Propagation of Uncertainty in Dynamic Systems

Robert Stengel
Optimal Control and Estimation MAE 546
Princeton University, 2015

- Propagation of the mean and variance in linear, time-varying discrete-time systems
- Markov processes and the transition function property
- White and colored noise inputs
- Sampled-data representation of continuous-time systems
- Propagation of the mean and variance in continuous-time systems

http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Uncertain Linear, Time-Varying (LTV) Dynamic Model

- Discrete-time LTV model with known coefficients

$$\boxed{\mathbf{x}_k = \mathbf{\Phi}_{k-1}\mathbf{x}_{k-1} + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1} + \mathbf{\Lambda}_{k-1}\mathbf{w}_{k-1}}$$

- Initial condition and disturbance inputs are not known precisely
- All random variables are Gaussian, i.e., they are fully described by means and covariances

Disturbance Input of the LTV Dynamic Model

$$\boxed{\mathbf{x}_k = \mathbf{\Phi}_{k-1}\mathbf{x}_{k-1} + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1} + \mathbf{\Lambda}_{k-1}\mathbf{w}_{k-1}}$$

Disturbance input ('process noise') is a white-noise sequence

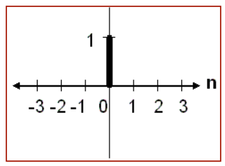
$$E(\mathbf{w}_k) = \mathbf{0}$$

$$E(\mathbf{w}_k\mathbf{w}_k^T) = \mathbf{Q}'_k; \quad E(\mathbf{w}_k\mathbf{w}_{k-l}^T) = \mathbf{0}, \quad l = \text{ any non-zero integer}$$

or

$$E(\mathbf{w}_j\mathbf{w}_k^T) = \mathbf{Q}'_k\,\delta_{jk}$$

$$\delta_{jk} \triangleq \textbf{Kronecker delta function} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$



Initial Condition and Control of the LTV Dynamic Model

Initial condition described by mean and covariance

$$E(\mathbf{x}_0) = \bar{\mathbf{x}}_0 \triangleq \mathbf{m}_o; \quad E\left[(\mathbf{x}_0 - \mathbf{m}_o)(\mathbf{x}_0 - \mathbf{m}_o)^T\right] = \mathbf{P}_0$$

$$\dim(\mathbf{x}_o) = n \times 1$$

Control input is known precisely

$$E[\mathbf{u}_k] = \bar{\mathbf{u}}_k = \mathbf{u}_k; \quad E\left[(\mathbf{u}_k - \bar{\mathbf{u}}_k)(\mathbf{u}_k - \bar{\mathbf{u}}_k)^T\right] = \mathbf{U}_k = \mathbf{0}$$

$$\dim(\mathbf{u}_k) = m \times 1$$

Cross-covariances are zero

$$E\left[(\mathbf{x}_k - \bar{\mathbf{x}}_k)\mathbf{w}_k^T\right] = \mathbf{M}_k = \mathbf{0}$$

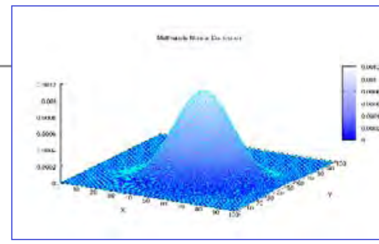
$$E\left[(\mathbf{x}_k - \bar{\mathbf{x}}_k)\mathbf{u}_k^T\right] = \mathbf{0}$$

$$E\left[\mathbf{w}_k\mathbf{u}_k^T\right] = \mathbf{0}$$

Probability Density Function of the LTV Dynamic Model

Initial probability density function depends only on the mean and covariance of the Gaussian distribution

$$\mathrm{pr}(\mathbf{x}_0) = \frac{1}{(2\pi)^{n/2}\left|\mathbf{P}_0\right|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}_0 - \mathbf{m}_0)^T \mathbf{P}_0^{-1}(\mathbf{x}_0 - \mathbf{m}_0)}$$



Expected Value of the State

First moment of x

$$E(\mathbf{x}_k) = E(\mathbf{\Phi}_{k-1}\mathbf{x}_{k-1} + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1} + \mathbf{\Lambda}_{k-1}\mathbf{w}_{k-1})$$

$$\bar{\mathbf{x}}_k = \mathbf{\Phi}_{k-1}\bar{\mathbf{x}}_{k-1} + \mathbf{\Gamma}_{k-1}\bar{\mathbf{u}}_{k-1} + \mathbf{\Lambda}_{k-1}(0)$$

$$\mathbf{m}_k = \mathbf{\Phi}_{k-1}\mathbf{m}_{k-1} + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1}$$

Expected Value of the State Covariance

Second central moment of x

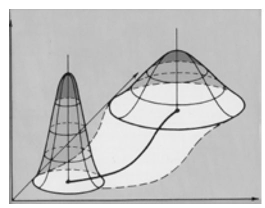
$$E\left[(\mathbf{x}_k-\mathbf{m}_k)(\mathbf{x}_k-\mathbf{m}_k)^T\right] \triangleq \mathbf{P}_k$$
$$= E\left\{\left[\boldsymbol{\Phi}_{k-1}(\mathbf{x}_{k-1}-\mathbf{m}_{k-1})+\boldsymbol{\Gamma}_{k-1}(\mathbf{u}_{k-1}-\bar{\mathbf{u}}_{k-1})+\boldsymbol{\Lambda}_{k-1}\mathbf{w}_{k-1}\right]\left[\boldsymbol{\Phi}_{k-1}(\mathbf{x}_{k-1}-\mathbf{m}_{k-1})+\boldsymbol{\Gamma}_{k-1}(\mathbf{u}_{k-1}-\bar{\mathbf{u}}_{k-1})+\boldsymbol{\Lambda}_{k-1}\mathbf{w}_{k-1}\right]^T\right\}$$

With negligible cross-covariance

$$\mathbf{P}_k = \boldsymbol{\Phi}_{k-1}E\left[(\mathbf{x}_{k-1}-\mathbf{m}_{k-1})(\mathbf{x}_{k-1}-\mathbf{m}_{k-1})^T\right]\boldsymbol{\Phi}_{k-1}{}^T + \boldsymbol{\Lambda}_{k-1}E\left(\mathbf{w}_{k-1}\mathbf{w}^T{}_{k-1}\right)\boldsymbol{\Lambda}_{k-1}{}^T$$
$$= \boldsymbol{\Phi}_{k-1}\mathbf{P}_{k-1}\boldsymbol{\Phi}_{k-1}{}^T + \boldsymbol{\Lambda}_{k-1}\mathbf{Q}'_{k-1}\boldsymbol{\Lambda}_{k-1}^T$$
$$\triangleq \boldsymbol{\Phi}_{k-1}\mathbf{P}_{k-1}\boldsymbol{\Phi}_{k-1}{}^T + \mathbf{Q}_{k-1}$$

$$\mathbf{Q}_{k-1} \triangleq \boldsymbol{\Lambda}_{k-1}\mathbf{Q}'_{k-1}\boldsymbol{\Lambda}_{k-1}^T$$

Conditional Probability Density Function of the State



$$\mathrm{pr}(\mathbf{x}_k) = \frac{1}{(2\pi)^{n/2}\left|\mathbf{P}_k\right|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}_k-\mathbf{m}_k)^T\mathbf{P}_k^{-1}(\mathbf{x}_k-\mathbf{m}_k)}$$

$$\mathbf{m}_k = \boldsymbol{\Phi}_{k-1}\mathbf{m}_{k-1} + \boldsymbol{\Gamma}_{k-1}\mathbf{u}_{k-1}$$
$$\mathbf{P}_k = \boldsymbol{\Phi}_{k-1}\mathbf{P}_{k-1}\boldsymbol{\Phi}_{k-1}{}^T + \mathbf{Q}_{k-1}$$

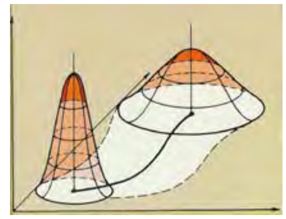
The density function is conditioned on the prior state

$$\mathrm{pr}(\mathbf{x}_k|\mathbf{x}_{k-1}) =$$
$$\frac{1}{(2\pi)^{n/2}\left|\boldsymbol{\Phi}_{k-1}\mathbf{P}_{k-1}\boldsymbol{\Phi}_{k-1}{}^T+\mathbf{Q}_{k-1}\right|^{1/2}} e^{-\frac{1}{2}\left[\mathbf{x}_k-(\boldsymbol{\Phi}_{k-1}\mathbf{m}_{k-1}+\boldsymbol{\Gamma}_{k-1}\mathbf{u}_{k-1})\right]^T\left(\boldsymbol{\Phi}_{k-1}\mathbf{P}_{k-1}\boldsymbol{\Phi}_{k-1}{}^T+\mathbf{Q}_{k-1}\right)^{-1}\left[\mathbf{x}_k-(\boldsymbol{\Phi}_{k-1}\mathbf{m}_{k-1}+\boldsymbol{\Gamma}_{k-1}\mathbf{u}_{k-1})\right]}$$

... and propagation is a Markov process

$$\mathrm{pr}(\mathbf{x}_k) = \mathrm{pr}(\mathbf{x}_k|\mathbf{x}_{k-1})\mathrm{pr}(\mathbf{x}_{k-1}) = \mathrm{pr}(\mathbf{x}_k|\mathbf{x}_{k-1}) \text{ if } \mathrm{pr}(\mathbf{x}_{k-1}) = 1$$

Gauss-Markov Sequence



$$\mathbf{m}_k = \mathbf{\Phi}_{k-1}\mathbf{m}_{k-1} + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1}$$
$$\mathbf{P}_k = \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}^T + \mathbf{Q}_{k-1}$$

- The mean and covariance are completely specified by the prior probability distribution
- The random sequence
  - has the transition function property
  - is a Gauss-Markov sequence

$$\mathrm{pr}(\mathbf{x}_k) = \mathrm{pr}(\mathbf{x}_k \mid \mathbf{x}_{k-1})\,\mathrm{pr}(\mathbf{x}_{k-1}) = \mathrm{pr}(\mathbf{x}_k \mid \mathbf{x}_{k-1})\,\mathrm{pr}(\mathbf{x}_{k-1} \mid \mathbf{x}_{k-2})\,\mathrm{pr}(\mathbf{x}_{k-2}) = \cdots$$
$$= \left[\prod_{i=1}^{k} \mathrm{pr}(\mathbf{x}_i \mid \mathbf{x}_{i-1})\right] \mathrm{pr}(\mathbf{x}_0)$$

Probability Mass and Density Functions

The random sequence

has the transition function property

is a Gauss-Markov sequence

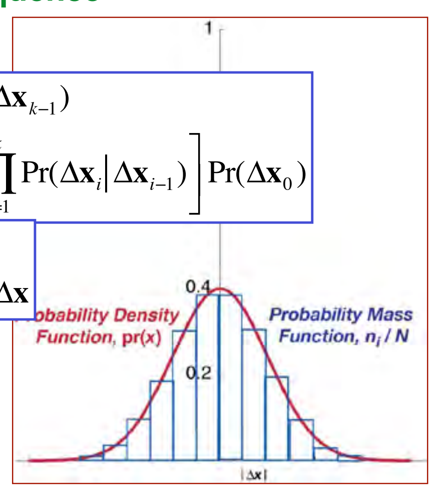
Probability Mass Function

$$\Pr(\Delta\mathbf{x}_k) = \Pr(\Delta\mathbf{x}_k \mid \Delta\mathbf{x}_{k-1})\,\Pr(\Delta\mathbf{x}_{k-1})$$
$$= \Pr(\Delta\mathbf{x}_k \mid \Delta\mathbf{x}_{k-1})\,\Pr(\Delta\mathbf{x}_{k-1} \mid \Delta\mathbf{x}_{k-2})\,\Pr(\Delta\mathbf{x}_{k-2}) = \left[\prod_{i=1}^{k} \Pr(\Delta\mathbf{x}_i \mid \Delta\mathbf{x}_{i-1})\right] \Pr(\Delta\mathbf{x}_0)$$

$$\Pr(\Delta\mathbf{x}_k) \approx \mathrm{pr}(\mathbf{x}_k)\Delta\mathbf{x}$$
$$\Pr(\Delta\mathbf{x}_k \mid \Delta\mathbf{x}_{k-1})\Pr(\Delta\mathbf{x}_{k-1}) \approx \mathrm{pr}(\Delta\mathbf{x}_k \mid \Delta\mathbf{x}_{k-1})\mathrm{pr}(\mathbf{x}_{k-1})\Delta\mathbf{x}$$

Probability Density Function

$$\lim_{\Delta\mathbf{x}\to 0} \Pr(\Delta\mathbf{x}_k) = \lim_{\Delta\mathbf{x}\to 0} \Pr(\Delta\mathbf{x}_k \mid \Delta\mathbf{x}_{k-1})\,\Pr(\Delta\mathbf{x}_{k-1})$$
$$\mathrm{pr}(\mathbf{x}_k) = \mathrm{pr}(\mathbf{x}_k \mid \mathbf{x}_{k-1})\,\mathrm{pr}(\mathbf{x}_{k-1})$$



Sampled-Data Representation of Continuous-Time Systems

Sampled-Data Representation of Continuous-Time Systems

Continuous-time LTV model with known coefficients

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t), \quad \mathbf{x}(t_o)\ given$$

$$\mathbf{x}(t) = \mathbf{x}(t_o) + \int_{t_o}^{t} \left[\mathbf{F}(\tau)\mathbf{x}(\tau) + \mathbf{G}(\tau)\mathbf{u}(\tau) + \mathbf{L}(\tau)\mathbf{w}(\tau)\right] d\tau$$

Incremental solution

$$\mathbf{x}(t_k) = \mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \left[\mathbf{F}\mathbf{x}(\tau) + \mathbf{G}\mathbf{u}(\tau) + \mathbf{L}\mathbf{w}(\tau)\right] d\tau$$

$$= \mathbf{\Phi}\left(t_k, t_{k-1}\right)\mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \mathbf{\Phi}\left(t_k, \tau\right)\left[\mathbf{G}(\tau)\mathbf{u}(\tau) + \mathbf{L}(\tau)\mathbf{w}(\tau)\right] d\tau$$

Descriptions of Random Variables

$$E(\mathbf{x}_0) = \mathbf{m}_o\ ; \quad E\left[(\mathbf{x}_0 - \mathbf{m}_o)(\mathbf{x}_0 - \mathbf{m}_o)^T\right] = \mathbf{P}_0$$

$$E\left[\mathbf{w}(t)\right] = \mathbf{0}$$

$$E\left[\mathbf{w}(t)\mathbf{w}^T(\tau)\right] = \mathbf{Q}'\delta(t-\tau)$$

$$E\left[\mathbf{u}(t)\right] = \mathbf{u}(t); \quad E\left\{\left[\mathbf{u}(t) - \bar{\mathbf{u}}(t)\right]\left[\mathbf{u}(t) - \bar{\mathbf{u}}(t)\right]^T\right\} = \mathbf{0}$$

Mean Value and Covariance Solutions

Mean value propagation from t_{k-1} to t_k

$$\mathbf{m}_k = \mathbf{\Phi}_k \mathbf{m}_{k-1} + \int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{G}(\tau)\mathbf{u}(\tau)\,d\tau$$

Covariance propagation

Double integration over time $(\tau \text{ and } \alpha)$

$$\begin{aligned}\mathbf{P}_k &= \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + E\left\{\left[\int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{L}(\tau)\mathbf{w}(\tau)\,d\tau\right]\left[\int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\alpha)\mathbf{L}(\alpha)\mathbf{w}(\alpha)\,d\alpha\right]^T\right\} \\ &= \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{L}(\tau)E\left[\mathbf{w}(\tau)\mathbf{w}^T(\alpha)\right]\mathbf{L}^T(\alpha)\mathbf{\Phi}^T(t_k,\alpha)\,d\tau\,d\alpha\end{aligned}$$

Covariance Propagation

$$\boxed{E\left[\mathbf{w}(\tau)\mathbf{w}^T(\alpha)\right] = \mathbf{Q'}_C\,\delta(\tau-\alpha)}$$

$$\begin{aligned}
\mathbf{P}_k &= \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{L}(\tau)\mathbf{Q'}_C\,\delta(\tau-\alpha)\mathbf{L}^T(\alpha)\mathbf{\Phi}^T(t_k,\alpha)\,d\tau\,d\alpha \\
&= \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + \int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{L}(\tau)\mathbf{Q'}_C\,\mathbf{L}^T(\tau)\mathbf{\Phi}^T(t_k,\tau)\,d\tau \\
&\triangleq \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + \mathbf{Q}_{k-1}
\end{aligned}$$

Relationship Between Discrete- and Continuous-Time Disturbance Models

$$\boxed{\mathbf{Q}_{k-1} = \int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{L}(\tau)\mathbf{Q'}_C\,\mathbf{L}^T(\tau)\mathbf{\Phi}^T(t_k,\tau)\,d\tau}$$

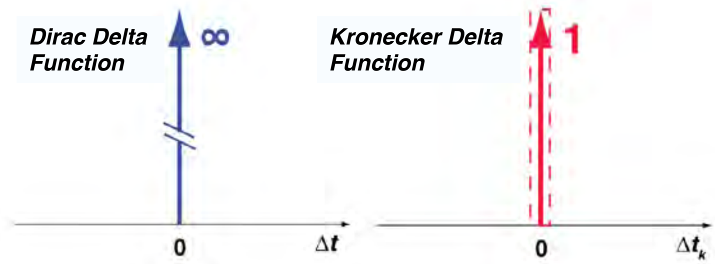
$\mathbf{Q}_{k-1}$: **Covariance matrix** $(n \times n)$

$\mathbf{Q'}_C$: **Spectral density matrix** $(s \times s)$

For small Δt

$$\boxed{\mathbf{Q}_{k-1} \approx \mathbf{L}(t_{k-1})\mathbf{Q'}_C(t_{k-1})\mathbf{L}^T(t_{k-1})\,\Delta t}$$

Autocovariance Functions for Continuous-and Discrete-Time White Noise



$$\boxed{\phi_{ww}(\Delta t) = \mathbf{Q}'_C\,\delta(\Delta t)}$$

$$\boxed{\phi_{ww}(\Delta t_k) = \mathbf{Q}_{k-1}\delta(\Delta t_k)}$$

$$\delta(\Delta t) = \begin{cases} \infty, & \Delta t = 0 \\ 0, & \Delta t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \lim_{\Delta t_1 \to 0} \int_{-\Delta t_1}^{\Delta t_1} \delta(\Delta t)\,d(\Delta t) = 1$$

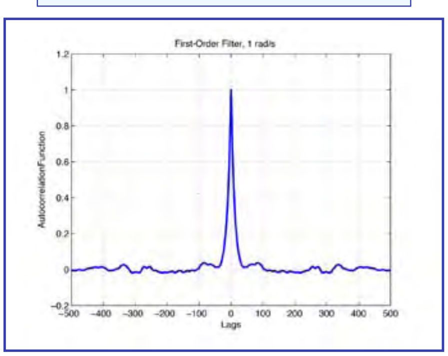
$$\delta_{jk} = \delta(\Delta t_k) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

Colored Noise, Variance, Spectral Density Matrices

- **Spreading of the autocovariance function is accompanied by**
  - **Finite variance** $\phi_{xx}(\tau)$
  - **Low-pass filtering of the power spectral density**

$$\boxed{\Phi_{xx}(\omega) = \int_{-\infty}^{\infty} \phi_{xx}(\tau)e^{-j\omega\tau}\,d\tau}$$

Autocovariance Function



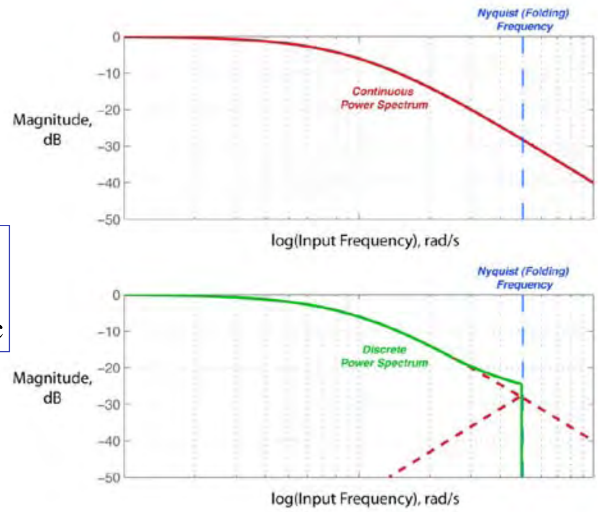
Power Spectral Density Function

Spectral Density Matrices

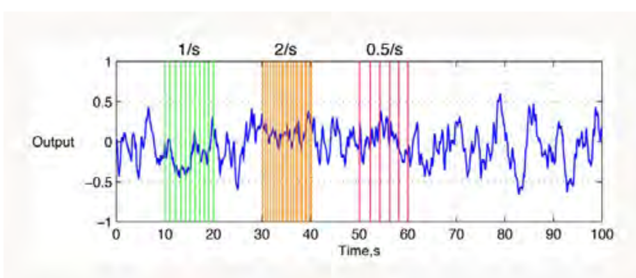
- Dynamic systems subject all white-noise inputs to low-pass filtering
- Nyquist (or folding) frequency

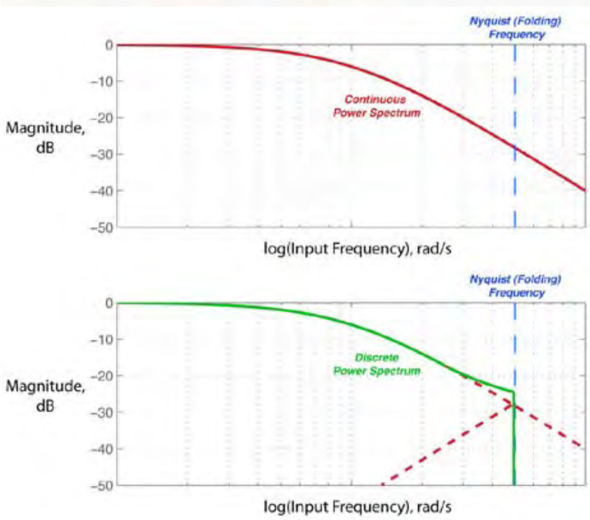
$$\omega_{Nyquist} = \frac{1}{2}\omega_{Sampling} = \frac{\pi}{T}$$
$$T = \text{ Sampling interval, sec}$$

- Signals above this frequency fold and corrupt lower-frequency sampled signals (aliasing)



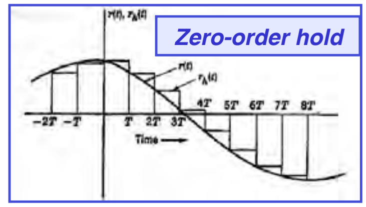
Caveat: Covariance and Spectra capture "stochastic equilibrium" effects but not transient effects



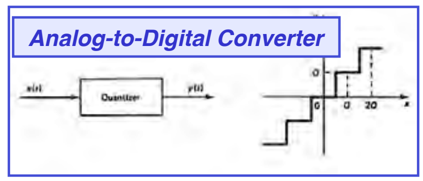


Frequency Folding (Aliasing)

Quantization, Zero-Order Hold, and Inter-sample Ripple



- Digital control subject to zero-order hold in D/A
- Sampled signal misses inter-sample transient response
- *Effective delay* of sampled signal



- Continuous signal sampled with finite precision in A/D

Next Time: Kalman Filter for Discrete-Time Systems

Supplemental Material

Colored Noise Disturbance

- **Disturbance may not be "white" noise**
  - **Power may vary with frequency**
- **Mean**

$$\boxed{E\left(\mathbf{w}_k\right) = \mathbf{0}}$$

$$\boxed{\dim\left(\mathbf{w}_k\right) = s \times 1}$$

- **Covariance**

$$\boxed{E\left(\mathbf{w}_k \mathbf{w}_k^T\right) = \mathbf{W}_k}$$

- **Correlation with adjacent signal**

$$\boxed{E\left(\mathbf{w}_k \mathbf{w}_{k-1}^T\right) = \mathbf{V}_k}$$

- **Linear model for disturbance propagation**
  - **Driven by white noise sequence**

$$\boxed{\mathbf{w}_k = \mathbf{A}_{k-1}\mathbf{w}_{k-1} + \boldsymbol{\eta}_{k-1}}$$

$$\boxed{\begin{aligned} &E\left(\boldsymbol{\eta}_k\right) = \mathbf{0} \\ &E\left(\boldsymbol{\eta}_j \boldsymbol{\eta}_k^T\right) = \mathbf{Q}_{\eta_k}\delta_{jk} \end{aligned}}$$

Calculation of Disturbance Process Parameters

Autocovariance of scalar Markov process

$$\phi_{xx}(1) = E(x_i x_{i+1}) = bE(x_i^2) = b\sigma_x^2$$
$$\phi_{xx}(k) = b^{|k|}\sigma_x^2 = b^{|k|}\sigma_u^2$$

Autocovariance of vector Markov process

$$\begin{aligned} E(\mathbf{w}_k\mathbf{w}_{k-1}^T) &= E\left[(\mathbf{A}_{k-1}\mathbf{w}_{k-1} + \boldsymbol{\eta}_{k-1})\mathbf{w}_{k-1}^T\right] \\ &= \mathbf{A}_{k-1}E(\mathbf{w}_{k-1}\mathbf{w}_{k-1}^T) \\ &= \mathbf{A}_{k-1}\mathbf{W}_{k-1} \\ &\triangleq \mathbf{V}_k \end{aligned}$$

Therefore, the state transition matrix of the noise model is

$$\mathbf{A}_{k-1} = \mathbf{V}_k\mathbf{W}_{k-1}^{-1}$$

Propagating the Disturbance

$$\mathbf{W}_k = \mathbf{A}_{k-1}\mathbf{W}_{k-1}\mathbf{A}_{k-1}^T + \mathbf{Q}_{\boldsymbol{\eta}_k}$$

- System state vector is augmented to include the disturbance

$$\mathbf{x}'_k \triangleq \begin{bmatrix} \mathbf{m}_k \\ \mathbf{w}_k \end{bmatrix}, \quad \dim(\mathbf{x}'_k) = (n+s)\times 1$$

- Augmented equation for the mean

$$\begin{bmatrix} \mathbf{m}_k \\ \mathbf{w}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{k-1} & \boldsymbol{\Lambda}_{k-1} \\ \mathbf{0} & \mathbf{A}_{k-1} \end{bmatrix} \begin{bmatrix} \mathbf{m}_{k-1} \\ \mathbf{w}_{k-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Gamma}_{k-1} \\ \mathbf{0} \end{bmatrix}\mathbf{u}_{k-1} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_s \end{bmatrix}\boldsymbol{\eta}_{k-1}$$

- Augmented covariance equation

$$\begin{bmatrix} \mathbf{P}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{k-1} & \boldsymbol{\Lambda}_{k-1} \\ \mathbf{0} & \mathbf{A}_{k-1} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{k-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_{k-1} & \boldsymbol{\Lambda}_{k-1} \\ \mathbf{0} & \mathbf{A}_{k-1} \end{bmatrix}^T + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_s \end{bmatrix}\mathbf{Q}_{\boldsymbol{\eta}_{k-1}}\begin{bmatrix} \mathbf{0} & \mathbf{I}_s \end{bmatrix}$$