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Module 10, Lecture

▼ 🏗️ Learning Objectives

- Recognize the differences between weak and strong induction
- Formally prove properties of non-negative integers (or a subset) that have appropriate self-referential structure using strong induction
- Identify the need to add more base cases in a strong induction proof

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Thursday, March 30

Reading Review

Principle of Strong Mathematical Induction
Let $P(n)$ be a property that is defined for integers n, and let a, b be fixed integers with $a \leq b$. Suppose the following two statements are true:
1. $P(a), P(a+1), ..., P(b)$ are all true. - **Basis Step**
2. For some integer $n \geq b$, if $P(k)$ is true for each integer i, $a \leq k \leq n$ (**inductive hypothesis)**, then $P(n+1)$ is true. - **Inductive Step**

Then, the statement:
For every integer $n \geq a, P(n)$
is true.

- We can also assume that $P(k)$ is true for all $k < n$ and demonstrate that $P(n)$ is true.

Strong Mathematical Induction

We use mathematical induction to analyze recursive algorithms.

Example:

```
(define (sum n)
  (if (= n 0)
    0
    (+ n (sum (- n 1)))))
```

- *If n is zero, produce zero, otherwise, sum the current n with the result of the function with (n-1)*
- For this example, the recursion can work by simply looking at the value one case behind (i.e. `(- n 1)`)

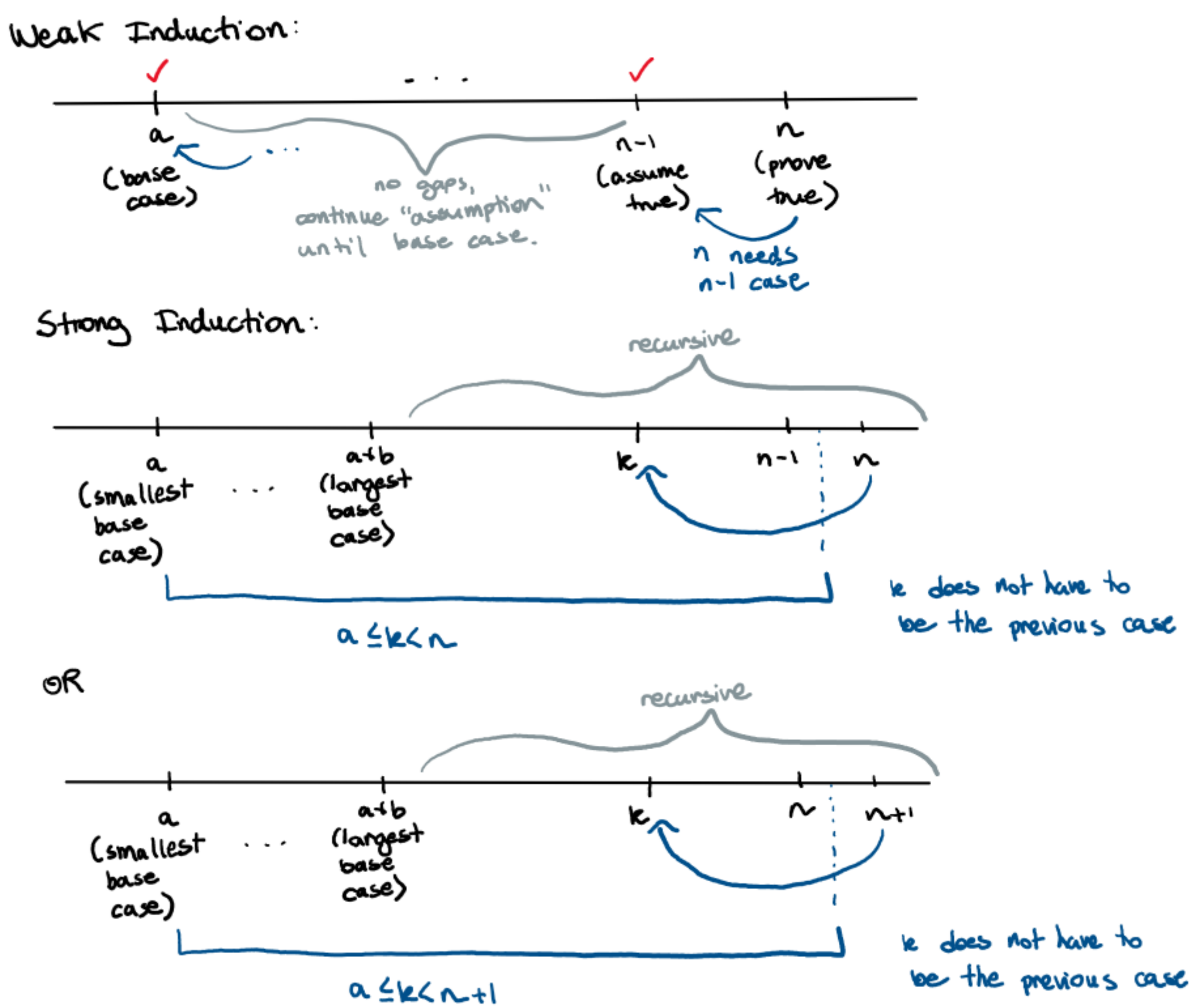
However, some recursions have recursive calls where the argument is not $(n-1)$. For instance:

- `(function1 (- n 2))`

- `(function2 (quotient n 2))`
- `(function 3 x)` where x is some integer in the range $0 < x < n$

How do we tackle these cases? We can use strong induction, and need:

- One or more base cases, when we can compute the answer directly
- An induction step
  - Computing recursively the solutions to one or more “smaller” sub-problems, and combining them to obtain the overall answer



💡 In other words, if the next case depends on a case that is not the one immediately before (i.e. a case further back), we use **strong induction**.

Hypothesis

Instead of proving that $\forall x \in \mathbb{Z}^+, Q(x) \to Q(x+1)$, as in weak induction, we will prove that

$$\forall x \in \mathbb{Z}^+, (Q(1) \wedge Q(2) \wedge ... \wedge Q(x)) \to Q(x+1)$$

The validity of this type of induction can be proven the same way by contradiction as we did for ordinary/weak induction.

📏 **Difference in Inductive Hypotheses:**
Weak Induction:
Suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$ *[**inductive hypothesis**]*
Then, show that $P(k+1)$ is true.

Strong Induction:
Suppose that for every integer $k \geq b$, if $P(i)$ is true for each integer i from a through k *[**inductive hypothesis**]*
Then, show that $P(k+1)$ is true.

ℹ️ Statements that can be proven with ordinary mathematical induction can be proven with strong mathematical induction.

It is also the case that any statement that can be proven with strong mathematical induction can be proven with ordinary mathematical induction.

Comparison

Weak Induction:

$$P(1) \to P(2) \to P(3) \to P(4) \to P(5) \to \dots$$

- If $P(1)$ is true and $P(1) \to P(2)$ is true, then $P(2)$ is true
- If $P(2)$ is true and $P(2) \to P(3)$ is true, then $P(3)$ is true
- and so on…

This creates a linear pattern:

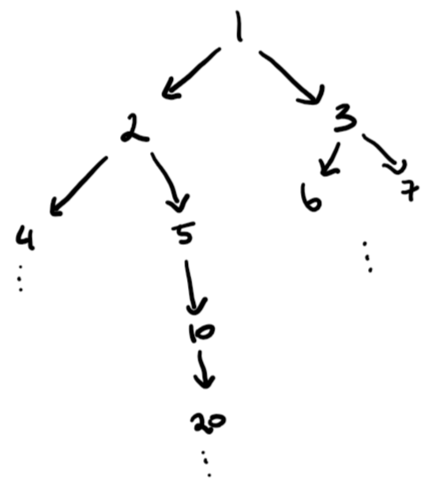
Strong Induction

Using $k = \lfloor \frac{n}{2} \rfloor$ as an example:

$$\begin{gathered} P(1) \to P(2) \to P(4) \to P(8) \to P(16) \to \dots \\ P(1) \to P(3) \to P(6) \to P(12) \to P(24) \to \dots \\ P(1) \to P(2) \to P(5) \to P(10) \to P(20) \to \dots \\ \dots \end{gathered}$$

- If $P(1)$ is true and $P(1) \to P(2)$ is true, then $P(2)$ is true
- If $P(2)$ is true and $P(2) \to P(4)$ is true, then $P(4)$ is true
- If $P(1)$ is true and $P(1) \to P(3)$ is true, then $P(3)$ is true
- If $P(3)$ is true and $P(3) \to P(6)$ is true, then $P(6)$ is true
- and so on…

This creates a tree-like pattern:



Deciding the Number of Base Cases

- **Look at the induction step:** Determine which values of k it can be used to prove $F(n)$
- The induction step might not be usable for some n:
  - because it needs $Q(\text{some } k < n)$ and k would be smaller than the smallest n for which Q holds
  - because some mathematical steps only work for large enough values of n
  - In these cases, then n needs to be a base case!

Examples

Example: Consider the relation:

$t_1 = 1$

$t_n = t_{\lfloor \frac{n}{2} \rfloor} + t_{\lceil \frac{n}{2} \rceil} + 1$, for all $n > 1$

Prove that $t_n = 2n - 1$ for all positive integers n

Proof by Strong Induction:

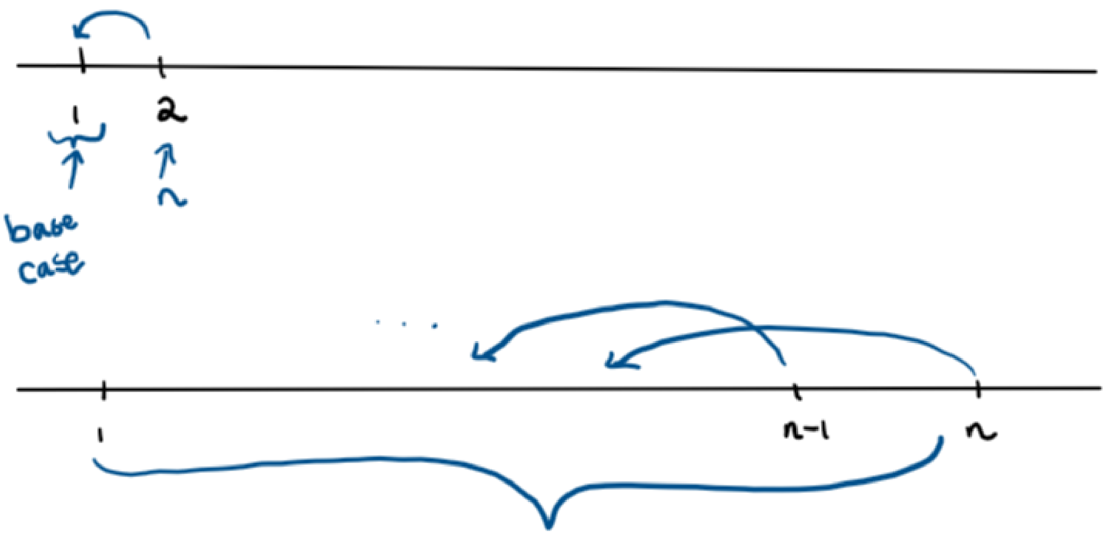
Base case: $n = 1$

$t_1 = 1 = 2 \cdot 1 - 1$, base case holds

Induction Hypothesis:

Consider an unspecified positive integer $n > 1$ ($n = 1$ is the largest base case)

Assume that $t_k = 2k - 1$, for any positive integer $1 \leq k < n$



Our base case is 1. Let's set a new $n = 2$, which uses our base case. Other values of n would use cases that have been proven in the past, but not necessarily the one immediately before it (e.g. $n = 4$ would use the $n = 2$ case as well)

Induction Step:

Because $n > 1$, the floor/ceiling of $\frac{n}{2}$ will always be less than n and will be within the range covered by the Inductive Hypothesis.

By the assumption from the IH, we know that:

- $t_{\lfloor \frac{n}{2} \rfloor} = 2 \cdot \lfloor \frac{n}{2} \rfloor - 1$
- $t_{\lceil \frac{n}{2} \rceil} = 2 \cdot \lceil \frac{n}{2} \rceil - 1$

Notice the idea that $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$ should give exactly n - we can convince ourselves by thinking about an odd and even number.

Using this fact, we can deduce:

- $t_n = t_{\lfloor \frac{n}{2} \rfloor} + t_{\lceil \frac{n}{2} \rceil} + 1$
- $t_n = (2 \cdot \lfloor \frac{n}{2} \rfloor - 1) + (2 \cdot \lceil \frac{n}{2} \rceil - 1) + 1$, by the assumption
- $t_n = 2(\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil) - 1 - 1 + 1$
- $t_n = 2n - 1$, as required.

Q.E.D

Example: **Abby asks Geoff to go to the post office to mail some paperwork for her. Postage stamps are sold in values of 4 cents and 7 cents.**

Prove that $F(x)$: Geoff can buy exactly *n* cents worth of postage stamps for any integer $n \geq 18$

First, we decide which fact we can use to prove $F(n)$:

Question: In the simplest induction step possible, which fact will we use to prove $F(n)$?

a. $F(n-1)$

b. $F(n-2)$

c. $F(n-3)$

d. $F(n-4)$

e. $F(x)$, where x is a value other than $n-1, n-2, n-3, n-4$

The correct answer is D. This is because if we have a denomination n, if we can prove that $n-4$ has previously worked, then we can just add a 4-cent stamp to reach n.

We could technically have used $F(n-7)$ as the fact, but it would not be the "simplest" induction step possible.

Using this fact, let's consider a base case now:

Question: what case(s) do we need to prove directly?

a. $n = 18$

b. $n = 18, n = 19$

c. $n = 18, n = 19, n = 20, n = 21$

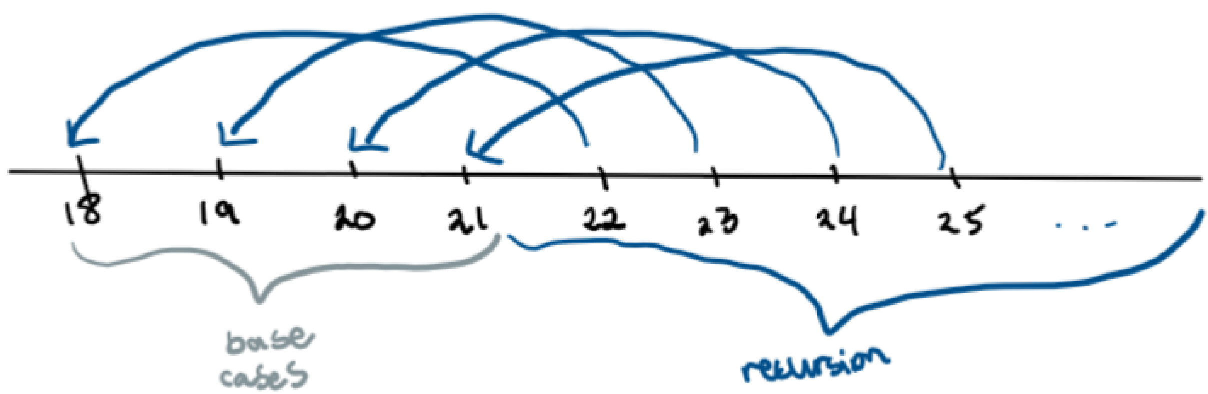
d. $n = 18$ up to $n = 26$

e. None of the above

The correct answer is C.

First, given in the problem, we must prove $n = 18$. Then, since we cannot guarantee that $n = 19, n = 20, n = 21$ could work (since we do not know for certain that $n - 4$ for those values work), we should also prove those.

As soon as we hit $n = 22$, however, notice that we can rewrite it as $18 + 4$, and we have proven $n = 18$. This is the same moving forward.



If we tried to go recursively before 18, notice that 21 cannot be represented by 17 + 4 (which is why the problem asks to start at 18!).

n	Solution
1	×
2	×
3	×
4	4
5	×
6	×
7	7
8	4+4
9	×
10	×
11	4+7
12	4+4+4
13	×
14	7+7
15	4+4+7
16	4+4+4+4
17	×
18	4+7+7
19	4+4+4+7
20	4+4+4+4+4
21	7+7+7

⋮

With this complete, let's write our proof:

Proof by Strong Induction:

Base Cases: $n = 18, 19, 20, 21$

$18 = 4 + 7 + 7$

$19 = 4 + 4 + 4 + 7$

$20 = 4 + 4 + 4 + 4 + 4$

$21 = 7 + 7 + 7$

The base case holds.

Induction Hypothesis:

Consider an unspecified integer $n > 21$ (largest base case)

Assume $F(k)$ holds when k is in the range $18 \leq k < n$ (18 is the smallest base case)

Induction Step:

By the IH, we make $n-4$ cents, since $n > 21$, $n-4$ will lie in the range of $18 \leq n-4 < n$

Then, we can make n cents by adding a one single 4 cent stamp.

Q.E.D.

End of Session
