Optimal Control Plan to Build Emergency Fund

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Abstract

A dynamic mathematical model that aims to accumulate a desired amount of capital in a savings bank account over a set number of years is presented. The model is constructed using the principles of Optimal Control and provides for variations in a number of factors affecting the amount of savings needed such as the monthly income of an individual, the interest rates that the federal banks offer, the national inflation rate and the monthly compounds of the money already in the account. Using these parameters, the goal was to construct and maximize a Hamiltonian function that would help us to determine how much capital is to be deposited in the savings account every month that would allow us to reach the desired sum at the end of all the years.

Introduction

Financial security has many levels. From managing your budget to how much money you are setting aside for retirement, there are many dimensions to your financial security. A person considered to have strong financial health has a steady income, seldom experiences changes in expenses, collects strong returns on investments and has an increasing cash balance. This financial state is what many if not most people aspire to obtain. There are three common suggestions to acquire strong financial health: create a budget, pay off debts and invest in an emergency fund [1]. This paper will focus on building an emergency savings fund that would enable a person to survive an unexpected personal financial crisis or accumulate enough capital for a secure retirement.

Most recently the outbreak of a global pandemic has sent many down the slopes of uncertainty. Before looking into the damages done by the Covid-19 pandemic, let's take a look at the overall financial health of Americans up to the pandemic. In March 2020, the National Foundation for Credit Counseling (NFCC) and Discover Financial Services collaborated with the Harris Poll to administer the 2020 Financial Literacy Survey. About

62% of American adults had credit card debt in the last twelve months and 27% did not pay their bills on time. The survey also showed that 58% struggle to minimize their debt and of those people about 19% claim that it is due to unforeseen financial emergencies. Before one can think of working on setting up their emergency fund, they need to be able to pay off their debt and be able to cover their monthly expenses without incurring more debt. These statistics show how many Americans are not on that level of minimal financial security. Some cheerful statistics that the report uncovered was that about 70% of American adults had some money saved for retirement and 70% reported having non-retirement savings. There was also a decrease from 2007 in the percentage of people concerned for saving for retirement and having emergency funds, showing that many Americans have been taking more precautions to assure their financial security. [2]

A survey was conducted by NPR, The Robert Wood Johnson Foundation, and Harvard T.H. Chan School of Public Health in July-August 2020 to outlay the financial problems facing Americans across the country after about six months of the outbreak of the pandemic. Results showed that of the 46% of Americans who are facing serious financial problems, 31% used all of most of their savings and about 10% had no savings initially. About one fifth of people reported that they were unable to make necessary payments on their mortgage, rent, credit card bills, and/or loans. In addition, about 46% of American adults reported that at least one person in the household lost some or all their income by either getting fired, losing their business, decrease in hours or wages or being furloughed. The annual income of Americans also displayed an impact, showing that about one fifth of Americans that made an annual income of $100,000 or more experienced serious financial problems, while about 54% of Americans that make less than that experienced serious financial problems. [3]

It is quite evident that regardless of where Americans secured their financial health prior to the Coronavirus Pandemic, such an unexpected occurrence has taken its toll on everyone regardless of prior economic standing. In the past few months, COVID-19 has put America at risk for future economic disaster as well. It is predicted that the country is facing a three percent decrease in GDP, due to the $16 trillion cost that will be shown over the next ten years [3]. If the country is unable to make a quick recovery through pandemic-related legislation, the American people can expect a future depression. This will make it increasingly difficult for an even greater percentage of Americans to be financially secure and possess the capability to manage their lively expenses. Financial preparation is vital for tragedies like we are currently facing, and those that may occur in the future. An emergency savings fund arms an individual with a power of financial security. The model that will be discussed is derived using optimal control theory and the construction of a specific Hamiltonian function. Several outside factors will be taken into consideration, such as a steady increase in the monthly income of the individual, the monthly U.S inflation rates, and monthly Constant Maturity Treasury rates.

Optimal Control Theory

Optimal Control Theory is a modern approach to dynamic optimization that uses control variables to either maximize or minimize a functional. There is a control variable denoted by u(t), and a state variable denoted by y(t). The control variable is subject to the researcher's choice and directly effects the output value of the state variable of interest. The state variable is a more dynamic version of the independent variable in static optimization problems and the state variable is like the dependent variable, just with more dimensions of output. The optimization of the control variables leads us to the derivation of the optimal state variables. There are many distinguishable characteristics of the optimal control. [4]

The relationship between the control and state variable is defined by the constraint $\dot{y} = f(t, y, u)$. In addition, at the initial time $t = 0$, and $y(0)$ so that $\dot{y}(0) = f(0, y(0), u(0))$, where we can choose $u(0)$. This makes the direction taken by the state variable, $\dot{y}$ clearly determined by the control variable. The control variable can be either in an open or closed and compact set, however, when control variable is in an open set, it often necessitates constraints. [4]

Another important aspect of the Optimal Control Theory is called the first order (necessary) condition or the Maximum Principle. This involves changing the constrained dynamic optimization problem into an unconstrained problem using the Hamiltonian function, denoted H, and $\lambda(t)$, which is the co-state variable, very similar to the Multiplier in the Lagrange Multiplier method, used as a shadow value between the state and control variables.[4]

$$H(t,y,u,\lambda) = F(t,y,u) + \lambda(t)f(t,y,u)$$

Furthermore, the first order conditions to the Hamiltonian is much like the Lagrangian since we differentiate with respect to the control, state and co-state variables.

$$\begin{aligned}
\max_u H(t,y,u,\lambda) \qquad & \forall \ t \in [0,T] \\
\dot{y} = \frac{\partial H}{\partial \lambda} \qquad & \text{Equation of Motion for } y \\
\dot{\lambda} = -\frac{\partial H}{\partial y} \qquad & \text{Equation of Motion for } \lambda \\
\lambda(T) = 0 \qquad & \text{Transversality Condition}
\end{aligned}$$

Mathematical Model

Parameters:

- E = Monthly earnings (increases 2% annually)
- ρ = Interest rate by the banks (inflation rate)
- r = Treasury bond rate
- t = Time in months
- c(t) = Monthly consumption
- u(t) = Control variable: monthly savings

- S(t) = State variable: total savings up to month t
- λ(t) = Co-state variable: shadow value

We need the current amount of savings already in our fund in order to compute the optimal amount of the savings deposits each month. To do so, we rely on the following equation:

$$\underbrace{S'(t)}_{\text{savings at time t}} = \underbrace{E}_{\text{salary}} + \underbrace{r * s(t)}_{\text{return from savings}} - \underbrace{c(t)}_{\text{consumption}}$$

The objective function for our model is the following integral over the *utility* function U, along with the discount factor $e^{-\rho t}$ and the boundary conditions:

$$max \int_0^T U\{c(t)\}\, e^{-\rho t} \;\; dt \quad such\, that,$$

$$S(0) = 0, \quad S(T) = S_T$$

Since $S'(t) = E + r * s(t) - c(t)$, we can solve for $c(t)$ as follows:

$$c(t) = E + r * s(t) - S'(t) \implies c(t) = E + r * s(t) - u(t)$$

This utility function depends, at each instant of time, on the value of the *control variable* $c(t)$ – which is the monthly consumption that we can directly control - and on the value of the *state variable* $S(t)$ – which is the required monthly savings that we cannot directly control as it is implied by the control variable.

We have the following objective integrand with the boundary conditions:

$$max \int_0^T U\{E + r * s(t) - u(t)\}\, e^{-\rho t} \;\; dt \quad such\, that,$$

$$S'(t) = u(t), \quad S(0) = 0, \quad S(T) = S_T$$

Optimal Solution

We used the natural log $(\ln)$ function to measure the utility.

$$U(c) = \ln(c)$$

The Hamiltonian function corresponding to our model is:

$$H(t,u,s,\lambda) = U(E + r * s(t) - u)\, e^{-\rho t} + \lambda(u)$$

$$H(t,u,s,\lambda) = \ln\left(E + r * s(t) - u\right) e^{-\rho t} + \lambda(u)$$

To reduce the complexity in our Hamiltonian, we can remove the discount factor from the function by dividing it throughout by $e^{\rho t}$. Then we redefine the Lagrange multiplier in the Hamiltonian as $m = \lambda\, e^{\rho t}$, which is our current shadow value.

$$H(m,u,s) = \ln\left(E + r * s(t) - u\right) + m(u), \quad where\; m = \lambda\, e^{\rho t}$$

Using the Maximum principal, we derived the following partial derivatives:

$$\boxed{\frac{\partial H}{\partial u} = 0 \iff \frac{\partial H}{\partial u} = 0 = \frac{1}{u} - m}$$

$$\boxed{\lambda'(t) = \frac{-\partial H}{\partial S} \iff m' - \rho m = \frac{\partial H}{\partial S} = m\, r}$$

$$\boxed{S' = E + r * s - c}$$

To determine the optimal trajectory of the state variable, $S(t)$, we solved the above system of equations and derived the following result:

$$Sopt(t,\rho,r,E) := \frac{E\,e^{(-t+10)\rho + r t} + (E + 40000\, r)\, e^{10\rho + r(t-10)} + (-E - 40000\, r)\, e^{(t-10)(r-\rho)} - E\left(e^{10\rho} + e^{r t} - 1\right)}{r\left(e^{10\rho} - 1\right)}$$

Sensitivity Analysis

We also analysed the sensitivity of the optimal solution using the sensitivity factors with respect to the inflation rate, ρ, and the return rate, r:

$$dSoptd\rho(t,\rho,r,E) = \frac{\partial}{\partial\rho}Sopt(t,\rho,r,E)$$

$$dSoptdr(t,\rho,r,E) = \frac{\partial}{\partial r}Sopt(t,\rho,r,E)$$

Finally, we adjusted the optimal savings amount for each month using the above sensitivity factors in the following equation:

$$\boldsymbol{Soptad(ti, \rho i, ri, E)} := Sopt(t, \rho, r, E) + dSoptd\rho(t, \rho, r) \cdot (\rho i - \rho) + dSoptdr(t, \rho, r) \cdot (ri - r)$$

Results

Our scenario included a person with a monthly salary of \$5,000 who is looking to set up an emergency fund with a target of \$40,000 by the end of ten year period. The assumption here is that the salary, E, is going to increase by 2% every year. To determine the amount of monthly savings and consumption in order to meet the targeted savings, we extracted the data on monthly Treasury Bond Rate, r, and monthly Inflation Rate, ρ, from September 2010 to September 2020. In the data set, the Treasury bond rate is showing a relatively more stable trend as compared to the Federal Reserve interest rate i.e. the inflation rate. This is because banks would prefer to have lesser volatility in their regular trading, while not altogether dismissing market conditions. The Fed Rate on the other hand is governed by the general equilibrium of macroeconomic conditions which include many idiosyncratic components which need to be accounted for.

These rates were then averaged for each year to derive the expected values for them. We then used these expected values to compute the *adjusted* values, as shown in Figure 1, for the monthly savings and consumption rates for each year using the $Soptad(t_i, \rho_i, r_i, E)$

adjusted savings function described previously.

$$S1adj(2) := 326.2729972$$
$$S1adj(3) := 318.9140654$$
$$S1adj(4) := 321.7051406$$
$$S1adj(5) := 311.3676345$$
$$S1adj(6) := 306.6193750$$
$$S1adj(7) := 303.6626868$$
$$S1adj(8) := 306.1827178$$
$$S1adj(9) := 318.2499496$$
$$S1adj(10) := 317.1331216$$
$$S1adj(11) := 316.2578138$$
$$S1adj(12) := 316.7249977$$
$$S1adj(13) := 337.183311$$

(a) Savings.

$$C(1) := 4673.727003$$
$$C1adj(2) := 4674.417614$$
$$C1adj(3) := 4682.569865$$
$$C1adj(4) := 4680.945755$$
$$C1adj(5) := 4692.243450$$
$$C1adj(6) := 4698.108848$$
$$C1adj(7) := 4701.703920$$
$$C1adj(8) := 4700.145405$$
$$C1adj(9) := 4688.388491$$
$$C1adj(10) := 4689.942147$$
$$C1adj(11) := 4691.608100$$
$$C1adj(12) := 4689.912592$$

(b) Consumption.

Figure 1: First year monthly savings and consumption.

As a general rule, when the inflation rate increases so does the treasury rate, and vice versa. Figure 2 shows this relationship and the volatility of Treasury Bond rate and Inflation rate.

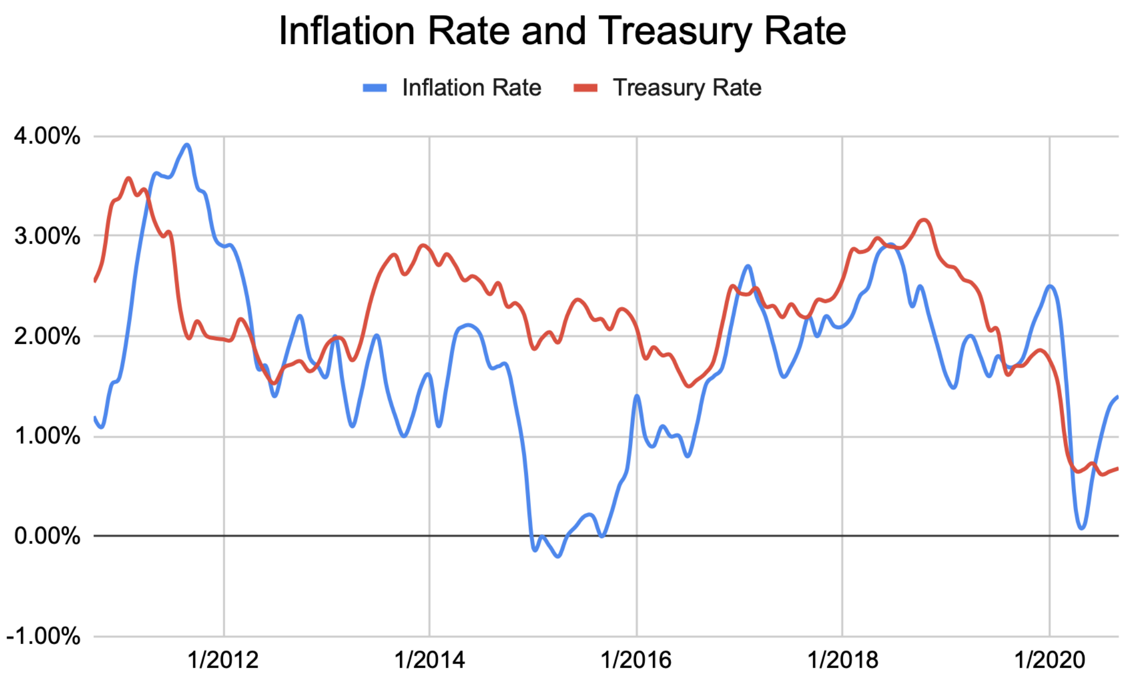

Figure 2: Ten years of Inflation and Treasury rates.

This difference can be best observed in 2015 where there is an economic deflation, while the treasury rate tends to stay relatively consistent around 2% and the inflation rate makes a larger jump from just below 0% to almost 1.5%. When the treasury rate increases, it is expected that one will save less and consume more. This is because money is more readily available, generating spending and ultimately stimulating the economy. As the return rate goes higher, more optimistic one is about the economy. For instance, as illustrated in Figure 3, in 2011 the treasury rate was increasing, while saving was low and consumption was high.

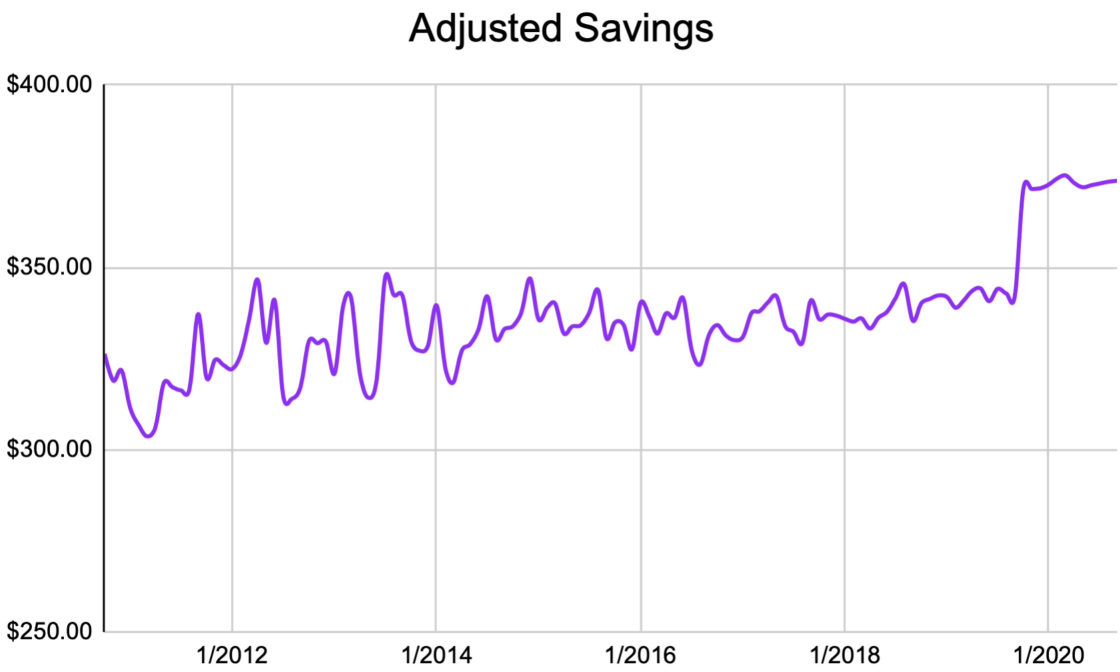

Figure 3: Required savings rates for ten years.

Alternatively, when the treasury rate decreases, it is expected that one will need to save more and consume less. This is well depicted in Figure 4, as in 2019 the treasury rate decreases while savings was high and consumption was low in order to meet our target. The savings and consumption amounts seem to be consistently high throughout 2020 as a result of Covid-19 pandemic which has severely affected the economy. We are assuming that the person is still employed during this recession period and is able to save the increased savings amount while consuming less. The consumption rates increase steadily with little volatility. This is because the salary is increasing each year while the target value of the savings, \$40,000, remains the same. As the salary increases, the person can consume more while needing to save about the same amount of money each month.

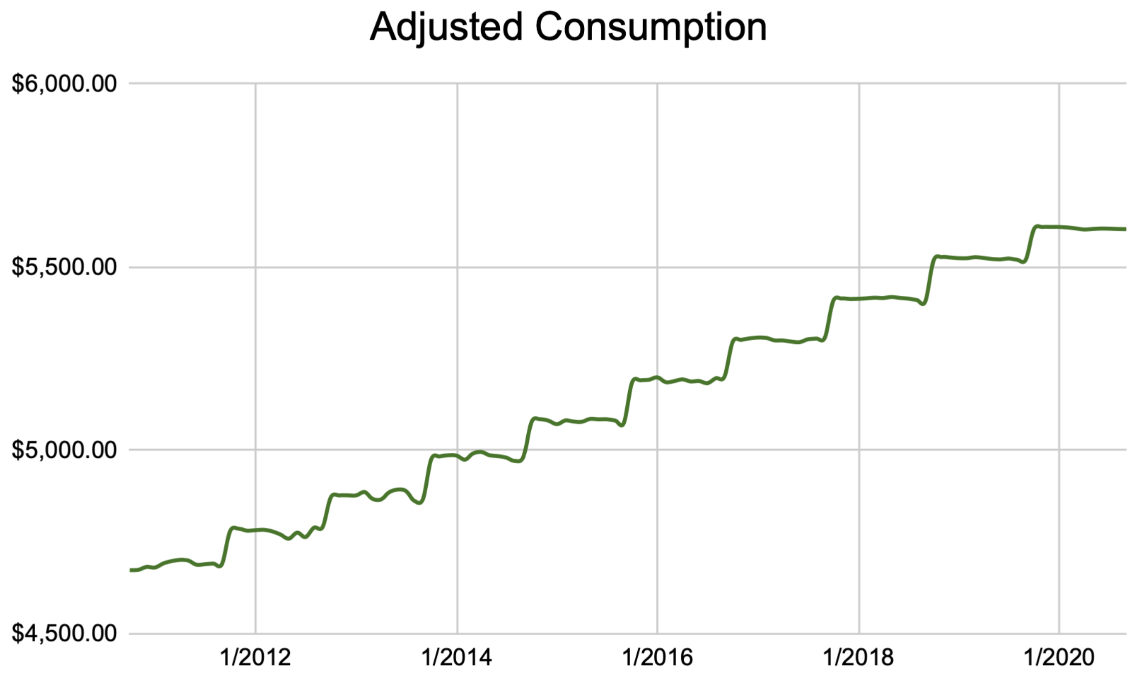

Figure 4: Allowed consummation rates for ten years.

Conclusion

Financial planning is incredibly important to abide by any uncertain economic condition. Having a financial security helps to slightly alleviate the many stresses one has in everyday life. This is why it is crucial to give it a thought in advance and not wait for a personal financial calamity. To some extent, the burden of financial stress is preventable with the right precautionary measures.

This project serves as an ideal example of how Optimal Control Theory can be particularly useful in financial planning. We were able to create a financial plan that achieved a significant savings fund while also optimizing consumption. Through the use of the mathematics software: Maple, we were able to construct and compute the Hamiltonian and do sensitivity analysis of the model which included the treasury yields, and inflation rates from 2010 to 2020, and a monthly salary. We were able to successfully solve for optimal trajectory of savings and consumption that allowed us to reach our target of $40,000 in savings in ten years.

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