1. [50 points] Solve the linear system

$$\boldsymbol{H}\boldsymbol{x} = \boldsymbol{b}$$

for $\boldsymbol{x} \in \mathbb{R}^N$ where the elements of $\boldsymbol{H}$ are defined by

$$h_{ij} = \frac{1}{i + j - 1}.$$

Write a function that solves the this linear system for a given vector $\boldsymbol{b}$.

(a) The function must be named `solve_hilbert` verbatim.

(b) The function must take one vector, $\boldsymbol{b}$ of length N as input and return a vector, $\boldsymbol{x}$ of length N as an output that is the solution to the linear system $\boldsymbol{H}\boldsymbol{x} = \boldsymbol{b}$.

(c) Use whatever appropriate linear solver you prefer.

2. [50 points] The finite difference method (FDM) approximation of the Laplacian operation in 1D can be written as the tridiagonal matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

Solve, using a banded matrix solver, the linear system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

for $\boldsymbol{x} \in \mathbb{R}^N$ where $b_i = -2/\left(N+1\right)^2$ for $i = 1, 2, \ldots, N$.

(a) The function must be named `solve_fdm` verbatim.

(b) The function must take a single non-negative integer N, which is the system size, as input and return **x** as a numpy array of length N as output.

(c) The function must use a banded matrix solver.