Gaussian Process

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This note aims to cover some materials on the Gaussian process. The primary references are Gaussian Process for Machine Learning by C. E. Rasmussen and CS-E4895 by Arno Solin.

1 Multivariate Normal Distribution

1.1 Linear transformation theorem for the multivariate normal distribution

Let x follow a multivariate normal distribution:

$$x \sim \mathcal{N}(\mu, \Sigma) \tag{1}$$

Then, any affine transformation of x is also multivariate normally distributed:

$$y = Ax + b \sim \mathcal{N}(A\mu + b, A\Sigma A^\top) \tag{2}$$

Proof:

The moment-generating function of random vector x is

$$M_x(t) = \mathbb{E}[\exp(t^T x)] \tag{3}$$

and therefore, the moment-generating function of the random vector y is given by

$$\begin{aligned}
M_y(t) &= \mathbb{E}\left[\exp(t^T(Ax+b))\right] \\
&= \mathbb{E}[\exp(t^\top Ax)\exp(t^\top b)] \\
&= \exp(t^\top b)\mathbb{E}[\exp(t^\top Ax)] \\
&= \exp(t^\top b)M_x(A^\top t)
\end{aligned} \tag{4}$$

The moment-generating function of the multivariate normal distribution is

$$M_x(t) = \exp(t^\top \mu + \frac{1}{2}t^\top \Sigma t) \tag{5}$$

and therefore the moment-generating function of random vector y becomes

$$M_y(t) = \exp(t^\top(A\mu + b) + \frac{1}{2}t^\top A\Sigma A^\top t) \tag{6}$$

Since the moment-generating function and the probability density function of a random variable are equivalent, this demonstrates that y is following a multivariate normal distribution with mean $A\mu + b$ and covariance $A\Sigma A^\top$.