Computations on case 2

Fixed sample sizes in period 1 and 2 with ncc

Set (and simplify) conditions

Note that here we assume r1+r2=1, then r3=0

In[1]:=
```
subst = { r01 -> r1-r11 , r02 -> (1 - r1) - r12 - r22 }
```

Out[1]=
$$\{r01 \to r1 - r11,\ r02 \to 1 - r1 - r12 - r22\}$$

In[2]:=
```
ex = {r1 → 0.4}
```

Out[2]=
$$\{r1 \to 0.4\}$$

Define terms to optimise

Note: *sigma*term1^(-1)/N* is the variance of the estimator of effect 1 (analogously *sigma*term2^(-1)/N* for effect 2). But since sigma and N are fixed, we simply work on term1 and term2 expressions.

In[3]:=
```
term1 = FullSimplify[(r11 * r01 / (r11 + r01)) + (r12 * r02 / (r12 + r02)) /. subst]
```

Out[3]=
$$r11 - \frac{r11^2}{r1} + r12 + \frac{r12^2}{-1 + r1 + r22}$$

In[4]:=
```
term2 = FullSimplify[
   (1 / r22 + 1 / r02 - ((1 / r02) ^ 2 / (1 / r01 + 1 / r02 + 1 / r11 + 1 / r12))) ^ (-1) /. subst]
```

Out[4]=
$$-\frac{\left(-r1^2\ (r11 + r12) + r1\ (r11 + r12)\ (1 + r11 - r12 - r22) + r11^2\ (-1 + r22)\right)\ r22}{r11^2 + r1^2\ (r11 + r12) - r1\ (1 + r11 - r12)\ (r11 + r12)}$$

In[5]:=
```
sol = Solve[term1 == term2, r12][[1]]
```

Out[5]=
$$\left\{r12 \to \frac{1}{2\ r1}\left(r1 - r1^2 - \sqrt{r1}\right.\right.$$
$$\left.\left.\sqrt{r1 - 2\ r1^2 + r1^3 + 4\ r1\ r11 - 4\ r1^2\ r11 - 4\ r11^2 + 4\ r1\ r11^2 - 4\ r1\ r22 + 4\ r1^2\ r22 + 4\ r1\ r22^2}\right)\right\}$$

In[6]:=
```
term3 = FullSimplify[term1 /. sol]
```

Out[6]=
$$\frac{1}{2\ r1\ (-1 + r1 + r22)}\ r22\ \left(r1^2 - 2\ r11^2 + r1\ (-1 + 2\ r11 + 2\ r22) -\right.$$
$$\left.\sqrt{r1}\ \sqrt{(-1 + r1)\ \left((-1 + r1)\ r1 - 4\ r1\ r11 + 4\ r11^2\right) + 4\ (-1 + r1)\ r1\ r22 + 4\ r1\ r22^2}\right)$$

In[7]:=
```
e1 = FullSimplify[D[term3, r22]]
```

Out[7]=
$$\frac{1}{2\,r1\,(-1+r1+r22)^2}\left(r22\,(-1+r1+r22)\left(2\,r1-\frac{2\,r1^{3/2}\,(-1+r1+2\,r22)}{\sqrt{(-1+r1)\left((-1+r1)\,r1-4\,r1\,r11+4\,r11^2\right)+4\,(-1+r1)\,r1\,r22+4\,r1\,r22^2}}\right)-\right.$$
$$r22\left(r1^2-2\,r11^2+r1\,(-1+2\,r11+2\,r22)-\right.$$
$$\left.\sqrt{r1}\,\sqrt{(-1+r1)\left((-1+r1)\,r1-4\,r1\,r11+4\,r11^2\right)+4\,(-1+r1)\,r1\,r22+4\,r1\,r22^2}\right)+$$
$$(-1+r1+r22)\left(r1^2-2\,r11^2+r1\,(-1+2\,r11+2\,r22)-\right.$$
$$\left.\left.\sqrt{r1}\,\sqrt{(-1+r1)\left((-1+r1)\,r1-4\,r1\,r11+4\,r11^2\right)+4\,(-1+r1)\,r1\,r22+4\,r1\,r22^2}\right)\right)$$

In[8]:=
```
e2 = FullSimplify[D[term3, r11]]
```

Out[8]=
$$\frac{r22\left(r1-2\,r11+\frac{(-1+r1)\,\sqrt{r1}\,(r1-2\,r11)}{\sqrt{(-1+r1)\left((-1+r1)\,r1-4\,r1\,r11+4\,r11^2\right)+4\,(-1+r1)\,r1\,r22+4\,r1\,r22^2}}\right)}{r1\,(-1+r1+r22)}$$

In[9]:=
```
sol2 = Solve[{r22 (-1 + r1 + r22)
    (2 r1 - (2 r1^(3/2) (-1 + r1 + 2 r22))/Sqrt[(-1 + r1) ((-1 + r1) r1 - 4 r1 r11 + 4 r11^2) + 4 (-1 + r1) r1 r22 + 4 r1 r22^2]) -
   r22 (r1^2 - 2 r11^2 + r1 (-1 + 2 r11 + 2 r22) -
     Sqrt[r1] Sqrt[(-1 + r1) ((-1 + r1) r1 - 4 r1 r11 + 4 r11^2) + 4 (-1 + r1) r1 r22 + 4 r1 r22^2]) +
   (-1 + r1 + r22) (r1^2 - 2 r11^2 + r1 (-1 + 2 r11 + 2 r22) -
     Sqrt[r1] Sqrt[(-1 + r1) ((-1 + r1) r1 - 4 r1 r11 + 4 r11^2) + 4 (-1 + r1) r1 r22 + 4 r1 r22^2]) ==
  0, r22 (r1 - 2 r11 +
    ((-1 + r1) Sqrt[r1] (r1 - 2 r11))/Sqrt[(-1 + r1) ((-1 + r1) r1 - 4 r1 r11 + 4 r11^2) + 4 (-1 + r1) r1 r22 + 4 r1 r22^2]) ==
  0}, {r11, r22}];
```

Solve: There may be values of the parameters for which some or all solutions are not valid.

The solutions are then the following

In[26]:=
```
sol2[[11]]
```

Out[26]=
$$\left\{r11 \to \frac{r1}{2},\right.$$

$$
\begin{aligned}
r22 \to 1 - r1 + \frac{1}{2}\sqrt{\left(4\,(-1+r1)^2 + \frac{1}{4}\left(-22+43\,r1-21\,r1^2\right) + \frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)} + \right.}\\
\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\right.\\
\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+\right.\\
139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+\right.}\\
3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+\\
232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-\\
264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\\
\left.\left.\left.r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}\right)+\\
\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+\right.\\
391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\\
\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+\right.}\\
55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-\\
315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+\\
162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-\\
\left.\left.5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}\Bigg)-\\
\frac{1}{2}\sqrt{\left(8\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)-\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}-\right.}\\
\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\right.\\
\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+\right.\\
139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+\right.}\\
3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+\\
232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-\\
264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\\
\left.\left.\left.r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}\right)-\\
\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+\right.\\
391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\\
\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+\right.}\\
55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-\\
315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+\\
162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-\\
\left.\left.5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}+\\
\left(-64\,(-1+r1)^3+4\,(-1+r1)\left(22-43\,r1+21\,r1^2\right)-6\left(-4+11\,r1-10\,r1^2+3\,r1^3\right)\right)\Big/\\
\left(4\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\right.}\right.\\
\left.\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\right.
\end{aligned}
$$

$$\left(3\,(-1+r1)\,\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)\right)\Bigg\}$$

In[24]:= `sol[[1]] /. sol2[[11]][[2]]`

Out[24]=

$$r12\to\frac{1}{2\,r1}\left(r1-r1^2-\sqrt{r1}\,\sqrt{\left(r1-2\,r1^2+r1^3-4\,r1\left(1-r1+\frac{1}{2}\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)-$$

$$\frac{1}{2}\sqrt{\left(8\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)-\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}-\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)-\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}+\left(-64\,(-1+r1)^3+4\,(-1+r1)\left(22-43\,r1+21\,r1^2\right)-6\left(-4+11\,r1-10\,r1^2+3\,r1^3\right)\right)\Big/\left(4\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)\right)\right)+$$

$$4\,r1^2\left(1-r1+\frac{1}{2}\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\right.\right.$$
$$\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/$$
$$\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+\right.\right.$$
$$391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+$$
$$\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+\right.}$$
$$55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-$$
$$315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+$$
$$162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-$$
$$\left.\left.\left.5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}\right)+$$
$$\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-\right.$$
$$304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+$$
$$\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+\right.}$$
$$55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-$$
$$315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+$$
$$162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-$$
$$\left.\left.5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}\Bigg)-$$
$$\frac{1}{2}\sqrt{\left(8\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)-\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}-\right.}$$
$$\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/$$
$$\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+\right.\right.$$
$$391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+$$
$$\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+\right.}$$
$$55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-$$
$$315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+$$
$$162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-$$
$$\left.\left.\left.5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}\right)-$$
$$\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-\right.$$
$$304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+$$
$$\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+\right.}$$
$$55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-$$
$$315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+$$
$$162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-$$
$$\left.\left.5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}+$$
$$\left(-64\,(-1+r1)^3+4\,(-1+r1)\left(22-43\,r1+21\,r1^2\right)-\right.$$
$$\left.6\left(-4+11\,r1-10\,r1^2+3\,r1^3\right)\right)\Big/$$

$$\left(4\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)\right)\right)\right)+$$

$$4\,r1\left(1-r1+\frac{1}{2}\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)-$$

$$\frac{1}{2}\sqrt{\left(8\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)-\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}-\right.}$$

$$\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/$$

$$\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+\right.\right.$$

$$391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+$$

$$\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\right.}$$

$$r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176$$

$$r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968$$

$$\left.\left.\left.r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\right.\right.\right.$$

$$\left.\left.\left.r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}\right)-$$

$$\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-\right.$$

$$304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+$$

$$\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+\right.}$$

$$55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-$$

$$315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+$$

$$162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-$$

$$\left.5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\Big)^{1/3}+$$

$$\left(-64\,(-1+r1)^3+4\,(-1+r1)\left(22-43\,r1+21\,r1^2\right)-\right.$$

$$\left.6\left(-4+11\,r1-10\,r1^2+3\,r1^3\right)\right)\Big/$$

$$\left(4\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\right.}\right.$$

$$\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/$$

$$\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-\right.\right.$$

$$304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-$$

$$34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+\right.}$$

$$3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-$$

$$130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-$$

$$315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968$$

$$r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+$$

$$24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-$$

$$\left.\left.47\,775\,744\,r1^{16}\right)\right)^{1/3}\Big)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-\right.$$

$$26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-$$

$$303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-\right.}$$

$$467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+$$

$$55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-$$

$$315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+$$

$$162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-$$

$$5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744$$

$$r1^{16}))^{1/3}))))^{2} + 4\, r1\, r11 - 4\, r1^{2}\, r11 - 4\, r11^{2} + 4\, r1\, r11^{2}))$$

In[23]:= `1 - r1 - sol2[[11]][[2]][[2]] - sol[[1]][[2]] /. sol2[[11]][[2]]`

Out[23]=

$$-\frac{1}{2}\sqrt{}\Bigg(\Bigg(4\,(-1+r1)^{2} + \frac{1}{4}\,(-22+43\,r1-21\,r1^{2}) + \frac{-22+65\,r1-64\,r1^{2}+21\,r1^{3}}{12\,(-1+r1)} +$$

$$(2^{1/3}\,(4-40\,r1+129\,r1^{2}-196\,r1^{3}+154\,r1^{4}-60\,r1^{5}+9\,r1^{6}))\Big/$$

$$(3\,(-1+r1)\,(1024-1536\,r1-26112\,r1^{2}+135040\,r1^{3}-304896\,r1^{4}+391296\,r1^{5}-$$

$$303104\,r1^{6}+139392\,r1^{7}-34560\,r1^{8}+3456\,r1^{9}+\sqrt{}(28311552\,r1-467140608\,r1^{2}+$$

$$3588489216\,r1^{3}-16978083840\,r1^{4}+55228760064\,r1^{5}-130682585088\,r1^{6}+$$

$$232168882176\,r1^{7}-315144732672\,r1^{8}+329334128640\,r1^{9}-264790867968$$

$$r1^{10}+162331361280\,r1^{11}-74439917568\,r1^{12}+24680595456$$

$$r1^{13}-5573836800\,r1^{14}+764411904\,r1^{15}-47775744\,r1^{16}))^{1/3}) +$$

$$\frac{1}{48\times 2^{1/3}\,(-1+r1)}\,(1024-1536\,r1-26112\,r1^{2}+135040\,r1^{3}-304896\,r1^{4}+$$

$$391296\,r1^{5}-303104\,r1^{6}+139392\,r1^{7}-34560\,r1^{8}+3456\,r1^{9}+$$

$$\sqrt{}(28311552\,r1-467140608\,r1^{2}+3588489216\,r1^{3}-16978083840\,r1^{4}+$$

$$55228760064\,r1^{5}-130682585088\,r1^{6}+232168882176\,r1^{7}-315144732672\,r1^{8}+$$

$$329334128640\,r1^{9}-264790867968\,r1^{10}+162331361280\,r1^{11}-74439917568\,r1^{12}+$$

$$24680595456\,r1^{13}-5573836800\,r1^{14}+764411904\,r1^{15}-47775744\,r1^{16}))^{1/3}) +$$

$$\frac{1}{2}\sqrt{}\Bigg(8\,(-1+r1)^{2} + \frac{1}{4}\,(-22+43\,r1-21\,r1^{2}) - \frac{-22+65\,r1-64\,r1^{2}+21\,r1^{3}}{12\,(-1+r1)} -$$

$$(2^{1/3}\,(4-40\,r1+129\,r1^{2}-196\,r1^{3}+154\,r1^{4}-60\,r1^{5}+9\,r1^{6}))\Big/$$

$$(3\,(-1+r1)\,(1024-1536\,r1-26112\,r1^{2}+135040\,r1^{3}-304896\,r1^{4}+391296\,r1^{5}-$$

$$303104\,r1^{6}+139392\,r1^{7}-34560\,r1^{8}+3456\,r1^{9}+\sqrt{}(28311552\,r1-467140608\,r1^{2}+$$

$$3588489216\,r1^{3}-16978083840\,r1^{4}+55228760064\,r1^{5}-130682585088\,r1^{6}+$$

$$232168882176\,r1^{7}-315144732672\,r1^{8}+329334128640\,r1^{9}-264790867968$$

$$r1^{10}+162331361280\,r1^{11}-74439917568\,r1^{12}+24680595456$$

$$r1^{13}-5573836800\,r1^{14}+764411904\,r1^{15}-47775744\,r1^{16}))^{1/3}) -$$

$$\frac{1}{48\times 2^{1/3}\,(-1+r1)}\,(1024-1536\,r1-26112\,r1^{2}+135040\,r1^{3}-304896\,r1^{4}+$$

$$391296\,r1^{5}-303104\,r1^{6}+139392\,r1^{7}-34560\,r1^{8}+3456\,r1^{9}+$$

$$\sqrt{}(28311552\,r1-467140608\,r1^{2}+3588489216\,r1^{3}-16978083840\,r1^{4}+$$

$$55228760064\,r1^{5}-130682585088\,r1^{6}+232168882176\,r1^{7}-315144732672\,r1^{8}+$$

$$329334128640\,r1^{9}-264790867968\,r1^{10}+162331361280\,r1^{11}-74439917568\,r1^{12}+$$

$$24680595456\,r1^{13}-5573836800\,r1^{14}+764411904\,r1^{15}-47775744\,r1^{16}))^{1/3} +$$

$$(-64\,(-1+r1)^{3}+4\,(-1+r1)\,(22-43\,r1+21\,r1^{2})-6\,(-4+11\,r1-10\,r1^{2}+3\,r1^{3}))\Big/$$

$$\left(4\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)\right)-\frac{1}{2\,r1}\left(r1-r1^2-\sqrt{r1}\sqrt{\left(r1-2\,r1^2+r1^3-4\,r1\left(1-r1+\frac{1}{2}\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)\right)\right)-$$

$$\frac{1}{2}\sqrt{\left(8\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)-\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}-\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)-\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}+\left(-64\,(-1+r1)^3+4\,(-1+r1)\left(22-43\,r1+21\,r1^2\right)-6\left(-4+11\,r1-10\,r1^2+3\,r1^3\right)\right)\Big/\left(4\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)\right)\right)\right)}+$$

$$4\,r1^2\left(1-r1+\frac{1}{2}\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}^{1/3}\right)\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}^{1/3}\right)-\frac{1}{2}\sqrt{\left(8\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)-\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}-\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}^{1/3}\right)\right)-\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}^{1/3}+\left(-64\,(-1+r1)^3+4\,(-1+r1)\left(22-43\,r1+21\,r1^2\right)-6\left(-4+11\,r1-10\,r1^2+3\,r1^3\right)\right)\Big/$$

$$\left(4\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)\right)\right)\right)+$$

$$4\,r1\left(1-r1+\frac{1}{2}\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)}\right)^{1/3}\right)-$$

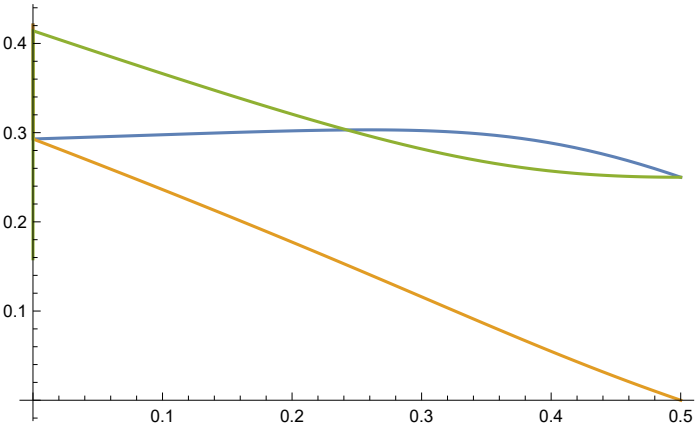
$$\frac{1}{2}\sqrt{\left(8\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)-\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}-\right.}$$
$$\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/$$
$$\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+\right.\right.$$
$$391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+$$
$$\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\right.}$$
$$r1^4+55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176$$
$$r1^7-315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968$$
$$r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456$$
$$\left.\left.\left.r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}\right)-$$
$$\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-\right.$$
$$304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+$$
$$\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+\right.}$$
$$55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-$$
$$315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+$$
$$162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-$$
$$\left.\left.5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744\,r1^{16}\right)\right)^{1/3}+$$
$$\left(-64\,(-1+r1)^3+4\,(-1+r1)\left(22-43\,r1+21\,r1^2\right)-\right.$$
$$\left.6\left(-4+11\,r1-10\,r1^2+3\,r1^3\right)\right)\Big/$$
$$\left(4\sqrt{\left(4\,(-1+r1)^2+\frac{1}{4}\left(-22+43\,r1-21\,r1^2\right)+\frac{-22+65\,r1-64\,r1^2+21\,r1^3}{12\,(-1+r1)}+\right.\right.}$$
$$\left(2^{1/3}\left(4-40\,r1+129\,r1^2-196\,r1^3+154\,r1^4-60\,r1^5+9\,r1^6\right)\right)\Big/$$
$$\left(3\,(-1+r1)\left(1024-1536\,r1-26\,112\,r1^2+135\,040\,r1^3-\right.\right.$$
$$304\,896\,r1^4+391\,296\,r1^5-303\,104\,r1^6+139\,392\,r1^7-$$
$$34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-467\,140\,608\,r1^2+\right.}$$
$$3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+55\,228\,760\,064\,r1^5-$$
$$130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-$$
$$315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968$$
$$r1^{10}+162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+$$
$$24\,680\,595\,456\,r1^{13}-5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-$$
$$\left.\left.\left.47\,775\,744\,r1^{16}\right)\right)^{1/3}\right)+\frac{1}{48\times 2^{1/3}\,(-1+r1)}\left(1024-1536\,r1-\right.$$
$$26\,112\,r1^2+135\,040\,r1^3-304\,896\,r1^4+391\,296\,r1^5-$$
$$303\,104\,r1^6+139\,392\,r1^7-34\,560\,r1^8+3456\,r1^9+\sqrt{\left(28\,311\,552\,r1-\right.}$$
$$467\,140\,608\,r1^2+3\,588\,489\,216\,r1^3-16\,978\,083\,840\,r1^4+$$
$$55\,228\,760\,064\,r1^5-130\,682\,585\,088\,r1^6+232\,168\,882\,176\,r1^7-$$
$$315\,144\,732\,672\,r1^8+329\,334\,128\,640\,r1^9-264\,790\,867\,968\,r1^{10}+$$
$$162\,331\,361\,280\,r1^{11}-74\,439\,917\,568\,r1^{12}+24\,680\,595\,456\,r1^{13}-$$
$$5\,573\,836\,800\,r1^{14}+764\,411\,904\,r1^{15}-47\,775\,744$$

$$r1^{16}\Big)\Big)^{1/3}\Big)\Big)\Big)\Big)^2 + 4\,r1\,r11 - 4\,r1^2\,r11 - 4\,r11^2 + 4\,r1\,r11^2\Big)\Big)$$

Plot solutions in period 2 with respect to r1

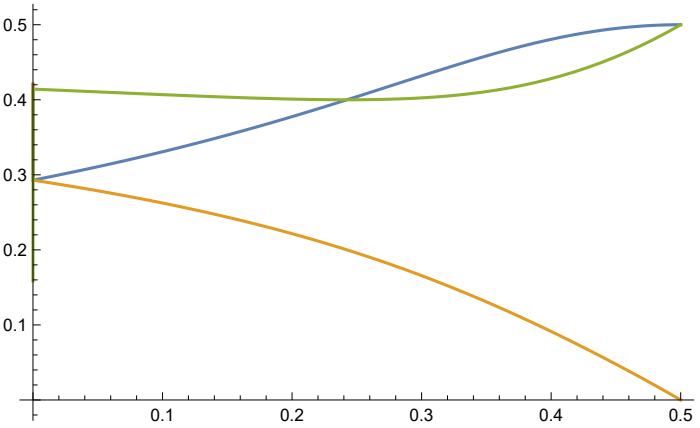
In[ ]:=
```
Plot[{sol2[[11]][[2]][[2]], sol[[1]][[2]] /. sol2[[11]][[2]] /. r11 → r1 / 2,
  1 - r1 - sol2[[11]][[2]][[2]] - sol[[1]][[2]] /. sol2[[11]][[2]] /. r11 → r1 / 2}, {r1, 0, .5}]
```

Out[ ]=



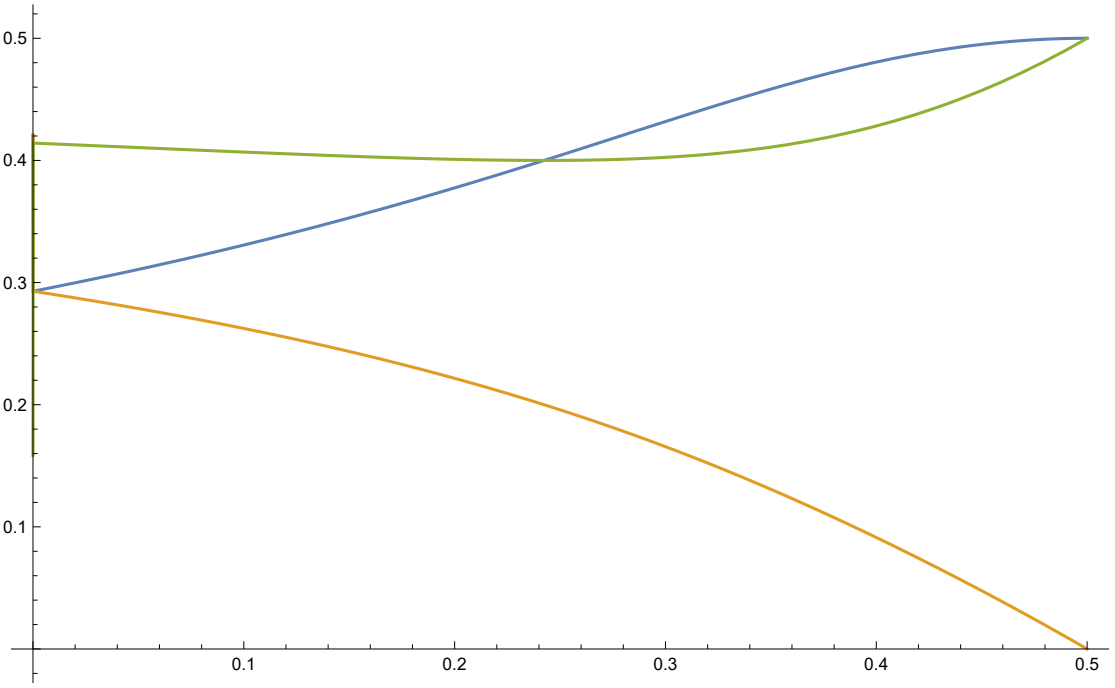
In[ ]:=
```
Plot[{sol2[[11]][[2]][[2]] / (1 - r1),
  (sol[[1]][[2]] /. sol2[[11]][[2]] /. r11 → r1 / 2) / (1 - r1), 1 - sol2[[11]][[2]][[2]] / (1 - r1) -
   (sol[[1]][[2]] /. sol2[[11]][[2]] /. r11 → r1 / 2) / (1 - r1)}, {r1, 0, .5}]
```

Out[ ]=



In[•]:=
```
Show[%222, ImageSize → Large]
```

Out[•]=



Numerical example:

In[•]:=
```
sol2 /. {r1 → 0.1}
```

Out[•]=
```
{{r22 → 0.875048 - 0.110736 i},
 {r22 → 0.875048 + 0.110736 i}, {r22 → 0.297642}, {r22 → 1.55226}}
```

In[•]:=
```
sol /. sol2 /. {r1 → 0.1}
```

Out[•]=
```
{{r12 → 0.00069231 + 0.104757 i},
 {r12 → 0.00069231 - 0.104757 i}, {r12 → 0.236194}, {r12 → -0.662421}}
```

Import solutions to R

```
CForm[sol2[[11]][[2]][[2]];
```

In[•]:=
```
CForm[sol[[1]][[2]]
```

Out[•]//CForm=
```
(r1 - Power(r1,2) - Sqrt(r1)*Sqrt(r1 - 2*Power(r1,2) + Power(r1,3) + 4*r1*r11 - 4*Powe
      4*r1*Power(r22,2)))/(2.*r1)
```

Computations on case 3. Three-period design using non-concurrent controls

We first define the treatment effect for arm 2 following the expressions presented in the supplementary material (see Section A.2). To do so, we define the matrices A, B, and C, and use equation (1.b) to to obtain point estimates

```
In[ ]:= nd1 = n01 + n11
        nd2 = n02 + n12 + n22
        nd3 = n03 + n23
        A = {{nd1, 0, 0}, {0, nd2, 0}, {0, 0, nd3}}
        B = {{n11, 0}, {n12, n22}, {0, n23}}
        Cm = {{n11 + n12, 0}, {0, n22 + n23}}
```

```
Out[ ]= n01 + n11

Out[ ]= n02 + n12 + n22

Out[ ]= n03 + n23

Out[ ]= {{n01 + n11, 0, 0}, {0, n02 + n12 + n22, 0}, {0, 0, n03 + n23}}

Out[ ]= {{n11, 0}, {n12, n22}, {0, n23}}

Out[ ]= {{n11 + n12, 0}, {0, n22 + n23}}
```

```
In[ ]:= M = FullSimplify[Inverse[Cm - Transpose[B].Inverse[A].B]]
        Nm = {{n11 * theta11 + n12 * theta12}, {n22 * theta22 + n23 * theta23}}
        Collect[FullSimplify[M.Nm][[2]], {theta11, theta12, theta22, theta23}]
        w11 = (n11 (n01 + n11) n12 n22 (n03 + n23) ) /
           (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
             n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
        w12 = ( (n01 + n11) n12^2 n22 (n03 + n23) ) /
           (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
             n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
        w22 =
          (n22 (n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) (n03 + n23) ) /
           (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
             n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
        w23 =
          ((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) n23 (n03 + n23) ) /
           (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
             n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
        sol = M.Nm
        True ===
         FullSimplify[sol[[2]][[1]] == w11 * theta11 + w12 * theta12 + w22 * theta22 + w23 * theta23]
```

Out[•]=
$$\{\{((n01+n11)\,(n03\,(n02+n12)\,n22+n03\,(n02+n12)\,n23+(n02+n03+n12)\,n22\,n23))\,/\,(n01\,n03\,(n11\,n12+n02\,(n11+n12))\,n22+n01\,(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22+n02\,(n11+n12)\,(n03+n22))\,n23+n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)),((n01+n11)\,n12\,n22\,(n03+n23))\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))\},\{((n01+n11)\,n12\,n22\,(n03+n23))\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))),((n11\,n12\,(n02+n22)+n01\,(n11\,n12+n02\,(n11+n12)+(n11+n12)\,n22))\,(n03+n23))\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))\}\}$$

Out[•]=
$$\{\{n11\,\text{theta11}+n12\,\text{theta12}\},\{n22\,\text{theta22}+n23\,\text{theta23}\}\}$$

Out[•]=
$$\Big\{(n11\,(n01+n11)\,n12\,n22\,(n03+n23)\,\text{theta11})\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))+\big((n01+n11)\,n12^2\,n22\,(n03+n23)\,\text{theta12}\big)\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))+(n22\,(n11\,n12\,(n02+n22)+n01\,(n11\,n12+n02\,(n11+n12)+(n11+n12)\,n22))\,(n03+n23)\,\text{theta22})\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))+((n11\,n12\,(n02+n22)+n01\,(n11\,n12+n02\,(n11+n12)+(n11+n12)\,n22))\,n23\,(n03+n23)\,\text{theta23})\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))\Big\}$$

Out[•]=
$$(n11\,(n01+n11)\,n12\,n22\,(n03+n23))\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))$$

Out[•]=
$$\big((n01+n11)\,n12^2\,n22\,(n03+n23)\big)\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))$$

Out[•]=
$$(n22\,(n11\,n12\,(n02+n22)+n01\,(n11\,n12+n02\,(n11+n12)+(n11+n12)\,n22))\,(n03+n23))\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))$$

```
Out[•]= ((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) n23 (n03 + n23)) /
         (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
           n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
              n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
```

```
Out[•]= {{((n01 + n11) (n03 (n02 + n12) n22 + n03 (n02 + n12) n23 + (n02 + n03 + n12) n22 n23)
              (n11 theta11 + n12 theta12)) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
             n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
             n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23)) +
           ((n01 + n11) n12 n22 (n03 + n23) (n22 theta22 + n23 theta23)) /
            (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
              n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))},
          {((n01 + n11) n12 n22 (n03 + n23) (n11 theta11 + n12 theta12)) /
            (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
              n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23)))) +
           ((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22))
              (n03 + n23) (n22 theta22 + n23 theta23)) /
            (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
              n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))}}
```

```
Out[•]= True
```

Variance computation

To compute the variance of treatment effect 2, first note

Var(theta2) = Var(w11*theta11+w12*theta12+w22*theta22+w23*theta23)

In[ ]:=
```
theta11 = n01 / (n01 + n11) * (y11 - y01);
theta12 = (n02 + n22) / (n02 + n12 + n22) * y12 -
    ((n02 / (n02 + n12 + n22)) * y02 + (n22 / (n02 + n12 + n22)) * y22);
theta22 = (n02 + n12) / (n02 + n12 + n22) * y22 -
    ((n02 / (n02 + n12 + n22)) * y02 + (n12 / (n02 + n12 + n22)) * y12);
theta23 = n03 / (n03 + n23) * (y23 - y03);
expr = w11 * theta11 + w12 * theta12 + w22 * theta22 + w23 * theta23;
Collect[FullSimplify[expr], {y01, y11, y02, y12, y22, y03, y23}];
expr01 = FullSimplify[
    (-n01 n03 n11 n12 n22 - n01 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr02 = FullSimplify[
    (-n01 n02 n03 n11 n22 - n01 n02 n03 n12 n22 - n02 n03 n11 n12 n22 - n01 n02 n11 n22 n23 -
       n01 n02 n12 n22 n23 - n02 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr03 = FullSimplify[
    (-n01 n03 n11 n12 n23 - n02 n03 n11 n12 n23 - n03 n11 n12 n22 n23 - n01 n03 n11 (n02 + n22)
        n23 - n01 n03 n12 (n02 + n22) n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr11 = FullSimplify[
    (n01 n03 n11 n12 n22 + n01 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr12 = FullSimplify[
    (-n01 n03 n11 n12 n22 - n01 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr22 = FullSimplify[
    (n02 n03 n11 n12 n22 + n01 n03 (n11 n12 + n02 (n11 + n12)) n22 + n01 n02 n12 n22 n23 + n02 n11
        n12 n22 n23 + n01 n11 (n02 + n12) n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr23 = FullSimplify[ (n02 n03 n11 n12 n23 + n03 n11 n12 n22 n23 +
       n01 n03 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22) n23) /
     (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
FullSimplify[
  Collect[FullSimplify[expr], {y01, y11, y02, y12, y22, y03, y23}] == expr01 * y01 +
    expr02 * y02 + expr03 * y03 + expr11 * y11 + expr12 * y12 + expr22 * y22 + expr23 * y23];
```

Variance expression is then term2s*sigma^2/N, where

```
In[ ]:= term2f = FullSimplify[expr01^2 * y01 + expr02^2 * y02 + expr03^2 * y03 + expr11^2 * y11 +
          expr12^2 * y12 + expr22^2 * y22 + expr23^2 * y23 /. {y01 → 1 / n01, y02 → 1 / n02,
          y03 → 1 / n03, y11 → 1 / n11, y12 → 1 / n12, y22 → 1 / n22, y23 → 1 / n23}];
        term2s =
          FullSimplify[term2f /. {n01 → r01 * Nt, n02 → r02 * Nt, n03 → r03 * Nt, n11 → r11 * Nt,
            n12 → r12 * Nt, n22 → r22 * Nt, n23 → r23 * Nt}] * Nt;
```

Define terms to optimise

```
In[ ]:= subst = {r11 → r1 / 2 , r01 → r1 / 2 , r23 → r3 / 2 , r03 → r3 / 2, r02 → r2 - r12 - r22 };
        term1 = FullSimplify[(r11 * r01 / (r11 + r01)) + (r12 * r02 / (r12 + r02)) /. subst];
        term2 = FullSimplify[(1 / term2s) /. subst];

In[ ]:= substg = {r01 → r1 - r11, r03 → r3 - r23, r02 → r2 - r12 - r22 };
        termg1 = FullSimplify[(r11 * r01 / (r11 + r01)) + (r12 * r02 / (r12 + r02)) /. substg];
        termg2 = FullSimplify[(1 / term2s) /. substg];
```

Numerical example: optimisation assuming balanced design in periods 1 and 3

```
In[ ]:= ex = {r1 → 0.1, r2 → 0.8, r3 → 0.1};
        FindMinimum[{(-term1) /. ex, term1 == term2 /. ex, r12 + r22 < 0.8, r12 > 0, r22 > 0},
         {{r12, r2 / 3 /. ex}, {r22, r2 / 3 /. ex}}]
Out[ ]= {-0.164091, {r12 → 0.24303, r22 → 0.231746}}
```

Optimisation (approach 1) - here we do not assume balanced design in periods 1 and 3 and thus also allocation rates in periods 1 and 3 are optimized

```
In[ ]:= ex = {r1 → 0.4, r2 → 0.4, r3 → 0.2};
        FindMinimum[{(-termg1) /. ex, termg1 == termg2 /. ex,
          r12 + r22 < 0.8, r12 > 0, r22 > 0, r11 > 0, r23 > 0, r11 < 1, r23 < 1},
         {{r11, r1 / 2 /. ex}, {r12, r2 / 3 /. ex}, {r22, r2 / 3 /. ex}, {r23, r3 / 2 /. ex}}]
Out[ ]= {-0.144071, {r11 → 0.2, r12 → 0.0615317, r22 → 0.183165, r23 → 0.1}}

In[ ]:= FindMinimum[{-termg1 /. ex, termg1 == termg2 /. ex,
          r12 + r22 < 0.8`, r12 > 0, r22 > 0, r11 > 0, r23 > 0, r11 < 1, r23 < 1},
         {{r11, r1/2 /. ex}, {r12, r2/3 /. ex}, {r22/3 /. ex}, {r23, r3/2 /. ex}}]

FindMinimum: The variable r22/3 /. ex cannot be localized so that it can be assigned to numerical values.

Out[ ]= FindMinimum[{-termg1 /. ex, termg1 == termg2 /. ex,
          r12 + r22 < 0.8, r12 > 0, r22 > 0, r11 > 0, r23 > 0, r11 < 1, r23 < 1},
         {{r11, r1/2 /. ex}, {r12, r2/3 /. ex}, {r22/3 /. ex}, {r23, r3/2 /. ex}}]

In[ ]:= termg2

Out[ ]= (r1 (-r12^2 + (r11 + r12) r2) r23^2 +
          r1 (r11 r22 (-r2 + r22) + r12 r22 (r12 - r2 + r22) + r12^2 r23 - (r11 + r12) r2 r23) r3 +
          r11^2 (-r2 r23^2 - r22^2 r3 + r2 (r22 + r23) r3)) / ((r11^2 r2 + r1 (r12^2 - (r11 + r12) r2)) r3)
```

```
In[ ]:= {{r11, r1/2 /. ex}, {r12, r2/3 /. ex}, {r22/3 /. ex}, {r23, r3/2 /. ex}}
```

$$\text{Out[ ]= } \left\{\{\text{r11}, 0.2\}, \{\text{r12}, 0.133333\}, \left\{\frac{\text{r22}}{3}\right\}, \{\text{r23}, 0.1\}\right\}$$

Note that we cannot find analytical solutions, but the numerical solutions satisfy that the optimal design follows a balanced design in periods 1 and 3.

Optimisation (approach 2) - assume balanced designs in periods 1 and 3

```
In[ ]:= constr = term1 - term2;
```

```
In[ ]:= e1 = FullSimplify[Solve[D[term1, r12] == l D[constr, r12], l]]
        e2 = FullSimplify[Solve[D[term1, r22] == l D[constr, r22], l]]
        e3 = e1[[1]][[1]][[2]] == e2[[1]][[1]][[2]]
```

$$\text{Out[ ]= } \left\{\left\{l \to \frac{-1+\frac{2\,\text{r12}}{\text{r2}-\text{r22}}}{-1+\frac{2\,\text{r12}}{\text{r2}-\text{r22}}-\frac{8\,\text{r12}\,(\text{r1}+2\,\text{r12})\,\text{r22}^2}{(\text{r1}\,\text{r2}+4\,\text{r12}\,(-\text{r12}+\text{r2}))^2}}\right\}\right\}$$

$$\text{Out[ ]= } \left\{\left\{l \to \frac{\text{r12}^2}{(\text{r2}-\text{r22})^2\left(1+\frac{\text{r12}^2}{(\text{r2}-\text{r22})^2}-\frac{2\,(\text{r1}+4\,\text{r12})\,\text{r22}}{\text{r1}\,\text{r2}+4\,\text{r12}\,(-\text{r12}+\text{r2})}\right)}\right\}\right\}$$

$$\text{Out[ ]= } \frac{-1+\frac{2\,\text{r12}}{\text{r2}-\text{r22}}}{-1+\frac{2\,\text{r12}}{\text{r2}-\text{r22}}-\frac{8\,\text{r12}\,(\text{r1}+2\,\text{r12})\,\text{r22}^2}{(\text{r1}\,\text{r2}+4\,\text{r12}\,(-\text{r12}+\text{r2}))^2}} == \frac{\text{r12}^2}{(\text{r2}-\text{r22})^2\left(1+\frac{\text{r12}^2}{(\text{r2}-\text{r22})^2}-\frac{2\,(\text{r1}+4\,\text{r12})\,\text{r22}}{\text{r1}\,\text{r2}+4\,\text{r12}\,(-\text{r12}+\text{r2})}\right)}$$

```
In[ ]:= sol2 = Solve[e3, {r12}];
```

```
In[ ]:= solsim = Simplify[sol2[[7]]]
```

$$\begin{aligned}\text{Out[ ]= } \Bigg\{\text{r12} \to \frac{1}{192\,(\text{r2}-\text{r22})}&\Big(32\left(3\,\text{r2}^2-6\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)+\left(16\,\text{i}\,2^{1/3}\left(\text{i}+\sqrt{3}\right)\left(3\,\text{r2}^4-12\,\text{r2}^3\,\text{r22}+18\,\text{r2}^2\,\text{r22}^2-\right.\right.\\&\left.\left.12\,\text{r2}\,\text{r22}^3+4\,\text{r22}^4+3\,\text{r1}\,\text{r2}\left(\text{r2}^2-3\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)\right)\right)\Big/\\&\Big(-9\,\text{r1}\,\text{r2}^3\,\text{r22}^2+18\,\text{r2}^4\,\text{r22}^2+27\,\text{r1}\,\text{r2}^2\,\text{r22}^3-72\,\text{r2}^3\,\text{r22}^3-18\,\text{r1}\,\text{r2}\,\text{r22}^4+108\,\text{r2}^2\,\text{r22}^4-\\&72\,\text{r2}\,\text{r22}^5+16\,\text{r22}^6+\sqrt{\Big(-4\left(3\,\text{r2}^4-12\,\text{r2}^3\,\text{r22}+18\,\text{r2}^2\,\text{r22}^2-12\,\text{r2}\,\text{r22}^3+4\,\text{r22}^4+\right.}\\&\left.3\,\text{r1}\,\text{r2}\left(\text{r2}^2-3\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)\right)^3+\text{r22}^4\left(9\,\text{r1}\,\text{r2}\left(\text{r2}^2-3\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)-\\&2\left(9\,\text{r2}^4-36\,\text{r2}^3\,\text{r22}+54\,\text{r2}^2\,\text{r22}^2-36\,\text{r2}\,\text{r22}^3+8\,\text{r22}^4\right)\Big)^2\Big)\Big)^{1/3}-\\&8\times 2^{2/3}\left(1+\text{i}\,\sqrt{3}\right)\Big(-9\,\text{r1}\,\text{r2}^3\,\text{r22}^2+18\,\text{r2}^4\,\text{r22}^2+27\,\text{r1}\,\text{r2}^2\,\text{r22}^3-72\,\text{r2}^3\,\text{r22}^3-\\&18\,\text{r1}\,\text{r2}\,\text{r22}^4+108\,\text{r2}^2\,\text{r22}^4-72\,\text{r2}\,\text{r22}^5+16\,\text{r22}^6+\\&\sqrt{\Big(-4\left(3\,\text{r2}^4-12\,\text{r2}^3\,\text{r22}+18\,\text{r2}^2\,\text{r22}^2-12\,\text{r2}\,\text{r22}^3+4\,\text{r22}^4+\\&3\,\text{r1}\,\text{r2}\left(\text{r2}^2-3\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)\right)^3+\text{r22}^4\left(9\,\text{r1}\,\text{r2}\left(\text{r2}^2-3\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)-\\&2\left(9\,\text{r2}^4-36\,\text{r2}^3\,\text{r22}+54\,\text{r2}^2\,\text{r22}^2-36\,\text{r2}\,\text{r22}^3+8\,\text{r22}^4\right)\Big)^2\Big)\Big)^{1/3}\Big)\Bigg\}\end{aligned}$$

```
In[ ]:= $Assumptions =
          r12 > 0 && r22 > 0 && r12 + r22 < r2 && Element[r12, Reals] && Element[r22, Reals]
```

$$\text{Out[ ]= } \text{r12} > 0 \,\&\&\, \text{r22} > 0 \,\&\&\, \text{r12} + \text{r22} < \text{r2} \,\&\&\, \text{r12} \in \mathbb{R} \,\&\&\, \text{r22} \in \mathbb{R}$$

In[•]:=
```
Re[solsim]
```

Out[•]=
```
{Re[
  r12 → 1/(192 (r2 - r22)) (32 (3 r2^2 - 6 r2 r22 + 2 r22^2) + (16 i 2^(1/3) (i + Sqrt[3]) (3 r2^4 - 12 r2^3 r22 + 18
        r2^2 r22^2 - 12 r2 r22^3 + 4 r22^4 + 3 r1 r2 (r2^2 - 3 r2 r22 + 2 r22^2)))/
    (-9 r1 r2^3 r22^2 + 18 r2^4 r22^2 + 27 r1 r2^2 r22^3 - 72 r2^3 r22^3 - 18 r1 r2 r22^4 + 108 r2^2 r22^4 -
      72 r2 r22^5 + 16 r22^6 + Sqrt[(-4 (3 r2^4 - 12 r2^3 r22 + 18 r2^2 r22^2 - 12 r2 r22^3 + 4 r22^4 +
          3 r1 r2 (r2^2 - 3 r2 r22 + 2 r22^2))^3 + r22^4 (9 r1 r2 (r2^2 - 3 r2 r22 + 2 r22^2) -
          2 (9 r2^4 - 36 r2^3 r22 + 54 r2^2 r22^2 - 36 r2 r22^3 + 8 r22^4))^2)]^(1/3) -
    8 × 2^(2/3) (1 + i Sqrt[3]) (-9 r1 r2^3 r22^2 + 18 r2^4 r22^2 + 27 r1 r2^2 r22^3 - 72 r2^3 r22^3 -
      18 r1 r2 r22^4 + 108 r2^2 r22^4 - 72 r2 r22^5 + 16 r22^6 +
      Sqrt[(-4 (3 r2^4 - 12 r2^3 r22 + 18 r2^2 r22^2 - 12 r2 r22^3 + 4 r22^4 +
          3 r1 r2 (r2^2 - 3 r2 r22 + 2 r22^2))^3 + r22^4 (9 r1 r2 (r2^2 - 3 r2 r22 + 2 r22^2) -
          2 (9 r2^4 - 36 r2^3 r22 + 54 r2^2 r22^2 - 36 r2 r22^3 + 8 r22^4))^2)]^(1/3))]}
```

In[•]:=
```
{r12 → 1/(192 (r2 - r22))
  (32 (3 r2^2 - 6 r2 r22 + 2 r22^2) + (16 i 2^(1/3) (i + Sqrt[3]) (3 r2^3 (r1 + r2) - 3 r2^2 (3 r1 + 4 r2)
        r22 + 6 r2 (r1 + 3 r2) r22^2 - 12 r2 r22^3 + 4 r22^4))/ (18 r2^4 r22^2 + 16 r22^6 +
    27 r2^2 r22^3 (r1 + 4 r22) - 18 r2 r22^4 (r1 + 4 r22) - 9 r2^3 r22^2 (r1 + 8 r22) +
    Sqrt[(-4 (3 r2^3 (r1 + r2) - 3 r2^2 (3 r1 + 4 r2) r22 + 6 r2 (r1 + 3 r2) r22^2 -
          12 r2 r22^3 + 4 r22^4)^3 + r22^4 (9 r1 r2 (r2 - 2 r22) (r2 - r22) -
          2 (3 r2^2 - 6 r2 r22 + 2 r22^2) (3 r2^2 - 6 r2 r22 + 4 r22^2))^2)]^(1/3) -
  8 × 2^(2/3) (1 + i Sqrt[3]) (18 r2^4 r22^2 + 16 r22^6 + 27 r2^2 r22^3 (r1 + 4 r22) -
    18 r2 r22^4 (r1 + 4 r22) - 9 r2^3 r22^2 (r1 + 8 r22) +
    Sqrt[(-4 (3 r2^3 (r1 + r2) - 3 r2^2 (3 r1 + 4 r2) r22 + 6 r2 (r1 + 3 r2) r22^2 - 12 r2 r22^3 +
          4 r22^4)^3 + r22^4 (9 r1 r2 (r2 - 2 r22) (r2 - r22) - 2 (3 r2^2 - 6 r2 r22 +
          2 r22^2) (3 r2^2 - 6 r2 r22 + 4 r22^2))^2)]^(1/3))} /. ex /. r22 → 0.24
```

Out[•]=
```
{r12 → -0.0336525 - 0.106938 i}
```

In[•]:=
```
sol2 /. ex /. r22 → 0.23
```

Out[•]=
```
{{r12 → -0.0828427}, {r12 → 0.482843}, {r12 → -0.0706438}, {r12 → 0.240644},
 {r12 → 0.143166}, {r12 → -0.0218773 + 0.0889001 i}, {r12 → -0.0218773 - 0.0889001 i}}
```

In[•]:=
```
eq = FullSimplify[term1 - term2]
```

Out[•]=
$$\frac{r1}{4} + r12 - r22 + \frac{(r1 + 4\,r12)\,r22^2}{r1\,r2 + 4\,r12\,(-r12 + r2)} + \frac{r12^2}{-r2 + r22} - \frac{r3}{4}$$

```
In[•]:= eq3 = FullSimplify[e3]
```

$$\text{Out[•]=}\quad \frac{-1+\frac{2\,r12}{r2-r22}}{-1+\frac{2\,r12}{r2-r22}-\frac{8\,r12\,(r1+2\,r12)\,r22^2}{(r1\,r2+4\,r12\,(-r12+r2))^2}} == \frac{r12^2}{(r2-r22)^2\left(1+\frac{r12^2}{(r2-r22)^2}-\frac{2\,(r1+4\,r12)\,r22}{r1\,r2+4\,r12\,(-r12+r2)}\right)}$$

```
In[•]:= NSolve[{eq == 0 /. ex, eq3 /. ex}, {r12, r22}]
```

```
Out[•]= {{r22 → 7755.25 - 1199.97 i, r12 → -0.100001 - 1.94884 × 10^-7 i},
  {r22 → -0.0124558 + 0.106249 i, r12 → -0.0846366 + 0.00206376 i},
  {r22 → -0.0124558 - 0.106249 i, r12 → -0.0846366 - 0.00206376 i},
  {r22 → 0.183165, r12 → 0.0615317}, {r22 → 0.736729, r12 → 0.7769},
  {r22 → 0.638789, r12 → -0.326643},
  {r22 → 0.28612 - 0.0820165 i, r12 → 0.0607311 + 0.062323 i},
  {r22 → 0.28612 + 0.0820165 i, r12 → 0.0607311 - 0.062323 i},
  {r22 → 0.28612 + 0.0820165 i, r12 → 0.0607311 - 0.062323 i},
  {r22 → 0.0469944 - 0.372019 i, r12 → 0.468011 + 0.288141 i},
  {r22 → 0.0469944 + 0.372019 i, r12 → 0.468011 - 0.288141 i}, {r22 → 0.2, r12 → 0.2}}
```

Solutions to plot in R

```
In[•]:= xx = {r1 → 0.2, r3 → 0.1}
Out[•]= {r1 → 0.2, r3 → 0.1}

In[•]:= x = {r1 → 0.2, r2 → 0.7, r3 → 0.1}
Out[•]= {r1 → 0.2, r2 → 0.7, r3 → 0.1}

In[•]:= FindMinimum[
          {(-term1) /. x, term1 == term2 /. x, r12 + r22 < 0.7, r12 > 0, r22 > 0}, {r12, r22}][[2]]
Out[•]= {r12 → 0.184831, r22 → 0.238117}

In[•]:= nsol = NSolve[{eq == 0 /. x, eq3 /. x, r12 > 0, r22 > 0, r12 + r22 < r2 /. x}, {r12, r22}]
Out[•]= {{r22 → 0.647259, r12 → 0.0329464}, {r22 → 0.238117, r12 → 0.184831}}

In[•]:= {term1, term2} /. x /. nsol
Out[•]= {{0.0623653, 0.0623653}, {0.160867, 0.160867}}

In[•]:= NSolve[{eq == 0 /. {r1 → 0.2, r2 → 0.7, r3 → 0.1},
            eq3 /. {r1 → 0.2, r2 → 0.7, r3 → 0.1}}, {r12, r22}];


In[•]:= term1
        term2
```

$$\text{Out[•]=}\quad \frac{r1}{4}+r12+\frac{r12^2}{-r2+r22}$$

$$\text{Out[•]=}\quad r22-\frac{(r1+4\,r12)\,r22^2}{r1\,r2+4\,r12\,(-r12+r2)}+\frac{r3}{4}$$

In[•]:=
```
v = Range[0.4, 0.9, 0.01]

f[r2_] := FindMinimum[{-(-r1/4 + r12 + r12^2/(-r2 + r22)) /. {r1 → 0.1},
    r1/4 + r12 + r12^2/(-r2 + r22) == r22 - (r1 + 4 r12) r22^2/(r1 r2 + 4 r12 (-r12 + r2)) + (1 - r1 - r2)/4 /. {r1 → 0.1},
    r12 + r22 < r2, r12 > 0, r22 > 0}, {{r12, r2 / 3 /. ex}, {r22, r2 / 3 /. ex}}][[2]]

results1 = ({r12, r22} /. Map[f, v]) / v

Export[
 "/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results1.csv", results1]
```

Out[•]=
```
{0.4, 0.41, 0.42, 0.43, 0.44, 0.45, 0.46, 0.47, 0.48, 0.49, 0.5, 0.51,
 0.52, 0.53, 0.54, 0.55, 0.56, 0.57, 0.58, 0.59, 0.6, 0.61, 0.62, 0.63, 0.64,
 0.65, 0.66, 0.67, 0.68, 0.69, 0.7, 0.71, 0.72, 0.73, 0.74, 0.75, 0.76, 0.77,
 0.78, 0.79, 0.8, 0.81, 0.82, 0.83, 0.84, 0.85, 0.86, 0.87, 0.88, 0.89, 0.9}
```

Out[•]=
```
{{0.5, 9.57139 × 10^-18}, {0.495012, 0.00989618}, {0.490046, 0.0195931},
 {0.485097, 0.0291022}, {0.480161, 0.0384342}, {0.475236, 0.0475985},
 {0.470317, 0.0566038}, {0.465402, 0.0654578}, {0.460488, 0.0741673},
 {0.455574, 0.0827387}, {0.450656, 0.0911774}, {0.445733, 0.0994882},
 {0.440804, 0.107675}, {0.435867, 0.115742}, {0.430922, 0.123692}, {0.425968, 0.131528},
 {0.421005, 0.13925}, {0.416032, 0.146862}, {0.411051, 0.154363}, {0.406061, 0.161754},
 {0.401065, 0.169036}, {0.396062, 0.176208}, {0.391056, 0.183269}, {0.386047, 0.190218},
 {0.381039, 0.197055}, {0.376034, 0.203777}, {0.371035, 0.210383},
 {0.366045, 0.216871}, {0.361068, 0.22324}, {0.356108, 0.229486}, {0.351167, 0.235609},
 {0.346252, 0.241606}, {0.341365, 0.247475}, {0.33651, 0.253214}, {0.331693, 0.258823},
 {0.326917, 0.2643}, {0.322186, 0.269644}, {0.317503, 0.274855}, {0.312874, 0.279931},
 {0.308301, 0.284874}, {0.303787, 0.289682}, {0.299336, 0.294358}, {0.29495, 0.298902},
 {0.290632, 0.303315}, {0.286384, 0.307599}, {0.282207, 0.311756}, {0.278103, 0.315788},
 {0.274074, 0.319697}, {0.27012, 0.323485}, {0.266241, 0.327157}, {0.262438, 0.330714}}
```

Out[•]=
```
/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results1.csv
```

In[•]:=
```
v = Range[0.2, 0.7, 0.01]

f[r2_] := FindMinimum[{-(-r1/4 + r12 + r12^2/(-r2 + r22)) /. {r1 → 0.3},
    r1/4 + r12 + r12^2/(-r2 + r22) == r22 - (r1 + 4 r12) r22^2/(r1 r2 + 4 r12 (-r12 + r2)) + (1 - r1 - r2)/4 /. {r1 → 0.3},
    r12 + r22 < r2, r12 > 0, r22 > 0}, {{r12, r2 / 3 /. ex}, {r22, r2 / 3 /. ex}}][[2]]

results2 = ({r12, r22} /. Map[f, v]) / v

Export["/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results2.csv",
 results2]
```

Out[•]=
```
{0.2, 0.21, 0.22, 0.23, 0.24, 0.25, 0.26, 0.27, 0.28, 0.29, 0.3, 0.31,
 0.32, 0.33, 0.34, 0.35, 0.36, 0.37, 0.38, 0.39, 0.4, 0.41, 0.42, 0.43, 0.44,
 0.45, 0.46, 0.47, 0.48, 0.49, 0.5, 0.51, 0.52, 0.53, 0.54, 0.55, 0.56, 0.57,
 0.58, 0.59, 0.6, 0.61, 0.62, 0.63, 0.64, 0.65, 0.66, 0.67, 0.68, 0.69, 0.7}
```

Out[•]=
```
{{0.5, 9.4495 × 10^-18}, {0.490186, 0.019471}, {0.480706, 0.0379681}, {0.47151, 0.0556027},
 {0.462555, 0.0724689}, {0.453806, 0.0886466}, {0.445229, 0.104204},
 {0.436797, 0.119199}, {0.428487, 0.133681}, {0.420277, 0.147693}, {0.412149, 0.161269},
 {0.404088, 0.17444}, {0.396082, 0.187229}, {0.388118, 0.199657}, {0.38019, 0.211738},
 {0.372291, 0.223484}, {0.364417, 0.234902}, {0.356568, 0.245995}, {0.348744, 0.256764},
 {0.34095, 0.267208}, {0.333192, 0.277321}, {0.325478, 0.287098}, {0.317819, 0.296532},
 {0.310229, 0.305614}, {0.302721, 0.314336}, {0.295311, 0.322692}, {0.288015, 0.330677},
 {0.280849, 0.338288}, {0.273828, 0.345522}, {0.266967, 0.352384}, {0.260278, 0.358877},
 {0.253771, 0.36501}, {0.247456, 0.370792}, {0.241337, 0.376236}, {0.235419, 0.381356},
 {0.229704, 0.386167}, {0.224191, 0.390686}, {0.218878, 0.394928}, {0.213763, 0.398912},
 {0.208841, 0.402652}, {0.204106, 0.406166}, {0.199552, 0.409468}, {0.195174, 0.412574},
 {0.190965, 0.415497}, {0.186917, 0.41825}, {0.183025, 0.420846}, {0.17928, 0.423295},
 {0.175676, 0.425609}, {0.172208, 0.427797}, {0.168868, 0.429868}, {0.16565, 0.43183}}
```

Out[•]=
```
/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results2.csv
```

In[•]:=
```
v = Range[0.1, 0.6, 0.01]

f[r2_] := FindMinimum[{-(-r1/4 + r12 + r12^2/(-r2 + r22)) /. {r1 → 0.4},
    r1/4 + r12 + r12^2/(-r2 + r22) == r22 - (r1 + 4 r12) r22^2/(r1 r2 + 4 r12 (-r12 + r2)) + (1 - r1 - r2)/4 /. {r1 → 0.4},
    r12 + r22 < r2, r12 > 0, r22 > 0}, {{r12, r2 / 3 /. ex}, {r22, r2 / 3 /. ex}}][[2]]

results3 = ({r12, r22} /. Map[f, v]) / v

Export[
 "/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results3.csv", results3]
```

Out[•]=
```
{0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21,
 0.22, 0.23, 0.24, 0.25, 0.26, 0.27, 0.28, 0.29, 0.3, 0.31, 0.32, 0.33, 0.34,
 0.35, 0.36, 0.37, 0.38, 0.39, 0.4, 0.41, 0.42, 0.43, 0.44, 0.45, 0.46, 0.47,
 0.48, 0.49, 0.5, 0.51, 0.52, 0.53, 0.54, 0.55, 0.56, 0.57, 0.58, 0.59, 0.6}
```

Out[•]=
```
{{0.5, 0.}, {0.48098, 0.0377298}, {0.463606, 0.0715795}, {0.447502, 0.102335},
 {0.432383, 0.130579}, {0.418029, 0.156753}, {0.404266, 0.181198}, {0.390951, 0.204176},
 {0.377971, 0.225889}, {0.365227, 0.24649}, {0.352641, 0.266093}, {0.34015, 0.284773},
 {0.327707, 0.302568}, {0.315285, 0.319486}, {0.302877, 0.335501}, {0.290505, 0.350562},
 {0.278221, 0.364598}, {0.266103, 0.377533}, {0.254253, 0.389306}, {0.242782, 0.399886},
 {0.231791, 0.409285}, {0.221362, 0.417557}, {0.211546, 0.42479}, {0.202362, 0.431094},
 {0.193807, 0.436583}, {0.185857, 0.441369}, {0.178476, 0.445556}, {0.171626, 0.449232},
 {0.165262, 0.452475}, {0.159343, 0.45535}, {0.153829, 0.457912}, {0.148684, 0.460205},
 {0.143875, 0.462267}, {0.13937, 0.464131}, {0.135143, 0.465822}, {0.131169, 0.467362},
 {0.127427, 0.468771}, {0.123897, 0.470064}, {0.120562, 0.471255}, {0.117405, 0.472354},
 {0.114413, 0.473373}, {0.111573, 0.474319}, {0.108874, 0.475201}, {0.106305, 0.476023},
 {0.103858, 0.476793}, {0.101522, 0.477514}, {0.099292, 0.478192},
 {0.0971595, 0.47883}, {0.0951185, 0.479432}, {0.0931631, 0.48}, {0.091288, 0.480537}}
```

Out[•]=
```
/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results3.csv
```

In[•]:=
```
v = Range[0.01, 0.49, 0.002]

f[r2_] := FindMinimum[{-(-r1/4 + r12 + r12^2/(-r2 + r22)) /. {r1 → 0.49},
    r1/4 + r12 + r12^2/(-r2 + r22) == r22 - (r1 + 4 r12) r22^2/(r1 r2 + 4 r12 (-r12 + r2)) + (1 - r1 - r2)/4 /. {r1 → 0.49},
    r12 + r22 < r2, r12 > 0, r22 > 0}, {{r12, r2 / 20 /. ex}, {r22, r2 / 2 /. ex}}][[2]]

results4 = ({r12, r22} /. Map[f, v]) / v

Export[
 "/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results4.csv", results4]
```

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```

Out[•]= /Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results4.csv

Computations on case 3 with Lagrange Multipliers

Fixed sample sizes in period 1 and 2

Set (and simplify) conditions

```
In[]:= subst = {r11 → r1 / 2 , r01 → r1 / 2 , r23 → r3 / 2 , r03 → r3 / 2, r02 → r2 - r12 - r22 }
```

$$\text{Out[]}= \left\{r11 \to \frac{r1}{2}, r01 \to \frac{r1}{2}, r23 \to \frac{r3}{2}, r03 \to \frac{r3}{2}, r02 \to -r12 + r2 - r22\right\}$$

```
In[]:= ex = {r1 → 0.1, r2 → 0.8, r3 → 0.1}
```

$$\text{Out[]}= \{r1 \to 0.1, r2 \to 0.8, r3 \to 0.1\}$$

Define terms to optimise

Note: *sigma*term1^(-1)/N* is the variance of the estimator of effect 1 (analogously *sigma*term2^(-1)/N* for effect 2). But since sigma and N are fixed, we simply work on term1 and term2 expressions. furthermore we set NT=1.

```
In[]:= term1 = FullSimplify[(r11 * r01 / (r11 + r01)) + (r12 * r02 / (r12 + r02)) /. subst]
```

$$\text{Out[]}= \frac{r1}{4} + r12 + \frac{r12^2}{-r2 + r22}$$

```
In[]:= term2 = FullSimplify[(r22 * r02 / (r22 + r02)) + (r23 * r03 / (r23 + r03)) /. subst]
```

$$\text{Out[]}= r22 + \frac{r22^2}{r12 - r2} + \frac{r3}{4}$$

```
In[]:= constr = term1 - term2;
```

```
In[]:= e1 = Solve[D[term1, r12] == l D[constr, r12], l]
       e2 = Solve[D[term1, r22] == l D[constr, r22], l]
```

$$\text{Out[]}= \left\{\left\{l \to \frac{(r12 - r2)^2 (2\,r12 - r2 + r22)}{2\,r12^3 - 5\,r12^2\,r2 + 4\,r12\,r2^2 - r2^3 + r12^2\,r22 - 2\,r12\,r2\,r22 + r2^2\,r22 - r2\,r22^2 + r22^3}\right\}\right\}$$

$$\text{Out[]}= \left\{\left\{l \to \frac{r12^2 (r12 - r2)}{r12^3 - r12^2\,r2 + r12\,r2^2 - r2^3 - 2\,r12\,r2\,r22 + 4\,r2^2\,r22 + r12\,r22^2 - 5\,r2\,r22^2 + 2\,r22^3}\right\}\right\}$$

```
In[]:= e3 = e1[[1]][[1]][[2]] == e2[[1]][[1]][[2]]
```

$$\text{Out[]}= \frac{(r12 - r2)^2 (2\,r12 - r2 + r22)}{2\,r12^3 - 5\,r12^2\,r2 + 4\,r12\,r2^2 - r2^3 + r12^2\,r22 - 2\,r12\,r2\,r22 + r2^2\,r22 - r2\,r22^2 + r22^3} == \frac{r12^2 (r12 - r2)}{r12^3 - r12^2\,r2 + r12\,r2^2 - r2^3 - 2\,r12\,r2\,r22 + 4\,r2^2\,r22 + r12\,r22^2 - 5\,r2\,r22^2 + 2\,r22^3}$$

In[]:=
```
sol = Solve[e3, {r12}][[2]]
```

Out[]=
$$\left\{\text{r12} \to \frac{\text{r2}\ (\text{r2} - 2\ \text{r22})}{2\ (\text{r2} - \text{r22})}\right\}$$

In[]:=
```
solr22 = NSolve[term1 - term2 /. sol /. ex][[6]]
```

Out[]=
```
{r22 → 0.234315}
```

In[]:=
```
solr12 = r12 /. sol /. solr22 /. ex
```

Out[]=
```
0.234315
```

In[]:=
```
r2 - r22 - solr12 /. solr22 /. ex
```

Out[]=
```
0.331371
```

In[]:=
```
eq = FullSimplify[term1 - term2 /. sol] == 0
```

Out[]=
$$\frac{1}{4}\left(\text{r1} + \frac{\left(\text{r2}^2 - 4\ \text{r2}\ \text{r22} + 2\ \text{r22}^2\right)\ \left(\text{r2}^2 - 2\ \text{r2}\ \text{r22} + 2\ \text{r22}^2\right)^2}{\text{r2}^2\ (\text{r2} - \text{r22})^3} - \text{r3}\right) == 0$$

Note that then the solutions are r22 satisfying "**eq**" and r12 when substituting r22 in "**sol**".

In[]:=
```
CForm[eq[[1]]]
```

Out[]//CForm=
```
(r1 + ((Power(r2,2) - 4*r2*r22 + 2*Power(r22,2))*Power(Power(r2,2) - 2*r2*r22 + 2*Powe
```

In[]:=
```
CForm[sol[[1, 2]]]
```

Out[]//CForm=
```
(r2*(r2 - 2*r22))/(2.*(r2 - r22))
```