Non-cooperative game theory

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Contents

Game theory is concerned with the analysis of conflictual situations involving various decision makers (called "players") having different aims or objectives.

The decision (called "strategy") of each player has a different cost depending on the strategies chosen by the other players.

Game theory studies the possibility to forecast the strategies that will be chosen by each player in order to minimize his cost.

Definition 1

A non-cooperative game (in normal form) is defined by a set of N players, where each player i has a set X_i of strategies and a cost function $f_i : X_1 \times \cdots \times X_N \to \mathbb{R}$.
The aim of each player i consists in solving the optimization problem

$$\begin{cases} \min \ f_i(x^1, x^2, \dots, x^{i-1}, x^i, x^{i+1}, \dots, x^N) \\ x^i \in X_i \end{cases}$$

We will consider non-cooperative games with two players:

Player 1: $\begin{cases} \min \; f_1(x,y) \\ x \in X \end{cases}$ Player 2: $\begin{cases} \min \; f_2(x,y) \\ y \in Y \end{cases}$

Definition 2

In a two-person non-cooperative game, a pair of strategies $(\bar{x}, \bar{y})$ is a **Nash equilibrium** if

$$f_1(\bar{x},\bar{y}) = \min_{x \in X} \; f_1(x,\bar{y}), \qquad f_2(\bar{x},\bar{y}) = \min_{y \in Y} \; f_2(\bar{x},y).$$

In other words, $(\bar{x}, \bar{y})$ is a **Nash equilibrium** if and only if

- $\bar{x}$ is the best response of player 1 to strategy $\bar{y}$ of player 2
- $\bar{y}$ is the best response of player 2 to strategy $\bar{x}$ of player 1

A **matrix game** is a two-person non-cooperative game where:

- X and Y are finite sets: $X = \{1, \ldots, m\}$, $Y = \{1, \ldots, n\}$;
- $f_2 = -f_1$ (zero-sum game).

It can be represented by a $m \times n$ matrix C, where $f_1(i,j) = c_{ij}$ is the amount of money player 1 pays to player 2 if player 1 chooses strategy i and player 2 chooses strategy j.

Remark 1

Notice that if a Nash equilibrium $(\bar{i}, \bar{j})$ exists it must be

$$f_1(\bar{i}, \bar{j}) = \min_{i \in X} f_1(i, \bar{j})$$

$$f_2(\bar{i}, \bar{j}) = \min_{j \in Y} f_2(\bar{i}, j) = \min_{j \in Y} -f_1(\bar{i}, j) = -\max_{j \in Y} f_1(\bar{i}, j), \quad i.e.,$$

$$f_1(\bar{i}, \bar{j}) = \max_{j \in Y} f_1(\bar{i}, j)$$

Example 1. Find the Nash equilibria of the matrix game

		Player 2		
		1	2	3
Player 1	1	1	-1	0
	2	3	-2	-1
	3	2	3	-2

For player 2, strategy 3 is worse than strategy 1 because his/her profit is less than the one obtained playing strategy 1 for any strategy of player 1. Hence, player 2 will never choose strategy 3, which can be deleted from the game. The game is equivalent to

		Player 2	
		1	2
Player 1	1	1	-1
	2	3	-2
	3	2	3

Now, for player 1 strategy 3 is worse than strategy 1.

The reduced game is

		Player 2	
		1	2
	1	1	-1
Player 1	2	3	-2

For player 2, strategy 2 is worse than strategy 1. Thus, player 2 will always choose strategy 1. The reduced game is

		Player 2
		1
	1	1
Player 1	2	3

Finally, for player 1, strategy 2 is worse than strategy 1. Therefore, player 1 will always choose strategy 1.
Hence $(1, 1)$ is a Nash equilibrium.

Strictly dominated strategies

Definition 3

Given a two-person non-cooperative game, a strategy $x \in X$ is strictly dominated by $\widetilde{x} \in X$ if

$$f_1(x, y) > f_1(\widetilde{x}, y) \qquad \forall\, y \in Y.$$

Similarly, a strategy $y \in Y$ is strictly dominated by $\widetilde{y} \in Y$ if

$$f_2(x, y) > f_2(x, \widetilde{y}) \qquad \forall\, x \in X.$$

Strictly dominated strategies can be deleted from the game.

Exercise 1

a) Find all the Nash equilibria of the following matrix game:

		Player 2				
		1	2	3	4	5
Player 1	1	1	−1	1	−2	−3
	2	2	−2	3	4	0
	3	1	0	1	−3	−4
	4	4	−3	2	−1	−1
	5	5	−2	4	−3	2

b) Prove that if (i, j) and (p, q) are Nash equilibria of a matrix game, then

- $c_{ij} = c_{pq}$
- (i, q) and (p, j) are Nash equilibria as well.

a) Strategies 2 and 5 of player 2 are dominated by Strategy 1 and can be deleted:

Player 1 \ Player 2	1	3	4
1	1	1	-2
2	2	3	4
3	1	1	-3
4	4	2	-1
5	5	4	-3

Strategies 2 and 4 of player 1 are dominated by Strategy 1 (or 3) and can be deleted:

Player 1 \ Player 2	1	3	4
1	1	1	-2
3	1	1	-3
5	5	4	-3

Strategy 4 of player 2 is dominated by the remaining ones and consequently Strategy 5 of player 1 is dominated by the remaining ones and can be deleted:

		Player 2	
		1	3
	1	1	1
Player 1	3	1	1

Clearly all the remaining strategies form pairs of Nash equilibria:

$$(1,1) \quad (1,3) \quad (3,1) \quad (3,3).$$

Two companies $C1$ and $C2$ want to build a new supermarket in one of the districts $D1$, $D2$ and $D3$ of a town.

$$X = \{1, 2, 3\} \quad Y = \{1, 2, 3\}$$

are the sets of the strategies of $C1$ and $C2$ where strategy i corresponds to the decision of building a supermarket in the district D_i.

The company $C1$ estimates that if she decides to build a supermarket in the district D_i, there is a loss due to the fact that the company $C2$ may build a supermarket in the district D_j, given by the cost function:

$$c(i,j) = 100 \frac{1}{1 + d_{ij}}$$

where d_{ij} is the (average) distance between D_i and D_j.

Assume that the distances (in minutes) between the districts are:

- 9 min between $D1$ and $D2$;
- 19 min between $D1$ and $D3$;
- 24 min between $D2$ and $D3$.

Obviously $d_{ii} = 0, \quad i = 1, 2, 3$.

We set $f_1(i,j) = c(i,j)$ and we assume that the company $C2$ has a profit equal to the loss of $C1$, i.e.,

$$f_2(i,j) = -f_1(i,j) \quad i = 1,2,3, \quad j = 1,2,3.$$

The matrix of the game is given by

$$C = \begin{pmatrix} 100 & 10 & 5 \\ 10 & 100 & 4 \\ 5 & 4 & 100 \end{pmatrix}$$

Remark

Notice that in this case no Nash equilibria exist.

Example 2. (Odds and evens)

		Player 2	
		1 (odd)	2 (even)
	1 (odd)	1	-1
Player 1	2 (even)	-1	1

- Are there strictly dominated strategies?
- Are there Nash equilibria?

In both cases the answer is NO

Definition 4

If C is a $m \times n$ matrix game, then a mixed strategy for player 1 is a m-vector of probabilities and we consider
$X = \{x \in \mathbb{R}^m : \ x \geq 0, \quad \sum_{i=1}^m x_i = 1\}$ the set of mixed strategies of player 1.
The vertices of X, i.e., $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ are pure strategies of player 1.

Similarly, a mixed strategy for player 2 is a n-vector of probabilities and $Y = \{y \in \mathbb{R}^n : \ y \geq 0, \quad \sum_{j=1}^n y_j = 1\}$ is the set of mixed strategies of player 2.

The expected costs are $f_1(x, y) = x^\mathsf{T} C y$ (player 1), $f_2(x, y) = -x^\mathsf{T} C y$ (player 2).

Note that

$$x^\mathsf{T} C y = \sum_{i=1}^m \sum_{j=1}^n x_i c_{ij} y_j .$$

Definition 5

If C is a $m \times n$ matrix game, then $(\bar{x}, \bar{y}) \in X \times Y$ is a mixed strategies Nash equilibrium if

$$\max_{y \in Y} \ \bar{x}^{\mathsf{T}} C y = \bar{x}^{\mathsf{T}} C \bar{y} = \min_{x \in X} x^{\mathsf{T}} C \bar{y},$$

or, equivalently,

$$\bar{x}^{\mathsf{T}} C y \leq \bar{x}^{\mathsf{T}} C \bar{y} \leq x^{\mathsf{T}} C \bar{y}, \quad \forall (x, y) \in X \times Y,$$

i.e., $(\bar{x}, \bar{y})$ is a saddle point of the function $f_1(x, y) = x^{\mathsf{T}} C y$ on $X \times Y$.

We recall the definition of a saddle point for a general function $F : X \times Y \to \mathbb{R}$.

Definition

Let $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$.
$(\bar{x}, \bar{y})$ is said to be a saddle point for the function $F : X \times Y \to \mathbb{R}$ if

$$F(\bar{x}, y) \leq F(\bar{x}, \bar{y}) \leq F(x, \bar{y}), \quad \forall (x, y) \in X \times Y. \tag{1}$$

Define

$$\psi(x) := \sup_{y \in Y} F(x,y), \quad x \in X$$

$$\phi(y) := \inf_{x \in X} F(x,y), \quad y \in Y$$

Theorem 1

$(\bar{x}, \bar{y}) \in X \times Y$ satisfies the saddle point condition (1) if and only if

1. $\bar{x}$ is an optimal solution of problem $\min_{x \in X} \psi(x)$;
2. $\bar{y}$ is an optimal solution of problem $\max_{y \in Y} \phi(y)$;
3. $\min_{x \in X} \psi(x) = \max_{y \in Y} \phi(y)$.

Remark

Notice that condition 3 can be written as

$$\min_{x \in X} \sup_{y \in Y} F(x,y) = \max_{y \in Y} \inf_{x \in X} F(x,y).$$

Theorem 2

Let $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$ and assume that

1. X and Y are nonempty compact convex sets;
2. $F(\cdot, y)$ is continuous and quasi convex on X, for every $y \in Y$;
3. $F(x, \cdot)$ is continuous and quasi concave on Y, for every $x \in X$.

Then F admits a saddle point on $X \times Y$.

As a consequence, we obtain the following characterization of a mixed strategies Nash equilibrium.

Corollary 1

Any matrix game has at least a mixed strategies Nash equilibrium.

$(\bar{x}, \bar{y})$ is a mixed strategies Nash equilibrium if and only if

$$\begin{cases} \bar{x} \text{ is an optimal solution of } \min\limits_{x \in X} \max\limits_{y \in Y} x^{\mathsf{T}} C y \\ \bar{y} \text{ is an optimal solution of } \max\limits_{y \in Y} \min\limits_{x \in X} x^{\mathsf{T}} C y \end{cases}$$

Theorem 3

1. The problem $\min_{x \in X} \max_{y \in Y} x^{\mathsf{T}} C y$ is equivalent to the linear programming problem

$$\begin{cases} \min \ v \\ v \geq \sum_{i=1}^{m} c_{ij} x_i \quad \forall \ j = 1, \dots, n \\ x \geq 0, \quad \sum_{i=1}^{m} x_i = 1 \end{cases} \tag{P_1}$$

2. The problem $\max_{y \in Y} \min_{x \in X} x^{\mathsf{T}} C y$ is equivalent to the linear programming problem

$$\begin{cases} \max \ w \\ w \leq \sum_{j=1}^{n} c_{ij} y_j \quad \forall \ i = 1, \dots, m \\ y \geq 0, \quad \sum_{j=1}^{n} y_j = 1 \end{cases} \tag{P_2}$$

Proposition 1

(P_2) is the dual of (P_1).

Remark

Notice that, by strong duality for linear programming it is also possible to prove that any matrix game has at least a mixed strategies Nash equilibrium.

Matlab solution

Let us formulate problem P_1 in matrix form, we obtain:

$$\begin{cases} \min \ v \\ (C^\top, -e_n) \begin{pmatrix} x \\ v \end{pmatrix} \leq 0 \\ (e_m^\top, 0) \begin{pmatrix} x \\ v \end{pmatrix} = 1 \\ x \geq 0, \end{cases} \tag{P_1}$$

where $e_n = (1, \dots, 1)^\top \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $v \in \mathbb{R}$.

Matlab solution

```
C=[.........] % Define C

m = size(C,1);

n = size(C,2);

c=[zeros(m,1);1];

A= [C', -ones(n,1)]; b=zeros(n,1);

Aeq=[ones(1,m),0]; beq=1;

lb= [zeros(m,1);-inf]; ub=[ ];

[sol,Val,exitflag,output,lambda] = linprog(c, A,b, Aeq, beq, lb, ub);

x = sol(1:m)

y = lambda.ineqlin
```

Example 3

(Example 2 continued: odds and evens)

		Player 2 1 (odd)	Player 2 2 (even)
Player 1	1 (odd)	1	−1
	2 (even)	−1	1

$$(P_1)\begin{cases} \min\ v \\ v \geq x_1 - x_2 \\ v \geq -x_1 + x_2 \\ x \geq 0 \\ x_1 + x_2 = 1 \end{cases} \quad \text{is equivalent to} \quad \begin{cases} \min\ v \\ v \geq 2x_1 - 1 \\ v \geq 1 - 2x_1 \\ 0 \leq x_1 \leq 1 \end{cases} \quad \Rightarrow \bar{x} = (1/2, 1/2)$$

$$(P_2)\begin{cases} \max\ w \\ w \leq y_1 - y_2 \\ w \leq -y_1 + y_2 \\ y \geq 0 \\ y_1 + y_2 = 1 \end{cases} \quad \text{is equivalent to} \quad \begin{cases} \max\ w \\ w \leq 2y_1 - 1 \\ w \leq 1 - 2y_1 \\ 0 \leq y_1 \leq 1 \end{cases} \quad \Rightarrow \bar{y} = (1/2, 1/2)$$

Exercise 2

Consider the following matrix game:

$$C = \begin{pmatrix} 7 & 15 & 2 & 3 \\ 4 & 2 & 3 & 10 \\ 5 & 3 & 4 & 12 \end{pmatrix}$$

a) Are there strictly dominated strategies?

b) Are there pure strategies Nash equilibria?

c) Find a mixed strategies Nash equilibrium.

a) Note that Strategy 3 of Player 1 is dominated by Strategy 2, while Strategy 3 of Player 2 is dominated by Strategy 1.
Therefore the third row and the third column can be deleted, i.e., $x_3 = 0$, $y_3 = 0$.
The reduced matrix results:

$$\begin{array}{cc} & \begin{array}{ccc} y_1 & y_2 & y_4 \end{array} \\ C_R = & \begin{pmatrix} 7 & 15 & 3 \\ 4 & 2 & 10 \end{pmatrix} \begin{array}{c} x_1 \\ x_2 \end{array} \end{array}$$

b) We observe that no pure strategy Nash equilibrium exist for the reduced game C_R. Indeed, not any of the minima evaluated on the columns, (i.e., 4,2,3) coincides with the maximum evaluated on the rows (i.e., 15,10).

c) Let us solve the linear programming problem associated with player 1.

$$(P_1) \qquad \begin{cases} \min \ v \\ v \geq 7x_1 + 4x_2 + 5x_3 \\ v \geq 15x_1 + 2x_2 + 3x_3 \\ v \geq 2x_1 + 3x_2 + 4x_3 \\ v \geq 3x_1 + 10x_2 + 12x_3 \\ x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

The previous problem can be solved by the Matlab function "linprog".

Matlab solution

```
C=[7 15 2 3; 4 2 3 10; 5 3 4 12]
m = 3;
n = 4;
c=[0 0 0 1]';
A= [C', -ones(n,1)]; b=[0;0;0;0];
Aeq=[1 1 1,0]; beq=1;
lb= [0;0;0;-inf]; ub=[ ];
[sol,Val,exitflag,output,lambda] = linprog(c, A,b, Aeq, beq, lb, ub);
x = sol(1:m)
y = lambda.ineqlin
```

Optimal solution

x = (0.4, 0.6, 0)

y = (0, 0.35, 0, 0.65)

is a mixed strategies Nash equilibrium.

A **bimatrix game** is a two-person non-cooperative game where:

- the sets of pure strategies are finite, hence the sets of mixed strategies are $X = \{x \in \mathbb{R}^m : \; x \geq 0, \quad \sum_{i=1}^m x_i = 1\}$ and $Y = \{y \in \mathbb{R}^n : \; y \geq 0, \quad \sum_{j=1}^n y_j = 1\}$;
- $f_2 \neq -f_1$ (non-zero-sum game), the cost functions are $f_1(x, y) = x^\mathsf{T} C_1 y$ and $f_2(x, y) = x^\mathsf{T} C_2 y$, where C_1 and C_2 are $m \times n$ matrices.

Theorem 3 (Nash)

Any bimatrix game has at least a mixed strategies Nash equilibrium.

Example: Prisoner's dilemma

Two persons have been arrested for the same severe crime and for small robbery. They are known to be guilty in the robbery but police has no evidence for the severe crime. They are interrogated separately.

Each of the two prisoners can choose: to confess (Strategy 1) or to stay quiet (Strategy 2).

If both stay quiet, they have 2 years for small robbery; if they both confess they are convicted to 5 years; if one and only one confesses, he will be convicted to 1 year and used as witness against the other who will spend 10 years in prison.

Bimatrix game associated with the prisoner's dilemma

$$C_1 = \begin{pmatrix} 5 & 1 \\ 10 & 2 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 5 & 10 \\ 1 & 2 \end{pmatrix}$$

Are there strictly dominated strategies?

Example 5

$$C_1 = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

(a) Are there strictly dominated strategies?

(b) Are there pure strategies Nash equilibria?

(c) Are there mixed strategies Nash equilibria?

(a) No row in C_1 is strictly greater than the other and similarly no column in C_2 is strictly greater than the other.

(b) Let us denote by (a, b) the couple of strategies chosen by the two players.
Consider player 1. By definition of (NE), if possible pure strategies exist, they may be:

$$(1, 1) \quad or \quad (2, 2)$$

since -5 and -1 are the minimum values in columns 1 and 2 of C_1, respectively.

Consider player 2. The cost related to couple (1,1) is -1 which is the minimum on the row 1 in C_2, so (1,1) is a pure strategies Nash equilibrium.

Similarly for the couple (2,2), -5 is the minimum on the row 2 in C_2, so (2,2) is a pure strategies Nash equilibrium.

(c) Are there mixed strategies Nash equilibria? How to compute them?

The considerations made in part (b) lead us to define a procedure to compute mixed strategies Nash equilibria, based on the definition of the best response mappings.

Theorem

If we define the best response mappings $B_1 : Y \to X$ and $B_2 : X \to Y$ as

$$B_1(y) = \left\{ \text{optimal solutions of } \min_{x \in X} \; x^{\mathsf{T}} C_1 y \right\},$$
$$B_2(x) = \left\{ \text{optimal solutions of } \min_{y \in Y} \; x^{\mathsf{T}} C_2 y \right\},$$

then $(\bar{x}, \bar{y})$ is a Nash equilibrium if and only if $\bar{x} \in B_1(\bar{y})$ and $\bar{y} \in B_2(\bar{x})$.

Example 5 (continued)

$$C_1 = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

Given $y \in Y$ we have to solve the problem

$$\begin{cases} \min \ x^{\mathsf{T}} C_1 y = -5x_1 y_1 - x_2 y_2 \\ x \in X \end{cases} \equiv \begin{cases} \min \ (1-6y_1)x_1 + y_1 - 1 \\ 0 \le x_1 \le 1 \end{cases}$$

hence the optimal solution is

$$B_1(y_1) = \begin{cases} 0 & \text{if } y_1 \in [0, 1/6] \\ [0,1] & \text{if } y_1 = 1/6 \\ 1 & \text{if } y_1 \in [1/6, 1] \end{cases}$$

Similarly, given $x \in X$ we have to solve the problem

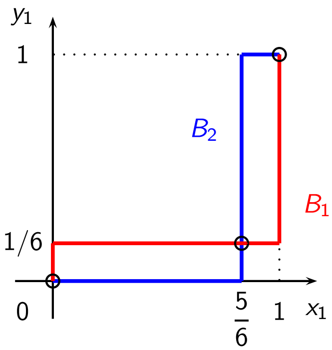
$$\begin{cases} \min \ x^{\mathsf{T}} C_2 y = -x_1 y_1 - 5x_2 y_2 \\ y \in Y \end{cases} \equiv \begin{cases} \min \ (5-6x_1)y_1 + 5x_1 - 5 \\ 0 \le y_1 \le 1 \end{cases}$$

hence the optimal solution is

$$B_2(x_1) = \begin{cases} 0 & \text{if } x_1 \in [0, 5/6] \\ [0,1] & \text{if } x_1 = 5/6 \\ 1 & \text{if } x_1 \in [5/6, 1] \end{cases}$$

Best response mappings

Nash equilibria are given by the intersections of the graphs of the best response mappings B_1 and B_2:



There are 3 Nash equilibria:

- $\bar{x} = (0, 1)$, $\bar{y} = (0, 1)$ (pure strategies)
- $\bar{x} = (5/6, 1/6)$, $\bar{y} = (1/6, 5/6)$ (mixed strategies)
- $\bar{x} = (1, 0)$, $\bar{y} = (1, 0)$ (pure strategies)

KKT conditions for bimatrix games

Consider the optimization problems associated with the two players:

$$P_1(y): \begin{cases} \min \ x^T C_1 y \\ \sum_{i=1}^m x_i = 1 \\ x \geq 0 \end{cases} \qquad P_2(x): \begin{cases} \min \ x^T C_2 y \\ \sum_{j=1}^n y_j = 1 \\ y \geq 0 \end{cases}$$

The KKT conditions for a bimatrix game are obtained by simultaneously considering the single KKT conditions associated with $P_1(y)$ and $P_2(x)$:

$$\begin{cases} C_1 y + \mu_1 e_m \geq 0 \\ x \geq 0, \quad \sum_{i=1}^m x_i = 1 \\ x_i(C_1 y + \mu_1 e_m)_i = 0, \quad i = 1,..,m \end{cases} \qquad \begin{cases} C_2^{\mathsf{T}} x + \mu_2 e_n \geq 0 \\ y \geq 0, \quad \sum_{j=1}^n y_j = 1 \\ y_j(C_2^{\mathsf{T}} x + \mu_2 e_n)_j = 0, \ j = 1,..,n \end{cases}$$

where $e_m = (1, \ldots, 1)^{\mathsf{T}} \in \mathbb{R}^m$ and $e_n = (1, \ldots, 1)^{\mathsf{T}} \in \mathbb{R}^n$.

Remark

Notice that $P_1(y)$ and $P_2(x)$ are parametric linear problems so that the KKT conditions are necessary and sufficient for optimality.

Theorem (KKT conditions for bimatrix games)

$(\bar{x}, \bar{y})$ is a Nash equilibrium if and only if there exist $\mu_1, \mu_2 \in \mathbb{R}$ such that

$$\begin{cases} C_1 \bar{y} + \mu_1 e_m \geq 0 \\ \bar{x} \geq 0, \quad \sum_{i=1}^{m} \bar{x}_i = 1 \\ \bar{x}_i (C_1 \bar{y} + \mu_1 e_m)_i = 0 \quad \forall\, i = 1, \dots, m \\ C_2^{\mathsf{T}} \bar{x} + \mu_2 e_n \geq 0 \\ \bar{y} \geq 0, \quad \sum_{j=1}^{n} \bar{y}_j = 1 \\ \bar{y}_j (C_2^{\mathsf{T}} \bar{x} + \mu_2 e_n)_j = 0 \quad \forall\, j = 1, \dots, n \end{cases} \tag{KS}$$

where $e_m = (1, \dots, 1)^{\mathsf{T}} \in \mathbb{R}^m$.

Exercise 3

Find all the Nash equilibria of the following bimatrix game by means of the KKT conditions:

$$C_1 = \begin{pmatrix} 3 & 3 \\ 4 & 1 \\ 6 & 0 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 3 & 4 \\ 4 & 0 \\ 3 & 5 \end{pmatrix}$$

The KKT conditions are given by:

$$\begin{cases} 3y_1 + 3y_2 + \mu_1 \geq 0 \\ 4y_1 + y_2 + \mu_1 \geq 0 \\ 6y_1 + \mu_1 \geq 0 \\ x \geq 0, \ x_1 + x_2 + x_3 = 1 \\ x_1(3y_1 + 3y_2 + \mu_1) = 0 \\ x_2(4y_1 + y_2 + \mu_1) = 0 \\ x_3(6y_1 + \mu_1) = 0 \\ 3x_1 + 4x_2 + 3x_3 + \mu_2 \geq 0 \\ 4x_1 + 5x_3 + \mu_2 \geq 0 \\ y \geq 0, \quad y_1 + y_2 = 1 \\ y_1(3x_1 + 4x_2 + 3x_3 + \mu_2) = 0 \\ y_2(4x_1 + 5x_3 + \mu_2) = 0 \end{cases}$$

Let us simplify the previous system by eliminating y_2 and x_1, i.e.,

$$y_2 = 1 - y_1, \quad x_1 = 1 - x_2 - x_3.$$

We obtain:

$$
\begin{cases}
3 + \mu_1 \geq 0 \\
3y_1 + 1 + \mu_1 \geq 0 \\
6y_1 + \mu_1 \geq 0 \\
x_2, x_3 \geq 0, \;\; x_2 + x_3 \leq 1 \\
(1 - x_2 - x_3)(3 + \mu_1) = 0 \\
x_2(3y_1 + 1 + \mu_1) = 0 \\
x_3(6y_1 + \mu_1) = 0 \\
3 + x_2 + \mu_2 \geq 0 \\
4 - 4x_2 + x_3 + \mu_2 \geq 0 \\
y_1 \geq 0, \quad y_1 \leq 1 \\
y_1(3 + x_2 + \mu_2) = 0 \\
(1 - y_1)(4 - 4x_2 + x_3 + \mu_2) = 0
\end{cases}
\qquad (S)
$$

We can consider the following three cases:

1. $y_1 = 0$,
2. $y_1 = 1$,
3. $0 < y_1 < 1$.

Case 1: $y_1 = 0$. The system becomes:

$$\begin{cases} 3 + \mu_1 \geq 0 \\ 1 + \mu_1 \geq 0 \\ \mu_1 \geq 0 \\ x_2, x_3 \geq 0, \; x_2 + x_3 \leq 1 \\ 1 - x_2 - x_3 = 0 \\ x_2 = 0 \\ x_3 \mu_1 = 0 \\ 3 + \mu_2 \geq 0 \\ 4 - 4x_2 + x_3 + \mu_2 \geq 0 \\ y_1 = 0, \\ 4 + x_3 + \mu_2 = 0 \end{cases}$$

The previous system is clearly impossible. Indeed, $x_3 = 1$ and by the last equation $\mu_2 = -5 \not\geq -3$.

Case 2: $y_1 = 1$. The system (S) becomes:

$$\begin{cases} \mu_1 \geq -3 \\ x_2, x_3 \geq 0, \;\; x_2 + x_3 \leq 1 \\ (1 - x_2 - x_3)(\mu_1 + 3) = 0 \\ x_2 = 0 \\ x_3 = 0 \\ 3 + \mu_2 \geq 0 \\ 4 + \mu_2 \geq 0 \\ 3 + \mu_2 = 0, \end{cases}$$

The previous system admits the solution $\mu_1 = \mu_2 = -3$, $x_2 = x_3 = 0$ which leads to the Nash Equilibrium:

$$\bar{x} = (1, 0, 0) \quad \bar{y} = (1, 0)$$

Case 3: $0 < y_1 < 1$. The system (S) becomes:

$$\begin{cases} 3 + \mu_1 \geq 0 \\ 3y_1 + 1 + \mu_1 \geq 0 \\ 6y_1 + \mu_1 \geq 0 \\ x_2, x_3 \geq 0, \ x_2 + x_3 \leq 1 \\ (1 - x_2 - x_3)(3 + \mu_1) = 0 \\ x_2(3y_1 + 1 + \mu_1) = 0 \\ x_3(6y_1 + \mu_1) = 0 \\ 3 + x_2 + \mu_2 = 0 \\ 4 - 4x_2 + x_3 + \mu_2 = 0 \\ y_1 > 0, \quad y_1 < 1 \end{cases} \tag{S3}$$

Note that $x_2 \neq 0$, indeed, otherwise, by the last two equalities

$$\mu_2 = -3, \quad x_3 = -1$$

Then system (S3) becomes:

$$\begin{cases} 3 + \mu_1 \geq 0 \\ 6y_1 + \mu_1 \geq 0 \\ x_2 > 0, \;\; x_3 \geq 0, \;\; x_2 + x_3 \leq 1 \\ (1 - x_2 - x_3)(3 + \mu_1) = 0 \\ 3y_1 + 1 + \mu_1 = 0 \\ x_3(6y_1 + \mu_1) = 0 \\ 3 + x_2 + \mu_2 = 0 \\ 4 - 4x_2 + x_3 + \mu_2 = 0 \\ y_1 > 0, \quad y_1 < 1 \end{cases}$$

We discuss the cases (a) $x_3 = 0$ and (b) $0 < x_3 \leq 1$.
(a) For $x_3 = 0$, by the last two equalities we obtain:

$$\mu_2 = -\frac{16}{5}, \quad x_2 = \frac{1}{5}$$

Consequently,

$$\mu_1 = -3, \quad y_1 = \frac{2}{3}$$

Therefore

$$\bar{x} = (\frac{4}{5}, \frac{1}{5}, 0) \quad \bar{y} = (\frac{2}{3}, \frac{1}{3}) \text{ is a Nash Equilibrium.}$$

(b) $0 < x_3 \leq 1$. The previous system becomes:

$$\begin{cases} 3 + \mu_1 \geq 0 \\ x_2 > 0, \ x_3 > 0, \ x_2 + x_3 \leq 1 \\ (1 - x_2 - x_3)(3 + \mu_1) = 0 \\ 3y_1 + 1 + \mu_1 = 0 \\ 6y_1 + \mu_1 = 0 \\ 3 + x_2 + \mu_2 = 0 \\ 4 - 4x_2 + x_3 + \mu_2 = 0 \\ 0 < y_1 < 1 \end{cases}$$

From the equalities

$$3y_1 + 1 + \mu_1 = 0, \quad 6y_1 + \mu_1 = 0$$

we obtain: $\mu_1 = -2, \quad y_1 = \frac{1}{3}$.
Since $\mu_1 = -2$, by the first equality it follows $x_2 + x_3 = 1$, i.e., $x_3 = 1 - x_2$ and substituting in the last equality, we have:

$$5 - 5x_2 + \mu_2 = 0, \quad 3 + x_2 + \mu_2 = 0,$$

which lead to

$$x_2 = \frac{1}{3}, \quad \mu_2 = -\frac{10}{3}$$

Therefore

$$\bar{x} = (0, \frac{1}{3}, \frac{2}{3}) \quad \bar{y} = (\frac{1}{3}, \frac{2}{3}) \text{ is a Nash Equilibrium.}$$

Let us solve system (KS) by using Matlab. To this aim we trasform it into an equivalent optimization problem defined on the set $X \times Y \times \mathbb{R}^2$. Note that (KS) can be written as:

$$\begin{cases} C_1\bar{y} + \mu_1 e_m \geq 0 \\ \bar{x} \geq 0, \quad \sum_{i=1}^m \bar{x}_i = 1 \\ \bar{x}^T(C_1\bar{y} + \mu_1 e_m) = 0 \\ C_2^{\mathsf{T}}\bar{x} + \mu_2 e_n \geq 0 \\ \bar{y} \geq 0, \quad \sum_{j=1}^n \bar{y}_j = 1 \\ \bar{y}^T(C_2^{\mathsf{T}}\bar{x} + \mu_2 e_n) = 0 \end{cases} \tag{KS}$$

where $e_m = (1, \dots, 1)^{\mathsf{T}} \in \mathbb{R}^m$.

Then

Proposition

$(\bar{x}, \bar{y}, \mu_1, \mu_2)$ is a solution of (KS) if and only if it is an optimal solution of the quadratic programming problem

$$\begin{cases} \min \psi(x, y, \mu_1, \mu_2) = [(x^T(C_1 y + \mu_1 e_m) + y^T(C_2^{\mathsf{T}} x + \mu_2 e_n)] \\ C_1 y + \mu_1 e_m \geq 0 \\ x \geq 0, \quad \sum_{i=1}^m x_i = 1 \\ C_2^{\mathsf{T}} x + \mu_2 e_n \geq 0 \\ y \geq 0, \quad \sum_{j=1}^n y_j = 1 \end{cases} \qquad (QP)$$

and $\psi(\bar{x}, \bar{y}, \mu_1, \mu_2) = 0$.

Remark

We observe that by Theorem 3 it follows that there exists at least one Nash Equilibrium for a bimatrix game so that the optimal value of (QP) is zero.

We have:

$$\nabla\psi(x,y,\mu_1,\mu_2) = \begin{pmatrix} C_1 y + \mu_1 e_m + C_2 y \\ C_1^T x + C_2^T x + \mu_2 e_n \\ e_m^T x \\ e_n^T y \end{pmatrix}$$

The Hessian matrix of ψ is given by:

$$H = \begin{pmatrix} O_{m\times m} & C_1 + C_2 & e_m & O_{m\times 1} \\ C_1^T + C_2^T & O_{n\times n} & O_{n\times 1} & e_n \\ e_m^T & O_{1\times n} & 0 & 0 \\ O_{1\times m} & e_n^T & 0 & 0 \end{pmatrix}$$

Let us write the constraints in the standard matrix form:

$$A_{in} = \begin{pmatrix} -C_2^T & O_{n\times n} & O_{n\times 1} & -e_n \\ O_{m\times m} & -C_1 & -e_m & O_{m\times 1} \end{pmatrix} \quad b_{in} = \begin{pmatrix} O_{n\times 1} \\ O_{m\times 1} \end{pmatrix}$$

$$A_{eq} = \begin{pmatrix} 1 & \cdot & \cdot & 1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & \cdot & \cdot & 0 & 1 & \cdot & \cdot & 1 & 0 & 0 \end{pmatrix} \quad b_{eq} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let

$$w^T = (x^T, y^T) \quad \mu^T = (\mu_1, \mu_2).$$

Problem (QP) can be written in the following matrix form:

$$\begin{cases} \min \psi(w, \mu) = \frac{1}{2}(w^T, \mu^T) H \begin{pmatrix} w \\ \mu \end{pmatrix} \\ A_{in} \begin{pmatrix} w \\ \mu \end{pmatrix} \leq b_{in} \\ A_{eq} \begin{pmatrix} w \\ \mu \end{pmatrix} = b_{eq} \\ w \geq 0, \end{cases} \qquad (QP)$$

Matlab commands

```
C1=[........]; C2=[........];

[m,n] = size(C1);

H=[zeros(m,m),C1+C2,ones(m,1), zeros(m,1);
C1'+C2',zeros(n,n),zeros(n,1),ones(n,1); ones(1,m), zeros(1,n+2);
zeros(1,m),ones(1,n),0,0];

X0=[............]; % m + n + 2 vector

Ain=[-C2', zeros(n,n),zeros(n,1),-ones(n,1);zeros(m,m),
-C1,-ones(m,1),zeros(m,1)]; bin=zeros(n+m,1);

Aeq=[ones(1,m),zeros(1,n+2);zeros(1,m),ones(1,n),0,0]; beq=[1;1];

LB=[zeros(m+n,1);-Inf;-Inf]; UB=[ones(m+n,1);Inf;Inf];

[sol,fval,exitflag,output]=fmincon(@(X) 0.5*X'*H*X, X0, Ain,bin,
Aeq,beq,LB,UB)

x = sol(1:m)

y = sol(m+1:m+n)
```

Exercise 4

Consider the problem defined in Exercise 3.

(a) Find by Matlab a mixed strategies Nash equilibrium.

(b) Try to find different Nash equilibria by varying the starting point $X0$ (multistart approach).

Exercise 5

Consider the problem defined in Example 5.

(a) Find by Matlab a mixed strategies Nash equilibrium.

(b) Try to find different Nash equilibria by a multistart approach.

For further exercises see the web page of prof. Mauro Passacantando:

https://people.unipi.it/mauro_passacantando/wp-content/uploads/sites/208/2020/05/exercises_games.pdf

Answer the questions from points (a) to (c) and (e).

Now, we consider a general two-person non-cooperative game

Player 1: $\begin{cases} \min_x \ f_1(x,y) \\ g_i^1(x) \leq 0 \ \ \forall \ i = 1,..,p \end{cases}$ Player 2: $\begin{cases} \min_y \ f_2(x,y) \\ g_j^2(y) \leq 0 \ \ \forall \ j = 1,..,q \end{cases}$

where f_1, g^1, f_2 and g^2 are continuously differentiable.
The game is said convex if the optimization problem of each player is convex.

Theorem

If the feasible regions $X = \{x \in \mathbb{R}^m : \ g_i^1(x) \leq 0 \ \ i = 1,\ldots,p\}$ and $Y = \{y \in \mathbb{R}^n : \ g_j^2(y) \leq 0 \ \ j = 1,\ldots,q\}$ are closed, convex and bounded, the cost function $f_1(\cdot, y)$ is quasiconvex for any $y \in Y$ and $f_2(x,\cdot)$ is quasiconvex for any $x \in X$, then there exists at least a Nash equilibrium.

Remark

The quasiconvexity of the cost functions is crucial. For example, the game defined as $X = Y = [0,1]$, $f_1(x,y) = -x^2 + 2xy$, $f_2(x,y) = y(1-2x)$ has no Nash equilibrium.

KKT conditions

Theorem

- If $(\bar{x}, \bar{y})$ is a Nash equilibrium and the Abadie constraints qualification holds both in $\bar{x}$ and $\bar{y}$, then there exist $\lambda^1 \in \mathbb{R}^p$, $\lambda^2 \in \mathbb{R}^q$ such that

$$\begin{cases} \nabla_x f_1(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \lambda_i^1 \nabla g_i^1(\bar{x}) = 0 \\ \lambda^1 \geq 0, \qquad g^1(\bar{x}) \leq 0 \\ \lambda_i^1 g_i^1(\bar{x}) = 0, \quad i = 1, \ldots, p \\ \nabla_y f_2(\bar{x}, \bar{y}) + \sum_{j=1}^{q} \lambda_j^2 \nabla g_j^2(\bar{y}) = 0 \\ \lambda^2 \geq 0, \qquad g^2(\bar{y}) \leq 0 \\ \lambda_j^2 g_j^2(\bar{y}) = 0, \quad j = 1, \ldots, q \end{cases}$$

- If $(\bar{x}, \bar{y}, \lambda^1, \lambda^2)$ solves the above system and the game is convex, then $(\bar{x}, \bar{y})$ is a Nash equilibrium.

Exercise 6

Consider the following convex game:

Player 1: $\begin{cases} \min\limits_{x} \ x^2 - x(2y+2) \\ -3 \le x \le 2 \end{cases}$ Player 2: $\begin{cases} \min\limits_{y} \ (x+2)(1-y) \\ -1 \le y \le 3 \end{cases}$

(a) Find the Nash equilibria by using KKT conditions.

(b) Find the Nash equilibria by using the best response mappings.

(a) The KKT conditions are:

$$\begin{cases} 2x - 2y - 2 - \lambda_1^1 + \lambda_2^1 = 0 \\ \lambda_1^1(-x-3) = \lambda_2^1(x-2) = 0 \\ \lambda^1 \ge 0, \ -3 \le x \le 2 \\ -x - 2 - \lambda_1^2 + \lambda_2^2 = 0 \\ \lambda_1^2(-1-y) = \lambda_2^2(y-3) = 0 \\ \lambda^2 \ge 0, \ -1 \le y \le 3 \end{cases} \qquad (KKT)$$

where $\lambda^1 = (\lambda_1^1, \lambda_2^1), \quad \lambda^2 = (\lambda_1^2, \lambda_2^2)$.

Consider the variable x; we have the following cases:

I) $-3 < x < 2$;

II) $x = -3$;

III) $x = 2$.

Case I) System (KKT) becomes:

$$\begin{cases} 2x - 2y - 2 = 0 \\ \lambda^1 = 0, -3 < x < 2 \\ -x - 2 - \lambda_1^2 + \lambda_2^2 = 0 \\ \lambda_1^2(-1 - y) = \lambda_2^2(y - 3) = 0 \\ \lambda^2 \geq 0, \; -1 \leq y \leq 3 \end{cases} \tag{$KKT1$}$$

By the first equation $x = y + 1$ and substituting in the other relations:

$$\begin{cases} x = y + 1 \\ \lambda^1 = 0, -4 < y < 1 \\ -y - 3 - \lambda_1^2 + \lambda_2^2 = 0 \\ \lambda_1^2(-1 - y) = \lambda_2^2(y - 3) = 0 \\ \lambda^2 \geq 0, \; -1 \leq y \leq 3 \end{cases}$$

Notice that $y < 1 \quad \Rightarrow \lambda_2^2 = 0$, so that the system becomes:

$$\begin{cases} x = y + 1 \\ \lambda^1 = 0, \\ -y - 3 - \lambda_1^2 = 0 \\ \lambda_1^2(-1 - y) = 0 \\ \lambda_1^2 \geq 0, \lambda_2^2 = 0 \ \ -1 \leq y < 1 \end{cases}$$

which turns out to be impossible as it can be easily checked (consider that $\lambda_1^2 = -y - 3$).

Case II) $x = -3$. System (KKT) becomes:

$$\begin{cases} -6 - 2y - 2 - \lambda_1^1 = 0, \\ \lambda_1^1 \geq 0, \lambda_2^1 = 0, \quad x = -3 \\ 1 - \lambda_1^2 + \lambda_2^2 = 0 \\ \lambda_1^2(-1 - y) = \lambda_2^2(y - 3) = 0 \\ \lambda^2 \geq 0, \ -1 \leq y \leq 3 \end{cases} \qquad (KKT2)$$

The previous system is impossible, indeed by the first equation

$$-2y - 8 = \lambda_1^1 \geq 0 \Rightarrow y \leq -4$$

which contradicts $y \geq -1$.

Case III) $x = 2$. System (KKT) becomes:

$$\begin{cases} 4 - 2y - 2 - \lambda_2^1 = 0, \\ \lambda_1^1 = 0, \lambda_2^1 \geq 0, \quad x = 2 \\ -4 - \lambda_1^2 + \lambda_2^2 = 0 \\ \lambda_1^2(-1 - y) = \lambda_2^2(y - 3) = 0 \\ \lambda^2 \geq 0, \ -1 \leq y \leq 3 \end{cases} \qquad (KKT3)$$

Clearly λ_1^2 and λ_2^2 cannot be simultaneously 0 (so we have the two possibilities $y = -1$ and $y = 3$). By the first equation

$$2y - 2 = \lambda_2^1 \geq 0 \Rightarrow y \geq 1$$

so that we have the solution $y = 3$, $x = 2$ with $\lambda_1^1 = \lambda_1^2 = 0$, $\lambda_2^1 = \lambda_2^2 = 4$.

Therefore $(\bar{x}, \bar{y}) = (2, 3)$ is a Nash equilibrium.

(b) Let us solve the problem by means of the best response mappings $B_1(y)$ and $B_2(x)$.

In order to find $B_1(y)$, given $y \in Y$ we have to solve the problem

$$P_1(y) : \begin{cases} \min\limits_{x} \; x^2 - x(2y+2) \\ -3 \leq x \leq 2 \end{cases}$$

Notice that the unconstrained minimum of $P_1(y)$ is in the point $x(y) = y + 1$. Then $x(y)$ is the global minimum of $P_1(y)$ if

$$-3 \leq y + 1 \leq 2, \quad i.e., \quad -4 \leq y \leq 1.$$

Similarly, the global minimum point of $P_1(y)$ is

- $x = -3$, for $y + 1 < -3$, $\quad i.e., \quad y < -4$;
- $x = 2$, $\quad$ for $y + 1 > 2$, $\quad i.e., \quad y > 1$.

Hence the optimal solutions of $P_1(y)$ are

$$B_1(y) = \begin{cases} y+1 & \text{if } y \in [-1,1] \\ 2 & \text{if } y \in [1,3] \end{cases}$$

In order to find $B_2(x)$, given $x \in X$ we have to solve the problem

$$P_2(x): \begin{cases} \min\limits_{y} \ (x+2)(1-y) \\ -1 \leq y \leq 3 \end{cases}$$

It is easy to see that the optimal solutions of $P_2(x)$ are:

- $y = 3$, for $x + 2 > 0$,
- $y \in [-1, 3]$ for $x = -2$,
- $y = -1$ for $x + 2 < 0$.

Hence,

$$B_2(x) = \begin{cases} -1 & \text{if } x \in [-3, -2) \\ [-1, 3] & \text{if } x = -2 \\ 3 & \text{if } x \in (-2, 2] \end{cases}$$

By drawing the respective graphs of B_1 and B_2, it can be checked that the only couple $(\bar{x}, \bar{y})$ such that $\bar{x} \in B_1(\bar{y})$, $\bar{y} \in B_2(\bar{x})$ is $(\bar{x}, \bar{y}) = (2, 3)$.