Supplementary Material of "Optimal allocation strategies in platform trials"

Design with concurrent and non-concurrent controls - Case 3

We first define the treatment effect for arm 2 following the expressions presented in the supplementary material (see Section A.2). To do so, we define the matrices A, B, and C, and use equation (1.b) to to obtain point estimates

```
In[ ]:= nd1 = n01 + n11
        nd2 = n02 + n12 + n22
        nd3 = n03 + n23
        A = {{nd1, 0, 0}, {0, nd2, 0}, {0, 0, nd3}}
        B = {{n11, 0}, {n12, n22}, {0, n23}}
        Cm = {{n11 + n12, 0}, {0, n22 + n23}}

Out[ ]= n01 + n11

Out[ ]= n02 + n12 + n22

Out[ ]= n03 + n23

Out[ ]= {{n01 + n11, 0, 0}, {0, n02 + n12 + n22, 0}, {0, 0, n03 + n23}}

Out[ ]= {{n11, 0}, {n12, n22}, {0, n23}}

Out[ ]= {{n11 + n12, 0}, {0, n22 + n23}}
```

In[ ]:=
```
M = FullSimplify[Inverse[Cm - Transpose[B].Inverse[A].B]]
Nm = {{n11 * theta11 + n12 * theta12}, {n22 * theta22 + n23 * theta23}}
Collect[FullSimplify[M.Nm][[2]], {theta11, theta12, theta22, theta23}]
w11 = (n11 (n01 + n11) n12 n22 (n03 + n23) ) /
   (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
     n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
         n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
w12 = ((n01 + n11) n12^2 n22 (n03 + n23) ) /
   (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
     n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
         n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
w22 =
  (n22 (n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) (n03 + n23) ) /
   (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
     n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
         n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
w23 =
  ((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) n23 (n03 + n23) ) /
   (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
     n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
         n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
sol = M.Nm
True ===
 FullSimplify[sol[[2]][[1]] == w11 * theta11 + w12 * theta12 + w22 * theta22 + w23 * theta23]
```

Out[ ]=
```
{{((n01 + n11) (n03 (n02 + n12) n22 + n03 (n02 + n12) n23 + (n02 + n03 + n12) n22 n23)) /
    (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
      n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
      n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23)),
   ((n01 + n11) n12 n22 (n03 + n23)) / (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
      n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
          n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))},
  {((n01 + n11) n12 n22 (n03 + n23)) / (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
      n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
          n02 (n11 + n12) (n22 n23 + n03 (n22 + n23)))),
   ((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) (n03 + n23)) /
    (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
      n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
          n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))}}
```

Out[ ]=
```
{{n11 theta11 + n12 theta12}, {n22 theta22 + n23 theta23}}
```

Out[•]=
```
{(n11 (n01 + n11) n12 n22 (n03 + n23) theta11) /
   (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
     n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
        n02 (n11 + n12) (n22 n23 + n03 (n22 + n23)))) +
  ((n01 + n11) n12^2 n22 (n03 + n23) theta12) /
   (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
     n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
        n02 (n11 + n12) (n22 n23 + n03 (n22 + n23)))) +
  (n22 (n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22))
      (n03 + n23) theta22) / (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
     n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
        n02 (n11 + n12) (n22 n23 + n03 (n22 + n23)))) +
  ((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22))
     n23 (n03 + n23) theta23) /
   (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
     n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
        n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))}
```

Out[•]=
```
(n11 (n01 + n11) n12 n22 (n03 + n23)) /
 (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
   n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
      n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
```

Out[•]=
```
((n01 + n11) n12^2 n22 (n03 + n23)) / (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
   n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
      n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
```

Out[•]=
```
(n22 (n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) (n03 + n23)) /
 (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
   n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
      n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
```

Out[•]=
```
((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) n23 (n03 + n23)) /
 (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
   n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
      n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
```

Out[ ]=
```
{{((n01 + n11) (n03 (n02 + n12) n22 + n03 (n02 + n12) n23 + (n02 + n03 + n12) n22 n23)
      (n11 theta11 + n12 theta12)) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
      n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
      n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23)) +
   ((n01 + n11) n12 n22 (n03 + n23) (n22 theta22 + n23 theta23)) /
    (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
      n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
         n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))},
 {((n01 + n11) n12 n22 (n03 + n23) (n11 theta11 + n12 theta12)) /
    (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
      n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
         n02 (n11 + n12) (n22 n23 + n03 (n22 + n23)))) +
   ((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22))
      (n03 + n23) (n22 theta22 + n23 theta23)) /
    (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
      n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
         n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))}}
```

Out[ ]=
```
True
```

Variance computation

To compute the variance of treatment effect 2, first note
Var(theta2) = Var(w11*theta11+w12*theta12+w22*theta22+w23*theta23)

```
In[ ]:= theta11 = n01 / (n01 + n11) * (y11 - y01);
theta12 =  (n02 + n22) / (n02 + n12 + n22) * y12 -
    ((n02 / (n02 + n12 + n22)) * y02 + (n22 / (n02 + n12 + n22)) * y22);
theta22 =  (n02 + n12) / (n02 + n12 + n22) * y22 -
    ((n02 / (n02 + n12 + n22)) * y02 + (n12 / (n02 + n12 + n22)) * y12);
theta23 = n03 / (n03 + n23) * (y23 - y03);
expr = w11 * theta11 + w12 * theta12 + w22 * theta22 + w23 * theta23;
Collect[FullSimplify[expr], {y01, y11, y02, y12, y22, y03, y23}];
expr01 = FullSimplify[
    (-n01 n03 n11 n12 n22 - n01 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
        n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
        n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr02 = FullSimplify[
    (-n01 n02 n03 n11 n22 - n01 n02 n03 n12 n22 - n02 n03 n11 n12 n22 - n01 n02 n11 n22 n23 -
        n01 n02 n12 n22 n23 - n02 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
        n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
        n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr03 = FullSimplify[
    (-n01 n03 n11 n12 n23 - n02 n03 n11 n12 n23 - n03 n11 n12 n22 n23 - n01 n03 n11 (n02 + n22)
         n23 - n01 n03 n12 (n02 + n22) n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
        n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
        n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr11 = FullSimplify[
    (n01 n03 n11 n12 n22 + n01 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
        n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
        n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr12 = FullSimplify[
    (-n01 n03 n11 n12 n22 - n01 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
        n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
        n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr22 = FullSimplify[
    (n02 n03 n11 n12 n22 + n01 n03 (n11 n12 + n02 (n11 + n12)) n22 + n01 n02 n12 n22 n23 + n02 n11
         n12 n22 n23 + n01 n11 (n02 + n12) n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
        n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
        n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr23 = FullSimplify[ (n02 n03 n11 n12 n23 + n03 n11 n12 n22 n23 +
        n01 n03 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22) n23) /
      (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
        n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
        n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
FullSimplify[
  Collect[FullSimplify[expr], {y01, y11, y02, y12, y22, y03, y23}] == expr01 * y01 +
     expr02 * y02 + expr03 * y03 + expr11 * y11 + expr12 * y12 + expr22 * y22 + expr23 * y23];
```

Variance expression is then term2s*sigma^2/N, where

In[ ]:=
```
term2f = FullSimplify[expr01^2 * y01 + expr02^2 * y02 + expr03^2 * y03 + expr11^2 * y11 +
     expr12^2 * y12 + expr22^2 * y22 + expr23^2 * y23 /. {y01 → 1 / n01, y02 → 1 / n02,
     y03 → 1 / n03, y11 → 1 / n11, y12 → 1 / n12, y22 → 1 / n22, y23 → 1 / n23}];
term2s =
  FullSimplify[term2f /. {n01 → r01 * Nt, n02 → r02 * Nt, n03 → r03 * Nt, n11 → r11 * Nt,
      n12 → r12 * Nt, n22 → r22 * Nt, n23 → r23 * Nt}] * Nt;
```

Define terms to optimise

In[ ]:=
```
substp =
  {r11 → r1 p11, r12 → r2 p12, r22 → r2 p22, r02 → r2 p02, r23 → r3 p23, r3 → 1 - r1 - r2};
```

In[ ]:=
```
subst = {r11 → r1 / 2 , r01 → r1 / 2 , r23 →  r3 / 2 , r03 → r3 / 2, r02 → r2 - r12 - r22 };
term1 = FullSimplify[ (r11 * r01 / (r11 + r01)) + (r12 * r02 / (r12 + r02)) /. subst /. substp]
term2 = FullSimplify[ (1 / term2s) /. subst /. substp]
```

Out[ ]= $\frac{r1}{4}+\frac{p12\ (-1+p12+p22)\ r2}{-1+p22}$

Out[ ]= $\left(r1^2+r1\ \left(-1+\left(-4\ (-1+p12)\ p12+(1-2\,p22)^2\right)\ r2\right)+4\,p12\,r2\ (-1+p12+r2+4\ (-1+p22)\ p22\,r2+p12\ (-1+4\,p22)\ r2)\right)\Big/\ (-4\,r1+16\ (-1+p12)\ p12\,r2)$

In[ ]:=
```
substg = {r01 → r1 - r11, r03 →  r3 - r23, r02 → r2 - r12 - r22 };
termg1 =
 FullSimplify[ (r11 * r01 / (r11 + r01)) + (r12 * r02 / (r12 + r02)) /. substg /. substp]
termg2 = FullSimplify[ (1 / term2s) /. substg /. substp]
```

Out[ ]= $p11\ (r1-p11\,r1)+\frac{p12\ (-1+p12+p22)\ r2}{-1+p22}$

Out[ ]= $\frac{p22\,r2\ (-((-1+p11)\ p11\ (-1+p22)\ r1)+p12\ (-1+p12+p22)\ r2)}{(-1+p11)\ p11\,r1+(-1+p12)\ p12\,r2}+p23\,r3\ \left(1+\frac{p23\,r3}{-1+r1+r2}\right)$

Numerical example: optimisation assuming balanced design in periods 1 and 3

In[ ]:=
```
ex = {r1 → 0.1, r2 → 0.8, r3 → 0.1};
FindMinimum[{(-term1) /. ex, term1 == term2 /. ex, p12 > 0,  p22 > 0},
 {{p12, r2 / 3 /. ex}, {p22, r2 / 3 /. ex}}]
```

Out[ ]= {-0.164091, {p12 → 0.303787, p22 → 0.289682}}

Optimisation (approach 1) - here we do not assume balanced design in periods 1 and 3 and thus also allocation rates in periods 1 and 3 are optimized

In[ ]:=
```
ex = {r1 → 0.4, r2 → 0.4, r3 → 0.2};
FindMinimum[{(-termg1) /. ex, termg1 == termg2 /. ex,
  p12 > 0, p22 > 0, p11 > 0, p23 > 0, p11 < 1, p23 < 1},
 {{p11, r1 / 2 /. ex}, {p12, r2 / 3 /. ex}, {p22, r2 / 3 /. ex}, {p23, r3 / 2 /. ex}}]
```

Out[ ]= {-0.144071, {p11 → 0.5, p12 → 0.153829, p22 → 0.457912, p23 → 0.5}}

```
In[∘]:= FindMinimum[
         {-termg1 /. ex, termg1 == termg2 /. ex, p12 > 0, p22 > 0, p11 > 0, p23 > 0, p11 < 1, p23 < 1},
         {{p11, r1/2 /. ex}, {p12, r2/3 /. ex}, {p22, r2/3 /. ex}, {p23, r3/2 /. ex}}]
```

```
Out[∘]= {-0.144071, {p11 → 0.5, p12 → 0.153829, p22 → 0.457912, p23 → 0.5}}
```

```
In[∘]:= termg2
```

$$\text{Out[∘]=}\ \frac{p22\ r2\ (-((-1+p11)\ p11\ (-1+p22)\ r1)+p12\ (-1+p12+p22)\ r2)}{(-1+p11)\ p11\ r1+(-1+p12)\ p12\ r2}+p23\ r3\left(1+\frac{p23\ r3}{-1+r1+r2}\right)$$

```
In[∘]:= {{p11, r1/2 /. ex}, {p12, r2/3 /. ex}, {p22, r2/3 /. ex}, {p23, r3/2 /. ex}}
```

```
Out[∘]= {{p11, 0.2}, {p12, 0.133333}, {p22, 0.133333}, {p23, 0.1}}
```

Note that we cannot find analytical solutions, but the numerical solutions satisfy that the optimal design follows a balanced design in periods 1 and 3.

Optimisation (approach 2) - assume balanced designs in periods 1 and 3

```
In[∘]:= constr = term1 - term2;
```

```
In[∘]:= e1 = FullSimplify[Solve[D[term1, p12] == l D[constr, p12], l]]
        e2 = FullSimplify[Solve[D[term1, p22] == l D[constr, p22], l]]
        e3 = e1[[1]][[1]][[2]] == e2[[1]][[1]][[2]]
```

$$\text{Out[∘]=}\ \{\{l\to((-1+2\,p12+p22)\,(r1-4\,(-1+p12)\,p12\,r2)^2)\,/\,((-1+2\,p12+p22)\,r1^2-8\,p12\,(2\,p12^2+p12\,(-3+p22)-(-1+p22)\,(1+p22^2))\,r1\,r2+16\,p12^2\,(-1+p12+p22)\,(1+2\,p12^2+p22^2-p12\,(3+p22))\,r2^2)\}\}$$

$$\text{Out[∘]=}\ \{\{l\to(p12^2\,(-r1+4\,(-1+p12)\,p12\,r2))\,/\,(-p12^2\,r1+(-1+p22)^2\,(-1+2\,p22)\,r1+4\,p12\,(-1+p12+p22)\,(1+p12^2-(3+p12)\,p22+2\,p22^2)\,r2)\}\}$$

$$\text{Out[∘]=}\ ((-1+2\,p12+p22)\,(r1-4\,(-1+p12)\,p12\,r2)^2)\,/\,((-1+2\,p12+p22)\,r1^2-8\,p12\,(2\,p12^2+p12\,(-3+p22)-(-1+p22)\,(1+p22^2))\,r1\,r2+16\,p12^2\,(-1+p12+p22)\,(1+2\,p12^2+p22^2-p12\,(3+p22))\,r2^2)==(p12^2\,(-r1+4\,(-1+p12)\,p12\,r2))\,/\,(-p12^2\,r1+(-1+p22)^2\,(-1+2\,p22)\,r1+4\,p12\,(-1+p12+p22)\,(1+p12^2-(3+p12)\,p22+2\,p22^2)\,r2)$$

```
In[∘]:= sol2 = Solve[e3, {p12}];
```

In[ ]:=
```
solsim = Simplify[sol2[[7]]]
```

Out[ ]=

$$
\begin{aligned}
\Big\{ p12 \to \; & \frac{1}{24\,(-1+p22)\,r2} \Big(-4\,(3-6\,p22+2\,p22^2)\,r2 + \big(2\times 2^{1/3}\,(1-i\,\sqrt{3})\,r2\,\big((3-9\,p22+6\,p22^2)\,r1 + \\
& (3-12\,p22+18\,p22^2-12\,p22^3+4\,p22^4)\,r2\big)\big) \Big/ \Big(-9\,p22^2\,r1\,r2^2+27\,p22^3\,r1\,r2^2- \\
& 18\,p22^4\,r1\,r2^2+18\,p22^2\,r2^3-72\,p22^3\,r2^3+108\,p22^4\,r2^3-72\,p22^5\,r2^3+16\,p22^6\,r2^3+ \\
& \sqrt{\Big(r2^3\,\Big(-4\,\big((3-9\,p22+6\,p22^2)\,r1+(3-12\,p22+18\,p22^2-12\,p22^3+4\,p22^4)\,r2\big)^3+} \\
& \quad p22^4\,r2\,\big(9\,(1-3\,p22+2\,p22^2)\,r1- \\
& \quad\quad 2\,(9-36\,p22+54\,p22^2-36\,p22^3+8\,p22^4)\,r2\big)^2\Big)\Big)\Big)^{1/3} + \\
& 2^{2/3}\,(1+i\,\sqrt{3})\,\Big(-9\,p22^2\,r1\,r2^2+27\,p22^3\,r1\,r2^2-18\,p22^4\,r1\,r2^2+18\,p22^2\,r2^3- \\
& 72\,p22^3\,r2^3+108\,p22^4\,r2^3-72\,p22^5\,r2^3+16\,p22^6\,r2^3+ \\
& \sqrt{\Big(r2^3\,\Big(-4\,\big((3-9\,p22+6\,p22^2)\,r1+(3-12\,p22+18\,p22^2-12\,p22^3+4\,p22^4)\,r2\big)^3+} \\
& \quad p22^4\,r2\,\big(9\,(1-3\,p22+2\,p22^2)\,r1- \\
& \quad\quad 2\,(9-36\,p22+54\,p22^2-36\,p22^3+8\,p22^4)\,r2\big)^2\Big)\Big)\Big)^{1/3}\Big)\Big\}
\end{aligned}
$$

In[ ]:=
```
$Assumptions = p12 > 0 && p22 > 0 && Element[p12, Reals] && Element[p22, Reals]
```

Out[ ]= $p12 > 0$ && $p22 > 0$ && $p12 \in \mathbb{R}$ && $p22 \in \mathbb{R}$

In[ ]:=
```
Re[solsim]
```

Out[ ]=

$$
\begin{aligned}
\Big\{ \mathrm{Re}\Big[p12 \to \; & \frac{1}{24\,(-1+p22)\,r2} \Big(-4\,(3-6\,p22+2\,p22^2)\,r2 + \big(2\times 2^{1/3}\,(1-i\,\sqrt{3})\,r2\,\big((3-9\,p22+6\,p22^2)\,r1 + \\
& (3-12\,p22+18\,p22^2-12\,p22^3+4\,p22^4)\,r2\big)\big) \Big/ \Big(-9\,p22^2\,r1\,r2^2+27\,p22^3\,r1\,r2^2- \\
& 18\,p22^4\,r1\,r2^2+18\,p22^2\,r2^3-72\,p22^3\,r2^3+108\,p22^4\,r2^3-72\,p22^5\,r2^3+16\,p22^6\,r2^3+ \\
& \sqrt{\Big(r2^3\,\Big(-4\,\big((3-9\,p22+6\,p22^2)\,r1+(3-12\,p22+18\,p22^2-12\,p22^3+4\,p22^4)\,r2\big)^3+} \\
& \quad p22^4\,r2\,\big(9\,(1-3\,p22+2\,p22^2)\,r1- \\
& \quad\quad 2\,(9-36\,p22+54\,p22^2-36\,p22^3+8\,p22^4)\,r2\big)^2\Big)\Big)\Big)^{1/3} + \\
& 2^{2/3}\,(1+i\,\sqrt{3})\,\Big(-9\,p22^2\,r1\,r2^2+27\,p22^3\,r1\,r2^2-18\,p22^4\,r1\,r2^2+18\,p22^2\,r2^3- \\
& 72\,p22^3\,r2^3+108\,p22^4\,r2^3-72\,p22^5\,r2^3+16\,p22^6\,r2^3+ \\
& \sqrt{\Big(r2^3\,\Big(-4\,\big((3-9\,p22+6\,p22^2)\,r1+(3-12\,p22+18\,p22^2-12\,p22^3+4\,p22^4)\,r2\big)^3+} \\
& \quad p22^4\,r2\,\big(9\,(1-3\,p22+2\,p22^2)\,r1- \\
& \quad\quad 2\,(9-36\,p22+54\,p22^2-36\,p22^3+8\,p22^4)\,r2\big)^2\Big)\Big)\Big)^{1/3}\Big)\Big]\Big\}
\end{aligned}
$$

In[ ]:=
```
sol2 /. ex /. p22 → 0.23 / 0.8
```

Out[ ]=

$$
\begin{aligned}
& \{\{p12 \to -0.207107\},\ \{p12 \to 1.20711\},\ \{p12 \to -0.196053\}, \\
& \{p12 \to 0.908553\},\ \{p12 \to 0.866963 - 5.55112\times 10^{-17}\,i\}, \\
& \{p12 \to -0.185879 - 2.77556\times 10^{-17}\,i\},\ \{p12 \to 0.329662 + 1.11022\times 10^{-16}\,i\}\}
\end{aligned}
$$

In[∘]:=
```
eq = FullSimplify[term1 - term2]
```

Out[∘]=
$$\frac{1}{4}\left(r1+\frac{4\,p12\,(-1+p12+p22)\,r2}{-1+p22}+\frac{1}{r1-4\,(-1+p12)\,p12\,r2}\left(r1^2+r1\left(-1+\left(-4\,(-1+p12)\,p12+(1-2\,p22)^2\right)r2\right)+4\,p12\,r2\,(-1+p12+r2+4\,(-1+p22)\,p22\,r2+p12\,(-1+4\,p22)\,r2)\right)\right)$$

In[∘]:=
```
eq3 = FullSimplify[e3]
```

Out[∘]=
$$(r1-4\,(-1+p12)\,p12\,r2)\left(p12^2\Big/\left(-p12^2\,r1+(-1+p22)^2\,(-1+2\,p22)\,r1+4\,p12\,(-1+p12+p22)\left(1+p12^2-(3+p12)\,p22+2\,p22^2\right)r2\right)+((-1+2\,p12+p22)\,(r1-4\,(-1+p12)\,p12\,r2))\Big/\left((-1+2\,p12+p22)\,r1^2-8\,p12\left(2\,p12^2+p12\,(-3+p22)-(-1+p22)\left(1+p22^2\right)\right)r1\,r2+16\,p12^2\,(-1+p12+p22)\left(1+2\,p12^2+p22^2-p12\,(3+p22)\right)r2^2\right)\right)=0$$

In[∘]:=
```
NSolve[{eq == 0 /. ex, eq3 /. ex}, {p12, p22}]
```

Out[∘]=
$$\{\{p22\to -28\,924.1+18\,543.2\,i,\ p12\to -0.249998+9.81706\times 10^{-7}\,i\},$$
$$\{p22\to -0.0311394+0.265623\,i,\ p12\to -0.211592+0.0051594\,i\},$$
$$\{p22\to -0.0311394-0.265623\,i,\ p12\to -0.211592-0.0051594\,i\},$$
$$\{p22\to -0.0311394+0.265623\,i,\ p12\to -0.211592+0.0051594\,i\},$$
$$\{p22\to 0.457912,\ p12\to 0.153829\},\ \{p22\to 0.457912,\ p12\to 0.153829\},$$
$$\{p22\to 0.7153+0.205041\,i,\ p12\to 0.151828-0.155807\,i\},$$
$$\{p22\to 0.7153-0.205041\,i,\ p12\to 0.151828+0.155807\,i\},$$
$$\{p22\to 1.84182,\ p12\to 1.94225\},\ \{p22\to 0.5,\ p12\to 0.5\},\ \{p22\to 0.5,\ p12\to 0.5\},$$
$$\{p22\to 0.117486-0.930047\,i,\ p12\to 1.17003+0.720353\,i\},$$
$$\{p22\to 0.117486+0.930047\,i,\ p12\to 1.17003-0.720353\,i\},$$
$$\{p22\to 1.59697,\ p12\to -0.816608\}\}$$