Exact solution methods for stochastic single-allocation hub location problems

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Abstract

This paper presents a single-allocation hub location problem with uncertainty that affects the amount of product sent between origins and destinations, the transportation costs and the capacities of the hubs. We present an integrated model where the three uncertain parameters are simultaneously considered in a scenario-based model. In order to minimize the expected overall costs of the system, two decisions have to be made: (i) the location of the hubs (this location does not change in the scenarios), and (ii) the allocation of each origin/destination to one of these hubs. We prove that some of these stochastic problems can be reduced to a classical deterministic single-allocation hub location problem. A mixed integer programming formulation and some valid inequalities are provided for the introduced problem. A branch-and-cut method considering a constraint-relaxed formulation, the inclusion of cuts through a separation method and a preprocessing phase, is designed. This procedure provides very competitive solution times compared to other existing solution procedure for solving a very particular type of these problems.

Keywords: stochastic; hub; single allocation; capacity; branch-and-cut.

1 Introduction

The Hub Location Problem (HLP) is concerned with determining the location of hubs and allocation of non-hubs nodes to these hubs to minimize the operating cost, which tipically consists of the setup cost for hubs and the total transportation cost for Origin-Destination (OD) demand (flow). The main feature of the HLPs is that OD flow are routed through hubs, instead directly routing OD flows from origin nodes to destination nodes, to exploit economies of scale and reduce operating costs (Alumur et al., 2021; Farahani et al., 2013).

Most existing research on HLPs assume that the system parameters (flows, costs, hub capacities, among others) are deterministic. Such assumptions may not be valid in many real-life problems because many parameters are uncertain due to many unavoidable variations, including population shifts, economic, environmental and political circumstances and quality of the provided services. Hence, some information required for planning is not available a priori, and the associated uncertainties are only resolved once the system is constructed and the hubs are installed.

The parameters that have been most studied under uncertainty within the existing hub location models are the demands and transportation costs/times (see Alumur et al. (2012); Eiselt and Marianov (2023) and references therein for the review of hub location problems under uncertainty). In the large majority of HLPs with uncertainty it is assumed that the hubs have unlimited capacity or even if capacity is considered, it is assumed the capacity is known in advance. The studies dealing with uncertainty in hubs nodes are focused to the disruption or unavailability of the hubs, and the models do not consider other uncertain parameters simultaneously.

The main motivation of this work is that demand, transportation costs and capacity of the hubs are being modified over time, and they should be considered simultaneously at the hub location models. To the best of our knowledge, there is no work that explicitly models the three uncertain parameters simultaneously in a scenario-based model. In this work, we present such an integrated model.

Some application of the proposed model appears in the area of designing transportation networks (public transportation networks, air transportation), delivery service (postal delivery, express delivery service, drone delivery service) or telecommunication systems (data transmission systems).

Various scenarios can arise in communication networks where demands, costs, and capacities may fluctuate. Examples include technical failures, environmental changes (such as temperature or humidity), or cyberattacks that impair equipment performance, overload certain interconnection nodes with malicious traffic, and necessitate the diversion of some traffic to other nodes.

In the context of air transportation, the incident that affected hundreds of airports worldwide in July 2024—due to an error in a security update in some systems—reduced their operational capacity and compelled modifications in airport planning. Severe weather conditions or strikes can also lead to the partial closure of runways, requiring the rerouting of some traffic to other airports. In these cases, as well as in any other unforeseen situations—such as unscheduled maintenance operations, problems in power supply, sabotage, or any other types of incidents at hubs—failures may occur that result in a total or partial decrease in their capacity, making it necessary to reassign demands to other hubs. It is important to consider that such situations can also affect demands and costs, which justifies the use of stochastic models that take into account variations in all three types of parameters.

This paper deals with a Single-Allocation Hub Location Problem (SAHLP) with three simultaneously uncertainty resources: capacity of hubs, OD demands and transportation costs. We assume that these parameters are modelled using random variables with realization only after the hubs are selected. Different scenarios are considered (with their probabilities) and two decisions have to be made: i) the location of the hubs (this location does not change from one scenario to another), and ii) the allocation of each OD to one of these hubs in order to minimize the expected cost of the system. This is a realistic practice because the hubs are located before knowing the real scenario and the allocations are determined when the actual parameters are realized. This is motivated by the necessity of the network operators to quickly react to the changes in overall system performance by adjusting the assignments of the OD to the hubs once the uncertainty is realized.

The contributions of this paper can be discussed from two point of views: its modelling approach, and its solution methodology.

From the modelling approach's points of view, this paper simultaneously considers hub, OD, and link uncertainties in designing the hub network. A stochastic scenario-based approach is used to model the uncertainty. We show that some of these problems with uncertainty, already studied in the literature as stochastic models, can be considered as a particular version of deterministic SAHLP. The paper also contributes to the literature by considering capacities varying in each scenario instead of considering total disruption, allowing hubs are unavailable in some scenarios but not in all of them. A compact integer programming formulation is proposed for the stochastic SAHLP.

From the solution approach's point of view, this paper contributes to existing literature by developing a branch-and-cut procedure for solving the proposed problems that considers a constraint-relaxed formulation, the inclusion of cuts through a separation method and a preprocessing phase. This procedure provides very competitive solution times compared to other existing solution procedure for solving a very particular type of these problems. Moreover, the family of cuts provided are added to the formulation by solving the dual of a transportation problem, These cuts are proved dominate the so-called *subgradient cuts* obtained in Rostami et al. (2021).

The remainder of this paper is structured as follows. In the next section we review the related literature. The description of the problem is given in the Section 3. Section 4 presents mathematical formulations for the problem. Different valid inequalities and their corresponding separation methods are proposed in Sections 5 and 6. A branch-and-cut procedure is described in Section 7. The computational experiments and the numerical results are presented in Section 8. Finally, some conclusions and some outlooks for future research are provided.

2 Literature review

The Hub Location Problem (HLP) has been addressed by a wide community of operations researchers due to its practical relevance. Many reviews about location problems show the extensive activity in this field and the applications of these problems, see Alumur and Kara (2008), Campbell and O'Kelly (2012), and Contreras and O'Kelly (2019) among others.

O'Kelly (1987) proposed the first quadratic integer programming formulation for the so-called p-hub median problem. Since then, many attempts have been made to linearize the objective function and many exact and heuristic algorithm have been proposed in the literature. Among the most efficient solution methods: Labbé and Yaman (2004); Labbé et al. (2005) projected the variables out from the path-based formulation proposed by Skorin-Kapov et al. (1996); Meier et al. (2016) used a row generation procedure; Ghaffarinasab and Motallebzadeh (2018) proposed exact algorithms based on Benders decomposition for solving large-scale instances. Rostami et al. (2023) provided a convex reformulation and a branch-and-cut algorithm based on outer approximation cuts.

Many variants of HLPs have been studied. However, the majority of the studies focus on deterministic formulations. The uncertainties in HLPs can be generally categorized into hub-side uncertainty, OD-side uncertainty, and connection-link uncertainty. The hub uncertainty is often caused by the randomness in hub capacity, the reliability of hubs in serving the shipments and disruptions (Azizi and Salhi, 2022; Ghaffarinasab and Motallebzadeh, 2018; Rostami et al., 2018; Tran et al., 2017). The OD uncertainty is often due to the randomness in demands (Correia et al., 2018; Ghaffari et al., 2016; Shahabi and Unnikrishnan, 2014). Link uncertainty could be due, for example, to the random travel time, random transportation cost or unreliable routes (Blanco et al., 2023; Sim et al., 2009).

The main approaches for dealing with uncertainty in the context of HLP include fuzzy programming, robust optimization, and stochastic optimization. The fuzzy-based methods for the HLP depend on the judgements of experts to define uncertain parameters as fuzzy numbers or variables (Eydi and Shirinbayan, 2023). In the presence of randomness, robust optimization is a suitable approach if no distributional information is available (Martins de Sá et al., 2018; Wang et al., 2020; Zetina et al., 2017).

Stochastic programming methods can be used when the input parameters are random variables with known probability distributions (estimated based upon the sufficient objective/ historical data). Contreras et al. (2011) studied stochastic uncapacitated multiple allocation HLP where uncertainty is associated with demands and transportation costs. The authors proved that stochastic problems with uncertain demands or dependent transportation costs are equivalent to a deterministic expected value problem. Chaharsooghi et al. (2016) studied a reliable multiple allocation HLP with hub failure possibility. Peiró et al. (2019) considered a stochastic version of the uncapacitated r-allocation p-hub median problem with direct shipment option in which the uncertainty is associated with OD demands and transportation costs. Shang et al. (2020) devised a stochastic hierarchical multi-modal HLP to capture uncertainty in hub construction costs and travel time at the strategic level. Zhang et al. (2022) studied a stochastic incomplete multi-modal HLP with uncertain demands and transportation costs. Ghaffarinasab et al. (2023) introduced risk-averse stochastic multiple allocation HLPs with conditional-mean and conditional value-at-risk criteria for measuring risk. Rostami et al. (2021) presented two-stage stochastic programming models for a SAHLP with demand uncertainty. They developed an alternative deterministic mixed-integer nonlinear programming formulation and a customized solution approach based on the cutting-plane method. In the paper that we present here, a compact integer programming formulation for a stochastic SAHLP is given.

The large majority of HLPs under uncertainty studied in the literature focus exclusively on uncertainties such as demands or transportation costs/times. Uncertainty on the capacity of the hubs have been studied only in a few papers and using a robust approach. In Habibzadeh et al. (2016) studied a robust capacitated HLPs under capacity and setup cost uncertainty. Other papers addressed capacitated HLPs with possibilities of hubs disruptions (Azizi and Salhi, 2022; Momayezi et al., 2021; Mohammadi et al., 2016, 2019). Some models considered a backup hub allocated to each demand point in case of failure of a hub (Azizi and Salhi, 2022; Mohammadi et al., 2019). In others, the probability that a traffic flow entering a hub will exceed the actual capacity and disrupt the flow process was given (Mohammadi et al., 2016). In this paper, the authors considered that hub may have a partial disruption and be available, and only the size of the flow that enters the hubs must be decreased.

The present paper considers a scenario-based approach where the capacity at hubs varies from one scenario to another, but there are not probabilities that each hub is disrupted in each scenario. We do not consider backup hubs, we want to locate a set of hubs, whose location does not change for all the scenarios, and allocate each OD to only one hub in each scenario, in order to minimize the expected total cost. Our model allows a hub to be unavailable in some scenarios (where its capacity is not enough for its self service) but available in the remaining scenarios to not discard desirable solutions with a lower expected total cost.

3 Description of the problem

Let $N := \{1, \ldots, n\}$ be the set of sites representing the origins and destinations of the product to be transported, and $K \subset N$ be the set of potential hub locations. We require the product to be sent from the origin i to destination j through one or, at most, two hubs for any $i, j \in N$. Moreover, every site i which is not a hub must be allocated to a single hub, so that the amount of product sent from/to i to/from any other site must pass through this hub.

Let $S := \{1, \ldots, n_s\}$ be a set of scenarios such that the probability that scenario $s \in S$ occurs is given by $p_s \geq 0$, with $\sum_{s=1}^{n_s} p_s = 1$. For any scenario $s \in S$, we define:

- ω_{ij}^s as the amount of product to be sent from site i to j, for any $i, j \in N$.
- C_{ijkm}^s as the unit transportation cost from site i to j through hubs k and m in this order, for any $i, j \in N$, $k, m \in K$. Although these costs are presented in many other papers as disaggregated in three components (origin-hub, hub-hub and hub-destination), here we do not assume any structure. In particular, we do not suppose either the costs can be disaggregated, or they are based on distances or satisfy the triangle inequality.
- τ_k^s as the capacity of a hub installed at site k.

Let f_k be the operational cost associated with the location of a hub at site k. Observe that this cost does not depend on the scenario.

The Stochastic Single-Allocation Hub Location Problem (S_SAHLP) aims to (i) locate a set of hubs (this location is the same for all the scenarios), and (ii) allocate each origin/destination to one of these hubs (in each scenario), in order to minimize the expected value of the transportation cost between origins and destinations through the hubs, plus the operational costs associated with the location of the hubs.

Depending on whether capacity constraints are considered or not, this problem is called Stochastic Capacitated Single-Allocation Hub Location Problem (S_CSAHLP) or Stochastic Uncapacitated Single-Allocation Hub Location Problem (S_USAHLP), respectively. Note that the classical capacitated and uncapacitated single-allocation hub location problems are particular cases of these problems by setting $n_s = 1$. Therefore, both S_CSAHLP and S_USAHLP are also NP-hard problems (O'Kelly, 1987).

The stochastic hub location problems proposed here also allow us to consider different uncertainty situations where hubs and established connections between origin and destinations are not always operational as initially planned. Accidental disruptions in hubs are covered by considering the zero capacity at the scenario where these hubs are disrupted. The situation where a connection between origins and destinations is broken or disrupted in some scenario is represented by considering the corresponding transportation cost large enough.

In addition, it is worth mentioning that some of these problems with uncertainty, already studied in the literature as stochastic models, can be considered as a particular version of deterministic single-allocation hub location problems, as we show in the following proposition.

Proposition 3.1. *S_USAHLP is equivalent to a deterministic uncapacitated single-allocation hub location problem.*

Proof. Let us consider the S_USAHLP as defined above. This problem is equivalent to a deterministic single-allocation hub location problem in a directed graph (N', A'). The elements of N' represent either an origin/destination site $i \in N$ in a scenario $s \in S$ or a hub site $k \in K$. Hence, $N' := \{i_s : i \in N \text{ and } s \in S'\}$, where $S' = S \cup \{0\}$. Thus, $i_s \in N$ represents origin/destination site i in scenario s, when $s \neq 0$, and a potential hub site when $s = 0$. Let $K' := \{i_0 \in N'\}$, the set of potential sites to locate hubs. The set of arcs A' is given for those arcs $(i_s, i'_{s'})$ where one of the situations is given: (a) $s = s'$ or (b) $s = 0$ and $s' \neq 0$ or (c) $s \neq 0$ and $s' = 0$. Moreover, the amount of product sent from i_s to j_t is

$$\omega'_{i_s j_t} := \begin{cases} \omega^s_{ij}, & \text{if } s = t \text{ and } s \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since we are considering single-allocation in each scenario $s \in S$, each site i_s is allocated to only one hub k_0. Let us consider $C'_{i_s j_t k_0 m_0}$, for $i_s, j_t \in N'$ and $k_0, m_0 \in K'$, as the cost of transporting the total amount of product between sites i_s and j_t using k_0 as the hub assigned to i_s and m_0 as the one assigned to j_t. This cost is defined as

$$C'_{i_s j_t k_0 m_0} := \begin{cases} p_s C^s_{ijkm}, & \text{if } s = t, \\ M, & \text{otherwise.} \end{cases}$$

where M is a large enough value.

The problem of opening a set of hubs at sites of K' and assigning every site of N' to a hub, in order to minimize the overall transportation cost of sending the products between origins and destinations through at least one hub and at most two hubs, is a deterministic hub location problem equivalent to the S_USAHLP. This follows from the definition of the graph, C' costs, and the fact that the location of the hubs is independent of the scenario. □

Remark 3.1. *In case of considering capacities for the hubs in each scenario, the result above is not longer valid. Indeed, in that case the reduction to a deterministic problem does not guarantee that the capacities of the hubs are not exceeded. In the S_CSAHLP we have, for each scenario $s \in S$, a capacity for each hub k defined by τ^s_k, and this capacity should not be exceeded in this scenario. However, in the single-allocation hub location problem defined in (N', A'), even defining $\tau'_{k_0} := \sum_{s\in S} \tau^s_k$, we have a global capacity constraint for each $k_0 \in K'$ but there is not way to guarantee the capacity constraint in each scenario.*

4 An MILP formulation for S_SAHLP

In this section we provide a mixed integer formulation for S_SAHLP. Two families of variables are used for location and allocation, respectively. The location of the hubs is independent of the scenarios and the allocations are decisions that will be taken depending on the observed scenario $s \in S$. For any $i \in N$, $k(\neq i) \in K$ and $s \in S$, let us define:

$$z_k = \begin{cases} 1, & \text{if a hub is located at site } k, \\ 0, & \text{otherwise,} \end{cases}$$

$$x^s_{ik} = \begin{cases} 1, & \text{if site } i \text{ is allocated to hub } k \text{ under scenario } s, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, let $O^s_i = \sum_{j\in N} \omega^s_{ij}$, $D^s_i = \sum_{j\in N} \omega^s_{ji}$ and $\hat{C}^s_{ijkm} := \omega^s_{ij} C^s_{ijkm} + \omega^s_{ji} C^s_{jimk}$.

Using these variables and parameters, S_USAHLP is formulated as

$$\min \quad \sum_{k\in K} f_k z_k + \sum_{s\in S} p_s \sum_{i\in N} \sum_{\substack{j\in N\\ j>i}} \left(\sum_{\substack{k\in K\\ k\neq i}} \sum_{\substack{m\in K\\ m\neq j}} \hat{C}^s_{ijkm} x^s_{ik} x^s_{jm} + \right.$$

$$\left. \sum_{\substack{m\in K\\ m\neq j}} \hat{C}^s_{ijim} z_i x^s_{jm} + \sum_{\substack{k\in K\\ k\neq i}} \hat{C}^s_{ijkj} x^s_{ik} z_j + \hat{C}^s_{ijij} z_i z_j \right), \tag{1}$$

$$\text{s.t.} \quad \sum_{\substack{k\in K\\ k\neq i}} x^s_{ik} = 1 - z_i, \qquad \forall i \in N, \forall s \in S, \tag{2}$$

$$x^s_{ik} \leq z_k, \qquad \forall i \in N,\ k(\neq i) \in K,\ s \in S, \tag{3}$$

$$x^s_{ik}, z_k \in \{0,1\}, \qquad \forall i \in N, k(\neq i) \in K, s \in S. \tag{4}$$

The objective (1) minimizes the expected transportation cost, calculated multiplying the overall transportation cost in each scenario times its probability of occurrence, plus the operational cost associated with the hubs. Constraints (2) force each node i to be allocated to one potential hub at each scenario (i.e. single allocation) or to be a hub itself. Constraints (3) ensure that i is not allocated to a site k in any scenario unless a hub is located at k. The binarity conditions are given by constraints (4).

In the capacitated version, additional constraints that limit the capacity of the installed hubs should be included in the formulation. For the SAHLP with uncertainty only in the demand, Rostami et al. (2021) propose the following capacity constraints:

$$\sum_{\substack{i\in N\\ i\neq k}} O^s_i x^s_{ik} \leq (\tau^s_k - O^s_k) z_k \qquad \forall k \in K, s \in S. \tag{5}$$

Constraints (5) ensure that the capacity of a hub in each scenario must be sufficient to its self service ($\tau^s_k > O^s_k$, for $k \in K$, $s \in S$). However, these constraints do not allow the possibility of having an inactive hub in a scenario but active in the rest, think for example the case where this scenario represents the disruption of this hub. This situation can be modeled by assuming null capacity of this hub in that scenario, but constraints (5) would prevent opening that hub, i.e., it would make some desirable solutions to be unfeasible. The same argument is valid even when the capacity for each hub is the same in all scenarios. In the following, we illustrate this fact with an example.

Example 4.1. *Consider* $N = \{(5, 21.65), (0, 13), (20, 21.65), (25, 13), (17.5, 0), (7.5, 0), (10, 13), (15, 13), (12.5, 8.66)\}$ *as the coordinates in the plane of the set of sites representing the origins, destinations and potential hub locations. Three scenarios are considered, and the situation in each one is the following.*

- *The origin-destination flows are defined by* $\omega^2_{71} = \omega^2_{72} = \omega^2_{73} = \omega^2_{74} = \omega^2_{75} = \omega^2_{76} = 5$, $\omega^3_{5j} = 2$, *for all* $j \in N$. *For the remaining origin-destination pairs, one unit of flow is considered in each scenario.*

- *The capacities of the potential hubs in each scenario are given by*

$$\tau^1 = \tau^2 = \tau^3 = (1, 1, 1, 1, 1, 1, 30, 60, 50).$$

- *The transportation costs between* i *and* j *are given by* $C^1_{ijkm} = C^2_{ijkm} = C^3_{ijkm} = d_{ik} + d_{km} + d_{mj}$ *where* d_{ij} *is the Euclidean distance between* i *and* j.

- *The operational costs are given by*

$$f = (50, 50, 50, 50, 50, 50, 10, 10, 10).$$

In scenario 2, $\tau^2_7 < O^2_7$ *holds. By solving* S_CSAHLP *with constraints* (5)*, the optimal solution locates hubs at* 8 *and* 9 *(see Figure 1) and its optimal value is* 3879.67*. Locating a hub at* 7 *is not possible because constraint* (5) *fixes* $z_7 = 0$ *in all scenarios. In Figures 1(b) and 2(b) to avoid misunderstandings, the allocation of site 1 to hub 9 has been represented by a curved arc.*

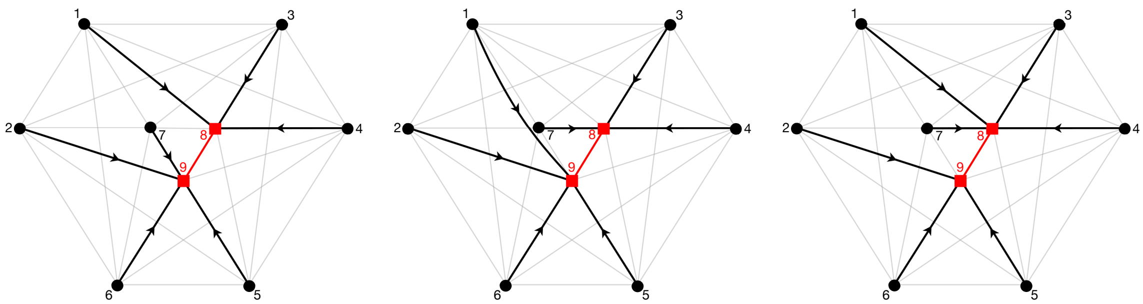
However, allowing a hub not to be active in some scenarios produces an optimal solution that locates hubs at 7*,* 8 *and* 9 *(see Figure 2), with a lower optimal value of* 3573.25 *(observe that in scenario* 2 *hub at site* 7 *is not active).*

In order to give a valid formulation for S_CSAHLP in the general case where, for a given $k \in K$, τ^s_k is not necessarily greater than O^s_k for any $s \in S$, we define the parameter I^s_k as follows (Chaharsooghi et al., 2016):

$$I^s_k = \begin{cases} 1, & \text{if } \tau^s_k \geq O^s_k, \\ 0, & \text{otherwise,} \end{cases} \qquad \forall k \in K, s \in S, \tag{6}$$

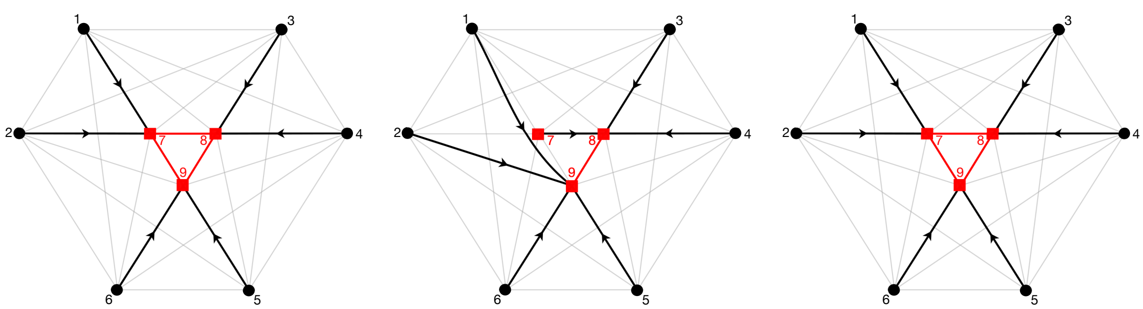
and to replace (5) by

$$\sum_{i(\neq k) \in N} O^s_i x^s_{ik} \leq (\tau^s_k - O^s_k) z_k I^s_k, \qquad \forall s \in S, k \in K. \tag{7}$$



(a) Scenario 1 (b) Scenario 2 (c) Scenario 3

Figure 1: Optimal solution of Example 4.1 using capacity constraints (5)



(a) Scenario 1 (b) Scenario 2 (c) Scenario 3

Figure 2: Optimal solution of Example 4.1 using capacity constraints (7)

Now, if $\tau_k^s < O_k^s$, then $x_{ik}^s = 0$, $\forall i(\neq k) \in N$, and $\sum_{j(\neq k)\in K} x_{kj}^s = 1$, but z_k would not be fixed to zero. Observe that $\tau_k^s < O_k^s$ is equivalent to assume that $\tau_k^s = 0$.

Hence, S_CSAHLP can be formulated as follows:

$$(F_{NL})\min \quad \sum_{k\in K} f_k z_k + \sum_{s\in S} p_s \sum_{i\in N} \sum_{\substack{j\in N\\ j>i}} \left(\sum_{\substack{k\in K\\ k\neq i}} \sum_{\substack{m\in K\\ m\neq j}} \hat{C}^s_{ijkm} x^s_{ik} x^s_{jm} + \right.$$

$$\left. \sum_{\substack{m\in K\\ m\neq j}} \hat{C}^s_{ijim} z_i x^s_{jm} I^s_i + \sum_{\substack{k\in K\\ k\neq i}} \hat{C}^s_{ijkj} x^s_{ik} z_j I^s_j + \hat{C}^s_{ijij} z_i z_j I^s_i I^s_j \right), \tag{8}$$

$$\text{s.t.} \quad \sum_{\substack{k\in K\\ k\neq i}} x^s_{ik} = 1 - z_i I^s_i, \qquad \forall i\in N, \forall s\in S, \tag{9}$$

$$x^s_{ik} \leq z_k I^s_k, \qquad \forall i\in N,\ k(\neq i)\in K,\ s\in S, \tag{10}$$

$$\sum_{\substack{i\in N\\ i\neq k}} O^s_i x^s_{ik} \leq (\tau^s_k - O^s_k) z_k I^s_k, \qquad \forall k\in K, s\in S, \tag{11}$$

$$x^s_{ik}, z_k \in \{0,1\}, \qquad \forall i\in N, k(\neq i)\in K, s\in S. \tag{12}$$

We need to include the parameter I^s_k in the objective function to make the transportation costs in scenario s associated to hub k equal to 0 whenever k is not active in scenario s. Moreover, it is also included in constraints (9) and (10) to avoid that $z_k = 0$ if for some scenario $s \in S$, $\tau^s_k < O^s_k$ and to ensure in that case that k is allocated to any other hub. Note that we can previously fix the variables x^s_{ik} to zero when $\tau^s_k < O^s_k$ for some scenario $s \in S$ in a preprocessing phase.

The objective function (8) is nonlinear due to the product of the variables. In order to provide a mixed integer linear programming formulation we define, for $i \in N$ and $s \in S$, the family of variables

$$y^s_i \;=\; \text{overall transportation cost with origin/destination at } i \text{ to/from any } j\in N, j>i \text{ under scenario } s.$$

For the sake of readability, we denote $z_k^s := z_k I_k^s$. Hence, the linear formulation is given by

$$
\begin{aligned}
(F_L)\ \ \min\ & \sum_{k\in K} f_k z_k + \sum_{s\in S} p_s \sum_{i\in N} y_i^s \\
\text{s.t.}\ & (9)-(12) \\
& y_i^s \geq \sum_{\substack{j\in N\\ j>i}} \Big(\sum_{\substack{m\in K\\ m\neq j}} \hat{C}_{ijkm}^s x_{jm}^s + \hat{C}_{ijkj}^s z_j^s + (x_{ik}^s - 1)\max_{m\in K}\{\hat{C}_{ijkm}^s\}\Big), \quad \forall i\in N, k(\neq i)\in K,\ \forall s\in S, && (13)\\
& y_i^s \geq \sum_{\substack{j\in N\\ j>i}} \Big(\sum_{\substack{m\in K\\ m\neq j}} \hat{C}_{ijim}^s x_{jm}^s + \hat{C}_{ijij}^s z_j^s + (z_i^s - 1)\max_{m\in K}\{\hat{C}_{ijim}^s\}\Big), \ \forall i\in N, s\in S, && (14)\\
& y_i^s \geq 0,\ \forall i\in N, s\in S. && (15)
\end{aligned}
$$

Proposition 4.1. *(F_L) is a formulation for S_CSAHLP.*

Proof. For any $i \in N$, $k(\neq i) \in K$ and $s \in S$ such that $x_{ik}^s = 1$, the corresponding constraint (13) is given by $y_i^s \geq \sum_{\substack{j\in N\\ j>i}} (\sum_{\substack{m\in K\\ m\neq j}} \hat{C}_{ijkm}^s x_{jm}^s + \hat{C}_{ijkj}^s z_j^s)$. The right hand side of this inequality accounts for the sum of the transportation costs with origin/destination in site i and destination/origin in any site $j > i$, under scenario s. Since, in the objective of the problem, variable y_i^s is multiplied by a positive coefficient, it will take the value of the right hand side in the optimal solution. Otherwise, if $x_{ik}^s = 0$, then the right hand side of (13) is non-positive, since by (9) it holds $\sum_{\substack{m\in K\\ m\neq j}} \hat{C}_{ijkm} x_{jm}^s + \hat{C}_{ijkj} z_j^s \leq \max_{m\in K}\{\hat{C}_{ijkm}\}$, i.e., the constraint is idle. Constraints (14) are a particular case of (13) for $i = k$. □

Observe that this new formulation (F_L) proposed for the S_CSAHLP has a smaller number of binary variables ($(n^2 - n)n_s + n$) and constraints ($(2n^2 + n)n_s$) in comparison with the flow-based mixed integer linear reformulation proposed by Ernst and Krishnamoorthy (1996) applied to (F_{NL}) ($n^2 n_s$ binary variables and $3n^2 n_s$ constraints).

(F_L) can be straightforwardly extended to the Stochastic Single-Allocation p-Hub Median Problem (S_SApHMP) where the operational costs associated with the location of the hubs are replaced by the requirement of opening p hubs.

Since (F_L) provides weak linear relaxations, several families of valid inequalities are proposed to strengthen it.

5 Valid inequalities

The first family of valid inequalities is given in the following proposition.

Proposition 5.1. *The following families of inequalities are valid for* (F_L)

i) For any $i \in N$, $k(\neq i) \in K$, $s \in S$, $\hat{K}_j \subseteq K$, *with* $j(>i) \in N$,

$$y_i^s \geq \sum_{\substack{j \in N \\ j>i}} H_{ijk}^s x_{ik}^s + \sum_{\substack{j \in N \\ j>i}} \Bigg(\sum_{m(\neq j) \in \hat{K}_j} (\hat{C}_{ijkm}^s - H_{ijk}^s)(x_{ik}^s + x_{jm}^s - 1) + (\hat{C}_{ijkj}^s - H_{ijk}^s)(x_{ik}^s + z_j^s - 1)\mathcal{I}_{\hat{K}_j}(j) \Bigg), \tag{16}$$

ii) For any $i \in N$, $s \in S$, $\hat{K}_j \subseteq K$, *with* $j(>i) \in N$,

$$y_i^s \geq \sum_{\substack{j \in N \\ j>i}} H_{iji}^s z_i^s + \sum_{\substack{j \in N \\ j>i}} \Bigg(\sum_{m(\neq j) \in \hat{K}_j} (\hat{C}_{ijim}^s - H_{iji}^s)(z_i^s + x_{jm}^s - 1) + (\hat{C}_{ijij}^s - H_{iji}^s)(z_i^s + z_j^s - 1)\mathcal{I}_{\hat{K}_j}(j) \Bigg), \tag{17}$$

where $H_{ijk}^s := \min_{m \in K}\{\hat{C}_{ijkm}^s\}$ *and* $\mathcal{I}_{\mathcal{W}}(j)$ *is the indicator function, i.e.,* $\mathcal{I}_{\mathcal{W}}(j) = 1$ *if* $j \in \mathcal{W}$ *and* 0 *otherwise.*

Proof. Let us consider (x, z, y) a feasible solution of (F_L). For the sake of readability, we define:

$$X_{ik}^s = \begin{cases} x_{ik}^s, & \text{if } i \neq k, \\ z_i^s, & \text{if } i = k, \end{cases} \qquad \forall\, i \in N, k \in K, s \in S. \tag{18}$$

Hence (16) and (17) can be rewritten as

$$y_i^s \geq \sum_{\substack{j \in N \\ j>i}} H_{ijk}^s X_{ik}^s + \sum_{\substack{j \in N \\ j>i}} \sum_{m \in \hat{K}_j} (\hat{C}_{ijkm}^s - H_{ijk}^s)(X_{ik}^s + X_{jm}^s - 1), \ \forall i \in N, k \in K, s \in S, \hat{K}_j \subseteq K \tag{19}$$

We distinguish two cases.

- If $\sum_{m \in \hat{K}_j} X_{jm}^s = 0$, then

$$y_i^s \geq \sum_{\substack{j \in N \\ j>i}} H_{ijk}^s X_{ik}^s + \sum_{\substack{j \in N \\ j>i}} \sum_{m \in \hat{K}_j} (\hat{C}_{ijkm}^s - H_{ijk}^s)(X_{ik}^s - 1). \tag{20}$$

  Hence, if $X_{ik}^s = 0$, the right hand side of these constraints is non-positive and the corresponding constraints are idle. Otherwise, if $X_{ik}^s = 1$, (20) is rewritten as $y_i^s \geq$

$\sum_{\substack{j \in N \\ j>i}} H_{ijk}^s$, and this holds since y_i^s is at least the sum of the minimum costs of the flows from/to site i to/from another site $j > i$ under scenario s.

- If $\sum_{m \in \hat{K}_j} X_{jm}^s = 1$, and $X_{ik}^s = 1$, then $y_i^s \geq \sum_{\substack{j \in N \\ j>i}} \hat{C}_{ijkm}^s$. That is, y_i^s is at least the transportation cost from/to i of the overall flow to/from $j > i$ and since we are minimizing, y_i^s represents that transportation cost for each $i \in N$. In the case where $X_{ik}^s = 0$, again the right hand side of (19) becomes a non-positive amount and then the corresponding constraints are idle.

□

Proposition 5.2. *The valid inequalities* (16) *and* (17) *can be reinforced by the following:*

i) For any $i \in N, k(\neq i) \in K, s \in S$, $\hat{K}_j \subseteq K$, *with* $j(> i) \in N$,

$$y_i^s \geq \sum_{\substack{j \in N \\ j>i}} H_{ijk}^s + \sum_{\substack{j \in N \\ j>i}} \Big(\sum_{m(\neq j) \in \hat{K}_j} (\hat{C}_{ijkm}^s - H_{ijk}^s) x_{jm}^s + (\hat{C}_{ijkj}^s - H_{ijk}^s) z_j^s \mathcal{I}_{\hat{K}_j}(j) + \max_{m \in \hat{K}_j} \{\hat{C}_{ijkm}^s\}(x_{ik}^s - 1) \Big). \tag{21}$$

ii) For any $i \in N$, $s \in S$, $\hat{K}_j \subseteq K$, *with* $j(> i) \in N$,

$$y_i^s \geq \sum_{\substack{j \in N \\ j>i}} H_{iji}^s + \sum_{\substack{j \in N \\ j>i}} \Big(\sum_{m(\neq j) \in \hat{K}_j} (\hat{C}_{ijim}^s - H_{iji}^s) x_{jm}^s + (\hat{C}_{ijij}^s - H_{iji}^s) z_j^s \mathcal{I}_{\hat{K}_j}(j) + \max_{m \in \hat{K}_j} \{\hat{C}_{ijim}^s\}(z_i^s - 1) \Big). \tag{22}$$

Proof. The proof of the validity of (21) and (22) is very similar to the one given in Proposition 5.1. We will prove that constraints (21) reinforce (16). The right hand side of (16) can be rewritten as

$$\sum_{\substack{j\in N\\ j>i}}\Big(H^s_{ijk}x^s_{ik} + \sum_{m(\neq j)\in \hat{K}_j}(\hat{C}^s_{ijkm} - H^s_{ijk})x^s_{jm} + (\hat{C}^s_{ijkj} - H^s_{ijk})z^s_j\mathcal{I}_{\hat{K}_j}(j) +$$
$$\sum_{m\in\hat{K}_j}(\hat{C}^s_{ijkm} - H^s_{ijk})(x^s_{ik}-1)\Big) \leq$$
$$\sum_{\substack{j\in N\\ j>i}}\Big(H^s_{ijk}x^s_{ik} + \sum_{m(\neq j)\in \hat{K}_j}(\hat{C}^s_{ijkm} - H^s_{ijk})x^s_{jm} + (\hat{C}^s_{ijkj} - H^s_{ijk})z^s_j\mathcal{I}_{\hat{K}_j}(j) +$$
$$(\max_{m\in\hat{K}_j}\{\hat{C}^s_{ijkm}\} - H^s_{ijk})(x^s_{ik}-1)\Big) =$$
$$\sum_{\substack{j\in N\\ j>i}}\Big(H^s_{ijk} + \sum_{m(\neq j)\in \hat{K}_j}(\hat{C}^s_{ijkm} - H^s_{ijk})x^s_{jm} + (\hat{C}^s_{ijkj} - H^s_{ijk})z^s_j\mathcal{I}_{\hat{K}_j}(j) + \max_{m\in\hat{K}_j}\{\hat{C}^s_{ijkm}\}(x^s_{ik}-1)\Big).$$

Hence, the result follows.

The reinforcement of (22) can be analogously proved. □

Proposition 5.3. *Given $(\bar{x},\bar{z},\bar{y})$ a fractional solution of (F_L) and $i\in N$, $k\in K$ and $s\in S$, the following sets of inequalities are the most violated within the families* (16) *and* (17)*:*

a) $i\neq k$:

$$y^s_i \geq \sum_{\substack{j\in N\\ j>i}} H^s_{ijk}x^s_{ik} + \sum_{\substack{j\in N\\ j>i}}\Bigg(\sum_{\substack{m(\neq j)\in K\\ \bar{x}^s_{ik}+\bar{x}^s_{jm}>1}}(\hat{C}^s_{ijkm} - H^s_{ijk})(x^s_{ik}+x^s_{jm}-1) +$$
$$(\hat{C}^s_{ijkj} - H^s_{ijk})(x^s_{ik}+z^s_j-1)\mathcal{I}_{\{j:\bar{x}^s_{ik}+\bar{z}_j>1\}}(j)\Bigg), \tag{23}$$

b) $i=k$:

$$y^s_i \geq \sum_{\substack{j\in N\\ j>i}} H^s_{iji}z^s_i + \sum_{\substack{j\in N\\ j>i}}\Bigg(\sum_{\substack{m(\neq j)\in K\\ \bar{z}_i+\bar{x}^s_{jm}-1}}(\hat{C}^s_{ijim} - H^s_{iji})(z^s_i+x^s_{jm}-1) +$$
$$(\hat{C}^s_{ijij} - H^s_{iji})(z^s_i+z^s_j-1)\mathcal{I}_{\{j:\bar{z}_i+\bar{z}_j>1\}}(j)\Bigg), \tag{24}$$

Proof. Let us consider $(\bar{x},\bar{z},\bar{y})$ a fractional solution of (F_L) and $i\in N$, $k\in K$, $s\in S$ and $\bar{X}$ defined by (18).

We will prove that the set of inequalities (23) are the most violated within the family (16) (the case of (24) can be analogously proved). For each $j(>i)\in N$, we want to find sets $\hat{K}_j\subseteq K$

with maximal violation of (16). The right hand side of (16) is given by:

$$y_i^s \geq \sum_{\substack{j\in N\\ j>i}} H_{ijk}^s \bar{X}_{ik}^s + \sum_{\substack{j\in N\\ j>i}} \sum_{m\in\hat{K}_j} (\hat{C}_{ijkm}^s - H_{ijk}^s)(\bar{X}_{ik}^s + \bar{X}_{jm}^s - 1), \tag{25}$$

Hence, the maximum of the right hand side of (25) for this solution will be obtained by considering $\hat{K}_j$ as the set of $j \in N$ such that $\bar{X}_{ik}^s + \bar{X}_{jm}^s - 1 > 0$, since $\hat{C}_{ijkm}^s - H_{ijk}^s \geq 0$.

Therefore, we define $\hat{K}_j = \{m \in K : \bar{X}_{ik}^s + \bar{X}_{jm}^s > 1\}$. □

Proposition 5.4. *Given $(\bar{x}, \bar{z}, \bar{y})$ a fractional solution of (F_L), $\bar{X}$ defined by* (18)*, and $i \in N$, $k \in K$ and $s \in S$, the following sets of inequalities are the most violated within the families* (21) *and* (22)*:*

a) $i \neq k$,

$$y_i^s \geq \sum_{\substack{j\in N\\ j>i}} H_{ijk}^s + \sum_{\substack{j\in N\\ j>i}} \Big(\sum_{\substack{m(\neq j)\in K\\ \hat{C}_{ijkm}^s \leq \hat{C}_{ijkm_0}^s}} (\hat{C}_{ijkm}^s - H_{ijk}^s) x_{jm}^s + (\hat{C}_{ijkj}^s - H_{ijk}^s) z_j^s \mathcal{I}_{\{j:\hat{C}_{ijkm}^s \leq \hat{C}_{ijkm_0}^s\}}(j) + \hat{C}_{ijkm_0}^s (x_{ik}^s - 1) \Big). \tag{26}$$

b) $i = k$,

$$y_i^s \geq \sum_{\substack{j\in N\\ j>i}} H_{iji}^s + \sum_{\substack{j\in N\\ j>i}} \Big(\sum_{\substack{m(\neq j)\in K\\ \hat{C}_{ijim}^s \leq \hat{C}_{ijim_0}^s}} (\hat{C}_{ijim}^s - H_{iji}^s) x_{jm}^s + (\hat{C}_{ijij}^s - H_{iji}^s) z_j^s \mathcal{I}_{\{j:\hat{C}_{ijim}^s \leq \hat{C}_{ijim_0}^s\}}(j) + \hat{C}_{ijim_0}^s (z_i^s - 1) \Big), \tag{27}$$

for $m_0 \in K$ such that

$$m_0 \;=\; \arg\max_{m'\in K} \Big\{ \sum_{\substack{m\in K\\ \hat{C}_{ijkm}^s \leq \hat{C}_{ijkm'}^s}} (\hat{C}_{ijkm}^s - H_{ijk}^s) \bar{X}_{jm}^s + \hat{C}_{ijkm'}^s (\bar{X}_{ik}^s - 1) \Big\}.$$

Proof. We will prove that the set of inequalities (26) are the most violated within the family (21). The right hand side of this family is given by:

$$\sum_{\substack{j\in N\\ j>i}} \Big(H_{ijk}^s + \sum_{m\in\hat{K}_j} (\hat{C}_{ijkm}^s - H_{ijk}^s) \bar{X}_{jm}^s + \max_{m\in\hat{K}_j} \{\hat{C}_{ijkm}^s\} (\bar{X}_{ik}^s - 1) \Big). \tag{28}$$

By maximizing (28), we will obtain the most violated version of (21). First, observe that the maximum value of (28), for a set $\hat{K}_j \subseteq K$, will be obtained by computing, for any $j \in N$ with $j > i$, the expression:

$$\max_{\hat{K}_j \subseteq K} \Big\{ \sum_{m\in\hat{K}_j} (\hat{C}_{ijkm}^s - H_{ijk}^s) \bar{X}_{jm}^s + \max_{m\in\hat{K}_j} \{\hat{C}_{ijkm}^s\} (\bar{X}_{ik}^s - 1) \Big\}. \tag{29}$$

Since $\hat{C}^s_{ijkm} - H^s_{ijk} \geq 0$, for any $m \in K$, assuming that $\max_{m \in \hat{K}_j} \hat{C}^s_{ijkm} = \hat{C}^s_{ijkm_0}$, $\hat{K}_j$ should contain any $m \in K$ such that $\hat{C}^s_{ijkm} \leq \hat{C}^s_{ijkm_0}$, therefore, (29) can be reformulated as:

$$\max_{m_0 \in K} \sum_{\substack{m \in K \\ \hat{C}^s_{ijkm} \leq \hat{C}^s_{ijkm_0}}} (\hat{C}^s_{ijkm} - H^s_{ijk}) \bar{X}^s_{jm} + \hat{C}^s_{ijkm_0} (\bar{X}^s_{ik} - 1).$$

Therefore, $\hat{K}_j = \{m \in K : \hat{C}^s_{ijkm} \leq \hat{C}^s_{ijkm_0}\}$ where

$$m_0 = \arg\max_{m' \in K} \Big\{ \sum_{\substack{m \in K \\ \hat{C}^s_{ijkm} \leq \hat{C}^s_{ijkm'}}} (\hat{C}^s_{ijkm} - H^s_{ijk}) \bar{X}^s_{jm} + \hat{C}^s_{ijkm'} (\bar{X}^s_{ik} - 1) \Big\}.$$

□

Another family of valid inequalities is developed.

Proposition 5.5. *For all $i \in N$ and $s \in S$, the following inequalities are valid for (F_L):*

$$y^s_i \geq \sum_{\substack{j \in N \\ j > i}} \left(\sum_{\substack{k \in K \\ k \neq i}} \gamma^s_{ijk} x^s_{ik} + \gamma^s_{iji} z^s_i + \sum_{\substack{m \in K \\ m \neq j}} \mu^s_{ijm} x^s_{jm} + \mu^s_{ijj} z^s_j \right), \tag{30}$$

with γ^s_{ijk}, $\mu^s_{ijm} \in \mathbb{R}$, $\forall j \in N, k, m \in K$ satisfying that:

$$\gamma^s_{ijk} + \mu^s_{ijm} \leq \hat{C}^s_{ijkm}. \tag{31}$$

Proof. Let us consider (x, z, y) a feasible solution of (F_L) and X defined by (18). Then, (30) can be rewritten as

$$y^s_i \geq \sum_{\substack{j \in N \\ j > i}} \left(\sum_{k \in K} \gamma^s_{ijk} X^s_{ik} + \sum_{m \in K} \mu^s_{ijm} X^s_{jm} \right).$$

Since $\sum_{k \in K} X^s_{ik} = 1$, for any $i \in N$, let us consider $k_i \in K$ such that $X^s_{ik_i} = 1$. Analogously, for each $j (> i) \in N$, there exists only one $m_j \in K$ such that $X^s_{jm_j} = 1$. The variable y^s_i is defined as the overall transportation cost with origin/destination at site i to/from any site $j \in N, j > i$ under scenario s, and using (31), then

$$y^s_i = \sum_{\substack{j \in N \\ j > i}} \hat{C}^s_{ijk_i m_j} \geq \sum_{\substack{j \in N \\ j > i}} (\gamma^s_{ijk_i} + \mu^s_{ijm_j}) = \sum_{\substack{j \in N \\ j > i}} \left(\sum_{k \in K} \gamma^s_{ijk} X^s_{ik} + \sum_{m \in K} \mu^s_{ijm} X^s_{jm} \right),$$

and the result follows. □

Since the set of constraints (30) is uncountable, in order to incorporate them to (F_L) a separation procedure is required. We do it in an efficient way obtaining in what follows the inequality that is the most violated by a given fractional solution of (F_L).

Proposition 5.6. *Given $(\bar{x}, \bar{z}, \bar{y})$ a fractional solution of (F_L), $\bar{X}$ given by* (18)*, and fixed values of $i \in N$ and $s \in S$, the most violated inequality in family* (30) *is given by the parameters $\gamma^s_{ijk}, \mu^s_{ijm} \in \mathbb{R}$, $\forall k, m \in K$ and $j(> i) \in N$ obtained by solving the dual problem of a transportation problem with origins $k \in K$ and destinations $m \in K$, supply of origin $k \in K$ equal to $\bar{X}^s_{ik}$, demand of destination $m \in K$ equal to $\bar{X}^s_{jm}$ and transportation costs from origin $k \in K$ to destination $m \in K$ equal to $\bar{C}^s_{ijkm}$.*

Proof. Let $(\bar{x}, \bar{z}, \bar{y})$ be a fractional solution of (F_L) and $\bar{X}$ given by (18). The maximal violation of (30) is given by solving the following linear subproblem, that can be separated for any $s \in S$, $i \in N$ and $j(> i) \in N$, as follows:

$$
\begin{array}{llll}
(TP_{ijs}) & \max\limits_{\boldsymbol{\gamma}^s_{ij}, \boldsymbol{\mu}^s_{ij}} & \sum\limits_{k \in K} \gamma^s_{ijk} \bar{X}^s_{ik} + \sum\limits_{m \in K} \mu^s_{ijm} \bar{X}^s_{jm} & \\
& \text{s.t.} & \gamma^s_{ijk} + \mu^s_{ijm} \leq \hat{C}^s_{ijkm}, & \forall k, m \in K, \\
& & \gamma^s_{ijk}, \mu^s_{ijm} \text{ unrestricted}, & \forall k, m \in K.
\end{array}
$$

(TP_{ijs}) is the dual of a transportation problem with the input data described in the statement of the proposition. □

6 Subgradient cuts

Rostami et al. (2021) studied the single-allocation hub location problem where only the demands are uncertain and the transportation costs are given in a disaggregated way: $C_{ijkm} = \chi c_{ik} + \alpha c_{km} + \delta c_{mj}$, where c_{ij} denotes the unit transportation cost between i and j and χ, α and δ are collection, transfer and distribution factors, respectively. Under these conditions, the following formulation was proposed:

$$
\begin{aligned}
\min \quad & \sum_{k\in K} f_k z_k + \sum_{s\in S}\sum_{i\in N}\sum_{\substack{k\in K\\k\neq i}} p_s c_{ik}(\chi O_i^s + \delta D_i^s) x_{ik}^s + \sum_{s\in S}\sum_{i\in N} p_s \alpha y_i^s \\
\text{s.t.} \quad & (2)-(4), (15), \\
& y_i^s \geq \sum_{j\in N} \omega_{ij}^s \Big(\sum_{\substack{k\in K\\k\neq i}} u_{ik}^s x_{ik}^s + u_{ii}^s z_i + \sum_{\substack{m\in K\\m\neq j}} v_{im}^s x_{jm}^s + v_{ij}^s z_j \Big), \ \forall i\in N, \forall s\in S, && (32)\\
& u_{ik}^s + v_{im}^s \geq c_{km}, \quad \forall i\in N, k,m\in K, s\in S, && (33)\\
& u, v \text{ unrestricted.} && (34)
\end{aligned}
$$

Notice that there is a product of variables in constraints (32). Rostami et al. (2021) provided a branch-and-cut algorithm for solving this problem first removing (32)-(34) from the formulation above and then by generating the following family of cuts that they called subgradient cuts:

$$
y_i^s \geq \sum_{j\in N} \omega_{ij}^s \Bigg(\sum_{\substack{k\in K\\k\neq i}} \bar{u}_{ik}^s x_{ik}^s + \bar{u}_{ii}^s z_i + \sum_{\substack{m\in K\\m\neq j}} \bar{v}_{im}^s x_{jm}^s + \bar{v}_{ij}^s z_j \Bigg), \quad \forall i\in N, s\in S \tag{35}
$$

where $(\bar{u}_i^s, \bar{v}_i^s)$ are the optimal values of the following cut generating subproblem:

$$
\begin{aligned}
(SP_{is}) \qquad & \max_{\boldsymbol{u_i^s}, \boldsymbol{v_i^s}} \quad \sum_{j\in N} \omega_{ij}^s \left(\sum_{k\in K} u_{ik}^s \bar{X}_{ik}^s + \sum_{m\in K} v_{im}^s \bar{X}_{jm}^s \right), \\
& \text{s.t.} \quad u_{ik}^s + v_{im}^s \leq c_{km}, && \forall k,m\in K, \quad (36)\\
& \qquad\ \ u_{ik}^s, v_{im}^s \text{ unrestricted,} && \forall k,m\in K, \quad (37)
\end{aligned}
$$

where $\bar{X} \in [0,1]^{N\times N}$ defined by (18) satisfies the corresponding constraints (2) and (3).

Rostami et al. (2021) claimed that an optimal solution of (SP_{is}) is given by

$$
\bar{v}_{im}^s = \sum_{k\in K} c_{km}\bar{X}_{ik}^s, \quad \forall m\in K; \qquad \bar{u}_{ik}^s = \min_{m\in K}\{c_{km} - \bar{v}_{im}^s\}, \quad \forall k\in K. \tag{38}
$$

However, the proof of Theorem 3 in Rostami et al. (2021) fails at their equality (48) if the solution is not integer. Actually, in a subsequent article, Rostami et al. (2023) stated in Theorem 5 that the solution obtained in (38) is optimal if $\bar{X}$ is integer. For this reason, to obtain stronger cuts, we consider to solve optimally (SP_{is}) even when $\bar{X}$ is not integer. To do that, we take advantage that (SP_{is}) is the dual formulation of a transportation problem in a scenario $s \in S$ with origins $k \in K$ and destinations $m \in K$, i.e.,

$$\begin{aligned} \max_{\boldsymbol{u_i^s}, \boldsymbol{v_i^s}} \quad & \left(\sum_{k \in K} O_i^s \bar{X}_{ik}^s u_{ik}^s + \sum_{m \in K} \sum_{j \in N} \omega_{ij}^s \bar{X}_{jm}^s v_{im}^s \right) \\ \text{s.t.} \quad & (36), (37). \end{aligned}$$

The supply of origin k is $O_i^s \bar{X}_{ik}^s$, for any $k \in K$, and the demand of destination m is $\sum_{j \in N} \omega_{ij} \bar{X}_{jm}^s$, for any $m \in K$. The transportation cost from origin k to destination m is c_{km}.

Hence, for a given fractional solution $\bar{X}$, we are providing the most violated cut of the family (35) which was not guaranteed in the methodology developed by Rostami et al. (2021). However, in spite of that, the next result shows that the cuts (30) obtained in Proposition 5.6 (adapted to disaggregated costs) dominate this strengthening. For the sake of simplicity in the comparison of these two type of cuts, we consider the following version of cuts (30) for the disaggregated costs where the first sum is for any $j \in N$ and not just for $j > i$, i.e.,

$$y_i^s \geq \sum_{j \in N} \omega_{ij} \left(\sum_{\substack{k \in K \\ k \neq i}} \gamma_{ijk}^s x_{ik}^s + \gamma_{iji}^s z_i^s + \sum_{\substack{m \in K \\ m \neq j}} \mu_{ijm}^s x_{jm}^s + \mu_{ijj}^s z_j^s \right), \tag{39}$$

with γ_{ijk}^s, $\mu_{ijm}^s \in \mathbb{R}$, $\forall j \in N, k, m \in K$ satisfying

$$\gamma_{ijk}^s + \mu_{ijm}^s \leq c_{km}^s.$$

Hence, the adaptation to disaggregated cost of the linear subproblem (TP_{ijs}) for any $s \in S$, $i, j \in N$, to cuts (39) is given by

$$\begin{aligned} (TP_{ijs}^D) \qquad & \max_{\boldsymbol{\gamma_{ij}^s}, \boldsymbol{\mu_{ij}^s}} \quad \left(\sum_{k \in K} \gamma_{ijk}^s \bar{X}_{ik}^s + \sum_{m \in K} \mu_{ijm}^s \bar{X}_{jm}^s \right) \\ & s.t \quad \gamma_{ijk}^s + \mu_{ijm}^s \leq c_{km}^s, && \forall k, m \in K, \\ & \qquad \gamma_{ijk}^s, \mu_{ijm}^s \text{ unrestricted}, && \forall k, m \in K. \end{aligned}$$

Let denote our formulation (F_L) adapted to use disaggregated costs by (F_{LD}).

Proposition 6.1. *Let $(\bar{x}, \bar{z}, \bar{y})$ be a fractional solution of (F_{LD}) and $\bar{X}$ given by* (18). *For any $s \in S$ and $i \in N$, the cut of family* (39) *obtained by solving (TP_{ijs}^D) dominates the cut of family* (35) *obtained by solving (SP_{is}).*

Proof. Let $(\bar{x}, \bar{z}, \bar{y})$ be a fractional solution of (F_{LD}) and $\bar{X}$ given by (18). For a given $s \in S$ and $i \in N$, let us consider an optimal solution of (SP_{is}), $\bar{u}^s_{ik}$ and $\bar{v}^s_{im}$ for any $k, m \in K$ and define $\gamma^s_{ijk} = \bar{u}^s_{ik}$ and $\mu^s_{ijm} = \bar{v}^s_{im}$, for all $j \in N$. Then, γ^s_{ijk} and μ^s_{ijm} constitute a feasible solution of (TP^D_{ijs}). On the other hand, for each $j \in N$, we denote $\bar{\gamma}^s_{ijk}, \bar{\mu}^s_{ijm} \in \mathbb{R}$ an optimal solution of (TP^D_{ijs}) for each $i, j \in N$, $s \in S$, then

$$\begin{aligned}\max_{\boldsymbol{u^s_i}, \boldsymbol{v^s_i}} \quad \sum_{j \in N} \omega^s_{ij} \left(\sum_{k \in K} u^s_{ik} \bar{X}^s_{ik} + \sum_{m \in K} v^s_{im} \bar{X}^s_{jm} \right) \quad &\leq \sum_{j \in N} \omega^s_{ij} \max_{\boldsymbol{u^s_i}, \boldsymbol{v^s_i}} \left(\sum_{k \in K} u^s_{ik} \bar{X}^s_{ik} + \sum_{m \in K} v^s_{im} \bar{X}^s_{jm} \right) \\ &= \sum_{j \in N} \omega^s_{ij} \left(\sum_{k \in K} \bar{\gamma}^s_{ijk} \bar{X}^s_{ik} + \sum_{m \in K} \bar{\mu}^s_{ijm} \bar{X}^s_{jm} \right),\end{aligned}$$

and the result follows. □

When the S_CSAHLP is considered and capacities are taken into account, these cuts have to include the parameter I^s_k, for any $k \in K$ and $s \in S$, defined in (6).

7 A branch-and-cut method for solving S_SAHLP

We develop a solution procedure for solving S_SAHLP that considers a constraint-relaxed formulation, the inclusion of cuts through a separation method and a preprocessing phase for the capacitated case.

7.1 Feasibility cut procedure

Formulation (F_L) is relaxed and constraints (13) and (14) are initially ignored. We will add these constraints in a branch-and-cut procedure to obtain feasible solutions. This method, called Feasibility Cut Procedure (FCP), consists of solving (F_L) with only a subset of constraints (13) and (14). For each new incumbent integer solution found in the branch-and-bound tree, we add the violated feasibility cuts of this type, and continue with the solution method. Thus, we guarantee that the solutions found are feasible for (F_L).

To reinforce the procedure, we have also added the violated cuts (13) and (14) whenever a fractional optimal solution is found at any node of the branch-and-bound tree (up to 4 nodes).

7.2 Adding cuts

The families of cuts given in Section 5 are added to (F_L) in the branch-and-cut procedure to speed up the solution procedure. Whenever an optimal solution is found at any node of the branch-and-bound tree, the corresponding violated cuts are added to the formulation.

Families of cuts (26)-(27) are deeper than (23)-(24). Therefore, the computational analyses of this paper have been carried out using (26)-(27).

Regarding the family (30), for each $i, j(> i) \in N$ and $s \in S$, we have to solve subproblem (TP_{ijs}). Note that (TP_{ijs}) is the dual of a transportation problem with origins $k \in K$ and destinations $m \in K$ where the supply of origin k is $\bar{X}^s_{ik}$, for any $k \in N$, and the demand of destination m is $\bar{X}^s_{jm}$, for any $m \in N$. The transportation cost from origin k to destination m is $\hat{C}^s_{ijkm}$. Therefore, for adding (30), we first solve the dual of these transportation problems to obtain the parameters $\bar{\gamma}^s_{ijk}, \bar{\mu}^s_{ijm}$, for any $k, m \in K$, $i, j(> i) \in N$ and $s \in S$, and then we add each violated cut of type (30) to (F_L). The algorithm for solving the transportation problem is based on the primal-dual algorithm with preprocessing described in Haddadi and Slimani (2012).

To improve the efficiency of our method, we reduce the number of subproblems to be solved by restricting ourselves to the scenarios $s \in S$ and origins/destinations $i, j \in N$ where $\omega^s_{ij} > 0$.

7.3 Preprocessing using the capacities

When capacities are considered in the scenarios, there might occur that the capacity of a hub would not be enough to cover the product sent from itself in a scenario. In this case, there would not be any origin/destination assigned to this hub in that scenario. This has been checked a priori to remove some variables and constraints from (F_L).

For each scenario $s \in S$ and hub $k \in K$, if $\tau^s_k < O^s_k$ then variables x^s_{ik} (and the constraints associated to these values of s and k) have not been included in (F_L).

8 Computational results

This section presents computational experiments to evaluate the solution procedure for solving S_SAHLP, S_CSAHLP and S_SApHMP.

First, details regarding the implementation are given. Then, the computational results are presented, where our branch-and-cut algorithm is compared with the one given in Rostami et al. (2021) for the case of uncertainty only in the demand, when the costs are given in a disaggregated way. The instances used in Rostami et al. (2021) for the capacitated version do not always verify $\tau_k^s \geq O_k^s$. However, we have checked that the optimal solution with our model for these instances satisfies that the hubs are active in all the scenarios, that is, the formulation by Rostami et al. (2021) and our formulation provide the same optimal values in the capacitated version.

8.1 Implementation and data

All formulations and solution methods have been implemented in C++ using Gurobi 10.0 (see Gurobi Optimization 2022) in a Windows 10 with an Intel Core i9-12900K processor at 3.2 GHz and 32 GB of RAM and setting by default all cuts and presolve strategies of the solver. A limit of two hours of CPU time was set in all the experiments.

The test instances used are the well-known AP instances introduced in Ernst and Krishnamoorthy (1996), based on a postal delivery in Sydney with 200 postal districts. The transportation costs are given in a disaggregated way where the transfer, collection and distribution factors are set to $\alpha = 0.75$, $\chi = 3.0$ and $\delta = 2.0$, respectively, as specified in the AP dataset. The sizes of the instances (n) considered in this computational study are $n \in \{25, 40, 50, 60, 75, 90, 100, 125, 150, 175, 200\}$.

There are two types of fixed hub establishment cost values and two types of hub capacity values in the AP data set, tight (T) and loose (L). These combinations of types are denoted as LL, LT, TL and TT. The first letter indicates whether loose or tight fixed costs apply (loose fixed costs are low, being dominated by transportation costs, while it is the other way around for tight fixed costs). The second letter indicates if the capacities are loose or tight.

Different scenarios were considered by generating demand uncertainty as described in Rostami et al. (2021). They used Poisson distribution with a multiplicative factor to avoid high or low demands.

8.2 Computational results

Tables 1, 2 and 3 show the computational results obtained by solving S_SAHLP, S_SApHMP and S_CSAHLP, respectively, when five scenarios are considered. The performance of the different formulation for more than five scenarios is analyzed in Tables 4, 5 and 6.

In Tables 1-3, our procedure is compared with the branch-and-cut framework implemented by Rostami et al. (2021), in the following denoted by (Ros_et_al_21). They add the violated cuts of type (35) with $\bar{u}_{ik}^s$ and $\bar{v}_{im}^s$, $\forall i \in N$, $k, m \in K$ and $s \in S$, given in (38) in two ways:

(i) in each new incumbent integer solution found in the branch-and-bound tree, and (ii) in the fractional solutions of the root node. Since the values of $\bar{u}_i^s$ and $\bar{v}_i^s$ given in (38) are not guaranteed to be optimal for fractional solutions of (SP_{is}), (see Section 6), their procedure can be improved in the root node by searching the optimal solution of the subproblem (SP_{is}). This solution can be obtained by using an adaptation of the algorithm for solving the transportation problem based on the primal-dual algorithm with preprocessing described in Haddadi and Slimani (2012). Moreover, taking into account that we are dealing with single allocation, the number of problems to be solved is reduced by considering only the values of j that are greater than i, as we do in our solution procedure described in Section 7. The results when all these improvements are carried out over the branch-and-cut framework implemented by Rostami et al. (2021) are shown in column "Ros_imp".

The structure of the first three tables is the following. The first column gives the value of n. The second column provides one of these three options: the type of fixed cost (for S_SAHLP, with values L or T), the type of fixed cost and capacity (for S_CSAHLP, with values LL, LT, TL or TT) or the value of p for S_SApHMP. The third column reports the optimal value when possible; otherwise, it shows the best value among those obtained with the solution procedures compared in the tables and its is labeled with '*'. The next five blocks of two columns give the relative gap (between the best lower bound and the best solution found) and the CPU time required to obtain the optimal solution (or the time limit if it is not found) for the following five solution procedures: (i) the one given by Rostami et al. (2021) (column (Ros_et_al_21)), (ii) our improvements described in Section 6 on the procedure by Rostami et al. (2021) (column Ros_imp), (iii) the feasibility cut procedure described in Section 7 (column *FCP*), (iv) the *FCP* and the valid inequalities (26) and (27) (column *FCP*+(26)+(27)), and (v) the *FCP* and the valid inequalities (26), (27) and (30) (column *FCP*+(26)+(27)+(30)) Observe that in (iii), (iv) and (v) we have adapted the corresponding cuts to disaggregated costs.

The last row of every table provides the overall computing time needed to solve all instances.

In general, the improvements on the method given in Rostami et al. (2021) considered in Section 6 (Ros_imp) show better performance in terms of computing times (23308/15662 seconds for S_SAHLP, 135380/120027 seconds for S_SApHMP and 185364/180225 seconds for S_CAHLP). With respect to *FCP*, high computing times are needed to obtain the solutions of the instances. Moreover, the optimal solutions are not obtained in 7200 seconds in most of the cases. Aiming at improving the performance of *FCP*, cuts (26) and (27) are included in the branch-and-cut procedure. These cuts provide a significant improvement, since the overall computing times are largely reduced (from 94914 to 59009 seconds for S_SAHLP, from

267530 to 238769 seconds for S_SApHMP and from 264680 to 219820 seconds for S_CAHLP). Finally, the last method consists in separating all cuts ((26)+(27)+(30)) when solving *FCP*. This formulation has reported the best performance, solving all the instances in 2175 seconds for S_SAHLP, 26739 seconds for S_SApHMP and 112263 seconds for S_CAHLP. In conclusion, our approach has improved Ros_et_al_21 for S_SAHLP and S_SApHMP in more than one order of magnitude and more than 40% for S_CSAHLP. A detailed description of the results obtained using *FCP*+(26)+(27)+(30) for S_SAHLP, S_SApHMP and S_CAHLP can be seen in Tables 7, 8 and 9 in the Appendix, where running times at root node (CPU0), number of nodes and number of cuts of each type are reported, as well as the upper bound and the lower bound when the optimal solution is not reached within the time limit.

This analysis has been carried out for more than five scenarios in Tables 4, 5 and 6. These tables report the gap and computing time of solving S_SAHLP, S_SApHMP and S_CSAHLP, respectively, using the solution method introduced in Rostami et al. (2021) and *FCP*+(26)+(27)+(30). The third column of these tables provides the number of scenarios (n_s) considered for each instance. Following the structure of the computational study for more than five scenarios considered in Rostami et al. (2021), we have solved the instances with 25 and 50 nodes. Our approach entails an improvement of two orders of magnitude for S_SAHLP, more than one order of magnitude for S_SAPHMP and around 32% for C_SAHLP. A detailed description of these results can be seen in Tables 10, 11 and 12 in the Appendix.

9 Conclusions

This paper presents a single-allocation hub location problem with uncertainty that affect hubs, origins/destinations and links. We assume that the amount of product sent from origins to destinations, the transportation costs and the capacities of the hubs are uncertain and modelled by the use of a set of different scenarios occurring with a given probability with realization only after the hubs are selected. The models proposed here also incorporate uncertainty situations where hubs and established connections are not always operational as planned (accidental disruptions at hubs and broken connections).

We prove that some of the already studied stochastic versions of the HLPs are reduced to a particular version of a classical deterministic SAHLP.

A compact integer programming formulation is proposed for the stochastic SAHLP with three variants: uncapacitated, capacitated and p-hub problem. Valid inequalities are developed to reinforce the formulation. Different separation procedures are proposed for these valid

n	Type	Opt	Ros_et_al_21 GAP	Ros_et_al_21 CPU	Ros_imp GAP	Ros_imp CPU	*FCP* GAP	*FCP* CPU	*FCP*+(26)+(27) GAP	*FCP*+(26)+(27) CPU	*FCP*+(26)+(27)+(30) GAP	*FCP*+(26)+(27)+(30) CPU
25	L	203271	0.00	1	0.00	1	0.00	5	0.00	13	0.00	2
25	T	240689	0.00	0	0.00	0	0.00	0	0.00	1	0.00	1
40	L	257241	0.00	11	0.00	20	0.00	6111	0.00	133	0.00	10
40	T	307257	0.00	6	0.00	10	0.00	3	0.00	2	0.00	7
50	L	219345	0.00	14	0.00	24	0.00	2261	0.00	61	0.00	15
50	T	277276	0.00	2	0.00	2	0.00	4	0.00	4	0.00	3
60	L	230305	0.00	14	0.00	22	0.18	7200	0.00	347	0.00	21
60	T	275062	0.00	20	0.00	24	0.18	7200	0.00	51	0.00	23
75	L	271648	0.00	418	0.00	449	1.94	7200	1.63	7200	0.00	68
75	T	325953	0.00	26	0.00	29	0.00	7200	0.00	62	0.00	34
90	L	244711	0.00	108	0.00	113	4.74	7200	8.67	7200	0.00	69
90	T	278762	0.00	29	0.00	55	0.00	56	0.00	32	0.00	37
100	L	251871	0.00	134	0.00	191	1.05	7200	0.31	7200	0.00	85
100	T	331972	0.00	22	0.00	26	0.00	4	0.00	5	0.00	10
125	L	233366	0.03	7200	0.02	7200	6.36	7200	5.59	7200	0.00	139
125	T	262691	0.00	52	0.00	52	0.00	32	0.00	99	0.00	87
150	L	227546	0.00	6268	0.00	1957	4.03	7200	4.01	7200	0.00	193
150	T	234622	0.00	94	0.00	128	0.00	38	0.00	24	0.00	125
175	L	232238	0.00	508	0.00	820	7.47	7200	9.11	7200	0.00	239
175	T	251119	0.00	205	0.00	300	0.08	7200	0.00	575	0.00	171
200	L	251050	0.00	4580	0.00	1824	8.14	7200	7.82	7200	0.00	478
200	T	260868	0.00	3597	0.00	2405	1.56	7200	3.18	7200	0.00	358
		Sum		23308		15652		94914		59009		2175

Table 1: Computational results for S_SAHLP with 5 scenarios

			Ros_et_al_21		Ros_imp		*FCP*		*FCP*+(26)+(27)		*FCP*+(26)+(27)+(30)	
n	p	Opt	GAP	CPU	GAP	CPU	GAP	CPU	GAP	CPU	GAP	CPU
25	2	146259	0.00	1	0.00	2	0.00	2	0.00	2	0.00	2
25	3	125011	0.00	3	0.00	3	0.00	15	0.00	15	0.00	2
25	4	110716	0.00	4	0.00	4	0.00	19	0.00	21	0.00	3
25	5	977340	0.00	2	0.00	3	0.00	96	0.00	22	0.00	3
40	2	197565	0.00	18	0.00	18	0.00	17	0.00	16	0.00	9
40	3	175623	0.00	163	0.00	221	1.04	7200	0.00	1358	0.00	15
40	4	159074	0.00	39	0.00	49	2.37	7200	0.49	7200	0.00	18
40	5	144945	0.06	7200	0.00	4266	3.65	7200	1.29	7200	0.00	17
50	2	157233	0.00	12	0.00	13	0.00	267	0.00	17	0.00	11
50	3	139851	0.00	13	0.00	21	1.08	7200	0.00	1794	0.00	19
50	4	124972	0.00	9	0.00	16	1.63	7200	0.00	1974	0.00	19
50	5	113818	0.00	10	0.00	16	3.67	7200	1.09	7200	0.00	15
60	2	181998	0.00	15	0.00	19	0.16	7200	0.00	282	0.00	22
60	3	162309	0.00	486	0.00	690	1.74	7200	0.61	7200	0.00	29
60	4	148832	0.00	164	0.00	198	9.20	7200	7.26	7200	0.00	31
60	5	137135	0.00	191	0.00	265	5.72	7200	7.13	7200	0.00	35
75	2	217498	0.00	43	0.00	59	0.00	712	0.00	297	0.00	35
75	3	192734	0.00	537	0.00	957	14.94	7200	4.19	7200	0.00	51
75	4	175058	0.00	80	0.00	114	11.98	7200	13.72	7200	0.00	46
75	5	159837	0.00	171	0.00	162	11.16	7200	12.78	7200	0.00	52
90	2	200327	0.00	97	0.00	132	1.43	7200	6.99	7200	0.00	62
90	3	176290	0.00	387	0.00	346	11.15	7200	10.73	7200	0.00	72
90	4	158724	0.00	3315	0.00	2811	14.12	7200	12.58	7200	0.00	72
90	5	149298	0.00	1483	0.00	1208	19.15	7200	19.89	7200	0.00	159
100	2	191461	0.00	92	0.00	108	0.62	7200	0.00	2571	0.00	71
100	3	173051	0.05	7200	0.00	2433	8.97	7200	13.41	7200	0.00	127
100	4	154798	0.00	4103	0.00	2102	16.91	7200	14.47	7200	0.00	128
100	5	144208	0.04	7200	0.00	7076	20.28	7200	21.86	7200	0.00	198
125	2	186248	0.05	7200	0.01	7200	5.48	7200	4.15	7200	0.00	99
125	3	166765	0.04	7200	0.04	7200	12.31	7200	13.82	7200	0.00	239
125	4	153478	0.14	7200	0.10	7200	16.75	7200	16.27	7200	0.00	272
125	5	143680	0.29	7200	0.26	7200	22.66	7200	22.35	7200	0.00	259
150	2	182843	0.00	676	0.00	663	1.53	7200	6.86	7200	0.00	177
150	3	161355	0.11	7200	0.11	7200	10.52	7200	13.07	7200	0.00	418
150	4	144348	0.15	7200	0.11	7200	14.78	7200	18.55	7200	0.00	188
150	5	135773	0.15	7200	0.39	7200	23.06	7200	21.55	7200	0.00	4689
175	2	186457	0.06	7200	0.00	3422	7.44	7200	9.01	7200	0.00	225
175	3	168659	0.78	7200	0.11	7200	15.24	7200	16.05	7200	0.00	1781
175	4	152744	0.10	7200	0.08	7200	18.19	7200	20.50	7200	0.00	864
175	5	143888	2.00	7200	2.17	7200	23.42	7200	22.08	7200	0.00	7200
200	2	203691	0.00	864	0.00	1264	8.64	7200	7.27	7200	0.00	300
200	3	178795	0.02	7200	0.00	4966	11.63	7200	11.12	7200	0.00	563
200	4	164928	1.03	7200	1.09	7200	21.88	7200	19.74	7200	0.00	1522
200	5	155072	2.53	7200	3.50	7200	22.54	7200	23.62	7200	0.00	6618
		Sum		135380		120027		267530		238769		26739

Table 2: Computational results for S_SApHMP with 5 scenarios

			Ros_et_al_21		Ros_imp		*FCP*		*FCP*+(26)+(27)		*FCP*+(26)+(27)+(30)	
n	Type	Opt	GAP	CPU	GAP	CPU	GAP	CPU	GAP	CPU	GAP	CPU
25	LL	203271	0.00	1	0.00	1	0.00	7	0.00	10	0.00	2
25	LT	229861	0.00	20	0.00	18	0.00	53	0.00	51	0.00	27
25	TL	262630	0.00	4	0.00	10	0.00	4	0.00	5	0.00	7
25	TT	289729	0.00	35	0.00	33	0.00	1941	0.00	231	0.00	35
40	LL	258881	0.00	13	0.00	21	0.38	7200	0.00	186	0.00	34
40	LT	288546	0.00	2635	0.00	3203	0.54	7200	0.43	7200	0.00	113
40	TL	309217	0.00	19	0.00	26	0.00	47	0.00	20	0.00	29
40	TT	379782	0.24	7200	0.20	7200	1.86	7200	0.79	7200	0.00	2807
50	LL	219852	0.00	11	0.00	18	0.17	7200	0.00	54	0.00	12
50	LT	251264	0.00	368	0.00	341	0.89	7200	0.07	7200	0.00	192
50	TL	296194	0.00	37	0.00	96	0.00	164	0.00	58	0.00	92
50	TT	374532	0.00	3748	0.00	5922	1.04	7200	0.41	7200	0.00	384
60	LL	229586	0.00	118	0.00	40	0.43	7200	0.00	341	0.00	99
60	LT	*274198	0.26	7200	0.00	7200	2.65	7200	1.42	7200	0.16	7200
60	TL	270405	0.00	128	0.00	95	0.00	3186	0.00	57	0.00	109
60	TT	415946	0.02	7200	0.00	7200	0.41	7200	0.25	7200	0.00	288
75	LL	284389	0.00	4543	0.00	2717	2.15	7200	1.65	7200	0.00	537
75	LT	303407	0.17	7200	0.18	7200	5.32	7200	3.34	7200	0.00	900
75	TL	339693	0.00	92	0.00	32	0.00	7200	0.00	28	0.00	47
75	TT	*429201	0.22	7200	0.20	7200	1.29	7200	0.84	7200	0.05	7200
90	LL	245927	0.00	291	0.00	379	6.11	7200	7.94	7200	0.00	349
90	LT	289033	0.24	7200	0.17	7200	7.60	7200	8.68	7200	0.00	1302
90	TL	*326213	0.03	7200	0.03	7200	0.81	7200	0.35	7200	0.01	7200
90	TT	*428397	0.08	7200	0.09	7200	1.02	7200	0.44	7200	0.04	7200
100	LL	258412	0.02	7200	0.00	1419	3.31	7200	6.80	7200	0.00	849
100	LT	279161	0.03	7200	0.03	7200	5.92	7200	8.94	7200	0.00	1544
100	TL	382020	0.00	5223	0.00	4971	0.18	7200	0.08	7200	0.00	236
100	TT	*517013	0.85	7200	0.84	7200	2.14	7200	1.37	7200	0.20	7200
125	LL	242470	0.03	7200	0.01	7200	7.52	7200	5.67	7200	0.00	141
125	LT	*268464	0.10	7200	0.10	7200	11.26	7200	10.48	7200	0.00	7200
125	TL	250400	0.00	63	0.00	63	0.00	78	0.00	30	0.00	101
125	TT	304256	0.00	359	0.00	360	0.04	7200	0.09	7200	0.00	649
150	LL	231722	0.07	7200	0.07	7200	7.79	7200	8.07	7200	0.00	2729
150	LT	241948	0.00	6166	0.00	6169	6.79	7200	9.85	7200	0.00	2518
150	TL	261916	0.02	7200	0.03	7200	0.41	7200	0.00	923	0.00	216
150	TT	*302821	0.21	7200	0.21	7200	1.02	7200	0.23	7200	0.02	7200
175	LL	238299	0.00	2753	0.00	2755	15.14	7200	21.30	7200	0.00	2137
175	LT	*254790	0.19	7200	0.19	7200	10.46	7200	-	7200	18.97	7200
175	TL	251065	0.00	339	0.00	339	0.00	7200	0.00	1822	0.00	179
175	TT	*294528	0.11	7200	0.11	7200	0.35	7200	0.32	7200	0.01	7200
200	LL	*279885	8.62	7200	8.62	7200	12.19	7200	12.64	7200	5.01	7200
200	LT	*313672	12.88	7200	12.88	7200	14.34	7200	18.00	7200	8.33	7200
200	TL	*270305	0.12	7200	0.12	7200	6.11	7200	9.87	7200	-	7200
200	TT	*316641	0.43	7200	0.43	7200	9.01	7200	13.53	7200	-	7200
		Sum		185364		180225		264680		219820		112263

Table 3: Computational results for S_CSAHLP with 5 scenarios

n	Type	n_s	Opt	Ros_et_al_21		FCP+(26)+(27)+(30)	
				GAP	CPU	GAP	CPU
25	L	10	182976	0.00	15	0.00	4
25	L	15	161730	0.00	0	0.00	2
25	L	20	139872	0.00	3	0.00	1
25	L	25	124325	0.00	0	0.00	0
25	T	10	207890	0.00	1	0.00	0
25	T	15	172080	0.00	0	0.00	0
25	T	20	149946	0.00	0	0.00	0
25	T	25	134399	0.00	0	0.00	0
50	L	10	249767	0.00	75	0.00	29
50	L	15	246810	0.02	7200	0.00	71
50	L	20	244274	0.00	3467	0.00	89
50	L	25	243173	0.15	7200	0.00	130
50	T	10	316072	0.01	7200	0.00	24
50	T	15	313654	0.20	7200	0.00	47
50	T	20	310341	0.14	7200	0.00	67
50	T	25	308267	0.01	7200	0.00	85
			Sum		46762		548

Table 4: Computational results for S_SAHLP with 10 to 25 scenarios

inequalities (some of them based on the resolution of transportation problems) and added in a branch-and-cut procedure.

The SAHLP with uncertain demand is analyzed and our branch-and-cut procedure is compared with the most recent solution approach for solving this problem, see Rostami et al. (2021). We also prove that our valid inequalities dominate their subgradient cuts.

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				Ros_et_al_21		FCP+(26)+(27)+(30)	
n	p	n_s	Opt	GAP	CPU	GAP	CPU
25	2	10	125965	0.00	7	0.00	3
25	2	15	107053	0.00	5	0.00	4
25	2	20	93963	0.00	6	0.00	6
25	2	25	82296	0.00	17	0.00	7
25	3	10	106689	0.00	27	0.00	5
25	3	15	89636	0.00	21	0.00	5
25	3	20	76346	0.00	42	0.00	6
25	3	25	66660	0.00	3308	0.00	8
25	4	10	86518	0.00	3	0.00	3
25	4	15	68729	0.00	5	0.00	4
25	4	20	58285	0.00	4	0.00	4
25	4	25	50715	0.00	4	0.00	5
25	5	10	76752	0.00	9	0.00	3
25	5	15	61494	0.00	36	0.00	5
25	5	20	52427	0.00	22	0.00	5
25	5	25	45648	0.00	16	0.00	7
50	2	10	186228	0.00	14	0.00	19
50	2	15	184856	0.05	7200	0.00	45
50	2	20	183156	0.18	7200	0.00	71
50	2	25	182568	0.26	7200	0.00	115
50	3	10	165238	0.00	312	0.00	28
50	3	15	165248	0.00	2559	0.00	68
50	3	20	163340	0.07	7200	0.00	114
50	3	25	162737	0.13	7200	0.00	146
50	4	10	145908	0.00	37	0.00	34
50	4	15	148475	0.00	3836	0.00	66
50	4	20	148675	0.09	7200	0.00	144
50	4	25	148328	0.07	7200	0.00	275
50	5	10	132449	0.00	42	0.00	30
50	5	15	134489	0.00	466	0.00	64
50	5	20	134437	0.04	7200	0.00	124
50	5	25	134321	0.10	7200	0.00	180
			Sum		75597		1604

Table 5: Computational results for S_SApHMP with 10 to 25 scenarios

				Ros_et_al_21		FCP+(26)+(27)+(30)	
n	Type	n_s	Opt	GAP	CPU	GAP	CPU
25	LL	10	182976	0.00	8	0.00	4
25	LL	15	164064	0.00	22	0.00	20
25	LL	20	148365	0.00	44	0.00	29
25	LL	25	134324	0.00	89	0.00	188
25	LT	10	203143	0.00	73	0.00	48
25	LT	15	175286	0.00	151	0.00	84
25	LT	20	157802	0.00	315	0.00	92
25	LT	25	145651	0.00	709	0.00	120
25	TL	10	226425	0.00	13	0.00	34
25	TL	15	189990	0.00	21	0.00	87
25	TL	20	169841	0.00	25	0.00	91
25	TL	25	155800	0.00	19	0.00	36
25	TT	10	250181	0.00	222	0.00	69
25	TT	15	216614	0.00	299	0.00	151
25	TT	20	198276	0.00	556	0.00	508
25	TT	25	185212	0.00	262	0.28	7200
50	LL	10	257444	0.00	151	0.00	288
50	LL	15	252576	0.00	4664	0.00	529
50	LL	20	248968	0.03	7200	0.00	185
50	LL	25	247226	0.08	7200	0.00	214
50	LT	10	293984	0.26	7200	0.03	7200
50	LT	15	289836	0.38	7200	0.08	7200
50	LT	20	286540	0.74	7200	0.11	7200
50	LT	25	285326	1.05	7200	0.47	7200
50	TL	10	329786	0.00	60	0.00	128
50	TL	15	325201	0.05	7200	0.00	81
50	TL	20	321351	0.10	7200	0.00	139
50	TL	25	319598	0.12	7200	0.00	814
50	TT	10	464981	0.44	7200	0.12	7200
50	TT	15	468986	5.57	7200	0.20	7200
50	TT	20	451449	3.61	7200	4.15	7200
50	TT	25	450379	4.38	7200	7.97	7200
			Sum		101303		68740

Table 6: Computational results for S_CSAHLP with 10 to 25 scenarios

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A Appendix

							Root node cuts			Total Cuts		
n	Type	Opt	GAP	CPU0	CPU	Nodes	*FCP*	(26)+(27)	(30)	*FCP*	(26)+(27)	(30)
25	L	203271	0.00	1.84	2.1	3	177	282	257	1059	301	275
25	T	240689	0.00	1	1	1	99	198	198	99	198	198
40	L	257241	0.00	4.73	9.9	7	176	495	556	215	791	699
40	T	307257	0.00	4.95	7.3	3	186	564	721	489	690	881
50	L	219345	0.00	10.2	15	4	263	700	537	3914	856	607
50	T	277276	0.00	3.02	3	1	216	149	147	216	149	147
60	L	230305	0.00	11	21	4	382	720	678	4226	1026	929
60	T	275062	0.00	15	23	7	594	775	729	2043	925	884
75	L	271648	0.00	18	68	163	297	954	1059	43770	1808	1587
75	T	325953	0.00	24.7	34	4	744	1112	794	812	1335	994
90	L	244711	0.00	26.4	69	9	664	1140	1212	9925	2048	1839
90	T	278762	0.00	24.7	37	3	393	996	1298	26191	1272	1574
100	L	251871	0.00	32.2	85	6	954	1224	1002	16604	2493	1590
100	T	331972	0.00	9.63	9.7	1	233	150	138	233	150	138
125	L	233366	0.00	57.6	139	13	781	1872	1667	19228	3066	2309
125	T	262691	0.00	56	87	7	828	1290	1237	8957	1474	1647
150	L	227546	0.00	81.6	193	11	1656	1686	1682	52039	2891	2316
150	T	234622	0.00	70.3	125	5	558	888	786	11285	1186	1374
175	L	232238	0.00	110	239	7	1330	2410	2190	4141	3157	2798
175	T	251119	0.00	97.5	171	6	1306	1255	1309	2576	1295	1346
200	L	251050	0.00	152	478	15	1891	2783	2729	1992	4527	2879
200	T	260868	0.00	150	358	7	1783	2638	1987	2074	4069	2219

Table 7: Computational results for S_SAHLP with 5 scenarios

							Root node cuts			Total Cuts		
n	p	Opt	GAP	CPU0	CPU	Nodes	*FCP*	(26)+(27)	(30)	*FCP*	(26)+(27)	(30)
25	2	146259	0.00	2	2	2	166	185	185	464	191	191
25	3	125011	0.00	2	2	2	207	358	272	1152	377	313
25	4	110716	0.00	2	3	5	163	418	233	596	567	299
25	5	97739.6	0.00	2	3	5	154	440	321	257	503	356
40	2	197565	0.00	6	9	5	103	440	292	1911	609	433
40	3	175623	0.00	5	15	47	341	477	504	1165	1146	906
40	4	159074	0.00	5	18	143	239	525	567	515	1210	921
40	5	144945	0.00	5	17	312	206	540	541	346	1229	712
50	2	157233	0.00	9	11	3	171	494	501	879	527	554
50	3	139851	0.00	8	19	5	261	648	768	777	1134	1039
50	4	124972	0.00	8	19	7	223	648	792	547	1185	1111
50	5	113818	0.00	8	15	6	324	648	723	373	910	824
60	2	181998	0.00	14	22	4	220	564	320	3705	874	613
60	3	162309	0.00	12	29	11	356	780	567	1453	1500	1055
60	4	148832	0.00	13	31	11	480	780	852	1088	1635	1193
60	5	137135	0.00	16	35	9	605	1090	866	971	2087	1131
75	2	217498	0.00	22	35	4	177	810	641	6335	1030	916
75	3	192734	0.00	18	51	20	305	867	714	723	1863	1072
75	4	175058	0.00	18	46	13	309	951	674	475	1895	1166
75	5	159837	0.00	19	52	9	465	1011	1059	1402	2142	1546
90	2	200327	0.00	33	62	7	610	1130	762	12514	1843	1687
0	3	176290	0.00	26	72	13	623	996	1044	1476	2195	1469
90	4	158724	0.00	26	72	21	677	1152	771	2591	2504	1224
90	5	149298	0.00	26	159	288	606	1146	1263	1100	2687	1830
100	2	191461	0.00	37	71	5	466	965	776	4242	1525	1394
100	3	173051	0.00	44	127	449	915	1716	1425	2100	3297	1560
100	4	154798	0.00	32	128	324	702	1260	1379	1412	2681	2097
100	5	144208	0.00	31	198	1330	665	1260	1341	1346	2710	1790
125	2	186248	0.00	53	99	11	371	1136	813	6617	1537	1635
125	3	166765	0.00	63	239	33	1605	1877	1830	6175	3572	2190
125	4	153478	0.00	62	272	218	1536	1926	1350	2261	3890	1566
125	5	143680	0.00	60	259	638	580	2029	604	1693	4050	777
150	2	182843	0.00	76	177	14	1309	1366	1415	16154	2320	1993
150	3	161355	0.00	88	418	115	1951	2084	2106	15977	4109	2392
150	4	144348	0.00	86	188	14	1026	2403	1300	5941	4152	1604
150	5	135773	0.00	87	4689	3206	806	2400	819	6887	4729	1126
175	2	186457	0.00	99	225	13	617	1054	671	9389	1318	1090
175	3	168659	0.00	118	1781	107	2076	2452	2711	5105	4918	3136
175	4	152744	0.00	121	864	75	1970	2605	2178	8304	4739	2746
175	5	143888	0.00	120	7200	14977	1446	2711	2014	12784	5178	2105
200	2	203691	0.00	134	300	11	874	1751	838	2762	2695	1696
200	3	178795	0.00	150	563	9	1244	3144	2334	12702	5997	3485
200	4	164928	0.00	164	1522	43	1560	3115	2087	2683	6048	2507
200	5	155072	0.00	156	6618	1427	1060	3204	1360	7117	6338	1671

Table 8: Computational results for S_SApHMP with 5 scenarios

							Root node cuts			Total Cuts				
n	Type	Opt	GAP	CPU0	CPU	Nodes	*FCP*	(26)+(27)	(30)	*FCP*	(26)+(27)	(30)	LB	UB
25	LL	203271	0.00	2	2	3	175	282	258	175	303	283		
25	LT	229861	0.00	2	27	962	170	466	409	426	847	576		
25	TL	262630	0.00	2	7	172	160	428	341	1366	743	469		
25	TT	289729	0.00	2	35	20596	194	427	280	329	647	341		
40	LL	258881	0.00	7	34	322	328	648	501	450	1132	779		
40	LT	288546	0.00	7	113	1437	373	736	530	1015	1383	756		
40	TL	309217	0.00	5	29	547	443	564	678	1706	951	967		
40	TT	379782	0.00	7	2807	792273	290	751	690	1336	1395	865		
50	LL	219852	0.00	10	12	3	257	754	444	3811	807	535		
50	LT	251264	0.00	11	192	1573	538	919	948	1046	1810	1228		
50	TL	296194	0.00	10	92	625	226	648	336	5349	1231	619		
50	TT	374532	0.00	12	384	29970	467	855	813	1906	1480	1247		
60	LL	229586	0.00	14	99	529	311	1022	432	6596	1502	655		
60	LT	274099	0.16	16	7200	581719	315	1119	709	1796	2044	904	[273667,	274099]
60	TL	270405	0.00	14	109	1172	161	964	375	1072	1539	653		
60	TT	415946	0.00	17	288	7494	680	1109	960	1209	2009	1331		
75	LL	284389	0.00	26	537	6957	514	1333	1095	4876	2509	1311		
75	LT	303407	0.00	27	900	27871	778	1379	1175	2843	2545	1374		
75	TL	339693	0.00	26	47	4	310	1306	1001	11515	1877	1364		
75	TT	429207	0.05	29	7200	558352	745	1145	1210	1055	1803	1573	[429009,	429207]
90	LL	245927	0.00	35	349	739	724	1495	1119	1056	2331	1466		
90	LT	289033	0.00	38	1302	26890	625	1701	1087	2143	3144	1218		
90	TL	326218	0.01	39	7200	1026819	865	1641	1478	3529	3011	1935	[326185,	326218]
90	TT	428403	0.04	41	7200	668295	797	1618	1580	1107	2837	2093	[428252,	428403]
100	LL	258412	0.00	44	849	2215	1469	1560	1567	14377	2458	2093		
100	LT	279161	0.00	47	1544	5203	443	1785	1369	21445	3361	1901		
100	TL	382020	0.00	48	236	7337	610	1734	1792	16867	2433	2241		
100	TT	517019	0.20	55	7200	391812	930	1774	1486	1429	3290	2085	[515988,	517019]
125	LL	242470	0.00	62	141	7	1292	1942	1401	8654	3117	1908		
125	LT	268464	0.00	70	7200	26418	1280	2053	1931	12475	3869	2535	[268455,	268464]
125	TL	250400	0.00	59	101	8	867	1476	1172	9107	1744	1509		
125	TT	304256	0.00	72	649	636	1045	1987	1623	5642	2703	2102		
150	LL	231722	0.00	86	2729	2856	1844	2088	2392	32350	4402	3036		
150	LT	241948	0.00	101	2518	7017	1209	2482	1716	34997	4506	1967		
150	TL	261916	0.00	124	216	12	680	2655	1128	15153	3204	1378		
150	TT	302812	0.02	108	7200	532248	1074	2414	2058	11328	4178	2644	[302741,	302812]
175	LL	238299	0.00	128	2137	261	995	2755	2253	128115	4783	2552		
175	LT	313015	18.97	138	7200	2020	1043	2784	2145	1533	4968	2293	[253631,	313015]
175	TL	251065	0.00	99	179	11	1305	1573	1379	1881	1749	1428		
175	TT	294485	0.01	167	7200	136863	1378	2721	2037	3125	4424	2691	[294461,	294485]
200	LL	272674	5.01	198	7200	1201	1509	3161	1816	2402	6057	1932	[259016,	272674]
200	LT	303805	8.32	236	7200	1360	698	3302	1609	2760	6233	2228	[278514,	303805]
200	TL	-	-	-	7200	-	-	-	-	-	-	-	-	-
200	TT	-	-	-	7200	-	-	-	-	-	-	-	-	-

Table 9: Computational results for S_CSAHLP with 5 scenarios

n	p	n_s	Opt	GAP	CPU0	CPU	Nodes	Root node cuts *FCP*	Root node cuts (26)+(27)	Root node cuts (30)	Total Cuts *FCP*	Total Cuts (26)+(27)	Total Cuts (30)
25	L	10	182976	0.00	3	4	3	357	766	705	505	786	756
25	L	15	161730	0.00	2	2	1	343	601	510	343	601	510
25	L	20	139872	0.00	1	1	1	78	78	0	78	78	0
25	L	25	124325	0.00	0	0	1	0	0	0	0	0	0
25	T	10	207890	0.00	0	0	1	0	0	0	0	0	0
25	T	15	172080	0.00	0	0	1	0	0	0	0	0	0
25	T	20	149946	0.00	0	0	1	0	0	0	0	0	0
25	T	25	134399	0.00	0	0	1	0	0	0	0	0	0
50	L	10	249767	0.00	15	29	6	529	1365	404	3855	2058	1051
50	L	15	246810	0.00	23	71	10	727	1728	1794	11377	3512	2843
50	L	20	244274	0.00	42	89	5	1270	2384	876	49121	4270	2283
50	L	25	243173	0.00	44	130	9	1165	2925	2172	2649	4890	3777
50	T	10	316072	0.00	13	24	6	547	737	565	1482	1306	1251
50	T	15	313654	0.00	27	47	6	679	1232	789	3913	1529	1580
50	T	20	310341	0.00	42	67	6	863	2072	1591	29651	3025	2623
50	T	25	308267	0.00	56	85	3	1084	2660	728	44646	3893	2022

Table 10: Computational results for S_SAHLP with 10 to 25 scenarios

Our formulation considers uncertainty in the flow sent from the origins to the destinations, the transportation cost and the capacities of the hubs. With the aim of highlighting the importance of considering uncertainty in flow, transportation costs and capacities simultaneously, let us consider a network of airports where flows of passengers travel from one airport (origin) to another one (destination) trough one or two hubs. The transportation costs and capacities of the hubs could be affected by various factors, such as weather conditions. Hub capacities may be modified due to damage to the airstrips or communication errors, reducing the number of planes that can land and take off. Moreover, the transportation costs could increase due to route modifications for planes connecting origins/destinations to hubs. ....

								Root node cuts			Total Cuts		
n	p	n_s	Opt	GAP	CPU0	CPU	Nodes	*FCP*	(26)+(27)	(30)	*FCP*	(26)+(27)	(30)
25	2	10	125965	0.00	3	3	2	366	388	414	367	431	442
25	2	15	107053	0.00	4	4	3	313	761	434	1571	839	527
25	2	20	93963.2	0.00	4	6	5	702	1041	1082	1861	1255	1296
25	2	25	82296.2	0.00	5	7	5	413	1223	428	2117	1655	935
25	3	10	106689	0.00	3	5	7	486	588	300	1679	807	527
25	3	15	89635.8	0.00	3	5	13	310	848	183	713	912	223
25	3	20	76346.3	0.00	4	6	6	426	887	315	625	1081	430
25	3	25	66660.2	0.00	6	8	4	500	1512	511	804	1683	598
25	4	10	86518.4	0.00	3	3	3	412	496	555	490	502	570
25	4	15	68728.5	0.00	4	4	3	578	660	628	633	698	634
25	4	20	58284.8	0.00	4	4	2	602	601	588	671	604	610
25	4	25	50714.8	0.00	4	5	2	716	710	695	872	743	730
25	5	10	76752	0.00	3	3	3	380	542	564	512	551	583
25	5	15	61493.8	0.00	4	5	4	460	706	734	829	739	772
25	5	20	52427.4	0.00	4	5	3	749	770	800	2566	922	890
25	5	25	45647.8	0.00	5	7	4	1003	961	973	1822	1121	1090
50	2	10	186228	0.00	15	19	3	298	1092	715	1237	1292	951
50	2	15	184856	0.00	29	45	4	458	1603	1045	4739	2054	1337
50	2	20	183156	0.00	30	71	6	748	1551	1557	8232	2241	2634
50	2	25	182568	0.00	40	115	7	969	1941	1971	8604	3240	4055
50	3	10	165238	0.00	16	28	6	431	1194	1117	4099	1917	1851
50	3	15	165248	0.00	33	68	7	635	2422	1948	1226	3715	2640
50	3	20	163340	0.00	36	114	7	1133	2334	2388	5886	4154	3799
50	3	25	162737	0.00	47	146	7	1195	2931	3526	5125	5040	5218
50	4	10	145908	0.00	13	34	7	668	1248	1272	2600	2475	1724
50	4	15	148475	0.00	24	66	7	1090	1761	2073	5209	3713	2817
50	4	20	148675	0.00	36	144	80	1337	2355	2400	1628	5708	4055
50	4	25	148328	0.00	48	275	390	1653	2952	3006	2804	7238	5742
50	5	10	132449	0.00	13	30	7	601	1248	1263	1631	2138	1911
50	5	15	134489	0.00	24	64	6	921	1908	1917	1575	3232	2587
50	5	20	134437	0.00	36	124	11	1239	2568	2649	1716	5094	3723
50	5	25	134321	0.00	48	180	9	1562	3228	3312	2406	7224	5494

Table 11: Computational results for S_SApHMP with 10 to 25 scenarios

								Root node cuts			Total Cuts				
n	Type	n_s	Opt	GAP	CPU0	CPU	Nodes	*FCP*	(26)+(27)	(30)	*FCP*	(26)+(27)	(30)	LB	UP
25	LL	10	182976	0.00	4	4	3	348	744	701	755	790	755		
25	LL	15	164064	0.00	4	20	157	419	869	582	4373	1856	932		
25	LL	20	148365	0.00	5	29	405	394	1430	977	13844	2244	1583		
25	LL	25	134324	0.00	8	188	14288	238	2448	625	16939	3504	823		
25	LT	10	203143	0.00	3	48	776	677	846	717	1021	1647	1042		
25	LT	15	175286	0.00	4	84	2181	963	1155	1026	3329	1877	1234		
25	LT	20	157802	0.00	5	92	1619	693	1416	1247	1703	2307	1491		
25	LT	25	145651	0.00	8	120	3184	1000	2304	1525	9300	2862	1853		
25	TL	10	226425	0.00	5	34	1782	101	1180	362	1950	1739	651		
25	TL	15	189990	0.00	7	87	1096	218	1844	580	14679	2668	1004		
25	TL	20	169841	0.00	6	91	781	194	1858	914	46502	2770	1410		
25	TL	25	155800	0.00	7	36	345	274	2319	566	16696	3329	1318		
25	TT	10	250181	0.00	4	69	8823	131	1009	656	1336	1611	997		
25	TT	15	216614	0.00	5	151	17655	116	1361	951	3203	2031	1365		
25	TT	20	198276	0.00	7	508	57684	171	1573	365	1521	2154	579		
25	TT	25	185219	0.28	6	7200	684758	222	1583	1036	2608	2535	1409	[184708,	185219]
50	LL	10	257444	0.00	15	288	959	809	1209	1245	2194	2222	1811		
50	LL	15	252576	0.00	35	529	1103	1557	1772	1660	9578	3319	2114		
50	LL	20	248966	0.00	51	185	94	2132	2422	2420	23808	4332	3520		
50	LL	25	247224	0.00	70	214	118	2952	3052	3164	16712	4646	4104		
50	LT	10	293980	0.03	21	7200	574696	637	1792	1122	2796	3167	1629	[293885,	293980]
50	LT	15	289671	0.08	43	7200	136978	1364	2618	2494	2235	4985	3469	[289436,	289671]
50	LT	20	286344	0.11	62	7200	99504	1804	3484	2954	2613	6641	4068	[286034,	286344]
50	LT	25	285160	0.47	85	7200	33040	2853	4577	4728	5508	8335	5966	[283820,	285160]
50	TL	10	329786	0.00	19	128	607	224	1256	383	32029	2115	1206		
50	TL	15	325194	0.00	40	81	72	677	1979	1589	52737	3096	2726		
50	TL	20	321340	0.00	54	139	129	904	2050	1869	64390	4141	2993		
50	TL	25	319595	0.00	73	814	1107	2160	2572	2204	76135	5552	3966		
50	TT	10	464284	0.12	21	7200	283566	764	1665	1403	4321	2971	2113	[463722,	464284]
50	TT	15	454607	0.20	42	7200	122552	1215	2549	2176	2341	4757	3330	[453684,	454607]
50	TT	20	456702	4.15	64	7200	21577	1636	3429	3139	3405	6383	4583	[437728,	456702]
50	TT	25	470352	7.97	84	7200	19348	2046	4325	3954	5343	7946	5942	[432861,	470352]

Table 12: Computational results for S_CSAHLP with 10 to 25 scenarios