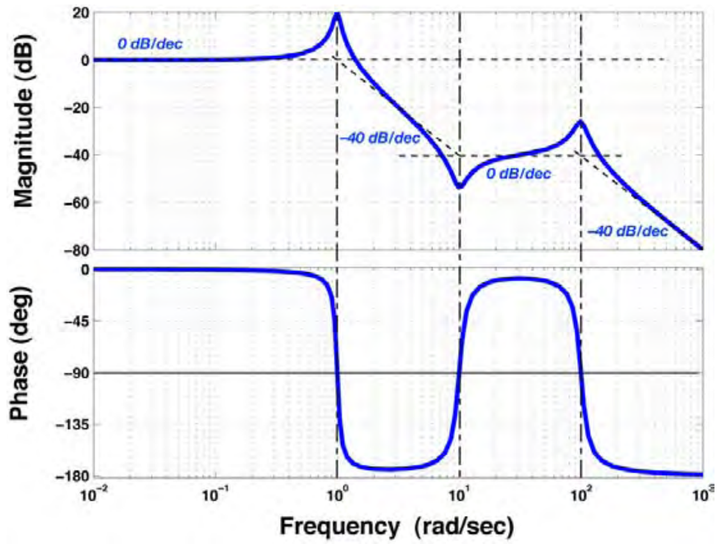
$$\boxed{H(j\omega) = \left(j\omega - z\right)\left(j\omega - z^*\right) = \left[\left(j\omega\right)^2 + 2\left(0.1\right)\left(100\right)\left(j\omega\right) + 100^2\right]}$$



- Complex pair of zeros produces an amplitude ratio "notch" at its "natural frequency"

4^{th}-Order Transfer Function with 2^{nd}-Order Zero

$$H(j\omega) = \frac{\left[(j\omega)^2 + 2(0.1)(10)(j\omega) + 10^2\right]}{\left[(j\omega)^2 + 2(0.05)(1)(j\omega) + 1^2\right]\left[(j\omega)^2 + 2(0.1)(100)(j\omega) + 100^2\right]}$$



Elevator-to-Normal-Velocity Frequency Response

$$\frac{\Delta w(s)}{\Delta \delta E(s)} = \frac{n_{\delta E}^{w}(s)}{\Delta_{Lon}(s)} \approx \frac{M_{\delta E}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{Approx\ Ph}\left(s - z_3\right)}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{Ph}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{SP}}$$

- $(n - q) = 1$
- Complex zero almost (but not quite) cancels phugoid response