

1 - Preliminary notions of convex analysis

G. Mastroeni

Department of Computer Science, University of Pisa

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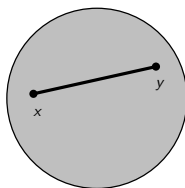
Contents of the lessons

- Convex sets
- Cones
- Convex functions
- Strictly and strongly convex functions

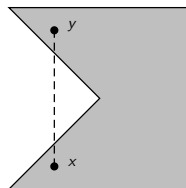
Definition (Convex set)

A set $C \subseteq \mathbb{R}^n$ is **convex** if, for every $x, y \in C$ and for every $\alpha \in [0, 1]$,

$$\alpha x + (1 - \alpha)y \in C.$$



convex set



non-convex set

Examples of convex sets: affine sets

Definition (Affine set)

A set $C \subseteq \mathbb{R}^n$ is **affine** if, for every $x, y \in C$ and every $\alpha \in \mathbb{R}$,

$$\alpha x + (1 - \alpha)y \in C.$$

Examples of affine sets:

- any single point $\{x\}$
- any line
- the solution set of a system of linear equations

$$C = \{x \in \mathbb{R}^n : Ax = b\},$$

where A is a $m \times n$ matrix and $b \in \mathbb{R}^m$

- any subspace

Examples of convex sets: subspaces

Note that a subspace is a particular affine set.

In fact, a set $S \subseteq \mathbb{R}^n$ is a **subspace** if, for every $x, y \in S$ and every $\alpha, \beta \in \mathbb{R}$,

$$\alpha x + \beta y \in S$$

Examples of subspaces:

- $\{0\}$
- any line which passes through zero
- the solution set of a homogeneous system of linear equations

$$S = \{x \in \mathbb{R}^n : Ax = 0\},$$

where A is a $m \times n$ matrix.

Definition

A **convex combination** of the points x^1, x^2, \dots, x^k is a point

$$y = \sum_{i=1}^k \alpha_i x^i \text{ where } \alpha_1, \dots, \alpha_k \in [0, 1] \text{ and } \sum_{i=1}^k \alpha_i = 1.$$

Remark

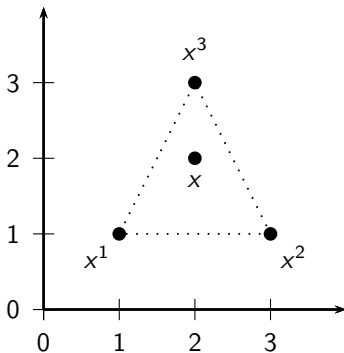
By definition, a set $C \subseteq \mathbb{R}^n$ is **convex** if it contains all the convex combinations of any two points in C .

Example. Consider the following 3 points in the plane:

$$x^1 = (1, 1), \quad x^2 = (3, 1), \quad x^3 = (2, 3).$$

$x = (2, 2)$ is a convex combination of x^1 , x^2 e x^3 , in fact:

$$x = \frac{1}{4}x^1 + \frac{1}{4}x^2 + \frac{1}{2}x^3.$$



A convex set contains any convex combination of its points.

Lemma 1

If C is convex, then for any $x^1, \dots, x^k \in C$ and $\alpha_1, \dots, \alpha_k \in [0, 1]$ s.t. $\sum_{i=1}^k \alpha_i = 1$,

$$\sum_{i=1}^k \alpha_i x^i \in C.$$

Proof. By induction on k . For $k = 2$, the thesis holds, by definition of convexity. Assume that the thesis holds for a given k and let us prove it holds for $k + 1$.

Let $x^1, \dots, x^{k+1} \in C$ and $\alpha_1, \dots, \alpha_{k+1} \in [0, 1]$ s.t. $\sum_{i=1}^{k+1} \alpha_i = 1$. With no loss of generality, we assume that $\alpha_1 \neq 0$.

$$\sum_{i=1}^{k+1} \alpha_i x^i = \alpha_1 x_1 + \sum_{i=2}^{k+1} \alpha_i x^i = \alpha_1 x_1 + \left(1 - \sum_{i=2}^{k+1} \alpha_i\right) \sum_{i=2}^{k+1} \frac{\alpha_i}{1 - \sum_{i=2}^{k+1} \alpha_i} x^i$$

Since $\sum_{i=2}^{k+1} \frac{\alpha_i}{1 - \sum_{i=2}^{k+1} \alpha_i} = 1$, by inductive assumption we have:

$$\bar{x} := \sum_{i=2}^{k+1} \frac{\alpha_i}{1 - \sum_{i=2}^{k+1} \alpha_i} x^i \in C$$

and finally, since C is convex,

$$\alpha_1 x_1 + \left(1 - \sum_{i=2}^{k+1} \alpha_i\right) \bar{x} \in C.$$

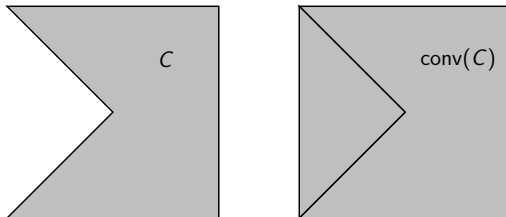
Proposition

If $\{C_i\}_{i \in I}$ is any (possibly infinite) family of convex sets, then $\bigcap_{i \in I} C_i$ is convex.

Definition (Convex hull)

The **convex hull** $\text{conv}(C)$ of a set C is the intersection of all the convex sets containing C .

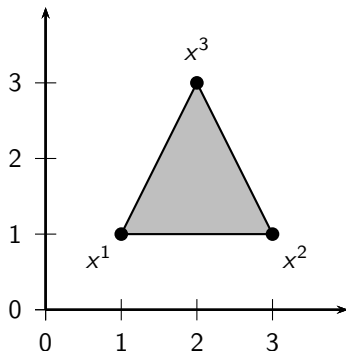
In other words, it is the smallest convex set containing C .



The convex hull of the points

$$x^1 = (1, 1), \quad x^2 = (3, 1), \quad x^3 = (2, 3).$$

is the grey triangle with vertexes the three points:



Proposition

$$\text{conv}(C) = \{\text{all convex combinations of points in } C\}$$

Proof. It can be proved that the set of convex combinations of points in C is a convex set containing C , so that

$$\text{conv}(C) \subseteq \{\text{all convex combinations of points in } C\}.$$

Since $C \subseteq \text{conv}(C)$ and $\text{conv}(C)$ is convex, by Lemma 1 it contains any convex combination of its points, and therefore

$$\text{conv}(C) \supseteq \{\text{all convex combinations of points in } C\}.$$

Remark

Observe that C is convex if and only if $C = \text{conv}(C)$.

Examples of convex sets: Polyhedra

Definition (Polyhedron)

A polyhedron P is the intersection of a finite number of closed halfspaces in \mathbb{R}^n .

A closed halfspace is the set of solutions of a linear inequality:

$$a^T x \leq \beta, \quad \text{where } a \in \mathbb{R}^n \text{ e } \beta \in \mathbb{R}.$$

Consequently, a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

is the solution set of a system of linear inequalities where A is a $m \times n$ matrix and $b \in \mathbb{R}^m$.

A polyhedron P is convex since any closed halfspace is a convex set and the intersection of convex sets is convex.

Examples of convex sets: Balls

- A ball is defined by $B(\bar{x}, r) := \{z \in \mathbb{R}^n : \|z - \bar{x}\| \leq r\}$, where $\|\cdot\|$ is any norm, e.g.

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} \text{ (Euclidean norm)}$$

$$\|x\|_1 = \sum_{i=1}^n |x_i| \text{ (Manhattan distance)}$$

$$\|x\|_\infty = \max_{i=1, \dots, n} |x_i| \text{ (Chebyshev norm)}$$

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}, \text{ with } 1 \leq p < +\infty$$

$$\|x\|_A = \sqrt{x^T A x}, \text{ where } A \text{ is a symmetric and positive definite matrix, i.e.,}$$

$$x^T A x > 0 \quad \forall x \neq 0.$$

Recall that a norm on a real vector space X is a function $p : X \rightarrow \mathbb{R}$ such that:

- ❶ $p(x + y) \leq p(x) + p(y), \quad \forall x, y \in X;$
- ❷ $p(\alpha x) = |\alpha|p(x), \quad \forall x \in X, \forall \alpha \in \mathbb{R};$
- ❸ $p(x) = 0 \iff x = 0.$

By the previous conditions it follows that $p(x) \geq 0, \forall x \in X$.

Exercise 1.1 Find the unit ball $B(0, 1)$ w.r.t. $\|\cdot\|_1$, $\|\cdot\|_\infty$ and $\|\cdot\|_A$, where

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

Operations that preserve convexity

Algebraic operations

Sum and product by a constant

If C_1 and C_2 are convex, then $C_1 + C_2 := \{x + y : x \in C_1, y \in C_2\}$ is convex.

If C is convex and $\alpha \in \mathbb{R}$, then $\alpha C := \{\alpha x : x \in C, \}$ is convex.

Consequently, if C_1 and C_2 are convex, then $C_1 - C_2 := \{x - y : x \in C_1, y \in C_2\}$ is convex.

Topological operations

Closure and interior

If C is convex, then $\text{cl}(C)$ is convex.

If C is convex, then $\text{int}(C)$ is convex, provided that $\text{int}(C) \neq \emptyset$.

Relative interior

Given a set $C \subseteq \mathbb{R}^n$ we denote by $\text{aff}(C)$ the smallest affine set containing C .

Definition (relative interior)

Let $C \subseteq \mathbb{R}^n$ be a convex set.

The relative interior points of C is defined by

$$\text{ri}(C) = \{x \in C : \exists \epsilon > 0 : \text{aff}(C) \cap B(x, \epsilon) \subseteq C\}$$

Examples

- Let $C := \{(x_1, x_2) \in \mathbb{R}^2 : 1 \leq x_1 \leq 3, x_2 = 0\}$. Then

$$\text{ri}(C) := \{(x_1, x_2) \in \mathbb{R}^2 : 1 < x_1 < 3, x_2 = 0\}.$$

- Let $C = \{\bar{x}\}$, then $\text{ri}(C) = C$.

Theorem

Let C be a nonempty convex set in \mathbb{R}^n . Then the relative interior of C is a nonempty convex set.

Separation of convex sets

The sets A and B in \mathbb{R}^n are said to be linearly separable if there exists $a \in \mathbb{R}^n$, $a \neq 0$, $\beta \in \mathbb{R}$, such that

$$a^T x \geq \beta \quad \forall x \in A, \quad a^T x \leq \beta \quad \forall x \in B,$$

The separation is said to be proper if strict inequality holds for at least one $x \in A \cup B$.

Theorem

Let A, B be nonempty convex sets in \mathbb{R}^n . Then A and B are properly linearly separable if and only if

$$ri(A) \cap ri(B) = \emptyset.$$

In particular two disjoint convex sets are always properly linearly separable.

Example Let $A := \{(x_1, x_2) \in \mathbb{R}^2 : 1 \leq x_1 \leq 2, x_2 = 0\}$,
 $B := \{(x_1, x_2) \in \mathbb{R}^2 : 2 \leq x_1 \leq 4, x_2 = 0\}$.

Then $ri(A) \cap ri(B) = \emptyset$ and the sets are properly separable by the hyperplane of equation $x_1 = 2$.

Operations that preserve convexity

Affine functions

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be affine, i.e. $f(x) = Ax + b$, with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

- If $C \subseteq \mathbb{R}^n$ is convex, then $f(C) = \{f(x) : x \in C\}$ is convex
- If $C \subseteq \mathbb{R}^m$ is convex, then $f^{-1}(C) = \{x \in \mathbb{R}^n : f(x) \in C\}$ is convex

Examples:

- $f(x) = \alpha x$, with $\alpha \in \mathbb{R}$
- $f(x) = x + b$, with $b \in \mathbb{R}^n$
- $f(x) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} x$, with $\theta \in (0, 2\pi)$ (rotation)

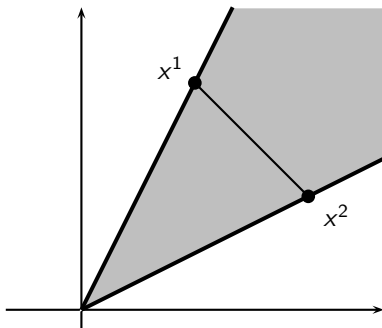
Definition (Cone)

A set $C \subseteq \mathbb{R}^n$ is a **cone** if, for every $x \in C$ and for every $\lambda \geq 0$, it results

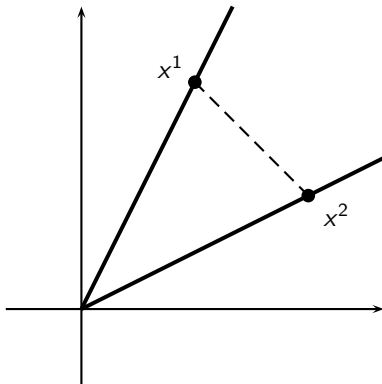
$$\lambda x \in C.$$

In other words, if C contains a point x different from 0, then it contains the whole halfline starting from 0 and passing through x .

Example. A cone may be convex



or non convex:



Examples of cones

- \mathbb{R}_+^n is a convex cone.
- $\{x \in \mathbb{R}^2 : x_1 x_2 = 0\}$ is a non-convex cone.
- Given a polyhedron $P = \{x : Ax \leq b\}$, the recession cone of P is defined as

$$\text{rec}(P) := \{d : x + \alpha d \in P \text{ for any } x \in P, \alpha \geq 0\}.$$

It can be proved that $\text{rec}(P) = \{d : Ad \leq 0\}$, thus it is a polyhedral cone.

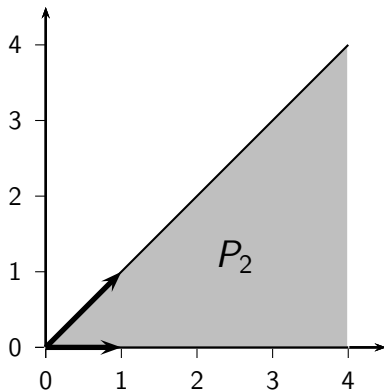
- $\{x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\}$ is a non-polyhedral cone.
- Given $\bar{x} \in \text{cl}(C) \subseteq \mathbb{R}^n$, the set

$$T_C(\bar{x}) = \left\{ d \in \mathbb{R}^n : \exists \{z_k\} \subset C, \exists \{t_k\} > 0, z_k \rightarrow \bar{x}, t_k \rightarrow 0, \lim_{k \rightarrow \infty} \frac{z_k - \bar{x}}{t_k} = d \right\}$$

is called the *tangent cone* to C at \bar{x} .

Example

$$P_1 = \{x \in \mathbb{R}^2 : x_2 \leq x_1, \quad x_2 \geq 0\}$$

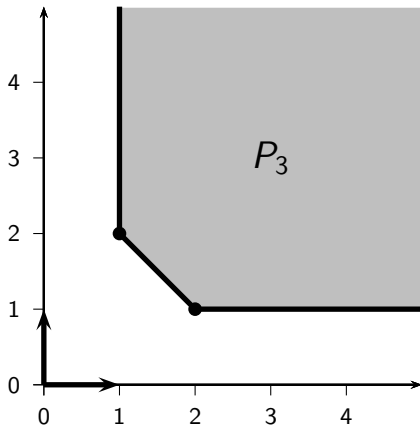


is a polyhedral cone.

$$\text{rec}(P_1) = P_1, \quad T_{P_1}((0,0)) = P_1.$$

Example

$$P_2 = \{x \in \mathbb{R}^2 : x_1 \geq 1, \quad x_2 \geq 1, \quad x_1 + x_2 \geq 3\}$$



$$\text{rec}(P_2) = \{d \in \mathbb{R}^2 : d_1 \geq 0, \quad d_2 \geq 0\}$$

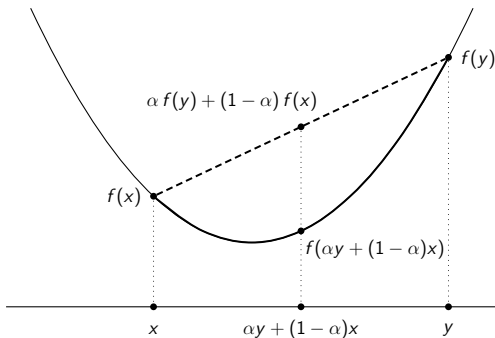
$$T_{P_2}((1, 2)) = \{d \in \mathbb{R}^2 : d_1 \geq 0, \quad d_1 + d_2 \geq 0\}$$

- 1.2** Let $P = \{x : Ax \leq b\}$ where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Prove that $\text{rec}(P) = \{d \in \mathbb{R}^n : Ad \leq 0\}$.
- 1.3** If C_1 and C_2 are convex, then is $C_1 \cup C_2$ convex?
- 1.4** Prove that $B(\bar{x}, r) := \{z \in \mathbb{R}^n : \|z - \bar{x}\| \leq r\}$, is a convex set, whatever the norm $\|\cdot\|$ may be.
- 1.5** Write the vector $(1, 1)$ as a convex combination of the vectors $(0, 0)$, $(3, 0)$, $(0, 2)$, $(3, 2)$.

Definition (Convex function)

Let $C \subseteq \mathbb{R}^n$ be convex. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **convex** on C if

$$f(\alpha y + (1 - \alpha)x) \leq \alpha f(y) + (1 - \alpha)f(x) \quad \forall x, y \in C, \forall \alpha \in [0, 1]$$



Remark

When $C = \mathbb{R}^n$ we will simply say that f is convex.

Theorem

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **convex** on \mathbb{R}^n if and only if the set

$$\text{epi } f_C := \{(x, y) \in C \times \mathbb{R} : y \geq f(x)\}$$

is convex.

Definition (Concave function)

Given a convex set $C \subseteq \mathbb{R}^n$, a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **concave** on C if $-f$ is convex, i.e.,

$$f(\alpha y + (1 - \alpha)x) \geq \alpha f(y) + (1 - \alpha)f(x) \quad \forall x, y \in C, \forall \alpha \in [0, 1]$$

Examples.

- A linear (affine) function $f(x) = c^T x + b$ is both convex and concave.
- Let $\|\cdot\|$ be any norm, then $f(x) = \|x\|$ is convex.

Theorem (continuity of convex functions)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex on the convex set $C \subseteq \mathbb{R}^n$. Then f is continuous on $\text{ri}(C)$.

Strictly convex and strongly convex functions

Definition (strictly convex function)

Given a convex set $C \subseteq \mathbb{R}^n$, a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **strictly convex** on C if

$$f(\alpha y + (1 - \alpha)x) < \alpha f(y) + (1 - \alpha)f(x) \quad \forall x, y \in C, x \neq y, \forall \alpha \in (0, 1)$$

Definition (strongly convex function)

Given a convex set $C \subseteq \mathbb{R}^n$, a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **strongly convex** on C if there exists $\tau > 0$ s.t.

$$f(\alpha y + (1 - \alpha)x) \leq \alpha f(y) + (1 - \alpha)f(x) - \frac{\tau}{2}\alpha(1 - \alpha)\|y - x\|_2^2 \\ \forall x, y \in C, \forall \alpha \in [0, 1]$$

Remark

Similarly to convex functions, we say that f is strictly (strongly) concave on C if $-f$ is strictly (strongly) convex on C .

Theorem

f is strongly convex if and only if $\exists \tau > 0$ such that $f(x) - \frac{\tau}{2} \|x\|_2^2$ is convex

Remark

By the previous theorem it follows that f is strongly convex if and only if there exists a convex function ψ and $\tau > 0$ such that $f(x) = \psi(x) + \frac{\tau}{2} \|x\|_2^2$.

Exercise 1.6

- Prove that: strongly convex \implies strictly convex \implies convex
- convex \implies strictly convex ?
- strictly convex \implies strongly convex ?

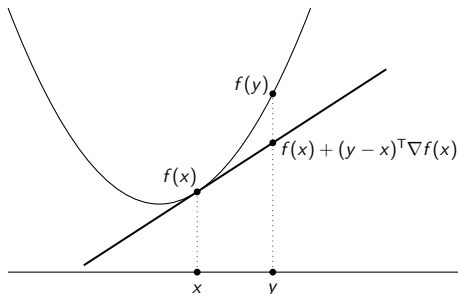
First order conditions

Assume that $C \subseteq \mathbb{R}^n$ is open and convex, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable on C .

Theorem

f is **convex** on C if and only if

$$f(y) \geq f(x) + (y - x)^T \nabla f(x) \quad \forall x, y \in C.$$



First-order approximation of f is a global **underestimator**

Theorem

- f is **strictly convex** on C if and only if

$$f(y) > f(x) + (y - x)^T \nabla f(x) \quad \forall x, y \in C, \text{ with } x \neq y.$$

- f is **strongly convex** on C if and only if there exists $\tau > 0$ such that

$$f(y) \geq f(x) + (y - x)^T \nabla f(x) + \frac{\tau}{2} \|y - x\|_2^2 \quad \forall x, y \in C.$$

Second order conditions

Assume that $C \subseteq \mathbb{R}^n$ is open and convex, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice continuously differentiable on C .

Theorem

- f is **convex** on C if and only if for all $x \in C$ the Hessian matrix $\nabla^2 f(x)$ is positive semidefinite, i.e.

$$v^T \nabla^2 f(x) v \geq 0 \quad \forall v \in \mathbb{R}^n, \forall x \in C,$$

or, equivalently, the eigenvalues of $\nabla^2 f(x)$ are ≥ 0 , $\forall x \in C$.

- If $\nabla^2 f(x)$ is positive definite for all $x \in C$, then f is **strictly convex** on C .
- f is **strongly convex** on C if and only if there exists $\tau > 0$ such that $\nabla^2 f(x) - \tau I$ is positive semidefinite for all $x \in C$, i.e.

$$v^T \nabla^2 f(x) v \geq \tau \|v\|_2^2 \quad \forall v \in \mathbb{R}^n, \quad \forall x \in C,$$

or, equivalently, the eigenvalues of $\nabla^2 f(x)$ are $\geq \tau$, $\forall x \in C$.

Convexity of quadratic functions

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{1}{2}x^T Qx + c^T x$$

where Q is a $n \times n$ symmetric matrix, $c \in \mathbb{R}^n$. It is easy to check that

- $\nabla f = \frac{1}{2}(Qx + (x^T Q)^T) + c = Qx + c$
- Q is the Hessian of f .

Then f is:

- convex iff Q is positive semidefinite
- strongly convex iff Q is positive definite
- concave iff Q is negative semidefinite
- strongly concave iff Q is negative definite

Examples

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $C := \mathbb{R}_+ \setminus \{0\}$.

- $f(x) = e^{px}$ for any $p \in \mathbb{R} \setminus \{0\}$ is strictly convex (on \mathbb{R}), but not strongly convex
- $f(x) = x^p$ is strictly convex on C if $p > 1$ or $p < 0$.
Is it strongly convex?
- $f(x) = x^p$ is strictly concave on C if $0 < p < 1$
- $f(x) = \log(x)$ is strictly concave, but not strongly concave on C

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

- $f(x) = \|x\|$ is convex, but not strictly convex
- $f(x) = \max\{x_1, \dots, x_n\}$ is convex, but not strictly convex

1.7 Prove that $f(x) = \|x\|$ is convex, whatever the norm $\|\cdot\|$ may be.

1.8 Prove that if f is convex, then for any $x^1, \dots, x^k \in C$ and $\alpha_1, \dots, \alpha_k \in (0, 1)$ s.t. $\sum_{i=1}^k \alpha_i = 1$, one has $f\left(\sum_{i=1}^k \alpha_i x^i\right) \leq \sum_{i=1}^k \alpha_i f(x^i)$.

Hint. Follow the proof given in Lemma 1.

1.9 Prove that $f(x_1, x_2) = \frac{1}{x_1 x_2}$ is convex on the set $\{x \in \mathbb{R}^2 : x_1, x_2 > 0\}$.

1.10 Analyse the convexity properties of the function

$$f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + x_3^2 + 3x_1x_2 + x_2x_3 - 6x_1 - 4x_2 - 3x_3$$

1.11 Let f_1 and f_2 be convex, then is the product $f_1 f_2$ convex?

Operations that preserve convexity

Theorem

- If f is convex and $\alpha > 0$, then αf is convex
- If f_1 and f_2 are convex, then $f_1 + f_2$ are convex
- If f is convex, then $f(Ax + b)$ is convex

Examples

- Log barrier for linear inequalities:

$$f(x) = - \sum_{i=1}^m \log(b_i - a_i^T x) \quad C = \{x \in \mathbb{R}^n : b_i - a_i^T x > 0 \quad \forall i = 1, \dots, m\}$$

- Norm of affine function: $f(x) = \|Ax + b\|$

Theorem

- If f_1, \dots, f_m are convex, then $f(x) = \max\{f_1(x), \dots, f_m(x)\}$ is convex.
- If $\{f_i\}_{i \in I}$ is a family of convex functions, then $f(x) = \sup_{i \in I} f_i(x)$ is convex.

Example. If $\psi(x, \lambda) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is convex in x and concave in λ , then

$$\begin{aligned} p(x) &= \sup_{\lambda} \psi(x, \lambda) && \text{is convex} \\ d(\lambda) &= \inf_x \psi(x, \lambda) && \text{is concave} \end{aligned}$$

Composition

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$.

Theorem

- If f is convex and g is convex and nondecreasing, then $g \circ f$ is convex.
- If f is concave and g is convex and nonincreasing, then $g \circ f$ is convex.
- If f is concave and g is concave and nondecreasing, then $g \circ f$ is concave.
- If f is convex and g is concave and nonincreasing, then $g \circ f$ is concave.

Examples Let $f : \mathbb{R} \rightarrow \mathbb{R}$.

- If f is convex, then $e^{f(x)}$ is convex
- If f is concave and positive, then $\log f(x)$ is concave
- If f is convex, then $-\log(-f(x))$ is convex on $\{x : f(x) < 0\}$
- If f is concave and positive, then $\frac{1}{f(x)}$ is convex
- If f is convex and nonnegative, then $f(x)^p$ is convex for all $p \geq 1$

Given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $k \in \mathbb{R}$, the set

$$S_k(f) = \{x \in \mathbb{R}^n : f(x) \leq k\}$$

is said the **k -sublevel set** of f .

Exercise 1.12 Prove that if f is convex, then $S_k(f)$ is a convex set for any $k \in \mathbb{R}$.

Is the converse true?

Definition (Quasiconvex convex function)

Given a convex set $C \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said **quasiconvex** on C if the sets

$$S_k(f) \cap C = \{x \in C : f(x) \leq k\}$$

are convex for all $k \in \mathbb{R}$.

f is said quasiconcave on C if $-f$ is quasiconvex on C .

Examples

- $f(x) = \sqrt{|x|}$ is quasiconvex on \mathbb{R}
- $f(x_1, x_2) = x_1 x_2$ is quasiconcave on $\{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$
- $f(x) = \log x$ is quasiconvex and quasiconcave