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Young Symmetry, the Flag Manifold, and Representations of $GL(n)$

JACOB TOWBER

DePaul University, Chicago, Illinois 60614

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INTRODUCTION

It has long been known that all the finite-dimensional irreducible polynomial representations of $GL(n)$ may be obtained by a simple and explicit construction which utilizes the coordinate ring of the flag manifold (cf. [4–6]. This suggests the following natural idea:

(i) Define the coordinate ring of the flag manifold by generators and relations over $\mathbb{Z}$.

(ii) Using the same description by generators and relations, associate a "flag algebra" Λ^+E to any module E over any commutative ring R.

The resulting "flag algebra" has a number of remarkable properties and applications, some of which are studied in the present paper; it was first defined and studied by the author in the earlier paper "Two New Functors from Modules to Algebras", *J. Algebra* **47** (1977), 80–104 [18] (which will be referred to as "Two functors" in the present paper).

The above two-step program has recently been carried out by Glenn Lancaster and the present author, for the flag manifolds of all the classical groups (cf. [11, 19]); the constructions for these utilize the algebra Λ^+E associated with $GL(n)$.

To study the coordinate ring of the flag manifold, it is necessary to begin by considering carefully the equations of the flag manifold. In the characteristic 0 case, Hodge [6, 7] has given a generating set for the prime ideal of the flag manifold, consisting of the left sides of Eqs. (1) and (2) below. It is proved in the present paper (Proposition 3.2) that, for nonzero characteristic, (1) and (2) no longer generate in general this prime ideal (even though their zero locus is still the flag manifold). The literature seems to contain no generating set (proved to be such) for the flag manifold in general characteristic; in the present paper two quite distinct such generating sets are given, one in Section 2 (based on determinantal identities of Sylvester explained in Section 1) and the other in Section 5 (based on determinantal identities given by Turnbull in [20, Chap. III], which were later used by Hodge in studying the flag manifold).

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(See also [16], in which Musili derives generating sets in arbitrary characteristic for the prime ideals of the Grassmann and Schubert varieties, related to the generating set in Section 5 of the present paper).

The finite-dimensional irreducible polynomial representations of $GL(n)$ may also be obtained by a well-known construction, quite different in appearance, due to Weyl, where the representations are furnished by symmetry classes of tensors, whose definition utilizes certain idempotents, constructed by Young, in the group algebra of the symmetric group. In this connection, the present paper studies two concepts which appear to be new, "Young symmetry" and "Young alternation" of functions of several variables with range a $\mathbb{Z}$-module A; when A is a $\mathbb{Q}$-module, these are the functions fixed by the action of the relevant Young idempotent. (The author sketched the second of these two concepts in "Two Functors," p. 82, and more recently in [19]; it was originally suggested to the author by the study of Carter and Lusztig [1, p. 211].) Some simple facts about these two concepts are obtained in Section 2, while in Section 4, properties of the flag manifold are used to prove these two less obvious facts:

(a) In characteristic 0, "Young alternation" indeed means (as asserted above) "fixed by the relevant Young indempotent."

(b) For any ring R, the class of functions of n variables, with range an R-module, which have Young alternation with respect to a given partitioning of the variables, coincides with the class of such functions which are left annihilated, by the left annihilator in $R[\mathfrak{S}_n]$ of the relevant Young quasi-indempotent.

If the two classical constructions mentioned above for the representations of $GL(n)$ are carefully compared, a remarkable result is obtained which appears to be new, namely, that the coordinate ring of the flag manifold has some extra structure, consisting of a (graded) alternating multiplication defined naturally; this extra structure only arises in the characteristic 0 case, however. This operation was first defined (without proofs) in "Two Functors," p. 101; the relevant proofs are provided in the present paper, especially in Sections 6 and 7.

In characteristic 0, it is well known what the generating set is, for the prime ideal of the flag manifold. We may break it up into two subsets, as follows. The first subset consists of the left sides of the following well-known quadratic relations, satisfied by the Plücker coordinates $\pi(i_1, ..., i_p)$ of a $(p-1)$-dimensional linear subspace L^{p-1} of projective $(n-1)$-space P^{n-1},

$$\sum_{s=0}^{p} (-1)^s \pi(i_1, ..., i_{p-1}, j_s)\, \pi(j_0, ..., \hat{j}_s \cdots j_p) = 0 \tag{1}$$

(all i's and j's between 1 and n). The second subset arises from the necessary and sufficient conditions

$$\sum_{s=0}^{p} (-1)^s \pi(i_1, ..., i_{q-1}, j_s)\, \pi(j_0, ..., \hat{j}_s \cdots j_p) = 0 \tag{2}$$

(all i's and j's between 1 and n) that a $(q-1)$-dimensional linear subspace L^{q-1} of P^{n-1} $(q < p)$ with Plücker coordinates $\pi(i_1, \dots, i_q)$ be a subspace of the preceding L^{p-1}.

How messy are (1) and (2)? They are quadratic equations with all coefficients 0, 1, or -1; whatever complexity they have is thus of a *combinatorial* nature, involving interchanges among the indices $i_1, \dots, i_{q-1}, j_0, \dots, j_p$ $(q \leqslant p)$. Indeed, Eqs. (1) and (2), as well as the "extra" equations obtained in Theorem 3.1, assert precisely that $\pi(i_1, \dots, i_{q-1}, j_0)\, \pi(j_1, \dots, j_p)$ has certain symmetry properties, considered as a function of its indices; these are indeed precisely the symmetry conditions studied in Section 2.

It is this observation (which appears to be new, and which is formulated more precisely as Corollary 3 to Theorem 3.1 below) which furnishes the basic connection between the flag manifold and "Young symmetry," and thus between the two classical constructions of the irreducible representations of the general linear group (that of Young–Weyl via "quantics" and that of DeRuyts–Littlewood–Hodge via "connexes").

An important extension of these constructions was initiated in [5] by Higman; utilizing the coordinate ring of the flag manifold, he constructed functors of vector spaces V, here denoted by $\Lambda^{a_1, \dots, a_s} V$, which yield the absolutely irreducible representations of $\mathrm{Aut}(V)$ (when the ground field has characteristic 0) via the homomorphism

$$\mathrm{Aut}(V) \to \mathrm{Aut}(\Lambda^{a_1, \dots, a_s} V), \qquad T \mapsto \Lambda^{a_1, \dots, a_s} T.$$

A different version of these functors (naturally isomorphic to those of Higman in the characteristic 0 case) was given by Epstein in [2], based on the Young–Weyl construction; Carter and Lusztig gave what seem to be the correct generalization of Epstein's functors, at least in the case of finitely generated projective modules over arbitrary commutative rings. In "Two Functors" the present author refined these two constructions in the following ways (some of which are further developed in the present paper):

(i) Utilizing (without proof) some of the results proved in the present paper, Higman's construction was extended to arbitrary modules over arbitrary commutative rings

(ii) It is immediate from his construction itself, that the functors constructed by Higman form the graded pieces of a commutative associative algebra; it is far from obvious that the same is true for the Carter–Lusztig functors, but in [18] it is shown how to put the structure of an associative graded anticommutative algebra, in a functorial way, on the direct sum of the Carter–Lusztig functors. More precisely, this structure is placed on the direct sum of *certain* functors, whose identity with the Carter–Lusztig functors is obtained in Section 6 of the present paper.

(iii) It is proved that the generalization of the Higman functors and the Carter–Lusztig functors stand in a relation of *duality*.

Let us finally note, on p. 81 of "Two Functors," the suggestion that these functors should prove useful in obtaining a free resolution of the ideal of $p \times p$ minors of a matrix with indeterminate entries; this possible area of application has recently attracted the attention of a number of researchers to these functors, including Buchsbaum, Eisenbud, Procesi, and others. In this connection should also be mentioned Alain Lascoux, who in his thesis has given a different construction for these functors [12] and has applied them to the above-mentioned problem of resolving the ideal of $p \times p$ minors; he has been kind enough to notify the author, in writing, that he had seen "Two Functors" prior to the completion of his thesis.

Some recent work of James ([9] *et al.*) is also of interest in the present connection, as is the work of Liulevicius in [15].

Appendix 1 concerns a combinatorial result, which played an important role in [18]. The author had originally planned to include this proof in [18], but inquiries made at that time seemed to indicate such a result was already in the literature, which (except for the well-known special case when the tableau has no repetitions; cf. Boerner) thefuthor no longer considers to be the case; the author now believes he may claim this result as original.

1. Sylvester's Theorem

In [17], Sylvester proved the following remarkable property of determinants: Let us denote by $[a_1 \cdots a_n]$ the $n \times n$ determinant whose ith column is a_i; then

$$[a_1 \cdots a_n][b_1 \cdots b_n] \tag{3}$$

remains unchanged in value, if we interchange b_1 with each of $a_1, \ldots, a_n$ in turn, and then add:

$$[a_1 \cdots a_n][b_1 \cdots b_n] = \sum_{i=1}^{n} [a_1 \cdots a_{i-1} b_1 a_{i-1} \cdots a_n][a_i b_2 \cdots b_n]. \tag{4}$$

More generally, he showed that, for $j_1, \ldots, j_r$ any distinct integers between 1 and n, expression (3) remains unchanged in value if, for all $\binom{n}{r}$ choices of $i_1 < \cdots < i_r$ between 1 and n, we interchange a_{i_1} with b_{j_1}, a_{i_2} with $b_{j_2}, \ldots, a_{i_r}$ with b_{j_r} in (3), and then add the $\binom{n}{r}$ results.

Example. With $n = 3, j_1 = 2, j_2 = 3$, we have

$$[a_1a_2a_3][b_1b_2b_3] = [b_2b_3a_3][b_1a_1a_2] + [b_2a_2b_3][b_1a_1a_3] + [a_1b_2b_3][b_1a_2a_3].$$

Sylvester's theorem, and his original proof, remain valid when the entries in the a's and b's lie in any commutative ring R. Here is a quick proof of (4): The expression

$$[a_1 \cdots a_n][b_1 \cdots b_n] - \sum_{i=1}^{n} [a_1 \cdots a_{i-1} b_1 a_{i+1} \cdots a_n][a_i b_2 \cdots b_n]$$

is an R-multilinear function of the variables $a_1 ,..., b_n$, all ranging over R^n, which is alternating in the $n + 1$ variables $a_1 ,..., a_n$, b_1 . Any such function must be 0. (The second half of Sylvester's theorem will be deduced from (4) in the next section; see Corollary 2 to Proposition 2.2.)

The quadratic relations (1) for the Grassmann manifold, given in the Introduction, are immediate consequences of (4). Indeed, suppose the linear subspace L^{p-1} of projective $(n - 1)$-space is spanned by p points whose coordinates are given, respectively, by the rows of the $p \times n$ matrix E with ith column e_i; then the Plücker coordinates of L^{p-1} are given by

$$\pi(i_1 ,..., i_r) = [e_{i_1} \cdots e_{i_r}],$$

and applying (4) to $[e_{j_1} \cdots e_{j_p}][e_{j_0} e_{i_1} \cdots e_{i_{p-1}}]$, it is readily seen that we obtain (1). Similarly, we may deduce (2) from (4): If L^{q-1} is a q-dimensional subspace of L^{p-1}, we may choose the above matrix E so L^{q-1} is generated by points whose coordinates are given by the first q rows of E, whence the Plücker coordinate $\pi(i_1 ,..., i_q)$ of L^{q-1} is the $q \times q$ subdeterminant of E formed by the first q rows and by columns $i_1 ,..., i_q$; then, applying (4) to

$$[e_{j_1} \cdots e_{j_p}][e_{j_0} e_{i_1} \cdots e_{i_{q-1}} \delta_{q+1} \cdots \delta_p]$$

(where δ_i is the column of length p with 1 in the ith place and 0's elsewhere) we obtain (2).

The second part of Sylvester's theorem yields, in a similar way, additional equations satisfied on the flag manifold; we shall see in Section 3 that these are precisely the "extra" equations mentioned in the Introduction, which together with (1) and (2) generate the prime ideal of the flag manifold (over any ground field).

We next proceed to abstract the symmetry properties asserted of expression (3) by Sylvester's theorem.

2. Symmetry Properties of Functions of Several Variables

It is desirable to be able to rewrite (4) in the form

$$[a_1 ,..., a_n][b_1 ,..., b_n] = \sum_{i=1}^{n} (a_i b_1) \circ [a_1 ,..., a_n][b_1 ,..., b_n], \tag{5}$$

where $\circ$ denotes the action of the symmetric group, on the $2n$ symbols a_1 through b_n, upon functions of those variables.

Let us now proceed to clarify the concepts involved in the notation (5). If E is any set, we denote by $S(E)$ the group of bijections of E; our convention will be to write functions on the *left* of elements, so that for α and β in $S(E)$, $\alpha\beta$ is the result of applying first β and then α. We wish to regard $S(E)$ as acting upon functions

$$f(\{X_i\}_{i\in E})$$

of E-indexed variables, all of which have the same domain of definition, say D. To be precise:

By a *function of E-indexed variables, with common domain D, taking values in T* (where E, D, T are any sets), will be meant an element of

$$T^{(D^E)}$$

(where as usual D^E means the set of maps $E \to D$), i.e., a map f assigning to each

$$E \to D, \qquad i \mapsto X_i \in D$$

an element $f(\{X_i\}_{i\in E})$ in T. Of course, if $E = \{1,..., n\}$, we recover the usual notion of a function $f(X_1, ..., X_n)$ of n variables.

There is a natural *left* action of $S(E)$ on such functions, defined as follows: if $f \in T^{(D^E)}$, $\pi \in S(E)$ then $\pi \circ f \in T^{(D^E)}$ is defined by $(\pi \circ f)(E \xrightarrow{X} D) = f(X \circ \pi)$ or equivalently by $(\pi \circ f)(\{X_i\}_{i\in E}) = f(\{Y_i\}_{i\in E})$ with $Y_i = X_{\pi i}$ (all X_i in D). (One may verify that this is a left action, either by a direct computation, or by noting that $T^{(D^E)}$ is a covariant functor of E). If T has the structure of an Abelian group, or module over a commutative ring R, this action extends in the usual way to a left action of the group algebra $\mathbb{Z}[S(E)]$ or $R[S(E)]$, respectively.

With these conventions (5) now makes sense, taking the indexing set E to consist of the variable symbols $a_1, ..., b_n$, taking T to be the ground ring R, and taking the common domain D to consist of columns of length n over R. Note that we are, as usual, free to regard (4) at will, either as an equation (equivalent to (5)) between two functions of $2n$ variables, or as an identity holding for all $a_1, ..., b_n$ in D. Note also that the second part of Sylvester's theorem may now be written

$$[a_1 \cdots a_n][b_1 \cdots b_n] = \sum_{i \leqslant i_1 < \cdots < i_r \leqslant n} (a_{i_1} b_{j_1}) \cdots (a_{i_r} b_{j_r}) \circ [a_1 \cdots a_n][b_1 \cdots b_n].$$

Throughout the remainder of this section, E and D will denote fixed sets, $(A, +)$ a fixed abelian group, and $f \in A^{(D^E)}$ a function of E-indexed variables with common domain D, taking values in A. We shall denote the cardinality of any set U by $\# U$. We assume throughout that E is finite.

DEFINITION 2.1. Let U be any subset of E. We say f is *symmetric in* U if $(jk) \circ f = f$ for all distinct j and k in U. We say f is *alternating in* U if, for all distinct j and k in U,

$$f(\{X_i\}_{i \in E}) = 0 \qquad \text{if} \quad X_j = X_k, \quad \text{and} \quad (jk) \circ f = -f.$$

Note. In particular, f is both symmetric and alternating in U if $\# U = 0$ or 1. If f is symmetric (or alternating) in U, it is such for each subset of U.

DEFINITION 2.2. Let

$$U = \{u_1, ..., u_n\}, \qquad V = \{v_1, ..., v_n\} \tag{6}$$

be disjoint subsets of E with the same cardinality n; assume that f is either symmetric, in U and in V, or alternating in U and in V. Then by

$$\mathrm{Int}(U, V) f,$$

the result of interchanging U *and* V *in* f, will be meant

$$(u_1 v_1) \cdots (u_n v_n) \circ f$$

(and will be meant f, if U and V are empty).

Note. $\mathrm{Int}(U, V) f$ is independent of the particular choice (6) of orderings for U and V, as the following computation suffices to show:

Let ϵ denote $+1$ or -1, according as f is symmetric or alternating in both U and V; then

$$(u_1 v_2)(u_2 v_1) \circ f = (u_1 v_1)(u_2 v_2)(u_1 u_2)(v_1 v_2) \circ f = \epsilon^2 (u_1 v_1)(u_2 v_2) \circ f$$

whence

$$(u_1 v_2)(u_2 v_1)(u_3 v_3) \cdots (u_n v_n) \circ f = (u_1 v_1)(u_2 v_2)(u_3 v_3) \cdots (u_n v_n) \circ f.$$

DEFINITION 2.3. Let U, V be disjoint subsets of E, and let f be alternating in U and in V. We say f is *symmetric from* V *to* U if

$$f = \sum_{\substack{U_1 \subseteq U \\ \# U_1 = \# V}} \mathrm{Int}(U_1, V) f$$

and that f *has Young symmetry from* V *to* U if f is symmetric from each subset of V to U.

Remark. Adopting the usual convention that a sum indexed by an empty set is 0, we see that, if $\# V > \# U$, f is only symmetric from V to U when $f = 0$. Thus, this concept is only interesting when $\# V \leqslant \# U$.

EXAMPLE 2.1. Sylvester's theorem asserts precisely that the function

$$f = [a_1 \cdots a_n][b_1 \cdots b_n]$$

has Young symmetry from $\{b_1, ..., b_n\}$ to $\{a_1, ..., a_n\}$.

EXAMPLE 2.2. Let $\| X_{ij} \|$ be an $n \times n$ matrix over a commutative ring R, and let $\pi(j_1, ..., j_p)$ denote the subdeterminant of this matrix formed from the first p rows and the $j_1, ..., j_p$ columns; then if $p \geqslant q$, the function

$$\pi(J_1, ..., J_p)\, \pi(K_1, ..., K_q)$$

of variables indexed by $E = \{J_1, ..., K_q\}$ with common domain $D = \{1, ..., n\}$, taking values in R, has Young symmetry from $\{K_1, ..., K_q\}$ to $\{J_1, ..., J_p\}$.

Proof. Apply Sylvester's theorem to the following product of $p \times p$ determinants:

$$\begin{vmatrix} X_{1,i_1} \cdots X_{1,i_p} \\ \vdots \\ X_{p,j_1} \cdots X_{p,j_p} \end{vmatrix} \cdot \begin{vmatrix} X_{1,j_1} \cdots X_{1,j_q} & \text{0's} \\ \vdots & \\ X_{p,j_1} \cdots X_{p,j_q} & I_{p-q} \end{vmatrix}$$

(cf. derivation of 2) in Section 1).

Remark. The assertion of Example 2.2 amounts to a number of equations between E-indexed functions; for example,

$$\pi(J_1J_2J_3)\, \pi(K_1K_2) = \sum_{1 \leqslant l_1 < l_2 \leqslant 3} (J_{l_1}K_1)(J_{l_2}K_2) \circ \pi(J_1J_2J_3)\, \pi(K_1K_2).$$

This in turn amounts to n^5 equations between elements of R, obtained by applying both sides to "specializations"

$$E \to D, J_1 \mapsto j_1, ..., K_2 \mapsto k_2 \qquad (j\text{'s and } k\text{'s in } D = \{1, ..., n\}),$$

and by an abuse of notation which will sometimes be convenient, these n^5 resulting equations will be indicated by

$$\pi(j_1j_2j_3)\, \pi(k_1k_2) = \sum_{1 \leqslant l_1 < l_2 \leqslant 3} (j_{l_1}k_1)(j_{l_2}k_2) \circ \pi(j_1j_2j_3)\, \pi(k_1k_2)$$

(j's and k's between 1 and n) which is equivalent to

$$\pi(j_1j_2j_3)\, \pi(k_1k_2) = \pi(k_1k_2j_3)\, \pi(j_1j_2) + \pi(k_1j_2k_2)\, \pi(j_1j_3) + \pi(j_1k_1k_2)\, \pi(j_2j_3).$$

EXAMPLE 2.3. The Riemann–Christoffel tensor R_{ijkl} satisfies:

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$$

and

$$R_{ijkl} = R_{kjil} + R_{ljki};$$

these relations assert precisely that R_{ijkl}, considered as a function of its indices, has Young symmetry from $\{i, j\}$ to $\{k, l\}$.

PROPOSITION 2.1. *Let f be alternating in each of two disjoint subsets U and V of E; let $V_1 \subseteq V$, $\# V_1 = r$. If f is symmetric from V_1 to U, then f is symmetric from every subset of V with r elements to U.*

Proof. It suffices to show that if $f(X_1, ..., X_p; Y_1, ..., Y_q)$ is a function of $p + q$ variables, with common domain D, taking values in A, which is alternating in the X's and in the Y's, and satisfies

$$f = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant p} (X_{i_1} Y_1) \cdots (X_{i_r} Y_r) \circ f \tag{7}$$

then f also satisfies

$$f = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant p} (X_{i_1} Y_{j_1}) \cdots (X_{i_r} Y_{j_r}) \circ f$$

for all distinct $j_1, ..., j_r$ between 1 and q. Indeed, letting π denote any permutation of

$$E = \{X_1, ..., X_p, Y_1, ..., Y_q\}$$

which fixes the X's and maps Y_i into Y_{j_i} for $1 \leqslant i \leqslant r$, and operating on both sides of (7) with $\pi\, \circ$, we obtain:

$$\begin{aligned}(\operatorname{sgn} \pi) f = \pi \circ f &= \sum_i \pi (X_{i_1} Y_1) \cdots (X_{i_r} Y_r) \circ f \\ &= (\operatorname{sgn} \pi) \sum_i \pi (X_{i_1} Y_1) \cdots (X_{i_r} Y_r)\, \pi^{-1} \circ f \\ &= (\operatorname{sgn} \pi) \sum_i (X_{i_1} Y_{j_1}) \cdots (X_{i_r} Y_{j_r}) \circ f.\end{aligned}$$

PROPOSITION 2.2. *Let f be alternating in the disjoint subsets U and V of E; let $v \in V$. If f is symmetric from $V - \{v\}$ to U and from $\{v\}$ to U, and if $\# V$ is not a zero divisor on A, then f is symmetric from V to U.*

Proof. Pick orderings

$$U = \{u_1, ..., u_m\}, \qquad V - \{v\} = \{v_1, ..., v_n\}.$$

By hypothesis, we have

$$f = \sum_{s=1}^{m} (u_s v) \circ f. \tag{8}$$

Let $U' = \{u_{i_1}, ..., u_{i_n}\}$ be any n-element subset of U. Applying $(u_{i_1} v_1) \cdots (u_{i_n} v_n) \circ$ to both sides of (8), and noting that, for $1 \leqslant t \leqslant n$,

$$(u_{i_t} v_t)(u_{i_t} v) = (v v_t)(u_{i_t} v_t)$$

we obtain

$$\begin{aligned} \mathrm{Int}(U', V - \{v\}) f &= (u_{i_1} v_1) \cdots (u_{i_n} v_n) \circ \sum_{s=1}^{m} (u_s v) f \\ &= \sum_{\substack{1 \leqslant s \leqslant m \\ s \neq i_1, \ldots, i_n}} \mathrm{Int}(U' U\{u_s\}, V) f + \sum_{t=1}^{n} (v v_t) \circ \mathrm{Int}(U', V - \{v\}) f. \end{aligned}$$

Summing this over all n-element subsets U' of U, and recalling that f is symmetric from $V - \{v\}$ to U (so the left side sums of f) we obtain

$$f = (n+1) \sum_{\substack{U_1 \subseteq U \\ \# U_1 = n+1}} \mathrm{Int}(U_1, V) f + \sum_{t=1}^{n} (v v_t) \circ f = (n+1) \sum_{U_1} \mathrm{Int}(U_1, V) f - nf$$

$$(n+1) f = (n+1) \sum_{\substack{U_1 \subseteq U \\ \# U_1 = \# V}} \mathrm{Int}(U_1, V) f.$$

Since $n + 1 = \# V$ is a nonzero divisor on A, we conclude that f is symmetric from V to U.

Corollary 1. *Let f be alternating in each of two disjoints subsets U and V of E. Suppose A is torsion-free. If f is symmetric from $\{v\}$ to U for some element v in V, then either $f = 0$, or else: $\# U \geqslant \# V$ and f has Young symmetry from V to U.*

Proof. Immediate from the two preceding propositions.

Note. It suffices to assume $(\# V)!$ is not a zero divisor on A.

Remarks. The conditions in the hypothesis of Corollary 1 are known to physicists as "Fock's cyclic symmetry conditions" (cf. [3]).

Corollary 2 (Sylvester's theorem). *Let R be a commutative ring; then (with notation as in Section 1) the function*

$$f = [a_1, ..., a_n][b_1, ..., b_n]$$

indexed by $\{a_1, \ldots, b_n\}$, *with common domain* R^n, *and taking values in* R, *has Young symmetry from* $\{b_1, \ldots, b_n\}$ *to* $\{a_1, \ldots, a_n\}$.

Proof. It suffices to prove the theorem under the assumption that R is a field of characteristic 0 (since Sylvester's theorem amounts to a set of identities between polynomials with integer coefficients). In this case, the preceding corollary reduces us to the assertion that f is symmetric from $\{b_1\}$ to $\{a_1, \ldots, a_n\}$, which is (4), and has already been proved.

Here is an example to show that the hypothesis that A is torsion-free cannot be dropped from Corollary 1 to Proposition 2.2:

EXAMPLE 2.4. Let f be the function indexed by $E = \{1, 2, 3, 4\}$ with common domain $D = \{1, 2, 3, 4\}$ and taking values in F_2, defined by: $f(x_1, x_2, x_3, x_4) = 0$ if two among the x's are equal or if one of x_1, x_2 equals 4; $f = 1$ otherwise. Let $U = \{1, 2\}$, $V = \{3, 4\}$. f is alternating in U and in V, but does not have Young symmetry from U to V, since

$$1 = f(1, 2, 3, 4) \neq f(3, 4, 1, 2) = 0.$$

On the other hand, f is symmetric from 1 to $\{3, 4\}$, i.e., satisfies

$$f(x_1, x_2, x_3, x_4) = f(x_3, x_2, x_1, x_4) + f(x_4, x_2, x_3, x_1) \qquad (\text{all } x_i \text{ in } D). \tag{9}$$

This is clear if two x's are equal; in the remaining cases, we note that (since we are working over F_2) (9) may be rewritten as

$$f(x_2x_1x_3x_4) + f(x_2x_3x_1x_4) + f(x_2x_4x_1x_3) = 0$$

and is thus one of the following readily verified set of four equations:

$$\begin{aligned} 0 &= f(1, 2, 3, 4) + f(1, 3, 2, 4) + f(1, 4, 2, 3) \\ &= f(2, 1, 3, 4) + f(2, 3, 1, 4) + f(2, 4, 1, 3) \\ &= f(3, 1, 2, 4) + f(3, 2, 1, 4) + f(3, 4, 1, 2) \\ &= f(4, 1, 2, 3) + f(4, 2, 1, 3) + f(4, 3, 1, 2). \end{aligned} \tag{10}$$

The following modification of the preceding example will prove useful later on (cf. Proposition 3.2):

EXAMPLE 2.5. Let g be the function indexed by $E = \{1, 2, 3, 4, 5\}$ with comlon domain $D = E$ and taking values in F_2, defined by

$$\begin{aligned} g(x_1, \ldots, x_s) &= 0 \qquad \text{if two } x\text{'s are equal;} \\ &= 0 \qquad \text{if } x_1 \text{ or } x_2 \text{ equals 4 or 5;} \\ &= 1 \qquad \text{otherwise.} \end{aligned}$$

Let $U = \{1, 2\}$, $V = \{3, 4, 5\}$. g is alternating in U and in V, but is not symmetric from U to V, since

$$1 = g(1, 2, 3, 4, 5) \neq g(3, 4, 1, 2, 5) + g(3, 5, 1, 4, 2) + g(4, 5, 3, 1, 2) = 0. \tag{11}$$

However, we claim g does have symmetry from $\{1\}$ to V, i.e., satisfies

$$g(x_1x_2x_3x_4x_5) = g(x_3x_2x_1x_4x_5) + g(x_4x_2x_3x_1x_5) + g(x_5x_2x_3x_4x_1). \tag{12}$$

This is clear if two x's are equal. If the x's are all distinct, there are three cases to consider:

1. *Neither x_1 nor x_2 equals* 5. Without loss of generality, we may assume $x_5 = 5$; then (12) reduces to (9), since in this case $g(x_1x_2x_3x_4x_5) = f(x_1x_2x_3x_4)$.

2. $x_2 = 5$. Here, both sides of (12) are 0.

3. $x_1 = 5$. Here (12) becomes (using the fact f is alternating, hence symmetric in U and V)

$$0 = f(x_2x_3x_4x_5) + f(x_2x_4x_3x_5) + f(x_2x_5x_3x_4)$$

which is one of Eqs. (10).

PROPOSITION 2.3. *Let f be alternating in each of two disjoint subsets U and V of E, with $\#U = \#V$. If f has Young symmetry from V to U, then f has Young symmetry from U to V.*

Proof. We may assume

$$f = f(X_1, ..., X_n, Y_1, ..., Y_n)$$

with $U = \{X_1, ..., X_n\}$, $V = \{Y_1, ..., Y_n\}$, and are then given that f is alternating in the X's and in the Y's, and that

$$f = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} (X_{i_1} Y_{j_1}) \cdots (X_{i_r} Y_{j_r}) \circ f \tag{13}$$

if $1 \leqslant r \leqslant n$ and $j_1, ..., j_r$ are distinct integers between 1 and n. Applying (13) with $r = n$ we see that

$$f(X_1, ..., X_n, Y_1, ..., Y_n) = f(Y_1, ..., Y_n, X_1, ..., X_n)$$

and using this to interchange X's and Y's in (13) yields

$$f = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} (X_{j_1} Y_{i_1}) \cdots (X_{j_r} Y_{i_r}) \circ f$$

as we may see more formally from the following computation:

$$\begin{aligned} f &= (X_1Y_1) \cdots (X_nY_n) \circ f = (X_1Y_1) \cdots (X_nY_n) \sum_i (X_{i_1}Y_{j_1}) \cdots (X_{i_1}Y_{j_r}) \circ f \\ &= \sum_i (X_{j_1}Y_{i_1}) \cdots (X_{j_r}Y_{i_r})(X_1Y_1) \cdots (X_nY_n) \circ f \\ &= \sum_i (X_{j_1}Y_{i_1}) \cdots (X_{j_r}Y_{i_r}) \circ f. \end{aligned}$$

Hence f has Young symmetry from U to V.

PROPOSITION 2.4. *Let f be alternating in each of three disjoint subsets U, V and W of E, and let f have Young symmetry from W to V and from V to U; then f has Young symmetry from W to U.*

Proof. We may assume

$$f = f(X_1, \ldots, X_m;\ Y_1, \ldots, Y_n;\ Z_1, \ldots, Z_p),$$

with

$$U = \{X_1, \ldots, X_m\}, \qquad V = \{Y_1, \ldots, Y_n\}, \qquad W = \{Z_1, \ldots, Z_p\};$$

the hypotheses then imply that, for $k_1, \ldots, k_r$ any distinct integers between 1 and p, we have

$$\begin{aligned} f &= \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} (Y_{j_1}Z_{k_1}) \cdots (Y_{j_r}Z_{k_r}) \circ f \\ &= \sum_{1 \leqslant j_1 < \cdots < j_r \leqslant n} \ \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant m} (Y_{j_1}Z_{k_1}) \cdots (Y_{j_r}Z_{k_r})(X_{i_1}Y_{j_1}) \cdots (X_{i_r}Y_{j_r}) \circ f \\ &= \sum_j \sum_i (X_{i_1}Z_{k_1}Y_{j_1}) \cdots (X_{i_r}Z_{k_r}Y_{j_r}) \circ f \\ &= \sum_j \sum_i (X_{i_1}Z_{k_1}) \cdots (X_{i_r}Z_{k_r})(Y_{j_1}Z_{k_1}) \cdots (Y_{j_r}Z_{k_r}) \circ f \\ &= \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant m} (X_{i_1}Z_{k_1}) \cdots (X_{i_r}Z_{k_r}) \circ f \end{aligned}$$

as was to be proved.

DEFINITION 2.4. Let U, V be disjoint subsets of E, and let f be symmetric in U and in V. We say that *f is alternating from V to U* provided the two following conditions are satisfied:

(i) $f = (-1)^{\#V} \sum_{U'\,U\subseteq,\, \#U=\#V} \mathrm{Int}(U', V) f.$

(ii) If U, V have the same cardinality n, say

$$U = \{u_1, \ldots, u_n\},\ V = \{v_1, \ldots, v_n\} \tag{14}$$

and if n is odd, then we also require: $f(\{X_i\}_{i\in E}) = 0$ whenever all $X_i \in D$ and $X_{u_1} = X_{v_1}, \ldots, X_{u_n} = X_{v_n}$.

We say that f *has Young alternation from* V *to* U if f is alternating from each subset of V to U.

Remarks. These two properties (dual to those of Definition 2.3, in a sense that will be discussed further in Section 6) are clearly independent of the choice of orderings (14) for U and V; this question does not come up if 2 is not a zero divisor on A, since then (ii) is an immediate consequence of (i). If $\#U < \#V$ and f is alternating from V to U, then $f = 0$.

The following four propositions are precise analogs of the preceding ones as are their proofs (except for some easy modifications involving (ii)), which will not be given here. In the first three of these propositions, we assume f is symmetric in each of two disjoint subsets U and V of E.

PROPOSITION 2.1a. *If $V_1 \subseteq V$, $\#V_1 = r$ and f is alternating from V_1 to U, then f is alternating from every r-element subset of V to U.*

PROPOSITION 2.2a. *If $v \in V$, f is alternating from $V - \{v\}$ to U and from $\{v\}$ to U, and $\#V$ is not a zero divisor on A, then f is alternating from V to U, provided also that* (ii) *is satisfied when $\#U = \#V$ is odd.*

COROLLARY. *If A is torsion-free, and f is alternating from $\{v\}$ to U for some $v \in V$, then either $f = 0$, or else: $\#U \geqslant \#V$, and (provided that* (ii) *is satisfied when $\#U = \#V$ is odd) f has Young alternation from V to U.*

PROPOSITION 2.3a. *If $\#U = \#V$ and f has Young alternation from V to U, then f has Young alternation from U to V.*

PROPOSITION 2.4a. *If f is symmetric in three disjoint subsets U, V, W of E, and has Young alternation from W to V and from V to U, then f has Young alternation from W to U.*

DEFINITION 2.5. By a *partitioning* of E will be meant a finite set

$$P = \{U_1, \ldots, U_r\}$$

of disjoint nonempty sets whose union is E. We say f *is symmetric* (*or alternating*) *in* P if it is such in each elemnt U_i of P. We say f *has Young alternation in* P if f is alternating in P, and has Young symmetry from U_i to U_j whenever U_i, U_j are distinct elements of P with $\#U_i \leqslant \#U_j$. Finally, we say f *has Young symmetry in* P if f is symmetric in P, and has Young alternation from U_i to U_j whenever U_i, U_j are distinct elements of P with $\#U_i \leqslant \#U_j$.

Remarks. The requirement $\# U_i \leqslant \# U_j$ in this definition is necessary, since otherwise f would be identically 0. Part of the interest of these symmetry conditions involving P is that, if A is torsion-free, they are maximal, i.e., imposing any further symmetry condition which is not a consequence of these forces f to be 0.

EXAMPLE 2.6. With notation as in Example 2.2, the function

$$F = \pi(J_{1,1} \cdots J_{1,a_1})\, \pi(J_{2,1} \cdots J_{2,a_2}) \cdots \pi(J_{s,1}, \ldots, J_{s,a_s})$$

with

$$E = \{J_{11}, \ldots, J_{s,a_s}\}, \qquad D = \{1, \ldots, n\}, \qquad A = R$$

has Young alternation in the partitioning P of E into the s disjoint subsets

$$U_i = \{J_{i,1}, \ldots, J_{i,a_i}\} \qquad (1 \leqslant i \leqslant s).$$

Note. This illustrates the connection between Young alternation and the coordinate ring of the flag manifold (which is spanned over R by expressions of the above type F).

We conclude this section by describing the ideal in $Z[S(E)]$ which annihilates functions with Young alternation in P.

PROPOSITION 2.5. *Let R be a commutative ring with* 1, *I a left ideal in* $R[S(E)]$, $\alpha \in R[S(E)]$.

If every E-indexed function f with values in an R-module, such that f is annihilated by I, is also annihilated by α, then $\alpha \in I$.

Proof. Apply the hypothesis to the E-indexed function $\bar{f}$ with common domain E, taking values in $R[S(E)]/I$, defined by:

$$\begin{aligned} \bar{f}(\sigma\colon E \to E) &= 0 \qquad && \text{if } \sigma \text{ is not a bijection,} \\ &= \sigma + I && \text{if } \sigma \in S(E). \end{aligned}$$

Note that, for $\alpha \in R[S(E)]$,

$$(\alpha \circ \bar{f})(1_E) = \alpha + I.$$

DEFINITION 2.6. Let

$$P = \{U_1, \ldots, U_s\}$$

be a partitionong of the finite set E into s disjoint nonempty subsets, and suppose that (for $1 \leqslant j \leqslant s$)

$$U_j = \{u_{j,1}, \ldots, u_{j,a_j}\}, \qquad a_j = \# U_j\,. \tag{15}$$

By the *Young alternation ideal* for P, denoted by $\mathcal{YA}(P)$, will be meant the left ideal in $\mathbb{Z}[S(E)]$ generated by all elements of the two following forms:

$$I + (u_j u_j') \qquad (u_j \text{ and } u_j' \text{ distinct elements of some } U_j \in P), \tag{16a}$$

$$G_{j,k,r} = I - \sum_{1 \leqslant m_1 < \cdots < m_r \leqslant a_j} (u_{j,m_1} u_{k,1}) \cdots (u_{j,m_r} u_{k,r}) \tag{16b}$$

(with j and k distinct integers between 1 and s, $1 \leqslant r \leqslant a_k \leqslant a_j$).

Remarks. Clearly, if f has Young alternation in P, it is left annihilated by $\mathcal{YA}(P)$. The converse is almost true, except for the troublesome distinction between alternation and skew symmetry; in any event, if f is alternating in P and annihilated by $\mathcal{YA}(P)$, then f has Young alternation in P (cf. Proposition 2.1).

Proposition 2.6. *Let R be a commutative ring with* 1, *P a partitioning of E, L a left ideal in* $R[S(E)]$ *which contains all elements of the form* (16a), *and let* $\alpha \in R[S(E)]$.

If α annihilates all E-indexed functions with values in an R-module which alternate in P and are annihilated by L, then $\alpha \in L$.

Proof. This follows upon observing that the function $\bar{f}$, constructed in the proof of Proposition 2.5, is here alternating in P (because of the additional assumption on L).

Corollary 1. *$R \cdot \mathcal{YA}(P)$ consists precisely of those elements in* $R[S(E)]$ *which annihilate all E-indexed functions f, such that f takes values in an R-module and has Young alternation in P.*

Corollary 2. *$\mathcal{YA}(P)$ is independent of the particular choice of orderings* (15) *for each U_i in P (used to define the elements* (16b)) *and depends only on P.*

Corollary 3. *If $a_1 \geqslant \cdots \geqslant a_s > 0$, those $G_{j,k,r}$ in* (16b) *with $j + 1 = k$, $1 \leqslant r \leqslant a_k$, together with the elements* (16a), *suffice to generate the left ideal $\mathcal{YA}(P)$ in* $\mathbb{Z}[S(E)]$.

Proof. It suffices to note that if f is alternating in P and annihilated by these $G_{i,i+1,k}$ then f has Young symmetry from U_{i+1} to U_i ($1 \leqslant i < s$); hence also from U_j to U_i ($1 \leqslant i < j \leqslant s$), (by Proposition 2.4); hence also from any U_j to any U_i provided $i \neq j$, $\#U_i \geqslant \#U_j$ (by Proposition 2.3); that is, f then has Young alternation in P.

3. Equations of the Flag Manifold

Let V be a finite-dimensional vector space over a field K, with basis $B = \{e_1, \ldots, e_n\}$. We denote by PV the associated projective space, by $[x]$ the image in PV of a nonzero element x of V, and by mV the direct sum of m copies of V.

If $\omega \in \Lambda^p V$, the coordinates $\Pi(j_1, \ldots, j_p) = \Pi(j_1, \ldots, j_p)(\omega)$ of ω with respect to B are defined by

$$\omega = \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} \Pi(j_1, \ldots, j_p)\, e_{j_1} \wedge \cdots \wedge e_{j_p}$$

when $j_1 < \cdots < j_p$, and are then specified uniquely for arbitrary $j_1, \ldots, j_p$ between 1 and n by the requirement that $\Pi(j_1, \ldots, j_p)$ be alternating in $j_1, \ldots, j_p$.

An element $\omega \in \Lambda^p V$ is called *pure*, or *completely reducible*, if there exist v_i in V such that

$$\omega = v_1 \wedge \cdots \wedge v_p. \tag{17}$$

In terms of the B-coordinates for ω, necessary and sufficient conditions for the complete reducibility of ω are given by the Eqs. (1) in the Introduction. If ω is nonzero and completely reducible, it may be associated with the p-dimensional subspace

$$\omega^\perp = \{v \in V: v \wedge \omega = 0\}$$

of V, which has $\{v_1, \ldots, v_p\}$ for K-basis if (17) holds. (For a proof of these assertions, cf. [14].)

The completely reducible ω in $\Lambda^p V$ form the K-rational points of the *affine Grassman variety* $G_p(V)$, which may be defined as the image of the morphism

$$PV \to \Lambda^p V,\ (v_1, \ldots, v_p) \mapsto v_1 \wedge \cdots \wedge v_p.$$

This forms a cone over the *projective Grassmann manifold* $G_{p-1}(PV)$ whose K-rational points parametrize the $(p-1)$-dimensional linear subspaces of PV.

Similarly, we define the *affine flag variety* $\mathrm{Flag}(V)$ to be the image of the morphism

$$\Phi_V: nV \to V,\ (v_1, \ldots, v_n) \mapsto v_1 + v_1 \wedge v_2 + \cdots + v_1 \wedge v_2 \wedge \cdots \wedge v_n;$$

this is a multicone over the *projective flag manifold*

$$\mathrm{Flag}(PV) \subseteq PV \times P\Lambda^2 V \times \cdots \times P\Lambda^{n-1} V.$$

The latter has for generic point

$$([v_1], [v_1 \wedge v_2], \ldots, [v_1 \wedge \cdots \wedge v_{n-1}])$$

(where $v_1, \ldots, v_{n-1}$ are independent generic vectors in V) and its K-rational points parametrize the *flags* in PV, i.e., the sequences

$$L^0 \subset L^1 \subset \cdots \subset L^{n-1} = PV,$$

where L^i is an i-dimensional linear subspace of PV. It is known that Flag(V), Flag(PV) are reduced irreducible subschemes of V, PV, respectively; note also that Flag(PV) is a subscheme of $G_0(PV) \times \cdots \times G_{n-2}(PV)$.

We shall find it more convenient to work with the affine flag variety rather than the projective flag manifold, because in Section 6 we shall wish to generalize these matters from vector spaces to modules; for the results in the present section it is a matter of indifference which of the two is studied. (Roughly speaking, the relation between Flag(V) and Flag(PV) is similar to the relation between the representation theory of $GL(n)$ and of $SL(n)$.)

We now turn to the problem of determining generators for the prime ideal of the affine flag variety Flag(V), i.e., for the kernel of the K-algebra homomorphism

$$\Phi_V^*: K[\Lambda V] \to K[nV]$$

dual to the morphism Φ_V . $K[\Lambda V]$ is the polynomial ring over K on the 2^n independent indeterminates:

$$\Pi(j_1, \ldots, j_p) \qquad (\text{for } 1 \leqslant p \leqslant n, 1 \leqslant j_1 < \cdots < j_p \leqslant n) \text{ and } \Pi(\ ),$$

which are the linear functionals on ΛV assigning to each

$$\omega = \omega_0 + \omega_1 + \cdots + \omega_n \qquad (\omega_i \in \Lambda^i V)$$

its B-coordinates

$$\Pi(j_1, \ldots, j_p)(\omega) = \Pi(j_1, \ldots, j_p)(\omega_p), \qquad \Pi(\ )(\omega) = \omega_0 .$$

Similarly, $K[nV]$ is the polynomial ring over K on the n^2 independent indeterminates X_{ij} ($1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$) defined by $X_{ij}(\sum a_{1k} f_k, \ldots, \sum a_{nk} f_k) = a_{ij}$. It is readily verified that the map Φ_V^* takes $\Pi(\ )$ to 0, and takes $\Pi(j_1, \ldots, j_p)$ to the subdeterminant of the $n \times n$ matrix $\| X_{ij} \|$ obtained from the first p rows and columns $j_1, \ldots, j_p$; its kernal is thus given by the following theorem:

THEOREM 3.1. *Let R be any commutative ring with* 1, *let n any positive integer, and let $R[X]$ be the polynomial ring over R in n^2 commuting independent indeterminates X_{ij} $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n)$. Let $R[\Pi]$ be the polynomial ring over R in* $2^n - 1$ *independent indeterminates*

$$\Pi(j_1, \ldots, j_p) \qquad (1 \leqslant p \leqslant n, 1 \leqslant j_1 < \cdots < j_p \leqslant n)$$

with the meaning of $\Pi(j_1, ..., j_p)$ *extended to all sequences* $j_1, ..., j_p$ *of integers between* 1 *and n by requiring that it be alternating in the j's. Finally, let*

$$\phi^*: R[\Pi] \to R[X]$$

be the R-algebra homomorphism which maps each $\Pi(j_1, ..., j_p)$ *into*

$$\pi(j_1, ..., j_p) = \begin{vmatrix} X_{1,j_1} & \cdots & X_{1,j_p} \\ \vdots & & \\ X_{p,j_1} & \cdots & X_{p,j_p} \end{vmatrix}. \tag{18}$$

Then the ideal $\operatorname{Ker}\phi^*$ *in* $R[\Pi]$ *is generated by the union of the images of the following functions*:

$$g_{p,q,r} = \left[I - \sum_{1 \leqslant s_1 < \cdots < s_r \leqslant p} (J_{s_1}K_1) \cdots (J_{s_r}K_r)\right] \circ \Pi(J_1 \cdots J_p)\, \Pi(K_1, ..., K_q) \tag{19}$$

($n \geqslant p \geqslant q \geqslant r \geqslant 1$, *all J's and K's having common domain* $\{1, ..., n\}$).

Remark. In other words, $\operatorname{Ker}\phi^*$ is generated as an ideal by the set of elements in $R[\Pi]$ which (employing the abuse of notation described in Example 2.2) may be written

$$\left[I - \sum_{1 \leqslant s_1 < \cdots < s_r \leqslant p} (j_{s_1}k_1) \cdots (j_{s_r}k_r)\right] \circ \Pi(j_1, ..., j_p)\, \Pi(k_1, ..., k_q) \tag{20}$$

($n \geqslant p \geqslant q \geqslant r \geqslant 1$, all j's and k's integers between 1 and n).

Proof of Theorem 3.1. Denote by L the ideal in $R[\Pi]$ generated by the union of the images of the functions (19). The fact

$$L \subseteq \operatorname{Ker}\phi^*$$

follows immediately from Example 2.2 in Section 2. Thus, to complete the proof of the theorem, it suffices to exhibit a subset H of $R[\Pi]$ with the two following properties:

(A) The image of H generates $R[\Pi]/L$ as an R-module

(B) The images under ϕ^* of the elements of H are linearly independent over R.

(Indeed, given (A), both maps in

$$RH \to R[\Pi]/L \to R[\Pi]/\operatorname{Ker}\phi^*$$

are epimorphisms of R-modules, while (B) implies the composite map is monic, whence the second map is an isomorphism and $L = \operatorname{Ker}\phi^*$.)

We define H to consist of 1 together with all monomials

$$\omega = \Pi(j_{1,1},\ldots, j_{1,a_1})\,\Pi(j_{2,1},\ldots, j_{2,a_2})\cdots\Pi(j_{s,1},\ldots, j_{s,a_s}), \tag{21}$$

all j's integers between 1 and n, with

(i) $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s \geqslant 1$,

(ii) $j_{r,1} < j_{r,2} < \cdots < j_{r,a_r} \qquad (1 \leqslant r \leqslant s)$,

(iii) $j_{1,r} \leqslant j_{2,r} \leqslant \cdots \qquad (1 \leqslant r \leqslant a_2)$.

These are the "standard power products" of Young [23], used by Hodge to study the Grassmann and flag manifolds in [7, p. 24]; in Section 5 of the same paper, Hodge proves that H has property (3). (Although dealing only with fields of characteristic 0, his proof is in fact valid for any commutative ring R.)

It remains to be proved that H has property (A). Thus, given any monomial ω of the form (21) (but not necessarily satisfying conditions (i), (ii), and (iii)), we must show ω differs from an R-linear combination of elements of H by an element in L. Unless two j's occurring in the same factor of ω are equal (in which case $\omega = 0$) we may assume (possibly replacing ω by $-\omega$) that ω satisfies conditions (i) and (ii); let $\Lambda(a_1,\ldots, a_s)$ denote the set of all expressions (21) satisfying (i) and (ii) for given $a_1,\ldots, a_s$.

If ω already has property (iii) we are done; if not, we have $j_{u,r} > j_{v,r}$ for some u, v, r satisfying $1 \leqslant u \leqslant v \leqslant s$, $1 \leqslant r \leqslant a_v$; then ω differs by one of the given generators of L from the sum which we write (by an abuse of notation)

$$\sum_{1\leqslant s_1<\cdots<s_r\leqslant p} (j_{u,s_1} j_{v,1})\cdots(j_{u,s_r} j_{v_r})\circ\omega.$$

This sum (dropping terms which are 0) may be written in the form

$$\pm\omega_1 \pm\cdots\pm\omega_N$$

with each ω_i in $\Lambda(a_1,\ldots, a_s)$; moreover, each ω_i occurs later than ω in the linear ordering on the finite set $\Lambda(a_1,\ldots, a_s)$ obtained by lexiocographically ordering the sequences

$$j_{s,a_s},\ldots, j_{1,2}, j_{1,1}$$

(associated with the elements (21) by reading their indices from right to left). Hence, a finite number of iterations of this process yields the desired representation of ω, which complete the proof of the theorem.

Remarks. The preceding proof that H has property (A) is a bit sketchy; the reader may find the details supplied in [18, Proposition 2.1]. This part of the proof is also essentially due to Hodge [6, Sect. 3], with some modifications needed because Hodge used a generating set slightly different from the one used here (cf. Corollary 3 to Theorem 5.3 in Section 5).

COROLLARY 1. *If K is a field of characteristic* 0, *Eqs.* (1) *and* (2) *of the Introduction generate the prime ideal of the flag variety.*

Proof. Immediate from Corollary 1 to Proposition 2.2 and the discussion immediately preceding Theorem 3.1.

The next two corollaries are most conveniently proved together.

COROLLARY 2. *Let $P = \{U_1, \ldots, U_s\}$ be a partitioning of a finite set E into s disjoint subsets; let*

$$U_j = \{u_{j,1}, \ldots, u_{j,a_j}\}, \qquad a_j = \# U_j \qquad (1 \leqslant j \leqslant s)$$

and assume

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s \geqslant 0.$$

Then a free $\mathbb{Z}$-basis for $\mathbb{Z}[S(E)]\mathscr{Y}\mathfrak{A}(P)$ (whence also a free R-basis for $R[S(E)]/R\mathscr{Y}\mathfrak{A}(P)$, R being any commutative ring with 1) *is furnished by the image of the set H' (with $H' \subseteq S(E) \subseteq \mathbb{Z}[S(E)]$) defined as follows:*

We first define a total ordering $<$ on E by

$$u_{1,1} < \cdots < u_{1,a_1} < u_{2,1} < \cdots < u_{2,a_2} < \cdots < u_{s,a_s}$$

and then define $\sigma \in S(E)$ to be standard *if it satisfies the two following conditions:*

(a) $\sigma u_{j,1} < \cdots < \sigma u_{j,a_j} \qquad (1 \leqslant j \leqslant s)$;

(b) $\sigma u_{1,k} < \sigma u_{2,k} < \cdots \qquad (1 \leqslant k \leqslant a_2)$;

finally we define H' to be the set of all standard $\sigma \in S(E)$.

COROLLARY 3. *Let R be a commutative ring with* 1, *n a positive integer, $R[X]$ as in Theorem* 3.1. *Define a left action of S_n on $R[X]$ by*

$$\sigma \circ f(X_{1,1}, \ldots, X_{i,j}, \ldots, X_{n,n}) = f(X_{1,\sigma 1}, \ldots, X_{i,\sigma j}, \ldots, X_{n,\sigma n}) \qquad (\sigma \in S_n)$$

Suppose $a_1 \geqslant \cdots \geqslant a_s \geqslant 0, \sum_j a_j = n$.

Then the left annihilator in $\mathbb{Z}[S_n]$ of the element

$$\omega = \pi(1, \ldots, a_1)\, \pi(a_1 + 1, \ldots, a_1 + a_2) \cdots \pi(a_1 + \cdots + a_{s-1} + 1, \ldots, n)$$

in $R[X]$ (notation as in Theorem 3.1) *is $\mathscr{Y}\mathfrak{A}(P)$, where $P = \{U_1, \ldots, U_s\}$ is the partitioning of $\{1, \ldots, n\}$ defined by*

$$U_1 = \{1, \ldots, a_1\}, \qquad U_2 = \{a_1 + 1, \ldots, a_1 + a_2\}, \qquad \textit{etc., i.e.,}$$
$$U_j = \{k\colon a_1 + \cdots + a_{j-1} < k \leqslant a_1 + \cdots + a_j\} \qquad (1 \leqslant j \leqslant s)$$

Proof of Corollary 2 *and Corollary* 3. By precisely the same reasoning used in the proof of Theorem 3.1 to show H satisfies Condition A, we may show (under the hypothesis of Corollary 2) that every $\sigma \in S(E)$ differs from a Z-linear combination of standard $t \in H'$ by an element in the left ideal in $\mathbb{Z}[S(E)]$ generated by the elements

$$G_{p,q,r} = I - \sum_{1 \leqslant s_1 < \cdots < s_r \leqslant a_p} (u_{p,s_1} u_{q,1}) \cdots (u_{p,s_r} u_{p,r}) \tag{22a}$$
$$(1 \leqslant p \leqslant s,\ 1 \leqslant q \leqslant s,\ p \neq q,\ a_p \geqslant a_q \geqslant r \geqslant 1)$$

together with the elements

$$I + (u_{j,p} u_{j,q}) \qquad (1 \leqslant j \leqslant s,\ 1 \leqslant p < q \leqslant a_j) \tag{22b}$$

(for we may use these generators to reduce σ to a linear combination of standard permutations, as the relations (20) were used in the proof of Theorem 3.1.). But the left ideal generated by (22a), (22b) is $\mathscr{YA}(P)$, which shows:

(1) the image of H' spans $\mathbb{Z}[S(E)]\mathscr{YA}(P)$ over $\mathbb{Z}$.

For the remainder of the proof we shall assume (as we may without loss of generality in Corollary 2) that $E = \{1,..., n\}$, that the partitionings in Corollaries 2 and 3 coincide, and that the ordering (22) is the usual one on $\{1,..., n\}$, which means

$$u_{j,1} = a_1 + \cdots + a_{j-1} + 1,\ u_{j,2} = a_1 + \cdots + a_{j-1} + 2, ..., u_{j,a_j} = a_1 + \cdots + a_j .$$

For $\sigma \in S_n$, we have

$$\sigma \circ \omega = \pi(\sigma 1, ..., \sigma a_1)\, \pi(\sigma(a_1 + 1), ..., \sigma(a_1 + a_2)) \cdots.$$

Now observe the two following facts:

(2) *ω is annihilated by $R \cdot \mathscr{YA}(P)$* (this follows from Theorem 3.1, and indeed from Sylvester's theorem).

(3) *$\sigma \mapsto \sigma \circ \omega$ maps H' injectively into the set H of Theorem* 3.1, *which was proved linearly independent over R.*

Corollaries 2 and 3 are both immediate consequences of (1), (2), and (3) (with $R = \mathbb{Z}$ for Corollary 2).

COROLLARY 4. *Let F denote the function in Example* 2.6; *then the annihilator of F in $R[S(E)]$ is $R \cdot \mathscr{YA}(P)$.*

Proof. With notation as in Example 2.6, let λ denote the map $D \to E$ which sends

$$J_{1,1}, ..., J_{1,a_1}, J_{2,1}, ..., J_{2,a_2}, ..., J_{s,a_s}$$

(in that order) into $1,..., n$.

It suffices to show the annihilator of F is contained in $R \cdot \mathcal{YO}(P)$; this follows, using the notation and result of the preceding Corollary 3, from

$$\alpha \in R[S(E)] \quad \text{and} \quad \alpha \circ F = 0 \Rightarrow 0 = (\alpha \circ F)(\lambda) = \alpha \circ \omega \Rightarrow \alpha \in R \cdot \mathcal{YO}(P).$$

The final result we shall prove in this section, is that the first corollary to Theorem 3.1 does not hold without the hypothesis char $K = 0$:

PROPOSITION 3.2. *If V is a 5-dimensional vector space over a field of characteristic 2, the left sides of the equations* (1), (2) *in the Introduction do not generate the prime ideal of* Flag(V).

Proof. In the present notation, the assertion to be proved is, that Ker Φ_V^* properly contains the ideal in $K[\Lambda V]$ generated by $\Pi(\ )$ together with the elements

$$\Pi(i_1, \ldots, i_p)\, \Pi(j_1, \ldots, j_q) - \sum_{s=1}^{p} \Pi(i_1 \cdots i_{s-1} j_1 i_{s+1} \cdots i_p)\, \Pi(i_s j_2 \cdots j_p) \qquad (23)$$

($5 \geqslant p \geqslant q \geqslant 1$, all i's and j's between 1 and 5). It suffices to show the element

$$\begin{aligned} G = {} & \Pi(3, 4, 5)\, \Pi(1, 2) \\ & - \Pi(1, 2, 5)\, \Pi(3, 4) - \Pi(1, 4, 2)\, \Pi(3, 5) - \Pi(3, 1, 2)\, \Pi(4, 5) \end{aligned}$$

in Ker Φ_V^* (which lies in the range of the function $g_{3,2,2}$; cf. (19)) is not a $K[\Lambda V]$-linear combination of $\Pi(\ )$ and the elements (23). By simple degree considerations, it suffices to show G is not a F_2-linear combination of the following special elements in (23):

$$\begin{aligned} \Pi(i_1, i_2, i_3)\, \Pi(j_1, j_2) - \Pi(j_1, i_2, i_3)\, \Pi(i_1, j_2) - \Pi(i_1, j_1, i_3)\, \Pi(i_2, j_2) \\ - \Pi(i_1, i_2, j_1)\, \Pi(i_3, j_2) \\ \text{(all } i\text{'s and } j\text{'s between 1 and 5).} \end{aligned} \qquad (24)$$

Let W denote the 100-dimensional subspace of $F_2[\Lambda V]$ generated over F_2 by all $\Pi(i_1 i_2 i_3)\, \Pi(j_1 j_2)$ (i's and j's between 1 and 5); then W contains G and the elements (24), and it suffices to define a linear functional Ψ on W which is 0 for the elements (24) and takes on the value 1 for G. Such a linear functional is supplied by the function g of Example 2.5 in Section 2; we define

$$\Psi\colon W \to F_2,\ \Pi(i_1, i_2, i_3)\, \Pi(j_1, j_2) \to g(j_1, j_2, i_1, i_2, i_3),$$

Ψ vanishes on the elements (24), because $g(j_1, j_2, i_1, i_2, i_3)$ is symmetric from $\{j_1\}$ to $\{i_1, i_2, i_3\}$ and $\Psi(G) = 1$ because of (11).

Note. By similar reasoning, the (still open) question of whether, for V an n-dimensional vector space over a field k, the prime ideal of the Grassmann variety $G_n(V)$ is generated by Eqs. (1) in the Introduction, may be seen to be equivalent to the following purely combinatorial question (involving the given field k and integer n).

PROBLEM. *Let $f(X_1,..., X_n, Y_1,..., Y_n)$ be a function of $\{X_1,..., Y_n\}$-indexed variables, with common domain D, taking values in a vector-space over k, and alternating in $U = \{X_1,..., X_n\}$ and $V = \{Y_1,..., Y_n\}$. If f is symmetric from U to V and from $\{X_1\}$ to V, must f have Young symmetry from U to V?*

The answer is in the affirmative if char $k = 0$. It is conceivable the answer is in the affirmative even if f is allowed to take values in an arbitrary Abelian group. There is no loss of generality involved in assuming $D = \{1,..., 2n\}$. It is easy to see that if f has symmetry from one (hence every, by Proposition 2.1) i-element subset of U to V, then f has symmetry from $(n - i)$-element subsets of U to V; hence the answer is in the affirmative if $n \leqslant 3$. The first nontrivial case is thus $n = 4$, char $k = 2$; some work involving the use of a computer (in which Kenneth Baclawski was kind enough to aid the author) seems to prove the answer is affirmative in this particular case also.[1]

4. The Relation between the Young Alternater and the Young Alternation Ideal

Throughout this section, we shall maintain the following assumptions of Section 2:

$f \in A^{(D^E)}$, $(A, +)$ is an Abelian group, E is a finite set, D is any set. Also, we shall adopt, for the remainder of this paper, the usual convention whereby U being any subset of E, we identify $S(U)$ in $S(E)$, $\mathbb{Z}[S(U)]$ in $\mathbb{Z}[S(E)]$, via the injection i: $S(U) \to S(E)$ defined by

$$\begin{aligned} i(\pi)e &= \pi e \qquad \text{if} \quad e \in U \qquad (\pi \in S(U), e \in E),\\ &= e \qquad \text{if} \quad e \notin U. \end{aligned}$$

DEFINITION 4.1. If $U \subseteq E$, we define

$$\mathrm{Sym}(U) = \sum_{\sigma \in S(U)} \sigma \in \mathbb{Z}[S(E)],$$

$$\mathrm{Alt}(U) = \sum_{\sigma \in S(U)} (\mathrm{sgn}\ \sigma)\sigma \in \mathbb{Z}[S(E)].$$

[1] E. Procesi and C. de Concini, after examining a preprint of the present paper, have notified the author that the case $n = 5$, char $= 2$ gives a counterexample.

If P is a partitioning of E (cf. Definition 2.5) we define:

$$\mathrm{Sym}(P) = \prod_{U \in P} \mathrm{Sym}(U), \qquad \mathrm{Alt}(P) = \prod_{U \in P} \mathrm{Alt}(U),$$

$$S(P) = \{\sigma \text{ in } S(E)\colon x \in U \in P \Rightarrow \sigma x \in U\} = S(U_1) \cdots S(U_s).$$

Remarks. If U, U' are disjoint subsets of E, $S(U)$ and $S(U')$ commute elementwise (with the above identifications); it follows that each of $\mathrm{Sym}(U)$, $\mathrm{Alt}(U)$ commutes with each of $\mathrm{Sym}(U')$, $\mathrm{Alt}(U')$, whence the above products make sense. Note that $\mathrm{Sym}(U) \circ f$, $\mathrm{Sym}(P) \circ f$ are symmetric in U and in P, respectively; similarly, $\mathrm{Alt}(U) \circ f$ and $\mathrm{Alt}(P) \circ f$ are alternating in U and in P, respectively.

While Young alternation, and the Young alternation ideal, are defined with respect to a partitioning $P = \{U_1, \ldots, U_s\}$ of E into disjoint sets, *the definition of the Young alternater requires in addition the assignment of a linear ordering to each of these sets*, as we shall now see. This discrepancy deserves further discussion, and we shall return to it.

Definition 4.2. Let $P_< = \{(U_1, <_1), \ldots, (U_s, <_s)\}$ be a set of linearly ordered disjoint nonempty sets whose union is E; for $1 \leqslant j \leqslant s$, let the elements of U_j in the ordering $<_j$ be $u_{j,1}, \ldots, u_{j,a_j}$ (so $a_j = \# U_j$).

We define the *jth row of* $P_<$ to be U_j $(1 \leqslant j \leqslant s)$ and let P denote the partitioning $(U_1, \ldots, U_s)$ of E.

We define the *kth column of* $P_<$ to be the (unordered) set V_k consisting of all U_{jk} in E with second index k (for $1 \leqslant k \leqslant \max a_j$), and we let $P_<^*$ denote the partitioning $\{V_1, V_2, \ldots\}$ of E.

Finally, we define the *Young alternater for* $P_<$ to be the element

$$YA(P_<) = \mathrm{Alt}(P)\,\mathrm{Sym}(P_<^*)$$

in $\mathbb{Z}[S(E)]$, and define the *Young symmetrizer for* $P_<$ to be

$$YS(P_<) = \mathrm{Sym}(P)\,\mathrm{Alt}(P_<^*).$$

Remark. Note that this definition does not depend on a choice of ordering in P, i.e., between the s elements $U_1, \ldots, U_s$ of P, as in the usual formulation via tableaux.

Theorem 4.1. If R is a commutative ring with 1, E, and $P_<$ as in Definition 4.2, *then the left annihilator of* $YA(P_<)$ *in* $R[S(E)]$ *is* $R \cdot \mathscr{Y}\mathscr{A}(P)$.

Proof. We assume, as we may without loss of generality, that (using the notation of Definition 4.2):

$$a_1 \geqslant \cdots \geqslant a_s > 0, \qquad E = \{1, \ldots, n\},$$

$$U_1 = \{1, \ldots, a_1\}, \qquad U_2 = \{a_1 + 1, \ldots, a_1 + a_2\}, \text{ etc.},$$

the ordering $<_j$ in

$$U_j = \{a_1 + \cdots + a_{j-1} + 1, \ldots, a_1 + \cdots + a_j\}$$

being the usual one.

To prove the theorem, it suffices to construct an element ω in some left $R[S_n]$-module with the two following properties:

(1) The annihilator of ω in $R[S_n]$ is $R \cdot \mathscr{Y}\mathscr{A}(P)$.

(2) If $\alpha \in R[S(E)]$, then $\alpha \circ \omega = 0 \Leftrightarrow \alpha YA(P_<) = 0$.

For this purpose, we use the element

$$\omega = \pi(1, \ldots, a_1)\pi(a_1 + 1, \ldots, a_1 + a_2) \cdots \pi(a_1 + \cdots + a_{s-1} + 1, \ldots, n) \in \Pi[X]$$

(using the notation of Theorem 3.1), the left action of $R[S_n]$ on $\Pi[X]$ being the "column action" defined by

$$\sigma \circ f(X_{1,1}, \ldots, X_{i,j}, \ldots) = f(X_{1,\sigma 1}, \ldots, X_{i,\sigma j}, \ldots) \qquad (\sigma \in S_n) \tag{25}$$

We know that (1) holds by Corollary 3 to Theorem 3.1, and must next verify (2).

We begin by noting that

$$\omega = \begin{vmatrix} X_{1,1} & \cdots & X_{1,a_1} \\ \vdots & & \\ X_{a_1,1} & \cdots & X_{a_1,a_1} \end{vmatrix} \cdot \begin{vmatrix} X_{1,a_1+1} & \cdots & X_{1\ a_1+a_2} \\ \vdots & & \vdots \\ X_{a_2,a_1+1} & \cdots & X_{a_2,a_1+a_2} \end{vmatrix}$$

$$\cdots \begin{vmatrix} X_{1,a_1+\cdots+a_{s-1}+1}, \ldots, X_{1,n} \\ \vdots \\ X_{a_s,a_1+\cdots+a_{s-1}+1}, \ldots, X_{a_s,n} \end{vmatrix}$$

may be rewritten as

$$\sum_{\sigma_1 \in S(U_1)} \cdots \sum_{\sigma_s \in S(U_s)} \left(\prod_{i=1}^{s} (\operatorname{sgn} \sigma_i)\, \sigma_i \right) \circ X_{1,1}X_{2,2} \cdots X_{a_1,a_1}X_{1,a_1+1} \cdots X_{a_2,a_1+a_2} \cdots X_{a_s,n}$$

$$= \operatorname{Alt}(P) \circ \prod_j \prod_{k \in V_j} X_{j,k},$$

where the action of $R[S_n]$ is that given by (25) and the V's are the columns of $P_<$ as given by Definition 4.2; if the reader wishes, these V's may be thought of as the columns of the Young's tableau

$$\begin{array}{l} 1, \ldots\ldots\ldots\ldots\ldots\ldots, a_1 \\ a_1 + 1, \ldots\ldots\ldots, a_1 + a_2 \\ \vdots \\ a_1 + \cdots + a_{s-1} + 1, \ldots, n \end{array}$$

Thus we have

$$\omega = \operatorname{Alt}(P) \circ \theta, \qquad \theta = \prod_{j} \prod_{k \in V_j} X_{j,k} \in \Pi[X]. \tag{26}$$

We next observe that if $\sigma \in S_n$,

$$\sigma \circ \theta = \theta \Leftrightarrow \theta \in S(V_1)\, S(V_2) \cdots = S(P_<^*)$$

whence (the r_σ denoting elements of R)

$$\left(\sum_{\sigma \in S_n} r_\sigma \sigma\right) \circ \theta = 0 \Leftrightarrow \sum_{\tau_1 \in S(P_<^*)} r_{\tau\tau_1} = 0 \quad \text{for all } \tau \in S_n \Leftrightarrow \left(\sum_{\sigma \in S_n} r_\sigma \sigma\right) \operatorname{Sym}(P_<^*) = 0.$$

Hence, if $\sigma \in R[S_n]$, then

$$0 = \alpha \circ \omega = [\alpha \operatorname{Alt}(P_<)] \circ \theta \Leftrightarrow 0 = \alpha \operatorname{Alt}(P_<) \operatorname{Sym}(P_<^*) = \alpha YA(P_<),$$

as was to be proved.

Remarks. The special case of this theorem when $R = \theta$ was essentially known to Young [24, Sect. 12]. Let us consider one consequence of the theorem just proved, in this special case that R is a field of characteristic 0. In this case one knows that $R[S(E)]$ is semisimple, and that the right ideal generated by $YA(P_<)$ in $R[S(E)]$ is minimal; recall also that $YA(P_<)$ is a quasi-idempotent:

$$[YA(P_<)]^2 = \gamma YA(P_<), \tag{27}$$

where γ is a positive integer (cf. [2] or [15]). Let A be an R-module, D any set; it then follows from Theorem 4.1 that $\mathcal{YA}(P)$ is a maximal left ideal, and that $YA(P_<) \circ$ projects $A^{(D^E)}$ onto the sub-R-module of functions which have Young alternation in P; thus, this class of functions may be characterized as

$$YA(P_<) \circ A^{(D^E)} = \left\{ f \in A^{(D^E)} : \frac{1}{\gamma} YA(P_<) \circ f = f \right\}.$$

There is, it seems to the author, one disadvantage possessed by this latter characterization (which is related to Weyl's definition of "quantics") as opposed to that by $\mathcal{YA}$ (which is related to the extension of Weyl's construction by Carter and Lusztig), namely, that the class of functions in question depends only on the partitioning P, not on the extra structure required in the definition of $P_<$ (i.e., the assignment of an ordering to each set in P). The topics we are now studying stand in the closest connection with the representation theory of the symmetric group; for this reason one must be more careful than usual about making nonnatural identifications between a set U_i of cardinality a_i and the set $\{1,\ldots, a_i\}$.

In general, the combinatorial structure of a Young tableau involves more than that of a partitioning into ordered sets (though only the latter is involved in the definition of the Young alternater), which in turn involves more structure than is required for the questions we have been studying, which only depend on the partitoning P. (This extra structure is, however, helpful in situations where a basis is required, and is implicit in the definition of the basis H of "standard" elements used in the proof of Theorem 3.1; it is also useful in defining the well-known "hookproduct" rule for the integer γ in (27).)

5. THE TURNBULL–HODGE GENERATORS

The reader is reminded of the classical convention for writing symmetric polynomials, whereby, for instance, the symmetrization

$$\sum{}' x_1{}^2x_2{}^2x_3$$

of the polynomial $x_1{}^2x_2{}^2x_3$ in $k[x_1, x_2, x_3]$ means

$$x_1{}^2x_2{}^2x_3 + x_1{}^2x_3{}^2x_2 + x_2{}^2x_3{}^2x_1,$$

rather than

$$\sum_{\sigma\in S_3} x_{\sigma1}^2x_{\sigma2}^2x_{\sigma3} = 2\sum{}' x_1{}^2x_2{}^2x_3,$$

the latter form being undesirable if char $k = 2$. It will now be useful to formalize this convention.

DEFINITION 5.1. Let E be a finite set, $P = \{U_1, \dots, U_s\}$ a partitioning of E into disjoint nonempty subsets, and V another subset of E. Let f be a function of E-indexed variables, which have a common domain, with f taking values in some Abelian group.

Pick any set L of left coset representatives for

$$S(P) \cap S(V) = S(U_1 \cap V) \cdots S(U_s \cap V) \text{ in } S(V):$$
$$S(V) = \bigcup_{\lambda\in L} \lambda[S(V) \cap S(P)].$$

If f is symmetric in P, we set

$$\mathrm{Sym}(V \mid P)f = \sum_{\lambda\in L} \lambda \circ f;$$

similarly, if f is alternating in P, we set

$$\mathrm{Alt}(V \mid P)f = \sum_{\lambda\in L} (\mathrm{sgn}\, \lambda)\lambda \circ f.$$

Remarks. These modified symmetrizers and alternators differ (when they are defined) from the more usual ones only by a numerical factor: we have $\mathrm{Sym}(V)\circ f = C\cdot \mathrm{Sym}(V \mid P)f$ if f is symmetric in P, $\mathrm{Alt}(V)\circ f = C\cdot \mathrm{Alt}(V\mid P)f$ if f is alternating in P, with

$$C = \prod_{i=1}^{s} (\#[U_i \cap V])! = \#[S(V)\cap S(P)]. \tag{28}$$

The point is this: If f is symmetric in P, then each term in $\mathrm{Sym}(V)\circ f = \sum_{\sigma\in S(V)} \sigma\circ f$ is repeated C times, since $\sigma\circ f = \sigma_1\circ f$ if $\sigma_1^{-1}\sigma \in S(P)$; by extending the sum only over L, we select one out of each set of C equal terms.

Note that Theorem 5.3 below and its corollaries all become false if this modified alternation is replaced by the usual kind.

$\mathrm{Sym}(V\mid P)f$ is only defined if f in symmetric in P, and is then symmetric in V (but not, in general, symmetric in P). Similarly, if f is alternating in P, then $\mathrm{Alt}(V\mid P)f$ is alternating in V.

$\mathrm{Sym}(V\mid P)$ and $\mathrm{Alt}(V\mid P)$ are easily seen to be independent of the particular choice L of left coset representatives. If we assume a particular total ordering

$$e_1 < e_2 < \cdots < e_n$$

for E, we may then choose L to be the set

$$L_< = L_<(P, V) = \{\sigma\in S(E)\colon \text{for } 1\leqslant i\leqslant s,\ \sigma \mid (U_i\cap V) \text{ is monotonic increasing}\}. \tag{29}$$

The earliest reference the author has for the operation $\mathrm{Alt}(V\mid P)$ on P-alternating functions is Turnbull [20, p. 27, 43], where this modified alternation is indicated by placing dots over the variables to be alternated.

EXAMPLE 5.1. With $E = \{1, 2, 3\}$, $U_1 = \{1, 2\}$, $U_2 = \{3\}$, we have

$$\mathrm{Sym}(E \mid U_1, U_2)\, x_1{}^2x_2{}^2x_3 = \textstyle\sum' x_1{}^2x_2{}^2x_3\,.$$

If $<$ is the usual ordering on E, $L_<$ here consists of those σ in S_3 for which $\sigma 1 < \sigma 2$, i.e., of $\binom{1\,2\,3}{1\,2\,3}$, $\binom{1\,2\,3}{1\,3\,2}$, $\binom{1\,2\,3}{2\,3\,1}$.

PROPOSITION 5.1. *Let R be a commutative ring, and let f denote the function*

$$f(\{C_{i,j}\}_{(i,j)\in E}) = [C_{1,1}\cdots C_{1,n}][C_{2,1}\cdots C_{2,n}]\cdots[C_{m,1}\cdots C_{m,n}]$$

of variables indexed by

$$E = \{(i,j)\colon 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n\}$$

with common domain R^n, and taking values in R (where $[C_1,\ldots, C_n]$ denotes, as in Section 1, the $n\times n$ determinant whose ith column is $C_i\in R^n$).

Let $P_0 = \{U_1, ..., U_m\}$ denote the partitioning of E into the m disjoint subsets

$$U_i = \{(i, j): 1 \leqslant j \leqslant n\} \qquad (1 \leqslant i \leqslant m).$$

If V is any subset of E of cardinality $> n$, then $\mathrm{Alt}(V \mid P_0)f = 0$ (*a fortiori*, $\mathrm{Alt}(V) \circ f = 0$).

Proof. Note first that f alternates in P, so $\mathrm{Alt}(V \mid P_0)f$ makes sense. The proof is now the same as that of (4) (which this generalizes):

Any R-multilinear function of variables with common domain R^n, which alternates in $n+1$ of the variables must be 0, and $\mathrm{Alt}(V \mid P_0)f$ has these properties.

Note. This result was proved (using a rather complicated argument which proves a bit more) by Turnbull [20, p. 45, 48].

For the remainder of this section, we resume the assumptions of Section 2: $f \in A^{(D^E)}$, $(A, +)$ is an Abelian group, E is a finite set, D is any set.

THEOREM 5.2. *Let f have Young alternation in a partitioning $P = \{U_1, ..., U_s\}$ of E. If V is a subset of E, of cardinality greater than that of any U_i it meets, then*

$$\mathrm{Alt}(V \mid P)f = 0$$

(*a fortiori*, $\mathrm{Alt}(V) \circ f = 0$).

Proof. Let L be a complete set of left coset representatives for $S(P) \cap S(V)$ in $S(V)$, and set

$$\alpha = \sum_{\lambda \in L} (\mathrm{sgn}\ \lambda)\lambda \in \mathbb{Z}[S(E)];$$

by definition,

$$\mathrm{Alt}(V \mid P)f = \alpha \circ f,$$

and it suffices to prove $\alpha \in \mathcal{YA}(P)$.

We may assume without loss of generality that V meets every U_i, since neither α nor the truth of the assertion to be proved are affected if we subtract from E and from P those U_i disjont from V. We may also assume that $E = \{1, ..., n\}$, and that

$$U_i = \{j: a_1 + \cdots + a_{i-1} < j \leqslant a_1 + \cdots + a_i\}, a_1 \geqslant \cdots \geqslant a_s > 0, \qquad \sum_{i=1}^{s} a_i = n.$$

We now invoke the element ω of Corollary 3 to Theorem 3.1 (with $R = \mathbb{Z}$) whose left annihilator in $\mathbb{Z}[S_n]$ is $\mathcal{YA}(P)$; thus, it suffices to prove that the following expression (using the notation of the corollary just cited) equals 0:

$$\alpha \circ \omega = \sum_{\lambda \in L} (\mathrm{sgn}\ \lambda)\ \pi(\lambda 1, ..., \lambda a_1)\ \pi(\lambda(a_1 + 1), ..., \lambda(a_1 + a_2)) \cdots \pi(\cdots, \lambda n).$$

Observe that $\pi(i_1, \ldots, i_a)$ is the determinant $[C_{i_1} \cdots C_{i_a} D_{a+1} \cdots D_n]$, where C_i is the ith column of the $n \times n$ matrix $\| X_{ij} \|$ used in the definition of ω, and D_i is the column of length n with 1 in the ith place and 0's elsewhere. Applying Proposition 5.1 with $R = \mathbb{Z}[X_{1,1}, \ldots, X_{n,n}]$ we see that the E-indexed function $f \in R^{(R^n)^E}$ defined by

$$f(\{C_{i,j}\}_{(i,j) \in E}) = \mathrm{Alt}(V \mid P_0) \prod_{i=1}^{s} [C(i, 1), \ldots, C(i, n)]$$

(with P_0 as in Proposition 5.1) is identically 0. Applying this to the map $C: E \to R^n$ defined by

$$\begin{aligned} C(i, j) &= C_{a_1 + \cdots + a_{i-1} + j} \qquad && (1 \leqslant j \leqslant a_i), \\ &= D_j && (a_i < j \leqslant n) \end{aligned}$$

we obtain $\alpha \circ \omega = 0$, which completes the proof. (cf. argument in Example 2.2)

Note. The same method may be used to extend a number of other determinantal identities of Turnbull [20, p. 45, p. 48] to properties of functions with Young alternation.

THEOREM 5.3. *Let E be the disjoint union of two nonempty subsets U and V with $\#U \geqslant \#V$. Assume f is alternating in U and in V. Then f has Young symmetry from V to U, if and only if the following condition is satisfied:*
For every subset of E, with $\#F = 1 + \#U$, we have

$$\mathrm{Alt}(F \mid \{U, V\}) f = 0. \tag{30}$$

Proof. Let the total ordering $<$ on E be defined by

$$\begin{gathered} U = \{J_1, \ldots, J_p\}, \qquad V = \{K_1, \ldots, K_q\}, \\ \#U = p, \qquad \#V = q, \\ J_1 < \cdots < J_p < K_1 < \cdots < K_q . \end{gathered}$$

Consider the left ideal $\mathscr{Y}$ in $\mathbb{Z}[S(E)]$ generated by all the following elements:

$$I + (J_i J_j) \qquad (1 \leqslant i < j \leqslant p), \tag{31a}$$

$$I + (K_i K_j) \qquad (1 \leqslant i < j \leqslant p), \tag{31b}$$

$$\Gamma_F = \sum \{(\mathrm{sgn}\, \lambda)\lambda : \lambda \in S(E), \lambda \mid F \cap U \text{ and } \lambda \mid F \cap V \text{ increasing}\}$$
$$(F \subseteq E, \#F = 1 + \#U). \tag{31c}$$

Then (30) holds precisely when f is left annihilated by $\mathscr{Y}$ and to prove the theorem it suffices to prove

$$\mathscr{Y} = \mathscr{Y}\mathscr{O}\!\mathscr{l}(P).$$

Note first that, if f has Young alternation from V to U, hence, has Young symmetry in $\{U, V\}$, then f satisfies (30) by Theorem 5.2; by Proposition 2.6, Corollary 1, it follows that

$$\mathscr{Y} \subseteq \mathscr{Y}\mathscr{A}(P).$$

Thus we have the canonical epimorphism of left $\mathbb{Z}[S(E)]$-modules

$$\iota\colon \mathbb{Z}[S(E)]/\mathscr{Y} \to \mathbb{Z}[S(E)]/\mathscr{Y}\mathscr{A}(P)$$

and are done if we can prove ι is monic.

By Corollary 2 to Theorem 3.1, $\mathbb{Z}[S(E)]/\mathscr{Y}\mathscr{A}(P)$ is $\mathbb{Z}$-free on the image of the subset H' of $S(E)$ consisting of all $\pi \in S(E)$ which satisfy:

$$\pi J_1 < \cdots < \pi J_p \quad \text{and} \quad \pi K_1 < \cdots < \pi K_q, \tag{32a}$$

$$\pi J_i < \pi K_i \quad \text{for} \quad 1 \leqslant i \leqslant q. \tag{32b}$$

Thus, we are done if we can prove the assertion that $\mathbb{Z}[S(E)]/\mathscr{Y}$ is spanned by the image of H' (since then ι is surely monic). This last assertion is proved by exactly the same argument we used to prove the corresponding assertion with $\mathscr{Y}\mathscr{A}(P)$ in place of $\mathscr{Y}$:

We must show every $\sigma \in S(E)$ is congruent mod $\mathscr{Y}$ to a $\mathbb{Z}$-linear combination of elements in H'. Without loss of generality, we may assume σ satisfies (32a), since at any rate σ differs from such a permutation by a $\mathbb{Z}[S(E)]$-linear combination of elements (31a), (31b). If $\sigma \in H'$ we are done; if not, for some i between 1 and q,

$$\sigma J_i > \sigma K_i .$$

To visualize the next step better, it is helpful to associate with σ the Young tableau

$$\begin{array}{ll} \sigma J_1, & \sigma K_1 \\ \vdots & \\ \sigma J_i, & \sigma K_i \\ \vdots & \vdots \\ \sigma K_p & \sigma K_q \end{array}$$

Note that both columns are increasing from top to bottom, with respect to the given ordering on E. We next make use of the element Γ_F in H' (cf. (31c)) with $F = \{K_1, ..., K_i, J_i, ..., J_p)$ to express σ as the sum of an element $\sigma\Gamma_F$ in $\mathscr{Y}$ and the linear combination

$$\sum \sigma_i' \equiv \sum \pm \sigma_i \qquad (\text{mod } \mathscr{Y})$$

with each σ_i having its tableau derived from the above tableau for σ by interchanging several elements in the second column with the same number of strictly larger elements in the first column (strictly larger, since

$$\sigma K_1 < \cdots < K_i < \sigma J_i < \cdots < \sigma J_q;$$

we again assume each σ_i satisfies (32a)).

It only remains to impose a total ordering on the set of all σ satisfying (32a), in such a way that σ always precedes each of the σ_i obtained above (since then repetition of this procedure terminates, with an expression for σ of the desired form, after a finite number of steps). We may do this by the lexicographic ordering described in the proof of Theorem 3.1, or, following Hodge [6] and assuming (as we may without loss of generality) that $E \subseteq \mathbb{Z}$ and so the J's and K's are distinct integers, it is sufficent to note that the sum of the integers in the first column of the Young's tableau decreases as we pass from σ to each σ_i.

COROLLARY 1. *Let f be alternating in a partitioning P of E; then the following condition is necessary and sufficient for f to have Young alternation in P:*

If F is any subset of E that meets precisely two subsets U, U' of E belonging to P, where furthermore

$$\#F = 1 + \max(\#U, \#U'),$$

then

$$\mathrm{Alt}(V \mid P)f = 0.$$

COROLLARY 2. *Let $P = \{U_1, ..., U_s\}$ be a partitioning of E into s disjoint sets, and let $<$ be a total ordering on E. Assume also*

$$\#U_1 \geqslant \cdots \geqslant \#U_s > 0.$$

Then the left ideal $\mathcal{YA}(P)$ in $\mathbb{Z}[S(E)]$ is generated by the set of all elements of the two following kinds:

(a) $I + (uu')$ (*u and u' distinct elements in the same $U_i \in P$*).

(b) $\Gamma_{j,F} = \sum\{(\operatorname{sgn} \lambda)\lambda\colon \lambda \in S(F), \lambda \mid F \cap U_j$ *and* $\lambda \mid F \cap U_{j+1}$ *increasing, with respect to the given ordering* $<$ *on* $E\}$ *with* $1 \leqslant j < s$, $F \subseteq U_j U U_{j+1}$, $\#F = I + \#U_j$.

Proof. Let f be a function of E-indexed variables with common domain, taking values in an Abelian group, and alternating in P. By Theorem 5.3, f is annihilated by all the above elements $\Gamma_{j,F}$, if and only if f has Young symmetry from U_j to U_{j+1} for $1 \leqslant j < s$; this is true if and only if f has Young alternation in P, by Propositions 2.3 and 2.4 (cf. proof of Corollary 3 to Proposition 2.6). The result now follows by Corollary 1 to Proposition 2.6.

COROLLARY 3. *With the notation of Theorem* 3.1, $\operatorname{Ker} \phi^*$ (*i.e., if R is a field, essentially the prime ideal of the flag variety*) *is generated by the union of the images of all the functions*

$$\Gamma_{p,q,V} = \operatorname{Alt}(V \mid P) \circ \Pi(J_1, \ldots, J_p)\, \Pi(K_1, \ldots, K_q),$$

where

$$n \geqslant p \geqslant q \geqslant 1,\ V \subseteq \{J_1, \ldots, K_q\},\ \#V = p + 1,$$
$$P = \{\{J_1, \ldots, J_p\}, \{K_1, \ldots, K_q\}\},$$

and the common domain of the J's and K's is $\{1, \ldots, n\}$.

Proof. Applying the special case of Corollary 2 when $s = 2$, we see it is equivalent to say that $\operatorname{Ker} \phi^*$ is generated by the ranges of all

$$\alpha \circ \Pi(J_1, \ldots, J_p)\, \Pi(K_1, \ldots, K_q)$$

with $n \geqslant p \geqslant q \geqslant 1$, $\alpha \in \mathscr{Y}\mathscr{O}\!\mathscr{l}(\{\{J_1, \ldots, J_p\}, \{K_1, \ldots, K_q\}\})$; we know this to be the case by Theorem 3.1.

Note. The equations of Corollary 3 (but with $\operatorname{Alt}(V \mid P)$ replaced by $\operatorname{Alt}(V)$, so that they no longer generate the prime ideal in characteristic p) are essentially those used by Hodge in [7], and by Hodge and Pedoe in [8], to study the Grassmann and flag manifolds. Hodge acknowledges Turnbull [20] as his source for these in [7, p. 25]; in Turnbull these equations occur in the $\operatorname{Alt}(V \mid P)$ form

6. RELATIONS BETWEEN THE "QUANTICS" AND "CONNEXES" CONSTRUCTIONS

Assume $V = W^*$ and W are dual finite-dimensional vector spaces over the field k, with cononical pairing denoted by $\langle , \rangle$. Recall that there is induced a duality pairing, which we shall also denote by $\langle , \rangle$, between $\Lambda^p V$ and $\Lambda^p W$, uniquely specified by

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \det \| \langle v_i, w_j \rangle \| \qquad (v\text{'s in } V,\ w\text{'s in } W).$$

Let

$$B = \{e_1, \ldots, e_n\}, \qquad B^* = \{f_1, \ldots, f_n\}$$

be dual k-bases for V and W respectively, so

$$\langle e_i, f_j \rangle = \delta_{ij}.$$

The coordinates

$$\Pi(i_1, \ldots, i_p)(\omega) \qquad (i\text{'s between 1 and } n)$$

with respect to B of an element

$$\omega = \sum_{p=0}^{n} \omega_p \qquad (\omega_p \in \Lambda^p V)$$

in ΛV, as described at the beginning of Section 3, are thus given by

$$\Pi(i_1, \ldots, i_p)(\omega) = \langle \omega_p, f_{i_1} \wedge \cdots \wedge f_{i_p} \rangle. \tag{33}$$

The coordinate ring $k[\mathrm{Flag}(V)]$ of the flag variety of V is then generated as a k-algebra by the restrictions

$$\pi(i_1, \ldots, i_p) = \Pi(i_1, \ldots, i_p) \mid \mathrm{Flag}(V)$$

of these coordinate functions to the flag variety, again as in Section 3. This algebra possesses the following natural gradation: if

$$n \geqslant a_1 \geqslant \cdots \geqslant a_s > 0$$

we define

$$C^{a_1, \ldots, a_s}(V) = \Lambda^{a_1, \ldots, a_s} W$$

to be the k-submodule of $k[\mathrm{Flag}(V)]$ generated by all elements of the form

$$\pi(i_{1,1} \cdots i_{1,a_1})\, \pi(i_{2,1} \cdots i_{2,a_2}) \cdots \pi(i_{s,1} \cdots i_{s,a_s}). \tag{34}$$

Clearly we have

$$k[\mathrm{Flag}(V)] = k \oplus \left(\bigoplus_{n \geqslant a_1 \geqslant \cdots \geqslant a_s} C^{a_1, \ldots, a_s} V \right).$$

Note. In the case $\operatorname{char} k = 0$, Hodge [7] called expressions of the form (34) (with $\pi(i_1, \ldots, i_k)$ interpreted as in Theorem 3.1 rather than as above, which is equivalent to taking $V = k^n$), "*connexes of type* $(b_1, \ldots, b_t)$" (where $b_1 + \cdots + b_t$ is the partition conjugate to $a_1 + \cdots + a_s$), and conjectured a formula for the dimension of the vector space $C^{a_1, \ldots, a_s}(k^n)$ of these connexes (i.e., a postulation formula for the flag manifold). Littlewood [13] pointed out that this conjecture followed immediately from known results on the representations of $GL(n)$; indeed, there is a natural action of $GL(n)$ on $C^{a_1, \ldots, a_s}(k^n)$, and in this way one obtains all finite-dimensional irreducible polynomial representations of $GL(n)$, with no repetitions. G. Higman [5] made the important further observation that, with no assumptions on the characteristic of k, since $C^{a_1, \ldots, a_s}(V) = \Lambda^{a_1, \ldots, a_s} W$ is a covariant functor of W, there is a (functorial) representation

$$GL(W) \to GL(\Lambda^{a_1, \ldots, a_s} W), \qquad T \mapsto \Lambda^{a_1, \ldots, a_s} T$$

of $GL(W)$ over k.

We now turn to the problem of obtaining generators and relations for the commutative k-algebra

$$k \oplus \left(\bigoplus \Lambda^{a_1,\ldots,a_s} W\right) = k[\mathrm{Flag}(W^*)].$$

In the first place, as noted by Higman [5] we may embed ΛW as a sub-k-module (*not* subalgebra) of $k[\mathrm{Flag}(W^*)]$, which generates $k[\mathrm{Flag}(W^*)]$ as a k-algebra, via the k-homomorphism

$$\iota \colon \Lambda W \to k[\mathrm{Flag}(W^*)]$$

defined by:

$$\iota(1_{\Lambda W}) = 1_{k[\mathrm{Flag}(W^*)]},$$

$$\iota(\phi_p) \colon \sum_{q=0}^{n} w_q \mapsto \langle \omega_p , \phi_p \rangle \qquad (\text{each } \omega_q \in \Lambda^q W^*)$$

for $1 \leqslant p \leqslant n$, $\phi_p \in \Lambda^p W$.

Indeed, Im ι contains the generators

$$\pi(i_1 ,\ldots, i_p) = \iota(f_{i_1} \wedge \cdots \wedge f_{i_p})$$

for $k[\mathrm{Flag}(W^*)]$, by (33) and ι is injective, since otherwise some nontrivial k-linear combination of these $\pi(i_1 ,\ldots, i_p)$ would be 0, which is impossible by Theorem 3.1.

We shall regard ι as an identification, and will denote the multiplication in $k[\mathrm{Flag}(W^*)]$ by a dot. Thus, $\Lambda^{a_1,\ldots,a_s} W$ is spanned over k by elements of the form

$$(w_{1,1} \wedge \cdots \wedge w_{1,a_1}) \cdot (w_{2,1} \wedge \cdots \wedge w_{2,a_2}) \cdots (w_{s,1} \wedge \cdots \wedge w_{s,a_s})$$

(all $w_{i,j}$ in W).

The relations on the generating set ΛW for $k[\mathrm{Flag}(W^*)]$, i.e., the kernel of the k-algebra epimorphism

$$\iota' \colon S\Lambda W \xrightarrow{\sim} k[\Lambda W^*] \xrightarrow{\iota} k[\mathrm{Flag}(W^*)]$$

induced by ι may be obtained by Theorem 3.1: this kernel is generated as an ideal in $S\Lambda W$ by the union of the images of the following functions (in which $*$ denotes the commutative product in the ring $S\Lambda W$):

$$g_{p,q,r} = \left[I - \sum_{1 \leqslant s_1 < \cdots < s_r \leqslant p} (u_{s_1} v_1) \cdots (u_{s_r} v_r) \right] \circ \{(u_1 \wedge \cdots \wedge u_p) * (v_1 \wedge \cdots \wedge v_q)\}$$

($n \geqslant p \geqslant q \geqslant r \geqslant 1$), the u's and v's having common domain $B^* = \{f_1 ,\ldots, f_n\}$.

A fortiori, this is still true if instead we take the u's and v's to have common domain W. Thus, we are led to the following definition.

DEFINITION 6.1. Let R be a commutative ring with 1, and W an R-module. By the *shape-algebra* Λ^+W of W will be meant the commutative associative unitary R-algebra, specified to within unique R-isomorphism by the three following properties:

(i) *Λ^+E contains ΛE as a generating R-submodule.*

(ii) $1_{\Lambda E} = 1_{\Lambda^+E}$.

(iii) If $p \geqslant q > 0$, the function $(X_1 \wedge \cdots \wedge X_p) \cdot (Y_1 \wedge \cdots \wedge Y_q)$ of variables indexed by $\{X_1 ,..., Y_q\}$, with common domain W, taking values in Λ^+W, has Young symmetry from $\{Y_1 ,..., Y_q\}$ to $\{X_1 ,..., X_p\}$ (where multiplication in Λ^+ is here indicated by a dot).

If $a_1 \geqslant \cdots \geqslant a_s > 0$, then $\Lambda^{a_1,...,a_s}W$ will denote the R-submodule of Λ^+W spanned by all

$$(w_{1,1} \wedge \cdots \wedge w_{1,a_1}) \cdot \cdots \cdot (w_{s,1} \wedge \cdots \wedge w_{s,a_s}),$$

and an element of $\Lambda^{a_1,...,a_s}W$ will be called a *shape in W of degree* $(a_1 ,..., a_s)$.

Observe that these $\Lambda^{a_1,...,a_s}$ coincide, when $R = k$ is a field and W is a finite k-module, with those of the earlier discussion, while Λ^+W is then naturally isomorphic to $k[\text{Flag}(W^*)]$. We note that, if $\sum_{i=1}^{s} a_1 = n$ and $e_1 ,..., e_n$ are linearly independent elements of W, we obtain a representation of S_n on the subspace

$$k[S_n] \circ (e_1 \wedge \cdots \wedge e_{a_1})(e_{a_1+1} \wedge \cdots \wedge e_{a_1+a_2}) \cdots (\cdots \wedge e_n)$$

of $\Lambda^{a_1,...,a_s}W$ (where S_n acts by permuting the subscripts); we shall omit here the verification that, if char $k = 0$, we obtain in this way all irreducible representations of S_n over k without repetitions (with the character of the above representations of S_n being $\chi^{(b_1,...,b_t)}$ in the notation of [9, p. 67, 70], where $b_1 + \cdots + b_t$ is the partition of n conjugate to $a_1 + \cdots + a_s$).

We next study the relationship between Definition 6.1 and the generalization of the Young–Weyl "quantics" construction given by Carter and Lusztig in [1, p. 211].

If W is a R-module over a commutative ring R, and $a_1 \geqslant \cdots \geqslant a_s > 0$, then Carter and Lusztig define an R-module which they call the "Weyl module", and denote by $W^{a_1,...,a_s}$; in the notation of the present paper, their definition may be stated as follows:

Let $N = \sum_{i=1}^{s} a_i$, and let $P = \{U_1 ,..., U_s\}$ denote the partitioning of $\{1,..., N\}$ defined by $U_1 = \{1,..., a_1\}$, $U_2 = \{a_1 + 1,..., a_1 + a_2\}$, etc., i.e.,

$U_i = \{j: a_1 + \cdots + a_{i-1} < j \leqslant a_1 + \cdots + a_i\}$; note that $\# U_i = a_i$. Define a left action of S_N (hence $R[S_N]$) on

$$W^{\otimes N} = W \otimes_R W \otimes_R \cdots \otimes_R W \qquad (N \text{ factors})$$

by

$$\sigma \circ (w_1 \otimes \cdots \otimes w_n) = w_{\sigma^{-1}1} \otimes \cdots \otimes w_{\sigma^{-1}n} \qquad (w\text{'s in } W,\ \sigma \in S_N).$$

Then $W^{a_1,\ldots,a_s}$ is the R-module consisting of all $T \in W^{\otimes N}$ with the two following properties:

$$T \text{ is left annihilated by } \mathscr{Y}\mathscr{O}\!\mathscr{l}(P) \text{ (cf. Definition 2.6)} \tag{34a}$$

$$\begin{gathered}\text{If } w_1^*,\ldots, w_N^* \text{ lie in } W^* = \operatorname{Hom}_R(W, R) \text{ and there exist}\\ \text{distinct } j, k \text{ in the same } U \in P \text{ such that } w_j^* = w_k^*, \text{ then}\\ \langle T, w_1^* \otimes \cdots \otimes w_n^* \rangle = 0\end{gathered} \tag{34b}$$

(where $\langle , \rangle$ denotes the canonical pairing between $W^{\otimes n}$ and $(W^*)^{\otimes n}$).

Note. In [1], condition (34a) is stated in a form which is later noted (p. 2.2, Eq. (31)) to be equivalent in the presence of (34b) to symmetry conditions on T, which in the present language assert that T is left annihilated by the generators for $\mathscr{Y}\mathscr{O}\!\mathscr{l}(P)$ given above in Corollary 3 to Proposition 2.6.

PROPOSITION 6.1. *If W is a finitely generated free R-module, there is a natural R-isomorphism*

$$W^{a_1,\ldots,a_s} \approx (\Lambda^{a_1,\ldots,a_s} W^*)^*.$$

Proof. Let S denote the R-submodule of $(W^*)^{\otimes n}$ spanned by those $w_1^* \otimes \cdots \otimes w_n^*$ as in (34b) (i.e., with all w^*'s in W^*, and with distinct i and j in the same $U \in P$ such that $w_i^* = w_j^*$); thus (34b) asserts that T lies in the orthogonal complement $S^\perp$ of S in $W^{\otimes n}$. Note that $S^\perp$ consists of all tensors T whose representation

$$T = \sum t_{i_1 \cdots i_N} f_{i_1} \otimes \cdots \otimes f_{i_N},$$

with respect to a basis $\{f_1, \ldots, f_n\}$ for W, is such that $t_{i_1 \cdots i_N}$ is alternating in P considered as a function of its indices. It follows that $I + (ij)$ left annihilates $S^\perp$, for all distinct i and j in the same $U \in P$.

If we set

$$A_i = a_1 + \cdots + a_{i-1} \qquad (=0 \text{ if } i = 1),$$

the generators for $\mathscr{Y}\mathscr{O}\!\mathscr{l}(P)$ given by (16b) in Section 2 become

$$G_{j,k,r} = I - \sum_{1 \leqslant m_1 < \cdots < m_r \leqslant a_j} (A_j + m_1, A_k + 1) \cdots (A_j + m_r, A_k + r) \quad (35)$$

(j and k distinct integers between 1 and s, $1 \leqslant r \leqslant a_k \leqslant a_j$) and we may write

$$W^{a_1,\ldots,a_s} = \left(\bigcap [0: G_{j,k,r}]_{W^{\otimes N}}\right) \cap S. \quad (36)$$

(Note: We have not used the generators (16a) in this intersection, since any submodule of $W^{\otimes n}$ they annihilate contains $S^{\perp}$).

Similarly, we have

$$\Lambda^{a_1,\ldots,a_s}(W^*) = (W^*)^{\otimes n} \Big/ \Big[S + \sum G_{j,k,r} \circ (W^*)^{\otimes n}\Big]. \quad (37)$$

Note that, for σ in $\mathfrak{S}_n$, $T \in (W)^{\otimes N}$, $T^* \in (W^*)^{\otimes N}$ we have

$$\langle \sigma \circ T, T^* \rangle = \langle T, \sigma^{-1} \circ T^* \rangle$$

whence (noting that each term of (35) is its own inverse)

$$\langle G_{j,k,r} \circ T, T^* \rangle = \langle T, G_{j,k,r} \circ T^* \rangle.$$

From this and (36) we obtain

$$\Big[S + \sum G_{j,k,r} \circ (W^*)^{\otimes N}\Big]^{\perp} = W^{a_1,\ldots,a_s}$$

which, together with (37), immediately implies the desired result.

Remarks. $\Lambda^{a_1,\ldots,a_s}W$ is a finitely generated free module (cf. [18, Theorem 2.4]) and the same is proved for $V_{a_1,\ldots,a_s}W$ in [1, p. 218]; thus, we may freely shift the *'s in the preceding proposition. Proposition 6.1 is not valid for arbitrary finitely generated R-modules.

There is one apparent advantage of the functors $\Lambda^{a_1,\ldots,a_s}W$ over the functors $V_{a_1,\ldots,a_s}W$; namely, the former, by virtue of their very construction, fit together as pieces of a graded commutative associative R-algebra Λ^+W. It is however, possible to give a "dual" construction, obtaining an associative algebra V_+W whose graded pieces are the modules $W^{a_1,\ldots,a_s}$; this construction is given in [18, Sects. 1.4 and 1.5]. One remarkable fact should be stressed: Although one would expect the dual of a commutative algebra would itself be commutative, V_+W in fact is a graded alternating algebra. It is this fact which gives Λ^+W the double structure mentioned in the Introduction; it may perhaps be regarded as another manifestation of the "duality" between symmetry and alternation, already observed in connection with these matters in Section 2. Indeed, if R is a field of characteristic 0, or more generally a $\mathbb{Q}$-algebra, there is a natural isomorphism between $\Lambda^{a_1,\ldots,a_s}W$ and $W^{a_1,\ldots,a_s}$; one may see this, either by a simple computation of characters, or better, by constructing an explicit isomorphism (cf. [18, Sect. 2.5]). This natural isomorphism will be used in the next section to carry over to Λ^+W the graded alternating multiplication of V_+W, thus giving Λ^+W the double structure in question, in a fashion unique up to multiplication by nonzero rationals.

The precise definition of the algebra

$$V_+W = k \oplus \left(\bigoplus_{a_1 \geqslant \cdots \geqslant a_s > 0} V_{a_1 \cdots a_s}W \right)$$

must be referred to [18], where it is already-proved (Theorem 2.5) that, when W is a finitely generated projective R-module, there is a natural isomorphism

$$V_{a_1,\ldots,a_s}W \approx (\Lambda^{b_1,\ldots,b_t}(W^*))^*.$$

Note that this result, together with Proposition 6.1, allows us to identify the functor $V_{a_1,\ldots,a_s}W$ defined in [18] with the Carter–Lusztig functor $W^{a_1,\ldots,a_s}$ for finitely generated free (or projective) R-modules W, though not for arbitrary finitely generated R-modules.

7. Extra Structure in the Coordinate Ring of the Flag Manifold

Throughout this section, R will denote a commutative $\mathbb{Q}$-algebra and E will denote any R-module. A natural multiplication on Λ^+E will be constructed, which extends the wedge product from the sub-R-module ΛE of Λ^+E.

7.1. *Notation and Terminology*

By a *numerical partitioning* will be meant an "unordered finite sequence of positive integers," i.e., an equivalence class with respect to the following equivalence relation on finite sequences of positive integers:

$$(a_1 ,\ldots, a_m) \sim (b_1 ,\ldots, b_n)$$

if $m = n$ and $\exists\ \pi \in S_n$ with $b_i = a_{\pi_i}$ for $1 \leqslant i \leqslant n$. We denote by $\langle a_1 ,\ldots, a_n \rangle$ the numerical partitioning which is the equivalence class of $(a_1 ,\ldots, a_n)$. Associated with every partitioning $P = \{U_1 ,\ldots, U_r\}$ in the sense of Definition 2.5, there is the numerical partitioning $P = \langle \# U_1 ,\ldots, \# U_r \rangle$.

The functors $\Lambda^{a_1,\ldots,a_s}$ and $V_{a_1,\ldots,a_s}$ of Section 6 do not depend on the ordering of the a's, and so may also be denoted by Λ^α and V_α respectively, where $\alpha = \langle a_1 ,\ldots, a_s \rangle$; this is the notation in [18].

Now let

$$\alpha = \langle a_1, \dots, a_s \rangle, \qquad a_1 \geqslant \cdots \geqslant a_s > 0 \tag{38}$$

be any partition, and define the *frame* of α to be

$$F_\alpha = \{(i, j): 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant a_i\}.$$

Assume the dual partiton to α is given by

$$\alpha^* = \langle \bar{a}_1, \dots, \bar{a}_{\bar{s}} \rangle,\ \bar{a}_1 \geqslant \cdots \geqslant \bar{a}_{\bar{s}} > 0, \tag{39}$$

so that

$$F_{\alpha^*} = \{(i, j): (j, i) \in F_\alpha\}.$$

There is a natural partitioning P_α of the frame F_α into the disjoint sets $U_1, \dots, U_\alpha$ with

$$U_i = \{(i, 1), \dots, (i, a_i)\}$$

and $|P_\alpha| = \alpha$. If we order these sets U_i in the obvious way:

$$(i, 1) < \cdots < (i, a_i)$$

we may construct as in Definition 4.2 the two elements

$$YA(\alpha) = YA((P_\alpha)_<) = \mathrm{Alt}(P_\alpha)\,\mathrm{Sym}((P_\alpha)_<^*)$$

and

$$YS(\alpha) = YS((P_\alpha)_<) = \mathrm{Sym}(P_\alpha)\,\mathrm{Alt}((P_\alpha)_<^*)$$

in the symmetric group algebra $R[S(F_\alpha)]$. It is well known there exists a positive integer $H(\alpha)$ (given by the "hookproduct Rule," cf. [4]) such that

$$[YS(\alpha)]^2 = H(\alpha)\, YS(\alpha), \qquad [YA(\alpha)]^2 = H(\alpha)\, YA(\alpha). \tag{40}$$

We shall denote by ω^α the F_α-indexed function

$$\omega^\alpha(\{e_{ij}\}) = \prod_{i=1}^{s} (e_{i,1} \wedge \cdots \wedge e_{i,a_i})$$

with common domain E taking values in $\Lambda_R{}^\alpha E$, and similarly denote by θ_α the F_α-indexed function

$$(e_{11} \circ \cdots \circ e_{1,a_1}) \wedge \cdots \wedge (e_{s,1} \circ \cdots \circ e_{s,a_s})$$

with common domain E taking values in $V_\alpha E$. Note that ω^α has Young alternation, and θ_α has Young symmetry, in the partitioning P_α (cf. Definition 2.5).

Let D and T be R-modules, M a set, and f an M-indexed function with common domain D taking values in T; then f will be called *R-multilinear* provided it satisfies the following condition:

$$f(F) = r_1 f(F_1) + r_2 f(F_2)$$

whenever r_1 and r_2 are in R, and F, F_1, F_2 are maps $M \to D$ which coincide at all elements of M except possibly m_0, where $F(m_0) = r_1 F_1(m_0) + r_2 F_2(m_0)$.

For example, the F_α-indexed functions ω^α and θ_α defined above, are R-multilinear.

If M is any set and D is any R-module, we denote by $D^{\otimes M}$ the universal target of all M-indexed R-multilinear maps with common domain D. (For our purposes, it is important to have the notation, which unlike the usual one does not assume an ordering on M.) In more detail:

(i) There is an R-mutilinear M-indexed map $\otimes_M$ with common domain D taking values in $D^{\otimes M}$, which assigns to each map $\{d_M\}_{m \in M}$ from M to D the element $\otimes_{m \in M} d_m \in D^{\otimes M}$.

(ii) For each R-multilinear M-indexed function f with common domain M taking values in any R-module T, there is a unique R-homomorphism

$$\tilde{f}\colon D^{\otimes M} \to T$$

such that

$$f(\{d_\alpha\}_{\alpha \in M}) = \tilde{f}(\otimes_{m \in M} d_m). \tag{41}$$

Note finally that there is a natural left action of $S(M)$ on R-homomorphisms $D^{\otimes M} \to T$, which is compatible via (41) with the left action of $S(M)$ on M-indexed functions; these extend to left actions of $R[S(M)]$.

7.2. *Construction of the Wedge Product on $\Lambda^+ E$*

PROPOSITION 7.1. *Let D, T be modules over the commutative $\mathbb{Q}$-algebra R, and let α be any partition; let*

$$f \in T^{(D^{F_\alpha})}$$

be an F_α-indexed R-multilinear function, with common domain D, taking values in T.

Then $YA(\alpha) \circ f$ and $YS(\alpha) \circ f$ factor through ω^α and θ_α, respectively, that is, there exist unique R-homomorphisms

$$f^\alpha\colon \Lambda^\alpha D \to T \qquad \textit{and} \qquad f_\alpha\colon V_\alpha D \to T$$

such that

$$YA(\alpha) \circ f = f^\alpha \circ \omega^\alpha, \qquad YS(\alpha) \circ f_\alpha = f_\alpha \circ \theta_\alpha$$

Proof. We shall only prove the assertion about $YA(\alpha) \circ f$, omitting the similar proof for $YS(\alpha) \circ f$. f factors through a unique R-homomorphism

$$\tilde{f}: D^{\otimes F_\alpha} \to T$$

and it suffices to verify that $YA(\alpha) \circ \tilde{f}$ vanishes on the kernel K of the R-homomorphism

$$\tilde{\omega}^\alpha: D^{\otimes F_\alpha} \to \Lambda^\alpha E$$

induced from ω^α. Assume now that α has the form (38). It follows from Definition 6.1 that K is spanned over R by the set of all $(G_{j,k,r} \circ \tilde{\omega}^\alpha)(\otimes_{(i,j)\in F_\alpha} d_{i,j})$ where $G_{j,k,r}$ is given by (35) and all $d_{i,j} \in D$. We are done since $G_{j,k,r} \circ YA(\alpha) = 0$ by Theorem 4.1.

DEFINITION AND PROPOSITION 7.2. *Let E be a module over the commutative $\mathbb{Q}$-algebra R, and let α be a partition satisfying* (38) *and* (39). *Denote by τ_α the bijection $F_\alpha \to F_{\alpha^*}$, $(i,j) \to (j,i)$.*

Then there exist R-isomorphisms

$$\delta^\alpha: \Lambda^\alpha E \xrightarrow{\approx} V_{\alpha^*}E, \qquad \delta_{\alpha^*}E \xrightarrow{\approx} \Lambda^\alpha E$$

uniquely defined by

$$\delta^\alpha \circ \omega^\alpha = \mathrm{Alt}(P_\alpha) \circ \tau_{\alpha^*} \circ \theta_{\alpha^*}, \tag{42}$$

$$\delta_\alpha \circ \theta_{\alpha^*} = \mathrm{Sym}(P_{\alpha^*}) \circ \tau_\alpha \circ \omega_\alpha . \tag{43}$$

Moreover,

$$\delta_\alpha \circ \delta_{\alpha^*} = H(\alpha)\,\mathrm{Id}(V_{\alpha^*}E),\ \delta_{\alpha^*} \circ \delta_\alpha = H(\alpha)\,\mathrm{Id}(\Lambda^\alpha E). \tag{44}$$

Finally, if

$$\omega_1 \in \Lambda^\alpha E,\ \omega_2 \in \Lambda^\beta E$$

then we define

$$\omega_1 \wedge \omega_2 = \delta_{\gamma^*}((\delta^\alpha \omega_1) \wedge (\delta^\beta \omega_2)) \in \Lambda^\gamma E,$$

where $\gamma = (\alpha^* + \beta^*)^*$.

Remark. (42) and (43) may be restated (less precisely but more intuitively) as

$$\delta^\alpha \prod_{i=1}^{s} \bigwedge_{j=1}^{a_i} e_{ij} = \left[\prod_i \mathrm{Alt}\{e_{i,1}, \ldots, e_{i,a_i}\}\right] \circ \bigwedge_{j=1}^{\bar{s}} \prod_{i=1}^{\bar{a}_i} e_{i,j},$$

$$\delta_{\alpha^*} \bigwedge_{j=1}^{\bar{s}} \prod_{i=1}^{\bar{a}_i} e_{i,j} = \left[\prod_j \mathrm{Sym}\{e_{1,j}, \ldots, e_{\bar{a}_j,j}\}\right] \circ \prod_{i=1}^{s} \bigwedge_{j=1}^{a_i} e_{i,j} \qquad (\text{all } e_{i,j} \in E).$$

Proof of Proposition 7.2. Since $\theta_{\alpha*}$ is symmetric in the partitioning $P_{\alpha*}$ of $F_{\alpha*}$, we have

$$n_1(RHS \text{ of } (42)) = YA(\alpha) \circ \tau_{\alpha*} \circ \theta_{\alpha*},$$

where

$$n_1 = \prod_{i=1}^{\bar{s}} (\bar{a}_i)!$$

and similarly

$$n_2(RHS \text{ of } (43)) = YS(\alpha) \circ \tau_{\alpha} \circ \omega_{\alpha},$$

where

$$n_2 = \prod_{i=1}^{s} (a_i)!.$$

The existence of R-homomorphisms δ^α, $\delta_{\alpha*}$ satisfying (42) and (43) is thus guaranteed by Proposition 4.1.

We have

$$\delta_\alpha \circ \delta_{\alpha*} \circ \omega^\alpha = \mathrm{Alt}(P_\alpha)\, \mathrm{Sym}((P_\alpha)^*_<)\, \omega^\alpha = YA(\omega^\alpha)$$

which equals $H(\alpha)\omega^\alpha$ by the remark following Theorem 4.1 and the fact that ω^α has Young alternation in P_α. This proves the first half of (44), and the proof of the second half is similar. Hence δ^α and $\delta_{\alpha*}$ are R-isomorphisms, as asserted.

Remarks. This Λ operation

$$\Lambda_{\alpha,\beta}: \Lambda^\alpha \otimes \Lambda^\beta \to \Lambda^\gamma \qquad (\gamma = (\alpha^* + \beta^*)^*) \tag{45}$$

is essentially unique, in the sense that if $R = \mathbb{C}$, the multiplicity of the representation $[\gamma]$ associated with Λ^γ, in $[\alpha] \otimes [\beta]$ is 1, so all natural transformations (45) coincide to within scalar multiples.

Appendix 1

Throughout this appendix, we shall assume that

$$\alpha = \langle a_1, \ldots, a_s \rangle, \qquad a_1 \geqslant \cdots \geqslant a_s > 0,$$

is a partition, with dual partition

$$\beta = \langle b_1, \ldots, b_t \rangle, \qquad b_1 \geqslant \cdots \geqslant b_t > 0.$$

By the *Young frame* of type α will be meant the set

$$F_\alpha = \{(i, j) \colon 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant Q_i\}.$$

We shall refer to the set

$$\{(i, 1), \ldots, (i, a_i)\} \qquad (1 \leqslant i \leqslant s)$$

as the ith *row of* F_α, and to the set

$$\{(1, j), \ldots, (b_j, j)\} \qquad (1 \leqslant j \leqslant t)$$

as the jth *column* of F_α. By a *tableau* (or *Young tableau*) *of type* α *over a set* B will be meant a map $T: F_\alpha \to B$; we then refer to $T(i, j)$ as the *entry in box* (i, j) *of* T.

There is a *left* action of $S(F_\alpha)$ on tableaux of type α over B, defined by

$$(\pi T)(i, j) = T(\pi^{-1}(i, j)).$$

Note. This action on tableaux is that employed in [22] with πT obtained from T by moving the T-entry in each box B of F_α into the box πB.

If $<$ is a total ordering on B, there is associated the usual lexicographical ordering $<_t$ on F_α, defined by $(i, j) <_t (i_1, j_1) \Leftrightarrow$ *Either* $i < i_1$ *or* $i = i_1$, $j < j_1$. Also, we associate to each tableau T of type α over B the sequence

$$T(1, 1), \ldots, T(1, a_1), T(2, 1), \ldots, T(2, a_2), \ldots, T(s, a_s)$$

and let $<_p$ denote the total ordering on the set of such tableaux T, obtained by ordering their associated sequences. In other words, $T <_p T'$ means, $\exists(i, j)$ in F_α such that

(1) $T(i, j) < T'(i, j)$.

(2) If $(i_1, j_1) <_t (i, j)$ then $T(i_1, j_1) = T'(i_1, j_1)$.

Associated with the partition α we have the two following subgroups of $S(F_\alpha)$: Row (α) = set of permutations which preserve the rows of

$$F_\alpha = \{\pi \in S(F_\alpha): \pi(i, j) = (i', j') \Rightarrow i = i'\},$$

and similarly

$$\mathrm{Col}(\alpha) = \{\pi \in F_\alpha: \pi(i, j) = (i', j') \Rightarrow j = j'\}.$$

One final definition, and we shall be ready to state the proposition: A tableau T of type α over the ordered set B is *row-strict standard* if it has the two following properties:

(A) $T(i, 1) < T(i, 2) < \cdots < T(i, a_i)$ $(1 \leqslant i \leqslant s)$; i.e., T is strictly increasing in each row of F.

(B) $T(1, j) \leqslant T(2, j) \leqslant \cdots \leqslant T(b_j, j)$ $(1 \leqslant j \leqslant t)$; i.e., T is nondecreasing in each column of F.

PROPOSITION A. *Let $(B, <)$ be a totally ordered set and let T be a row-strict standard tableau of type α over B. Let*

$$\sigma \in \mathrm{Row}\,(\alpha), \qquad \pi \in \mathrm{Col}\,(\alpha);$$

then $T \leqslant_p \sigma\pi T$, and if equality holds then σ is the identity permutation.

Proof. Set

$$(1) \qquad T' = \sigma\pi T.$$

Since $\pi \in \mathrm{Col}(\alpha)$, there exists for $1 \leqslant j \leqslant t$ a unique $\pi_j \in S(b_j)$ such that

$$(2) \qquad \pi(i,j) = (\pi_j i, j), \qquad \text{all } (i,j) \in F_\alpha .$$

We next modify π, by means of the following considerations.

By an *entry-column for* T will be meant a nonempty subset $C_{j,l}$ of the jth column of F_α of the form

$$C_{j,l} = \{(i,j) \in F_\alpha \colon T(i,j) = l\}$$

(i.e., $C_{j,l}$ is the set of boxes in the jth column of F_α whose T-entries have a given value l which actually occurs in that column of T.) Clearly, F_α is the disjoint union of these $C_{j,l}$.

If $\pi, \pi' \in \mathrm{Col}(\alpha)$ then

$$\pi \cdot T = \pi' \cdot T$$

holds, if and only if $\pi C = \pi' C$ for every entry-column C for T. (This is because $\pi C_{j,l}$ is precisely the set of all boxes in the jth column, whose entry in πT is l.) Thus, given $\pi \in \mathrm{Col}(\alpha)$, the most general $\pi' \in \mathrm{Col}(\alpha)$ with $\pi T = \pi' T$ is obtained by picking arbitrary bijections

$$\pi'_{j,l} \colon C_{j,l} \to \pi(C_{jl})$$

and piecing them together to a permutation π' of Σ_α. Among all such π' there is a unique one, which we may without loss of generaity assume to be π itself, satisfying the following condition:

For all entry columns $C_{j,l}$ for T, the map

$$\pi \mid C_{jl} \colon C_{jl} \to \pi(C_{j,l})$$

between subsets of the jth column of Σ_α, is order-preserving with respect to the vertical ordering

$$(1, j) < \cdots < (b_j, j)$$

for that column; equivalently:

$$(3) \qquad \text{If } T(i,j) = T(i',j) \text{ and } i < i' \text{ then } \pi_j i < \pi_j i'.$$

For the remainder of this proof, it will be assumed that π satisfies (3).

We next prove the following consequence of this normalization of π:

Claim. If $(i,j) \in F_\alpha$, and if T and T' coincide on all boxes of F_α which precede (i,j) in the lexicographic ordering $<_t$ defined above, then both the following statements hold:

(i) Both σ and π fix all boxes $<_t (i,j)$

(ii) *Either* σ and π fix (i,j) (whence $T(i,j) = T'(i,j)$ by (1)), *or* $T(i,j) < T'(i,j)$

Remark. Both the hypothesis of this claim, and (i), are vacuously satisfied if $(i,j) = (1, 1)$; the general argument below for proving (ii) works in this case also.

Proof of claim. We argue by induction on the ordering $<_t$ of Σ_α; i.e., we assume the claim holds for all boxes $<_t(i,j)$.

If $(i,j) \neq (1, 1)$ then (i,j) has an immediate predecessor, say (i',j'), in the ordering $<_t$. Applying the induction hypothesis to (i',j') we obtain:

(ia) π and σ fix all boxes $<_t(i',j')$.

(iia) Since $T(i',j') = T'(i',j')$, π and σ fix (i',j').

The preceding two statements clearly imply (i) holds for (i,j) (if $(i,j) \neq (1, 1)$; (i) holds vacuously if $(i,j) = (1, 1)$).

We next prove (ii). Let

$$(4) \qquad \sigma^{-1}(i,j) = (i, q), \qquad \pi^{-1}(i, q) = (p, q)$$

whence

$$T'(i,j) = (\sigma^{-1}T')(i, q) = (\pi T)(i, q) = T(p, q)$$

Since we now know (i) holds, we have:

$$\sigma(i,j') = (i,j') \text{ if } j' < j; \qquad \pi(i', q) = (i', q) \text{ if } i' < i$$

which with (4) implies

$$(5) \qquad p \geqslant i, \qquad q \geqslant j.$$

Since T is row-strict standard, it follows that

$$(6) \qquad T(i,j) \leqslant T(i, q) \leqslant T(p, q) = T'(i,j).$$

If σ does not fix (i,j) then, by (4),

$$q > j \qquad \text{whence } T(i,j) < T(i, q) \leqslant T'(i,j)$$

(using (6) and the fact T is row-strict standard); i.e., (ii) holds in the case.

Thus, to complete the proof of the claim, it suffices to prove (ii) under the assumption that

$$T(i,j) \geqslant T'(i,j).$$

Indeed, it then follows from (6) that $T(i,j) = T'(i,j)$, so by what has just been shown, σ fixes (i,j); then (4) shows $j = q$, so (4) and (6) become

$$(7) \qquad T(i,j) = T(p,j) = T'(i,j), \pi(p,j) = (i,j); \qquad \text{i.e., } i = \pi_j p.$$

Hence, (i,j) and (p,j) lie in the same entry-column for T. This shows $p = i$, for otherwise by (5) we would have $p > i$ which by (3) and (7) would imply $i = \pi_j p > \pi_j i$, whence

$$\pi(i,j) = (i',j) \qquad \text{with} \quad i' < i$$

which contradicts (i) (which has been proved to hold for (i,j)). This completes the induction, and so proves the preceding claim.

We may now readily complete the proof of Proposition A, as follows. In the first place, if $T = T'$, then the claim, applied to the last box (s, a_s) of F_α, shows that $\sigma = I$. On the other hand, if $T \neq T'$, pick the $<_t$ earliest box (i,j) of F_α for which T and T' differ; the claim then shows that $T(i,j) < T'(i,j)$ whence $T <_p T'$. Q.E.D.

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References

1. R. W. Carter and G. Lusztig, On the modular representations of the general linear and symmetric groups, *Math. Z.* **136** (1974), 193–242.
2. D. B. A. Epstein, Group representations and functors, *Amer. J. Math.* **91** (1969), 395–414.
3. V. Fock, *Sov. JETP* **10** (1940), 961.
4. J. S. Frame, G. de B. Robinson, and R. M. Thrall, The hookgraphs of the symmetric groups, *Canad. J. Math.* **6** (1954), 316–323.
5. G. Higman, Representations of general linear groups and varieties of p-groups, *in* "Proc. Internat. Conf. Theory of Groups, Austral. Nat. Univ. Canberra, Aug. 1965," pp. 167–173, Gordon & Breach, New York, 1967.
6. W. V. D. Hodge, A note on connexes, *Proc. Cambridge Philos. Soc.* **38** (1942), 129–143.
7. W. V. D. Hodge, Some enumerative results in the theory of forms, *Proc. Cambridge Philos. Soc.* **39** (1943), 22–30.

8. W. V. D. HODGE AND D. PEDOE, "Methods of Algebraic Geometry," Vol. II, Cambridge Univ. Press, London/New York, 1952.
9. G. D. JAMES, A characteristic-free approach to the representation theory of $\mathfrak{S}_n$, *J. Algebra* **46** (1977), 430–450.
10. S. L. KLEIMAN AND D. LAKSOV, Schubert calculus, *Amer. Math. Monthly* **79** (1972), 1061–1082.
11. G. LANCASTER AND J. TOWBER, Representation-functors and flag-algebras for the classical groups I, *J. Algebra* **59** (1979), 16–39; II, *J. Algebra*, in press.
12. A. LASCOUX, thesis (unpublished).
13. D. E. LITTLEWOOD, On the number of terms in a simple algebraic form, *Proc. Cambridge Philos. Soc.* **39** (1942), 394–396.
14. D. E. LITTLEWOOD, "The Theory of Group Characters," 2nd ed., Oxford Univ. Press (Clarendon), London, 1950.
15. A. LIULEVICIUS, Representation rings of the symmetric groups—a Hopf algebra approach, Aarhus Univ. preprint series No. 29, 1975.
16. C. MUSILI, Postulation formula for Schubert varieties, *J. Indian Math. Soc.* **36** (1972), 143–171.
17. J. J. SYLVESTER, *Phil. Mag.* **4**, No. II (1851), 142–145.
18. J. TOWBER, Two new functors from modules to algebras, *J. Algebra* **47** (1977), 80–104.
19. J. TOWBER, Young alternation and the flag-manifold, *in* "Proceedings of Young Day Conference, University of Waterloo, June 2, 1978."
20. H. W. TURNBULL, "The Theory of Determinants, Matrices and Invariants, Blackie, London/Glasgow, 1929.
21. B. L. VAN DER WAERDEN, "Einfuhrung in die Algebraische Geometrie," Berlin, 1939.
22. H. WEYL, "The Classical Groups, Their Invariants and Representations," Princeton Univ. Press, Princeton, N. J., 1946.
23. A. YOUNG, On quantitative substitutional analysis (second paper), *Proc. London Math. Soc. (1)* **34** (1902), 261–397.
24. A. YOUNG, On quantitative substitutional analysis (third paper), *Proc. London Math. Soc. (2)* **28** (1927), 255–292.