Deterministic Modeling - Project 2 (Biology)

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1 Introduction

The aim of this project is making use of mathematics in biology. Precisely - modeling the process of originating the pattern on animal's skin. We will use the Gierer - Meinhardt system of equations, so called "reaction-diffusion model".

In order to carry out the experiment, we will start with drawing the scheme of the animal which is going to be the object of study. I have chosen a bear. In the picture below there are some parameters describing the exact location of each body part.

(I will put a picture here)

2 Creating an animal

The next step is writing a class in Python, which will "create" an animal. We can use it to see the scheme of the bear - it will be useful later, during the calculations. The class is stored in the file called *class_bear.py*.

(I will put the picture here)

3 The mathematical part

First of all we will analyse the equations of Gierer - Meinhardt model. There are many modifications of this system of equations. The model which we are going to use is one of them. It is shown below:

$$u_t = D_u \Delta u - C_1 u + \frac{C_2 u^2}{v} + C_3$$

$$v_t = D_v \Delta v + C_4 u^2 - C_5 v$$

The model is related to the process of diffusion. The pattern on the skin is the result of two substances diffusing with different rates.

The variable u describes the activator (short-range autocatalytic substance) and the variable v means a long-range inhibitor, which is an antagonist to the activator. The activator regulates the production of the inhibitor. The derivative u_t characters a change of activator concentration in some unit of time and similarly v_t shows a change of inhibitor.

The number of molecules that decay in some time unit is proportional to the decay rate C_1 and to the number of present molecules (u). The $\frac{C_2 u^2}{v}$ part describes the production rate, depending in a non-linear way on the activator concentration and is slowed down by the inhibitor. The $D_u u_{xx}$ term enables the exchange of molecules. The C_3 parameter is a small activator-independent production rate of the activator. Its task is initiating the activator autocatalysis at very low activator concentration, for example when the organism needs to regenerate. The parameters of the second equation can be analogously determined in case of the inhibitor change.

We assume that there are no special changes of the pigment on the "edges" - that is on the fragments connecting body parts together. It means that every part of the body is individual; for example the pigment on the head doesn't "mix" with the pigment on the belly. We can treat the edges as blockades separating the body parts and their amounts of the pigment.

We need to write a numerical scheme which will be used to solve the system of equations. First of all we notice that:

$$\Delta u(x,y,t) \approx \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \approx$$

$$\approx \frac{u(x+h_x,y,t) - 2u(x,y,t) + u(x-h_x,y,t)}{h_x^2} + \frac{u(x,y+h_y,t) - 2u(x,y,t) + u(x,y-h_y,t)}{h_y^2}$$

We assume that $h_x = h_y$, so:

$$\Delta u(x,y,t) \approx \frac{1}{h_x^2}\Big(u(x+h_x,y,t)+u(x-h_x,y,t)+u(x,y-h_x,t)+u(x,y+h_x,t)-4u(x,y,t)\Big)$$

Similarly:

$$\Delta v(x,y,t) \approx \frac{1}{h_x^2}\Big(v(x+h_x,y,t)+v(x-h_x,y,t)+v(x,y-h_x,t)+v(x,y+h_x,t)-4v(x,y,t)\Big)$$

Now we can rewrite the first equation using the definition of the derivative.

$$u_t = \lim_{h_t \to 0} \frac{u(x,y,t+h_t) - u(x,y,t)}{h_t}$$

$$\frac{u(x,y,t+h_t) - u(x,y,t)}{h_t} = \frac{D_u}{h_x^2}\Big(u(x+h_x,y,t)+u(x-h_x,y,t)+u(x,y-h_x,t)+u(x,y+h_x,t)-4u(x,y,t)\Big)-$$

$$-C_1 u(x,y,t) + \frac{C_2 u(x,y,t)^2}{v(x,y,t)} + C_3$$

Differently:

$$\frac{u_{i,j,k+1} - u_{i,j,k}}{h_t} = \frac{D_u}{h_x^2}\Big(u_{i+1,j,k}+u_{i-1,j,k}+u_{i,j-1,k}+u_{i,j+1,k}-4u_{i,j,k}\Big)-C_1 u_{i,j,k}+\frac{C_2 u_{i,j,k}^2}{v_{i,j,k}}+C_3$$

After some modifications:

$$u_{i,j,k+1} = h_t\Big(\frac{D_u}{h_x^2}\Big(u_{i+1,j,k}+u_{i-1,j,k}+u_{i,j-1,k}+u_{i,j+1,k}-4u_{i,j,k}\Big)-C_1 u_{i,j,k}+\frac{C_2 u_{i,j,k}^2}{v_{i,j,k}}+C_3\Big)+u_{i,j,k}$$

Similarly we can get the second equation:

$$\frac{v(x,y,t+h_t) - v(x,y,t)}{h_t} = \frac{D_v}{h_x^2}\Big(v(x+h_x,y,t)+v(x-h_x,y,t)+v(x,y-h_x,t)+v(x,y+h_x,t)-4v(x,y,t)\Big)+$$

$$+C_4 u(x,y,t)^2 - C_5 v(x,y,t)$$

$$\frac{v_{i,j,k+1} - v_{i,j,k}}{h_t} = \frac{D_v}{h_x^2}\Big(v_{i+1,j,k}+v_{i-1,j,k}+v_{i,j-1,k}+v_{i,j+1,k}-4v_{i,j,k}\Big)+C_4 u_{i,j,k}^2-C_5 v_{i,j,k}$$

$$v_{i,j,k+1} = h_t\Big(\frac{D_v}{h_x^2}\Big(v_{i+1,j,k}+v_{i-1,j,k}+v_{i,j-1,k}+v_{i,j+1,k}-4v_{i,j,k}\Big)+C_4 u_{i,j,k}^2-C_5 v_{i,j,k}\Big)+v_{i,j,k}$$

The above equations for u and v work almost for every case, but there are some exceptions, therefore we have to add some conditions.
(to be continued)