

10 - Noncooperative game theory

Mauro Passacantando

Department of Computer Science, University of Pisa
mauro.passacantando@unipi.it

Optimization Methods and Game Theory
Master of Science in Artificial Intelligence and Data Engineering
University of Pisa – A.Y. 2020/21

Game theory

Game theory deals with the analysis of conflictual situations among different decision makers (players) which have different interests.

The decision (strategy) of each player can produce different results depending on the strategies chosen by the other players.

Game theory studies the possibility to forecast the strategies that will be chosen by each player which is assumed to be “rational”.

Definition

A noncooperative game in normal form is given by a set of N players, each player i has a set Ω_i of strategies and a cost function $f_i : \Omega_1 \times \cdots \times \Omega_N \rightarrow \mathbb{R}$.

The aim of each player i consists in solving the optimization problem

$$\begin{cases} \min f_i(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N) \\ x_i \in \Omega_i \end{cases}$$

Nash equilibrium

From now on, we will consider noncooperative games with two players:

$$\text{Player 1: } \begin{cases} \min f_1(x, y) \\ x \in X \end{cases}$$

$$\text{Player 2: } \begin{cases} \min f_2(x, y) \\ y \in Y \end{cases}$$

How to define an *equilibrium* notion?

Nash equilibrium

From now on, we will consider noncooperative games with two players:

$$\text{Player 1: } \begin{cases} \min f_1(x, y) \\ x \in X \end{cases}$$

$$\text{Player 2: } \begin{cases} \min f_2(x, y) \\ y \in Y \end{cases}$$

How to define an *equilibrium* notion?

Definition

In a two-person noncooperative game, a pair of strategies (\bar{x}, \bar{y}) is a **Nash equilibrium** if no player can decrease his/her cost by unilateral deviation, i.e.,

$$f_1(\bar{x}, \bar{y}) = \min_{x \in X} f_1(x, \bar{y}),$$

$$f_2(\bar{x}, \bar{y}) = \min_{y \in Y} f_2(\bar{x}, y).$$

Equivalent definition: (\bar{x}, \bar{y}) is a **Nash equilibrium** if and only if

- ▶ \bar{x} is the best response of player 1 to strategy \bar{y} of player 2
- ▶ \bar{y} is the best response of player 2 to strategy \bar{x} of player 1

Matrix games

A **matrix game** is a two-person noncooperative game where:

- ▶ X and Y are finite sets: $X = \{1, \dots, m\}$, $Y = \{1, \dots, n\}$;
- ▶ $f_2 = -f_1$ (**zero-sum game**).

It can be represented by a $m \times n$ matrix C , where c_{ij} is the amount of money player 1 pays to player 2 if player 1 chooses strategy i and player 2 chooses strategy j .

Example.

| | | Player 2 | | |
|----------|---|----------|----|----|
| | | 1 | 2 | 3 |
| Player 1 | 1 | 1 | -1 | 0 |
| | 2 | 3 | -2 | -1 |
| | 3 | 2 | 3 | -2 |

Are there Nash equilibria?

Example

| | | Player 2 | | |
|----------|---|----------|----|----|
| | | 1 | 2 | 3 |
| Player 1 | 1 | 1 | -1 | 0 |
| | 2 | 3 | -2 | -1 |
| | 3 | 2 | 3 | -2 |

For player 2, strategy 3 is worse than strategy 1 because his/her profit is less than the one obtained playing strategy 1 for any strategy of player 1. Hence, player 2 will never choose strategy 3, which can be deleted from the game. The game is equivalent to

| | | Player 2 | |
|----------|---|----------|----|
| | | 1 | 2 |
| Player 1 | 1 | 1 | -1 |
| | 2 | 3 | -2 |
| | 3 | 2 | 3 |

Now, for player 1 strategy 3 is worse than strategy 1.

Example

The reduced game is

| | | Player 2 | |
|----------|---|----------|----|
| | | 1 | 2 |
| Player 1 | 1 | 1 | -1 |
| | 2 | 3 | -2 |

For player 2, strategy 2 is worse than strategy 1. Thus, player 2 will always choose strategy 1. The reduced game is

| | | Player 2 |
|----------|---|----------|
| | | 1 |
| Player 1 | 1 | 1 |
| | 2 | 3 |

Finally, for player 1, strategy 2 is worse than strategy 1. Therefore, player 1 will always choose strategy 1.

Hence $(1, 1)$ is a Nash equilibrium.

Strictly dominated strategies

Definition

Given a two-person noncooperative game, a strategy $x \in X$ is strictly dominated by $\tilde{x} \in X$ if

$$f_1(x, y) > f_1(\tilde{x}, y) \quad \forall y \in Y.$$

Similarly, a strategy $y \in Y$ is strictly dominated by $\tilde{y} \in Y$ if

$$f_2(x, y) > f_2(x, \tilde{y}) \quad \forall x \in X.$$

Strictly dominated strategies can be deleted from the game.

Exercises

Exercise 10.1.

a) Find all the Nash equilibria of the following matrix game:

| | | Player 2 | | | | |
|----------|---|----------|----|---|----|----|
| | | 1 | 2 | 3 | 4 | 5 |
| Player 1 | 1 | 1 | -1 | 1 | -2 | -3 |
| | 2 | 2 | -2 | 3 | 4 | 0 |
| | 3 | 1 | 0 | 1 | -3 | -4 |
| | 4 | 4 | -3 | 2 | -1 | -1 |
| | 5 | 5 | -2 | 4 | -3 | 2 |

b) Prove that if (i, j) and (p, q) are Nash equilibria of a matrix game, then

- ▶ $c_{ij} = c_{pq}$
- ▶ (i, q) and (p, j) are Nash equilibria as well.

Mixed strategies

Example. (Odds and evens)

| | | Player 2 | |
|----------|----------|------------|-------------|
| | | 1 (odd) | 2 (even) |
| Player 1 | 1 (odd) | 1 | -1 |
| | 2 (even) | -1 | 1 |

Are there strictly dominated strategies?

Mixed strategies

Example. (Odds and evens)

| | | Player 2 | |
|----------|----------|----------|--------|
| | | 1 | 2 |
| | | (odd) | (even) |
| Player 1 | 1 (odd) | 1 | -1 |
| | 2 (even) | -1 | 1 |

Are there strictly dominated strategies? NO

Mixed strategies

Example. (Odds and evens)

| | | Player 2 | |
|----------|----------|------------|-------------|
| | | 1 (odd) | 2 (even) |
| Player 1 | 1 (odd) | 1 | -1 |
| | 2 (even) | -1 | 1 |

Are there strictly dominated strategies? NO

Are there Nash equilibria?

Mixed strategies

Example. (Odds and evens)

| | | Player 2 | |
|----------|----------|------------|-------------|
| | | 1 (odd) | 2 (even) |
| Player 1 | 1 (odd) | 1 | -1 |
| | 2 (even) | -1 | 1 |

Are there strictly dominated strategies? NO

Are there Nash equilibria? NO

Mixed strategies

Example. (Odds and evens)

| | | Player 2 | |
|----------|----------|------------|-------------|
| | | 1 (odd) | 2 (even) |
| Player 1 | 1 (odd) | 1 | -1 |
| | 2 (even) | -1 | 1 |

Are there strictly dominated strategies? NO

Are there Nash equilibria? NO

Definition

If C is a $m \times n$ matrix game, then a mixed strategy for player 1 is a m -vector of probabilities and we consider $X = \{x \in \mathbb{R}^m : x \geq 0, \sum_{i=1}^m x_i = 1\}$ the set of mixed strategies of player 1.

The vertices of X , i.e., $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ are pure strategies of player 1.

$Y = \{y \in \mathbb{R}^n : y \geq 0, \sum_{j=1}^n y_j = 1\}$ is the set of mixed strategies of player 2.

The expected costs are $f_1(x, y) = x^T C y$ (player 1), $f_2(x, y) = -x^T C y$ (player 2).

Mixed strategies Nash equilibria

Definition

If C is a $m \times n$ matrix game, then $(\bar{x}, \bar{y}) \in X \times Y$ is a mixed strategies Nash equilibrium if

$$\max_{y \in Y} \bar{x}^T C y = \bar{x}^T C \bar{y} = \min_{x \in X} x^T C \bar{y},$$

i.e., (\bar{x}, \bar{y}) is a saddle point of the function $x^T C y$.

Mixed strategies Nash equilibria

Definition

If C is a $m \times n$ matrix game, then $(\bar{x}, \bar{y}) \in X \times Y$ is a mixed strategies Nash equilibrium if

$$\max_{y \in Y} \bar{x}^T C y = \bar{x}^T C \bar{y} = \min_{x \in X} x^T C \bar{y},$$

i.e., (\bar{x}, \bar{y}) is a saddle point of the function $x^T C y$.

Theorem

(\bar{x}, \bar{y}) is a mixed strategies Nash equilibrium if and only if

$$\begin{cases} \bar{x} \text{ is an optimal solution of } \min_{x \in X} \max_{y \in Y} x^T C y \\ \bar{y} \text{ is an optimal solution of } \max_{y \in Y} \min_{x \in X} x^T C y \end{cases}$$

Mixed strategies Nash equilibria

Theorem

1. The problem $\min_{x \in X} \max_{y \in Y} x^T Cy$ is equivalent to the linear programming problem

$$\begin{cases} \min v \\ v \geq \sum_{i=1}^m c_{ij} x_i \quad \forall j = 1, \dots, n \\ x \geq 0, \quad \sum_{i=1}^m x_i = 1 \end{cases} \quad (P_1)$$

2. The problem $\max_{y \in Y} \min_{x \in X} x^T Cy$ is equivalent to the linear programming problem

$$\begin{cases} \max w \\ w \leq \sum_{j=1}^n c_{ij} y_j \quad \forall i = 1, \dots, m \\ y \geq 0, \quad \sum_{j=1}^n y_j = 1 \end{cases} \quad (P_2)$$

3. (P_2) is the dual of (P_1) .

Corollary. Any matrix game has at least a mixed strategies Nash equilibrium.

Mixed strategies Nash equilibria

Example. (Odds and evens)

| | | Player 2 | |
|----------|----------|----------|----|
| | | 1 | 2 |
| Player 1 | 1 (odd) | 1 | -1 |
| | 2 (even) | -1 | 1 |

$$(P_1) \begin{cases} \min v \\ v \geq x_1 - x_2 \\ v \geq -x_1 + x_2 \\ x \geq 0 \\ x_1 + x_2 = 1 \end{cases} \text{ is equivalent to } \begin{cases} \min v \\ v \geq 2x_1 - 1 \\ v \geq 1 - 2x_1 \\ 0 \leq x_1 \leq 1 \end{cases} \Rightarrow \bar{x} = (1/2, 1/2)$$

$$(P_2) \begin{cases} \max w \\ w \leq y_1 - y_2 \\ w \leq -y_1 + y_2 \\ y \geq 0 \\ y_1 + y_2 = 1 \end{cases} \text{ is equivalent to } \begin{cases} \max w \\ w \leq 2y_1 - 1 \\ w \leq 1 - 2y_1 \\ 0 \leq y_1 \leq 1 \end{cases} \Rightarrow \bar{y} = (1/2, 1/2)$$

Mixed strategies Nash equilibria

Exercise 10.2. (Penalty kick) Consider the following matrix game:

| | | Player 2 (forward) | | |
|--------------------------|---|-----------------------|----|---|
| | | L | C | R |
| Player 1 (goalkeeper) | L | 1 | 2 | 3 |
| | C | 3 | -1 | 3 |
| | R | 3 | 2 | 1 |

- a) Are there strictly dominated strategies?
- b) Are there pure strategies Nash equilibria?
- c) Find a mixed strategies Nash equilibrium.

Bimatrix games

A **bimatrix game** is a two-person noncooperative game where:

- ▶ the sets of pure strategies are finite, hence the sets of mixed strategies are $X = \{x \in \mathbb{R}^m : x \geq 0, \sum_{i=1}^m x_i = 1\}$ and $Y = \{y \in \mathbb{R}^n : y \geq 0, \sum_{j=1}^n y_j = 1\}$;
- ▶ $f_2 \neq -f_1$ (**non-zero-sum game**), the cost functions are $f_1(x, y) = x^T C_1 y$ and $f_2(x, y) = x^T C_2 y$, where C_1 and C_2 are $m \times n$ matrices.

Theorem (Nash)

Any bimatrix game has at least a mixed strategies Nash equilibrium.

Bimatrix games

A **bimatrix game** is a two-person noncooperative game where:

- ▶ the sets of pure strategies are finite, hence the sets of mixed strategies are $X = \{x \in \mathbb{R}^m : x \geq 0, \sum_{i=1}^m x_i = 1\}$ and $Y = \{y \in \mathbb{R}^n : y \geq 0, \sum_{j=1}^n y_j = 1\}$;
- ▶ $f_2 \neq -f_1$ (**non-zero-sum game**), the cost functions are $f_1(x, y) = x^T C_1 y$ and $f_2(x, y) = x^T C_2 y$, where C_1 and C_2 are $m \times n$ matrices.

Theorem (Nash)

Any bimatrix game has at least a mixed strategies Nash equilibrium.

Example. (Prisoner's dilemma)

$$C_1 = \begin{pmatrix} 5 & 1 \\ 10 & 2 \end{pmatrix} \quad C_2 = \begin{pmatrix} 5 & 10 \\ 1 & 2 \end{pmatrix}$$

Are there strictly dominated strategies?

Best response mappings

Example. (Battle of the buddies)

$$C_1 = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \quad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

Are there strictly dominated strategies?

Are there pure strategies Nash equilibria?

Are there mixed strategies Nash equilibria? How to compute them?

Best response mappings

Example. (Battle of the buddies)

$$C_1 = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \quad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

Are there strictly dominated strategies?

Are there pure strategies Nash equilibria?

Are there mixed strategies Nash equilibria? How to compute them?

Theorem

If we define the best response mappings $B_1 : Y \rightarrow X$ and $B_2 : X \rightarrow Y$ as

$$B_1(y) = \left\{ \text{optimal solutions of } \min_{x \in X} x^T C_1 y \right\},$$
$$B_2(x) = \left\{ \text{optimal solutions of } \min_{y \in Y} x^T C_2 y \right\},$$

then (\bar{x}, \bar{y}) is a Nash equilibrium if and only if $\bar{x} \in B_1(\bar{y})$ and $\bar{y} \in B_2(\bar{x})$.

Best response mappings

Example (continued).

$$C_1 = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \quad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

Given $y \in Y$ we have to solve the problem

$$\begin{cases} \min_{x \in X} x^T C_1 y = -5x_1 y_1 - x_2 y_2 \\ \end{cases} \equiv \begin{cases} \min (1 - 6y_1)x_1 + y_1 - 1 \\ 0 \leq x_1 \leq 1 \end{cases}$$

hence the optimal solution is

$$B_1(y_1) = \begin{cases} 0 & \text{if } y_1 \in [0, 1/6] \\ [0, 1] & \text{if } y_1 = 1/6 \\ 1 & \text{if } y_1 \in [1/6, 1] \end{cases}$$

Similarly, given $x \in X$ we have to solve the problem

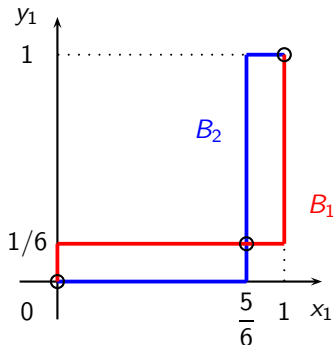
$$\begin{cases} \min_{y \in Y} x^T C_2 y = -x_1 y_1 - 5x_2 y_2 \\ \end{cases} \equiv \begin{cases} \min (5 - 6x_1)y_1 + 5x_1 - 5 \\ 0 \leq y_1 \leq 1 \end{cases}$$

hence the optimal solution is

$$B_2(x_1) = \begin{cases} 0 & \text{if } x_1 \in [0, 5/6] \\ [0, 1] & \text{if } x_1 = 5/6 \\ 1 & \text{if } x_1 \in [5/6, 1] \end{cases}$$

Best response mappings

Nash equilibria are given by the intersections of the graphs of the best response mappings B_1 and B_2 :



There are 3 Nash equilibria:

- ▶ $\bar{x} = (0, 1)$, $\bar{y} = (0, 1)$ (pure strategies)
- ▶ $\bar{x} = (5/6, 1/6)$, $\bar{y} = (1/6, 5/6)$ (mixed strategies)
- ▶ $\bar{x} = (1, 0)$, $\bar{y} = (1, 0)$ (pure strategies)

KKT conditions for Nash equilibria

Theorem

(\bar{x}, \bar{y}) is a Nash equilibrium if and only if there are $\mu_1, \mu_2 \in \mathbb{R}$ such that

$$\left\{ \begin{array}{l} C_1 \bar{y} + \mu_1 e \geq 0 \\ \bar{x} \geq 0, \quad \sum_{i=1}^m \bar{x}_i = 1 \\ \bar{x}_i (C_1 \bar{y} + \mu_1 e)_i = 0 \quad \forall i = 1, \dots, m \\ C_2^T \bar{x} + \mu_2 e \geq 0 \\ \bar{y} \geq 0, \quad \sum_{j=1}^n \bar{y}_j = 1 \\ \bar{y}_j (C_2^T \bar{x} + \mu_2 e)_j = 0 \quad \forall j = 1, \dots, n \end{array} \right.$$

where $e = (1, \dots, 1)^T$.

Exercise 10.3. Find all the Nash equilibria of the following bimatrix game by exploiting the KKT conditions:

$$C_1 = \begin{pmatrix} 3 & 3 \\ 4 & 1 \\ 6 & 0 \end{pmatrix} \quad C_2 = \begin{pmatrix} 3 & 4 \\ 4 & 0 \\ 3 & 5 \end{pmatrix}$$

Characterization of Nash equilibria

Theorem

Assume that $C_1 < 0$ and $C_2 < 0$ (this is not a restrictive assumption).

- If (\bar{x}, \bar{y}) is a Nash equilibrium then there are $u > 0$, $v > 0$ such that $\tilde{x} = \bar{x}/u$ and $\tilde{y} = \bar{y}/v$ solve the following system:

$$\begin{cases} \tilde{x} \geq 0, & C_1 \tilde{y} + e \geq 0, & \tilde{x}_i (C_1 \tilde{y} + e)_i = 0 & \forall i = 1, \dots, m \\ \tilde{y} \geq 0, & C_2^T \tilde{x} + e \geq 0, & \tilde{y}_j (C_2^T \tilde{x} + e)_j = 0 & \forall j = 1, \dots, n \end{cases} \quad (S)$$

- If (\tilde{x}, \tilde{y}) solves system (S), with $\tilde{x} \neq 0$ and $\tilde{y} \neq 0$, then $\left(\frac{\tilde{x}}{\sum_{i=1}^m \tilde{x}_i}, \frac{\tilde{y}}{\sum_{j=1}^n \tilde{y}_j} \right)$ is a Nash equilibrium.

Characterization of Nash equilibria

Define the polyhedra

$$P = \left\{ (x_1, \dots, x_m) : \begin{array}{ll} x_i \geq 0 & \forall i = 1, \dots, m \\ (C_2^T x + e)_j \geq 0 & \forall j = m+1, \dots, m+n \end{array} \right\}$$

$$Q = \left\{ (y_{m+1}, \dots, y_{m+n}) : \begin{array}{ll} (C_1 y + e)_i \geq 0 & \forall i = 1, \dots, m \\ y_j \geq 0 & \forall j = m+1, \dots, m+n \end{array} \right\}$$

Theorem

- ▶ (\tilde{x}, \tilde{y}) solves system (S) if and only if $\tilde{x} \in P$, $\tilde{y} \in Q$ and for any $k = 1, \dots, m+n$ either the k -th constraint of P is active in \tilde{x} or the k -th constraint of Q is active in \tilde{y} .
- ▶ If the vertices of P and Q are non-degenerate and (\tilde{x}, \tilde{y}) solves system (S), then \tilde{x} is a vertex of P and \tilde{y} is a vertex of Q .

Therefore, if $C_1 < 0$, $C_2 < 0$ and vertices of P and Q are non-degenerate, then we can find all the Nash equilibria analyzing all the pairs (x, y) of vertices of P and Q , checking if for any $k = 1, \dots, m+n$ either the k -th constraint of P is active in x or the k -th constraint of Q is active in y .

Characterization of Nash equilibria

Example. (Battle of the buddies)

$$C_1 = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix} \quad C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

Since the elements of C_1 and C_2 are not all negative, we can reformulate the game setting $(C_1)_{ij} = (C_1)_{ij} - 1$ and $(C_2)_{ij} = (C_2)_{ij} - 1$:

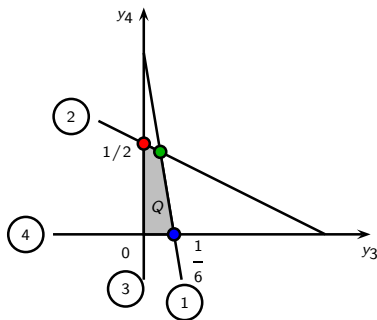
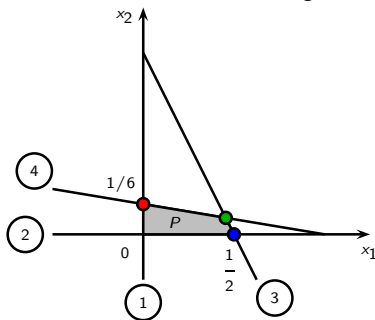
$$C_1 = \begin{pmatrix} -6 & -1 \\ -1 & -2 \end{pmatrix} \quad C_2 = \begin{pmatrix} -2 & -1 \\ -1 & -6 \end{pmatrix}$$

Characterization of Nash equilibria

$$P = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, -2x_1 - x_2 + 1 \geq 0, -x_1 - 6x_2 + 1 \geq 0\}$$

$$Q = \{(y_3, y_4) : -6y_3 - y_4 + 1 \geq 0, -y_3 - 2y_4 + 1 \geq 0, y_3 \geq 0, y_4 \geq 0\}$$

Vertices of P and Q are non-degenerate.



Solutions of system (S)

$$x = (0, 1/6) \text{ (active const. 1, 4)}$$

$$y = (0, 1/2) \text{ (active const. 2, 3)}$$

$$x = (5/11, 1/11) \text{ (active const. 3, 4)}$$

$$y = (1/11, 5/11) \text{ (active const. 1, 2)}$$

$$x = (1/2, 0) \text{ (active const. 2, 3)}$$

$$y = (1/6, 0) \text{ (active const. 1, 4)}$$

Nash equilibria

$$x = (0, 1) \quad y = (0, 1)$$

$$x = (5/6, 1/6) \quad y = (1/6, 5/6)$$

$$x = (1, 0) \quad y = (1, 0)$$

Lemke-Howson algorithm

When $C_1 < 0$, $C_2 < 0$ and vertices of P and Q are non-degenerate, a Nash equilibrium can be found by using the Lemke-Howson algorithm:

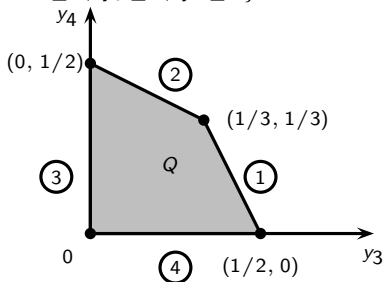
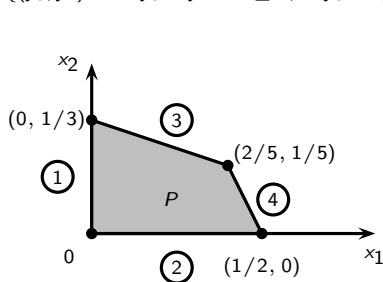
1. Set $x = 0$, $y = 0$. Define $I_x = \{1, \dots, m\}$, $I_y = \{m+1, \dots, m+n\}$. Choose an exiting index $h \in \{1, \dots, m\}$.
2. In the polyhedron P , move from x along the edge given by constraints in $I_x \setminus \{h\}$ to the adjacent vertex x' .
3. Find the entering index k corresponding to the new active constraint at x' . Set $x = x'$ and $I_x = (I_x \setminus \{h\}) \cup \{k\}$.
4. If $k \notin I_y$ then STOP: (x, y) solves (S) \rightarrow find a Nash equilibrium.
else set $h = k$ and in the polyhedron Q move from y along the edge given by constraints in $I_y \setminus \{h\}$ to the adjacent vertex y' .
5. Find the entering index k corresponding to the new active constraint at y' . Set $y = y'$ and $I_y = (I_y \setminus \{h\}) \cup \{k\}$.
6. If $k \notin I_x$ then STOP: (x, y) solves (S) \rightarrow find a Nash equilibrium.
else set $h = k$ and go to step 2.

Lemke-Howson algorithm

Example. Consider the following bimatrix game: $C_1 = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$ $C_2 = \begin{pmatrix} -1 & -2 \\ -3 & -1 \end{pmatrix}$

$$P = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, -x_1 - 3x_2 + 1 \geq 0, -2x_1 - x_2 + 1 \geq 0\}$$

$$Q = \{(y_3, y_4) : -2y_3 - y_4 + 1 \geq 0, -y_3 - 2y_4 + 1 \geq 0, y_3 \geq 0, y_4 \geq 0\}$$



| Iter | h | k | x | I_x | y | I_y |
|-------------|-----|-----|------------|--------|------------|--------|
| 1 | | | (0, 0) | {1, 2} | (0, 0) | {3, 4} |
| 2 | 1 | 4 | (1/2, 0) | {2, 4} | (0, 0) | {3, 4} |
| 3 | 4 | 2 | (1/2, 0) | {2, 4} | (0, 1/2) | {2, 3} |
| 4 | 2 | 3 | (2/5, 1/5) | {3, 4} | (0, 1/2) | {2, 3} |
| 5 | 3 | 1 | (2/5, 1/5) | {3, 4} | (1/3, 1/3) | {1, 2} |
| Nash equil. | | | (2/3, 1/3) | | (1/2, 1/2) | |

Convex games

Now, we consider a two-person noncooperative game

$$\text{Player 1: } \begin{cases} \min_x f_1(x, y) \\ g_i^1(x) \leq 0 \quad \forall i = 1, \dots, p \end{cases} \quad \text{Player 2: } \begin{cases} \min_y f_2(x, y) \\ g_j^2(y) \leq 0 \quad \forall j = 1, \dots, q \end{cases}$$

where f_1 , g^1 , f_2 and g^2 are continuously differentiable.

The game is said convex if the optimization problem of each player is convex.

Theorem

If the feasible regions $X = \{x \in \mathbb{R}^m : g_i^1(x) \leq 0 \quad i = 1, \dots, p\}$ and $Y = \{y \in \mathbb{R}^n : g_j^2(y) \leq 0 \quad j = 1, \dots, q\}$ are closed, convex and bounded, the cost function $f_1(\cdot, y)$ is quasiconvex for any $y \in Y$ and $f_2(x, \cdot)$ is quasiconvex for any $x \in X$, then there exists at least a Nash equilibrium.

The **quasiconvexity** of the cost functions is crucial.

Example. The game defined as $X = Y = [0, 1]$, $f_1(x, y) = -x^2 + 2xy$, $f_2(x, y) = y(1 - 2x)$ has no Nash equilibrium.

KKT conditions

Theorem

- If (\bar{x}, \bar{y}) is a Nash equilibrium and the Abadie constraints qualification holds both in \bar{x} and \bar{y} , then there are $\lambda^1 \in \mathbb{R}^p$, $\lambda^2 \in \mathbb{R}^q$ such that

$$\left\{ \begin{array}{l} \nabla_x f_1(\bar{x}, \bar{y}) + \sum_{i=1}^p \lambda_i^1 \nabla g_i^1(\bar{x}) = 0 \\ \lambda^1 \geq 0, \quad g^1(\bar{x}) \leq 0 \\ \lambda_i^1 g_i^1(\bar{x}) = 0, \quad i = 1, \dots, p \\ \nabla_y f_2(\bar{x}, \bar{y}) + \sum_{j=1}^q \lambda_j^2 \nabla g_j^2(\bar{y}) = 0 \\ \lambda^2 \geq 0, \quad g^2(\bar{y}) \leq 0 \\ \lambda_j^2 g_j^2(\bar{y}) = 0, \quad j = 1, \dots, q \end{array} \right.$$

- If $(\bar{x}, \bar{y}, \lambda^1, \lambda^2)$ solves the above system and the game is convex, then (\bar{x}, \bar{y}) is a Nash equilibrium.

KKT conditions

Exercise 10.4. Consider the following convex game:

$$\text{Player 1: } \begin{cases} \min_x x^2 - x(4y + 6) \\ -4 \leq x \leq 6 \end{cases} \quad \text{Player 2: } \begin{cases} \min_y (x + 3)(4 - y) \\ -5 \leq y \leq 5 \end{cases}$$

- a) Find all the Nash equilibria by using KKT conditions.
- b) Find all the Nash equilibria by using the best response mappings.

Merit functions

Merit functions allow reformulating the Nash equilibrium problem into an equivalent optimization problem.

Assume that the game is convex. Consider the Nikaido-Isoda function

$$f(x, y, u, v) = f_1(u, y) - f_1(x, y) + f_2(x, v) - f_2(x, y),$$

where $x, u \in \mathbb{R}^m$ and $y, v \in \mathbb{R}^n$. Define the **gap function** as

$$\psi(x, y) = \max_{u \in X, v \in Y} [-f(x, y, u, v)].$$

Then:

- ▶ The problem defining ψ is convex
- ▶ $\psi(x, y) \geq 0$ for any $(x, y) \in X \times Y$
- ▶ (\bar{x}, \bar{y}) is a Nash equilibrium if and only if $(\bar{x}, \bar{y}) \in X \times Y$ and $\psi(\bar{x}, \bar{y}) = 0$

Therefore, finding Nash equilibria is equivalent to solve the constrained optimization problem

$$\begin{cases} \min \psi(x, y) \\ (x, y) \in X \times Y \end{cases}$$

Merit functions

In general ψ is not differentiable, but it is possible to regularize it.

Given a parameter $\alpha > 0$, the **regularized gap function** is defined as

$$\psi_{\alpha}(x, y) = \max_{u \in X, v \in Y} \left[-f(x, y, u, v) - \frac{\alpha}{2} \|(x, y) - (u, v)\|^2 \right].$$

Then:

- ▶ The problem defining ψ_{α} is convex and has a unique optimal solution
- ▶ ψ_{α} is continuously differentiable
- ▶ $\psi_{\alpha}(x, y) \geq 0$ for any $(x, y) \in X \times Y$
- ▶ (\bar{x}, \bar{y}) is a Nash equilibrium if and only if $(\bar{x}, \bar{y}) \in X \times Y$ and $\psi_{\alpha}(\bar{x}, \bar{y}) = 0$.

Therefore, finding Nash equilibria is equivalent to solve the smooth constrained optimization problem

$$\begin{cases} \min & \psi_{\alpha}(x, y) \\ & (x, y) \in X \times Y \end{cases}$$

Merit functions

It is possible to reformulate the problem of finding Nash equilibria as an **unconstrained** optimization problem.

Given two parameters $\beta > \alpha > 0$, the **D-gap function** is defined as

$$\psi_{\alpha,\beta}(x, y) = \psi_{\alpha}(x, y) - \psi_{\beta}(x, y).$$

Then:

- ▶ $\psi_{\alpha,\beta}$ is continuously differentiable
- ▶ $\psi_{\alpha,\beta}(x, y) \geq 0$ for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$
- ▶ (\bar{x}, \bar{y}) is a Nash equilibrium if and only if $\psi_{\alpha,\beta}(\bar{x}, \bar{y}) = 0$.

Therefore, finding Nash equilibria is equivalent to solve the smooth, unconstrained optimization problem

$$\begin{cases} \min & \psi_{\alpha,\beta}(x, y) \\ & (x, y) \in \mathbb{R}^m \times \mathbb{R}^n \end{cases}$$

Merit functions

Exercise 10.5. Consider the following bimatrix game:

$$C_1 = \begin{pmatrix} 3 & 3 \\ 4 & 1 \\ 6 & 0 \end{pmatrix} \quad C_2 = \begin{pmatrix} 3 & 4 \\ 4 & 0 \\ 3 & 5 \end{pmatrix}$$

- a) Implement in MATLAB the gap function, the regularized gap function and the D-gap function.
- b) Exploit the gap function ψ to check if the point $w = (x, y)$, where $x = (1/3, 1/3, 1/3)$ and $y = (1/2, 1/2)$, is a Nash equilibrium.
- c) Find a local minimum of the regularized gap function ψ_α with $\alpha = 1$ starting from w .
- d) Try to find a global minimum of the regularized gap function ψ_α with a multistart approach.
- e) Try to find a global minimum of the D-gap function $\psi_{\alpha,\beta}$, with $\alpha = 1$ and $\beta = 10$, with a multistart approach.