Numerical Solution of the Incompressible Navier-Stokes Equations

The incompressible Navier-Stokes equations describe a wide range of problems in fluid mechanics. They are composed of an equation in mass conservation and two momentum conservation equations, one for each Cartesian velocity component. The dependent variables will be the pressure *p* and the velocity components *u* and *v* in the *x* and *y* directions respectively. The use of numerical methods to solve the governing equations will follow the same methods as used in the previous discussion on numerical solution of non-linear equations. The three governing equations in non-dimensional conservative form are,

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{\partial u^*}{\partial t^*} + \frac{\partial}{\partial x^*}(u^{*2} + p^*) + \frac{\partial}{\partial y^*}(u^* v^*) = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$\frac{\partial v^*}{\partial t^*} + \frac{\partial}{\partial x^*}(u^* v^*) + \frac{\partial}{\partial y^*}(v^{*2} + p^*) = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

The variables have been non-dimensionalized using the free stream velocity V_∞, density ρ_∞, viscosity μ_∞, and a length scale *L*, shown below.

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad t^* = \frac{tV_\infty}{L}, \quad u^* = \frac{u}{V_\infty}, \quad v^* = \frac{v}{V_\infty}, \quad p^* = \frac{p}{\rho_\infty V_\infty^2}$$

$$\mathrm{Re} = \frac{\rho_\infty V_\infty L}{\mu_\infty}$$

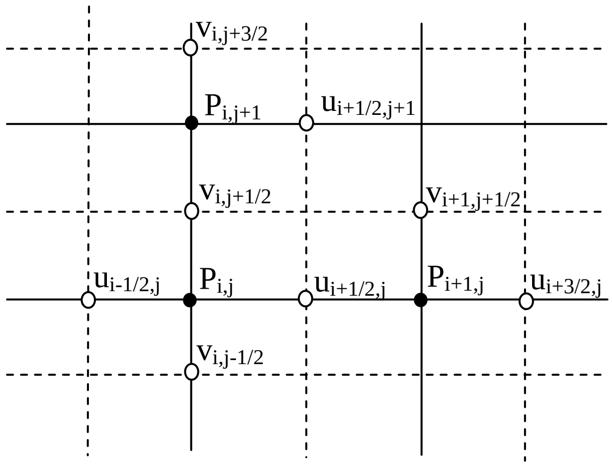
The superscript denoting non-dimensional format will be subsequently dropped from here on. Unlike the compressible Navier-Stokes equations, the incompressible form does not have a time dependent term in the mass conservation equation. In the compressible form, the time dependent term provided a direct reference to density, allowing for a numerical scheme to solve for density directly. In the incompressible for, the momentum equations can be used to solve for *u* and *v*, but the continuity equation does not reference *p* because of the absence to this term. An artificial compressibility term must be added to the continuity equation to allow for the solution of *p*, shown below.

$$\frac{\partial p}{\partial t} + a^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

The term *a* denotes a pseudo speed of sound. As the numerical scheme progresses towards a steady state solution, the new time dependent term tends to zero magnitude. This implies that in the steady state the original continuity equation is recovered.

Explicit Solution

Explicit solution of the incompressible Navier-Stokes is relatively straightforward given the previous notes on non-linear terms. Numerous methodologies make be used, forward time- central space (FTCS), Dufort Frankel, MacCormack, Crank-Nicolson to name a few. If central differencing is used on the convective terms, artificial dissipation may be needed to ensure stability if the Reynolds number is fairly large. Another method of improving stability for incompressible flows is with the use of a staggered grid. An example of a staggered grid is shown below. This grid arrangement provides stronger coupling between the pressure and velocity variables thus improving stability. The original primary grid is denoted with solid lines while the secondary grid is shown with dashed lines. The pressure is assigned to the nodes on the primary while the velocities are defined on the secondary nodes. More specifically, the u velocity is defined on the 1/2 grid line between the nodes on the primary in the x direction. While the v velocity is at the 1/2 lines in the y direction, as shown.



An explicit algorithm based on the staggered grid as shown can begin with the discretization of the momentum equations using a first order difference in time and a central scheme in space. The velocity components are solved first using,

$$\frac{u^{n+1}_{i+1/2,j}-u^{n}_{i+1/2,j}}{\Delta t}+\frac{\left(u^2\right)^n_{i+1,j}-\left(u^2\right)^n_{i,j}}{\Delta x}+\frac{p^n_{i+1,j}-p^n_{i,j}}{\Delta x}+\frac{(uv)^n_{i+1/2,j+1/2}-(uv)^n_{i+1/2,j-1/2}}{\Delta y}$$

$$=\frac{1}{\mathrm{Re}}\left[\frac{u^n_{i-1/2,j}-2u^n_{i+1/2,j}+u^n_{i+3/2,j}}{\Delta x^2}+\frac{u^n_{i+1/2,j-1}-2u^n_{i+1/2,j}+u^n_{i+1/2,j+1}}{\Delta y^2}\right]$$

$$\frac{v^{n+1}_{i,j+1/2}-v^{n}_{i,j+1/2}}{\Delta t}+\frac{\left(v^2\right)^n_{i,j+1}-\left(v^2\right)^n_{i,j}}{\Delta y}+\frac{p^n_{i,j+1}-p^n_{i,j}}{\Delta y}+\frac{(uv)^n_{i+1/2,j+1/2}-(uv)^n_{i-1/2,j+1/2}}{\Delta x}$$

$$=\frac{1}{\mathrm{Re}}\left[\frac{v^n_{i-1,j+1/2}-2v^n_{i,j+1/2}+v^n_{i+1,j+1/2}}{\Delta x^2}+\frac{v^n_{i,j-1/2}-2v^n_{i,j+1/2}+v^n_{i,j+3/2}}{\Delta y^2}\right]$$

Once the velocity components are computed for the time level *n+1*, the modified continuity expression can be solved for the pressure,

$$\frac{p^{n+1}_{i,j}-p^n_{i,j}}{\Delta t}+a^2\left(\frac{u^{n+1}_{i+1/2,j}-u^{n+1}_{i-1/2,j}}{\Delta x}+\frac{v^{n+1}_{i,j+1/2}-v^{n+1}_{i,j-1/2}}{\Delta y}\right)=0$$

Since the velocities are only available on the secondary grid, an interpolation must be used to obtain velocities at points on the primary grid. The following approximations can be used,

$$(u^2)_{i+1,j}=\frac{1}{4}(u_{i+3/2,j}+u_{i+1/2,j})^2$$

$$(u^2)_{i,j}=\frac{1}{4}(u_{i+1/2,j}+u_{i-1/2,j})^2$$

$$(v^2)_{i,j+1} = \frac{1}{4}(v_{i,j+3/2} + v_{i,j+1/2})^2$$

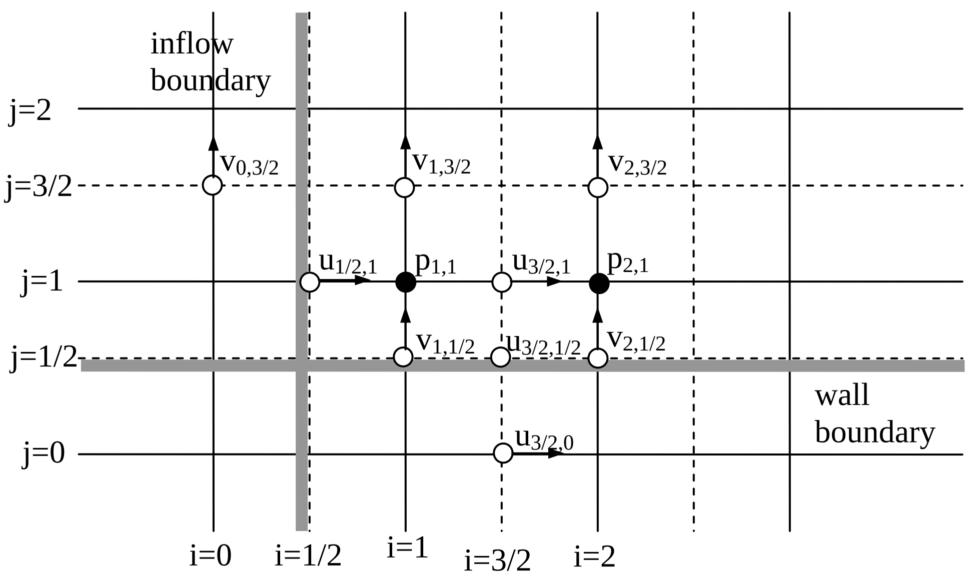
$$(v^2)_{i,j} = \frac{1}{4}(v_{i,j+1/2} + v_{i,j-1/2})^2$$

$$(uv)_{i+1/2,j+1/2} = \frac{1}{4}(u_{i+1/2,j} + u_{i+1/2,j+1})(v_{i,j+1/2} + v_{i+1,j+1/2})$$

$$(uv)_{i+1/2,j-1/2} = \frac{1}{4}(u_{i+1/2,j} + u_{i+1/2,j-1})(v_{i,j-1/2} + v_{i+1,j-1/2})$$

$$(uv)_{i-1/2,j+1/2} = \frac{1}{4}(u_{i-1/2,j} + u_{i-1/2,j+1})(v_{i,j+1/2} + v_{i-1,j+1/2})$$

In a large number of cases a pressure boundary condition will not be available at a given boundary. This means that the pressure at a boundary will be unknown. In this instance, the algorithm just outlined, often called the Marker and Cell approach, will alleviate the problem. If the boundaries are selected to coincide with the secondary grid, pressure on the boundaries will not be needed. For example, consider an inflow boundary and a wall, shown below. If the wall is aligned with $j = ½$, and the inflow with $i = ½$, the pressure does not appear on the surface and is calculated only in the interior. However, values of $u_{3/2,0}$ and $v_{0,3/2}$ must be specified.



Since the wall is solid, no slip conditions prevail, therefore,

$$v_{1,1/2} = v_{2,1/2} = v_{3,1/2} = ... = 0$$

$$u_{3/2,1/2} = u_{5/2,1/2} = u_{7/2,1/2} = ... = 0$$

The values such as $u_{3/2,0}$ will be required in the analysis, which can be found from,

$$u_{3/2,1/2} = \frac{1}{2}(u_{3/2,0} + u_{3/2,1}) = 0$$

which provides,

$$u_{3/2,0} = -u_{3/2,1}$$

At the inflow boundary the u velocities are specified directly from the input conditions. Values such as $u_{1/2,1}$, $u_{1/2,2}$ are know from any problem definition and need no special treatment. The v components of velocity outside of the boundary, such as $v_{0,3/2}$, $v_{0,5,2}$ .. can be found from extrapolation,

$$v_{0,3/2} = v_{1,3/2} - \frac{\partial v}{\partial x}\Delta x \quad = v_{1,3/2} - \frac{(v_{2,3/2} - v_{1,3/2})}{\Delta x}\Delta x \quad = 2v_{1,3/2} - v_{2,3/2}$$

At an outflow boundary, one could use the same approach to calculate v or even u. Both the time step size and the value of the pseudo speed of sound define the stability requirement for this method. The time step size is limited by

$$\Delta t \le \frac{2 \cdot \Delta_{\min}}{N^{1/2} a(1+\sqrt{5})}$$

Here, $\Delta_{\min}$ means the smallest Δx or Δy on the grid, N is the number of dimensions in the domain, and a is the speed of sound used in the pseudo continuity equation. This value is defined by the maximum velocity,

$$\frac{V_{\max}}{a} < 1 \qquad\qquad V = \sqrt{u^2 + v^2}$$

Note that it is not necessary to use a staggered grid but it does naturally enforce conservation at the main grid points. Also note that all variables are non-dimensional, meaning that the incoming velocity at the inflow boundary in the above example will be $u/V_\infty = 1$.

Implicit Solution

The implicit solution of the incompressible Navier-Stokes is more easily accomplished if the equations are written in conservative form,

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = \frac{1}{\mathrm{Re}}[M]\nabla^2 Q$$

$$Q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \qquad E = \begin{bmatrix} a^2 u \\ u^2 + p \\ uv \end{bmatrix}, \qquad F = \begin{bmatrix} a^2 v \\ uv \\ v^2 + p \end{bmatrix}, \qquad M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In an implicit scheme, the non-linearity of the flux terms must be addressed if a system of linear algebraic equations is to be developed. Using Newton's method,

$$E^{n+1} = E^n + \frac{\partial E}{\partial t}\Delta t + O(\Delta t^2)$$

$$E^{n+1} = E^n + \frac{\partial E}{\partial Q}\frac{\partial Q}{\partial t}\Delta t + O(\Delta t^2)$$

$$E^{n+1} = E^n + \frac{\partial E}{\partial Q}\frac{\Delta Q}{\Delta t}\Delta t + O(\Delta t^2)$$

$$E^{n+1} = E^n + \frac{\partial E}{\partial Q}\Delta Q + O(\Delta t^2)$$

$$\Delta Q = Q^{n+1} - Q^n$$

The terms ∂E/∂x and ∂F/∂y are Jacobian matrices and will be denoted as A and B from this point on. As an example, matrix A is defined as,

$$A = \begin{bmatrix} \frac{\partial E_1}{\partial Q_1} & \frac{\partial E_1}{\partial Q_2} & \frac{\partial E_1}{\partial Q_3} \\ \frac{\partial E_2}{\partial Q_1} & \frac{\partial E_2}{\partial Q_2} & \frac{\partial E_2}{\partial Q_3} \\ \frac{\partial E_3}{\partial Q_1} & \frac{\partial E_3}{\partial Q_2} & \frac{\partial E_3}{\partial Q_3} \end{bmatrix}$$

These values can be determined by re-writing the original flux vectors in terms of the components of the solution vector, Q_1, Q_2, Q_3. The flux vectors are written in terms of these values and differentiated,

$$Q = \begin{bmatrix} p \\ u \\ v \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \qquad E = \begin{bmatrix} a^2 u \\ u^2 + p \\ uv \end{bmatrix} = \begin{bmatrix} a^2 Q_2 \\ Q_2^2 + Q_1 \\ Q_2 Q_3 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

and

$$\frac{\partial E_1}{\partial Q_1} = 0, \qquad \frac{\partial E_1}{\partial Q_2} = a^2, \qquad \frac{\partial E_1}{\partial Q_3} = 0, \qquad \frac{\partial E_2}{\partial Q_1} = 1, \qquad \frac{\partial E_2}{\partial Q_2} = 2u \quad ....$$

$$A = \begin{bmatrix} 0 & a^2 & 0 \\ 1 & 2u & 0 \\ 0 & v & u \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & v & u \\ 1 & 0 & 2v \end{bmatrix}$$

Inserting the linear approximations into the vector form of the Navier-Stokes equations, and approximating the time derivative term,

$$\frac{\Delta Q}{\Delta t} + \frac{\partial}{\partial x}(E^n + A\Delta Q) + \frac{\partial}{\partial y}(F^n + B\Delta Q) = \frac{1}{\mathrm{Re}}[M]\nabla^2 Q^{n+1}$$

using

$$Q^{n+1} = \Delta Q + Q^n$$

$$\left(I + \Delta t\left[\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} - \frac{M}{\mathrm{Re}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right] \right)\Delta Q = \Delta t\left[-\frac{\partial E^n}{\partial x} - \frac{\partial F^n}{\partial y} + \frac{M}{\mathrm{Re}}\left(\frac{\partial^2 Q^n}{\partial x^2} + \frac{\partial^2 Q^n}{\partial y^2} \right) \right]$$

Using approximate factorization, the equation above in the form Ax=B, can be separated into equations involving *x* and *y* directions involving tri-diagonal matrices

$$\left[I + \Delta t\left(\frac{\partial A}{\partial x} - \frac{M}{\mathrm{Re}}\frac{\partial^2}{\partial x^2} \right) \right]\left[1 + \Delta t\left(\frac{\partial B}{\partial y} - \frac{M}{\mathrm{Re}}\frac{\partial^2}{\partial y^2} \right) \right]\Delta Q = RHS$$

x direction tri-diagonal equation,

$$\left[I+\Delta t\left(\frac{\partial A}{\partial x}-\frac{M}{\mathrm{Re}}\frac{\partial^2}{\partial x^2}\right)+\varepsilon_{2i}(\Delta x^2)\frac{\partial^2}{\partial x^2}\right]\Delta Q^* = RHS-\varepsilon_{4i}\left[(\Delta x^4)\frac{\partial^4}{\partial x^4}+(\Delta y^4)\frac{\partial^4}{\partial y^4}\right]Q^n$$

y direction,

$$\left[I+\Delta t\left(\frac{\partial B}{\partial y}-\frac{M}{\mathrm{Re}}\frac{\partial^2}{\partial y^2}\right)+\varepsilon_{2i}(\Delta y^2)\frac{\partial^2}{\partial y^2}\right]\Delta Q=\Delta Q^*$$

In these two tri-diagonal equations, second order artificial dissipation has been added to the left hand side (LHS) and fourth order to the RHS to ensure stability. The star super script on the ΔQ indicates an intermediate solution preceding the final solution of the *y* equation. A staggered grid is not necessary in this approach due to the coupling afforded by the implicit scheme. If a central difference is applied to all spatial terms the *x* direction equation would become,

$$\Delta Q^*_{i,j}+\frac{\Delta t}{2\Delta x}(A^n_{i+1,j}\Delta Q^*_{i+1,j}-A^n_{i-1,j}\Delta Q^*_{i-1,j})-\frac{M}{\mathrm{Re}}\frac{\Delta t}{\Delta x^2}(\Delta Q^*_{i+1,j}-2\Delta Q^*_{i,j}+\Delta Q^*_{i-1,j})$$

$$+\varepsilon_{2i}(\Delta Q^*_{i+1,j}-2\Delta Q^*_{i,j}+\Delta Q^*_{i-1,j})=$$

$$\Delta t\left[-\frac{E^n_{i+1,j}-E^n_{i-1,j}}{2\Delta x}-\frac{F^n_{i,j+1}-F^n_{i,j-1}}{2\Delta y}+\frac{M}{\mathrm{Re}}\frac{Q^n_{i+1,j}-2Q^n_{i,j}+Q^n_{i-1,j}}{\Delta x^2}+\frac{M}{\mathrm{Re}}\frac{Q^n_{i,j+1}-2Q^n_{i,j}+Q^n_{i,j-1}}{\Delta y^2}\right]$$

$$-\varepsilon_{4i}\left[\left(Q^n_{i-2,j}-4Q^n_{i-1,j}+6Q^n_{i,j}-4Q^n_{i+1,j}+Q^n_{i+2,j}\right)+\left(Q^n_{i,j-2}-4Q^n_{i,j-1}+6Q^n_{i,j}-4Q^n_{i,j+1}+Q^n_{i,j+2}\right)\right]$$

in tri-diagonal form the *x* equation is,

$$\left[-\left(\frac{\Delta t}{2\Delta x}\right)A^n_{i-1,j}-\left(\frac{M}{\mathrm{Re}}\frac{\Delta t}{\Delta x^2}\right)+\varepsilon_{2i}\right]\Delta Q^*_{i-1,j}+\left[I+\left(\frac{2M}{\mathrm{Re}}\frac{\Delta t}{\Delta x^2}\right)-2\varepsilon_{2i}\right]\Delta Q^*_{i,j}$$

$$+\left[\left(\frac{\Delta t}{2\Delta x}\right)A^n_{i+1,j}-\left(\frac{M}{\mathrm{Re}}\frac{\Delta t}{\Delta x^2}\right)+\varepsilon_{2i}\right]\Delta Q^*_{i+1,j}=RHS_{i,j}$$

the *y* equation is

$$\left[-\left(\frac{\Delta t}{2\Delta y}\right)B^n_{i,j-1}-\left(\frac{M}{\mathrm{Re}}\frac{\Delta t}{\Delta y^2}\right)+\varepsilon_{2i}\right]\Delta Q_{i,j-1}+\left[I+\left(\frac{2M}{\mathrm{Re}}\frac{\Delta t}{\Delta y^2}\right)-2\varepsilon_{2i}\right]\Delta Q_{i,j}$$

$$+\left[\left(\frac{\Delta t}{2\Delta y}\right)B^n_{i,j+1}-\left(\frac{M}{\mathrm{Re}}\frac{\Delta t}{\Delta y^2}\right)+\varepsilon_{2i}\right]\Delta Q_{i,j+1}=\Delta Q^*_{i,j}$$

Reference

Hoffmann, K.A., Chiang, S.T., Computational Fluid Dynamics, Vol. 1, Engineering Education System, Wichita, Kansas, 1998, ISBN 0-9623731-1-7.