Obtaining Brachistochrone Curve With Metaheuristic Algorithms

Kivanc Guckiran
Dept. of Electronics and Comm. Eng.
Yildiz Technical University
Istanbul, Turkey
kivancguckiran@gmail.com

Tulay Yildirim
Dept. of Electronics and Comm. Eng.
Yildiz Technical University
Istanbul, Turkey
tulay.yildirim@ieee.org

***Abstract*—Brachistochrone means "shortest time" in Ancient Greek. This curve defines the path a bead travels only with gravitational force between point A and a lower point B not directly under, in least amount of time. In this article, with the help of Snell's Law and the Fermat's principle, different metaheuristic algorithms are used to obtain Brachistochrone curve and compared with each other. Fermat's Principle states that wave chooses the shortest travel path when passing between different mediums. Snell's Law indicates the refraction angle of the wave on the surface of the media correlates to the intrinsic quality of the medium. As for metaheuristic algorithms, Genetic Algorithm (GA), Harmony Search (HS), Artificial Bee Colony (ABC) and Differential Evolution (DE) are used.**

***Index Terms*—Metaheuristics, Genetic Algorithm, Harmony Search, Artificial Bee Colony, Differential Evolution, Snell's Law, Fermat's Principle, Brachistochrone Curve**

I. Introduction

Refraction of light is observed in everyday life. This phenomenon has been a research topic for scientists from early era of the civilization. Even though reflection of light as a term is encountered within the studies of Euclid and followed by Ptolemy, first manuscripts on refraction of light belong to Ibn Al-Haytham.

While studying on refraction of light, Ibn Al-Haytham connected the dots between the curvature of the glass and the magnifying process. He also noticed that, refraction phenomena occurs on the surface of the medium, not inside. Although, Ibn Al-Haytham has many works on the refraction of light, there is no formulation on the light's refraction angle, reached today [1].

Following Ibn Al-Haytham, Pierre de Fermat observed that the waves passing between mediums choose the shortest path regarding intrinsic quality of the medium.

$$T = \int_a^b n ds$$

- T : Total time
- a : Starting point
- b : Destination point
- n : Refraction index
- ds : Differential position change

Snell's law, which defines the refraction angle of the waves passing between different mediums, is officially formulated by Willebrord Sinellius. According to Snell's law, the ratio between the sinuses of the refraction angle between mediums is equal to ratio between the refraction index or the corresponding wave's propagation velocities inside the mediums. Figure 1 shows refraction phenomenon between different mediums.

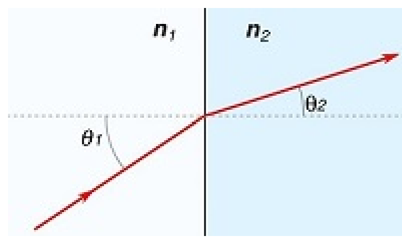

Fig. 1. Refraction

$$\frac{n_1}{n_2} = \frac{\sin(\theta_1)}{\sin(\theta_2)}$$

- n_1 : First medium's refraction index
- n_2 : Second medium's refraction index
- θ_1 : First medium's refraction angle
- θ_2 : Second medium's refraction angle

Despite widely belief, there are studies claiming that the formulation is first presented by Ibn Sahl [2].

In June 1696, Johann Bernoulli posed a question to the readers and participants of the Acta Eruditorum journal: "Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time?". This question challenged the scientists of that era, like Newton, Leibniz.

The answer to the problem is the Brachistochrone curve. Figure 2 depicts this curve, which is the path travelled by a point on a circle which revolves on a plane. This is called a Cycloid curve. Additionally, this curve is also called Tautochrone curve, which means "same time" in Ancient Greek.

978-1-5386-7786-5/18/$31.00 2018 IEEE

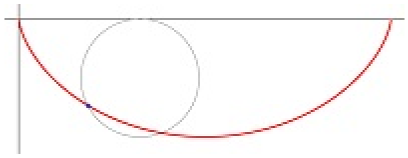

Fig. 2. Brachistochrone curve

Erlichson stated in his study that, Brachistochrone curve can be obtained by using Fermat's principle [3]. Using Erlichson's proof, the curve is obtained by using a wave propagating between mediums which have different intrinsic qualities. In this study, the points which the wave follows are optimized by multiple metaheuristic algorithms, to acquire the Brachistochrone curve,

II. Problem Definition

The problem is to obtain Brachistochrone curve by creating mediums in which the wave travels with different velocities. These mediums are placed on top of another, creating a map consists of patterns with an increasing velocity from top to bottom. Wave starts traveling from top left of the map trying to reach bottom right.

Fig. 3. Problem map

Figure 3 shows that, there are 12 different mediums with different propagation velocities. Speeds are chosen using the gravitational force. While mimicking the gravitational force, to obtain a high amount of curvature, speeds are increased logarithmically.

Since starting and end points are constant, there are 11 points left to optimize. Every point is on the edge of medium change, the refraction point. The steps are taken from one point to the nearest medium's point.

Even though shortest path is diagonal, the path created is expected to be like Brachistochrone curve.

III. Metaheuristic Algorithms

A. Genetic Algorithm (GA)

Genetic algorithm is inspired from natural selection. It is the most used and cited metaheuristic algorithm.

Algorithm has three mechanism; selection, crossover and mutation. Parameters used in this study for Genetic algorithm are:

- Population size : 100
- Gene length : 11
- Breeder count : 30
- Mutation chance : 5%

B. Harmony Search (HS)

In 2001, Zong Woo Geem, Joong Hoon Kim, and G. V. Loganathan theorized Harmony Search, which is inspired from jazz musician's improvisation techniques [4]. It is also argued that, Harmony Search is just a subset of the Evolutionary Strategy [5]. Parameters used are defined below:

- Harmony memory size : 100
- Melody size : 11
- Harmony memory consideration rate : 99%
- Pitch adjust rate : 5%
- Bandwidth : 0.005

C. Differential Evolution (DE)

Differential Evolution algorithm, theorized in 1997, is based on optimizing each solution iteratively. For each solution, two other random solutions are picked and with mutation and crossover operators, the solution is improved. Parameters used for this algorithm are:

- Solution candidate size : 100
- Differential constant (F) : 0.5
- Crossover constant (CR) : 0.5

D. Artificial Bee Colony (ABC)

Artificial Bee Colony is inspired from work distribution of the bee colony. This metaheuristic algorithm, which is theorized by Derviş Karaboğa, has found wide usage throughout academia [7]. There are three phases for the algorithm:

- Worker bees work: Food sources are improved.
- Onlooker bees work: With the information from worker bees, best food sources are further improved.
- Scout bees work: Food sources which cannot be improved are replaced.

Parameters used are:

- Food source count : 100
- Food source size : 11
- Scout replace limit : 50
- Worker bee count : 20
- Onlooker bee count : 20

IV. Simulation

Simulation process starts with defining a fitness function, which is used to calculate errors of the solution candidates that are produced from metaheuristic algorithms. This function computes time by calculating the distance between each point in the candidate and dividing the distance by the speed corresponding to that medium. These time values are added to calculate the total time. Total time traveled is the one which algorithms need to optimize.

Similar to above method, after the optimization process calculates the curve, using each line angle and propagation velocities of the mediums, the errors are calculated with Snell's law. Error values are used to compare between metaheuristic algorithms.

For these comparisons, each algorithm optimized the solutions in 10 epochs with different iteration counts. Iteration counts are 500 and 5000. After each epoch, an error is calculated. Finally, average errors and average execution times for the iterations are calculated and compared.

A. Genetic Algorithm

1) 500 iteration: Best iteration was epoch 8 for Genetic Algorithm. Each and average error values are shown in Table I.

TABLE I
GA ERRORS

Epoch	Error
1	1.951
2	2.032
3	2.287
4	1.278
5	1.634
6	2.012
7	1.818
8	**1.274**
9	2.494
10	1.398
Average error	1.818
Average exec. time	00:05.100

Figure 4 shows the best path found with Genetic algorithm.

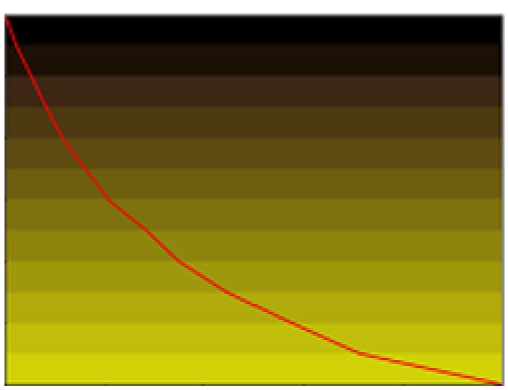

Fig. 4. GA Best Path

Figure 5 shows the error values over iteration for epoch number 8.

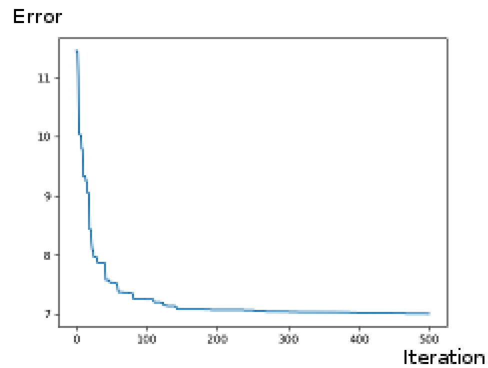

Fig. 5. GA error over iteration

2) 5000 iteration: After 10 times of 5000 iteration, the average error is **0.246**.

B. Harmony Search

1) 500 iteration: Best iteration was epoch 1 for Harmony Search. Each and average error values are shown in Table II.

TABLE II
HS ERRORS

Epoch	Error
1	**0.461**
2	1.935
3	0.504
4	0.613
5	1.865
6	2.202
7	1.650
8	4.708
9	2.017
10	2.207
Average error	1.816
Average exec. time	00:04.875

Figure 6 shows the best path found with Harmony Search algorithm.

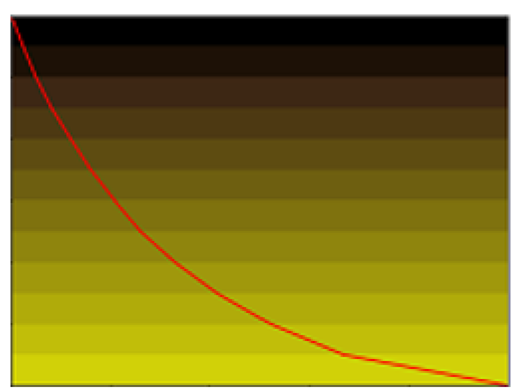

Fig. 6. HS Best Path

Figure 7 shows the error values over iteration for epoch number 1.

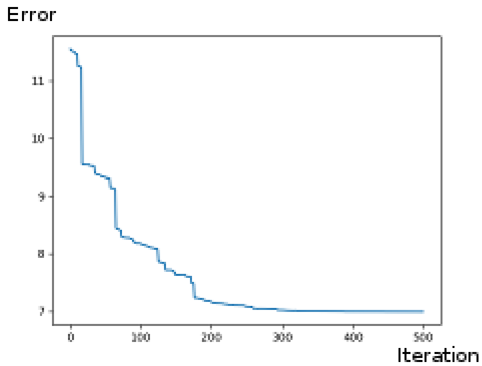

Fig. 7. HS error over iteration

2) 5000 iteration: After 10 times of 5000 iteration, the average error is **0.289**.

C. Artificial Bee Colony

1) 500 iteration: Best iteration was epoch 1 for Artificial Bee Colony. Each and average error values are shown in Table III.

TABLE III
ABC ERRORS

Epoch	Error
1	**1.451**
2	1.800
3	2.522
4	5.590
5	1.739
6	3.554
7	1.453
8	3.458
9	1.651
10	1.812
Average error	2.503
Average exec. time	00:04.270

Figure 8 shows the best path found with Artificial Bee Colony algorithm.

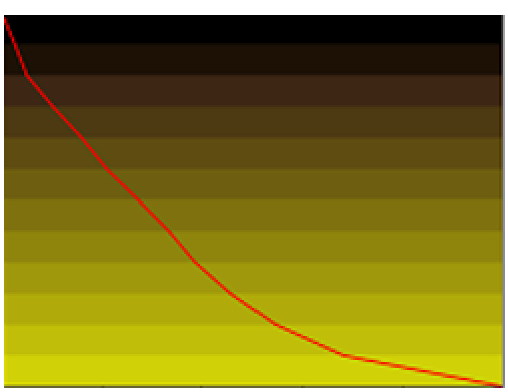

Fig. 8. ABC best path

Figure 9 shows the error values over iteration for epoch number 1.

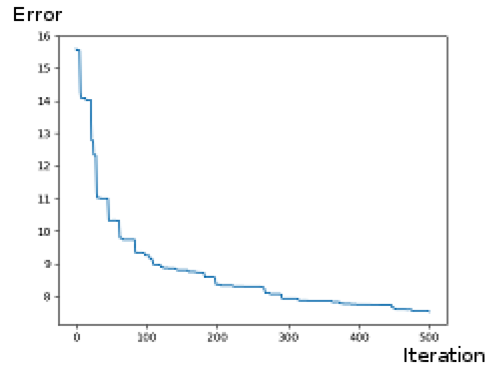

Fig. 9. ABC error over iteration

2) 5000 iteration: After 10 times of 5000 iteration, the average error is **0.152**.

D. Differential Evolution

1) 500 iteration: Best iteration was epoch 2 for Differential Evolution. Each and average error values are shown in Table IV.

TABLE IV
DE ERORRS

Epoch	Error
1	0.186
2	**0.136**
3	0.182
4	0.161
5	0.161
6	0.208
7	0.178
8	0.159
9	0.344
10	0.154
Average error	0.187
Average exec. time	00:16.318

Figure 10 shows the best path found with Differential Evolution algorithm.

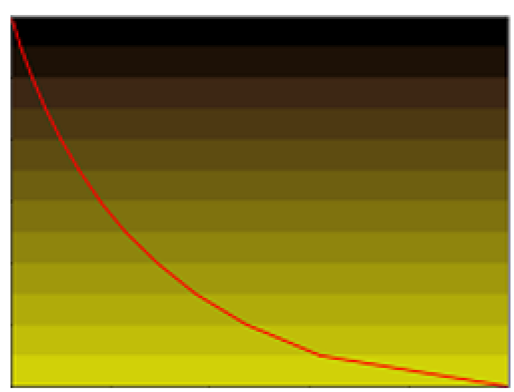

Fig. 10. DE best path

Figure 11 shows the error values over iteration for epoch number 2.

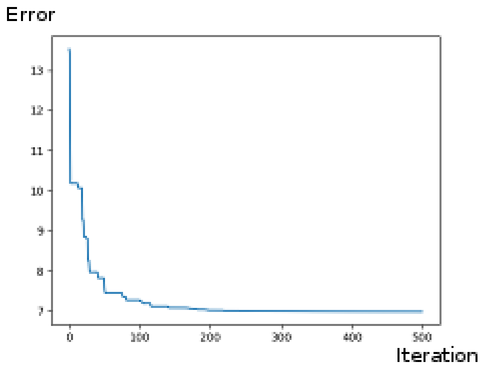

Fig. 11. DE error over iteration

2) 5000 iteration: After 10 times of 5000 iteration, the average error is $4 * 10^{-7}$. Table V shows each metaheuristic algorithm's error values for comparison.

TABLE V
COMPARISON

Algorithm	**Avg. Err. It. 500**	**Avg. Err. It. 5000**
Genetic Algorithm	1.818	0.246
Harmony Search	1.816	0.289
Artificial Bee Colony	2.503	0.152
Differential Evolution	**0.187**	$4 * 10^{-7}$

Simulations are done on a i7-4710HQ CPU and a 8GB RAM machine. Programming language was Python.

V. CONCLUSION

In this study, Brachistochrone curve is obtained with help of Snell's law and Fermat's principle, using metaheuristic optimization algorithms. Results are compared with each other after iterations, using the error of the curve and time spent in the simulation.

Best optimization algorithm for the Brachistochrone problem is Differential Evolution for both iteration counts. But, compared to other algorithms, Differential Algorithm is taking substantially more time to optimize. This disadvantage may be eliminated with the multi-threading capability of the algorithm [9].

Another observation is, for 500 iterations, Artificial Bee Colony algorithm has the highest error rate. But, after 5000 iterations, it surpasses Harmony Search and Genetic Algorithm.

If error over iteration graphs are carefully examined, it can be seen that the errors are declining abruptly at the start, then slowly decays over iterations. To get a better error over iteration graph, it might be better to tighten the solution range of each point after each epoch. Szarkowicz, in his study [8], converged to Brachistochrone curve using this tightening strategy to acquire a better error graph.

Brachistochrone curves can be used in winter sports, especially for designing acceleration slopes. Same goes for the amusement or aqua parks, to design a roller coaster or a water slide.

REFERENCES

[1] Zghal, Mourad, et al. "The first steps for learning optics: Ibn Sahl's, Al-Haytham's and Young's works on refraction as typical examples." Education and Training in Optics and Photonics. Optical Society of America, 2007, pp. 56.
[2] Kwan, A., Dudley, J., & Lantz, E. (2002). Who really discovered Snell's law?. Physics World, 15(4), 64, p. 2.
[3] Erlichson, H. (1999). Johann Bernoulli's brachistochrone solution using Fermat's principle of least time. European journal of physics, 20(5), 299, pp. 301305
[4] Geem, Z. W., Kim, J. H., & Loganathan, G. V. (2001). A new heuristic optimization algorithm: harmony search. simulation, 76(2), 60-68.
[5] Weyland, D. (2015). A critical analysis of the harmony search algorithmHow not to solve sudoku. Operations Research Perspectives, 2, 97-105.
[6] Storn, R., & Price, K. (1997). Differential evolutiona simple and efficient heuristic for global optimization over continuous spaces. Journal of global optimization, 11(4), 341-359.
[7] Karaboga, D., & Basturk, B. (2007). A powerful and efficient algorithm for numerical function optimization: artificial bee colony (ABC) algorithm. Journal of global optimization, 39(3), 459-471.
[8] Szarkowicz, D. S. (1995). Investigating brachistochrone trajectories with a multistage realparameter genetic algorithm. International Journal of Mathematical Education in Science and Technology, 26(5), 709-720.
[9] D. Izzo, M. Rucinski and C. Ampatzis, (2009) Parallel global optimisation meta-heuristics using an asynchronous island-model. 2009 IEEE Congress on Evolutionary Computation, Trondheim, pp. 2301-2308.