

Omni-MATH

Problem

Let $h(x,y)$ be a real-valued function that is twice continuously differentiable throughout \mathbb{R}^2 , and define $\rho(x,y) = y h_x - x h_y$. Prove or disprove: For any positive constants d and r with $d > r$, there is a circle S of radius r whose center is a distance d away from the origin such that the integral of ρ over the interior of S is zero.

Reference answer

Proven: such a circle S exists.

Solution

We prove the given statement. For any circle S of radius r whose center is at distance d from the origin, express the integral in polar coordinates s, ϑ : $\iint_S \rho = \int_{s_1}^{s_2} \int_{\vartheta_1(s)}^{\vartheta_2(s)} (y h_x - x h_y)(s \sin(\vartheta), s \cos(\vartheta)) s d\vartheta ds$. F or fixed s , the integral over ϑ is a line integral of $\text{grad } h$, which evaluates to $h(P_2) - h(P_1)$, where P_1, P_2 are the endpoints of the endpoints of the arc of the circle of radius s centered at the origin lying within S . If we now fix r and d and integrate $\iint_S \rho$ over all choices of S (this amounts to a single integral over an angle in the range $[0, 2\pi]$), we may interchange the order of integration to first integrate over ϑ , then over the choice of S , and at this point we get 0 for every s . We conclude that the integral of \iint_S over all choices of S vanishes; since the given integral varies continuously in S , by the intermediate value theorem there must be some S where the given integral is 0 .

Metadata

Id: 3239 · Domain: Calculus · Difficulty: 8 · Source: Putnam

Model

Model generation

True. Proof: Let $c_0 = (d, 0)$ and let D_0 be the disk of radius r centered at c_0 . For $\vartheta \in [0, 2\pi)$, let R_ϑ be the rotation by angle ϑ about the origin and let $D_\vartheta = R_\vartheta D_0$; these are precisely the disks of radius r with centers at distance d from the origin. Define $F(\vartheta) = \int_{D_\vartheta} h(x, y) dx dy$. Then F is 2π -periodic. Using the change of variables $x = R_\vartheta y$ (Jacobian 1), $F(\vartheta) = \int_{D_0} h(R_\vartheta y) dy$. Since $h \in C^2$, we may differentiate under the integral sign: $F'(\vartheta) = \int_{\partial D_\vartheta} \nabla h(R_\vartheta y) \cdot (J R_\vartheta y) dy$, where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the generator of rotations. Noting that $\nabla h(z) \cdot (J z) = y h_x(z) - x h_y(z) = \rho(z)$, we get $F'(\vartheta) = \int_{D_0} \rho(R_\vartheta y) dy = \int_{D_\vartheta} \rho(x, y) dx dy$. Thus, if we set $I(\vartheta) = \int_{D_\vartheta} \rho$, we have $I(\vartheta) = F'(\vartheta)$. Because F is 2π -periodic, $\int_0^{2\pi} I(\vartheta) d\vartheta = F(2\pi) - F(0) = 0$. Since I is continuous, either $I \equiv 0$ or it takes both positive and negative values; in either case there exists ϑ_0 with $I(\vartheta_0) = 0$. Therefore, the circle $S = \partial D_{\vartheta_0}$ of radius r with center at distance d from the origin satisfies $\int_{\text{int}(S)} \rho(x, y) dx dy = 0$.

Token use

Reasoning tokens: 3520 – Output tokens: 437

Omni-Judge

Inputs: problem, model generation, reference answer

Extracted final answer

True

Judge: Incorrect

Although the student's answer is also "True", it does not capture the same meaning as the reference answer, which explicitly states that such a circle exists and can be proven to exist. The student's solution concludes with the statement "True" but fails to demonstrate or discuss the existence of the desired circle as required by the problem. Therefore, despite sharing the same truth value "True", the student's answer does not align with the reference answer in terms of its substance and explanation.