Computations on case 3. Three-period design using non-concurrent controls

We first define the treatment effect for arm 2 following the expressions presented in the supplementary material (see Section A.2). To do so, we define the matrices A, B, and C, and use equation (1.b) to to obtain point estimates

```
In[ ]:= nd1 = n01 + n11
        nd2 = n02 + n12 + n22
        nd3 = n03 + n23
        A = {{nd1, 0, 0}, {0, nd2, 0}, {0, 0, nd3}}
        B = {{n11, 0}, {n12, n22}, {0, n23}}
        Cm = {{n11 + n12, 0}, {0, n22 + n23}}
```

```
Out[ ]= n01 + n11

Out[ ]= n02 + n12 + n22

Out[ ]= n03 + n23

Out[ ]= {{n01 + n11, 0, 0}, {0, n02 + n12 + n22, 0}, {0, 0, n03 + n23}}

Out[ ]= {{n11, 0}, {n12, n22}, {0, n23}}

Out[ ]= {{n11 + n12, 0}, {0, n22 + n23}}
```

```
In[ ]:= M = FullSimplify[Inverse[Cm - Transpose[B].Inverse[A].B]]
        Nm = {{n11 * theta11 + n12 * theta12}, {n22 * theta22 + n23 * theta23}}
        Collect[FullSimplify[M.Nm][[2]], {theta11, theta12, theta22, theta23}]
        w11 = (n11 (n01 + n11) n12 n22 (n03 + n23) ) /
           (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
             n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
        w12 = ( (n01 + n11) n12^2 n22 (n03 + n23) ) /
           (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
             n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
        w22 =
          (n22 (n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) (n03 + n23) ) /
           (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
             n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
        w23 =
          ((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) n23 (n03 + n23) ) /
           (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
             n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
        sol = M.Nm
        True ===
         FullSimplify[sol[[2]][[1]] == w11 * theta11 + w12 * theta12 + w22 * theta22 + w23 * theta23]
```

Out[•]=
$$\{\{((n01+n11)\,(n03\,(n02+n12)\,n22+n03\,(n02+n12)\,n23+(n02+n03+n12)\,n22\,n23))\,/\,(n01\,n03\,(n11\,n12+n02\,(n11+n12))\,n22+n01\,(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22+n02\,(n11+n12)\,(n03+n22))\,n23+n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)),((n01+n11)\,n12\,n22\,(n03+n23))\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))\},\{((n01+n11)\,n12\,n22\,(n03+n23))\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))),((n11\,n12\,(n02+n22)+n01\,(n11\,n12+n02\,(n11+n12)+(n11+n12)\,n22))\,(n03+n23))\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))\}\}$$

Out[•]=
$$\{\{n11\,\text{theta11}+n12\,\text{theta12}\},\{n22\,\text{theta22}+n23\,\text{theta23}\}\}$$

Out[•]=
$$\Big\{(n11\,(n01+n11)\,n12\,n22\,(n03+n23)\,\text{theta11})\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))+\big((n01+n11)\,n12^2\,n22\,(n03+n23)\,\text{theta12}\big)\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))+(n22\,(n11\,n12\,(n02+n22)+n01\,(n11\,n12+n02\,(n11+n12)+(n11+n12)\,n22))\,(n03+n23)\,\text{theta22})\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))+((n11\,n12\,(n02+n22)+n01\,(n11\,n12+n02\,(n11+n12)+(n11+n12)\,n22))\,n23\,(n03+n23)\,\text{theta23})\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))\Big\}$$

Out[•]=
$$(n11\,(n01+n11)\,n12\,n22\,(n03+n23))\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))$$

Out[•]=
$$\big((n01+n11)\,n12^2\,n22\,(n03+n23)\big)\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))$$

Out[•]=
$$(n22\,(n11\,n12\,(n02+n22)+n01\,(n11\,n12+n02\,(n11+n12)+(n11+n12)\,n22))\,(n03+n23))\,/\,(n11\,n12\,(n02\,n03\,n22+n03\,n22\,n23+n02\,(n03+n22)\,n23)+n01\,(n03\,n11\,n12\,n22+(n03\,n11\,n12+n11\,n12\,n22+n03\,(n11+n12)\,n22)\,n23+n02\,(n11+n12)\,(n22\,n23+n03\,(n22+n23))))$$

```
Out[•]= ((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22)) n23 (n03 + n23)) /
         (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
           n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
              n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))
```

```
Out[•]= {{((n01 + n11) (n03 (n02 + n12) n22 + n03 (n02 + n12) n23 + (n02 + n03 + n12) n22 n23)
              (n11 theta11 + n12 theta12)) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
             n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
             n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23)) +
           ((n01 + n11) n12 n22 (n03 + n23) (n22 theta22 + n23 theta23)) /
            (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
              n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))},
          {((n01 + n11) n12 n22 (n03 + n23) (n11 theta11 + n12 theta12)) /
            (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
              n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23)))) +
           ((n11 n12 (n02 + n22) + n01 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22))
              (n03 + n23) (n22 theta22 + n23 theta23)) /
            (n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23) +
              n01 (n03 n11 n12 n22 + (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22) n23 +
                 n02 (n11 + n12) (n22 n23 + n03 (n22 + n23))))}}
```

```
Out[•]= True
```

Variance computation

To compute the variance of treatment effect 2, first note

Var(theta2) = Var(w11*theta11+w12*theta12+w22*theta22+w23*theta23)

In[ ]:=
```
theta11 = n01 / (n01 + n11) * (y11 - y01);
theta12 = (n02 + n22) / (n02 + n12 + n22) * y12 -
    ((n02 / (n02 + n12 + n22)) * y02 + (n22 / (n02 + n12 + n22)) * y22);
theta22 = (n02 + n12) / (n02 + n12 + n22) * y22 -
    ((n02 / (n02 + n12 + n22)) * y02 + (n12 / (n02 + n12 + n22)) * y12);
theta23 = n03 / (n03 + n23) * (y23 - y03);
expr = w11 * theta11 + w12 * theta12 + w22 * theta22 + w23 * theta23;
Collect[FullSimplify[expr], {y01, y11, y02, y12, y22, y03, y23}];
expr01 = FullSimplify[
    (-n01 n03 n11 n12 n22 - n01 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr02 = FullSimplify[
    (-n01 n02 n03 n11 n22 - n01 n02 n03 n12 n22 - n02 n03 n11 n12 n22 - n01 n02 n11 n22 n23 -
       n01 n02 n12 n22 n23 - n02 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr03 = FullSimplify[
    (-n01 n03 n11 n12 n23 - n02 n03 n11 n12 n23 - n03 n11 n12 n22 n23 - n01 n03 n11 (n02 + n22)
        n23 - n01 n03 n12 (n02 + n22) n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr11 = FullSimplify[
    (n01 n03 n11 n12 n22 + n01 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr12 = FullSimplify[
    (-n01 n03 n11 n12 n22 - n01 n11 n12 n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr22 = FullSimplify[
    (n02 n03 n11 n12 n22 + n01 n03 (n11 n12 + n02 (n11 + n12)) n22 + n01 n02 n12 n22 n23 + n02 n11
        n12 n22 n23 + n01 n11 (n02 + n12) n22 n23) / (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
expr23 = FullSimplify[ (n02 n03 n11 n12 n23 + n03 n11 n12 n22 n23 +
       n01 n03 (n11 n12 + n02 (n11 + n12) + (n11 + n12) n22) n23) /
     (n01 n03 (n11 n12 + n02 (n11 + n12)) n22 +
       n01 (n03 n11 n12 + n11 n12 n22 + n03 (n11 + n12) n22 + n02 (n11 + n12) (n03 + n22)) n23 +
       n11 n12 (n02 n03 n22 + n03 n22 n23 + n02 (n03 + n22) n23))];
FullSimplify[
  Collect[FullSimplify[expr], {y01, y11, y02, y12, y22, y03, y23}] == expr01 * y01 +
    expr02 * y02 + expr03 * y03 + expr11 * y11 + expr12 * y12 + expr22 * y22 + expr23 * y23];
```

Variance expression is then term2s*sigma^2/N, where

```
In[ ]:= term2f = FullSimplify[expr01^2 * y01 + expr02^2 * y02 + expr03^2 * y03 + expr11^2 * y11 +
          expr12^2 * y12 + expr22^2 * y22 + expr23^2 * y23 /. {y01 → 1 / n01, y02 → 1 / n02,
          y03 → 1 / n03, y11 → 1 / n11, y12 → 1 / n12, y22 → 1 / n22, y23 → 1 / n23}];
        term2s =
          FullSimplify[term2f /. {n01 → r01 * Nt, n02 → r02 * Nt, n03 → r03 * Nt, n11 → r11 * Nt,
            n12 → r12 * Nt, n22 → r22 * Nt, n23 → r23 * Nt}] * Nt;
```

Define terms to optimise

```
In[ ]:= subst = {r11 → r1 / 2 , r01 → r1 / 2 , r23 → r3 / 2 , r03 → r3 / 2, r02 → r2 - r12 - r22 };
        term1 = FullSimplify[(r11 * r01 / (r11 + r01)) + (r12 * r02 / (r12 + r02)) /. subst];
        term2 = FullSimplify[(1 / term2s) /. subst];

In[ ]:= substg = {r01 → r1 - r11, r03 → r3 - r23, r02 → r2 - r12 - r22 };
        termg1 = FullSimplify[(r11 * r01 / (r11 + r01)) + (r12 * r02 / (r12 + r02)) /. substg];
        termg2 = FullSimplify[(1 / term2s) /. substg];
```

Numerical example: optimisation assuming balanced design in periods 1 and 3

```
In[ ]:= ex = {r1 → 0.1, r2 → 0.8, r3 → 0.1};
        FindMinimum[{(-term1) /. ex, term1 == term2 /. ex, r12 + r22 < 0.8, r12 > 0, r22 > 0},
         {{r12, r2 / 3 /. ex}, {r22, r2 / 3 /. ex}}]
Out[ ]= {-0.164091, {r12 → 0.24303, r22 → 0.231746}}
```

Optimisation (approach 1) - here we do not assume balanced design in periods 1 and 3 and thus also allocation rates in periods 1 and 3 are optimized

```
In[ ]:= ex = {r1 → 0.4, r2 → 0.4, r3 → 0.2};
        FindMinimum[{(-termg1) /. ex, termg1 == termg2 /. ex,
          r12 + r22 < 0.8, r12 > 0, r22 > 0, r11 > 0, r23 > 0, r11 < 1, r23 < 1},
         {{r11, r1 / 2 /. ex}, {r12, r2 / 3 /. ex}, {r22, r2 / 3 /. ex}, {r23, r3 / 2 /. ex}}]
Out[ ]= {-0.144071, {r11 → 0.2, r12 → 0.0615317, r22 → 0.183165, r23 → 0.1}}

In[ ]:= FindMinimum[{-termg1 /. ex, termg1 == termg2 /. ex,
          r12 + r22 < 0.8`, r12 > 0, r22 > 0, r11 > 0, r23 > 0, r11 < 1, r23 < 1},
         {{r11, r1/2 /. ex}, {r12, r2/3 /. ex}, {r22/3 /. ex}, {r23, r3/2 /. ex}}]

FindMinimum: The variable r22/3 /. ex cannot be localized so that it can be assigned to numerical values.

Out[ ]= FindMinimum[{-termg1 /. ex, termg1 == termg2 /. ex,
          r12 + r22 < 0.8, r12 > 0, r22 > 0, r11 > 0, r23 > 0, r11 < 1, r23 < 1},
         {{r11, r1/2 /. ex}, {r12, r2/3 /. ex}, {r22/3 /. ex}, {r23, r3/2 /. ex}}]

In[ ]:= termg2

Out[ ]= (r1 (-r12^2 + (r11 + r12) r2) r23^2 +
          r1 (r11 r22 (-r2 + r22) + r12 r22 (r12 - r2 + r22) + r12^2 r23 - (r11 + r12) r2 r23) r3 +
          r11^2 (-r2 r23^2 - r22^2 r3 + r2 (r22 + r23) r3)) / ((r11^2 r2 + r1 (r12^2 - (r11 + r12) r2)) r3)
```

```
In[ ]:= {{r11, r1/2 /. ex}, {r12, r2/3 /. ex}, {r22/3 /. ex}, {r23, r3/2 /. ex}}
```

$$\text{Out[ ]= } \left\{\{\text{r11}, 0.2\}, \{\text{r12}, 0.133333\}, \left\{\frac{\text{r22}}{3}\right\}, \{\text{r23}, 0.1\}\right\}$$

Note that we cannot find analytical solutions, but the numerical solutions satisfy that the optimal design follows a balanced design in periods 1 and 3.

Optimisation (approach 2) - assume balanced designs in periods 1 and 3

```
In[ ]:= constr = term1 - term2;
```

```
In[ ]:= e1 = FullSimplify[Solve[D[term1, r12] == l D[constr, r12], l]]
        e2 = FullSimplify[Solve[D[term1, r22] == l D[constr, r22], l]]
        e3 = e1[[1]][[1]][[2]] == e2[[1]][[1]][[2]]
```

$$\text{Out[ ]= } \left\{\left\{l \to \frac{-1+\frac{2\,\text{r12}}{\text{r2}-\text{r22}}}{-1+\frac{2\,\text{r12}}{\text{r2}-\text{r22}}-\frac{8\,\text{r12}\,(\text{r1}+2\,\text{r12})\,\text{r22}^2}{(\text{r1}\,\text{r2}+4\,\text{r12}\,(-\text{r12}+\text{r2}))^2}}\right\}\right\}$$

$$\text{Out[ ]= } \left\{\left\{l \to \frac{\text{r12}^2}{(\text{r2}-\text{r22})^2\left(1+\frac{\text{r12}^2}{(\text{r2}-\text{r22})^2}-\frac{2\,(\text{r1}+4\,\text{r12})\,\text{r22}}{\text{r1}\,\text{r2}+4\,\text{r12}\,(-\text{r12}+\text{r2})}\right)}\right\}\right\}$$

$$\text{Out[ ]= } \frac{-1+\frac{2\,\text{r12}}{\text{r2}-\text{r22}}}{-1+\frac{2\,\text{r12}}{\text{r2}-\text{r22}}-\frac{8\,\text{r12}\,(\text{r1}+2\,\text{r12})\,\text{r22}^2}{(\text{r1}\,\text{r2}+4\,\text{r12}\,(-\text{r12}+\text{r2}))^2}} == \frac{\text{r12}^2}{(\text{r2}-\text{r22})^2\left(1+\frac{\text{r12}^2}{(\text{r2}-\text{r22})^2}-\frac{2\,(\text{r1}+4\,\text{r12})\,\text{r22}}{\text{r1}\,\text{r2}+4\,\text{r12}\,(-\text{r12}+\text{r2})}\right)}$$

```
In[ ]:= sol2 = Solve[e3, {r12}];
```

```
In[ ]:= solsim = Simplify[sol2[[7]]]
```

$$\begin{aligned}\text{Out[ ]= } \Bigg\{\text{r12} \to \frac{1}{192\,(\text{r2}-\text{r22})}&\Big(32\left(3\,\text{r2}^2-6\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)+\left(16\,\text{i}\,2^{1/3}\left(\text{i}+\sqrt{3}\right)\left(3\,\text{r2}^4-12\,\text{r2}^3\,\text{r22}+18\,\text{r2}^2\,\text{r22}^2-\right.\right.\\&\left.\left.12\,\text{r2}\,\text{r22}^3+4\,\text{r22}^4+3\,\text{r1}\,\text{r2}\left(\text{r2}^2-3\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)\right)\right)\Big/\\&\Big(-9\,\text{r1}\,\text{r2}^3\,\text{r22}^2+18\,\text{r2}^4\,\text{r22}^2+27\,\text{r1}\,\text{r2}^2\,\text{r22}^3-72\,\text{r2}^3\,\text{r22}^3-18\,\text{r1}\,\text{r2}\,\text{r22}^4+108\,\text{r2}^2\,\text{r22}^4-\\&72\,\text{r2}\,\text{r22}^5+16\,\text{r22}^6+\sqrt{\Big(-4\left(3\,\text{r2}^4-12\,\text{r2}^3\,\text{r22}+18\,\text{r2}^2\,\text{r22}^2-12\,\text{r2}\,\text{r22}^3+4\,\text{r22}^4+\right.}\\&\left.3\,\text{r1}\,\text{r2}\left(\text{r2}^2-3\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)\right)^3+\text{r22}^4\left(9\,\text{r1}\,\text{r2}\left(\text{r2}^2-3\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)-\\&2\left(9\,\text{r2}^4-36\,\text{r2}^3\,\text{r22}+54\,\text{r2}^2\,\text{r22}^2-36\,\text{r2}\,\text{r22}^3+8\,\text{r22}^4\right)\Big)^2\Big)\Big)^{1/3}-\\&8\times 2^{2/3}\left(1+\text{i}\,\sqrt{3}\right)\Big(-9\,\text{r1}\,\text{r2}^3\,\text{r22}^2+18\,\text{r2}^4\,\text{r22}^2+27\,\text{r1}\,\text{r2}^2\,\text{r22}^3-72\,\text{r2}^3\,\text{r22}^3-\\&18\,\text{r1}\,\text{r2}\,\text{r22}^4+108\,\text{r2}^2\,\text{r22}^4-72\,\text{r2}\,\text{r22}^5+16\,\text{r22}^6+\\&\sqrt{\Big(-4\left(3\,\text{r2}^4-12\,\text{r2}^3\,\text{r22}+18\,\text{r2}^2\,\text{r22}^2-12\,\text{r2}\,\text{r22}^3+4\,\text{r22}^4+\\&3\,\text{r1}\,\text{r2}\left(\text{r2}^2-3\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)\right)^3+\text{r22}^4\left(9\,\text{r1}\,\text{r2}\left(\text{r2}^2-3\,\text{r2}\,\text{r22}+2\,\text{r22}^2\right)-\\&2\left(9\,\text{r2}^4-36\,\text{r2}^3\,\text{r22}+54\,\text{r2}^2\,\text{r22}^2-36\,\text{r2}\,\text{r22}^3+8\,\text{r22}^4\right)\Big)^2\Big)\Big)^{1/3}\Big)\Bigg\}\end{aligned}$$

```
In[ ]:= $Assumptions =
          r12 > 0 && r22 > 0 && r12 + r22 < r2 && Element[r12, Reals] && Element[r22, Reals]
```

$$\text{Out[ ]= } \text{r12} > 0 \,\&\&\, \text{r22} > 0 \,\&\&\, \text{r12} + \text{r22} < \text{r2} \,\&\&\, \text{r12} \in \mathbb{R} \,\&\&\, \text{r22} \in \mathbb{R}$$

In[•]:=
```
Re[solsim]
```

Out[•]=
```
{Re[
  r12 → 1/(192 (r2 - r22)) (32 (3 r2^2 - 6 r2 r22 + 2 r22^2) + (16 i 2^(1/3) (i + Sqrt[3]) (3 r2^4 - 12 r2^3 r22 + 18
        r2^2 r22^2 - 12 r2 r22^3 + 4 r22^4 + 3 r1 r2 (r2^2 - 3 r2 r22 + 2 r22^2)))/
    (-9 r1 r2^3 r22^2 + 18 r2^4 r22^2 + 27 r1 r2^2 r22^3 - 72 r2^3 r22^3 - 18 r1 r2 r22^4 + 108 r2^2 r22^4 -
      72 r2 r22^5 + 16 r22^6 + Sqrt[(-4 (3 r2^4 - 12 r2^3 r22 + 18 r2^2 r22^2 - 12 r2 r22^3 + 4 r22^4 +
          3 r1 r2 (r2^2 - 3 r2 r22 + 2 r22^2))^3 + r22^4 (9 r1 r2 (r2^2 - 3 r2 r22 + 2 r22^2) -
          2 (9 r2^4 - 36 r2^3 r22 + 54 r2^2 r22^2 - 36 r2 r22^3 + 8 r22^4))^2)]^(1/3) -
    8 × 2^(2/3) (1 + i Sqrt[3]) (-9 r1 r2^3 r22^2 + 18 r2^4 r22^2 + 27 r1 r2^2 r22^3 - 72 r2^3 r22^3 -
      18 r1 r2 r22^4 + 108 r2^2 r22^4 - 72 r2 r22^5 + 16 r22^6 +
      Sqrt[(-4 (3 r2^4 - 12 r2^3 r22 + 18 r2^2 r22^2 - 12 r2 r22^3 + 4 r22^4 +
          3 r1 r2 (r2^2 - 3 r2 r22 + 2 r22^2))^3 + r22^4 (9 r1 r2 (r2^2 - 3 r2 r22 + 2 r22^2) -
          2 (9 r2^4 - 36 r2^3 r22 + 54 r2^2 r22^2 - 36 r2 r22^3 + 8 r22^4))^2)]^(1/3))]}
```

In[•]:=
```
{r12 → 1/(192 (r2 - r22))
  (32 (3 r2^2 - 6 r2 r22 + 2 r22^2) + (16 i 2^(1/3) (i + Sqrt[3]) (3 r2^3 (r1 + r2) - 3 r2^2 (3 r1 + 4 r2)
        r22 + 6 r2 (r1 + 3 r2) r22^2 - 12 r2 r22^3 + 4 r22^4))/ (18 r2^4 r22^2 + 16 r22^6 +
    27 r2^2 r22^3 (r1 + 4 r22) - 18 r2 r22^4 (r1 + 4 r22) - 9 r2^3 r22^2 (r1 + 8 r22) +
    Sqrt[(-4 (3 r2^3 (r1 + r2) - 3 r2^2 (3 r1 + 4 r2) r22 + 6 r2 (r1 + 3 r2) r22^2 -
          12 r2 r22^3 + 4 r22^4)^3 + r22^4 (9 r1 r2 (r2 - 2 r22) (r2 - r22) -
          2 (3 r2^2 - 6 r2 r22 + 2 r22^2) (3 r2^2 - 6 r2 r22 + 4 r22^2))^2)]^(1/3) -
  8 × 2^(2/3) (1 + i Sqrt[3]) (18 r2^4 r22^2 + 16 r22^6 + 27 r2^2 r22^3 (r1 + 4 r22) -
    18 r2 r22^4 (r1 + 4 r22) - 9 r2^3 r22^2 (r1 + 8 r22) +
    Sqrt[(-4 (3 r2^3 (r1 + r2) - 3 r2^2 (3 r1 + 4 r2) r22 + 6 r2 (r1 + 3 r2) r22^2 - 12 r2 r22^3 +
          4 r22^4)^3 + r22^4 (9 r1 r2 (r2 - 2 r22) (r2 - r22) - 2 (3 r2^2 - 6 r2 r22 +
          2 r22^2) (3 r2^2 - 6 r2 r22 + 4 r22^2))^2)]^(1/3))} /. ex /. r22 → 0.24
```

Out[•]=
```
{r12 → -0.0336525 - 0.106938 i}
```

In[•]:=
```
sol2 /. ex /. r22 → 0.23
```

Out[•]=
```
{{r12 → -0.0828427}, {r12 → 0.482843}, {r12 → -0.0706438}, {r12 → 0.240644},
 {r12 → 0.143166}, {r12 → -0.0218773 + 0.0889001 i}, {r12 → -0.0218773 - 0.0889001 i}}
```

In[•]:=
```
eq = FullSimplify[term1 - term2]
```

Out[•]=
$$\frac{r1}{4} + r12 - r22 + \frac{(r1 + 4\,r12)\,r22^2}{r1\,r2 + 4\,r12\,(-r12 + r2)} + \frac{r12^2}{-r2 + r22} - \frac{r3}{4}$$

```
In[•]:= eq3 = FullSimplify[e3]
```

$$\text{Out[•]=}\quad \frac{-1+\frac{2\,r12}{r2-r22}}{-1+\frac{2\,r12}{r2-r22}-\frac{8\,r12\,(r1+2\,r12)\,r22^2}{(r1\,r2+4\,r12\,(-r12+r2))^2}} == \frac{r12^2}{(r2-r22)^2\left(1+\frac{r12^2}{(r2-r22)^2}-\frac{2\,(r1+4\,r12)\,r22}{r1\,r2+4\,r12\,(-r12+r2)}\right)}$$

```
In[•]:= NSolve[{eq == 0 /. ex, eq3 /. ex}, {r12, r22}]
```

```
Out[•]= {{r22 → 7755.25 - 1199.97 i, r12 → -0.100001 - 1.94884 × 10^-7 i},
  {r22 → -0.0124558 + 0.106249 i, r12 → -0.0846366 + 0.00206376 i},
  {r22 → -0.0124558 - 0.106249 i, r12 → -0.0846366 - 0.00206376 i},
  {r22 → 0.183165, r12 → 0.0615317}, {r22 → 0.736729, r12 → 0.7769},
  {r22 → 0.638789, r12 → -0.326643},
  {r22 → 0.28612 - 0.0820165 i, r12 → 0.0607311 + 0.062323 i},
  {r22 → 0.28612 + 0.0820165 i, r12 → 0.0607311 - 0.062323 i},
  {r22 → 0.28612 + 0.0820165 i, r12 → 0.0607311 - 0.062323 i},
  {r22 → 0.0469944 - 0.372019 i, r12 → 0.468011 + 0.288141 i},
  {r22 → 0.0469944 + 0.372019 i, r12 → 0.468011 - 0.288141 i}, {r22 → 0.2, r12 → 0.2}}
```

Solutions to plot in R

```
In[•]:= xx = {r1 → 0.2, r3 → 0.1}
Out[•]= {r1 → 0.2, r3 → 0.1}

In[•]:= x = {r1 → 0.2, r2 → 0.7, r3 → 0.1}
Out[•]= {r1 → 0.2, r2 → 0.7, r3 → 0.1}

In[•]:= FindMinimum[
          {(-term1) /. x, term1 == term2 /. x, r12 + r22 < 0.7, r12 > 0, r22 > 0}, {r12, r22}][[2]]
Out[•]= {r12 → 0.184831, r22 → 0.238117}

In[•]:= nsol = NSolve[{eq == 0 /. x, eq3 /. x, r12 > 0, r22 > 0, r12 + r22 < r2 /. x}, {r12, r22}]
Out[•]= {{r22 → 0.647259, r12 → 0.0329464}, {r22 → 0.238117, r12 → 0.184831}}

In[•]:= {term1, term2} /. x /. nsol
Out[•]= {{0.0623653, 0.0623653}, {0.160867, 0.160867}}

In[•]:= NSolve[{eq == 0 /. {r1 → 0.2, r2 → 0.7, r3 → 0.1},
            eq3 /. {r1 → 0.2, r2 → 0.7, r3 → 0.1}}, {r12, r22}];


In[•]:= term1
        term2
```

$$\text{Out[•]=}\quad \frac{r1}{4}+r12+\frac{r12^2}{-r2+r22}$$

$$\text{Out[•]=}\quad r22-\frac{(r1+4\,r12)\,r22^2}{r1\,r2+4\,r12\,(-r12+r2)}+\frac{r3}{4}$$

In[•]:=
```
v = Range[0.4, 0.9, 0.01]

f[r2_] := FindMinimum[{-(-r1/4 + r12 + r12^2/(-r2 + r22)) /. {r1 → 0.1},
    r1/4 + r12 + r12^2/(-r2 + r22) == r22 - (r1 + 4 r12) r22^2/(r1 r2 + 4 r12 (-r12 + r2)) + (1 - r1 - r2)/4 /. {r1 → 0.1},
    r12 + r22 < r2, r12 > 0, r22 > 0}, {{r12, r2 / 3 /. ex}, {r22, r2 / 3 /. ex}}][[2]]

results1 = ({r12, r22} /. Map[f, v]) / v

Export[
 "/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results1.csv", results1]
```

Out[•]=
```
{0.4, 0.41, 0.42, 0.43, 0.44, 0.45, 0.46, 0.47, 0.48, 0.49, 0.5, 0.51,
 0.52, 0.53, 0.54, 0.55, 0.56, 0.57, 0.58, 0.59, 0.6, 0.61, 0.62, 0.63, 0.64,
 0.65, 0.66, 0.67, 0.68, 0.69, 0.7, 0.71, 0.72, 0.73, 0.74, 0.75, 0.76, 0.77,
 0.78, 0.79, 0.8, 0.81, 0.82, 0.83, 0.84, 0.85, 0.86, 0.87, 0.88, 0.89, 0.9}
```

Out[•]=
```
{{0.5, 9.57139 × 10^-18}, {0.495012, 0.00989618}, {0.490046, 0.0195931},
 {0.485097, 0.0291022}, {0.480161, 0.0384342}, {0.475236, 0.0475985},
 {0.470317, 0.0566038}, {0.465402, 0.0654578}, {0.460488, 0.0741673},
 {0.455574, 0.0827387}, {0.450656, 0.0911774}, {0.445733, 0.0994882},
 {0.440804, 0.107675}, {0.435867, 0.115742}, {0.430922, 0.123692}, {0.425968, 0.131528},
 {0.421005, 0.13925}, {0.416032, 0.146862}, {0.411051, 0.154363}, {0.406061, 0.161754},
 {0.401065, 0.169036}, {0.396062, 0.176208}, {0.391056, 0.183269}, {0.386047, 0.190218},
 {0.381039, 0.197055}, {0.376034, 0.203777}, {0.371035, 0.210383},
 {0.366045, 0.216871}, {0.361068, 0.22324}, {0.356108, 0.229486}, {0.351167, 0.235609},
 {0.346252, 0.241606}, {0.341365, 0.247475}, {0.33651, 0.253214}, {0.331693, 0.258823},
 {0.326917, 0.2643}, {0.322186, 0.269644}, {0.317503, 0.274855}, {0.312874, 0.279931},
 {0.308301, 0.284874}, {0.303787, 0.289682}, {0.299336, 0.294358}, {0.29495, 0.298902},
 {0.290632, 0.303315}, {0.286384, 0.307599}, {0.282207, 0.311756}, {0.278103, 0.315788},
 {0.274074, 0.319697}, {0.27012, 0.323485}, {0.266241, 0.327157}, {0.262438, 0.330714}}
```

Out[•]=
```
/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results1.csv
```

In[•]:=
```
v = Range[0.2, 0.7, 0.01]

f[r2_] := FindMinimum[{-(-r1/4 + r12 + r12^2/(-r2 + r22)) /. {r1 → 0.3},
    r1/4 + r12 + r12^2/(-r2 + r22) == r22 - (r1 + 4 r12) r22^2/(r1 r2 + 4 r12 (-r12 + r2)) + (1 - r1 - r2)/4 /. {r1 → 0.3},
    r12 + r22 < r2, r12 > 0, r22 > 0}, {{r12, r2 / 3 /. ex}, {r22, r2 / 3 /. ex}}][[2]]

results2 = ({r12, r22} /. Map[f, v]) / v

Export["/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results2.csv",
 results2]
```

Out[•]=
```
{0.2, 0.21, 0.22, 0.23, 0.24, 0.25, 0.26, 0.27, 0.28, 0.29, 0.3, 0.31,
 0.32, 0.33, 0.34, 0.35, 0.36, 0.37, 0.38, 0.39, 0.4, 0.41, 0.42, 0.43, 0.44,
 0.45, 0.46, 0.47, 0.48, 0.49, 0.5, 0.51, 0.52, 0.53, 0.54, 0.55, 0.56, 0.57,
 0.58, 0.59, 0.6, 0.61, 0.62, 0.63, 0.64, 0.65, 0.66, 0.67, 0.68, 0.69, 0.7}
```

Out[•]=
```
{{0.5, 9.4495 × 10^-18}, {0.490186, 0.019471}, {0.480706, 0.0379681}, {0.47151, 0.0556027},
 {0.462555, 0.0724689}, {0.453806, 0.0886466}, {0.445229, 0.104204},
 {0.436797, 0.119199}, {0.428487, 0.133681}, {0.420277, 0.147693}, {0.412149, 0.161269},
 {0.404088, 0.17444}, {0.396082, 0.187229}, {0.388118, 0.199657}, {0.38019, 0.211738},
 {0.372291, 0.223484}, {0.364417, 0.234902}, {0.356568, 0.245995}, {0.348744, 0.256764},
 {0.34095, 0.267208}, {0.333192, 0.277321}, {0.325478, 0.287098}, {0.317819, 0.296532},
 {0.310229, 0.305614}, {0.302721, 0.314336}, {0.295311, 0.322692}, {0.288015, 0.330677},
 {0.280849, 0.338288}, {0.273828, 0.345522}, {0.266967, 0.352384}, {0.260278, 0.358877},
 {0.253771, 0.36501}, {0.247456, 0.370792}, {0.241337, 0.376236}, {0.235419, 0.381356},
 {0.229704, 0.386167}, {0.224191, 0.390686}, {0.218878, 0.394928}, {0.213763, 0.398912},
 {0.208841, 0.402652}, {0.204106, 0.406166}, {0.199552, 0.409468}, {0.195174, 0.412574},
 {0.190965, 0.415497}, {0.186917, 0.41825}, {0.183025, 0.420846}, {0.17928, 0.423295},
 {0.175676, 0.425609}, {0.172208, 0.427797}, {0.168868, 0.429868}, {0.16565, 0.43183}}
```

Out[•]=
```
/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results2.csv
```

In[•]:=
```
v = Range[0.1, 0.6, 0.01]

f[r2_] := FindMinimum[{-(-r1/4 + r12 + r12^2/(-r2 + r22)) /. {r1 → 0.4},
    r1/4 + r12 + r12^2/(-r2 + r22) == r22 - (r1 + 4 r12) r22^2/(r1 r2 + 4 r12 (-r12 + r2)) + (1 - r1 - r2)/4 /. {r1 → 0.4},
    r12 + r22 < r2, r12 > 0, r22 > 0}, {{r12, r2 / 3 /. ex}, {r22, r2 / 3 /. ex}}][[2]]

results3 = ({r12, r22} /. Map[f, v]) / v

Export[
 "/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results3.csv", results3]
```

Out[•]=
```
{0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21,
 0.22, 0.23, 0.24, 0.25, 0.26, 0.27, 0.28, 0.29, 0.3, 0.31, 0.32, 0.33, 0.34,
 0.35, 0.36, 0.37, 0.38, 0.39, 0.4, 0.41, 0.42, 0.43, 0.44, 0.45, 0.46, 0.47,
 0.48, 0.49, 0.5, 0.51, 0.52, 0.53, 0.54, 0.55, 0.56, 0.57, 0.58, 0.59, 0.6}
```

Out[•]=
```
{{0.5, 0.}, {0.48098, 0.0377298}, {0.463606, 0.0715795}, {0.447502, 0.102335},
 {0.432383, 0.130579}, {0.418029, 0.156753}, {0.404266, 0.181198}, {0.390951, 0.204176},
 {0.377971, 0.225889}, {0.365227, 0.24649}, {0.352641, 0.266093}, {0.34015, 0.284773},
 {0.327707, 0.302568}, {0.315285, 0.319486}, {0.302877, 0.335501}, {0.290505, 0.350562},
 {0.278221, 0.364598}, {0.266103, 0.377533}, {0.254253, 0.389306}, {0.242782, 0.399886},
 {0.231791, 0.409285}, {0.221362, 0.417557}, {0.211546, 0.42479}, {0.202362, 0.431094},
 {0.193807, 0.436583}, {0.185857, 0.441369}, {0.178476, 0.445556}, {0.171626, 0.449232},
 {0.165262, 0.452475}, {0.159343, 0.45535}, {0.153829, 0.457912}, {0.148684, 0.460205},
 {0.143875, 0.462267}, {0.13937, 0.464131}, {0.135143, 0.465822}, {0.131169, 0.467362},
 {0.127427, 0.468771}, {0.123897, 0.470064}, {0.120562, 0.471255}, {0.117405, 0.472354},
 {0.114413, 0.473373}, {0.111573, 0.474319}, {0.108874, 0.475201}, {0.106305, 0.476023},
 {0.103858, 0.476793}, {0.101522, 0.477514}, {0.099292, 0.478192},
 {0.0971595, 0.47883}, {0.0951185, 0.479432}, {0.0931631, 0.48}, {0.091288, 0.480537}}
```

Out[•]=
```
/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results3.csv
```

In[•]:=
```
v = Range[0.01, 0.49, 0.002]

f[r2_] := FindMinimum[{-(-r1/4 + r12 + r12^2/(-r2 + r22)) /. {r1 → 0.49},
    r1/4 + r12 + r12^2/(-r2 + r22) == r22 - (r1 + 4 r12) r22^2/(r1 r2 + 4 r12 (-r12 + r2)) + (1 - r1 - r2)/4 /. {r1 → 0.49},
    r12 + r22 < r2, r12 > 0, r22 > 0}, {{r12, r2 / 20 /. ex}, {r22, r2 / 2 /. ex}}][[2]]

results4 = ({r12, r22} /. Map[f, v]) / v

Export[
 "/Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results4.csv", results4]
```

Out[•]= {0.01, 0.012, 0.014, 0.016, 0.018, 0.02, 0.022, 0.024, 0.026, 0.028, 0.03, 0.032, 0.034,
0.036, 0.038, 0.04, 0.042, 0.044, 0.046, 0.048, 0.05, 0.052, 0.054, 0.056, 0.058,
0.06, 0.062, 0.064, 0.066, 0.068, 0.07, 0.072, 0.074, 0.076, 0.078, 0.08, 0.082,
0.084, 0.086, 0.088, 0.09, 0.092, 0.094, 0.096, 0.098, 0.1, 0.102, 0.104, 0.106,
0.108, 0.11, 0.112, 0.114, 0.116, 0.118, 0.12, 0.122, 0.124, 0.126, 0.128, 0.13,
0.132, 0.134, 0.136, 0.138, 0.14, 0.142, 0.144, 0.146, 0.148, 0.15, 0.152, 0.154,
0.156, 0.158, 0.16, 0.162, 0.164, 0.166, 0.168, 0.17, 0.172, 0.174, 0.176, 0.178,
0.18, 0.182, 0.184, 0.186, 0.188, 0.19, 0.192, 0.194, 0.196, 0.198, 0.2, 0.202,
0.204, 0.206, 0.208, 0.21, 0.212, 0.214, 0.216, 0.218, 0.22, 0.222, 0.224, 0.226,
0.228, 0.23, 0.232, 0.234, 0.236, 0.238, 0.24, 0.242, 0.244, 0.246, 0.248, 0.25,
0.252, 0.254, 0.256, 0.258, 0.26, 0.262, 0.264, 0.266, 0.268, 0.27, 0.272, 0.274,
0.276, 0.278, 0.28, 0.282, 0.284, 0.286, 0.288, 0.29, 0.292, 0.294, 0.296, 0.298,
0.3, 0.302, 0.304, 0.306, 0.308, 0.31, 0.312, 0.314, 0.316, 0.318, 0.32, 0.322,
0.324, 0.326, 0.328, 0.33, 0.332, 0.334, 0.336, 0.338, 0.34, 0.342, 0.344, 0.346,
0.348, 0.35, 0.352, 0.354, 0.356, 0.358, 0.36, 0.362, 0.364, 0.366, 0.368, 0.37,
0.372, 0.374, 0.376, 0.378, 0.38, 0.382, 0.384, 0.386, 0.388, 0.39, 0.392, 0.394,
0.396, 0.398, 0.4, 0.402, 0.404, 0.406, 0.408, 0.41, 0.412, 0.414, 0.416, 0.418,
0.42, 0.422, 0.424, 0.426, 0.428, 0.43, 0.432, 0.434, 0.436, 0.438, 0.44, 0.442,
0.444, 0.446, 0.448, 0.45, 0.452, 0.454, 0.456, 0.458, 0.46, 0.462, 0.464, 0.466,
0.468, 0.47, 0.472, 0.474, 0.476, 0.478, 0.48, 0.482, 0.484, 0.486, 0.488, 0.49}

Out[•]= {{0.5, 0.}, {0.464563, 0.0707543}, {0.435977, 0.127583}, {0.411972, 0.175037},
{0.39117, 0.215865}, {0.372665, 0.251848}, {0.355823, 0.284207}, {0.340164, 0.313821},
{0.325292, 0.341351}, {0.310833, 0.36731}, {0.296374, 0.392106}, {0.281348, 0.416028},
{0.264799, 0.439095}, {0.245038, 0.460364}, {0.221218, 0.476415}, {0.198127, 0.484979},
{0.179457, 0.489153}, {0.164604, 0.491475}, {0.15247, 0.492943}, {0.142295, 0.493959},
{0.133587, 0.494705}, {0.126014, 0.49528}, {0.119346, 0.495736}, {0.113414, 0.496109},
{0.108092, 0.496419}, {0.103284, 0.496681}, {0.0989129, 0.496907},
{0.0949187, 0.497102}, {0.0912516, 0.497274}, {0.087871, 0.497426},
{0.0847428, 0.497562}, {0.0818386, 0.497683}, {0.0791342, 0.497793},
{0.0766089, 0.497892}, {0.0742449, 0.497983}, {0.0720267, 0.498066},
{0.0699407, 0.498143}, {0.0679753, 0.498213}, {0.0661199, 0.498279},
{0.0643653, 0.498339}, {0.0627035, 0.498396}, {0.0611269, 0.498448},
{0.0596292, 0.498498}, {0.0582043, 0.498544}, {0.0568472, 0.498587},
{0.0555528, 0.498628}, {0.054317, 0.498667}, {0.0531357, 0.498703},
{0.0520054, 0.498738}, {0.0509228, 0.49877}, {0.0498849, 0.498801},
{0.048889, 0.498831}, {0.0479325, 0.498859}, {0.0470131, 0.498885},
{0.0461287, 0.498911}, {0.0452773, 0.498935}, {0.044457, 0.498958},
{0.0436662, 0.498981}, {0.0429033, 0.499002}, {0.0421668, 0.499022},
{0.0414553, 0.499042}, {0.0407677, 0.499061}, {0.0401026, 0.499079},
{0.0394591, 0.499096}, {0.0388361, 0.499113}, {0.0382325, 0.499129},
{0.0376475, 0.499145}, {0.0370803, 0.49916}, {0.03653, 0.499174}, {0.035996, 0.499188},
{0.0354773, 0.499202}, {0.0349736, 0.499215}, {0.0344839, 0.499228},
{0.0340079, 0.49924}, {0.0335449, 0.499252}, {0.0330944, 0.499263},
{0.032656, 0.499275}, {0.032229, 0.499285}, {0.0318131, 0.499296},
{0.0314078, 0.499306}, {0.0310128, 0.499316}, {0.0306276, 0.499326},
{0.030252, 0.499335}, {0.0298854, 0.499344}, {0.0295277, 0.499353},
{0.0291785, 0.499362}, {0.0288374, 0.49937}, {0.0285043, 0.499378},
{0.0281788, 0.499386}, {0.0278607, 0.499394}, {0.0275497, 0.499401},
{0.0272456, 0.499409}, {0.0269482, 0.499416}, {0.0266572, 0.499423},
{0.0263724, 0.49943}, {0.0260937, 0.499437}, {0.0258208, 0.499443},

```
{0.0255536, 0.49945}, {0.0252919, 0.499456}, {0.0250355, 0.499462},
{0.0247842, 0.499468}, {0.024538, 0.499474}, {0.0242966, 0.499479},
{0.0240599, 0.499485}, {0.0238278, 0.49949}, {0.0236002, 0.499496},
{0.0233769, 0.499501}, {0.0231577, 0.499506}, {0.0229427, 0.499511},
{0.0227316, 0.499516}, {0.0225244, 0.499521}, {0.0223209, 0.499526},
{0.0221211, 0.49953}, {0.0219248, 0.499535}, {0.021732, 0.499539},
{0.0215425, 0.499544}, {0.0213564, 0.499548}, {0.0211734, 0.499552},
{0.0209936, 0.499556}, {0.0208168, 0.49956}, {0.0206429, 0.499564},
{0.0204719, 0.499568}, {0.0203038, 0.499572}, {0.0201383, 0.499576},
{0.0199756, 0.49958}, {0.0198155, 0.499583}, {0.0196579, 0.499587},
{0.0195028, 0.49959}, {0.0193502, 0.499594}, {0.0191999, 0.499597},
{0.0190519, 0.499601}, {0.0189063, 0.499604}, {0.0187628, 0.499607},
{0.0186215, 0.49961}, {0.0184823, 0.499613}, {0.0183451, 0.499617},
{0.01821, 0.49962}, {0.0180769, 0.499623}, {0.0179457, 0.499626},
{0.0178164, 0.499628}, {0.017689, 0.499631}, {0.0175633, 0.499634},
{0.0174395, 0.499637}, {0.0173174, 0.49964}, {0.0171969, 0.499642},
{0.0170782, 0.499645}, {0.0169611, 0.499648}, {0.0168455, 0.49965},
{0.0167316, 0.499653}, {0.0166191, 0.499655}, {0.0165082, 0.499658},
{0.0163987, 0.49966}, {0.0162907, 0.499662}, {0.0161841, 0.499665},
{0.0160789, 0.499667}, {0.0159751, 0.499669}, {0.0158726, 0.499672},
{0.0157714, 0.499674}, {0.0156714, 0.499676}, {0.0155728, 0.499678},
{0.0154753, 0.49968}, {0.0153791, 0.499682}, {0.0152841, 0.499685},
{0.0151903, 0.499687}, {0.0150975, 0.499689}, {0.015006, 0.499691},
{0.0149155, 0.499693}, {0.0148261, 0.499695}, {0.0147378, 0.499696},
{0.0146505, 0.499698}, {0.0145642, 0.4997}, {0.014479, 0.499702},
{0.0143947, 0.499704}, {0.0143115, 0.499706}, {0.0142292, 0.499708},
{0.0141478, 0.499709}, {0.0140673, 0.499711}, {0.0139878, 0.499713},
{0.0139092, 0.499715}, {0.0138314, 0.499716}, {0.0137545, 0.499718},
{0.0136785, 0.49972}, {0.0136033, 0.499721}, {0.0135289, 0.499723},
{0.0134553, 0.499724}, {0.0133825, 0.499726}, {0.0133105, 0.499727},
{0.0132393, 0.499729}, {0.0131688, 0.499731}, {0.0130991, 0.499732},
{0.0130301, 0.499734}, {0.0129619, 0.499735}, {0.0128943, 0.499736},
{0.0128275, 0.499738}, {0.0127613, 0.499739}, {0.0126959, 0.499741},
{0.0126311, 0.499742}, {0.0125669, 0.499743}, {0.0125034, 0.499745},
{0.0124405, 0.499746}, {0.0123783, 0.499748}, {0.0123167, 0.499749},
{0.0122557, 0.49975}, {0.0121953, 0.499751}, {0.0121355, 0.499753},
{0.0120763, 0.499754}, {0.0120176, 0.499755}, {0.0119595, 0.499756},
{0.011902, 0.499758}, {0.011845, 0.499759}, {0.0117886, 0.49976},
{0.0117327, 0.499761}, {0.0116774, 0.499762}, {0.0116225, 0.499764},
{0.0115682, 0.499765}, {0.0115143, 0.499766}, {0.011461, 0.499767},
{0.0114082, 0.499768}, {0.0113558, 0.499769}, {0.011304, 0.49977},
{0.0112526, 0.499772}, {0.0112016, 0.499773}, {0.0111512, 0.499774},
{0.0111011, 0.499775}, {0.0110516, 0.499776}, {0.0110024, 0.499777},
{0.0109537, 0.499778}, {0.0109055, 0.499779}, {0.0108576, 0.49978},
{0.0108102, 0.499781}, {0.0107632, 0.499782}, {0.0107166, 0.499783},
{0.0106704, 0.499784}, {0.0106246, 0.499785}, {0.0105791, 0.499786},
{0.0105341, 0.499787}, {0.0104895, 0.499788}, {0.0104452, 0.499789},
{0.0104013, 0.49979}, {0.0103578, 0.49979}, {0.0103146, 0.499791}}
```

Out[•]= /Users/mbofi/Dropbox/CeMSIIS/GitHub/Allocation/optimisation/results4.csv