Vector Spaces: Foundations

What is a vector space?

Def. A **vector space** over a field $\mathbb{F}$ is a set V with two operations:

1. **Addition:** $+ : V \times V \to V$
2. **Scalar multiplication:** $\cdot : \mathbb{F} \times V \to V$

satisfying 8 axioms (closure, associativity, commutativity, identities, inverses, distributivity).

Key examples?

Ex. Common vector spaces:

- $\mathbb{R}^n$ over $\mathbb{R}$ (Euclidean space)
- $\mathcal{P}_n(\mathbb{R})$ = polynomials of degree $\leq n$
- $\mathcal{M}_{m\times n}(\mathbb{R}) = m \times n$ matrices
- $C[a, b]$ = continuous functions on $[a, b]$

What is a subspace?

Def. A subset $W \subseteq V$ is a **subspace** if:

1. $\mathbf{0} \in W$ (contains zero vector)
2. $\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$ (closed under addition)
3. $c \in \mathbb{F}, \mathbf{v} \in W \Rightarrow c\mathbf{v} \in W$ (closed under scalar mult.)

Subspace test shortcut?

Prop. $W \neq \emptyset$ is a subspace $\Leftrightarrow \forall \mathbf{u}, \mathbf{v} \in W, \forall c, d \in \mathbb{F} : c\mathbf{u} + d\mathbf{v} \in W$.

Linear Independence & Span

Linear combo?

Def. A **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \quad c_i \in \mathbb{F}$$

Span = ?

Def. The **span** of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is the set of all linear combinations:

$$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \left\{ \sum_{i=1}^{n} c_i\mathbf{v}_i : c_i \in \mathbb{F} \right\}$$

This is always a subspace of V.

Linear independence?

Def. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **linearly independent** if:

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

Otherwise, they are **linearly dependent**.

How to test?

Rmk. For $\mathbb{R}^n$: Put vectors as columns in matrix A. They are linearly independent $\Leftrightarrow \det(A) \neq 0$ (if square) or $\operatorname{rank}(A) = n$.

Basis & Dimension

What is a basis?

Def. A **basis** for V is a set $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ such that:

1. $\mathcal{B}$ is linearly independent
2. $\operatorname{span}(\mathcal{B}) = V$

Standard basis?

Ex. Standard basis for $\mathbb{R}^3$: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Standard basis for $\mathcal{P}_2(\mathbb{R})$: $\{1, x, x^2\}$

Dimension?

Thm. (Dimension Theorem) All bases of a finite-dimensional vector space V have the same number of elements. This number is called the **dimension** of V, denoted $\dim(V)$.

Key dimensions?

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathcal{P}_n(\mathbb{R})) = n + 1$

Extend/reduce?

- $\dim(\mathcal{M}_{m\times n}) = mn$

Prop. • Any linearly independent set can be **extended** to a basis.

- Any spanning set can be **reduced** to a basis.

Coordinates & Change of Basis

Coordinate vector?

Def. If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis and $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, the **coordinate vector** of $\mathbf{x}$ w.r.t. $\mathcal{B}$ is:

$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{F}^n$$

Change of basis?

Thm. Let $\mathcal{B}$ and $\mathcal{C}$ be two bases. The **change of basis matrix** from $\mathcal{B}$ to $\mathcal{C}$ is $P_{\mathcal{C}\leftarrow\mathcal{B}}$ such that:

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}} \cdot [\mathbf{x}]_{\mathcal{B}}$$

Inverse?

Columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are the $\mathcal{C}$-coordinates of the $\mathcal{B}$-basis vectors.

Cor. $P_{\mathcal{B}\leftarrow\mathcal{C}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$

Rank-Nullity Theorem

Kernel/Image?

Def. For linear map $T : V \to W$:

- **Kernel:** $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$ (nullspace)
- **Image:** $\mathrm{Im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$ (range)

Rank-Nullity?

Thm. (Rank-Nullity) For $T : V \to W$ linear with $\dim(V) < \infty$:

$$\boxed{\dim(\ker T) + \dim(\mathrm{Im}\ T) = \dim(V)}$$

For matrices?

Equivalently: $\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V)$

Rmk. For $A \in \mathcal{M}_{m\times n}$: $\mathrm{nullity}(A) + \mathrm{rank}(A) = n$ (number of columns).

SUMMARY

Vector space: Set with addition & scalar mult. satisfying 8 axioms. **Subspace:** Contains $\mathbf{0}$, closed under $+$ and $\cdot$. **Span:** All linear combos. **Linearly independent:** Only trivial solution to $\sum c_i\mathbf{v}_i = 0$. **Basis:** Linearly indep. + spans V. **Dimension:** # of basis vectors. **Rank-Nullity:** $\dim(\ker T) + \dim(\mathrm{Im}\ T) = \dim(V)$.