On the Role of an Evolutionary Solution for the Brachistochrone-Problem

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***Abstract*— An evolutionary solution for the approximation of the Brachistochrone problem is presented by a specific configuration and operator setup of an evolutionary strategy (ES). To evaluate and compare the quality of the solution, an approximation of the theoretical cycloid-solution is determined by an approximation of n points with similar distance. The outperforming quality of the evolutionary determined shape of the approximated curve between a given starting- and end-point is verified by the time an idealized particle needs. All empirical results are carried out in a Monte Carlo simulation study. The dependency of the quality enhancement of the evolutionary solution on the number of approximating points is analyzed. Further the role of the evolutionary approach is discussed to encourage the usage of evolutionary computation for linear approximating polygons of well solved analytical problems.**

I. Introduction

In recent years many different new optimization approaches have been suggested. An appropriate optimization method which performs superior for the continuum of all problems cannot be provided. This can be followed as a strong implication of the no-free-lunch-theorem (NFL) [20]. Therefore, the optimizer's role for certain classes of problems is the one of applying approaches and collecting optimization quality [3].

Besides this well known fact, the continuum of problems taken into account for evolutionary optimization, are the one known as np-complete e.g. with a computational complexity growing with exponential or polynomial order. A strong indication of an np-complete problem is the order of the space of solutions. If the space of solutions is growing already for small configuration parameters in polynomial or exponential order, the brute force approaches become unfeasible. From the theoretical point of view, solutions are feasible, if an analytical solution does exist independent for all parameter settings.

This paper aims at proving that even if an analytical solution exists, any linear approximation in the finite space of approximations gives potentials for an evolutionary solution which outperforms the approximation of the analytical solution.

In the case of this paper an important problem in the history of optimization and mathematics has been chosen. This problem has been used to introduce the origin of the calculus of variations [18] formulated by Bernoulli [5] and finalized as a mathematical method by Euler-Lagrange [8] based on the first solution. Our proof is based on thoroughly, statistical justified empirical simulation results. The consequences of the results are discussed in comparison to the analytical solution. The paper is organized as follows. Section II introduces the historical definition of the Brachistochrone and the curve of a linear approximating polygon is defined as a approximated Brachistochrone as well. The type of an evolutionary solution, the formal description of the generation loop, the development of the fitness function and an example of the outperforming quality is introduced in section III. The analytical shape of the cycloid approximated by an infinite and a finite number of interpolation points and the hypothesis for the uniqueness of an evolutionary Brachistochrone for any Brachistochrone is part of section IV. In section V the results of the proposed approach are presented and compared.

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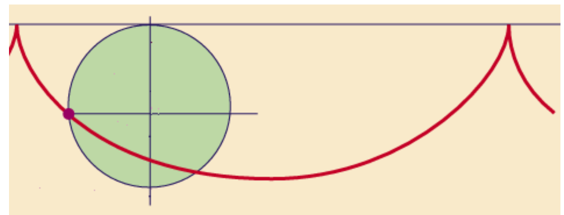

Fig. 1. Cycloid on an imagined wall by a winding wheel on the ceiling.

II. Models of Brachistochrone

As far as known from history [19], [4], the problem has been firstly formulated to address the scientific community as follows:

Definition Original Brachistochrone Statement:
Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time ?
Definition End

The well known solution is the curve of a cycloid, named by Galileo [9], [19], which has been studied before in a different mathematical context. Starting at a point A marked at the ceiling of an imagined wall matching the point on an outer perimeter of a wheel with an appropriate outer radius d, the curve of a cycloid from A to B in a vertical plane spanned by A and B, is given by winding the wheel on the ceiling towards point B. The curve traced out by the marked point on the wheel matches point B before the curve reaches the ceiling again (see figure 1).

Hereby, the problem is reconsidered in the sense of an approximated solution for practical application, called

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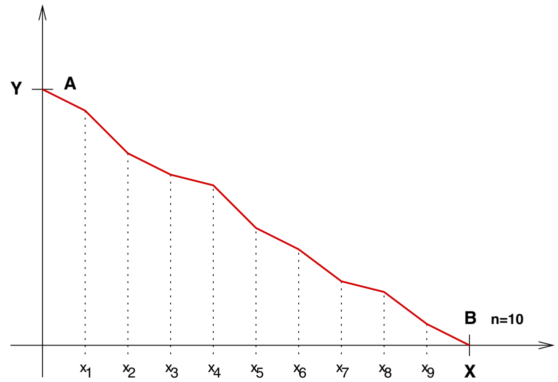

Fig. 2. An Approximated Brachistochrone (AB)

Approximated Brachistochrone (AB) in the following.

Definition Approximated Brachistochrone (AB).
The horizontal distance between point A and B is defined as the horizontal extent $X \in \mathbb{R}^+$ of the Brachistochrone (see def. of the Original Brachistochrone) and is divided in n equal distances:

$$x_0 = 0, x_1 = \frac{x}{n}, x_2 = \frac{2*x}{n}, \ldots, x_n = \frac{n*x}{n} = X. \quad (1)$$

The Approximated Brachistochrone (AB) is defined for any given problem of the Original Brachistochrone and a number $n \in \mathbb{N}, n \geq 1$ corresponding to $(n-1)$ approximating points, with an according vertical height of y_i at each horizontal point x_i, $i \in \mathbb{N}$. Sought after are the vertical components y_i of the approximating points $P_i(x_i, y_i)$ at x_i,

$$\{y_i \,|\, i = 1, \ldots, (n-1)\}. \quad (2)$$

The vertical position $y_0 = Y \in \mathbb{R}^+$ and y_n of the points at x_0 and x_n are apriori given by the Original Brachistochrone problem def. as the height of the starting point A (A_{vert}) and the height of the endpoint B (zero) as follows:

$$y_0 := A_{vert}, \; y_n := 0. \quad (3)$$

The definition of the AB is finalized by the approximating curve, consisting of straight lines connecting all direct sequences of points:

$$\{(x_i, y_i), (x_{i+1}, y_{i+1}) \,|\, \forall i = 1, \ldots, (n-1)\}. \quad (4)$$

Definition End

An example of an AB is illustrated in figure 2, but it has to be remarked, that the height of each point $y_i \in \mathbb{R}$ is not limited to $\mathbb{R}^+$. In the following section an evolutionary approach to AB is developed. In section V an analysis of the solutions for the AB is provided. At the end of this section, Bernoulli's introduction to the hereby reconsidered problem is worthy citing: *"I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise"* [4].

III. Evolutionary Solution

The evolutionary approach to solve the AB is described in most reproductive way by the coding scheme, a formal description of the generation-loop and the fitness function. To gain a most intuitive coding, each individual (or element) of the population is coded by a real valued, $(n-1)$-dimensional vector $\vec{y}$, which is a coding scheme of the evolutionary strategy (ES) [17]. Each component y_i, $i = 1, \ldots, (n-1)$ is the corresponding height of the AB (refer to the def. of the Approximated Brachistochrone, III). The dimension $(n-1)$ is given by the number of approximation points of any given AB. Accordingly, each individual is coding a different or the same curve given by approximating points. As a generation loop the well known evolutionary strategy [17] is selected. Therefore, a reproductive generation loop is defined by the number of individuals in the population μ and the number of individuals generated during the generation loop λ. The selection pressure is therefore given by $\frac{\mu}{\lambda}$. The formal definition of the generation loop is as follows.

Definition Formal description of the generation loop.

$$(\mu \,/ \rho \,, \, \lambda). \quad (5)$$

During the generation loop from generation t_g to $(t_g + 1)$, for the purpose of a λ-times multi-recombination, ρ individuals are selected randomly and separately for each single multi-recombination, in the predecessor population (generation t_g). A successor population of μ individuals at generation $(t_g + 1)$ is generated based on the applied evolutionary operators on the μ-individuals of the population at generation t_g. According to the defined description of the generation loop, each of the λ-times applied multi-recombinations is based on ρ randomly selected predecessors. Each multi-recombination generates one successor, where each component y_i^{suc}, $i = 1, \ldots, (n-1)$, of the successor individual $\vec{y}^{suc}$ is the average $\frac{1}{\rho}\sum_{j=1}^{\rho} \vec{y}^j$, with the randomly chosen individuals for a multi-recombination $\vec{y}^j$, $j \in [1, \ldots, \rho]$. All the successors of the λ-times multi-recombinations are mutated and under all those λ mutated individuals the μ best individuals ($\lambda > \mu$) are selected to achieve the successor population $(t_g + 1)$.
Definition End

This thoroughly formal description, first introduced in [17] is used to achieve an exact and intuitive reproducibility. However, the exact generation loop solution is exchangeable by any other evolutionary generation loop, like genetic algorithm (GA) [10], genetic programming [12], evolutionary programming [12] or nature inspired optimization techniques [3]. The necessary output determining fact is the cost- or fitness minimization-function (fitness). It will be developed as follows.

Definition Development of an AB related Fitness.
For any given AB (refer to the def. of the Approximated Brachistochrone, section II), the fitness is defined by the time a point needs to traverse all vertex positions

$$P_i(x_i, y_i),\ i = 0, \ldots, n, \tag{6}$$

from the starting point $A = P_0(x_0, y_0) = P_0(0, Y)$ to the endpoint $B = P_n(x_n, y_n) = P_n(X, 0)$ (see figure 2 for horizontal X and vertical Y distances). All points in direct order P_i, P_{i+1}, $i = 0, \ldots, (n-1)$, are traversed by straight lines. The traversed distance $s_i, i = 0, \ldots, (n-1)$, between two points P_i, P_{i+1}, $i = 0, \ldots, (n-1)$, is determined as the geometrical distance:

$$s_i = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}. \tag{7}$$

The velocity of the moving point at the starting position $A = P_0(0, Y)$ is zero, and is acted on by the gravity (summarized by a constant acceleration factor of gravity g) as follows:

$$a_i = g\frac{y_i - y_{i+1}}{s_i}, \tag{8}$$

By an assumed uniformly accelerated movement of the moving point the following holds for a straight distance s_i:

$$s_i = \frac{a_i}{2}t_i^2 + v_i * t_i, \tag{9}$$

A solution for the time t_i needed to traverse one of the straight lines connecting two points in direct order P_i, P_{i+1}, $i = 0, \ldots, (n-1)$, is obtained as follows: The velocity v_i at each position P_i, P_{i+1}, $i = 0, \ldots, (n-1)$, is the sum of the accelerations at each edge:

$$v_{i+1} = v_i + a_i * t_i. \tag{10}$$

From equ. (9) and (10) we obtain

$$s_i = \frac{v_i + v_{i+1}}{2} * t_i, \tag{11}$$

thus,

$$t_i = \frac{2 * s_i}{v_i + v_{i+1}}. \tag{12}$$

Moreover, by equivalence of kinetic and potential energy, we have

$$\frac{1}{2}v_i^2 = g * (y_0 - y_i), \tag{13}$$

that is,

$$v_i = \sqrt{2g} * \sqrt{y_0 - y_i}. \tag{14}$$

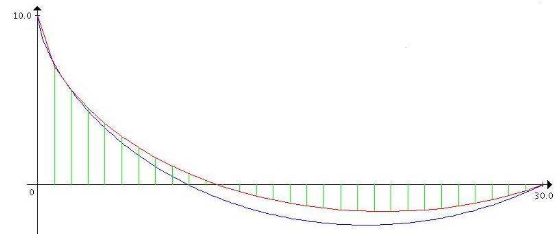

Fig. 3. Evolutionary solution in comparison to the cycloid solution

Combining equ. (7), (12) and (14), we get for the time t_i :

$$t_i = \frac{2 * \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}}{\sqrt{2g} * (\sqrt{y_0 - y_i} + \sqrt{y_0 - y_{i+1}})}. \tag{15}$$

The fitness function of any AB, that is, the sum of the single times needed to traverse each edge between two points P_i, P_{i+1}, $i = 0, \ldots, (n-1)$, of direct order in AB, is therefore given by the following close form formula:

$$F(\vec{y}) = \sqrt{\frac{2}{g}} \sum_{i=0}^{n-1} \frac{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}}{\sqrt{y_0 - y_i} + \sqrt{y_0 - y_{i+1}}}. \tag{16}$$

Definition End

We point out that F is defined if and only if $y_i \leq y_0$ and either $y_i < y_0$ or $y_{i+1} < y_0$ for all $i \in [1, (n-1)]$. This corresponds to our assumption that the kinetic energy (13) at level y_0 is zero, hence $P_i(x_i, y_i)$ can only be reached if $y_i < y_0$ and cannot be reached if $y_i = y_{i-1}$.

In figure 3 an evolutionary determined AB is illustrated in comparison to the analytical cycloid solution. The time needed for the evolutionary curve is $t = 3.23446s$ and that for the cycloid is $3.22074s$ (determined according to def. AB related fitness, equ. 16) and a horizontal distance of $X = 30$ and 30 equidistant approximating points P_i, $i = 1, ..., 30$. The parameter of the AB are a height of $Y = 10$ and a horizontal distance of $X = 30$ and 30 equidistant approximating points P_i, $i = 0, \ldots, (n-1)$. The solution illustrated has been obtained after 10000 generations, with $\rho = 5$ and $\lambda = 50$ (refer to def. generation loop, especially def. 5). An appropriate comparison of the evolutionary AB solution is obtained by an AB solution, where the height y_i of the approximating points P_i, P_{i+1}, $i = 0, \ldots, (n-1)$, is taken from the optimum solution of the cycloid. This solution will be referred as Approximated Cycloid (AC) in the following. The comparison of an evolutionary determined AB to an AC is illustrated in figure 4. The time necessary for the AC solution is $3.34054s$ and the time for the evolutionary obtained AB is $3.33159s$. The euclidean distance between the approximating points of the AC solution and the evolutionary AB is 2.45656. It follows, that the evolutionary solution is faster than the AC-solution. To verify these results, the exact cycloid solution is reconsidered and is used to rank the quality of the evolutionary AB in the next section. It is worthy to emphasize, that in the case of an AB, an approximated solution based on the height of the approximation

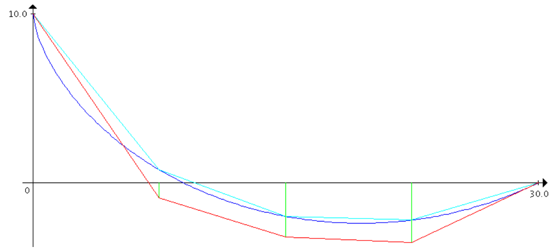

Fig. 4. Evolutionary AB-solution in comparison to the AC-solution

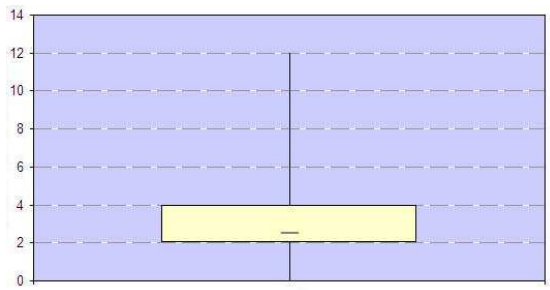

Fig. 5. Necessary generations to determine an AB-solution which outperforms an AC-solution for 3 approximating points

points taken from the cycloid solution is outperformed by the evolutionary AB solution.

IV. AB-Solution Synthesis

One of the earliest contributions to the Brachistochrone problem is known from Galileo [9]. Other famous contributors, like Newton [15] and Leibniz [13] published solutions in response to Bernoulli's famous problem formulation (refer to def. of the Original Brachistochrone (section II), [19] and the citation at the end of section 1). In terms of the so far defined AB (refer to def. Approximated Brachistochrone (AB), section II), Galileo concluded that all AB based on one approximation point (besides the start and the end point) that is below the direct connection between A and B is faster than the straight solution. Moreover, he assumed that an optimum solution for the one point approximation is given by all points following the curve of a perimeter of a (quarter) circle, including A and B. Beside the correctness of the first conclusion, the latter one is close to the final solution but has been corrected by the cycloid solution some centuries later [4].

Definition Cycloid-Solution (CS).
The curve of a cycloid as the solution of the Brachistochrone (def. Original Brachistochrone, section II) is described by a moving point on the perimeter of a wheel with radius r as follows: The horizontal coordinate of a moving point on the perimeter of a wheel is described as the dependency $x(\omega)$ of a radian measure of the wheel's angle position ω relative to the starting angle ($\omega = 0$, see figure 2). The vertical coordinate of the moving point on the perimeter of the wheel is described as the dependency $y(\omega)$ and the maximum height Y for the angle position $\omega = 0$.

$$\begin{pmatrix} x(\omega) \\ y(\omega) \end{pmatrix} = \begin{pmatrix} r \cdot (\omega - \sin(\omega)) \\ Y - r \cdot (1 - \cos(\omega)) \end{pmatrix} \tag{17}$$

For any given Brachistochrone, the maximum horizontal movement is obtained by a necessary rotation of the wheel by an angle $\omega_B < 2\pi$. The intersection of the moving point B and position of $x(\omega_B) = X$ is given by the horizontal distance of the points A and B (see def. of the Original Brachistochrone (section II) and figure 2 for distances X and Y). Moreover, the vertical position of the cycloid is $y(\omega_B) = 0$. Therefore, an equation to determine the angle ω_B is obtained as follows. From the horizontal position of the moving point (see equ. 17) at position B it can be followed for r:

$$r = \frac{X}{\omega_B - \sin(\omega_B)}. \tag{18}$$

Based on a height of $y(\omega_B) = 0$ the radian angle of the wheel can be obtained by the numerical solution (for instance by maple or mathematica) of the following equation (X and Y are assumed as given by the def. of the Original Brachistochrone (section II)).

$$\frac{Y}{X} = \frac{1 - \cos(\omega_B)}{\omega_B - \sin(\omega_B)} \tag{19}$$

Based on the radian angle ω_B for the cycloid at the point B, the radius r of the moving wheel as a main parameter of the cycloid is obtained by the equation for the horizontal position (equ. 18, $x(\omega_B) = X$).
Definition End

The hereby defined cycloid is also the optimum solution for the AB (see def. Approximated Brachistochrone, section II, especially def. 1) if an asymptotic value of the number of approximation points $\lim_{n\to\infty}$ is assumed. For the realistic case of a finite number n, an intuitive approximated cycloid (AC) is defined as follows.

Definition Approximated Cycloid (AC).
For any given Brachistochrone (see def. of the Original Brachistochrone, section II) and AB (def. Approximated Brachistochrone, section II), the AC is obtained at height $y_i, i = 1, \ldots, (n-1)$ of the $(n-1)$ approximating points $p_i, i = 1, \ldots, n-1$ by the height of a CS for the approximating points.

$$\forall i = 1, \ldots, (n-1) : y_i = y(\omega_{x_i}) = y(\omega) \tag{20}$$

Definition End

The most important result, which has been evaluated by the example in the last section can be summarized as follows. The AC-solution is outperformed by the evolutionary AB (see figure 4). In principle, one correct example proofs the following hypothesis.

TABLE I

PARAMETER SETTINGS OF THE EVOLUTIONARY AB

$n-1$	μ	ρ	λ	σ
3	10	5	100	7
20	10	5	100	7
30	10	5	100	7
40	11	5	100	8
60	12	5	100	9
...				
140	16	5	100	13
160	17	5	100	14
$Y_{max} = 10, Y_{min} = -5, X = 30$				
$(\mu/\rho_d + \lambda)$				

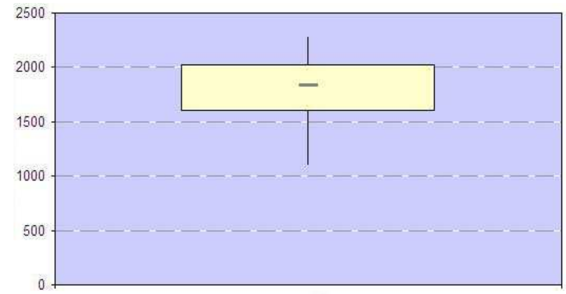

Fig. 6. Necessary generations to determine an AB-solution which outperforms an AC-solution for 30 approximating points

Definition AB-Hypothesis.

For any given Brachistochrone (see def. of the Original Brachistochrone, section II) which is approximated by a finite number ($(n-1)$) of approximating points $(x_i, \hat{y}_i)$ of an AB (def. Approximated Brachistochrone, section II), an evolutionary AB exists, which outperforms the AC-solution in a smaller value of the necessity of overall time for traversing the AB and in a different absolute value for the minimum of the overall height $\hat{y}_i$ of the approximating points. The euclidean distance of AC and AB is defined as follows:

$$d(\hat{y}_i, y(\omega)) = || \hat{y}_i, y(\omega) || = \sqrt{\sum_{i=1}^{n-1} (y(\omega) - \hat{y}_i)^2}. \quad (21)$$

For a rising number of approximating points $(n-1)$, the evolutionary AB turns into the curve of the CS and the euclidean distance to the AC solution is reducing to zero.

Definition End

The proof of the hypothesis for a single AB is given by the example of figure 4. The numerical differences to an AC solution and the configuration of the ES are part of the following section. Besides the earliest mathematical approaches for an analytical solution referred in section III and at the beginning of this section, the problem of Brachistochrone has been addressed in evolutionary optimization by a comparison of the time for the AB to the time needed for the cycloid solution [11], [2], [7], [21]. In the following section the hypothesis (see def. AB-Hypothesis) is backed up by an empirical proof based on several examples in a Monte Carlo simulation study [14].

V. PROOF BY SIMULATION

Let us start by the example of two approximating points illustrated in figure 4. It is obvious, that the evolutionary AB-solution of the lower polygon approximation implies different heights y_i of the approximation points than the ones taken from the AC-solution. As already mentioned, the necessary time for a moving point to traverse the AC-solution is $t_{AC} = 3.34054s$ and for the evolutionary AB solution is $t_{AB} = 3.33159s$. Therefore, the evolutionary solution outperforms the AC-solution. In figure 5 up to figure 7 the necessary generations to determine an evolutionary AB solution which outperforms the AC are illustrated. In the case of two approximating points, the median of necessary generations is below three. Therefore, the first successor population of $\mu = 10$ random AB is often a sufficient solution which outperforms the approximation taken from the CS.

All solutions have been determined by a Monte Carlo simulation [14] study with 30 recurrents of each simulation. The results are presented by boxplots. The configuration parameters are illustrated in table I for two up to 100 approximating points. The configuration for each dimension has been obtained according to idealized parameters for a universal theoretical problem of the same dimension [6], [16], [1] and a maximum convergence speed by adaptation. The mutation step size interval has been randomly modified [17].

Two general experiences are concluded from the simulations. At first, the so called ","-operator for the selection obtained a much more unstable convergence behavior in the simulation. Like reported in table I, the "+"-operator obtained an appropriate convergence behavior. Therefore it can be concluded from the simulations, that a centroid from the preceding generation and the number of λ-times individuals generated in the recent generation t is the appropriate configuration choice. This is indicated by the so called "+"-operator in the notation of the generation loop. Like reported in the generation loop notation in the last row of table I, this choice has been used for all simulations. Secondly, the convergence speed of improvements (notated φ in [6]) is much higher for the discrete variant of a multi-recombination. The usage of this operator is indicated by the index d in the notation of the generation loop. From the simulation experiences it can be reported that the necessary generations for the same accuracy quality is obtained within half the number of generations by the usage of discrete recombination.

VI. CONCLUSIONS

As a main consequence it cannot be followed that the evolutionary solution is superior to other optimization meth-

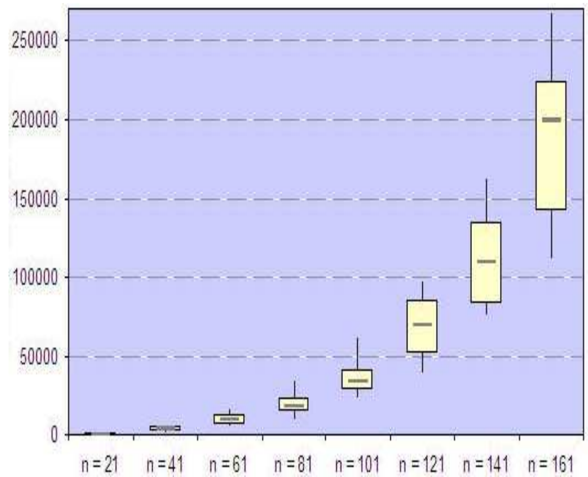

Fig. 7. Necessary generations for 20 up to 160 approximating points $(n-1)$ of AB-solutions which outperform an AC-solution

ods, not even to other analytical or numerical approaches in general, but in the sense that any analytical solution which is approximated by finite states in any practical application case leave potentials for an evolutionary solution which outperforms the quality of an approximation derived from the analytical solution. An empirical comparison of different optimization methods to the proposed Approximated Brachistochrone (AB) is hereby strongly recommended. Moreover, for all discretized version of problems of the calculus of variations.

As result of this paper, the universal type of the evolutionary solution driven by an appropriate fitness function is capable to obtain an AB-solution which is superior and therefore a solution for the huge amount of related practical problems known from calculus of variations. The analytical solution for a finite number of approximation points remains unsolved, but is under current investigation by the authors.

The results of this paper imply further investigations of analytically well solved problems for practical dimensions of approximations. Each engineering or architectural construction, e.g. a roof construction, is often realized based on homogeneous straight parts in a piecewise linear manner, due to construction cost and aesthetic considerations.

Therefore, a paradigm shift in the view of appropriate problems for evolutionary processes is strongly recommended. The results have been verified by an empirical simulation study. This paper has proven, that the famous solution of the cycloid is for a finite and low dimensions of approximating points not the appropriate height y_i of an AB-polygon. Of course, for a rising number of approximating points the evolutionary AB-solution tends into the cycloid solution quality. It depends on the accuracy which quality difference can still be achieved for the approximating polygon for a high number of approximating points.

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