9 - Multiobjective optimization

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Optimization Methods and Game Theory
Master of Science in Artificial Intelligence and Data Engineering
University of Pisa – A.Y. 2020/21

Pareto order

In a multiobjective optimization problem the objective function f is a vector of p elements: $f(x) = (f_1(x), f_2(x), \dots, f_p(x))$.

There are often conflicting objectives $\longrightarrow$ definition of *optimality* is not obvious. We need to define an order in $\mathbb{R}^p$.

Pareto order

Given $x, y \in \mathbb{R}^p$, we say that

$$x \geq y \quad \Longleftrightarrow \quad x_i \geq y_i \quad \text{for any } i = 1, \dots, p.$$

This relation is a partial order in $\mathbb{R}^p$: it is

- reflexive: $x \geq x$
- asymmetric: if $x \geq y$ and $y \geq x$ then $x = y$
- transitive: if $x \geq y$ and $y \geq z$ then $x \geq z$

but it is not a total order: if $x = (1, 4)$ and $y = (3, 2)$ then $x \not\geq y$ and $y \not\geq x$

Minimum definitions for a set of vectors

Definition Given a subset $A \subseteq \mathbb{R}^p$, we say

- $x \in A$ is a Pareto ideal minimum (or ideal efficient point) of A if $y \geq x$ for any $y \in A$.
- $x \in A$ is a Pareto minimum (or efficient point) of A if there is no $y \in A$, $y \neq x$ such that $x \geq y$.
- $x \in A$ is a Pareto weak minimum (or weakly efficient point) of A if there is no $y \in A$, $y \neq x$ such that $x > y$, i.e., $x_i > y_i$ for any $i = 1, \dots, p$.

$IMin(A)$, $Min(A)$ and $WMin(A)$ denote the set of ideal minima, minima, weak minima of A, respectively.

Example. $A = \{x \in \mathbb{R}^2 : \ x_1 \geq 0, \ x_2 \geq 0, \ x_1 + x_2 \leq 1\}$.
$IMin(A) = Min(A) = \{(0,0)\}$, $WMin(A) = \{x \in A : \ x_1 = 0 \text{ or } x_2 = 0\}$.

Example. $B = \{x \in \mathbb{R}^2 : \ 0 \leq x_1 \leq 2, \ 0 \leq x_2 \leq 2, \ x_1 + x_2 \geq 1\}$.
$IMin(B) = \emptyset$, $Min(B) = \{x \in B : x_1 + x_2 = 1\}$,
$WMin(B) = \{x \in B : x_1 = 0 \text{ or } x_2 = 0 \text{ or } x_1 + x_2 = 1\}$.

Proposition. $IMin(A) \subseteq Min(A) \subseteq WMin(A)$.
If $IMin(A) \neq \emptyset$, then $IMin(A) = Min(A) = \{\bar{x}\}$.

Minimum definitions for an optimization problem

Definition

Given a multiobjective optimization problem

$$\begin{cases} \min \ f(x) = (f_1(x), f_2(x), \dots, f_p(x)) \\ x \in \Omega \end{cases} \tag{P}$$

- $x^* \in \Omega$ is a Pareto ideal minimum of (P) if $f(x^*)$ is an Pareto ideal minimum of $f(\Omega)$, i.e., $f(x) \geq f(x^*)$ for any $x \in \Omega$.
- $x^* \in \Omega$ is a Pareto minimum of (P) if $f(x^*)$ is a Pareto minimum of $f(\Omega)$, i.e., if there is no $x \in \Omega$ such that

$$\begin{array}{ll} f_i(x^*) \geq f_i(x) & \text{for any } i = 1, \dots, p, \\ f_j(x^*) > f_j(x) & \text{for some } j \in \{1, \dots, p\}. \end{array}$$

- $x^* \in \Omega$ is a Pareto weak minimum of (P) if $f(x^*)$ is a Pareto weak minimum of $f(\Omega)$, i.e., if there is no $x \in \Omega$ such that

$$f_i(x^*) > f_i(x) \qquad \text{for any } i = 1, \dots, p.$$

Minimum definitions for an optimization problem

Example. Consider

$$\begin{cases} \min\ (x_1 - x_2,\ -2x_1 + x_2) \\ x_1 \leq 1 \\ -x_1 \leq 0 \\ -x_1 + x_2 \leq 2 \\ 2x_1 - x_2 \leq 0 \end{cases} \tag{P}$$

The image $f(\Omega) = \{(y_1, y_2) :\ y_1 = x_1 - x_2,\ y_2 = -2x_1 + x_2,\ x \in \Omega\}$.
We obtain $x_1 = -y_1 - y_2$ and $x_2 = -2y_1 - y_2$, hence
$f(\Omega) = \{(y_1, y_2) :\ -y_1 - y_2 \leq 1,\ y_1 + y_2 \leq 0,\ -y_1 \leq 2,\ -y_2 \leq 0\}$.
$IMin(f(\Omega)) = \emptyset$. $Min(f(\Omega)) = \{y \in f(\Omega) :\ -y_1 - y_2 = 1\}$, thus

$$\{\text{minima of (P)}\} = \{x \in \Omega :\ -x_1 + x_2 + 2x_1 - x_2 = 1\} = \{x \in \Omega :\ x_1 = 1\}.$$

$WMin(f(\Omega)) = \{y \in f(\Omega) :\ -y_1 - y_2 = 1 \text{ or } y_1 = -2 \text{ or } y_2 = 0\}$, thus

$$\{\text{weak minima of (P)}\} = \{x \in \Omega :\ x_1 = 1 \text{ or } x_1 - x_2 = -2 \text{ or} -2x_1 + x_2 = 0\}.$$

Existence results

Generalized Weierstrass Theorem

If f_i is continuous for any $i = 1 \ldots, p$ and Ω is closed and bounded, then there exists a minimum of (P).

Theorem

If f_i is continuous for any $i = 1 \ldots, p$, Ω is closed and there are $v \in \mathbb{R}$ and $j \in \{1, \ldots, p\}$ such that the sublevel set

$$\{x \in \Omega : \ f_j(x) \leq v\}$$

is nonempty and bounded, then there exists a minimum of (P).

Corollary. If f_i is continuous for any $i = 1 \ldots, p$, Ω is closed and f_j is coercive for some $j \in \{1, \ldots, p\}$, then there exists a minimum of (P).

Optimality conditions

Theorem

$x^* \in \Omega$ is a minimum of (P) if and only if the auxiliary optimization problem

$$\begin{cases} \max \ \sum_{i=1}^{p} \varepsilon_i \\ f_i(x) + \varepsilon_i \leq f_i(x^*) & \forall\ i = 1, \ldots, p \\ x \in \Omega \\ \varepsilon \geq 0 \end{cases}$$

has optimal value equal to 0.

Theorem

$x^* \in \Omega$ is a weak minimum of (P) if and only if the auxiliary optimization problem

$$\begin{cases} \max \ v \\ v \leq \varepsilon_i & \forall\ i = 1, \ldots, p \\ f_i(x) + \varepsilon_i \leq f_i(x^*) & \forall\ i = 1, \ldots, p \\ x \in \Omega \\ \varepsilon \geq 0 \end{cases}$$

has optimal value equal to 0.

Optimality conditions

Exercise 9.1. Consider the linear multiobjective problem

$$\begin{cases} \min\ (x_1 + 2x_2 - 3x_3\ ,\ -x_1 - x_2 - x_3\ ,\ -4x_1 - 2x_2 + x_3) \\ x_1 + x_2 + x_3 \leq 10 \\ x_3 \leq 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Check if the points $u = (5, 0, 5)$, $v = (4, 4, 2)$ and $w = (1, 4, 4)$ are minima or weak minima by solving the corresponding auxiliary problems.

First-order optimality conditions: unconstrained problems

Consider an unconstrained multiobjective problem

$$\begin{cases} \min \ f(x) = (f_1(x), f_2(x), \dots, f_p(x)) \\ x \in \mathbb{R}^n \end{cases} \tag{P}$$

where f_i is continuously differentiable for any $i = 1, \dots, p$.

Necessary optimality condition

If x^* is a weak minimum of (P), then there exists $\xi^* \in \mathbb{R}^p$ such that

$$\begin{cases} \sum_{i=1}^{p} \xi_i^* \, \nabla f_i(x^*) = 0 \\ \xi^* \geq 0, \quad \sum_{i=1}^{p} \xi_i^* = 1 \end{cases} \tag{S}$$

Sufficient optimality condition

If the problem (P) is convex, i.e., f_i is convex for any $i = 1, \dots, p$, and (x^*, ξ^*) is a solution of the system (S), then x^* is a weak minimum of (P).

First-order optimality conditions: unconstrained problems

Exercise 9.2. Consider the nonlinear multiobjective problem

$$\begin{cases} \min\ (x_1^2 + x_2^2,\ (x_1 - 1)^2 + (x_2 - 1)^2) \\ x \in \mathbb{R}^2 \end{cases}$$

a) Find the set of weak minima exploiting the first-order optimality conditions.

b) Find the set of minima.

First-order optimality conditions: constrained problems

Consider a constrained multiobjective problem

$$\begin{cases} \min \ f(x) = (f_1(x), f_2(x), \dots, f_p(x)) \\ g_j(x) \le 0 \qquad \forall\ j = 1, \dots, m \\ h_k(x) = 0 \qquad \forall\ k = 1, \dots, q \end{cases} \tag{P}$$

where f_i, g_j and h_k are continuously differentiable for any i, j, k.

Necessary optimality condition

If x^* is a weak minimum of (P) and ACQ holds at x^*, then there exist $\xi^* \in \mathbb{R}^p$, $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^q$ such that $(x^*, \xi^*, \lambda^*, \mu^*)$ solves the KKT system

$$\begin{cases} \sum_{i=1}^{p} \xi_i^* \nabla f_i(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla g_j(x^*) + \sum_{k=1}^{q} \mu_k^* \nabla h_k(x^*) = 0 \\ \xi^* \ge 0, \quad \sum_{i=1}^{p} \xi_i^* = 1 \\ \lambda^* \ge 0 \\ \lambda_j^* \, g_j(x^*) = 0 \qquad \forall\ j = 1, \dots, m \end{cases}$$

Sufficient optimality condition

If (P) is convex, i.e., f_i convex, g_j convex and h_k affine, and $(x^*, \xi^*, \lambda^*, \mu^*)$ solves the KKT system, then x^* is a weak minimum of (P).

First-order optimality conditions: constrained problems

Exercise 9.3. Consider the nonlinear multiobjective problem

$$\begin{cases} \min\ (x_1 + x_2,\ -x_1 + x_2) \\ x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

a) Find the set of weak minima by solving the KKT system.

b) Find the set of minima.

Scalarization method

Define a vector of weights associated to the objectives:

$$\alpha = (\alpha_1, \ldots, \alpha_p) \geq 0 \quad \text{such that} \quad \sum_{i=1}^{p} \alpha_i = 1$$

and consider the following scalar optimization problem

$$\begin{cases} \min \displaystyle\sum_{i=1}^{p} \alpha_i f_i(x) \\ x \in \Omega \end{cases} \qquad (\mathrm{P}_\alpha)$$

Let S_α be the set of optimal solutions of (P_α).

Theorem

- $\displaystyle\bigcup_{\alpha \geq 0} S_\alpha \subseteq \{\text{weak minima of (P)}\}$
- $\displaystyle\bigcup_{\alpha > 0} S_\alpha \subseteq \{\text{minima of (P)}\}$

Scalarization method

Solving (P_α) for any possible choice of α does not allow finding all the minima and weak minima.

Example. Consider

$$\begin{cases} \min\ (x_1,\ x_2) \\ x_1^2 + x_2^2 - 4 \le 0 \\ -x_1^2 - x_2^2 + 1 \le 0 \\ x_1 \ge 0,\ x_2 \ge 0 \end{cases}$$

$$\bigcup_{\alpha \ge 0} S_\alpha = \{(0, x_2) :\ x_2 \in [1, 2]\} \cup \{(x_1, 0) :\ x_1 \in [1, 2]\},$$

while

$$\{\text{weak minima of (P)}\} =$$
$$\{(0, x_2) :\ x_2 \in [1, 2]\} \cup \{(x_1, 0) :\ x_1 \in [1, 2]\} \cup \{x \in \mathbb{R}^2_+ :\ x_1^2 + x_2^2 = 1\}.$$

Furthermore,

$$\bigcup_{\alpha > 0} S_\alpha = \{(0, 1), (1, 0)\},$$

while

$$\{\text{minima of (P)}\} = \{x \in \mathbb{R}^2_+ :\ x_1^2 + x_2^2 = 1\}.$$

Scalarization method

Theorem

- If (P) is linear, then $\{\text{weak minima of (P)}\} = \bigcup_{\alpha \geq 0} S_\alpha$ and $\{\text{minima of (P)}\} = \bigcup_{\alpha > 0} S_\alpha$
- If (P) is convex, then $\{\text{weak minima of (P)}\} = \bigcup_{\alpha \geq 0} S_\alpha$
- If (P) is convex and f_i is strongly convex for any $i = 1, \dots, p$, then $\{\text{minima of (P)}\} = \{\text{weak minima of (P)}\} = \bigcup_{\alpha \geq 0} S_\alpha$

Exercise 9.4. Consider the linear multiobjective problem

$$\begin{cases} \min\ (x_1 - x_2\ ,\ x_1 + x_2) \\ -2\,x_1 + x_2 \leq 0 \\ -x_1 - x_2 \leq 0 \\ 5\,x_1 - x_2 \leq 6 \end{cases}$$

Find the set of minima and weak minima by means of the scalarization method.

Scalarization method

Exercise 9.5. Consider the nonlinear multiobjective problem

$$\begin{cases} \min \ (x_1 \ , \ x_1^2 + x_2^2 - 2x_1) \\ -x_1 \leq 0 \\ -x_2 \leq 0 \\ x_1 + x_2 \leq 2 \end{cases}$$

a) Find the set of weak minima by means of the scalarization method.

b) What is the set of minima?

Exercise 9.6. Consider the nonlinear multiobjective problem

$$\begin{cases} \min \ (x_1^2 + x_2^2 + 2x_1 - 4x_2 \ , \ x_1^2 + x_2^2 - 6x_1 - 4x_2) \\ -x_2 \leq 0 \\ -2x_1 + x_2 \leq 0 \\ 2x_1 + x_2 \leq 4 \end{cases}$$

Find the set of minima and weak minima by means of the scalarization method.

Goal method

In the objective space $\mathbb{R}^p$ define the ideal point z as

$$z_i = \min_{x \in \Omega} f_i(x), \qquad \forall\ i = 1, \dots, p.$$

Since very often (P) has no ideal minimum, i.e., $z \notin f(\Omega)$, we want to find the point of $f(\Omega)$ which is as close as possible to z:

$$\begin{cases} \min \ \|f(x) - z\|_s \\ x \in \Omega \end{cases} \qquad \text{with} \ \ s \in [1, +\infty]. \tag{G}$$

Theorem

- If $s \in [1, +\infty)$, then any optimal solution of (G) is a minimum of (P).
- If $s = +\infty$, then any optimal solution of (G) is a weak minimum of (P).

Goal method

Assume that (P) is a linear multiobjective optimization problem, i.e.,

$$\begin{cases} \min \ Cx \\ Ax \leq b \end{cases} \tag{P}$$

where C is a $p \times n$ matrix.

If $s = 2$, then (G) is equivalent to a quadratic programming problem:

$$\begin{cases} \min \ \frac{1}{2}\|Cx - z\|_2^2 = \frac{1}{2}x^\mathsf{T} C^\mathsf{T} Cx - x^\mathsf{T} C^\mathsf{T} z + \frac{1}{2} z^\mathsf{T} z \\ Ax \leq b \end{cases} \tag{G$_2$}$$

Goal method

If $s = 1$, then (G) is equivalent to the linear programming problem

$$\begin{cases} \min\limits_{x,y} \ \sum\limits_{i=1}^{p} y_i \\ y_i \geq C_i x - z_i & \forall\ i = 1, \dots, p \\ y_i \geq z_i - C_i x & \forall\ i = 1, \dots, p \\ Ax \leq b \end{cases} \tag{G_1}$$

If $s = +\infty$, then (G) is equivalent to the linear programming problem

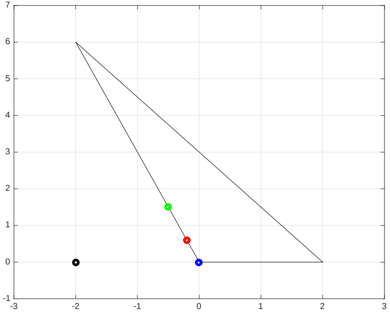
$$\begin{cases} \min\limits_{x,y} \ y \\ y \geq C_i x - z_i & \forall\ i = 1, \dots, p \\ y \geq z_i - C_i x & \forall\ i = 1, \dots, p \\ Ax \leq b \end{cases} \tag{G_∞}$$

Goal method

Example. Consider

$$\begin{cases} \min\ (x_1 - x_2\ ,\ x_1 + x_2) \\ -2\,x_1 + x_2 \leq 0 \\ -x_1 - x_2 \leq 0 \\ 5\,x_1 - x_2 \leq 6 \end{cases}$$

The set $f(\Omega)$ is shown in the figure below.
The ideal point is $z = (-2, 0)$ (black point).
The optimal solution of (G_2) is $x^* = (1/5, 2/5)$ and $f(x^*) = (-1/5, 3/5)$.
The optimal solution of (G_1) is $\widetilde{x} = (0, 0)$ and $f(\widetilde{x}) = (0, 0)$.
The optimal solution of (G_∞) is $\bar{x} = (1/2, 1)$ and $f(\bar{x}) = (-1/2, 3/2)$.



Goal method

Exercise 9.7. Consider the linear multiobjective problem

$$\begin{cases} \min \ (x_1 + 2x_2 - 3x_3 \ , \ -x_1 - x_2 - x_3 \ , \ -4x_1 - 2x_2 + x_3) \\ x_1 + x_2 + x_3 \leq 10 \\ x_3 \leq 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

a) Find the ideal point.

b) Apply the goal method with $s = 1$.

c) Apply the goal method with $s = 2$.

d) Apply the goal method with $s = +\infty$. Is the found point a minimum?