Foundations of Statistical Modeling

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Exercise sheet 3, submit on Monday March 18th, 2024 on Teams
Your name:

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1. Basic Operations on Sets [5 points]

Let $S_1 = \{2,3,4\}$ and $S_2 = \{3,4,5\}$ which are both in the DVS $S = \{1,2,3,4,5,6\}$.
a) Find their union $S_1 \cup S_2$ and their intersection $S_1 \cap S_2$.
b) Find the union and the intersection of S_1^c and S_2^c. Check de Morgan's laws: 1. $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$. 2. $(S_1 \cap S_2)^c = S_1^c \cup S_2^c$.
c) Define $S_a = S_1 \times S_2$ and $S_b = S_2 \times S_1$. Find the union and the intersection of S_a and S_b.

2. Sigma Fields [5 points]

Let $S = \{3,4,5,6\}$.
a) Give the power set $\mathbf{Pot}(S)$.
b) Consider the sets of sets $F_1 = \{\emptyset, S\}$ and $F_2 = \{\emptyset, \{3,4\}, \{5,6\}, S\}$ on S. Show that they are both sigma fields on S. Find the union and the intersection of the sigma fields F_1 and F_2.
c) Find two other sigma fields F_3 and F_4 on S such that their union $G = F_3 \cup F_4$ is not a sigma field. Check whether their intersection is then a sigma field.

3. Generation of Sigma Fields and Borel Sigma Fields [5 P]

Let $S = [0,1]$ be the data value space.
a) Generate a sigma field on S from the set $G_1 = \{\emptyset, S, (a,b)$ with $a < b \in S\}$.
b) Generate a sigma field on S from the set $G_2 = \{\emptyset, S, (a,b), (c,d),$ with $a < b < c < d \in S\}$.
c) Compare $\sigma(G_1)$ and $\sigma(G_2)$ with the Borel σ-field on S.

4. Sigma Fields and Measurable Spaces [2.5 P]

Let $\Omega = \{a,b,c,d\}$ be the universe and $S = \{3,4,5,6\}$ the data value space. Construct a non-trivial sigma field $\mathscr{F}$ on S ($\{0,S\}$ is trivial!) and a sigma field A on Ω and an RV-function $X : \Omega \to S$ such that X is $A - \mathscr{F}$- measurable.

5. Sigma Fields and Measurable Functions [2,5 P]

Let $(S_1, \mathscr{F}_1)$, $(S_2, \mathscr{F}_2)$, $(S_3, \mathscr{F}_3)$ be measurable spaces. If $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$ are respectively $\mathscr{F}_1 - \mathscr{F}_2$ and $\mathscr{F}_2 - \mathscr{F}_3$-measurable functions, prove that $f_2 \circ f_1 : S_1 \to S_3$, where $f_2 \circ f_1(x) := f_2(f_1(x))$ is $\mathscr{F}_1 - \mathscr{F}_3$ measurable.