Exercici 3 Computació i criptografia quàntica

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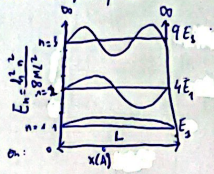
FQ verifica el principi d'incertesa de Heisenberg: parelles de quantitats observables per quals mai podrem assegurar que el sistema es troba en valors determinats simultàniament.



Ex important: posició - quantitat moviment

$$\Delta_x \Delta_p \geq \frac{\hbar}{2}$$

Avaluem incerteses, coneguda la funció d'ona ψ i és (on $p = -i\hbar\frac{d}{dx}$): $\Delta_x\psi = \sqrt{\langle x^2\rangle_\psi - \langle x\rangle_\psi^2}$ on:

$$\Delta_p\psi = \sqrt{\langle p^2\rangle_\psi - \langle p\rangle_\psi^2}$$

$\langle x\rangle_\psi = \int dx\, \psi^*(x,t)\, x\, \psi(x,t)$; $\langle x^2\rangle_\psi = \int dx\, \psi^*(x,t)\, x^2 \psi(x,t)$; $\langle p\rangle_\psi = \int dx\, \psi^*(x,t)\,(-i\hbar)\frac{d}{dx}\psi(x,t)$; $\langle p^2\rangle_\psi = \int dx\, \psi^*(x,t)\,(-\hbar^2)\frac{d^2}{dx^2}\psi(x,t)$

1. Quin seria el "valor esperat" valor mitjà que obtindríem si tiréssim un dau de sis cares moltes vegades? I quina la seva incertesa?

Si tirem un dau de sis cares varies vegades obtindrem una mitjana (μ) de:

$$\mu = \frac{\sum_{i=1}^{6} i}{6} = \boxed{3.5}$$

La seva incertesa seria de 2.5

2. Quant val $\langle x\rangle_{\psi_1}$ i $\langle p\rangle_{\psi_1}$, essent ψ_1 l'estat fonamental del pou unidimensional (1-d) infinit, d'amplada L. (A classe vàrem considerar que la partícula podia trobar-se a l'interval $0 \leq x \leq L$).

Per resoldre aquest apartat, primer resoldré l'apartat 3, ja que si ho demostrem $\forall n \in \mathbb{N} - \{0\}$, l'apartat 2 quedaria completament justificat. Per senzillesa, resoldré primer l'apartat 3.

3. Generalitzeu el resultat anterior a qualsevol dels estats propis de l'energia ψ_n.

Comencem resolent aquest apartat que ens permetrà resoldre també l'apartat 2.

Veiem que: $\psi_n = \sqrt{\frac{2}{L}}\sin\left(\underbrace{n\frac{\pi}{L}}_{k}x\right)e^{-i\frac{E_n t}{\hbar}}$, $\forall n \in \mathbb{N} - \{0\}$

Sabent que: $\int_a^b \psi_n^*\psi_n$:= Pr que de la partícula de trobar-se en un rang $[a,b]$. Trobarem $\langle x\rangle_n$ fent la mitjana ($\int$) de tots les x amb la seva Pr, podem veure que $\langle x\rangle_{\psi_n}$ queda com:

def int. trigonomètrica ($\int x\sin^2(ax)\,dx$) on $a = \frac{n\pi}{L}$

$$\langle x\rangle_{\psi_n} = \int_0^L \psi_n^*(x,t)\,\psi_n(x,t)\,x\,dx = \int_0^L |\psi_n(x,t)|^2 x\,dx = \int_0^L \frac{2}{L}\sin^2\left(n\frac{\pi}{L}x\right)x\,dx = \frac{2}{L}\int_0^L \sin^2\left(n\frac{\pi}{L}x\right)x\,dx =$$

$$= \frac{2}{L}\left[\frac{x^2}{4} - x\cdot\frac{\sin\left(2\cdot\frac{n\pi}{L}x\right)}{4\cdot\left(\frac{n\pi}{L}\right)} - \frac{\cos\left(2\cdot\frac{n\pi}{L}x\right)}{8\cdot\left(\frac{n\pi}{L}\right)^2}\right]_0^L = \frac{2}{L}\cdot\left[\left(\frac{L^2}{4} - L\cdot\frac{\sin\left(\frac{2\pi n\cancel{L}}{\cancel{L}}\right)}{4\cdot\left(\frac{n\pi}{L}\right)} - \frac{\cos\left(\frac{2\cdot n\pi \cancel{L}}{\cancel{L}}\right)}{8\cdot\left(\frac{n\pi}{L}\right)^2}\right) - \left(\frac{x^2}{4} - x\cdot\frac{\sin\left(\frac{2\pi n\cdot 0}{L}\right)}{4\cdot\left(\frac{n\pi}{L}\right)} - \frac{\cos\left(\frac{2n\pi\cdot 0}{L}\right)}{8\cdot\left(\frac{n\pi}{L}\right)^2}\right)\right]$$

$$= \frac{2}{L}\cdot\left[\left(\frac{L^2}{4} - \cancel{\frac{1}{8\cdot\left(\frac{n\pi}{L}\right)^2}}\right) + \left(\cancel{\frac{1}{8\cdot\left(\frac{n\pi}{L}\right)^2}}\right)\right] = \frac{2}{L}\cdot\frac{L^2}{4} = \boxed{\frac{L}{2}}$$

$$\langle p\rangle_{\psi_n} = \int_0^L \psi_n^*(x,t)(-i\hbar)\frac{\partial}{\partial x}\left(\psi_n(x,t)\right)dx = -i\hbar\int_0^L \psi_n^*(x,t)\frac{\partial}{\partial x}\left(\psi_n(x,t)\right)dx = -i\hbar\int_0^L\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\underbrace{e^{+i\frac{E_n t}{\hbar}}}_{1}\right)\frac{d}{dx}\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\underbrace{e^{-i\frac{E_n t}{\hbar}}}_{1}\right)dx$$

fora a part l'estat

1 → no té cap importància

def int trigonomètrica

$$= -i\hbar\int_0^L\left(\frac{\sqrt{2}}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right)\right)\left(\sqrt{\frac{2}{L}}\cdot\left(\frac{n\pi}{L}\right)\cdot\cos\left(\frac{n\pi x}{L}\right)\right)dx = -i\hbar\,\frac{2}{L}\left(\frac{n\pi}{L}\right)\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cdot\cos\left(\frac{n\pi x}{L}\right)dx = -i\hbar\,\frac{2}{L}\left(\frac{n\pi}{L}\right)\cdot\left[\frac{\sin^2\left(\frac{n\pi x}{L}\right)}{2\cdot\left(\frac{n\pi}{L}\right)}\right]_0^L =$$

$$= -i\hbar\,\frac{2}{L}\left(\frac{n\pi}{L}\right)\cdot\left[\frac{\sin^2\left(\frac{n\pi \cancel{L}}{\cancel{L}}\right)}{2\cdot\left(\frac{n\pi}{L}\right)} - \frac{\sin^2\left(\frac{n\pi}{L}\cdot 0\right)}{2\cdot\left(\frac{n\pi}{L}\right)}\right] = -i\hbar\,\frac{2}{L}\cdot\left(\frac{n\pi}{L}\right)\cdot 0 = \boxed{0}$$

Per tant, com hem vist $\forall n \in \mathbb{N}-\{0\}$ es compleix $\boxed{\langle x\rangle_{\psi_n} = \frac{L}{2}}$ i $\boxed{\langle p\rangle_{\psi_n} = 0}$

Un cop generalitzat a qualsevol dels estats propis de l'energia, veiem què passa a l'apartat 2.

2. Notem que tal i com hem vist a l'apartat anterior resulta (3) el valor de n no influeix en la mitjana ni a la probabilitat. Per tant tenim que per l'estat fonamental del pou 1-d de amplada L, $\langle x\rangle_{\psi_1}$ i $\langle p\rangle_{\psi_1}$, valen:

$$\boxed{\langle x\rangle_{\psi_1} = \langle x\rangle_{\psi_n} = \frac{L}{2}} \quad \text{i} \quad \boxed{\langle p\rangle_{\psi_1} = \langle p\rangle_{\psi_n} = 0}$$

4. De forma semblant, calculem $\langle x^2\rangle_{\psi_n}$ i $\langle p^2\rangle_{\psi_n}$

$$\langle x^2\rangle_{\psi_n} = \int_0^L \psi_n^*(x,t)\, x^2\, \psi_n(x,t)\,dx = \int_0^L |\psi_n(x,t)|^2 x^2\,dx = \int_0^L \left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) e^{-\frac{iE_nt}{\hbar}}\right)^2 x^2\,dx = \frac{2}{L}\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) \cancel{e^{-\frac{iE_nt}{\hbar}}}\, x^2\,dx$$

cte aplicant prop. trigon. $\int x^2 \sin^2(ax)\,dx$

$$= \frac{2}{L}\left[-\frac{x^2\cos\left(\frac{n\pi x}{L}\right)\cdot\sin\left(\frac{n\pi x}{L}\right)}{2\cdot\left(\frac{n\pi}{L}\right)} + \frac{x^3}{6} - \frac{x\cdot\cos^2\left(\frac{n\pi x}{L}\right)}{2\cdot\left(\frac{n\pi}{L}\right)^2} + \frac{\cos\left(\frac{n\pi x}{L}\right)\cdot\sin\left(\frac{n\pi x}{L}\right)}{4\cdot\left(\frac{n\pi}{L}\right)^3} + \frac{x}{4\cdot\left(\frac{n\pi}{L}\right)^2}\right]_0^L = \frac{2}{L}\left[\left(-\frac{L^2\cdot\cos(n\pi)\cdot\overset{0}{\sin(n\pi)}}{2\cdot\left(\frac{n\pi}{L}\right)} + \frac{L^3}{6} - \frac{L\cdot\overset{1}{\cos^2(n\pi)}}{2\cdot\left(\frac{n\pi}{L}\right)^2} + \frac{\cos(n\pi)\cdot\overset{0}{\sin(n\pi)}}{4\left(\frac{n\pi}{L}\right)^3} + \frac{L}{4\cdot\left(\frac{n\pi}{L}\right)^2}\right) - (0+0-0+0)\right]$$

$$= \frac{2}{L}\cdot\left(\frac{L^3}{6} - \frac{L}{2\cdot\left(\frac{n\pi}{L}\right)^2} + \frac{L}{4\left(\frac{n\pi}{L}\right)^2}\right) = \frac{2}{L}\cdot\left(\frac{L^3}{6} - \frac{L^3}{2n^2\pi^2} + \frac{L^3}{4n^2\pi^2}\right) = \boxed{L^2\cdot\left(\frac{1}{3} - \frac{1}{2\pi^2n^2}\right)}$$

$$\langle p^2\rangle_{\psi_n} = \int_0^L \psi_n^*(x,t)\,(-\hbar^2)\,\frac{\partial^2}{\partial x^2}\psi(x,t)\,dx = -\hbar^2\int_0^L \psi_n^*(x,t)\cdot\frac{\partial^2}{\partial x^2}\psi(x,t)\,dx = -\hbar^2\int_0^L\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\cdot\cancel{e^{\frac{iE_nt}{\hbar}}}\right)\frac{\partial^2}{\partial x^2}\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\cancel{e^{-\frac{iE_nt}{\hbar}}}\right)dx =$$

(cte)

$$= +\hbar^2\cdot\frac{2}{L}\frac{n^2\pi^2}{L^2}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cdot\sin\left(\frac{n\pi x}{L}\right)dx \overset{!!!}{=} +\hbar^2\frac{\pi^2 n^2}{L^2}\cdot\underbrace{\frac{2}{L}\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx}_{\text{per normalització} = 1} = +\hbar^2\frac{n^2\pi^2}{L^2}\cdot 1 = \boxed{\frac{\hbar^2 n^2\pi^2}{L^2}}$$

5. Comprovem que es compleix efectivament el principi de Heisenberg per cadascun dels estats d'energia continguts del pou infinit:

$$\Delta x_{\psi_n}\,\Delta p_{\psi_n} \geq \frac{\hbar}{2}, \quad \forall n\in\mathbb{N}-\{0\}$$

Tenim: $\Delta x_{\psi_n} = \sqrt{\langle x^2\rangle_{\psi_n} - \langle x\rangle^2_{\psi_n}} = \sqrt{\left(L^2\cdot\left(\frac{1}{3} - \frac{1}{2\pi^2n^2}\right)\right) - \left(\frac{L}{2}\right)^2} = \sqrt{\frac{L^2}{3} - \frac{L^2}{2\pi^2n^2} - \frac{L^2}{4}} = \boxed{\frac{L}{2n\pi}\sqrt{\frac{n^2\pi^2}{3} - 2}}$

$\Delta p_{\psi_n} = \sqrt{\langle p^2\rangle_{\psi_n} - \langle p\rangle^2_{\psi_n}} = \sqrt{\left(\frac{\hbar^2 n^2\pi^2}{L^2}\right) - 0^2} = \sqrt{\frac{\hbar^2n^2\pi^2}{L^2}} = \boxed{\frac{\hbar n\pi}{L}}$

Per tant:

$$\Delta x_{\psi_n}\cdot\Delta p_{\psi_n} = \left(\frac{L}{2n\pi}\cdot\sqrt{\frac{n^2\pi^2}{3} - 2}\right)\cdot\left(\frac{\hbar\cdot n\pi}{L}\right) = \frac{\hbar}{2}\underbrace{\sqrt{\frac{n^2\pi^2}{3} - 2}}_{>1} \geq \frac{\hbar}{2} \quad \blacksquare$$

>1 perquè $n\in\mathbb{N}-\{0\}$, en concret, per $n=1$ tindríem que l'arrel valdria 1.1357 i veuríem que: $\left(\frac{\hbar}{2}\right)\cdot 1.1357 \geq \frac{\hbar}{2}$ que ens verifica el principi de Heisenberg.