Quantitative Macroeconomics: Final Project

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Simple Variant of Krusell-Smith Algorithm

1. Proof of proposition 3 in Harenberg and Ludwig 2015

Proposition 3 in the paper states that, under assumptions 1 and 4, the equilibrium dynamics in general equilibrium are given by:

$$k_{t+1} = \frac{s(\tau)(1-\tau)(1-\alpha)\zeta_t k_t^{\alpha}}{(1+g)(1+\lambda)}$$

And the savings rate is given by:

$$s(\tau) = \frac{\beta\Phi(\tau)}{1+\beta\Phi(\tau)}$$

where $\Phi(\tau) = \mathbb{E}_t(\frac{1}{1+\frac{1-\alpha}{\alpha(1+\lambda)\varrho_{t+1}}(\lambda\eta_{i,2,t+1}+\tau(1+\lambda(1-\eta_{i,2,t+1})))}) \leq 1$

To prove this, the paper follows a guess and verify method. Due to the households all being identical ex-ante, our guess for their savings at young age is the following:

$$a_{2,t+1} = s(1-\tau)w_t$$

Where you can further substitute the wages with the ones resulting from the FOCs of the firm's problem ($w_t = (1-\alpha)\Upsilon_t\zeta_t k_t^{\alpha}$). Then, the equilibrium dynamics will be given by the following expression, which is the market clearing condition for capital considering that the population size of each age is normalized to 1:

$$K_{t+1} = a_{2,t+1} = s(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^{\alpha}$$

Written in terms of capital intensity (capital stock per efficient unit of labour), we can rewrite the last expression as follows, using the growth rate of the technology level in the second equality ($\Upsilon_{t+1} = (1+g)\Upsilon_t$)):

$$k_{t+1} = \frac{K_{t+1}}{\Upsilon_{t+1}(1+\lambda)} = \frac{s(1-\tau)(1-\alpha)\zeta_t k_t^{\alpha}}{(1+g)(1+\lambda)}$$

To verify the first equation in proposition 3 using our guess $a_{2,t+1}$, plug it into the household's budget constraints and also substitute the interest rate obtained from the firm's FOC ($(1+r_t) = \alpha k_t^{\alpha-1}\zeta_t \varrho_t$) as well as the social security budget constraint ($b_t = \lambda w_t \frac{1+\lambda}{1-\lambda}$).

$$c_{1t} + a_{2,t+1} = (1-\tau)w_t$$
$$c_{1t} = (1-s)(1-\tau)(1-\alpha)\Upsilon_t \zeta_t k_t^\alpha$$

$$c_{i,2,t+1} \leq a_{2,t+1}(1+r_{t+1}) + \lambda\eta_{i,2,t+1} w_{t+1}(1-\tau) + (1+\lambda)b_{t+1}$$

$$\begin{aligned} c_{i,2,t+1} = s(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha \alpha k_{t+1}^{\alpha-1}\zeta_{t+1}\varrho_{t+1} + \tau(1-\alpha)\Upsilon_{t+1}\varrho_{t+1}k_{t+1}^\alpha \\ ((1+\lambda)+\lambda\eta_{i,2,t+1}) + \lambda\eta_{i,2,t+1}(1-\alpha)\Upsilon_{t+1}\varrho_{t+1}k_{t+1}^\alpha \end{aligned}$$

$$\begin{aligned} c_{i,2,t+1} = s(1-\tau)(1-\alpha)\frac{\Upsilon_{t+1}}{1+g}\zeta_t k_t^\alpha \alpha k_{t+1}^{\alpha-1}\zeta_{t+1}\varrho_{t+1} \\ + (1-\alpha)\Upsilon_{t+1}\varrho_{t+1}k_{t+1}^\alpha(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))) \end{aligned}$$

$$\begin{aligned} c_{i,2,t+1} = \Upsilon_{t+1}\varrho_{t+1}k_{t+1}^\alpha(s(1-\tau)(1-\alpha)\frac{\Upsilon_1}{1+g}\zeta_t k_t^\alpha \alpha k_{t+1}^{-1}\varrho_{t+1} \\ + (1-\alpha)(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1})))) \end{aligned}$$

From there, we can substitute using the expression for k_{t+1} that we obtained from our guess to simplify $c_{i,2,t+1}$:

$$c_{i,2,t+1} = \Upsilon_{t+1}\varrho_{t+1}k_{t+1}^\alpha(\alpha\frac{\varrho_{t+1}}{1+\lambda} + (1+\alpha)(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))))$$

Finally, use the FOC of the household's problem and substitute in to get the expression for savings that is in the proposition:

$$1 = \beta\mathbb{E}_t(\frac{c_{1t}(1+r_{t+1})}{c_{i,2,t+1}})$$

$$1 = \beta\mathbb{E}_t(\frac{c_{1t}\alpha k_{t+1}^{\alpha-1}\zeta_{t+1}\varrho_{t+1}}{k_{t+1}^\alpha\Upsilon_{t+1}\zeta t + 1(\alpha\frac{\varrho_{t+1}}{1+\lambda} + (1-\alpha)(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))))})$$

$$1 = \beta\mathbb{E}_t(\frac{(1+s)(1-\tau)(1-\alpha)\Upsilon_t\varrho_t k_t^\alpha \alpha k_{t+1}^{-1}\varrho_{t+1}}{(1+g)\Upsilon_t(\alpha\frac{\varrho_{t+1}}{1+\lambda} + (1-\alpha)(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))))})$$

$$1 = \beta\mathbb{E}_t(\frac{(1-s)\alpha(1+g)(1+\lambda)s^{-1}\varrho_{t+1}}{(1+g)(\alpha\frac{\varrho_{t+1}}{1+\lambda} + (1-\alpha)(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))))})$$

$$1 = \beta\mathbb{E}_t(\frac{(1-s)}{s(1+\frac{(1-\alpha)}{\alpha(1+\lambda)\varrho_{t+1}}(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))))})$$

Where we can define $\Phi = \mathbb{E}_t \frac{1}{1+\frac{(1-\alpha)}{\alpha(1+\lambda)\varrho_{t+1}}((\lambda\eta_{i,2,t+1}+\tau(1+\lambda(1-\eta_{i,2,t+1})))}$ just like in the proposition and from there it is straightforward to get the expression for savings by isolating s:

$$1 = \beta\frac{1-s}{s}\Phi$$

$$s = \frac{\beta\Phi}{1+\beta\Phi}$$

2. Write code to perform a simulation of this first order difference equation in logs

$$ln(k_{t+1}) = ln(s(\tau)) + ln(1-\tau) + ln(1-\alpha) + ln(\zeta_t) - ln(1+\lambda) + \alpha k_t \quad (1)$$

To do this, I start by setting up the shocks: for the aggregate shocks (ζ_t and ρ_t) I create their two possible states as their respective mean $\pm$ their standard deviations, such that I have a good and a bad state for each. For the idiosyncratic shock I generate the eleven possible states and their transition probabilities using Gaussian quadrature through the function *qnwnorm*.

After that, I compute Φ as defined in the previous question and use it to calculate the savings (s). Then I calculate the steady state capital by isolating $k_t = k_{t+1} = k$ in equation 1, where I have added the term $ln(1+\lambda)$ as I think it was missing based on the expression for k_{t+1} we obtained in the previous question. I get $\Phi = 0.55, s = 0.26, k^{ss} = 0.05$

I run the simulation T=50000 times starting with an initial capital equal to the steady state value, and obtain values which oscillate between 0.04 and 0.06. These are around the value of the steady state but vary due to the different values taken by the aggregate shock to wages (ζ_t).

3. Simple Krusell-Smith algorithm

(a) Compute the theoretical values of the coefficients:

Using the Krusell-Smith algorithm requires first making an assumption on the policy function of assets tomorrow (a_{t+1}), such that the households will be able to forecast tomorrow's aggregate capital stock and the interest rate tomorrow. Therefore, assume that the level of assets tomorrow is a linear function of the constant marginal propensities to consume out of current assets and income conditional on the state of the economy today:

$$a_{t+1} = \psi_0 + \psi_1 a_t + \psi_3 \eta_t$$

And as a result, the aggregate capital stock tomorrow will take the following form:

$$K' = \psi_0 + \psi_1 \int a_t \delta\phi + \psi_3 \int \eta_t \delta\phi = \tilde{\psi_0(z)} + \psi_1(z)K$$

From there, it is easy to compute the theoretical values of the coefficients based on what would correspond to the intercept and the slope in equation (1), using the values of the parameters which were set out in the previous question:

$$\psi_0(z) = ln(s(\tau)) + ln(1-\tau) + ln(1-\alpha) + ln(\zeta_t) - ln(1+\lambda)$$

$$\psi_0(z) = ln(0.26) + ln(1) + ln(0.7) + ln(1) = (-2.2; -1.9)$$

$$\psi_1(z) = \alpha = (0.3; 0.3)$$

Note that the first value is for the recession state and the second one is for the boom. These theoretical values will be my initial guesses for the values of ψ_i in the following algorithm.

(b) Carry out the simple version of the Krusell-Smith algorithm to convergence:

Having created a grid on the logarithm of capital with 5 points, I follow the steps below in each iteration to find a solution for this problem using the Krusell-Smith approach:

1. For all values of the capital grid and all aggregate states (z), compute the predicted logarithm of capital tomorrow using the approximation ($k' = \psi_0(z)+\psi_1(z)K$). Also calculate the current wage and next period wages and interest rates, using the FOCs from the firm's problem:

$$w_{t+1} = (1-\alpha)\Upsilon_{t+1} k_{t+1}^{\alpha} \zeta_{t+1}$$

$$R_{t+1} = \alpha k_{t+1}^{\alpha-1} \zeta_{t+1} \varrho_{t+1}$$

2. Solve the household's problem for $a_{2,t+1}$, then compute savings and consumption. To do this, plug the budget constraints and FOCs of the firm into the Eurler equation. The expression I solve obtain analytically and then solve for in the code is the one below. I also included a non-negativity constraint for assets in the code because in that manner I avoid complex numbers in the calculations that follow.

$$\beta c_{1,t} = \mathbb{E}_t(\frac{c_{2,t+1}}{R_{t+1}})$$

$$\beta((1-\tau)w_t - a_{2,t+1}) = \mathbb{E}_t(\frac{a_{2,t+1}R_{t+1} + \lambda\eta_{i,2,t+1}w_{t+1}(1-\tau) + (1+\lambda)b_{t+1}}{R_{t+1}})$$

$$\beta((1-\tau)w_t - a_{2,t+1}) = a_{2,t+1} + \mathbb{E}_t(\frac{\lambda\eta_{i,2,t+1}w_{t+1}(1-\tau) + (1+\lambda)b_{t+1}}{R_{t+1}})$$

$$a_{2,t+1} = \frac{\beta(1-\tau)w_t - \lambda(1-\tau)\mathbb{E}_t(\frac{\eta_{i,2,t+1}w_{t+1}}{R_{t+1}}) - (1-\lambda)\mathbb{E}_t(\frac{b_{t+1}}{R_{t+1}})}{1+\beta}$$

$$a_{2,t+1} = \frac{\beta(1-\tau)w_t - \mathbb{E}_t(\frac{w_{t+1}}{R_{t+1}})(\lambda(1-\tau))\mathbb{E}_t(\eta_{i,2,t+1}) - \tau(1+\lambda))}{1+\beta}$$

Where the last equality is obtained by using the definition of pension benefits from the paper ($b_{t+1} = \tau w_{t+1}\frac{1+\lambda}{1-\lambda}$) and taking the common factor in the expectation term in order to simplify the computation. Note that in the code I define the terms $B = \mathbb{E}_t(\frac{w_{t+1}}{R_{t+1}})$ and $C = \mathbb{E}_t(\eta_{i,2,t+1})$.

3. Run a simulation for T=50000 periods in which I iterate forward on the law of motion of capital in order to get a sequence of values for capital next period. The initial value for capital is the third point in the grid, because it is the closest to the steady state. I use the same sequence of aggregate shocks ζ_t which I already created for the simulation in the previous question.

4. After getting rid of the first 500 observations in the simulated capital sequence, run two regressions of k' on k: one for those observations where the aggregate state was a boom and another where the state was a recession. The resulting coefficients will be estimated $\tilde{\psi_i(z)}$:

$$k^i_{t+1} = \psi_0 + \psi_1 k^i_t$$

$$\tilde{\psi^i} = (k^{i'}_t k^i_t)^{-1} k^{i'}_t k^i_{t+1}$$

Where $i \in \{R, B\}$ to denote the aggregate states of recession or boom.

5. Check for convergence: if $\|\psi(z) - \tilde{\psi(z)}| < \varepsilon$ there is convergence and the algorithm can be stopped. Otherwise, update the guesses of $\psi(z)$ in the following way and repeat from step 1:

$$\psi^{m+1} = \omega\tilde{\psi^m} + (1+\omega)\psi^m$$

I chose the tolerance level for convergence to be $\varepsilon = 0.0001$ and the update coefficient for ψ to be $\omega = 0.2$.

(c) Compare the analytical and numerical solutions:

After convergence of the previously defined algorithm, the resulting coefficients for the approximation of the law of motion of capital are $\psi_0(z) = (-3.2, -3.12)$ and $\psi_1(z) = (0.279, 0.288)$. The values for $\psi_0(z)$ are both smaller than the initially predicted theoretical values, but they are not very far off. Those for $\psi_1(z)$ are also smaller than the theoretical 0.3, but they are closer to the initial prediction. In both cases the value for the recession state is smaller than that for the boom, but the values of $\psi_0(z)$ for both states are closer together than I initially predicted.

The average savings obtained from the household problem in the last iteration range between 0.15 and 0.2 for the different points in the grid, those being the averages over both possible aggregate states. These values are slightly lower than the analytical value I obtained in the first exercise, which was s=0.25. When not averaged over the aggregate states, the value of savings in case of a recession ranges between 0 and 0.08 and in case of a boom they are between 0.3 and 0.32.

(d) Repeat for $\tau = 0.1$ and compare welfare measures:

In order to obtain the expected utility I run a simulation from which I obtain a sequence of values for$c_{1,t}$ and $\mathbb{E}_t(c_{i,2,t+1})$. To do so, I use the sequence of capital obtained in the simulation from the previous question to calculate the wages in the current period (for the realized aggregate shock ζ_t) and then use that to compute the sequence of consumption for the young generation ($c_{1,t}$). As the following step, I get the possible interest rate and wages in the next period and use them to obtain the expected consumption when old of the generation that is young in period t:

$$\mathbb{E}_t(c_{i,2,t+1}) = \mathbb{E}_t(a_{2,t+1}(1 + r_{t+1}) + \lambda\eta_{i,2,t+1}w_{t+1}(1 - \tau) + (1 + \lambda)b_{t+1})$$

$$\mathbb{E}_t(c_{i,2,t+1}) = a_{2,t+1}\mathbb{E}_t(R_{t+1}) + \lambda(1 - \tau)\mathbb{E}_t(\eta_{1,2,t+1}w_{t+1}) + \tau(1 + \lambda)\mathbb{E}_t(w_{t+1})$$

From there, I calculate expected lifetime utility of each generation born in period t as $EU_t = \mathbb{E}_t((1-\tilde{\beta})u(c_{1,t}) + \tilde{\beta}u(c_{i,2,t+1}))$. Finally, I calculate the average expected lifetime utility over the simulation, discarding the first 500 observations, as $E\bar{U}(\tau) = \frac{1}{T-500}\Sigma_{t=500}^{T}\mathbb{E}_t(U_t(\tau))$. In the case without a tax I get an average expected utility of -2.099, while after the introduction of the tax the average expected utility is -1.977. This implies that the individuals in this economy are, on average, better off after the introduction of the tax, as the benefit from the pension they receive when retired offsets the reduction in income while they are working.

To conclude, I compute the consumption equivalent variation, which is a measure of welfare which calculates the percentage increase in consumption necessary to make an individual indifferent between two scenarios (with or without the tax). It is calculated in the following manner:

$$g = exp(\frac{V_1 - V_0}{\beta}) - 1$$

Where V_0 is the average expected lifetime utility in the simulation with $\tau = 0$ and V_1 is the average expected lifetime utility in the simulation with

$\tau = 0.1$. In my case, I obtain $g = 0.19$, which reflects the improvement the individuals experience after the introduction of the tax.

Complex Variant of Krusell-Smith Algorithm

Starting from my code from general equilibrium OLG code for project 3, I modified it to include aggregate technology shocks in the production function ($Y_t = z_t F(K_t, L_t)$). Specifically, the aggregate shock takes two possible states ($z_t \in Z = [1.03, 0.97]'$) and has the following transition matrix:

$$\Pi_z = \begin{pmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{pmatrix}$$

I could not completely implement the Krusell-Smith algorithm in this case because I ran into some problems with aggregation and ran out of time. However, I did modify the code so that the calibration and the household problem run for the model with both idiosyncratic and aggregate risks, and I will explain how I would implement the rest of the algorithm.

To start with, we again need some guesses for the ψ coefficients of the approximated law of motion of capital, $k' = \psi_0 + \psi_1 k$. I suppose in this case the theoretical values would be obtained from the constraint $a' = s = a(1 + r) + \eta w - c$, but I was not sure how to calculate them so I made up an initial guess so that the code would run.

Due to the introduction of the aggregate shock in the production function, the problem of the firm and its' FOCs would change - they now have to include the shock, z. Therefore, the expressions for wages and interest rate become:

$$R = z\alpha K^{\alpha-1} L^{1-\alpha}$$

$$w = z(1 - \alpha) L^{-\alpha} K^{\alpha}$$

Another main change in the problem is that now the state space for all variables is increased to 5-dimensional: $(j, a, \eta; K, z)$. All policy functions will depend on these 5 states, and the value function will depend on both the idiosyncratic shock (η) and the aggregate shock (z).

On the code itself, I modified the calibration function so that it sets up the aggregate shock with its corresponding transition matrix, and I also generate a grid for aggregate capital in the same way as I did in the previous exercise (5 equally spaced grid points, with the minimum being $0.5K^{ss}$ and the maximum $1.5K^{ss}$). In the function that solves the household problem, I made sure to change all variables to now have 5 dimensions, and I added some loops to account for those additional state variables. I also updated the way in which the wages and interest rate were computed such that they include the technology shock, and took into account that due to the shock these variables can change in every period, so they had to be computed for the current and next periods.

My code currently runs the following steps:

1. Calibrate the model with the updated *func_calibr*
2. Set up the initial guess of the ψ coefficients
3. Approximate the logarithm of capital using the guess
4. Solve the household problem using the updated *func_hh*

To properly finish implementing the Krusell-Smith algorithm, I would need to add a loop that iterates over guesses of ψ, and inside this loop I would have to follow the same steps as in the previous question. In every iteration, after having solved the household problem, I would need to aggregate and then run a simulation whose values I could use to obtain estimated $\tilde{\psi}$ coefficients from a regression of K' on K. This would all need to take into account the increased dimensionality of the problem. Finally, I would check convergence by comparing ψ and $\tilde{\psi}$, and update the initial guess accordingly if there was no convergence.