Foundation of Statistical Modeling Assignment 3

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exercise sheet 3

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1. Basic Operations on sets

Given: S1={2,3,4} S2={3,4,5} S={1,2,3,4,5,6}

U: Union

I: Intersection

a) Union (S1 U S2): S1 S2={2,3,4}U{3,4,5}={2,3,4,5}

Intersection (S1 I S2): S1 I S2={2,3,4}I{3,4,5}={3,4}

b)Union and Intersection of Complements of S1 and S2:

Given: S1 ^c =S S1={1,5,6}

S2 ^c =S S2={1,2,6}

Union of Complements (S1c u S2c): S1^c uS2^c ={1,5,6}U{1,2,6}={1,2,5,6}

Intersection of Complements (S1c I S2c): S1 c I S2 c ={1,5,6}I{1,2,6}={1,6}

Verification of de Morgan's Laws:

1. (S1uS2) c ={1,2,3,4,5,6} {2,3,4,5}={1,6}

S1^c I S2^c ={1,6}

2. (S1 I S2) c ={2,3,4} {3,4}={2}

S1 c S2 c ={1,2,6}

The results confirm de Morgan's laws.

c)Union and Intersection of Cartesian Products Sa and Sb:

Given: Sa=S1×S2 Sb=S2×S1

Sa: Sa={(2,3),(2,4),(2,5),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5)}

Sb: Sb={(3,2),(3,3),(3,4),(4,2),(4,3),(4,4),(5,2),(5,3),(5,4)}

Union of Cartesian Products (Sa u Sb): SauSb= {(2,3),(2,4),(2,5),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4),(4,5),(5,2

Intersection of Cartesian Products (Sa I Sb): SaISb={(3,3),(3,4),(4,3),(4,4)}

2. Sigma Fields

a) Power Set of S: Given set S={3,4,5,6}, he power set, denoted as Pot(S) or P(S), is the set of all subsets of S, including the empty set and S itself.

Pot(S)= {0,{3},{4},{5},{6},{3,4},{3,5},{3,6},{4,5},{4,6},{5,6},{3,4,5},{3,4,6},{3,5,6},{4,5,6},S}

b) Showing F1 and F2 as Sigma Fields on S: Sigma Field F1:

F1={o,{3,4,5,6},{3,4},{5,6}}

Sigma Field F2:

F2={o,{3,4,5,6},{3,4},{5,6}}

Both F1 and F2 satisfy the three properties of a sigma field:

1. Contains the sample space S.
2. Closed under complementation.
3. Closed under countable unions.

Union of Sigma Fields (F1 F2): F1 F2={o,{3,4,5,6},{3,4},{5,6}}

Intersection of Sigma Fields (F1 F2): F1 F2={o,{3,4,5,6},{3,4},{5,6}}

c) Finding Other Sigma Fields F3 and F4: Sigma Field F3:

F3={o,{3,4,5,6},{3,4}}

Sigma Field F4:

F4={o,{3,4,5,6},{5,6}}

Both F3 and F4 satisfy the properties of a sigma field.

Union of F3 and F4 (G = F3 F4): G={o,{3,4,5,6},{3,4},{5,6}}

Although the union G contains all elements of a sigma field, it is not closed under complementation. For example, { 3 , 4 } {3,4} is in G, but its complement { 5 , 6 } {5,6} is not in G.

Intersection of F3 and F4: F3 F4={o,{3,4,5,6}}

The intersection of F3 and F4 contains the empty set and the sample space S, which satisfies the properties of a sigma field. Therefore, 3 4 F3 F4 is indeed a sigma field.

3.Generation of Sigma Fields and Borel Sigma Fields

a)Generating Sigma Field from G1: Given G1={o,S,(a,b) with a<b S},we need to generate the sigma field generated by G1, denoted as (G1).

Contains Empty Set and Sample Space: o,S (G1)

Closed under Complementation: if (a,b) G1, then [0,a) [b,1] =S (a,b) (G1)

Closed under Countable Unions: Any countable union of sets in G1 should be in (G1).

b) Generating Sigma Field from G2: Given

G2={o,S,(a,b),(c,d) with a<b<c<d S}, we aim to generate the sigma field (G2).

Contains Empty Set and Sample Space: o,S (G2)

Closed under Complementation: If (a,b) G2, then [0,a) [b,1]=S (a,b) (G2). Similarly, if (c,d) G2, then [0,c) [d,1]=S (c,d) (G2).

Closed under Countable Unions: Any countable union of sets in G2 should be in (G2).

c) Comparing (G1) and (G2) with Borel Sigma Field: The Borel sigma field on S=[0,1],denoted as B[0,1], contains all open intervals in S and all countable unions of these intervals.

Both (G1) and (G2)are sigma fields that contain open intervals, but they may also contain additional sets not in the Borel sigma field. Their relationship with the Borel sigma field depends on the specific sets G1 and G2.

Without detailed information about the specific intervals in G1 and G2, it's not possible to definitively compare (G1) and (G2) with the Borel sigma field. However, in general, (G1) and (G2) may contain subsets not present in the Borel sigma field, making them potentially larger than the Borel sigma field.

4. Sigma Fields and Measurable Spaces construct a non-trivial sigma field F on S={3,4,5,6} and a sigma field A on Ω={a,b,c,d}, along with an RV-function X:Ω→S such that X is A−F-measurable, we need to define F, A, and X accordingly.

Sigma Field F on S: We can construct a non-trivial sigma field F on S by considering subsets of S and their complements.

For instance, one possible choice for F could be: F={o,{3,4},{5,6},S}

Sigma Field A on Ω: Since we want to construct a non-trivial sigma field A on Ω, we can consider subsets of Ω and their complements.

For instance, one possible choice for A could be: A={o,{a},{b,c,d},Ω}

RV-function X from Ω to S: To define X, we need to specify the mapping from elements of Ω to elements of S. We want X to be A−F-measurable, meaning that pre-images of sets in F are in A.

One possible definition for X could be:

X(a)=3,X(b)=4,X(c)=5,X(d)=6

With this definition, the pre-image of any set in F (which consists of subsets of S) would be in A, satisfying the condition for X to be A−F-measurable.

Verification Let's verify that X is A−F-measurable:

For any set B F:

if B=o,then X^−1 (B)=o A

if B={3,4},then X^−1 (B) ={a,b} A.

if B={5,6}, then X^-1(B) = {c,d} A.

if B = S, then X^-1(B) = Ω A.

Since the pre-images of all sets in F belong to A, X is indeed A−F-measurable.

This construction satisfies the requirements of the problem by providing a non-trivial sigma field F on S, a sigma field A on Ω, and an RV-function X that is A−F-measurable.

5.Sigma Fields and Measurable Functions if f2 f1 : S1→S3 is F1 - F3 measurable, where f2 f1(x) :=f2(f1(x)), we need to show that pre-image of every set in F3 under f2 f1 is in F1.

let B F3 be an arbitray sey in S3. we want to show that (f2 f1)^-1(B) F1. since f2 is F2 - F3 measurable, teh pre-image of every set in F3 under f2 is in F2. similarly since f1 is F1 - F2 measurable, teh pre-image of every set in F2 under f1 is in F1

consider the pre-image of B under f2: f2^−1 (B) F 2

since f1 is F1 - F2 measurable, the pre-image of f2^-1(B) under f1 is in F1: $f1^{-1(f2}-1$ (B)) F1

Now, consider the composition f2 f1. the pre-image of B under f2 f1 is given by: (f2 f1)^-1(B) = $f1^{-1(f2}$-2(B))

since $f1^{-1(f2}$-1(B)) F1,we conclude that (f2 f1)^-1 (B) F1 therefore, f2 f1 is F1 - F3 measurable