1) Consider the (univariate) regression task defined by the data

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	-284	-168	-88	-38	-12	-4	-8	-18	-28	-32	-24

Write a `Matlab` code that solves the polynomial interpolation problem for degree $k = 3$ and succinctly describe it (the fundamental steps, leaving aside the unnecessary details). Then run it on the data above and report the obtained solution (the coefficients of the polynomial) and the Mean Square Error (MSE) between the interpolated polynomial and the original data, commenting on the quality of the obtained interpolation.

SOLUTION

We want to construct the third degree polynomial

$$c_3p^3 + c_2p^2 + c_1p + c_0$$

that best approximates the given y values on the given x ones in the Mean Square Error sense, i.e., minimizing

$$\sum_{i=1}^{11} (\, y_i - c_3x_i^3 + c_2x_i^2 + c_1x_i + c_0 \,)^2$$

In order to solve this polynomial interpolation problem we first construct, given the vector $x \in \mathbb{R}^11$, the matrix $X \in \mathbb{R}^{4\times11}$ whose i-th row is

$$[\, 1 \,,\, x_i \,,\, x_i^2 \,,\, x_i^3 \,]$$

This can be done, e.g., with the `Matlab` code

```
X = ones( 11 , 4 );
X( : , 2 ) = x;
for i = 3 : 4
    X( : , i ) = X( : , i - 1  ) .* x;
end
```

assuming that $x \in \mathbb{R}^{11}$ is a column vector (i.e., `size( x ) = { 11 , 1 }`). Then we solve the Linear Least Square problem

$$\min \left\{ \, \| \, Xc - y \, \|_2^2 \; : \; c \in \mathbb{R}^4 \, \right\}$$

This requires computing the pseudo-inverse of X, and it can be performed with just the single command

```
c = y' / X';
```

that produces a row vector $c \in \mathbb{R}^4$ (i.e., `size( c ) = { 1 , 4 }`) containing the coefficients of the powers of x in increasing order, i.e., $c[1] = c_0$ is the constant coefficient, $c[2] = c_1$ the first-order coefficient, and so on. So doing produces the answer

```
c = -4.0000     1.0000    -6.0000     1.0000
```

corresponding to the polynomial

$$p^3 - 6p^2 + p - 4$$

To compute the MSE one can just compute the values of the polynomial as

```
z = x.^3 - 6 * x.^2 + x - 4
```

and then the error as `y - z`: this results in the all-0 vector, and hence in a MSE of 0. In other words, the obtained polynomial exactly fits the provided data.

2) Solve the box-constrained quadratic optimization problem

$$\min\left\{\, x^T Q x/2 + qx \,:\, 0 \leq x \leq u \,\right\} \quad \text{with data} \quad Q = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad , \quad q = \begin{bmatrix} -3 \\ -9 \end{bmatrix} \quad , \quad u = \begin{bmatrix} 2.0 \\ 3.1 \end{bmatrix}$$

using the Frank-Wolfe (conditional gradient) method starting from the point "in the middle of the box" $x^0 = u/2$. Write a `Matlab` code that implements the algorithm and succinctly describe it (the fundamental steps, leaving aside the unnecessary details), then run it on the instance above and report the obtained (approximate) solution (detailing the stopping condition and what tolerances have been used in it). Optionally comment on the number of iterations and the convergence rate of the approach.

SOLUTION

Starting from the initial iterate `x = u / 2`, the Frank-Wolfe algorithm computes the current function value and gradient as

```
v = 0.5 * x' * Q * x + q' * x;
g = Q * x + q;
```

One then needs to solve the problem

$$y = \operatorname{argmin}\left\{\, gx \,:\, 0 \leq x \leq u \,\right\}$$

which trivially decomposes for each component: $y_i = 0$ if $g_i \geq 0$, and $y_i = u_i$ otherwise. This can be obtained, e.g., by the simple code

```
y = zeros( n , 1 );
ind = g < 0;
y( ind ) = u( ind );
```

It is then useful to compute the lower bound corresponding to the linearization, i.e., $l = f(x) + g(y - x)$, or simply

```
l = v + g' * ( y - x );
```

We know that $v \geq f_* \geq l$, hence one can stop when the upper estimate on the relative error of x given by

$$(\, v - l \,) \,/\, |\, l \,|$$

is smaller than some threshold, say `1e-6` (some adjustment is clearly needed for the case where $l = 0$ can happen, but this is not an issue in our instance). Since l is not necessarily nondecreasing over the iterations one may also want to keep the best (largest) value of l and use it for the comparison. Conversely, we know that v is necessarily decreasing: indeed, if the algorithms does not stop we know that `d = y - x` is a descent direction. One can then compute the unconstrained minimum along it with the usual formula

```
alpha = ( - g' * d ) / d' * Q * d;
```

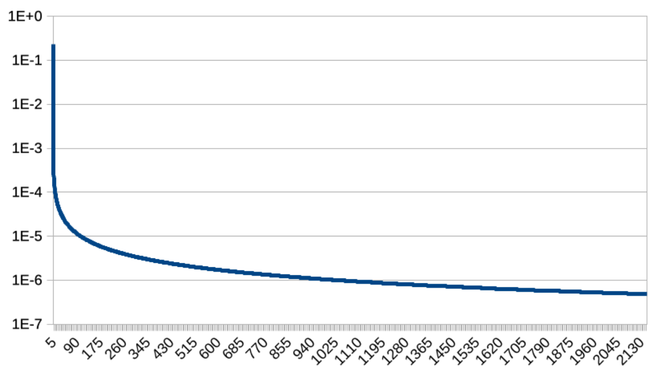
(where in general one should account for the possibility of the denumerator is zero, but again this is not happening in our instance). However, any step greater than 1 would lead to an unfeasible solution, so one actually has to take the minimum between `alpha` as computed above and 1. The corresponding next iterate

```
x = x + alpha * d;
```

is guaranteed to be feasible and the process can be repeated, eventually leading to convergence. With a tolerance of `1e-6` the process should take around 2000 iterations and return a solution close to

```
x = [ 0.0022 ; 2.9993 ] , v = -1.34999935e+01 , l = -1.35000070e+01
```

For the optional part one just has to have all the values of v at every iteration written down and compute the gap $(v - f_*)/f_*$ (where clearly $f_* = -13.5$). Then, plotting the resulting values in logarithmic scale one should obtain something like



The fact that the curve is not "straight" but it "flattens" (the derivative is negative but it clearly goes to 0) means that the convergence of the Frank-Wolfe approach in this instance is sublinear, as the theory predicts.

3) Consider the following multiobjective optimization problem:

$$\begin{cases} \min\ (x_1 + 2x_2^2,\ x_1^2 + 4x_2^2) \\ x_1^2 + 2x_2^2 - 4 \le 0 \end{cases}$$

(a) Prove that the problem admits a Pareto minimum point.

(b) Find the set of all weak Pareto minima.

(c) Find a suitable subset of Pareto minima.

SOLUTION

(a) Since the feasible set X is compact and the objective function is continuous then the problem admits a (Pareto) minimum point.

(b) - (c) We preliminarly observe that the problem is convex, since the objective and the constraint functions are convex. Therefore the set of weak minima coincides with the set of solutions of the scalarized problems (P_{α_1}), where $0 \le \alpha_1 \le 1$, i.e.

$$\begin{cases} \min\ \alpha_1(x_1 + 2x_2^2) + (1-\alpha_1)(x_1^2 + 4x_2^2) =: \psi_{\alpha_1}(x) \\ x_1^2 + 2x_2^2 - 4 \le 0 \end{cases}$$

For $0 \le \alpha_1 < 1$, ψ_{α_1} is strongly convex so that (P_{α_1}) admits a unique optimal solution which is a minimum of (P).

We note that (P_{α_1}) is convex and differentiable and ACQ holds at any $x \in X$; then the KKT system provides a necessary and sufficient condition for an optimal solution of (P_{α_1}). KKT system is given by:

$$\begin{cases} (1-\alpha_1)2x_1 + \alpha_1 + 2\lambda x_1 = 0 \\ 2(4 - 2\alpha_1)x_2 + 4\lambda x_2 = 0 \\ \lambda(x_1^2 + 2x_2^2 - 4) = 0 \\ 0 \le \alpha_1 \le 1,\ \lambda \ge 0 \\ x_1^2 + 2x_2^2 - 4 \le 0 \end{cases}$$

We obtain:

$$\begin{cases} x_1 = -\dfrac{\alpha_1}{2(1-\alpha_1+\lambda)},\ 1 - \alpha_1 + \lambda \ne 0 \\ x_2 = 0 \\ \lambda(x_1^2 - 4) = 0 \\ 0 \le \alpha_1 \le 1,\ \lambda \ge 0 \\ x_1^2 - 4 \le 0 \end{cases}$$

Notice that $1 - \alpha_1 + \lambda = 0$ implies $\lambda = 0$, $\alpha_1 = 1$, which is impossible by the first equation in the KKT system.

From the complementarity condition $\lambda(x_1^2 - 4) = 0$, we have the two cases: $I)$ $\lambda = 0$, $II)$ $x_1 = \pm 2$.

In case $I)$, $\lambda = 0$, $0 \le \alpha_1 < 1$, we obtain:

$$x_1 = -\frac{\alpha_1}{2(1-\alpha_1)},\ x_2 = 0$$

so that $Min(P) \supseteq \{(x_1, x_2) : -2 \le x_1 \le 0,\ x_2 = 0\}$, noticing that P_0 has a unique optimal solution.

In case $II)$, for $x_1 = 2$, the first equation is impossible, taking into account that $\lambda \ge 0$.

For $x_1 = -2$, we obtain the point $(-2, 0)$, already considered.

It remains to consider the case $\alpha_1 = 1$, $\lambda \ne 0$. The system becomes:

$$\begin{cases} 1 + 2\lambda x_1 = 0 \\ 4x_2(1+\lambda) = 0 \\ x_1^2 + 2x_2^2 - 4 = 0 \\ \lambda > 0 \end{cases}$$

with the unique solution $(x_1, x_2) = (-2, 0)$, $\lambda = \frac{1}{4}$.

In conclusion: $Weak\ Min(P) = Min(P) = \{(x_1, x_2) : -2 \le x_1 \le 0,\ x_2 = 0\}$.

4) Consider the following bimatrix game:

$$C_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 0 \end{pmatrix}$$

(a) Find the set of pure strategies Nash equilibria, if any. Alternatively, show that no pure strategy Nash equilibrium exists.

(b) Find a mixed strategy Nash equilibrium.

SOLUTION

(a) Strategy 3 of Player 1 is dominated by Strategy 1, so that row 3 in the two matrices can be deleted. Now Strategy 3 of Player 2 is dominated by Strategy 1 and column 3 in the two matrices can be deleted. The reduced game is given by the matrices

$$C_1^R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C_2^R = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$$

Now, it is simple to show that no pure strategies Nash equilibria exist. Indeed, considering Player 1, the possible couples of pure strategies Nash equilibria could be $(2,1)$ and $(1,2)$, while considering Player 2 the pure strategies Nash equilibria could be $(1,1)$ and $(2,2)$. Since there are no common couples, no pure strategies Nash equilibria exist.

This will also be shown in part (b) in the wider context of mixed strategies Nash equilibria.

(b) Consider the reduced game obtained in (a). The optimization problem associated with Player 1 is

$$\begin{cases} \min\ x^T C_1^R y = x_1 y_1 + x_2 y_2 \\ x_1 + x_2 = 1 \\ x_1, x_2 \geq 0 \end{cases} \equiv \begin{cases} \min\ (2y_1 - 1)x_1 + 1 - y_1 \\ 0 \leq x_1 \leq 1 \end{cases} \qquad (P_1(y_1))$$

Then, the best response mapping associated with $P_1(y_1)$ is:

$$B_1(y_1) = \begin{cases} 0 & \text{if } y_1 \in [1/2, 1) \\ [0,1] & \text{if } y_1 = 1/2 \\ 1 & \text{if } y_1 \in [0, 1/2) \end{cases}$$

Similarly, the optimization problem associated with Player 2 is

$$\begin{cases} \min\ x^T C_2^R y = 2x_2 y_1 + (2x_1 + x_2) y_2 \\ y_1 + y_2 = 1 \\ y_1, y_2 \geq 0 \end{cases} \equiv \begin{cases} \min\ (1 - 3x_1)y_1 + x_1 + 1 \\ 0 \leq y_1 \leq 1 \end{cases} \qquad (P_2(x_1))$$

Then, the best response mapping associated with $P_2(x_1)$ is:

$$B_2(x_1) = \begin{cases} 0 & \text{if } x_1 \in [0, 1/3) \\ [0,1] & \text{if } x_1 = 1/3 \\ 1 & \text{if } x_1 \in (1/3, 1] \end{cases}$$

The only couple $(\bar{x}_1, \bar{y}_1)$ such that $\bar{x}_1 \in B_1(\bar{y}_1)$ and $\bar{y}_1 \in B_2(\bar{x}_1)$ is $\bar{x}_1 = \frac{1}{3}, \bar{y}_1 = \frac{1}{2}$, so that

$$\bar{x} = (\frac{1}{3}, \frac{2}{3}, 0), \quad \bar{y} = (\frac{1}{2}, \frac{1}{2}, 0)$$

is the unique mixed strategy Nash equilibrium for the given game, which also shows that no pure strategies Nash equilibrium exists.