Big picture (slides logic)
Hadrons are **composite** (made of quarks), so in EM processes the hadronic EM current is a QCD current:

$$j^\mu_{\mathrm{QCD}} = \sum_{j=u,d,s,\dots} q_j\,\bar\psi_j\gamma^\mu\psi_j, \qquad \partial_\mu j^\mu_{\mathrm{QCD}} = 0.$$

We handle compositeness by:
- **Form factors** (low/intermediate energies): parameterize matrix elements using symmetries.
- **DIS/parton model** (large Q^2): photons see quasi-free point-like constituents (partons).
- **QCD** (underlying theory): local $SU(3)$ color gauge theory with gluons (neutral partons), asymptotic freedom, chiral symmetry (approx.).

6.1 Composite particles: EM current & form factors

Elementary QED (many flavors):

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{j=1}^{n}\bar\psi_j(i\gamma^\mu D^j_\mu - m_j)\psi_j, \quad D^j_\mu = \partial_\mu + iq_jA_\mu.$$

Global rephasings $\psi_j \to e^{i\theta_j}\psi_j$ ($\theta_j \neq \theta_j(x)$) $\Rightarrow$ n conserved flavor charges.
Interaction:

$$\mathcal{L}_I = -\sum_{j=1}^{n} q_j\,\bar\psi_j\gamma^\mu A_\mu\psi_j \equiv -j^\mu A_\mu.$$

For hadrons: replace "hadron EM current" by j^μ_{QCD} and **parameterize** unknown QCD matrix elements.

Pion EM form factor: $e^-e^+ \to \pi^+\pi^-$
Need matrix element (unknown in QCD):

$$\langle\pi^+(\vec p_1)\pi^-(\vec p_2)|\, j^\nu_{\mathrm{QCD}}(x)\,|0\rangle.$$

Use translation invariance (P^μ momentum operator):

$$\langle\pi\pi|j^\nu_{\mathrm{QCD}}(x)|0\rangle = e^{-i(p_1+p_2)\cdot x}\langle\pi\pi|j^\nu_{\mathrm{QCD}}(0)|0\rangle.$$

Lorentz invariance:

$$\langle\pi\pi|j^\nu_{\mathrm{QCD}}(0)|0\rangle = e(p_1+p_2)^\nu\tilde F_\pi(s) + e(p_2-p_1)^\nu F_\pi(s), \quad s=(p_1+p_2)^2.$$

Current conservation $(p_1+p_2)_\nu j^\nu = 0$:

$$0 = (p_1+p_2)_\nu\langle\pi\pi|j^\nu|0\rangle = 2s\,\tilde F_\pi(s) \;\Rightarrow\; \tilde F_\pi(s)=0.$$

Result (pion)

$$\boxed{\langle\pi^+\pi^-|j^\nu_{\mathrm{QCD}}(0)|0\rangle = e(p_2-p_1)^\nu F_\pi(s)}$$

All compositeness is encoded in $\boxed{F_\pi(s)}$.
Low energy: pion looks pointlike $\Rightarrow F_\pi(s)\simeq 1$ for $s \lesssim 4m_\pi^2$ (slides statement).

Proton form factors: $e^-p \to e^-p$
Need

$$\langle p(\vec p_2,\lambda_2)|\, j^\nu_{\mathrm{QCD}}(x)\,|p(\vec p_B,\lambda_B)\rangle = e^{-i(p_2-p_B)\cdot x}\,\bar u_{\lambda_2}(\vec p_2)\,D^\nu(p_2,p_B)\,u_{\lambda_B}(\vec p_B),$$

with $q \equiv p_2 - p_B$ ($q^2 = t$).
Most general Dirac structure + discrete symmetries; parity $\Rightarrow$ remove γ_5 terms; current conservation $q_\nu j^\nu = 0$ reduces to **two** independent form factors. Using Gordon identity, choose:

$$\boxed{D^\nu(p_2,p_B) = e\,F_1(q^2)\gamma^\nu + \frac{e}{2m_p}\,F_2(q^2)\,i\sigma^{\nu\rho}q_\rho}.$$

Low $q^2 \simeq 0$:

$$F_1(0)\simeq 1, \quad F_2(0)\simeq\kappa_p = \frac{g_p}{2}-1 \simeq 1.79 \quad (\text{neutron: } F_1(0)\simeq 0,\ F_2(0)\simeq\kappa_n\simeq -1.91).$$

Non-minimal Pauli term (unique dim-5, gauge/Lorentz/discrete symm.):

$$\delta\mathcal{L} = \frac{e\,\kappa_p}{2m_p}\,\bar\psi\,\sigma_{\mu\nu}F^{\mu\nu}\psi.$$

Unpolarized $e^-p\to e^-p$: hadronic tensor form

Instead of amplitude-level form factors, for unpolarized scattering define:

$$e^2L_p^{\mu\nu}(p_B,p_2) = \frac{1}{2}\sum_{\lambda_B=\pm}\sum_{\lambda_2=\pm}\langle p_2,\lambda_2|j^\mu_{\mathrm{QCD}}(0)|p_B,\lambda_B\rangle\langle p_2,\lambda_2|j^\nu_{\mathrm{QCD}}(0)|p_B,\lambda_B\rangle^*.$$

Lorentz + parity:

$$L_p^{\mu\nu} = -2m_p^2G_1g^{\mu\nu} + G_2p^\mu p^\nu + G_3q^\mu q^\nu + G_4(p^\mu q^\nu + q^\mu p^\nu), \quad p\equiv p_B+p_2.$$

Current conservation $q_\mu L_p^{\mu\nu}=0$:

$$G_4 = 0, \qquad G_3 = \frac{2m_p^2}{q^2}G_1,$$

so

$$\boxed{L_p^{\mu\nu} = 2m_p^2G_1\Big(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\Big) + G_2\,p^\mu p^\nu}.$$

Leptonic tensor:

$$L_e^{\mu\nu}(p_A,p_1) = -q^2\Big(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\Big) + (p_A+p_1)^\mu(p_A+p_1)^\nu, \quad q = p_A - p_1.$$

Matrix element squared:

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\,L_e^{\mu\nu}L^p_{\mu\nu}.$$

LAB cross section and Sachs form factors
Slides give (unpolarized, LAB):

$$L_e^{\mu\nu}L^p_{\mu\nu}\Big|_{\mathrm{LAB}} = 16m_p^2E_AE_1\Big(G_1\sin^2\frac{\theta}{2} + G_2\cos^2\frac{\theta}{2}\Big),$$

$$\boxed{\Big(\frac{d\sigma}{d\Omega}\Big)_{\mathrm{LAB}} = \frac{\alpha^2E_1^2}{4E_A^3\sin^4(\theta/2)}\Big(G_2\cos^2\frac{\theta}{2} + G_1\sin^2\frac{\theta}{2}\Big)}.$$

Often reparameterize:

$$G_2 = \frac{G_E^2 - \frac{q^2}{4m_p^2}G_M^2}{1-\frac{q^2}{4m_p^2}}, \qquad G_1 = -\frac{q^2}{2m_p^2}G_M^2, \qquad Q^2\equiv -q^2 > 0.$$

Empirical (slides):

$$G_E \simeq \frac{1}{1+Q^2/Q_0^2}, \qquad G_M \simeq \frac{g_p}{2}\,G_E, \qquad Q_0^2 \simeq 0.71\ \mathrm{GeV}^2.$$

6.2 Deep Inelastic Scattering (DIS): $e^-p \to e^- + X$

Final hadronic state $|f\rangle$ is **inclusive** (unobserved). Replace single-proton phase space by a sum over hadronic states:

$$\sum_f\int\Big(\prod_f\frac{d^3\vec p_f}{(2\pi)^32E_f}\Big)\frac{1}{2}\sum_{\lambda_B}\sum_{\lambda_f}\langle f|j^\mu_{\mathrm{QCD}}(0)|p_B,\lambda_B\rangle\,\langle f|j^\nu_{\mathrm{QCD}}(0)|p_B,\lambda_B\rangle^* \times (2\pi)^4\delta^{(4)}\Big(p_A+p_B-p_1-\sum_f p_f\Big) \equiv 4\pi m_p\,W^{\mu\nu}(p_B,q). \tag{1}$$

with

$$q = p_A - p_1, \qquad \nu \equiv \frac{q\cdot p_B}{m_p} = q^0 = E_A - E_1\ (\mathrm{LAB}), \qquad Q^2\equiv -q^2.$$

Lorentz + parity + current conservation:

$$\boxed{\begin{aligned} W^{\mu\nu} = {} & W_1(Q^2,\nu)\Big(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\Big) \\ & + W_2(Q^2,\nu)\Big(p_B^\mu - \frac{p_B\cdot q}{q^2}q^\mu\Big)\Big(p_B^\nu - \frac{p_B\cdot q}{q^2}q^\nu\Big). \end{aligned}} \tag{2}$$

Then (slides):

$$L_e^{\mu\nu}W_{\mu\nu} = 4E_AE_1\Big(W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2}\Big),$$

$$\boxed{\Big(\frac{d\sigma}{dE_1\,d\Omega}\Big)_{\mathrm{LAB}} = \frac{4\alpha^2E_1^2}{q^4}\Big(W_2(Q^2,\nu)\cos^2\frac{\theta}{2} + 2W_1(Q^2,\nu)\sin^2\frac{\theta}{2}\Big)} \tag{3}$$

Bjorken scaling and variables

For elementary fermion target (muon) the structure functions depend on

$$\omega \equiv -\frac{2m\nu}{q^2} = \frac{2m\nu}{Q^2} \quad \Rightarrow \quad \nu W_2,\ 2mW_1 \text{ depend only on } \omega.$$

SLAC data for $e^- p \to e^- +$ hadrons show approximate scaling at large Q^2.

Define DIS variables (slides):

$$x \equiv \frac{Q^2}{2m_p\nu}, \qquad y \equiv \frac{p_B \cdot q}{p_B \cdot p_A}.$$

Parton model (slides)

Proton = free partons with momentum fraction x:

$p_i = x\,p_B, \qquad f_i(x) \equiv$ probability density that parton i carries fraction x.

For spin-1/2 parton (slides) one finds the proton structure functions:

$$F_2(x) = \sum_i f_i(x)\,Q_i^2\,x, \qquad F_1(x) = \sum_i f_i(x)\,Q_i^2\,\frac{1}{2},$$

and

$\boxed{2xF_1(x) = F_2(x)}$ (Callan–Gross, characteristic of spin-1/2 partons).

If partons had spin 0 (slides note): $F_1(x) = 0$.

At large $s \gg m_p$ (slides exercise result):

$$\boxed{\frac{d\sigma}{dx\,dy} \simeq \frac{2\pi\alpha^2\,s}{q^4}\,F_2(x)\Big(1 + (1-y)^2\Big)}.$$

PDFs and quark content (slides)

Assume partons are quarks and only u, d, s relevant:

$$\frac{F_2(x)}{x} = \left(\frac{2}{3}\right)^2\!\big(u + \bar{u}\big) + \left(\frac{1}{3}\right)^2\!\big(d + \bar{d}\big) + \left(\frac{1}{3}\right)^2\!\big(s + \bar{s}\big), \quad i(x) \equiv f_i(x).$$

Split into valence + sea: $i(x) = i_v(x) + i_s(x)$. For proton/neutron: $s_v = 0$ and (SU(3) sea assumption in slides) $u_s = d_s = s_s = \bar{u}_s = \bar{d}_s = \bar{s}_s \equiv s(x)$.

Then (slides):

$$\frac{F_2^p(x)}{x} = \left(\frac{2}{3}\right)^2 u_v(x) + \left(\frac{1}{3}\right)^2 d_v(x) + \frac{4}{3}s(x),$$

$$\frac{F_2^n(x)}{x} = \left(\frac{2}{3}\right)^2 d_v(x) + \left(\frac{1}{3}\right)^2 u_v(x) + \frac{4}{3}s(x),$$

$$\frac{F_2^p(x)}{F_2^n(x)} \xrightarrow{x\to 0} 1, \qquad \frac{F_2^p(x)}{F_2^n(x)} \xrightarrow{x\to 1} \frac{4u_v + d_v}{4d_v + u_v}, \qquad \frac{1}{4} \le \frac{F_2^p}{F_2^n} \le 4,$$

$$\frac{F_2^p(x)}{x} - \frac{F_2^n(x)}{x} = \frac{1}{3}\big(u_v(x) - d_v(x)\big).$$

Normalization & momentum sum rules (slides)

Valence number:

$$\int_0^1 dx\,u_v(x) = 2, \qquad \int_0^1 dx\,d_v(x) = 1.$$

Momentum:

$$\boxed{1 = \int_0^1 dx\,x\big(u + \bar{u} + d + \bar{d} + s + \bar{s} + g(x)\big) \equiv I_{uv} + I_{dv} + 6I_s + I_g}.$$

Slides use measured

$$I_p \equiv \int_0^1 dx\,F_2^p(x) \simeq 0.18, \qquad I_n \simeq 0.12$$

to infer $\boxed{4I_g > I_{uv} + I_{dv}}$: electrically neutral partons carry lots of momentum $\Rightarrow$ gluons.

Scaling violations: PDFs develop mild Q^2 dependence; strong at very low x (slides remark).

6.3 QCD (slides)

Requirements listed in slides (summary):

- Quarks carry color $SU(3)$ and flavor; only color singlets physical.
- Approx. chiral symmetry $SU_L(N_f) \otimes SU_R(N_f)$ for small quark masses.
- Spontaneous chiral breaking to diagonal $SU(N_f)$.
- Asymptotic freedom: $\alpha_s(\mu) \to 0$ as $\mu \to \infty$ (quarks look free at high Q^2).
- Need neutral constituents (gluons) to satisfy momentum sum rule.

Color triplet quark field:

$$q(x) = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \qquad q \to g(x)q, \quad \bar{q} \to \bar{q}\,g^\dagger(x), \quad g(x) \in SU(3).$$

Covariant derivative:

$$D_\mu = \partial_\mu + ig_s\,G_\mu, \qquad G_\mu = T^a G_\mu^a, \quad G_\mu \to gG_\mu g^\dagger - \frac{i}{g_s} g\partial_\mu g^\dagger.$$

Field strength (Yang–Mills):

$$ig_s\,G_{\mu\nu} \equiv [D_\mu, D_\nu], \qquad G_{\mu\nu} = T^a G_{\mu\nu}^a, \quad G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g_s f^{abc} G_\mu^b G_\nu^c,$$

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2}\text{tr}(G_{\mu\nu}G^{\mu\nu}) = -\frac{1}{4}G_{\mu\nu}^a G^{a\,\mu\nu}.$$

$$\boxed{\mathcal{L}_{\text{QCD}} = -\frac{1}{4}G_{\mu\nu}^a G^{a\,\mu\nu} + \sum_{j=u,d,s,\ldots} \bar{q}_j(i\not{D} - m_j)q_j}.$$

Unlike QED, QCD has gluon self-interactions. Flavor rephasings $q_j \to e^{i\theta_j} q_j$ (global) $\Rightarrow$ flavor conserved. For $m_j = 0$, chiral symmetry $SU_L(N_f) \otimes SU_R(N_f)$.

$R(e^-e^+ \to \text{hadrons})$ **(duality)**

Define (slides):

$$\boxed{R \equiv \frac{\sigma(e^-e^+ \to \text{hadrons})}{\sigma(e^-e^+ \to \mu^+\mu^-)}}.$$

Duality hypothesis at high energy:

$$\sigma(e^-e^+ \to \text{hadrons}) \simeq \sigma(e^-e^+ \to \text{quarks} + \text{gluons}) \simeq \sum_j \sigma(e^-e^+ \to q_j\bar{q}_j) \quad (\sqrt{s}$$

Hence

$$\boxed{R \simeq \sum_{j=u,d,s,\ldots} Q_j^2\,N_c}.$$

Slides list stepwise predictions as thresholds open (e.g. u, d, s then add c, then b, then t).

Heavy quarks: NRQCD, HQET, quarkonium (slides)

For heavy $m_Q \gg \Lambda_{\text{QCD}}$, $\alpha_s(m_Q)$ small and motion non-relativistic:

$$\boxed{\mathcal{L}_{\text{NRQCD}} = \psi^\dagger\Big(iD_0 + \frac{\vec{D}^{\,2}}{2m_Q} + \vec{\mu}\cdot\vec{B} + \cdots\Big)\psi, \qquad \vec{\mu} \sim \frac{g_s}{m_Q}\vec{S}}.$$

Heavy–light hadron $H = (Q\,l)$ (slides):

$$\mathcal{L}_{\text{HQET}} \simeq \psi^\dagger iD_0\psi, \qquad \boxed{M_H = m_Q + \Lambda_l^0 + \mathcal{O}\Big(\frac{1}{m_Q}\Big)}, \qquad \boxed{M_{H^*} - M_H = \frac{\Lambda_l^2}{m_Q} +$$

Heavy quarkonium $H = (Q\bar{Q})$: non-relativistic potential picture; decays to light hadrons via gluons. Slides state:

$$J^{P+} \to gg, \qquad J^{P-} \to ggg, \qquad \frac{\Gamma(J^{P-} \to ggg)}{\Gamma(J^{P+} \to gg)} \sim \alpha_s(m_Q) \ll 1,$$

and quote that experimentally the corresponding ratios for $J/\psi, \Upsilon$ vs η_c, η_b are very small (numerical accident mentioned in slides).

Ultra-compact checklist (Topic 6)

- Composite hadrons $\Rightarrow$ use QCD current $j^\mu_{\text{QCD}} = \sum q_j \bar{\psi}_j \gamma^\mu \psi_j$ and symmetries to parameterize matrix elements.
- Pion: $\langle \pi\pi | j^\mu | 0\rangle = e(p_2 - p_1)^\mu F_\pi(s)$; low-$s$: $F_\pi \simeq 1$.
- Proton: $\langle p | j^\mu | p\rangle = \bar{u}\Big(eF_1\gamma^\mu + \frac{e}{2m_p}F_2 i\sigma^{\mu\nu}q_\nu\Big)u$; $F_2(0) = \kappa_p$.
- Unpolarized ep: hadronic tensor depends on two functions G_1, G_2; often rewrite via Sachs G_E, G_M.
- DIS: inclusive tensor $W^{\mu\nu}$ with W_1, W_2; scaling at large Q^2.
- Partons: PDFs $f_i(x)$; $F_2(x) = \sum f_i Q_i^2 x$; $\boxed{2xF_1 = F_2}$ for spin-1/2 partons.
- Momentum sum rule forces neutral partons $\Rightarrow$ gluons.
- QCD: local $SU(3)$ with gluons; asymptotic freedom explains scaling; non-perturbative for chiral breaking.
- $R(e^+e^- \to \text{hadrons}) \simeq \sum Q_j^2 N_c$ at high energy (duality).
- Heavy quarks: NRQCD/HQET mass and spin-splitting scaling with $1/m_Q$.