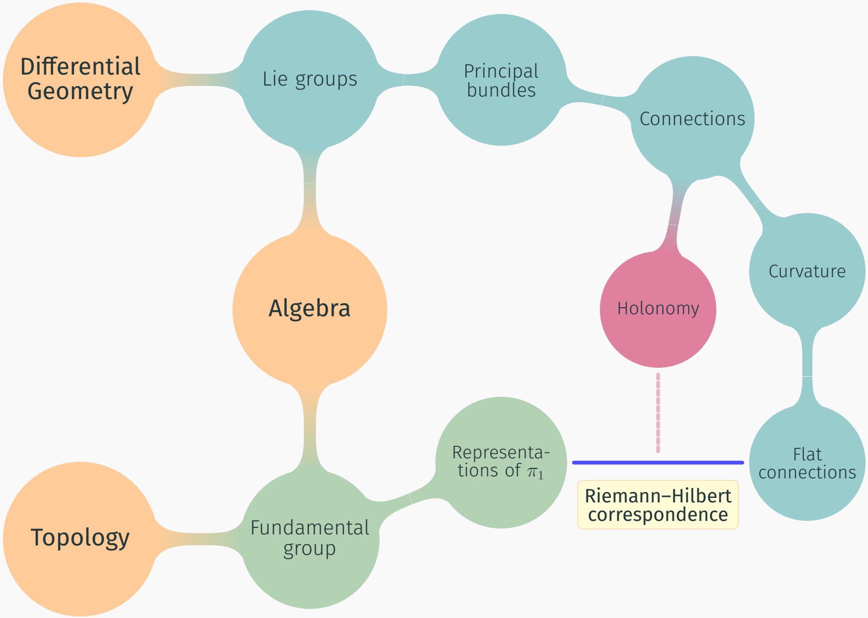
Universitat de Barcelona
Facultat de Matemàtiques i Informàtica

The Riemann–Hilbert Correspondence for Flat Connections on Principal Bundles

Mathematics Bachelor's Thesis

Marçal Herraiz Bayó
Supervisor: Dr. Ignasi Mundet Riera

June 27, 2025



Differential Geometry
Lie groups
Principal bundles
Connections
Algebra
Holonomy
Curvature
Representations of π_1
Flat connections
Riemann–Hilbert correspondence
Topology
Fundamental group

Lie Groups

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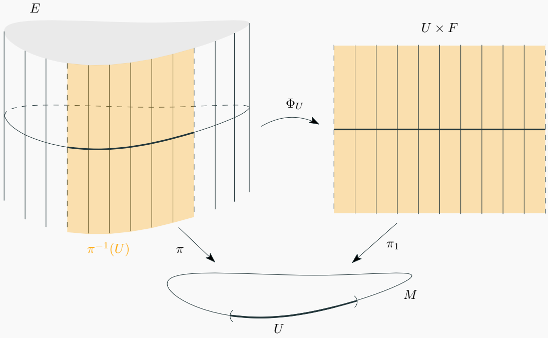
- The Lie algebra of G viewed as $T_e G$ will be written as $\mathfrak{g}$.
- The adjoint representation:

$$\mathrm{Ad} \colon G \longrightarrow \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{Ad}_g := \mathrm{Ad}(g).$$

Principal Bundles

Fiber Bundles



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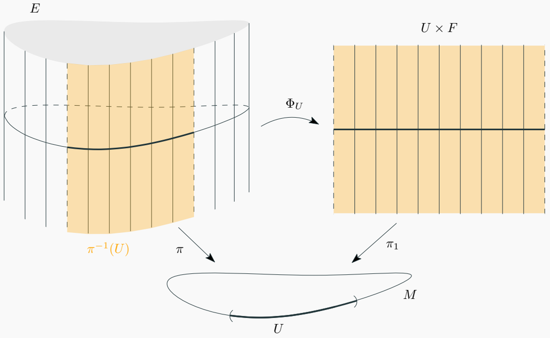

Definition (smooth fiber bundle)

- $\pi \colon E \to M$ smooth and **surjective**.
- **Locally trivial**: every $x \in M$ has a neighborhood $U \subseteq M$ and a map

$$\Phi_U \colon \pi^{-1}(U) \xrightarrow{\cong} U \times F, \quad \text{with} \quad \begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi_U"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd} .$$

What if each fiber is a **Lie group**?

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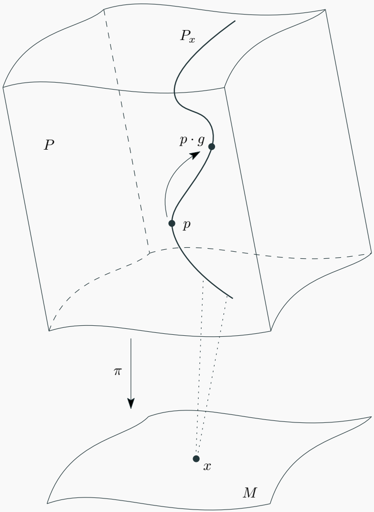


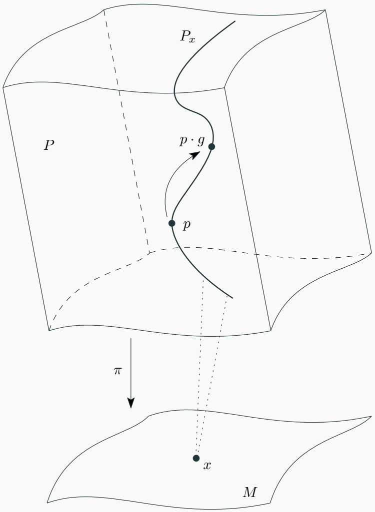
Principal Bundles

Definition (Principal G-bundle)

- Fiber bundle $\pi \colon P \to M$, fiber G.
- **Right action** of G on P:
  (i) Smooth and **free**.
  (ii) **Fiber-preserving**: $\pi(p \cdot g) = \pi(p)$.
  (iii) Local trivializations

$$\Phi : \pi^{-1}(U) \longrightarrow U \times G$$

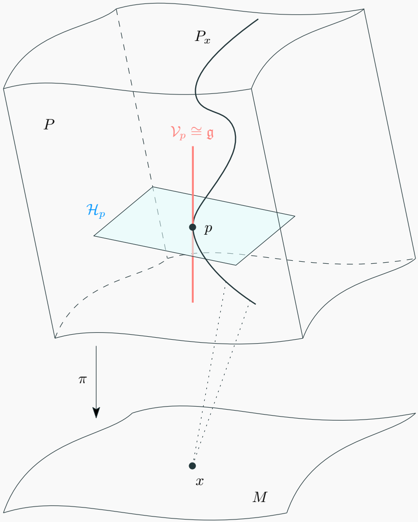
are **G-equivariant**.



Verticality:

$$\mathcal{V}_p := \ker d_p\pi \subset T_pP.$$

In fact, $\mathcal{V}_p \cong \mathfrak{g}$.



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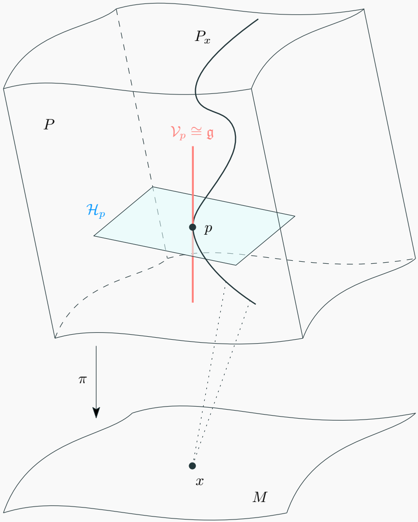
Decompose T_pP as

$$T_pP = \mathcal{V}_p \oplus \mathcal{H}_p.$$

$\mathcal{H}_p$: horizontal tangent subspace at p.



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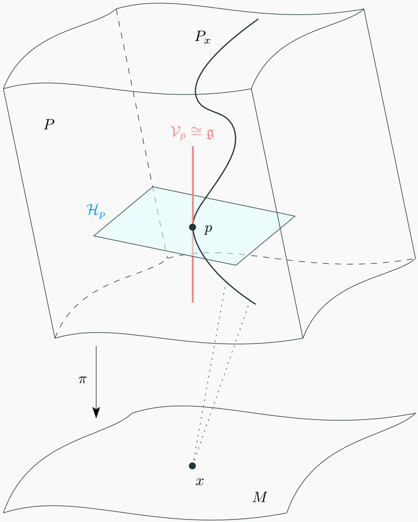
NOT UNIQUE!



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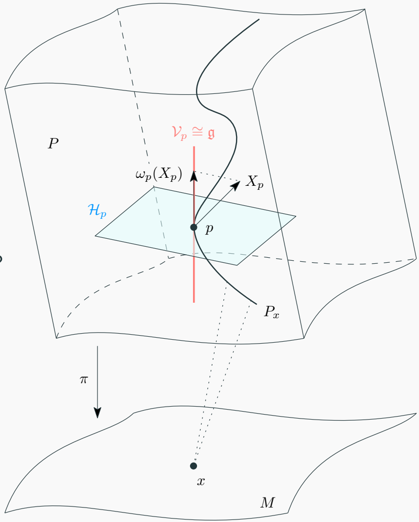

Connections on Principal Bundles

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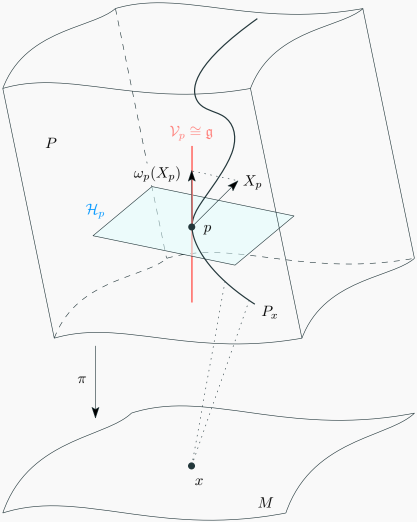
Definition (Connection)

- Principal G-bundle $\pi\colon P \to M$.
- $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$:
  (i) $\omega(\underline{A}) = A$, for all $A \in \mathfrak{g}$.
  (ii) $R_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$.



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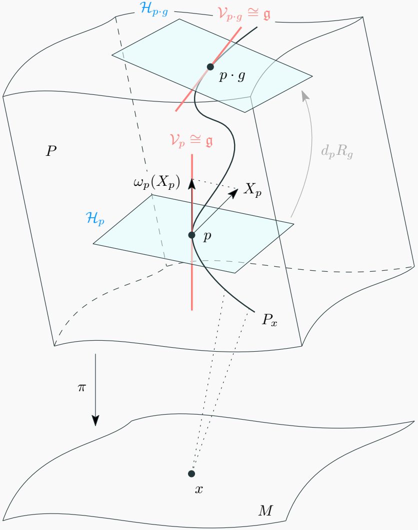
We can **recover horizontal spaces**:

Theorem

$\mathcal{H} := \ker\omega$ defines a right-invariant horizontal distribution on P.



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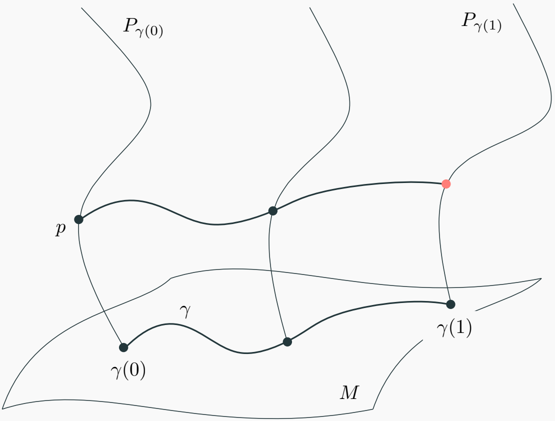
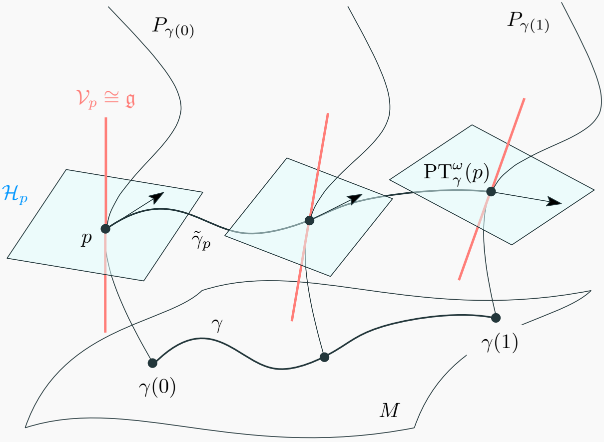


Parallel transport



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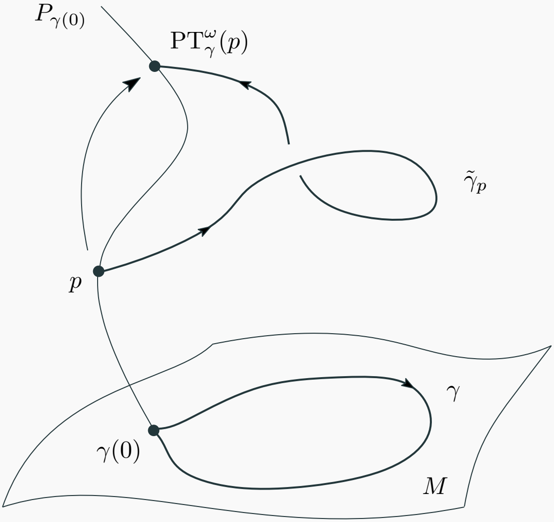

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If γ is a closed curve:

- $\mathrm{PT}^{\omega}_{\gamma}$ is an automorphism.
- $\exists!\,\mathrm{hol}^{\omega}_{p}(\gamma) \in G$ with

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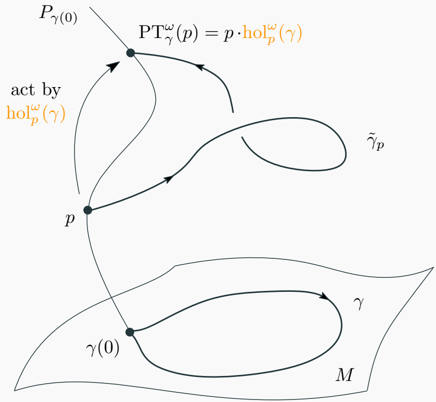

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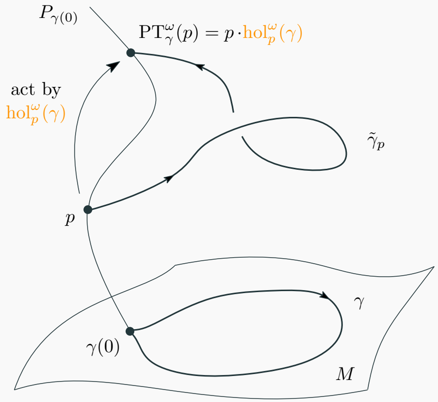
Moreover,

$$\mathrm{hol}^\omega_{p\cdot g}(\gamma) = g^{-1}\mathrm{hol}^\omega_p(\gamma) g$$



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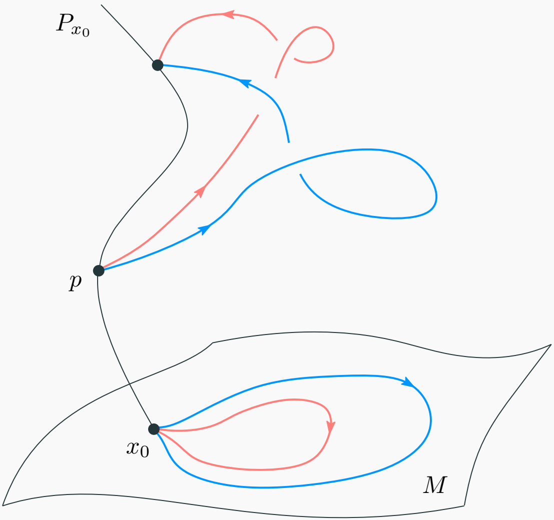

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- ω is flat if $\Omega \equiv 0$.

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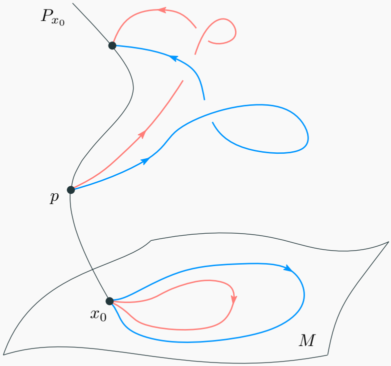

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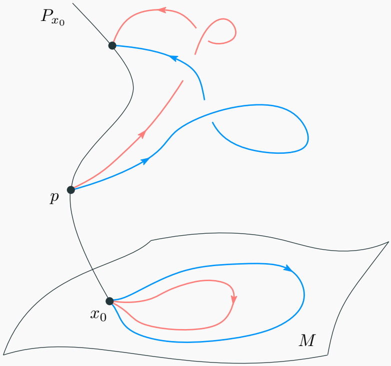

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Holonomy representation for flat connections: for $p \in P_{x_0}$,

$$\rho_p^\omega \colon \pi_1(M, x_0) \longrightarrow G, \quad \rho_p^\omega([\gamma]) := \mathrm{hol}_p^\omega(\gamma).$$

Curvature and Holonomy

This gives a map

$$\mathcal{F}(P) \longrightarrow \frac{\mathrm{Hom}\big(\pi_1(M, x_0), G\big)}{G}, \qquad \omega \longmapsto [\rho_p^\omega], \text{ for some } p \in P_{x_0}.$$

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Theorem (Riemann–Hilbert correspondence)

Let G be a Lie group, M a connected smooth manifold, and $x_0 \in M$. Then:

$$\frac{\{\textit{flat principal } G\textit{-bundles over } M\}}{\textit{isomorphism}} \longleftrightarrow \frac{\mathrm{Hom}\big(\pi_1(M, x_0), G\big)}{G}$$

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$$\begin{tikzcd}
\widetilde{P} = \widetilde{M} \times G \arrow[r, "\tilde{\pi}"] \arrow[d, "\widetilde{\Pi}"'] & \widetilde{M} \arrow[d, "\Pi"] \\
P_\rho = \widetilde{P}/\Gamma \arrow[r, "\pi"'] & M \cong \widetilde{M}/\Gamma.
\end{tikzcd}$$

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- Extend it via G-equivariance:

$$F(q \cdot g) := F(q) \cdot g.$$

$\square$

“Mathematics is the art of giving the same name to different things.”

— *Henri Poincaré*

Acknowledgements

Acknowledgements

- Dr. Ignasi Mundet.
- All professors and faculty members in Mathematics and Physics.
- Family and friends.
- Ministry of Education for the “Beca de col·laboració” 2024-2025.

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Backup Slides

Flat $\mathbb{S}^1$-bundles over oriented 2-manifolds

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- By the classification of closed, connected surfaces:
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Flat $\mathbb{S}^1$-bundles over oriented 2-manifolds

- M a **closed**, **connected**, 2-dimensional smooth manifold of genus g.
- By the classification of closed, connected surfaces:
  - If $g = 0$, $M \cong \mathbb{S}^2$.
  - If $g \geqslant 1$, $M \cong \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ (connected sum of g tori). Then

$$\Gamma := \pi_1(M, x_0) = \left\langle a_1, b_1, \ldots, a_g, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle,$$

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Example: Orientable 2-manifolds of genus 0

Only one flat $\mathbb{S}^1$-bundle up to isomorphism (the trivial bundle).

Parallel Transport

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Proposition (Horizontal lift)

Given a p. s. curve $\gamma : I \to M$ *and* $p \in P_{\gamma(0)}$*, there exists a unique p. s. curve:*

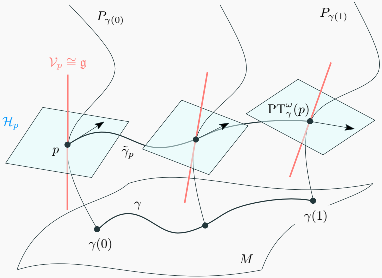
$$\widetilde{\gamma}_p : I \longrightarrow P$$

such that:

(i) $\pi \circ \widetilde{\gamma}_p = \gamma$ (**lift** *of* γ)

(ii) $\widetilde{\gamma}'_p(t) \in \mathcal{H}_{\widetilde{\gamma}_p(t)}$ *for all* t (**horizontal**)

(iii) $\widetilde{\gamma}_p(0) = p$ (**starts at** p)



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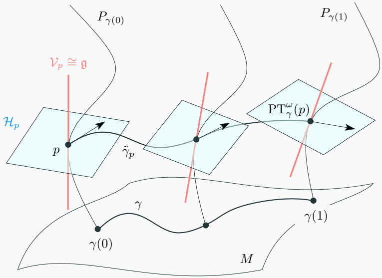

Definition (Parallel transport)

Defined as:

$$\mathrm{PT}^{\omega}_{\gamma} : P_{\gamma(0)} \to P_{\gamma(1)}, \qquad \mathrm{PT}^{\omega}_{\gamma}(p) := \widetilde{\gamma}_p(1).$$

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- **Form the quotient:**
  - Define $P_\rho := \widetilde{P}/\Gamma$.
  - Then $\pi \colon P_\rho \to \widetilde{M}/\Gamma \cong M$ is a principal G-bundle over M.

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  - Extend F to the fibers with G-equivariance: $F(q \cdot g) = F(q) \cdot g$.

$\square$