Zeros of functions

Def. $f\colon I_0=[a_0,b_0]\to\mathbb{R}$ with $f(a_0)\,f(b_0)<0$. Assume $\exists!\alpha\in I_0$ s.t. $f(\alpha)=0$.

- Bisection: sequence of intervals $I_k=[a_k,b_k]$, $x_k=\frac{a_k+b_k}{2}$, if $f(x_k)=0$ we are done, otherwise
  - $I_{k+1}=[a_k,x_k]$ if $f(a_k)\,f(x_k)<0$
  - $I_{k+1}=[x_k,b_k]$ if $f(b_k)\,f(x_k)<0$

  If we approximate α by x_k, the error is $|E|<\frac{b_0-a_0}{2^{k+1}}$. Geometric convergence.
- Secant: given $(x_{k-1},f(x_{k-1}))$ and $(x_k,f(x_k))$, let $g(x)$ be the line through them. Take x_{k+1} where $g(x)=0$, $x_{k+1}=x_k-f(x_k)\frac{x_{k+1}-x_k}{f(x_{k+1})-f(x_k)}$. Not always convergent, faster than bisection.
- Regula falsi: $I_k=[x_{k-1},x_k]$, compute x_{k+1} as in the secant method, choose I_{k+1} as in bisection. Convergent if in $I_1=[x_0,x_1]$, $f(x_0)f(x_1)<0$. Slower than secant.
- Newton: given $(x_k,f(x_k))$, let $g(x)$ be the line through the point with slope $f'(x_k)$. Take x_{k+1} where $g(x)=0$, $x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$. Not always convergent.

Thm. $f\colon[a,b]\to\mathbb{R}$, class $\mathscr{C}^2$ s.t.:
(i) $f(a)f(b)<0$
(ii) $f'(x)\neq 0\ \forall x\in[a,b]$
(iii) $f''(x)\geq 0$ (or $f''(x)\leq 0$) $\forall x\in[a,b]$
(iv) $c\in\{a,b\}$ is where $|f'(x)|$ is smaller $\Rightarrow\left|\frac{f(c)}{f'(c)}\right|\leq b-a$

Then $\exists!\alpha\in[a,b]$ such that $f(\alpha)=0$ and Newton's method converges to α $\forall x_0\in[a,b]$.

Def. $\{x_k\}_{k\geq 0}$ converging to α has order of convergence at least p if $\exists C>0$, $N\geq 0$ s.t. $|x_{k+1}-\alpha|\leq C|x_k-\alpha|^p$ $\forall k\geq N$. Equivalently, if $\exists\lim_{k\to\infty}\frac{x_{k+1}-\alpha}{(x_k-\alpha)^p}$ we say it has ord. conv. at least p. If $p=1$, we require $L<1$. L is the asymptotic error constant.

Prop. Iteration $x_{k+1}=g(x_k)$, s a fixed point. If $g\in\mathscr{C}^\infty(s-\varepsilon,s+\varepsilon)$, $g^{(j)}(s)=0$ for $j=0\div p-1$ and $g^{(p)}(s)\neq 0$, then $\{x_k\}_{k\geq 0}$ converges with order p.

Obs. (Aitken acceleration) Assume $x_k\to s$. Define $\Delta x_k:=x_{k+1}-x_k$, $\Delta^2 x_k:=\Delta x_{k+1}-\Delta x_k$. Let $g(x)$ be the line through $(x_k,\Delta x_k)$, $(x_{k+1},\Delta x_{k+1})$. Take x'_k where $g(x)=0$, $x'_k=x_k-\frac{(\Delta x_k)^2}{\Delta^2 x_k}$.

Prop. Assume $\lim_{k\to\infty}x_k=s$, $x_k\neq 0\ \forall k$, and $\exists C$, $|C|<1$ s.t. $x_{k+1}-s=(C+\delta_k)(x_k-s)$, with $\lim_{k\to\infty}\delta_k=0$. Then $\{x'_k\}_{k\geq 0}$ is well-defined for k large enough and $\lim_{k\to\infty}\frac{x'_k-s}{x_k-s}=0\Rightarrow\{x'_k\}_{k\geq 0}$ converges faster than $\{x_k\}_{k\geq 0}$.

Prop. (Steffensen acceleration) $g\colon[a,b]\to\mathbb{R}$ of class $\mathscr{C}^2$, $s\in[a,b]$ with $s=g(s)$ and $g'(s)\neq 1$. Let $G(x)=x-\frac{(g(x)-x)^2}{g(g(x))-2g(x)+x}$. If $x_{k+1}=g(x_k)$ conv. at least linearly to s, then $y_{k+1}=G(y_k)$ conv. at least quadratically to s. If $x_{k+1}=g(x_k)$ conv. with order $p>1$, then $y_{k+1}=G(y_k)$ conv. with order $2p-1$.

Obs. Aitken works on the original sequence, Steffensen builds the accelerated sequence.

Thm. (Fixed point) $T\subset\mathbb{R}^n$ closed, $G\colon T\to\mathbb{R}^n$ s.t.:
(i) $G(T)\subset T$ (contractive)
(ii) $\exists L\in(0,1)$ constant s.t. $\|G(x_1)-G(x_2)\|\leq L\|x_1-x_2\|$ $\forall x_1,x_2\in T$ (Lipschitz).

Then:
(a) $\exists!s\in I$ s.t. $s=G(x)$ (unique fixed point)
(b) $\forall x_0\in T$, $\{x_{k+1}\}_{k\geq 0}=\{G(x_k)\}_{k\geq 0}$ converges to s.
(c) $\|x_k-s\|\leq\frac{L^k}{1-L}\|x_1-x_0\|$, $\|x_k-s\|\leq\frac{L}{1-L}\|x_k-x_{k-1}\|$

Obs. (Newton for nonlinear systems) Let $f\colon\mathbb{R}^n\to\mathbb{R}^n$, given $x_k\in\mathbb{R}^n$, take Taylor $f(x)\approx f(x_k)+Df(x_k)(x-x_k)$. Let $\Delta x_k=x_{k+1}-x_k$, choose x_{k+1} s.t. $f(x)=0$. Then solve $Df(x_k)\Delta x_k=-f(x_k)$ and $x_{k+1}=x_k+\Delta x_k$.

Obs. If $p(x)\in\mathbb{R}_n[x]$, we write $p(x)=\sum_{k=0}^n a_kx^{n-k}$.

Obs. (Horner's rule) Let $p(x)=\sum_{k=0}^n a_kx^{n-k}$, group it as $p(x)=(\cdots(a_0x+a_1)x+a_2)x+\cdots a_{n-2})x+a_{n-1})x+a_n$. We want $p(\alpha)$. Define $b_0=a_0$, $b_i=b_{i-1}\alpha+a_i$, $i=1\div n$. Then $p(\alpha)=b_n$.

Obs. (Synthetic division) Horner's rule performs the division of $p(x)$ by $x-\alpha$, $p(x)=(x-\alpha)q(x)+r_p$ where $q(x)=\sum_{k=0}^{n-1}b_kx^{n-1-k}$ and $r_p=b_n=p(\alpha)$.

Obs. (Derivatives at α) Expand p in Taylor around α, $p(x)=p(\alpha)+p'(\alpha)(x-\alpha)+\cdots\frac{p^{(n)}(\alpha)}{n!}(x-\alpha)^n$. Then $q(x)=\frac{p(x)-p(\alpha)}{x-\alpha}=p'(\alpha)+(x-\alpha)\sum_{j=2}^n\frac{p^{(j)}(\alpha)}{j!}(x-\alpha)^{j-2}$. Synthetic division of $q(x)\Rightarrow q(x)=(x-\alpha)\sum_{k=0}^{n-2}c_kx^{n-2-k}+c_{n-1}$ and $q(\alpha)=p'(\alpha)=c_{n-1}$. Iterate:

$$\begin{array}{llll} b_0=a_0 & b_i=b_{i-1}\alpha+a_i & i=1\div n & p(\alpha)=b_n\\ c_0=b_0 & c_i=c_{i-1}\alpha+b_i & i=1\div n-1 & p'(\alpha)=c_{n-1}\\ d_0=c_0 & d_i=d_{i-1}\alpha+c_i & i=1\div n-2 & p''(\alpha)=2!\,d_{n-2}\\ e_0=d_0 & e_i=e_{i-1}\alpha+d_i & i=1\div n-3 & p'''(\alpha)=3!\,e_{n-3}\end{array}$$

Prop. (Laguerre's rule) Let $L>0$ and perform synthetic division of $p(x)$ by $x-L$, $p(x)=(x-L)\sum_{k=0}^{n-1}b_kx^{n-1-k}+b_n$. If $b_i>0$ (or $b_i<0$) $\forall i$, then L is an upper bound for the real roots of $p(x)$.

Prop. (Newton's rule) If $p(x)\in\mathbb{R}_n[x]$ and $L\in\mathbb{R}$ satisfies that $p(L)$, $p'(L),\ldots,p^{(n)}(L)$ are positive (or negative), then L is an upper bound for the real roots of $p(x)$.

Prop. Let $p(x)\in\mathbb{C}_n[x]$ and $z\in\mathbb{C}$ be a root of $p(x)$. Then $|z|\leq\max\left\{1,\sum_{i=1}^n\left|\frac{a_i}{a_0}\right|\right\}$.

Prop. Let $p(x)\in\mathbb{C}_n[x]$ and $z\in\mathbb{C}$ be a root of $p(x)$. Then $|z|\leq 1+\max\left\{\left|\frac{a_1}{a_0}\right|,\cdots,\left|\frac{a_{n-1}}{a_0}\right|\right\}$.

Obs. Bounds for other real roots of $p(x)$:

- Lower bound for positive roots: $\overline{p}(x):=p\left(\frac{1}{x}\right)$, $x\neq 0$. $\overline{L}>0$ upper bound for positive real roots of $\overline{p}(x)\Rightarrow\frac{1}{\overline{L}}$ lower bound for positive real roots of $p(x)$.
- Lower bound for negative roots: $\overline{p}(x):=p(-x)$. $\overline{L}>0$ upper bound for real roots of $\overline{p}(x)\Rightarrow-\overline{L}$ lower bound for negative real roots of $p(x)$.
- Upper bound for negative roots: $\overline{p}(x):=p\left(-\frac{1}{x}\right)$. $\overline{L}>0$ upper bound for real roots of $\overline{p}(x)\Rightarrow-\frac{1}{\overline{L}}$ upper bound for negative real roots of $p(x)$.

Polynomial interpolation of functions

Grid (x_k,f_k), $k=0\div n$ of $n+1$ points, $x_i\neq x_j\ \forall i\neq j$

Thm. (Existence and uniqueness) $\exists!\,p_n(x)$ polynomial of degree $\leq n$ that interpolates (x_k,f_k), i.e. $p_n(x_k)=f_k$, $k=0\div n$. Idea: $p_n(x)=\sum_{i=0}^n\left(C_i\prod_{j=0}^{i-1}(x-x_j)\right)$.

Def. $\langle x_0,\ldots,x_n\rangle:=(\min\{x_0,\ldots,x_n\},\max\{x_0,\ldots,x_n\})$.

Thm. (Error formula) If $f\in\mathscr{C}^{n+1}(a,b)$, $x_k\in(a,b)$ $k=0\div n$ and $p_n(x)$ is the interpolating polynomial, then $f(x)-p_n(x)=\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)\ldots(x-x_n)$ with $\xi(x)\in\langle x_0,\ldots,x_n,x\rangle$.

Def. (Lagrange pol.) $\ell_k(x)=\prod_{i\neq k}(x-x_i)/\prod_{i\neq k}(x_k-x_i)$, where $\ell_k(x_j)=\delta_{jk}$.

Obs. (Lagrange method) If $p_n(x)$ is the interpolating polynomial of (x_k,f_k) with $k=0\div n$, then $p_n(x)=\sum_{k=0}^n f_k\ell_k(x)$.

Def. Grid (x_k,f_k) with $k=0\div n$, define $f[x_i]:=f_i$ and $f[x_i,\ldots,x_{i+j+1}]=\frac{f[x_{i+1},\ldots,x_{i+j},x_{i+j+1}]-f[x_i,x_{i+1},\ldots,x_{i+j}]}{x_{i+j+1}-x_i}$.

Obs. (Newton method) If $p_n(x)$ is the interpolating polynomial of (x_k,f_k) with $k=0\div n$, then $C_i=f[x_0,\ldots,x_i]$ $i=0\ldots n$ and $p_n(x)=\sum_{i=0}^n\left(C_i\prod_{j=0}^{i-1}(x-x_j)\right)$.

Prop. $\forall\sigma\in S_{j+1}$ permutation of the symmetric group of degree $j+1$, $f[x_0,\ldots,x_j]=f[x_{\sigma(0)},\ldots,x_{\sigma(j)}]$.

Def. Grid (x_k,f_k) $k=0\div n$, define $\Delta^0 f(x_k)=f(x_k)$, $\Delta f(x_k)=f(x_{k+1})-f(x_k)$, $\Delta(f(x_i)-f(x_k))=\Delta f(x_i)-\Delta f(x_k)$, $\ldots$, $\Delta^n f(x_k)=\Delta^{n-1}(\Delta f(x_k))$.

Prop. If $h\in\mathbb{R}$ and we have the equally spaced grid $\{x_0,\ x_1=x_0+h,\ \ldots,x_n=x_0+nh\}$, then $f[x_0,\ldots,x_k]=\frac{1}{k!h^k}\Delta^k f(x_0)$.

Inverse interpolation

We know (x_k,f_k) $k=0\div n$, we want to solve $f(x)=c$. Assume f invertible and compute $x=g(c)$, $g=f^{-1}$. Interpolate $(y_i,g_i)=(f_i,x_i)$ $i=0\div n$ by an interpolating polynomial $q_n(y)$ and take $x_{\text{approx}}=q_n(c)$.From the error formula,

$$\max_{x\in[a,b]}|f(x)-p_n(x)|\leq$$
$$\leq\max_{x\in[a,b]}\left\{\frac{\left|f^{(n+1)}(x)\right|}{(n+1)!}\right\}\cdot\max_{x\in[a,b]}\left|\prod_{i=0}^n(x-x_i)\right|$$

We want to minimize $\max_{x\in[a,b]}|x-x_0|\cdots|x-x_n|$.

Thm. The best choice of $y_0,\dots,y_n \in [-1,1]$ to minimize $\max_{y\in[a,b]}|y-y_0|\cdots|y-y_n|$ is given by the roots of the Chebyshev polynomials of degree $n+1$ and equals $\frac{1}{2^n}$.
Def. Chebyshev pol. degree n: $T_n(x)=\cos(n\arccos(x))$.
Prop. For Chebyshev polynomials:

(i) $T_0(x)=1$, $T_1(x)=x$, $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$
(ii) Coefficient of x^n in $T_n(x)$ is 2^{n-1}
(iii) $T_n(x)$, $n\ge 1$ has n zeros in $[-1,1]$, of the form $x_k=\cos\left(\frac{2k+1}{2}\frac{\pi}{2}\right)$ $k=0\div n-1$
(iv) $T_n(x)$, $n\ge 1$ has $n+1$ extrema in $[-1,1]$, of the form $\overline{x}_k=\cos\left(\frac{k\pi}{n}\right)$ $k=0\div n$, $T_n(\overline{x}_k)=(-1)^k$.

Thm. Let $P_{n+1}=\sum_{i=0}^{n+1}a_ix^i$ be any monic polynomial of degree $n+1$ and $m=\max_{x\in[-1,1]}|P_{n+1}(x)|$. Then $\frac{T_{n+1}(x)}{2^n}$ is monic of degree $n+1$ and minimizes m, and $\min_{P_{n+1}(x)} m=\frac{1}{2^n}$.

Hermite interpolation

Goal: find a polynomial matching $f(x)$ and $f'(x)$ on the grid (x_k,f_k), (x_k,f'_k) $k=0\div m$.
Prop. The unique polynomial interpolating $f(x)$ and $f'(x)$ at (x_k,f_k), (x_k,f'_k) $k=0\div m$ is the Hermite polynomial of degree $2m+1$, $H_{2m+1}(x)=\sum_{i=0}^m f_i\phi_i(x)+\sum_{i=0}^m f_i\psi_i(x)$ where:

(i) $\phi_i(x)=[1-2\ell'_i(x_i)(x-x_i)]\,\ell_i^2(x)$
(ii) $\psi_i(x)=(x-x_i)\ell_i^2(x)$

with $\ell_i(x)$ the Lagrange interpolating polynomial of degree m.
Prop. (Error formula) $I\subset\mathbb{R}$ and $f\in\mathscr{C}^{2m+2}(I)$ with $x_k\in I$ for $k=0\div m$. $f(x)-H_{2m+1}(x)=\frac{f^{(2m+2)}(\xi(x))}{(2m+2)!}(x-x_0)^2\cdots(x-x_m)^2$ $\forall x\in I$ with $\xi(x)\in\langle x_0,\dots,x_m,x\rangle$.
Def. (Generalized divided differences) Given x_i and f, $\lim_{x\to x_i}f[x_i,x]=\lim_{x\to x_i}\frac{f(x)-f(x_i)}{x-x_i}=f'(x_i)=f[x_i,x_i]$.
Prop. Computation scheme for $H_{2m+1}(x)$:

$$\begin{aligned}H_{2m+1}(x)&=f_0+f[x_0,x_0](x-x_0)+f[x_0,x_0,x_1](x-x_0)^2+\\&+f[x_0,x_0,x_1,x_1](x-x_0)^2(x-x_1)+\dots\\&+f[x_0,x_0,\dots,x_m,x_m](x-x_0)^2\cdots(x-x_{m-1})^2(x-x_m)\end{aligned}$$

Spline interpolation

Set of points $x_0<x_1<\cdots<x_n$ and $f_k=f(x_k)$ $k=0\div n$.
Def. A spline $s(x)$ of degree p interpolating f at the nodes x_i $i=0\div n$ satisfies:

(i) Interpolates: $s(x_i)=f_i$ $i=0\div n$
(ii) $\forall[x_i,x_{i+1}]$ $i=0\div n-1$, $s(x)$ is a polynomial of degree p
(iii) $s(x)\in\mathscr{C}^{p-1}([x_0,x_n])$

In practice, natural cubic splines are used ($s''(x_0)=s''(x_n)=0$).

Obs. We build natural cubic splines. Define $s_i(x)=s(x)|_{[x_i,x_{i+1}]}$ $i=0\div n-1$. We need $4n$ coefficients.

$$\left.\begin{aligned}s_i(x_i)&=f_i, & i&=0\div n-1\\ s_{n-1}(x_n)&=f_n\\ s_i(x_{i+1})&=s_{i+1}(x_{i+1}) & i&=0\div n-2\\ s'_i(x_{i+1})&=s'_{i+1}(x_{i+1}) & i&=0\div n-2\\ s''_i(x_{i+1})&=s''_{i+1}(x_{i+1}) & i&=0\div n-2\end{aligned}\right\}\ 4n-2\text{ conditions}$$

Impose $s''(x_0)=s''(x_n)=0$ (natural) $\Rightarrow 4n$ conditions. Define: $h_i=x_{i+1}-x_i$ $i=0\div n-1$, $M_i=s''(x_i)$ $i=0\div n$. We have $s''_i(x)=M_i+\frac{M_{i+1}-M_i}{h_i}(x-x_i)$ $i=0\div n-1$. Integrate twice between x_i and x: $s_i(x)=M_i\frac{(x-x_i)^2}{2}+\frac{M_{i+1}-M_i}{h_i}\frac{(x-x_i)^3}{6}+B_i(x-x_i)+A_i$. Imposing $s_i(x_i)=f_i=A_i$
$A_i=f_i=s_i(x_i)$ $i=0\div n$

$$B_i=\frac{f_{i+1}-f_i}{h_i}-(M_{i+1}-M_i)\frac{h_i}{6}-\frac{M_ih_i}{2}\quad i=0\div n-1$$

$$h_iM_i+2(h_i+h_{i+1})M_{i+1}+h_{i+1}M_{i+2}=6d_{i+1}\quad i=0\div n-2$$

$$d_{i+1}=\frac{f_{i+2}-f_{i+1}}{h_{i+1}}-\frac{f_{i+1}-f_i}{h_i}\quad i=0\div n-2$$

$M_0=M_n=0$
The d_i are ordinary differences. We have the tridiagonal system $T[M_1\cdots M_{n-1}]'=[d_1\cdots d_{n-1}]'$ where

$$T=\begin{bmatrix}2(h_0+h_1) & h_1 & 0 & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ 0 & h_2 & 2(h_2+h_3) & h_3 & \\ \vdots & \vdots & \ddots & \ddots & \ddots\end{bmatrix}$$

Compute M_i, then B_i and $s_i(x)$.

Trigonometric interpolation

Interpolate a 2π-periodic function $f\colon[0,2\pi]\to\mathbb{R}$ by a trigonometric interpolating polynomial $\hat f$ of f at $x_j=\frac{2\pi}{n+1}j$ $j=0\div n$. We use $e^{ikx}=\cos(kx)+i\sin(kx)$. The interpolating polynomial is

$$\hat f(x)=\frac{a_0}{2}+\sum_{k=1}^{M+\mu}(a_k\cos(kx)+b_k\sin(kx))=\sum_{k=-M-\mu}^{M+\mu}c_ke^{ikx}$$

If n is even, $\mu=0$, $M=\frac{n}{2}$. If n is odd, $\mu=1$, $M=\frac{n-1}{2}$ and we impose $c_{M+1}=c_{-(M+1)}$ ($\Rightarrow a_{M+1}=2c_{M+1}$, $b_{M+1}=0$). Coefficient computation:

$$c_l=\frac{1}{n+1}\sum_{j=0}^n f(x_k)e^{-ijhl}$$

$$a_k=c_k+c_{-k}=\frac{2}{n+1}\sum_{j=0}^n f(x_k)\cos(jhk)$$

$$b_k=i(c_k-c_{-k})=\frac{2}{n+1}\sum_{j=0}^n f(x_k)\sin(jhk)$$

$$a_{M+1}=2c_{M+1}=\frac{2}{n+1}\sum_{j=0}^n f(x_j)e^{-ijh(M+1)}\qquad b_{M+1}=0$$

Function approximation

Given $\varphi_0(x),\dots,\varphi_n(x)$ l.i. on I, $\mathcal{F}_n=\{f_n=a_0\varphi_0(x)+\dots+a_n\varphi_n(x)\mid a_0,\dots,a_n\in\mathbb{R}\}$. Given f on I, we want $f_n^*\in\mathcal{F}_n$ s.t. $\|f-f_n^*\|_2=\min_{f_n\in\mathcal{F}_n}\|f-f_n\|_2$. $\|f\|$.
Thm. Given I and l.i. $\{\varphi_i(x)\}_{i=0,\dots,n}$, $\exists! f_n^*=\sum_{k=0}^n a_k^*\varphi_k(x)$ minimizing $\|f-f_n\|_2$ in $\mathcal{F}_n$.
Obs. From LA we know f_n^* is characterized by $\langle f-f_n^*,\varphi_i\rangle=0$.
From $\{\varphi_j\}_{j=0,\dots,n}$ we obtain orthogonal $\{\psi_j\}_{j=0,\dots,n}$ (Gram-Schmidt). Setting up the system

$$(\varphi_0(x),\dots,\varphi_n(x))=(\psi_0(x),\dots,\psi_n(x))\begin{pmatrix}r_{00} & r_{01} & \cdots & r_{0n}\\ & r_{11} & \cdots & r_{1n}\\ & & \ddots & \vdots\\ & & & r_{nn}\end{pmatrix}$$

with $r_{ij}=\frac{\langle\varphi_j,\psi_i\rangle}{d_i}$, where $d_i=\langle\psi_j,\psi_j\rangle$. We get $\psi_0(x)=\varphi_0(x)$, $\psi_j(x)=\varphi_j(x)-\sum_{i=0}^{j-1}\frac{\langle\varphi_j,\psi_i\rangle}{\langle\psi_i,\psi_i\rangle}\psi_i(x)$, $j=1\div n$.
Obs. Procedure: from $\{\varphi_j\}$ compute $\{\psi_j\}$, compute $\langle\psi_i,\psi_i\rangle$, solve the normal-eqs. LS system; now diagonal $c_i^*=\frac{\langle\psi_i,f\rangle}{\langle\psi_i,\psi_i\rangle}$.
$f_n^*=c_0^*\psi_0(x)+\dots+c_n^*\psi_n(x)$.
Prop. If $\psi_0(x),\dots,\psi_n(x)$ are orthogonal, $f_n^*(x)=\sum_{j=0}^n c_j^*\psi_j(x)$ is the least-squares function, and $\|f-f_n^*\|^2=\|f\|^2-\|f_n^*\|^2$.
Obs. The normal-eqs. LS system can be ill-conditioned. We want a diagonal LS system.

Polynomial approximation case: orthogonal polynomials

Prop. We have:

- $\langle xu,v\rangle=\langle u,xv\rangle\quad\forall u,v\in\mathcal{P}_n$.
- Given $\psi_0,\dots,\psi_i$ with degrees $\{\psi_j(x)\}=j$, orthogonal $\{\psi_j\}_j$ ($\Rightarrow$ l.i.), any polynomial p_i of degree i can be written uniquely as $p_i(x)=c_0\psi_0(x)+\dots+c_i\psi_i(x)$.
- $\langle\psi_j,p_i\rangle=0\quad\forall p_i(x)$ s.t. $i<j$.

Prop. Least-squares approx. solution given by $p_n^*(x)=\sum_{j=0}^n c_j^*\psi_j(x)$ where c_j^* is the solution of the normal-eqs. LS system (now diagonal) $c_j^*=\frac{\langle\psi_j,f\rangle}{\langle\psi_j,\psi_j\rangle}$, $j=0\div n$. Orthogonal polynomials $\psi_j(x)$ are given by the recurrence

$$\text{(RAPO)}\begin{cases}\psi_0(x)=A_0 & (\psi_{-1}(x)=0)\\ \psi_{j+1}(x)=\alpha_j(x-\beta_j)\psi_j(x)-\gamma_j\psi_{j-1}(x), & j\ge 0.\end{cases}$$

with

$$\begin{cases}\alpha_j=\frac{A_{j+1}}{A_j}, & j\ge 0\\ \beta_j=\frac{\langle\psi_j,x\psi_j\rangle}{\langle\psi_j,\psi_j\rangle}=\frac{\langle\psi_j,x\psi_j\rangle}{d_j}, & j\ge 0\\ \gamma_j=\frac{\alpha_j}{\alpha_{j-1}}\frac{\langle\psi_j,\psi_j\rangle}{\langle\psi_{j-1},\psi_{j-1}\rangle}=\frac{\alpha_jd_j}{\alpha_{j-1}d_{j-1}}, & j\ge 1\end{cases}$$