

Computations on case 2

Fixed sample sizes in period 1 and 2 with ncc

Set (and simplify) conditions

Note that here we assume $r_1 + r_2 = 1$, then $r_3 = 0$

In[1]:= **subst** = { **r01** -> **r1** - **r11** , **r02** -> (1 - **r1**) - **r12** - **r22** }

Out[1]= { **r01** -> **r1** - **r11**, **r02** -> 1 - **r1** - **r12** - **r22** }

In[2]:= **ex** = {**r1** -> 0.4}

Out[2]= { **r1** -> 0.4 }

Define terms to optimise

Note: $\sigma^2 \text{term1}^{(-1)}/N$ is the variance of the estimator of effect 1 (analogously $\sigma^2 \text{term2}^{(-1)}/N$ for effect 2). But since σ and N are fixed, we simply work on **term1** and **term2** expressions.

In[3]:= **term1** = FullSimplify[(**r11** * **r01** / (**r11** + **r01**)) + (**r12** * **r02** / (**r12** + **r02**)) /. **subst**]

Out[3]= **r11** - $\frac{r_{11}^2}{r_1}$ + **r12** + $\frac{r_{12}^2}{-1 + r_1 + r_{22}}$

In[4]:= **term2** = FullSimplify[
(1 / **r22** + 1 / **r02** - ((1 / **r02**) ^ 2 / (1 / **r01** + 1 / **r02** + 1 / **r11** + 1 / **r12**))) ^ (-1) /. **subst**]

Out[4]= $-\frac{(-r_1^2 (r_{11} + r_{12}) + r_1 (r_{11} + r_{12}) (1 + r_{11} - r_{12} - r_{22}) + r_{11}^2 (-1 + r_{22})) r_{22}}{r_{11}^2 + r_1^2 (r_{11} + r_{12}) - r_1 (1 + r_{11} - r_{12}) (r_{11} + r_{12})}$

In[5]:= **sol** = Solve[**term1** == **term2**, **r12**][[1]]

Out[5]= $\left\{ r_{12} \rightarrow \frac{1}{2 r_1} \left(r_1 - r_1^2 - \sqrt{r_1} \sqrt{r_1 - 2 r_1^2 + r_1^3 + 4 r_1 r_{11} - 4 r_1^2 r_{11} - 4 r_{11}^2 + 4 r_1 r_{11}^2 - 4 r_1 r_{22} + 4 r_1^2 r_{22} + 4 r_1 r_{22}^2} \right) \right\}$

In[6]:= **term3** = FullSimplify[**term1** /. **sol**]

Out[6]= $\frac{1}{2 r_1 (-1 + r_1 + r_{22})} r_{22} \left(r_1^2 - 2 r_{11}^2 + r_1 (-1 + 2 r_{11} + 2 r_{22}) - \sqrt{r_1} \sqrt{(-1 + r_1) ((-1 + r_1) r_1 - 4 r_1 r_{11} + 4 r_{11}^2) + 4 (-1 + r_1) r_1 r_{22} + 4 r_1 r_{22}^2} \right)$

In[7]:= **e1 = FullSimplify[D[term3, r22]]**

$$\text{Out[7]} = \frac{1}{2 r_1 (-1 + r_1 + r_{22})^2} \left(r_{22} (-1 + r_1 + r_{22}) \right. \\ \left. \left(2 r_1 - \frac{2 r_1^{3/2} (-1 + r_1 + 2 r_{22})}{\sqrt{(-1 + r_1) ((-1 + r_1) r_1 - 4 r_1 r_{11} + 4 r_{11}^2) + 4 (-1 + r_1) r_1 r_{22} + 4 r_1 r_{22}^2}} \right) - \right. \\ \left. r_{22} (r_1^2 - 2 r_{11}^2 + r_1 (-1 + 2 r_{11} + 2 r_{22}) - \sqrt{r_1} \sqrt{(-1 + r_1) ((-1 + r_1) r_1 - 4 r_1 r_{11} + 4 r_{11}^2) + 4 (-1 + r_1) r_1 r_{22} + 4 r_1 r_{22}^2}) + \right. \\ \left. (-1 + r_1 + r_{22}) (r_1^2 - 2 r_{11}^2 + r_1 (-1 + 2 r_{11} + 2 r_{22}) - \sqrt{r_1} \sqrt{(-1 + r_1) ((-1 + r_1) r_1 - 4 r_1 r_{11} + 4 r_{11}^2) + 4 (-1 + r_1) r_1 r_{22} + 4 r_1 r_{22}^2}) \right)$$

In[8]:= **e2 = FullSimplify[D[term3, r11]]**

$$\text{Out[8]} = \frac{r_{22} \left(r_1 - 2 r_{11} + \frac{(-1 + r_1) \sqrt{r_1} (r_1 - 2 r_{11})}{\sqrt{(-1 + r_1) ((-1 + r_1) r_1 - 4 r_1 r_{11} + 4 r_{11}^2) + 4 (-1 + r_1) r_1 r_{22} + 4 r_1 r_{22}^2}} \right)}{r_1 (-1 + r_1 + r_{22})}$$

In[9]:= **sol2 = Solve[{r22 (-1 + r1 + r22)**

$$\left(2 r_1 - \frac{2 r_1^{3/2} (-1 + r_1 + 2 r_{22})}{\sqrt{(-1 + r_1) ((-1 + r_1) r_1 - 4 r_1 r_{11} + 4 r_{11}^2) + 4 (-1 + r_1) r_1 r_{22} + 4 r_1 r_{22}^2}} \right) - \\ r_{22} (r_1^2 - 2 r_{11}^2 + r_1 (-1 + 2 r_{11} + 2 r_{22}) - \sqrt{r_1} \sqrt{(-1 + r_1) ((-1 + r_1) r_1 - 4 r_1 r_{11} + 4 r_{11}^2) + 4 (-1 + r_1) r_1 r_{22} + 4 r_1 r_{22}^2}) + \\ (-1 + r_1 + r_{22}) (r_1^2 - 2 r_{11}^2 + r_1 (-1 + 2 r_{11} + 2 r_{22}) - \sqrt{r_1} \sqrt{(-1 + r_1) ((-1 + r_1) r_1 - 4 r_1 r_{11} + 4 r_{11}^2) + 4 (-1 + r_1) r_1 r_{22} + 4 r_1 r_{22}^2}) == \\ 0, r_{22} \left(r_1 - 2 r_{11} + \frac{(-1 + r_1) \sqrt{r_1} (r_1 - 2 r_{11})}{\sqrt{(-1 + r_1) ((-1 + r_1) r_1 - 4 r_1 r_{11} + 4 r_{11}^2) + 4 (-1 + r_1) r_1 r_{22} + 4 r_1 r_{22}^2}} \right) == \\ 0\}, \{r_{11}, r_{22}\}];$$

 **Solve:** There may be values of the parameters for which some or all solutions are not valid.

The solutions are then the following

In[26]:= **sol2[[11]]**

$$\text{Out[26]} = \left\{ r_{11} \rightarrow \frac{r_1}{2}, \right.$$

$$\begin{aligned}
r_{22} \rightarrow & 1 - r_1 + \frac{1}{2} \sqrt{\left(4(-1 + r_1)^2 + \frac{1}{4}(-22 + 43r_1 - 21r_1^2) + \frac{-22 + 65r_1 - 64r_1^2 + 21r_1^3}{12(-1 + r_1)} + \right.} \\
& \left. \left(2^{1/3} (4 - 40r_1 + 129r_1^2 - 196r_1^3 + 154r_1^4 - 60r_1^5 + 9r_1^6) \right) \right) / \left(3(-1 + r_1) \right. \\
& \left. \left(1024 - 1536r_1 - 26112r_1^2 + 135040r_1^3 - 304896r_1^4 + 391296r_1^5 - 303104r_1^6 + \right. \right. \\
& \left. \left. 139392r_1^7 - 34560r_1^8 + 3456r_1^9 + \sqrt{\left(28311552r_1 - 467140608r_1^2 + \right. \right. \right. \\
& \left. \left. \left. 3588489216r_1^3 - 16978083840r_1^4 + 55228760064r_1^5 - 130682585088r_1^6 + \right. \right. \right. \\
& \left. \left. \left. 232168882176r_1^7 - 315144732672r_1^8 + 329334128640r_1^9 - \right. \right. \right. \\
& \left. \left. \left. 264790867968r_1^{10} + 162331361280r_1^{11} - 74439917568r_1^{12} + 24680595456 \right. \right. \right. \\
& \left. \left. \left. r_1^{13} - 5573836800r_1^{14} + 764411904r_1^{15} - 47775744r_1^{16} \right) \right)^{1/3} \right) + \\
& \frac{1}{48 \times 2^{1/3}(-1 + r_1)} \left(1024 - 1536r_1 - 26112r_1^2 + 135040r_1^3 - 304896r_1^4 + \right. \\
& \left. 391296r_1^5 - 303104r_1^6 + 139392r_1^7 - 34560r_1^8 + 3456r_1^9 + \right. \\
& \left. \sqrt{\left(28311552r_1 - 467140608r_1^2 + 3588489216r_1^3 - 16978083840r_1^4 + \right. \right. \right. \\
& \left. \left. \left. 55228760064r_1^5 - 130682585088r_1^6 + 232168882176r_1^7 - \right. \right. \right. \\
& \left. \left. \left. 315144732672r_1^8 + 329334128640r_1^9 - 264790867968r_1^{10} + \right. \right. \right. \\
& \left. \left. \left. 162331361280r_1^{11} - 74439917568r_1^{12} + 24680595456r_1^{13} - \right. \right. \right. \\
& \left. \left. \left. 5573836800r_1^{14} + 764411904r_1^{15} - 47775744r_1^{16} \right) \right)^{1/3} \right) - \\
& \frac{1}{2} \sqrt{\left(8(-1 + r_1)^2 + \frac{1}{4}(-22 + 43r_1 - 21r_1^2) - \frac{-22 + 65r_1 - 64r_1^2 + 21r_1^3}{12(-1 + r_1)} - \right.} \\
& \left. \left(2^{1/3} (4 - 40r_1 + 129r_1^2 - 196r_1^3 + 154r_1^4 - 60r_1^5 + 9r_1^6) \right) \right) / \left(3(-1 + r_1) \right. \\
& \left. \left(1024 - 1536r_1 - 26112r_1^2 + 135040r_1^3 - 304896r_1^4 + 391296r_1^5 - 303104r_1^6 + \right. \right. \\
& \left. \left. 139392r_1^7 - 34560r_1^8 + 3456r_1^9 + \sqrt{\left(28311552r_1 - 467140608r_1^2 + \right. \right. \right. \\
& \left. \left. \left. 3588489216r_1^3 - 16978083840r_1^4 + 55228760064r_1^5 - 130682585088r_1^6 + \right. \right. \right. \\
& \left. \left. \left. 232168882176r_1^7 - 315144732672r_1^8 + 329334128640r_1^9 - \right. \right. \right. \\
& \left. \left. \left. 264790867968r_1^{10} + 162331361280r_1^{11} - 74439917568r_1^{12} + 24680595456 \right. \right. \right. \\
& \left. \left. \left. r_1^{13} - 5573836800r_1^{14} + 764411904r_1^{15} - 47775744r_1^{16} \right) \right)^{1/3} \right) - \\
& \frac{1}{48 \times 2^{1/3}(-1 + r_1)} \left(1024 - 1536r_1 - 26112r_1^2 + 135040r_1^3 - 304896r_1^4 + \right. \\
& \left. 391296r_1^5 - 303104r_1^6 + 139392r_1^7 - 34560r_1^8 + 3456r_1^9 + \right. \\
& \left. \sqrt{\left(28311552r_1 - 467140608r_1^2 + 3588489216r_1^3 - 16978083840r_1^4 + \right. \right. \right. \\
& \left. \left. \left. 55228760064r_1^5 - 130682585088r_1^6 + 232168882176r_1^7 - \right. \right. \right. \\
& \left. \left. \left. 315144732672r_1^8 + 329334128640r_1^9 - 264790867968r_1^{10} + \right. \right. \right. \\
& \left. \left. \left. 162331361280r_1^{11} - 74439917568r_1^{12} + 24680595456r_1^{13} - \right. \right. \right. \\
& \left. \left. \left. 5573836800r_1^{14} + 764411904r_1^{15} - 47775744r_1^{16} \right) \right)^{1/3} + \\
& \left(-64(-1 + r_1)^3 + 4(-1 + r_1)(22 - 43r_1 + 21r_1^2) - 6(-4 + 11r_1 - 10r_1^2 + 3r_1^3) \right) / \\
& \left(4 \sqrt{\left(4(-1 + r_1)^2 + \frac{1}{4}(-22 + 43r_1 - 21r_1^2) + \frac{-22 + 65r_1 - 64r_1^2 + 21r_1^3}{12(-1 + r_1)} + \right. \right. \\
& \left. \left. \left(2^{1/3} (4 - 40r_1 + 129r_1^2 - 196r_1^3 + 154r_1^4 - 60r_1^5 + 9r_1^6) \right) \right) / \right.
\end{aligned}$$

$$\begin{aligned}
& \left(3 (-1 + r_1) \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \right. \\
& \quad 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \\
& \quad \sqrt{\left(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right.} \\
& \quad \quad 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \\
& \quad \quad 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \\
& \quad \quad 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \\
& \quad \quad \left. \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16} \right) \right)^{1/3} \Bigg) + \\
& \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \\
& \quad 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \\
& \quad \sqrt{\left(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right.} \\
& \quad \quad 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \\
& \quad \quad 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \\
& \quad \quad 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \\
& \quad \quad \left. \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16} \right) \right)^{1/3} \Bigg) \Bigg) \Bigg) \Bigg\}
\end{aligned}$$

In[24]:= sol[[1]] /. sol2[[11]][2]

Out[24]= $r_{12} \rightarrow \frac{1}{2 r_1}$

$$\begin{aligned}
& \left(r_1 - r_1^2 - \sqrt{r_1} \sqrt{\left(r_1 - 2 r_1^2 + r_1^3 - 4 r_1 \left(1 - r_1 + \frac{1}{2} \sqrt{\left(4 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) + \right.} \right. \right. \right. \\
& \quad \left. \left. \left. \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} + \left(2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - \right. \right. \right. \right. \\
& \quad \left. \left. \left. 60 r_1^5 + 9 r_1^6) \right) \right) \right) \Bigg) \Bigg) \Bigg) \Bigg/ \left(3 (-1 + r_1) \left(1024 - 1536 r_1 - 26112 r_1^2 + \right. \right. \\
& \quad 135040 r_1^3 - 304896 r_1^4 + 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - \\
& \quad 34560 r_1^8 + 3456 r_1^9 + \sqrt{\left(28311552 r_1 - 467140608 r_1^2 + \right.} \\
& \quad \quad 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 r_1^5 - \\
& \quad \quad 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + \\
& \quad \quad 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - \\
& \quad \quad 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + \\
& \quad \quad \left. \left. 764411904 r_1^{15} - 47775744 r_1^{16} \right) \right)^{1/3} \Bigg) + \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \\
& \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + 391296 r_1^5 - \right. \\
& \quad 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \\
& \quad \sqrt{\left(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right.} \\
& \quad \quad 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \\
& \quad \quad 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \\
& \quad \quad 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \\
& \quad \quad \left. \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16} \right) \right)^{1/3} \Bigg) -
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \sqrt{\left(8 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) - \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} - \right.} \\
& \quad \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) / \\
& \quad \left(3 (-1 + r_1) (1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \\
& \quad 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \\
& \quad \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \\
& \quad 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \\
& \quad 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \\
& \quad 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \\
& \quad 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})}^{1/3} \Big) - \\
& \quad \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - \right. \\
& \quad 304896 r_1^4 + 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \\
& \quad \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \\
& \quad 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \\
& \quad 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \\
& \quad 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \\
& \quad 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})}^{1/3} + \\
& \quad \left. (-64 (-1 + r_1)^3 + 4 (-1 + r_1) (22 - 43 r_1 + 21 r_1^2) - \right. \\
& \quad \left. 6 (-4 + 11 r_1 - 10 r_1^2 + 3 r_1^3)) \right) / \\
& \quad \left(4 \sqrt{\left(4 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) + \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} + \right.} \right. \\
& \quad \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) / \\
& \quad \left(3 (-1 + r_1) (1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \\
& \quad 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \\
& \quad \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - \\
& \quad 16978083840 r_1^4 + 55228760064 r_1^5 - 130682585088 r_1^6 + \\
& \quad 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - \\
& \quad 264790867968 r_1^{10} + 162331361280 r_1^{11} - \\
& \quad 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + \\
& \quad 764411904 r_1^{15} - 47775744 r_1^{16})}^{1/3} \Big) + \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \\
& \quad \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + 391296 \right. \\
& \quad r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \sqrt{(28311552} \\
& \quad r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \\
& \quad 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \\
& \quad 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \\
& \quad 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \\
& \quad 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})}^{1/3} \Big) \Big) \Big) \Big) +
\end{aligned}$$

$$\begin{aligned}
& 4 r_1^2 \left(1 - r_1 + \frac{1}{2} \sqrt[3]{\left(4 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) + \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} + \right. \right. \\
& \quad \left. \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) \right) / \\
& \quad \left(3 (-1 + r_1) \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \right. \\
& \quad \left. \left. 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \right. \\
& \quad \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right. \\
& \quad \left. 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \right. \\
& \quad \left. 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \right. \\
& \quad \left. 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \right. \\
& \quad \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} \Big) + \\
& \quad \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - \right. \\
& \quad \left. 304896 r_1^4 + 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \\
& \quad \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right. \\
& \quad \left. 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \right. \\
& \quad \left. 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \right. \\
& \quad \left. 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \right. \\
& \quad \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} \Big) - \\
& \quad \frac{1}{2} \sqrt[3]{\left(8 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) - \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} - \right. \\
& \quad \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) / \\
& \quad \left(3 (-1 + r_1) \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \right. \\
& \quad \left. \left. 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \right. \\
& \quad \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right. \\
& \quad \left. 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \right. \\
& \quad \left. 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \right. \\
& \quad \left. 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \right. \\
& \quad \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} \Big) - \\
& \quad \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - \right. \\
& \quad \left. 304896 r_1^4 + 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \\
& \quad \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right. \\
& \quad \left. 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \right. \\
& \quad \left. 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \right. \\
& \quad \left. 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \right. \\
& \quad \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} + \\
& \quad (-64 (-1 + r_1)^3 + 4 (-1 + r_1) (22 - 43 r_1 + 21 r_1^2) - \\
& \quad \left. 6 (-4 + 11 r_1 - 10 r_1^2 + 3 r_1^3) \right) /
\end{aligned}$$

$$\begin{aligned}
& \left(4 \sqrt[3]{4(-1+r_1)^2 + \frac{1}{4}(-22+43r_1-21r_1^2) + \frac{-22+65r_1-64r_1^2+21r_1^3}{12(-1+r_1)}} + \right. \\
& \quad \left. (2^{1/3}(4-40r_1+129r_1^2-196r_1^3+154r_1^4-60r_1^5+9r_1^6)) \right) / \\
& \quad \left(3(-1+r_1) \left(1024-1536r_1-26112r_1^2+135040r_1^3-304896r_1^4 + \right. \right. \\
& \quad \quad 391296r_1^5-303104r_1^6+139392r_1^7-34560r_1^8+3456r_1^9 + \\
& \quad \quad \sqrt{(28311552r_1-467140608r_1^2+3588489216r_1^3-} \\
& \quad \quad \quad 16978083840r_1^4+55228760064r_1^5-130682585088r_1^6+ \\
& \quad \quad \quad 232168882176r_1^7-315144732672r_1^8+329334128640r_1^9- \\
& \quad \quad \quad 264790867968r_1^{10}+162331361280r_1^{11}- \\
& \quad \quad \quad 74439917568r_1^{12}+24680595456r_1^{13}-5573836800r_1^{14}+ \\
& \quad \quad \quad \left. \left. 764411904r_1^{15}-47775744r_1^{16}\right) \right)^{1/3} \Bigg) + \frac{1}{48 \times 2^{1/3}(-1+r_1)} \\
& \quad \left(1024-1536r_1-26112r_1^2+135040r_1^3-304896r_1^4+391296 \right. \\
& \quad \quad r_1^5-303104r_1^6+139392r_1^7-34560r_1^8+3456r_1^9 + \sqrt{(28311552} \\
& \quad \quad \quad r_1-467140608r_1^2+3588489216r_1^3-16978083840r_1^4+ \\
& \quad \quad \quad 55228760064r_1^5-130682585088r_1^6+232168882176r_1^7- \\
& \quad \quad \quad 315144732672r_1^8+329334128640r_1^9-264790867968r_1^{10}+ \\
& \quad \quad \quad 162331361280r_1^{11}-74439917568r_1^{12}+24680595456r_1^{13}- \\
& \quad \quad \quad \left. \left. 5573836800r_1^{14}+764411904r_1^{15}-47775744r_1^{16}\right) \right)^{1/3} \Bigg) \Bigg) + \\
& 4r_1 \left(1-r_1 + \frac{1}{2} \sqrt[3]{4(-1+r_1)^2 + \frac{1}{4}(-22+43r_1-21r_1^2) + \frac{-22+65r_1-64r_1^2+21r_1^3}{12(-1+r_1)}} + \right. \\
& \quad \left. (2^{1/3}(4-40r_1+129r_1^2-196r_1^3+154r_1^4-60r_1^5+9r_1^6)) \right) / \\
& \quad \left(3(-1+r_1) \left(1024-1536r_1-26112r_1^2+135040r_1^3-304896r_1^4 + \right. \right. \\
& \quad \quad 391296r_1^5-303104r_1^6+139392r_1^7-34560r_1^8+3456r_1^9 + \\
& \quad \quad \sqrt{(28311552r_1-467140608r_1^2+3588489216r_1^3-16978083840} \\
& \quad \quad \quad r_1^4+55228760064r_1^5-130682585088r_1^6+232168882176 \\
& \quad \quad \quad r_1^7-315144732672r_1^8+329334128640r_1^9-264790867968 \\
& \quad \quad \quad r_1^{10}+162331361280r_1^{11}-74439917568r_1^{12}+24680595456 \\
& \quad \quad \quad \left. \left. r_1^{13}-5573836800r_1^{14}+764411904r_1^{15}-47775744r_1^{16}\right) \right)^{1/3} \Bigg) + \\
& \quad \frac{1}{48 \times 2^{1/3}(-1+r_1)} \left(1024-1536r_1-26112r_1^2+135040r_1^3- \right. \\
& \quad \quad 304896r_1^4+391296r_1^5-303104r_1^6+139392r_1^7-34560r_1^8+3456r_1^9 + \\
& \quad \quad \sqrt{(28311552r_1-467140608r_1^2+3588489216r_1^3-16978083840r_1^4+} \\
& \quad \quad \quad 55228760064r_1^5-130682585088r_1^6+232168882176r_1^7- \\
& \quad \quad \quad 315144732672r_1^8+329334128640r_1^9-264790867968r_1^{10}+ \\
& \quad \quad \quad 162331361280r_1^{11}-74439917568r_1^{12}+24680595456r_1^{13}- \\
& \quad \quad \quad \left. \left. 5573836800r_1^{14}+764411904r_1^{15}-47775744r_1^{16}\right) \right)^{1/3} \Bigg) -
\end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \sqrt{\left(8 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) - \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} - \right.} \\ & \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) / \\ & \left(3 (-1 + r_1) \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \right. \\ & \left. \left. 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \right. \\ & \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})} \right)^{1/3} \Big) - \\ & \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - \right. \\ & \left. 304896 r_1^4 + 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \\ & \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})} \right)^{1/3} + \\ & (-64 (-1 + r_1)^3 + 4 (-1 + r_1) (22 - 43 r_1 + 21 r_1^2) - \\ & 6 (-4 + 11 r_1 - 10 r_1^2 + 3 r_1^3)) / \\ & \left(4 \sqrt{\left(4 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) + \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} + \right. \right. \\ & \left. \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) / \right. \\ & \left. \left(3 (-1 + r_1) \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - \right. \right. \right. \\ & \left. \left. 304896 r_1^4 + 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - \right. \right. \\ & \left. \left. 34560 r_1^8 + 3456 r_1^9 + \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})} \right)^{1/3} \right) + \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \left(1024 - 1536 r_1 - \right. \\ & \left. 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + 391296 r_1^5 - \right. \\ & \left. 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})} \right)^{1/3} \Big) \end{aligned}$$

$$r1^{16})^{1/3})^2 + 4 r1 r11 - 4 r1^2 r11 - 4 r11^2 + 4 r1 r11^2)^2$$

In[23]:= **1 - r1 - sol2[[11]][2] - sol[[1]][2] /. sol2[[11]][2]**

$$\begin{aligned} \text{Out[23]} = & -\frac{1}{2} \sqrt{\left(4(-1+r1)^2 + \frac{1}{4}(-22+43r1-21r1^2) + \frac{-22+65r1-64r1^2+21r1^3}{12(-1+r1)} + \right.} \\ & \left. (2^{1/3}(4-40r1+129r1^2-196r1^3+154r1^4-60r1^5+9r1^6))\right) / \\ & \left(3(-1+r1) \left(1024-1536r1-26112r1^2+135040r1^3-304896r1^4+391296r1^5 - \right. \right. \\ & \quad 303104r1^6+139392r1^7-34560r1^8+3456r1^9 + \sqrt{(28311552r1-467140608r1^2+} \\ & \quad 3588489216r1^3-16978083840r1^4+55228760064r1^5-130682585088r1^6+ \\ & \quad 232168882176r1^7-315144732672r1^8+329334128640r1^9-264790867968 \\ & \quad r1^{10}+162331361280r1^{11}-74439917568r1^{12}+24680595456 \\ & \quad \left. r1^{13}-5573836800r1^{14}+764411904r1^{15}-47775744r1^{16})^{1/3}\right) + \\ & \frac{1}{48 \times 2^{1/3}(-1+r1)} \left(1024-1536r1-26112r1^2+135040r1^3-304896r1^4 + \right. \\ & \quad 391296r1^5-303104r1^6+139392r1^7-34560r1^8+3456r1^9 + \\ & \quad \sqrt{(28311552r1-467140608r1^2+3588489216r1^3-16978083840r1^4+} \\ & \quad 55228760064r1^5-130682585088r1^6+232168882176r1^7-315144732672r1^8+ \\ & \quad 329334128640r1^9-264790867968r1^{10}+162331361280r1^{11}-74439917568r1^{12}+ \\ & \quad \left. 24680595456r1^{13}-5573836800r1^{14}+764411904r1^{15}-47775744r1^{16})^{1/3}\right) + \\ & \frac{1}{2} \sqrt{\left(8(-1+r1)^2 + \frac{1}{4}(-22+43r1-21r1^2) - \frac{-22+65r1-64r1^2+21r1^3}{12(-1+r1)} - \right.} \\ & \left. (2^{1/3}(4-40r1+129r1^2-196r1^3+154r1^4-60r1^5+9r1^6))\right) / \\ & \left(3(-1+r1) \left(1024-1536r1-26112r1^2+135040r1^3-304896r1^4+391296r1^5 - \right. \right. \\ & \quad 303104r1^6+139392r1^7-34560r1^8+3456r1^9 + \sqrt{(28311552r1-467140608r1^2+} \\ & \quad 3588489216r1^3-16978083840r1^4+55228760064r1^5-130682585088r1^6+ \\ & \quad 232168882176r1^7-315144732672r1^8+329334128640r1^9-264790867968 \\ & \quad r1^{10}+162331361280r1^{11}-74439917568r1^{12}+24680595456 \\ & \quad \left. r1^{13}-5573836800r1^{14}+764411904r1^{15}-47775744r1^{16})^{1/3}\right) - \\ & \frac{1}{48 \times 2^{1/3}(-1+r1)} \left(1024-1536r1-26112r1^2+135040r1^3-304896r1^4 + \right. \\ & \quad 391296r1^5-303104r1^6+139392r1^7-34560r1^8+3456r1^9 + \\ & \quad \sqrt{(28311552r1-467140608r1^2+3588489216r1^3-16978083840r1^4+} \\ & \quad 55228760064r1^5-130682585088r1^6+232168882176r1^7-315144732672r1^8+ \\ & \quad 329334128640r1^9-264790867968r1^{10}+162331361280r1^{11}-74439917568r1^{12}+ \\ & \quad \left. 24680595456r1^{13}-5573836800r1^{14}+764411904r1^{15}-47775744r1^{16})^{1/3}\right) + \\ & (-64(-1+r1)^3 + 4(-1+r1)(22-43r1+21r1^2) - 6(-4+11r1-10r1^2+3r1^3)) / \end{aligned}$$

$$\begin{aligned}
& \left(4 \sqrt[3]{4 (-1 + r1)^2 + \frac{1}{4} (-22 + 43 r1 - 21 r1^2) + \frac{-22 + 65 r1 - 64 r1^2 + 21 r1^3}{12 (-1 + r1)}} + \right. \\
& \quad \left(2^{1/3} (4 - 40 r1 + 129 r1^2 - 196 r1^3 + 154 r1^4 - 60 r1^5 + 9 r1^6) \right) \Big/ \\
& \quad \left(3 (-1 + r1) (1024 - 1536 r1 - 26112 r1^2 + 135040 r1^3 - 304896 r1^4 + \right. \\
& \quad \quad 391296 r1^5 - 303104 r1^6 + 139392 r1^7 - 34560 r1^8 + 3456 r1^9 + \\
& \quad \quad \sqrt{(28311552 r1 - 467140608 r1^2 + 3588489216 r1^3 - 16978083840 r1^4 + \\
& \quad \quad \quad 55228760064 r1^5 - 130682585088 r1^6 + 232168882176 r1^7 - \\
& \quad \quad \quad 315144732672 r1^8 + 329334128640 r1^9 - 264790867968 r1^{10} + \\
& \quad \quad \quad 162331361280 r1^{11} - 74439917568 r1^{12} + 24680595456 r1^{13} - \\
& \quad \quad \quad 5573836800 r1^{14} + 764411904 r1^{15} - 47775744 r1^{16})}^{1/3}) \Big) + \\
& \quad \frac{1}{48 \times 2^{1/3} (-1 + r1)} \left(1024 - 1536 r1 - 26112 r1^2 + 135040 r1^3 - 304896 \right. \\
& \quad \quad r1^4 + 391296 r1^5 - 303104 r1^6 + 139392 r1^7 - 34560 r1^8 + 3456 r1^9 + \\
& \quad \quad \sqrt{(28311552 r1 - 467140608 r1^2 + 3588489216 r1^3 - 16978083840 r1^4 + \\
& \quad \quad \quad 55228760064 r1^5 - 130682585088 r1^6 + 232168882176 r1^7 - \\
& \quad \quad \quad 315144732672 r1^8 + 329334128640 r1^9 - 264790867968 r1^{10} + \\
& \quad \quad \quad 162331361280 r1^{11} - 74439917568 r1^{12} + 24680595456 r1^{13} - \\
& \quad \quad \quad 5573836800 r1^{14} + 764411904 r1^{15} - 47775744 r1^{16})}^{1/3}) \Big) \Big) - \\
& \quad \frac{1}{2 r1} \left(r1 - r1^2 - \sqrt{r1} \sqrt[3]{r1 - 2 r1^2 + r1^3 - 4 r1 \left(1 - r1 + \frac{1}{2} \sqrt[3]{4 (-1 + r1)^2 + \right.} \right. \right. \\
& \quad \quad \frac{1}{4} (-22 + 43 r1 - 21 r1^2) + \frac{-22 + 65 r1 - 64 r1^2 + 21 r1^3}{12 (-1 + r1)} + \\
& \quad \quad \left. \left. \left(2^{1/3} (4 - 40 r1 + 129 r1^2 - 196 r1^3 + 154 r1^4 - 60 r1^5 + 9 r1^6) \right) \right/ \right. \\
& \quad \quad \left. \left(3 (-1 + r1) (1024 - 1536 r1 - 26112 r1^2 + 135040 r1^3 - 304896 r1^4 + \right. \right. \\
& \quad \quad \quad 391296 r1^5 - 303104 r1^6 + 139392 r1^7 - 34560 r1^8 + 3456 r1^9 + \\
& \quad \quad \quad \sqrt{(28311552 r1 - 467140608 r1^2 + 3588489216 r1^3 - 16978083840 r1^4 + \\
& \quad \quad \quad \quad 55228760064 r1^5 - 130682585088 r1^6 + 232168882176 r1^7 - \\
& \quad \quad \quad \quad 315144732672 r1^8 + 329334128640 r1^9 - 264790867968 r1^{10} + \\
& \quad \quad \quad \quad 162331361280 r1^{11} - 74439917568 r1^{12} + 24680595456 r1^{13} - \\
& \quad \quad \quad \quad 5573836800 r1^{14} + 764411904 r1^{15} - 47775744 r1^{16})}^{1/3}) \Big) \Big) + \\
& \quad \frac{1}{48 \times 2^{1/3} (-1 + r1)} \left(1024 - 1536 r1 - 26112 r1^2 + 135040 r1^3 - \right. \\
& \quad \quad 304896 r1^4 + 391296 r1^5 - 303104 r1^6 + 139392 r1^7 - 34560 r1^8 + 3456 r1^9 + \\
& \quad \quad \sqrt{(28311552 r1 - 467140608 r1^2 + 3588489216 r1^3 - 16978083840 r1^4 + \\
& \quad \quad \quad 55228760064 r1^5 - 130682585088 r1^6 + 232168882176 r1^7 - \\
& \quad \quad \quad 315144732672 r1^8 + 329334128640 r1^9 - 264790867968 r1^{10} + \\
& \quad \quad \quad 162331361280 r1^{11} - 74439917568 r1^{12} + 24680595456 r1^{13} - \\
& \quad \quad \quad 5573836800 r1^{14} + 764411904 r1^{15} - 47775744 r1^{16})}^{1/3}) \Big) \Big) -
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \sqrt{\left(8 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) - \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} - \right.} \\
& \quad \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) / \left(3 (-1 + r_1) \right. \\
& \quad \left. (1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + 391296 r_1^5 - \right. \\
& \quad \left. 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \sqrt{(28311552 r_1 - \right. \\
& \quad \left. 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 \right. \\
& \quad \left. r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + \right. \\
& \quad \left. 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - \right. \\
& \quad \left. 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + \right. \\
& \quad \left. 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} - \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \\
& \quad \left. (1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + 391296 r_1^5 - \right. \\
& \quad \left. 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \sqrt{(28311552 r_1 - \right. \\
& \quad \left. 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 \right. \\
& \quad \left. r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + \right. \\
& \quad \left. 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - \right. \\
& \quad \left. 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + \right. \\
& \quad \left. 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} + (-64 (-1 + r_1)^3 + \\
& \quad \left. 4 (-1 + r_1) (22 - 43 r_1 + 21 r_1^2) - 6 (-4 + 11 r_1 - 10 r_1^2 + 3 r_1^3)) \right) / \\
& \quad \left(4 \sqrt{\left(4 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) + \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} + \right. \right. \\
& \quad \left. \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) \right) / \\
& \quad \left(3 (-1 + r_1) (1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \\
& \quad \left. 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \\
& \quad \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - \right. \\
& \quad \left. 16978083840 r_1^4 + 55228760064 r_1^5 - 130682585088 r_1^6 + \right. \\
& \quad \left. 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - \right. \\
& \quad \left. 264790867968 r_1^{10} + 162331361280 r_1^{11} - \right. \\
& \quad \left. 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + \right. \\
& \quad \left. 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} \right) + \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \\
& \quad \left. (1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + 391296 \right. \\
& \quad \left. r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \sqrt{(28311552 \right. \\
& \quad \left. r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right. \\
& \quad \left. 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \right. \\
& \quad \left. 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \right. \\
& \quad \left. 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \right. \\
& \quad \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} \right) \right) \right) +
\end{aligned}$$

$$\begin{aligned}
& 4 r_1^2 \left(1 - r_1 + \frac{1}{2} \sqrt[3]{\left(4 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) + \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} + \right. \right. \\
& \quad \left. \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) \right) / \\
& \quad \left(3 (-1 + r_1) \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \right. \\
& \quad \left. \left. 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \right. \\
& \quad \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right. \\
& \quad \left. 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \right. \\
& \quad \left. 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \right. \\
& \quad \left. 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \right. \\
& \quad \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} \Big) + \\
& \quad \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - \right. \\
& \quad \left. 304896 r_1^4 + 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \\
& \quad \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right. \\
& \quad \left. 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \right. \\
& \quad \left. 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \right. \\
& \quad \left. 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \right. \\
& \quad \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} \Big) - \\
& \quad \frac{1}{2} \sqrt[3]{\left(8 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) - \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} - \right. \\
& \quad \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) / \\
& \quad \left(3 (-1 + r_1) \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \right. \\
& \quad \left. \left. 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \right. \\
& \quad \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right. \\
& \quad \left. 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \right. \\
& \quad \left. 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \right. \\
& \quad \left. 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \right. \\
& \quad \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} \Big) - \\
& \quad \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - \right. \\
& \quad \left. 304896 r_1^4 + 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \right. \\
& \quad \left. \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + \right. \\
& \quad \left. 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - \right. \\
& \quad \left. 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + \right. \\
& \quad \left. 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - \right. \\
& \quad \left. 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16}) \right)^{1/3} + \\
& \quad (-64 (-1 + r_1)^3 + 4 (-1 + r_1) (22 - 43 r_1 + 21 r_1^2) - \\
& \quad \left. 6 (-4 + 11 r_1 - 10 r_1^2 + 3 r_1^3) \right) /
\end{aligned}$$

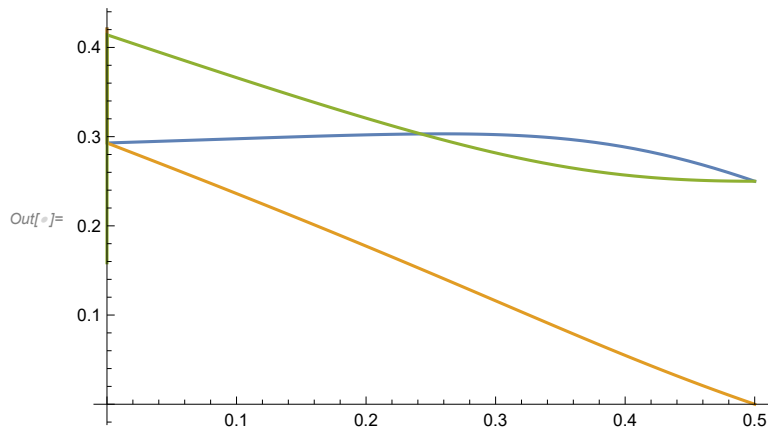
$$\begin{aligned}
& \left(4 \sqrt[3]{4(-1+r_1)^2 + \frac{1}{4}(-22+43r_1-21r_1^2) + \frac{-22+65r_1-64r_1^2+21r_1^3}{12(-1+r_1)}} + \right. \\
& \quad \left. (2^{1/3}(4-40r_1+129r_1^2-196r_1^3+154r_1^4-60r_1^5+9r_1^6)) \right) / \\
& \quad \left(3(-1+r_1) (1024-1536r_1-26112r_1^2+135040r_1^3-304896r_1^4+ \right. \\
& \quad 391296r_1^5-303104r_1^6+139392r_1^7-34560r_1^8+3456r_1^9+ \\
& \quad \sqrt{(28311552r_1-467140608r_1^2+3588489216r_1^3- \\
& \quad 16978083840r_1^4+55228760064r_1^5-130682585088r_1^6+ \\
& \quad 232168882176r_1^7-315144732672r_1^8+329334128640r_1^9- \\
& \quad 264790867968r_1^{10}+162331361280r_1^{11}- \\
& \quad 74439917568r_1^{12}+24680595456r_1^{13}-5573836800r_1^{14}+ \\
& \quad \left. 764411904r_1^{15}-47775744r_1^{16}))^{1/3}} + \frac{1}{48 \times 2^{1/3}(-1+r_1)} \right. \\
& \quad \left. (1024-1536r_1-26112r_1^2+135040r_1^3-304896r_1^4+391296 \right. \\
& \quad r_1^5-303104r_1^6+139392r_1^7-34560r_1^8+3456r_1^9+ \sqrt{(28311552 \\
& \quad r_1-467140608r_1^2+3588489216r_1^3-16978083840r_1^4+ \\
& \quad 55228760064r_1^5-130682585088r_1^6+232168882176r_1^7- \\
& \quad 315144732672r_1^8+329334128640r_1^9-264790867968r_1^{10}+ \\
& \quad 162331361280r_1^{11}-74439917568r_1^{12}+24680595456r_1^{13}- \\
& \quad \left. 5573836800r_1^{14}+764411904r_1^{15}-47775744r_1^{16}))^{1/3}} \right) \Bigg) + \\
& 4r_1 \left(1-r_1 + \frac{1}{2} \sqrt[3]{4(-1+r_1)^2 + \frac{1}{4}(-22+43r_1-21r_1^2) + \frac{-22+65r_1-64r_1^2+21r_1^3}{12(-1+r_1)}} + \right. \\
& \quad \left. (2^{1/3}(4-40r_1+129r_1^2-196r_1^3+154r_1^4-60r_1^5+9r_1^6)) \right) / \\
& \quad \left(3(-1+r_1) (1024-1536r_1-26112r_1^2+135040r_1^3-304896r_1^4+ \right. \\
& \quad 391296r_1^5-303104r_1^6+139392r_1^7-34560r_1^8+3456r_1^9+ \\
& \quad \sqrt{(28311552r_1-467140608r_1^2+3588489216r_1^3-16978083840 \\
& \quad r_1^4+55228760064r_1^5-130682585088r_1^6+232168882176 \\
& \quad r_1^7-315144732672r_1^8+329334128640r_1^9-264790867968 \\
& \quad r_1^{10}+162331361280r_1^{11}-74439917568r_1^{12}+24680595456 \\
& \quad r_1^{13}-5573836800r_1^{14}+764411904r_1^{15}-47775744r_1^{16}))^{1/3}} + \\
& \quad \frac{1}{48 \times 2^{1/3}(-1+r_1)} \left(1024-1536r_1-26112r_1^2+135040r_1^3- \right. \\
& \quad 304896r_1^4+391296r_1^5-303104r_1^6+139392r_1^7-34560r_1^8+3456r_1^9+ \\
& \quad \sqrt{(28311552r_1-467140608r_1^2+3588489216r_1^3-16978083840r_1^4+ \\
& \quad 55228760064r_1^5-130682585088r_1^6+232168882176r_1^7- \\
& \quad 315144732672r_1^8+329334128640r_1^9-264790867968r_1^{10}+ \\
& \quad 162331361280r_1^{11}-74439917568r_1^{12}+24680595456r_1^{13}- \\
& \quad \left. 5573836800r_1^{14}+764411904r_1^{15}-47775744r_1^{16}))^{1/3}} \right) -
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \sqrt{\left(8 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) - \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} - \right.} \\
& \quad \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) / \\
& \quad \left(3 (-1 + r_1) (1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + \right. \\
& \quad 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \\
& \quad \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})}^{1/3} \Big) - \\
& \quad \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \left(1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - \right. \\
& \quad 304896 r_1^4 + 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \\
& \quad \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})}^{1/3} + \\
& \quad \left. (-64 (-1 + r_1)^3 + 4 (-1 + r_1) (22 - 43 r_1 + 21 r_1^2) - \right. \\
& \quad \left. 6 (-4 + 11 r_1 - 10 r_1^2 + 3 r_1^3) \right) / \\
& \quad \left(4 \sqrt{\left(4 (-1 + r_1)^2 + \frac{1}{4} (-22 + 43 r_1 - 21 r_1^2) + \frac{-22 + 65 r_1 - 64 r_1^2 + 21 r_1^3}{12 (-1 + r_1)} + \right.} \right. \\
& \quad \left. (2^{1/3} (4 - 40 r_1 + 129 r_1^2 - 196 r_1^3 + 154 r_1^4 - 60 r_1^5 + 9 r_1^6)) \right) / \\
& \quad \left(3 (-1 + r_1) (1024 - 1536 r_1 - 26112 r_1^2 + 135040 r_1^3 - \right. \\
& \quad 304896 r_1^4 + 391296 r_1^5 - 303104 r_1^6 + 139392 r_1^7 - \\
& \quad 34560 r_1^8 + 3456 r_1^9 + \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})}^{1/3} \Big) + \frac{1}{48 \times 2^{1/3} (-1 + r_1)} \left(1024 - 1536 r_1 - \right. \\
& \quad 26112 r_1^2 + 135040 r_1^3 - 304896 r_1^4 + 391296 r_1^5 - \\
& \quad 303104 r_1^6 + 139392 r_1^7 - 34560 r_1^8 + 3456 r_1^9 + \sqrt{(28311552 r_1 - 467140608 r_1^2 + 3588489216 r_1^3 - 16978083840 r_1^4 + 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16})}^{1/3} \Big) \\
& \quad \left. 55228760064 r_1^5 - 130682585088 r_1^6 + 232168882176 r_1^7 - 315144732672 r_1^8 + 329334128640 r_1^9 - 264790867968 r_1^{10} + 162331361280 r_1^{11} - 74439917568 r_1^{12} + 24680595456 r_1^{13} - 5573836800 r_1^{14} + 764411904 r_1^{15} - 47775744 r_1^{16} \right)
\end{aligned}$$

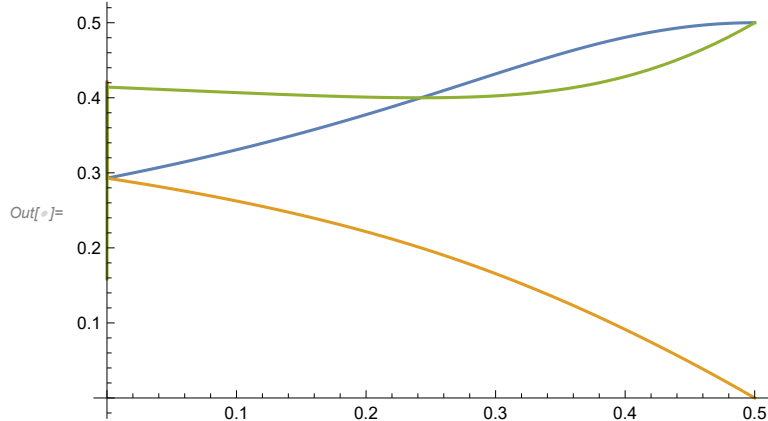
$$\left. r1^{16} \right)^{1/3} \left. \right)^2 + 4 r1 r11 - 4 r1^2 r11 - 4 r11^2 + 4 r1 r11^2 \left. \right)$$

Plot solutions in period 2 with respect to r1

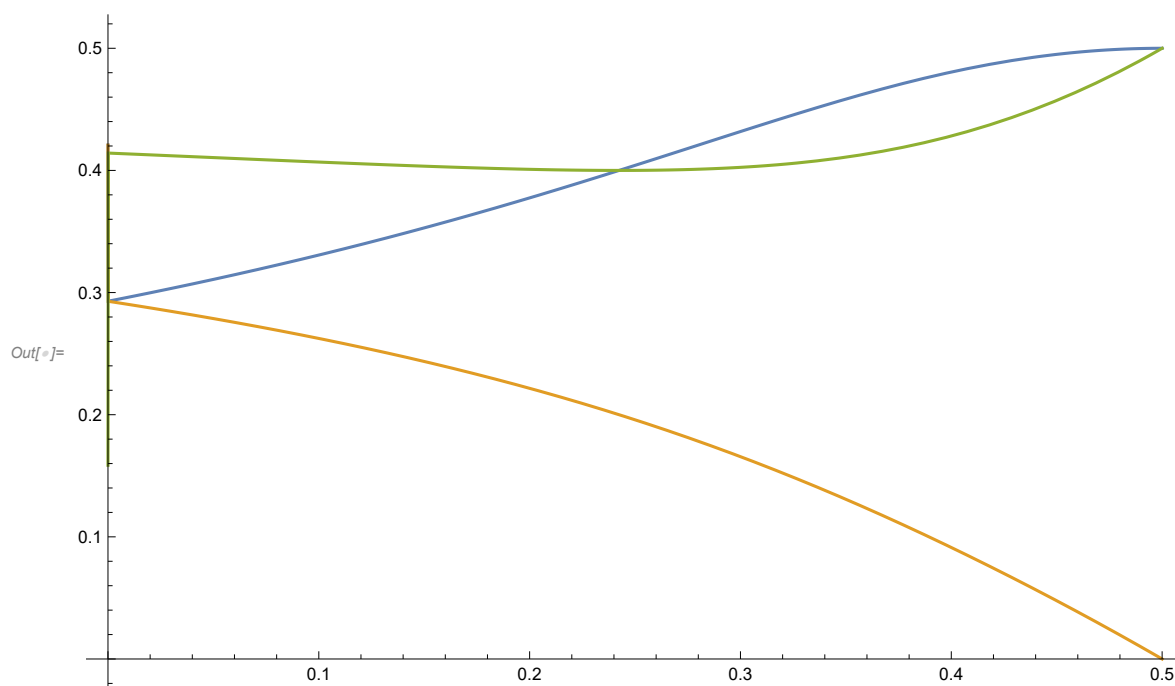
```
In[ ]:= Plot[{sol2[[11]][2][2], sol[[1][2] /. sol2[[11]][2] /. r11 -> r1/2,
  1 - r1 - sol2[[11]][2][2] - sol[[1][2] /. sol2[[11]][2] /. r11 -> r1/2}], {r1, 0, .5}]
```



```
In[ ]:= Plot[{sol2[[11]][2][2] / (1 - r1),
  (sol[[1][2] /. sol2[[11]][2] /. r11 -> r1/2) / (1 - r1), 1 - sol2[[11]][2][2] / (1 - r1) -
  (sol[[1][2] /. sol2[[11]][2] /. r11 -> r1/2) / (1 - r1)}, {r1, 0, .5}]
```



```
In[ ]:= Show[%222, ImageSize → Large]
```



Numerical example:

```
In[ ]:= sol2 /. {r1 → 0.1}
```

```
Out[ ]:= {{r22 → 0.875048 - 0.110736 i},
          {r22 → 0.875048 + 0.110736 i}, {r22 → 0.297642}, {r22 → 1.55226}}
```

```
In[ ]:= sol /. sol2 /. {r1 → 0.1}
```

```
Out[ ]:= {{r12 → 0.00069231 + 0.104757 i},
          {r12 → 0.00069231 - 0.104757 i}, {r12 → 0.236194}, {r12 → -0.662421}}
```

Import solutions to R

```
CForm[sol2[[1]] [[2]] [[2]]];
```

```
In[ ]:= CForm[sol[[1]] [[2]]]
```

```
Out[ ]//CForm=
(r1 - Power(r1,2) - Sqrt(r1)*Sqrt(r1 - 2*Power(r1,2) + Power(r1,3) + 4*r1*r11 - 4*Powe
4*r1*Power(r22,2)))/(2.*r1)
```