

Omni-MATH

Problem

Find in explicit form all ordered pairs of positive integers (m, n) such that $mn-1$ divides $m^2 + n^2$.

Reference answer

$(2, 1), (3, 1), (1, 2), (1, 3)$

Solution

To find all ordered pairs of positive integers (m, n) such that $mn-1$ divides $m^2 + n^2$, we start by considering the condition: $(m^2 + n^2)/(mn-1) = c$ where $c \in \mathbb{Z}$. This implies: $m^2 + n^2 = c(mn-1)$. Rewriting, we get: $m^2 - cmn + n^2 + c = 0$. Let (m, n) be a solution where $m + n$ is minimized. If (m, n) is a solution, then (m', n) must also be a solution, where: $m' = cn - m = (n^2 + c)/m$. Since m' is positive, $cn - m > 0$, and since m' is an integer, $cn - m \geq 1$. Assuming $m \neq n$ and without loss of generality, $m > n$, we claim $n = 1$. For contradiction, assume $n \neq 1$. Then $n > 1$ implies $m > n > 1$. By minimality of $m + n$, we must have: $m + n \leq m' + n \Rightarrow m \leq m'$. However, since $m > n > 1$: $n(m-n) \geq 2 \Rightarrow mn-2 \geq n^2 \Rightarrow m(mn-2) \geq mn^2 > n^3$, $n(m^2 - n^2) > 2m \Rightarrow m^2n > 2m + n^3$, $2m^2n - 2m > m^2n + n^3 \Rightarrow 2m(mn-1) > n(m^2 + n^2) \Rightarrow m > cn - m = m'$, a contradiction. Thus, $n = 1$. For $n = 1$, we have: $(m^2 + 1)/(m-1) = (m+1) + 2/(m-1)$. This is an integer if and only if $m-1$ divides 2. Hence, $m = 2$ or $m = 3$. Therefore, the solutions for $m > n$ are $(2, 1)$ and $(3, 1)$. Since the expression is symmetric in m and n , the pairs (m, n) that satisfy $(mn-1) | m^2 + n^2$ are: $(2, 1), (3, 1), (1, 2), (1, 3)$.

Metadata

Id: 30 · Domain: Number Theory · Difficulty: 6.5
· Source: USA team selection test

Model

Model generation

Let $mn-1$ divide $m^2 + n^2$. Then there is an integer $k \geq 1$ with $m^2 + n^2 = k(mn-1)$. By AM-GM, $m^2 + n^2 \geq 2mn$, so $k \geq 3$. Rewrite as a quadratic in m : $m^2 - knm + (n^2 + k) = 0$. Let m be a positive integer root with $m \geq n$. Then the other root is $m' = kn - m = (n^2 + k)/m$, hence m' is a positive integer. Compute the discriminant $\Delta = k^2n^2 - 4(n^2 + k)$. For $n \geq 2$ and $k \geq 3$ we have $\Delta - (k-2)^2n^2 = 4(k-2)n^2 - 4k > 0$, so $\sqrt{\Delta} > (k-2)n$, which implies $m' = (kn - \sqrt{\Delta})/2 < n$. Thus, from any solution with $n \geq 2$ we can produce a new solution with a smaller second coordinate. Iterating, we eventually reach a solution with $n = 1$. For $n = 1$ the equation becomes $m^2 + 1 = k(m-1) \Leftrightarrow m^2 - km + (k+1) = 0$. Its discriminant $k^2 - 4k - 4$ must be a perfect square. Solving $k^2 - 4k - 4 = t^2 \Leftrightarrow (k-2-t)(k-2+t) = 8$ gives the only positive integer solution $k = 5$. Then $m^2 - 5m + 6 = 0$, so $m = 2$ or 3 . Therefore, for all solutions we must have $k = 5$, $m^2 + n^2 = 5(mn-1)$. This Diophantine equation can be solved explicitly. It admits two infinite families generated from the base solutions $(1,2)$ and $(1,3)$ by the transformations $(m, n) \mapsto (n, 5n-m)$ or $(m, 5m-n)$. [...] The first few solutions are: $(1,2), (2,1), (1,3), (3,1), (2,9), (9,2), (3,14), (14,3), (9,43), (43,9), (14,67), (67,14), \dots$ Hence the complete set of ordered pairs is exactly the set of positive integer solutions to $m^2 + n^2 = 5(mn-1)$. [...] Equivalently, define sequences by a recurrence: Start with $a_0 = 1, a_1 \in \{2, 3\}$, and set $a_{t+1} = 5a_t - a_{t-1}$. Then every consecutive pair (a_t, a_{t+1}) (and its swap) is a solution, and all solutions arise this way.

Token use

Reasoning tokens: 21504 – Output tokens: 796

Omni-Judge

Inputs: problem, model generation, reference answer

Extracted final answer

All (m,n) in $\mathbb{Z}_{>0}$ with $m^2 + n^2 = 5(mn-1)$. Equivalently, the solutions are generated by $a_0=1, a_1 \in \{2,3\}, a_{t+1}=5a_t-a_{t-1}$ and taking consecutive pairs (and swaps).

Omni-Judge: Correct

The student's answer accurately describes all ordered pairs of positive integers (m, n) such that $mn-1$ divides $m^2 + n^2$. The reference solution is $(2, 1), (3, 1), (1, 2)$, and $(1, 3)$. Upon examination, it can be seen that these solutions are indeed among the ordered pairs generated by the Diophantine equation presented in the student's work, which corresponds to $m^2 + n^2 = 5(mn-1)$. This indicates that the student's answer shares the same meaning as the reference solution.

GPT-5 mini: Incorrect

The response's final answer asserts an infinite family: all positive integer solutions of $m^2 + n^2 = 5(mn-1)$, generated by the recurrence from seeds $(1,2)$ and $(1,3)$. The given `correct_answer` lists only the four ordered pairs $(2,1), (3,1), (1,2), (1,3)$. These do not agree: the response includes additional pairs (for example $(2,9), (9,2), (3,14), \dots$) that are not in the `correct_answer`. Thus the extracted final answer does not match the `correct_answer`.